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LIMITS OF CERTAIN SUBHOMOGENEOUS C^*-ALGEBRAS

Klaus Thomsen

Abstract. — It is shown that the Elliott invariant is a complete invariant for the simple unital C^*-algebras which can be realized as an inductive limit of a sequence of finite direct sums of algebras of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N$ is an arbitrary (finite) set on the circle $\mathbb{T}$ and d is a natural number dividing n. The corresponding range of invariants is identified and the classification result is extended to the non-unital case. A series of results about the structure of these C^*-algebras and the maps between them are also obtained.

Résumé. — On prouve que l'invariant d'Elliott est un invariant complet des C^*-algèbres simples à élément unité qui peuvent être réalisées comme limite inductive d'une suite de sommes finies d'algèbres de la forme

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \cdots, N\},$$

où $\{x_1, x_2, \cdots, x_N\} \subset \mathbb{T}$ est un sous-ensemble arbitraire et d un entier divisant n. On détermine l'ensemble des valeurs prises par l'invariant et on étend la classification aux algèbres sans unité. Par ailleurs on donne une série de résultats sur la structure de ces C^*-algèbres.

Contents

INTRODUCTION

Dette arbejde blev færdiggjort i mindet om Birger Iversen

The purpose of this paper is to introduce a new type of building block into the classification of inductive limit C^*-algebras and show that the Elliott invariant is also a complete invariant for the simple unital C^*- algebras which are inductive limits of finite direct sums of these building blocks. The building blocks we consider are of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d, i = 1, 2, \ldots, N\}$$

where $x_1, x_2, \ldots, x_N$ is an arbitrary finite set of elements on the circle $\mathbb{T}$ and $n, d \in \mathbb{N}$ are natural numbers such that d divides n. Such C^*-algebras will be referred to as *building blocks of type* 2. By taking $d = n$ we just get an ordinary circle algebra, but in general a building block of type 2 will have torsion in its K_1-group. This allows us to introduce torsion in the K_1-group without having more than one kind of building block. This is unlike the approach of Elliott in **[E1]**, where torsion was introduced by adding an additional type of building block, the socalled dimension-drop C^*-algebras. Note that the identity map of the dimension-drop algebra $\{f \in C[0,1] \otimes M_n : f(0), f(1) \in M_d\}$ factors through $\{f \in C(\mathbb{T}) \otimes M_n : f(1), f(-1) \in M_d\}$ which is a building block of type 2. Hence an inductive limit of a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras is also the limit of a sequence of finite direct sums of building blocks of type 2. Therefore the following theorem, which is our main result, unifies and generalizes the classification result for simple unital inductive limits of finite direct sums of circle algebras, **[E3]**, **[NT]**, and for simple real rank zero limits of finite direct sums of (matrix algebras over) dimension-drop C^*-algebras in **[E1]**, **[DL2]**.

THEOREM 0.1. — *Let A and B be simple, unital inductive limits of sequences of finite direct sums of building blocks of type 2. Assume that $\varphi_1\colon K_1(A) \to K_1(B)$ is an isomorphism, $\varphi_0\colon K_0(A) \to K_0(B)$ an isomorphism of partially ordered abelian groups with order units and $\varphi_T\colon T(B) \to T(A)$ an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A), \quad \omega \in T(B).$$

It follows that there is a $*$-isomorphism $\varphi\colon A \to B$ such that $\varphi_* = \varphi_1$ on $K_1(A)$, $\varphi_* = \varphi_0$ on $K_0(A)$ and $\varphi^* = \varphi_T$ on $T(B)$.

The maps r_A and r_B in this theorem are the canonical continuous affine surjections from the tracial state space onto the state space of the K_0-group of A and B, respectively.

Let us emphasize one particular consequence of this result. Consider

$$\{f \in C(\mathbb{T}) \otimes M_n : f(1) \in M_d\},$$

which is clearly a building block of type 2. It has exactly the same Elliott invariant as the circle algebra $C(\mathbb{T}) \otimes M_d$, although the algebra seems to be much closer to $C(\mathbb{T}) \otimes M_n$. It would therefore seem tempting to try to use this kind of building blocks to construct two non-isomorphic simple, unital inductive limits of type I C^*-algebras with the same Elliott invariant. This is not possible by the above theorem, in fact a corollary of it says such inductive limits, build only on these very special building blocks of type 2, will automatically be inductive limits of finite direct sums of circle algebras, and hence be subsumed under existing classification results, **[E3]**, **[NT]**. This observation gives some support to the belief that the Elliott invariant will turn out to be a complete invariant for simple inductive limits of more general sub-homogeneous C^*-algebras. It is very challenging to try for such an extension of the existing classification results because even very elementary sub-homogenous C^*-algebras give rise to simple inductive limits which display features that do not arise by using homogeneous building blocks, see **[ET]**, **[Th5]**, **[Th6]**; specifically, the K_0-group can be an arbitrary unperforated simple, (countable) partially ordered abelian group and the restriction map $r_A\colon T(A) \to SK_0(A)$ an arbitrary continuous affine surjection. However, these phenomena do not show up here since we stick to building blocks of type 2. Indeed, if Elliotts conjecture is true, the simple limits we build must also be inductive limits of a sequence of finite direct sums of homogenous C^*- algebras.

In very broad outline, the method of proof we use here is a combination of the methods developed in **[E1]**, **[Th2]**, **[E2]**, **[E3]**, **[DL2]** and **[NT]**. The key words are eigenvalue functions (or characteristic functions as we prefer to call them), determinants, KK-theory and unitary commutators. This paper is the first to handle a case where all these ingredients come into play at the same time. The KK-theory, which is an indispensable ingredient of the classification result in the (non-simple) real rank zero case, **[DL2]**, and the algebraic K_1-group, in the guise of the unitary group modulo the closure of its commutator subgroup, which is needed to determine the approximate inner equivalence class of maps lifted from the Elliott invariant, **[NT]**, play so prominent a role in the development presented here that it almost seems as a miracle that they do not show up in the classification result. They both leave the stage, elegantly we hope, just before the curtain.

On the way we establish several results which are of interest beyond their role in the proof of the classification result. One is that a simple unital inductive limit of a sequence of building blocks of type 2 is approximately divisible (Theorem 5.1), a notion introduced in **[BKR]** and of crucial importance in the previous classification results based on the Elliott invariant which go beyond the real rank zero case, **[E2]**, **[E3]**, **[NT]**. Another important step is the result that two unital $*$-homomorphisms between building blocks of type 2 are approximately inner equivalent when they agree on the tracial states (Theorem 1.4). At first sight it may seem surprising that no K_1-information is needed to reach this conclusion. It shows that exact equality on traces is a strong assumption, although it is of course a necessary condition. The K_1-information first becomes crucial when we consider, as we must, a case where the two maps only agree approximately on the trace level. A third theorem (Theorem B of Chapter 7) gives sufficient (and necessary) conditions for unital $*$-homomorphisms between unital limits of sums of building blocks of type 2 to be approximately inner equivalent when the domain algebra is simple, and we show that a map between the Elliott invariants of the two algebras can be lifted to a $*$-homomorphism when the target algebra is approximately divisible (Corollary A2 of Chapter 7). In fact, we show that the lift can be chosen to be compatible with any KK-element and any map between the unitary groups modulo the closure of their commutator subgroups, which is consistent with the map between the Elliott invariants (Theorem A of Chapter 7).

In the chapters following Chapter 7, which contains the main results, we prove a series of results which relate to the classification result and which are more or less direct consequences of that result and the methods leading to it. In Chapter 8 we describe the quotient group $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ of approximate inner equivalence classes of automorphisms of A when A is a simple unital limit of sums of building blocks of type 2. The main new feature appearing here, when compared with the previous chapters, is the introduction of the quotient $KL(A,A)$ of $KK(A,A)$. By using this device together with some recent results of Dadarlat and Loring, **[DL3]**, we show that $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is the semi-direct product of the group of automorphisms of the Elliott invariant by an abelian group, specifically that

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \simeq$$
$$[\mathrm{ext}(K_1(A), K_0(A)) \oplus \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)] \rtimes \mathrm{Aut}(\mathcal{E}_A).$$

In this expression the third component, $\mathrm{Aut}(\mathcal{E}_A)$, represents the expected part, namely the group of automorphisms of the Elliott invariant. The first component,

$$\mathrm{ext}(K_1(A), K_0(A)),$$

was discovered by Dadarlat and Loring in the real rank zero case, **[DL3]**, in which case the third piece, $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$, is zero (because $\mathrm{Aff}\, T(A) = \overline{\rho(K_0(A))}$). In the case where A is the limit of sums of circle algebras, $\mathrm{ext}(K_1(A), K_0(A))$ is zero, while

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$$

is zero if and only if A has real rank zero or $K_1(A)$ is a torsion group.

In Chapter 9 we describe the range of the Elliott invariant classified by the main result. The range consists of the quadruples $\big(\Delta, r, G, H\big)$ where Δ is a metrizable Choquet simplex, G is a countable dimension group ($\neq \mathbb{Z}$) with order unit, H a countable abelian group and $r\colon \Delta \to SG$ a continuous affine extreme-point preserving surjection. This characterisation is fairly easily obtained from the work of Villadsen **[V1]**. In order to tie the present work up with previous work dealing with the classification of direct sums of circle algebras and matrix algebras over the dimension-drop C^*-algebras, **[E1]**, **[DL1]** (in the real rank zero case), we show that all the invariants are realized by simple unital inductive limits of sequences of finite direct sums of circle algebras and matrix algebras over dimension drop C^*-algebras. In this way it becomes a corollary of the classification result that any simple unital limit of sums of building blocks of type 2 is also the limit of a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras.

In Chapter 10 we show how to extend the classification result to the non-unital case. While this is fairly straightforward and follows the line laid out in **[Th8]**, the other results from the unital case seem more difficult to generalize. In particular, it is not straightforward to describe $\mathrm{Aut}(A)/\overline{\mathrm{Inn}\, A}$ in the non-unital case, and we make no attempts here.

Finally, in Chapter 11 we have gathered a series of consequences of our main results for the structure of the class of C^*-algebras we consider. They all follow fairly straightforwardly by comparing the classification theorem we obtain here with previous work of others, except for the following result which is also of interest for other classes of C^*-algebras. Namely, we prove that the non-stable K-theory is trivial for all unital approximately divisible C^*-algebras, in the sense that the homotopy groups of the unitary group of such a C^*-algebra agree with the K-theory of the algebra, or equivalently, that the unitary group is homotopy equivalent to the 'unitary group' of the stabilized C^*-algebra, see Theorem 11.6.

The first seven chapters of this paper has been circulated in preprint form with the title "Limits of certain subhomogeneous C^*-algebras I".

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CHAPTER 1

THE BUILDING BLOCKS

Let $n, d_1, d_2, \dots, d_N \in \mathbb{N}$ be natural numbers such that d_i divides n for all i. Then M_{d_i} can be considered as a unital C^*-subalgebra of M_n. Let X be either the interval $[0,1]$ or the circle $\mathbb{T}$. Let $x_1, x_2, \dots, x_N$ be distinct points in X. Set

$$A = A(n, d_1, \dots, d_N) = \{f \in C(X) \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \dots, N\}.$$

We are going to consider the following cases:

- $X = \mathbb{T}$ - in this case we call A *a building block of type* 1.
- $X = \mathbb{T}$ and $d_1 = d_2 = \dots = d_N = d$ - in this case we call A *a building block of type* 2.
- $X = [0,1]$ - in this case we call A *a building block of type* 3.
- $X = [0,1]$ and $d_1 = d_2 = \dots = d_N = d$ - in this case we call A *a building block of type* 4.

In all cases the points $x_j \in X, j = 1, 2, \dots, N$, where the dimension of the fiber drops, will be called *the exceptional points* of A.

Let A be a building block of type 1. By renumbering the x_i's if necessary, we can assume that there are points $t_1 < t_2 < \dots < t_N$ in $]0,1]$ such that $x_k = e^{2\pi i t_k}, k = 1, 2, \dots, N$, and we can identify A with

$$\{f \in C[0,1] \otimes M_n : f(t_i) \in M_{d_i}, i = 1, 2, \dots, N, f(0) = f(1)\}.$$

Thus A is in a natural way a C^*-subalgebra of a building block of type 3. Similarly, if A is a building block of type 2, then can identify it with a C^*-subalgebra of a building block of type 4. In this picture the exceptional points are $t_1, t_2, \dots, t_N \in [0,1]$.

A building block A (of any type) comes equipped with N inequivalent irreducible representations, $\Lambda_j^A \colon A \to M_{d_j}$, with kernel $\{f \in A : f(x_j) = 0\}$, $j = 1, 2, \dots, N$. These representations will be called the *exceptional representations* of A and they shall play an important role in the following. When no confusion can arise from it, we will omit the superscript A and just write Λ_j.

Now we consider two building blocks of type 1,

$$A = A(n, d_1, \ldots, d_N) \quad \text{and} \quad B = A(m, e_1, \ldots, e_M),$$

with exceptional points $x_1, x_2, \ldots, x_N \in \mathbb{T}$ and $t_1, t_2, \ldots, t_M \in [0,1]$, respectively, and a unital $*$-homomorphism $\varphi\colon A \to B$ between them. Let z denote the identity function on $\mathbb{T}$ which we can consider as the canonical unitary generator of the center of a building block of type 1. There are then continuous functions $\lambda_i\colon [0,1] \to \mathbb{T}$ such that $\{\lambda_i(t) : i = 1, 2, \ldots, m\}$ are the eigenvalues of $\varphi(z)(t)$, counting multiplicities, for all $t \in [0,1]$, cf. **[K]**, Theorem II 5.2. For each $t \in [0,1]$, let

$$M_k^t = \#\left\{i : \lambda_i(t) = x_k\right\}, \quad k = 1, 2, \ldots, N,$$

and note that M_k^t is divisible by d_k. Indeed, if a_k^t denotes the multiplicity of Λ_k^A in the representation $A \ni f \mapsto \varphi(f)(t)$, then $a_k^t d_k = M_k^t, k = 1, 2, \ldots, N$. We let $r_k^t \in \{0, 1, 2, \ldots, n/d_k - 1\}$ denote the remainder obtained by dividing M_k^t/d_k by n/d_k, i.e. we write $M_k^t/d_k = m_k^t n/d_k + r_k^t$, $m_k^t \in \mathbb{N}$.

LEMMA 1.1. — *For each $k \in \{1, 2, \ldots, N\}$, the function $t \mapsto r_k^t$ is constant.*

Proof. — Let $t \in [0,1]$ and choose $\delta > 0$ so small that $\delta < |a - b|$ for any pair a, b of distinct elements from $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)\} \cup \{x_1, x_2, \ldots, x_N\}$. Let $g\colon \mathbb{T} \to [0,1]$ be a continuous function such that $g(\lambda) = 1$ when $|\lambda - x_k| < \delta/4$ and $g(\lambda) = 0$ when $|\lambda - x_k| > \delta/2$, and consider g as a central element of A. Then $\varphi(g)(s)$ is a projection in M_m for all s in a neighbourhood V of t, and by continuity the rank of $\varphi(g)(s)$ is the same as of $\varphi(g)(t)$. Thus we have that

$$M_k^s + X_s = M_k^t, \ s \in V,$$

where $X_s = \#\left\{i : \lambda_i(t) = x_k, \lambda_i(s) \neq x_k\right\}$. The crucial observation is that X_s must be divisible by n, indeed

$$X_s \quad = \sum_{\lambda \in \mathbb{T} \setminus \{x_1, \ldots, x_N\}} a_\lambda n,$$

where a_λ is the multiplicity of the representation $f \mapsto f(\lambda)$ in $f \mapsto \varphi(fg)(s)$. Hence

$$\frac{M_k^t}{d_k} - \frac{M_k^s}{d_k} = \frac{X_s}{d_k}$$

is divisible by n/d_k, and $r_k^s = r_k^t$ for all $s \in V$. Thus $t \mapsto r_k^t$ is locally constant and hence constant. $\square$

We denote the constant value of $r_k^t, t \in [0,1]$, by r_k^φ. For every $x \in \mathbb{T}\setminus\{x_1, x_2, \dots, x_N\}$ the number $\#\{i : \lambda_i(t) = x\}$ must be divisible by n (for all $t \in [0,1]$). It follows that $m - \sum_{i=1}^N M_i^t$ is divisible by n. Thus

$$m - \sum_{i=1}^{N} r_i^\varphi d_i = m - \sum_{i=1}^{N} M_i^t + n \sum_{i=1}^{N} m_i^t$$

is n-divisible and we set

$$N_\varphi = \frac{m - \sum_{i=1}^N r_i^\varphi d_i}{n}.$$

For each $i \in \{1, 2, \dots, N\}$ and each $k \in \mathbb{N}$, we denote by Λ_i^k the direct sum representation of k copies of Λ_i $(= \Lambda_i^A)$.

LEMMA 1.2. — *There are continuous functions $\mu_1, \mu_2, \dots, \mu_{N_\varphi} : [0,1] \to \mathbb{T}$ with the following property: For every $t \in [0,1]$ there is a unitary $u_t \in M_m$ such that*

$$u_t \varphi(f)(t) u_t^*$$
$$= \operatorname{diag}\left(f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_{N_\varphi}(t)), \Lambda_1^{r_1^\varphi}(f), \Lambda_2^{r_2^\varphi}(f), \dots, \Lambda_N^{r_N^\varphi}(f)\right), \quad f \in A.$$

Proof. — Fix first a $t \in [0,1]$. There are then elements

$$\kappa_1(t), \kappa_2(t), \dots, \kappa_L(t) \in \mathbb{T}\setminus\{x_1, x_2, \dots, x_N\}$$

and $s_1, s_2, \dots, s_N \in \mathbb{N}$ such that $A \ni f \mapsto \varphi(f)(t)$ is unitarily equivalent to

$$A \ni f \mapsto \operatorname{diag}\left(f(\kappa_1(t)), \dots, f(\kappa_L(t)), \Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f)\right).$$

Then $s_k = M_k^t / d_k$ and we write $s_k = m_k n / d_k + r_k^\varphi$, $m_k \in \mathbb{N}$. Set $m_0 = 0$ and define

$$\kappa_i(t) = x_k,$$

when

$$i = L + \sum_{j=0}^{k-1} m_j + 1, L + \sum_{j=0}^{k-1} m_j + 2, \dots, L + \sum_{j=0}^{k-1} m_j + m_k,$$
$$k = 1, 2, \dots, N.$$

Note that $L + \sum_{j=1}^N m_j = N_\varphi$. Then $A \ni f \mapsto \varphi(f)(t)$ is unitarily equivalent to

$$A \ni f \mapsto \operatorname{diag}\left(f(\kappa_1(t)), \dots, f(\kappa_{N_\varphi}(t)), \Lambda_1^{r_1^\varphi}(f), \dots, \Lambda_N^{r_N^\varphi}(f)\right).$$

It suffices now to show that there are continuous functions $\mu_1, \dots, \mu_{N_\varphi} \colon [0,1] \to \mathbb{T}$ such that

$$(\kappa_1(t), \dots, \kappa_{N_\varphi}(t)) = (\mu_1(t), \dots, \mu_{N_\varphi}(t))$$

as unordered N_φ-tuples for all $t \in [0,1]$. By **[K]**, Theorem II 5.2, it suffices for this purpose to show that the map

$$t \mapsto (\kappa_1(t), \dots, \kappa_{N_\varphi}(t))$$

is continuous into the unordered N_φ-tuples from $\mathbb{T}$, endowed with the metric

$$d\big((t_i),(s_i)\big) = \min_{\sigma \in \Sigma_{N_\varphi}} \max_i |t_i - s_{\sigma(i)}|,$$

where Σ_{N_φ} denotes the symmetric group of order $N_\varphi!$. To do this, let $t \in [0,1]$ and $\varepsilon > 0$ be given. Let $\mu_1, \mu_2, \dots, \mu_R$ be the mutually distinct elements of $\mathbb{T}$ such that $\{\mu_1, \mu_2, \dots, \mu_R\} = \{\kappa_i(t) : i = 1, 2, \dots, N_\varphi\}$. Let $\delta > 0$ be smaller than both 2ε and $|a - b|$ for any pair a, b of distinct elements of $\{\mu_1, \mu_2, \dots, \mu_R\} \cup \{x_1, x_2, \dots, x_N\}$. For each $k \in \{1, 2, \dots, R\}$, let $g_k \colon \mathbb{T} \to [0,1]$ be a continuous function with support in $\{\lambda \in \mathbb{T} : |\lambda - \mu_k| < \delta/2\}$ such that $g_k(\mu_k) = 1$. Consider g_k as a central element of A. Then $\varphi(g_k)(t)$ is a projection in M_m of rank a_k, where

$$a_k = \#\{i : \kappa_i(t) = \mu_k\}\, n$$

when $\mu_k \notin \{x_1, \dots, x_N\}$, and

$$a_k = \#\{i : \kappa_i(t) = \mu_k\}\, n + r_j^\varphi d_j$$

when $\mu_k = x_j$. Choose $\chi > 0$ so small that $\|\varphi(g_k)(s) - \varphi(g_k)(t)\| < 1$ for all k when $|s - t| < \chi$. Then $\varphi(g_k)(s)$ must be a positive element of M_m of rank $\geq a_k$ for all such s. Since $\varphi(g_k)(s)$ is unitarily equivalent to

$$\operatorname{diag}\big(g_k(\kappa_1(s)), \dots, g_k(\kappa_{N_\varphi}(s)), \Lambda_1^{r_1^\varphi}(g_k), \dots, \Lambda_N^{r_N^\varphi}(g_k)\big),$$

we conclude that

$$\#\{i : |\kappa_i(s) - \mu_k| < \delta/2\}\, n \geq a_k$$

when $\mu_k \notin \{x_1, \dots, x_N\}$, and

$$\#\{i : |\kappa_i(s) - \mu_k| < \delta/2\}\, n + r_j^\varphi d_j \geq a_k$$

when $\mu_k = x_j$. Thus

$$\#\{i : |\kappa_i(s) - \mu_k| < \delta/2\} \geq \#\{i : \kappa_i(t) = \mu_k\}$$

for all $k = \{1, 2, \ldots, R\}$. But

$$N_\varphi \geq \sum_{k=1}^{R} \# \left\{ i : |\kappa_i(s) - \mu_k| < \frac{\delta}{2} \right\} \geq \sum_{k=1}^{R} \# \{ i : \kappa_i(t) = \mu_k \} = N_\varphi,$$

so we see that $\# \{i : |\kappa_i(s) - \mu_k| < \delta/2\} = \# \{i : \kappa_i(t) = \mu_k\}$ for all k. It follows that there is a permutation $\sigma \in \Sigma_{N_\varphi}$ such that $|\kappa_{\sigma(i)}(s) - \kappa_i(t)| < \delta/2 < \varepsilon$ for all $i = 1, 2, \ldots, N_\varphi$. □

LEMMA 1.3. — *There is a continuous function $\xi \colon [0, \sqrt{2} - 1[\to \mathbb{R}$ with $\xi(0) = 0$ and the following property: When A is unital C^*-algebra containing a finite dimensional unital C^*-subalgebra B spanned by the matrix units*

$$\left\{ e^d_{ij} : i, j = 1, 2, \ldots, n_m, d = 1, 2, \ldots, m \right\}$$

and w is a unitary in A such that

$$\left(\sum_{l=1}^{m} n_l\right) \|w e^d_{ij} - e^d_{ij} w\| \leq t \in [0, \sqrt{2} - 1[$$

for all i, j, d, then there is a unitary $u \in A \cap B'$ such that $\|u - w\| \leq \xi(t)$.

Proof. — We can take $\xi(t) = t + (1 + t)(1 - (1 - 2t - t^2)^{-1/2})$. To see this note that there is a conditional expectation $P \colon A \to A \cap B'$ given by the formula

$$P(x) = \sum_{d=1}^{m} \sum_{i,j=1}^{n_d} \frac{1}{n_d} e^d_{ij} x e^d_{ji}, \quad x \in A.$$

Our assumption implies that $\|w - P(w)\| \leq t$. Standard arguments then show that $P(w)$ is invertible and that the unitary from its polar decomposition is a unitary $u \in A \cap B'$ such that $\|u - w\| \leq \xi(t)$. □

Recall that two $*$-homomorphism $\varphi, \psi \colon A \to B$ between unital C^*-algebras are *approximately inner equivalent* when there is a sequence $\{u_n\} \subset B$ of unitaries in B such that $\varphi(a) = \lim_{n \to \infty} u_n \psi(a) u_n^*$, $a \in A$.

THEOREM 1.4. — *Let*

$$\varphi, \psi \colon A(n, d_1, \ldots, d_N) \to A(m, e_1, \ldots, e_M)$$

be unital $*$-homomorphisms. Assume that $\varphi^* = \psi^*$ on $T(A(m, e_1, \ldots, e_M))$.

Then φ and ψ are approximately inner equivalent.

Proof. — Let $F \subset A(n, d_1, d_2, \ldots, d_N)$ be a finite subset and let $\varepsilon > 0$. As $A(n, d_1, d_2, \ldots, d_N)$ is separable, it suffices to find a unitary $u \in A(m, e_1, \ldots, e_M)$ such that

$$\|u\varphi(f)u^* - \psi(f)\| \leq \varepsilon$$

for all $f \in F$.

Let $y_1, y_2, \ldots, y_M \in \mathbb{T}$ be the exceptional points of $A(m, e_1, e_2, \ldots, e_M)$. We first reduce to the case where $\varphi(f)(y_i) = \psi(f)(y_i)$, $i = 1, 2, \ldots, M$, for all $f \in A(n, d_1, d_2, \ldots, d_N)$. Fix $i \in \{1, 2, \ldots, M\}$. Take points $a_1, a_2 \ldots, a_R$ in $\mathbb{T} \setminus \{x_1, \ldots, x_N\}$ and elements $j_1, j_2, \ldots, j_N \in \mathbb{N}$ such that $f \mapsto \varphi(f)(y_i)$ is unitarily equivalent to the representation

$$f \mapsto \operatorname{diag}\big(f(a_1), f(a_2), \ldots, f(a_R), \Lambda_1^{j_1}(f), \ldots, \Lambda_N^{j_N}(f)\big).$$

Similarly, there are points $b_1, b_2, \ldots, b_S \in \mathbb{T} \setminus \{x_1, \ldots, x_N\}$ and elements $i_1, i_2, \ldots, i_N \in \mathbb{N}$ such that $f \mapsto \psi(f)(y_i)$ is unitarily equivalent to the representation

$$f \mapsto \operatorname{diag}\big(f(b_1), f(b_2), \ldots, f(b_S), \Lambda_1^{i_1}(f), \ldots, \Lambda_N^{i_N}(f)\big).$$

Since $\varphi^* = \psi^*$ on $T(A(m, e_1, e_2, \ldots, e_M))$ we know that

$$\frac{1}{m}\operatorname{Tr}(\varphi(f)(y_i)) = \frac{1}{m}\operatorname{Tr}(\psi(f)(y_i)), \quad f \in A(n, d_1, \ldots, d_N).$$

Hence

$$\sum_{j=1}^{R}\operatorname{Tr}(f(a_j)) + \sum_{k=1}^{N}\operatorname{Tr}(\Lambda_k^{j_k}(f)) = \sum_{j=1}^{S}\operatorname{Tr}(f(b_j)) + \sum_{k=1}^{N}\operatorname{Tr}(\Lambda_k^{i_k}(f)),$$

$f \in A(n, d_1, \ldots, d_N)$. By inserting various types of such f it follows that $i_k = j_k, k = 1, 2, \ldots, N$, and that there is a bijection $\sigma\colon \{1, 2, \ldots, S\} \to \{1, 2, \ldots, R\}$ such that $b_i = a_{\sigma(i)}$ for all i. Thus the representations $f \mapsto \varphi(f)(y_i)$ and $f \mapsto \psi(f)(y_i)$ of $A(n, d_1, \ldots, d_N)$ are equivalent. This must therefore also be the case of the representations $f \mapsto \Lambda_i^B(\varphi(f))$ and $f \mapsto \Lambda_i^B(\psi(f))$, where $B = A(m, e_1, e_2, \ldots, e_M)$. Consequently there is a unitary $w_i \in M_{e_i} \subset M_m$ such that

$$w_i\varphi(f)(y_i)w_i^* = \psi(f)(y_i)$$

for all $f \in A(n, d_1, \ldots, d_N), i = 1, 2, \ldots, M$. Let $w \in C(\mathbb{T}) \otimes M_m$ be a unitary such that $w(y_i) = w_i, i = 1, 2, \ldots, M$. Then $w \in A(m, e_1, \ldots, e_M)$ and $\operatorname{Ad} w \circ \varphi(f)(y_i) = \psi(f)(y_i)$, $f \in A(n, d_1, \ldots, d_N)$, $i = 1, 2, \ldots, M$. So for the present purpose we may assume to begin with that $\varphi(f)(y_i) = \psi(f)(y_i)$, $f \in A(n, d_1, \ldots, d_N)$, $i = 1, 2, \ldots, M$.

Let G be a finite set containing F in $A(n, d_1, \dots, d_N)$ such that

$$\{\varphi(g)(y_i) : g \in G\}$$

contains a full set of matrix units for

$$\mathcal{A}_i = \varphi(A(n, d_1, \dots, d_N))(y_i),$$

$i = 1, 2, \dots, M$. Let $\delta \in \,]0, \varepsilon[$. The main problem in the proof will be to construct a unitary $W \in C(\mathbb{T}) \otimes M_m$ such that $\|W\varphi(f)W^* - \psi(f)\| < \delta$ for all $f \in G$. Assume for a moment that this has been achieved. Set $M = \sup_{f \in G} \|f\|$. If δ is small enough, Lemma 1.3 gives us unitaries $w_i \in M_m \cap \mathcal{A}_i'$ such that $\|W(y_i) - w_i\| < \varepsilon/(4M)$. Let $\rho > 0$ be so small that

$$\begin{aligned}
&\|W(t) - w_i\| < \varepsilon/(4M),\\
&\|\psi(f)(t) - \psi(f)(y_i)\| < \varepsilon/4, \text{ and}\\
&\|\varphi(f)(t) - \varphi(f)(y_i)\| < \varepsilon/4, \ f \in F,
\end{aligned}$$

for all $t \in \mathbb{T}$ with $|t - y_i| < \rho$, $i = 1, 2, \dots, M$. Let $u \colon \mathbb{T} \to M_m$ be a continuous path of unitaries such that

$$\begin{aligned}
&u(t) = W(t) \text{ when } |t - y_i| \geq \rho,\\
&u(t) = w_i \text{ when } |t - y_i| = \rho/2,\\
&u(y_i) = 1,\\
&u(t) \in M_m \cap \mathcal{A}_i', |t - y_i| \leq \delta/2, \text{ and}\\
&\sup\{\|u(t) - w_i\| : \rho/2 \leq |t - y_i| \leq \rho\} \leq \varepsilon/(4M),
\end{aligned}$$

$i = 1, 2, \dots, M$. Assuming, as we may, that $2\rho < \min\{|y_i - y_j| : i \neq j\}$, we have that $u \in A(m, e_1, \dots, e_M)$ and $\|u\varphi(f)u^* - \psi(f)\| < \varepsilon$ for all $f \in F$.

We have now reduced the problem to the following: Assuming, in addition, that $\varphi(f)(y_i) = \psi(f)(y_i), i = 1, 2, \dots, M$, for all $f \in A(n, d_1, \dots, d_N)$, construct a unitary $W \in C(\mathbb{T}) \otimes M_m$ such that $\|W\varphi(f)W^* - \psi(f)\| < \varepsilon$ for all $f \in F$. This is done as follows. By Lemma 1.2 there are continuous functions

$$\mu_i^{\varphi} \colon [0, 1] \to \mathbb{T},$$

$i = 1, 2, \dots, N_\varphi$, and numbers

$$r_1^{\varphi}, \dots, r_N^{\varphi} \in \mathbb{N},$$

such that the representation $f \mapsto \varphi(f)(t)$ is unitarily equivalent to

$$f \mapsto \operatorname{diag}\big(f(\mu_1^{\varphi}(t)), \dots, f(\mu_{N_\varphi}^{\varphi}(t)), \Lambda_1^{r_1^{\varphi}}(f), \dots, \Lambda_N^{r_N^{\varphi}}(f)\big)$$

for all $t \in [0, 1]$. For fixed $t \in [0, 1]$, these data are determined, up to permutations of $\mu_1(t), \mu_2(t), \dots, \mu_{N_\varphi}(t)$, by the action of φ^* on $T(A(m, e_1, \dots, e_M))$.

Thus, since we assume that $\varphi^* = \psi^*$ on $T(A(m, e_1, \dots, e_M))$, we have that the data of Lemma 1.2 for ψ are the same as for φ. This means that there are common numbers, $L, r_1, r_2, \dots, r_N \in \mathbb{N}$, and continuous functions $\mu_i\colon [0,1] \to \mathbb{T}$, $i = 1, 2, \dots, L$, such that both

$$f \mapsto \varphi(f)(t), \text{ and } f \mapsto \psi(f)(t)$$

are unitarily equivalent to

$$f \mapsto \operatorname{diag}\big(f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_L(t)), \Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f)\big)$$

for each $t \in [0,1]$. Choose a unitary $S \in C[0,1] \otimes M_m$ such that

$$S(t) \operatorname{diag}\big(f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_L(t)), \Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f)\big) S(t)^*$$

takes the same value at $t = 0$ and $t = 1$ for all $f \in A(n, d_1, \dots, d_N)$. This is possible because $(\mu_1(0), \dots, \mu_L(0)) = (\mu_1(1), \dots, \mu_L(1))$ as unordered tuples. It suffices to construct a unitary $W \in \{f \in C[0,1] \otimes M_m : f(0) = f(1)\}$ such that

$$\begin{aligned}\|W\varphi(f)W^*(t) - S(t) \operatorname{diag}\big(&f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_L(t)),\\ &\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f)\big) S(t)^*\| < \frac{1}{2}\varepsilon\end{aligned}$$

for all $t \in [0,1], f \in F$. To simplify notation, set

$$\begin{aligned}&D(g)(t) =\\ &S(t) \operatorname{diag}\big(g(\mu_1(t)), g(\mu_2(t)), \dots, g(\mu_L(t)), \Lambda_1^{r_1}(g), \dots, \Lambda_N^{r_N}(g)\big) S(t)^*,\end{aligned}$$

$g \in A(n, d_1, \dots, d_N), t \in [0,1]$. Set $\delta_0 = 1/2 \min\{|x_i - x_j| : i \neq j\}$ when $N \geq 2$ and $\delta_0 = 1$ when $N \leq 1$, and define $G_i\colon \mathbb{T} \to [0,1]$ by

$$G_i(z) = \max\big\{0, 1 - \delta_0^{-1} \operatorname{dist}(z, x_i)\big\},$$

$i = 1, 2, \dots, N$, and $z_A\colon \mathbb{T} \to \mathbb{C}$ by

$$z_A(z) = z \operatorname{dist}(z, \{x_1, x_2, \dots, x_N\}).$$

Let $\{e_{ij}\}$ and $\left\{p_{ij}^k\right\}$ be the standard matrix units in M_n and M_{d_k}, respectively, $k = 1, 2, \dots, N$. Then $F_0 = \{z_A \otimes e_{ij}\} \cup \left\{G_k \otimes p_{ij}^k\right\}$ generates A as a C^*-algebra, so for the present purpose we can assume that $F = F_0$.

Let $\delta > 0$ and $0 < \kappa < 1$. We shall require that

$$3(L + N)\kappa < \delta_0, \tag{1}$$

$$4C\kappa < \sqrt{2} - 1, \tag{2}$$

$$\delta + \kappa < \delta_0, \tag{3}$$

$$3(L+N)\kappa + 2\kappa < \delta, \tag{4}$$

$$2\xi(4C\kappa) + \kappa < \varepsilon/2, \tag{5}$$

$$\begin{aligned}&\kappa + (L+N)\max\left\{\delta_0^{-1}(6(L+N)+8)\kappa, (18(L+N)+24)\kappa, 4\delta_0^{-1}\delta, 4\delta\right\}\\ &< \varepsilon/2.\end{aligned} \tag{6}$$

Here $C = (L+N)n$ and $\xi\colon [0, \sqrt{2}-1[\to \mathbb{R}$ is the continuous function of Lemma 1.3. These conditions can be met by first choosing δ and then κ subsequently.

For each closed non-empty set $S \subset \mathbb{T}$ we define $g_S\colon \mathbb{T} \to [0,1]$ by

$$g_S(t) = \max\left\{0, 1 - \kappa^{-1}\operatorname{dist}(t, S)\right\}.$$

We call g_S a κ-test function. With the Hausdorff distance as metric the closed non-empty subsets of $\mathbb{T}$ form a compact metric space. By using this, it follows easily that there is a finite set H of κ-test functions such that each κ-test function is within the distance κ of an element of H, measured by the supremum norm of $C(\mathbb{T})$. An alternative proof of this fact can be found in [**S**], Lemma 2.4. Set

$$F_1 = \left\{h \otimes e_{ij}, h \otimes p_{ij}^k : h \in H, \text{ all } k, i, j\right\} \cap A.$$

Let $\mu > 0$ be so small that $|\mu_i(t) - \mu_i(s)| < \kappa$, $i = 1, 2, \ldots, L$, whenever $|t - s| < \mu$. There are points $0 = z_0 < z_1 < \cdots < z_K = 1$ and unitaries $u_i \in M_m$, $i = 1, \ldots, K-1$, such that

$$\|u_i\varphi(g)(t)u_i^* - D(g)(t)\| < \kappa, \tag{7}$$

$t \in [z_{i-1}, z_{i+1}]$, $i = 1, 2, \ldots, K-1$, for all $g \in F_1 \cup F$. We may assume that $u_{K-1} = u_1$. Let $J = [a, b]$ be a small interval of length $< \mu$ centered around z_i for some $i \in \{1, 2, \ldots, K-2\}$, not containing z_{i-1} or z_{i+1}. It now suffices to construct a path $V\colon J \to M_m$ of unitaries such that $V(a) = u_i, V(b) = u_{i+1}$ and $\|V\varphi(f)V^*(t) - D(f)(t)\| < \varepsilon/2$, $t \in J, f \in F$. Note that

$$\|u_{i+1}u_i^* D(g)(t) u_i u_{i+1}^* - D(g)(t)\| < 2\kappa, \quad t \in J, g \in F_1 \cup F,$$

if the interval $J = [a, b]$ is chosen small enough. Set $s = (a+b)/2 = z_i$. Group $\mu_1(s), \mu_2(s), \ldots, \mu_L(s), x_1, x_2, \ldots, x_N$ into disjoint sets, $S_1, S_2, \ldots, S_Q$, such that every point of S_i is at least 3κ apart from any element of S_j when $i \neq j$ and, on the other hand, no subset of S_i is 3κ isolated from the rest of S_i. Since the length of J is less than μ, it follows that $\mu_j(s) \in S_i \Rightarrow \operatorname{dist}(\mu_j(t), S_i) < \kappa$,

$t \in J$. Set $T_k = \{\lambda \in \mathbb{T} : \operatorname{dist}(\lambda, S_k) \leq \kappa\}$. The κ-test function g_{T_k} is within κ of an element h_k of H. For each $k \in \{1, 2, \dots, Q\}$, either

$$S_k \cap \{x_1, \dots, x_N\} = \varnothing, \text{ or} \tag{8}$$

$$x_{i_k} \in S_k \text{ for exactly one } i_k \in \{1, 2, \dots, N\}. \tag{9}$$

This is because $\operatorname{diam}(S_k) \leq 3(L+N)\kappa < \delta_0$ by (1). In the first case $g_{T_k} \otimes e_{ij}, h_k \otimes e_{ij} \in A$ for all i, j, because the S_i's are 3κ-separated, and in the second case $g_{T_k} \otimes p^{i_k}_{ij}, h_k \otimes p^{i_k}_{ij} \in A$ for all i, j, for the same reason. To simplify notation, set $d_0 = n, p^0_{ij} = e_{ij}$ and $i_k = 0$, when $S_k \cap \{x_1, \dots, x_N\} = \varnothing$. Then we have that $g_{T_k} \otimes p^{i_k}_{ij}, h_k \otimes p^{i_k}_{ij} \in A$ for all $k = 1, 2, \dots, Q$, $i, j = 1, 2, \dots d_{i_k}$, and since $\|g_{T_k} - h_k\| < \kappa$ for all k, we see that

$$\|u_{i+1}u_i^* D(g_{T_k} \otimes p^{i_k}_{rs})(t) u_i u_i + 1^* - D(g_{T_k} \otimes p^{i_k}_{rs})(t)\| < 4\kappa, \tag{10}$$

$\forall k, r, s$, $t \in J$. Set $f^k_{ij}(t) = D(g_{T_k} \otimes p^{i_k}_{ij})(t)$, $t \in J$. Then $\left\{f^k_{ij}\right\}$ are matrix units for a finite dimensional unital C^*-subalgebra $\mathcal{B}$ of $C(J) \otimes M_m$. By combining (2) and (10) with Lemma 1.3, we get a unitary $w \in \big(C(J) \otimes M_m\big) \cap \mathcal{B}'$ such that $\|w(t) - u_{i+1}u_i^*\| < \xi(4C\kappa), t \in J$. Set $q_k = D(g_{T_k} \otimes 1)|_J = \sum_i f^k_{ii}$ and $w_k = wq_k$.

Consider first the case (9). Using (4) we find that $\operatorname{diam}(T_k) \leq (L+N)3\kappa + 2\kappa < \delta$ so that

$$|x - x_{i_k}| < \delta \tag{11}$$

for all $x \in T_k$. Since the unitary group of $q_k\big(C(J) \otimes M_m \cap \mathcal{B}'\big)q_k$ is connected we can find a continuous path $\gamma_k(t), t \in [a, s]$, of unitaries in $q_k\big(C(J) \otimes M_m \cap \mathcal{B}'\big)q_k$ such that $\gamma_k(a) = q_k$ and $\gamma_k(s) = w_k$. We claim that

$$\|\gamma_k(t,t) D(z_A \otimes e_{ij})(t) - D(z_A \otimes e_{ij})(t)\gamma_k(t,t)\| \leq 4\delta, \tag{12}$$

for all $t \in [a, s], i, j = 1, 2, \dots, n$. To see this note first that

$$\begin{aligned} &\|\gamma_k(t,t) D(z_A \otimes e_{ij})(t) - D(z_A \otimes e_{ij})(t)\gamma_k(t,t)\| \\ &\leq 2\ \|q_k(t) D(z_A \otimes e_{ij})(t)\| \ = \ 2\|D(g_{T_k} z_A \otimes e_{ij})(t)\| \ \leq 2\ \|g_{T_k} z_A\|. \end{aligned}$$

When $\operatorname{dist}(u, T_k) \geq \kappa$, $g_{T_k}(u) z_A(u) = 0$. When $\operatorname{dist}(u, T_k) < \kappa$, there is a $x \in T_k$ with $|x - u| \leq \kappa$ and hence

$$\begin{aligned} |u - x_{i_k}| &< \kappa + \delta \quad \text{(by (11))} \\ &\leq 2\delta \qquad \text{(since } \kappa \leq \delta \text{ by (4))}, \end{aligned}$$

so that $|g_{T_k}(u)z_A(u)| \leq 2\delta$ for all u, i.e. $\|g_{T_k}z_A\| \leq 2\delta$, proving (12). We claim also that

$$\|\gamma_k(t,t)D(G_l \otimes p^l_{ij})(t) - D(G_l \otimes p^l_{ij})(t)\gamma_k(t,t)\| \leq 4\delta_0^{-1}\delta, \tag{13}$$

for all $t \in [a,s]$ and all l, i, j. To see this, consider first the case where $l \neq i_k$. Note that

$$q_k(t)D(G_l \otimes p^l_{ij})(t) = D(g_{T_k}G_l \otimes p^l_{ij})(t)$$

for all $t \in [a,s]$. However, g_{T_k} and G_l have disjoint supports when $l \neq i_k$ since there is no element $y \in \mathbb{T}$ such that $\operatorname{dist}(y, T_k) \leq \kappa$ and $\operatorname{dist}(y, x_l) \leq \delta_0$. (If there was such a y there would be an $x \in T_k$ with $|y - x| \leq \kappa$, so that $|y - x_{i_k}| < \delta + \kappa < \delta_0$ by (11) and (3). This would imply that $|x_l - x_{i_k}| < 2\delta_0$ and hence that $l = i_k$, contrary to our present assumption.) Therefore

$$\begin{aligned}\gamma_k(t,t)D(G_l \otimes p^l_{ij})(t) &= D(G_l \otimes p^l_{ij})(t)\gamma_k(t,t)\\ &= \gamma_k(t,t)q_k(t)D(G_l \otimes p^l_{ij})(t) = 0\end{aligned}$$

for all $t \in [a,s]$, and (13) is certainly true in this case. In the case $l = i_k$ we use that $\gamma_k(t,t)$ commutes with $D(g_{T_k} \otimes p^{i_k}_{ij})(t)$ to get the estimate

$$\begin{aligned}&\|\gamma_k(t,t)D(G_l \otimes p^l_{ij})(t) - D(G_l \otimes p^l_{ij})(t)\gamma_k(t,t)\| \leq\\ &\|D(g_{T_k} \otimes p^{i_k}_{ij})(t) - D(G_{i_k} \otimes p^{i_k}_{ij})(t)q_k(t)\| +\\ &\qquad\qquad \|D(g_{T_k} \otimes p^{i_k}_{ij})(t) - q_k(t)D(G_{i_k} \otimes p^{i_k}_{ij})(t)\|\\ &\leq 2\|g_{T_k} - G_{i_k}g_{T_k}\|,\end{aligned}$$

for all $t \in [a,s]$. If $u \in \mathbb{T}$ is in the support of g_{T_k}, we have that $\operatorname{dist}(u, T_k) \leq \kappa$ and hence that

$$|u - x_{i_k}| < \kappa + \delta < 2\delta$$

(by (11) and (4)), so that

$$|g_{T_k}(u)(1 - G_{i_k}(u))| \leq |1 - G_{i_k}(u)| = \delta_0^{-1}|u - x_{i_k}| < 2\delta_0^{-1}\delta.$$

Hence $\|g_{T_k} - G_{i_k}g_{T_k}\| \leq 2\delta_0^{-1}\delta$ and (13) follows.

Next we establish estimates similar to (12) and (13) in the case (8). So assume that (8) holds. Let $c \in S_k$. The diameter of the support of g_{T_k} is $\leq (3(L+N)+4)\kappa$ so that $\|z_A g_{T_k} - z_A(c)g_{T_k}\| \leq (9(L+N)+12)\kappa$. Since we are in case (8) the support of g_{T_k} does not contain any of the x_i's and hence

$$\|D(z_A \otimes e_{ij})q_k(t) - z_A(c)D(g_{T_k} \otimes e_{ij})q_k(t)\| < (9(L+N)+12)\kappa, \quad t \in J. \tag{14}$$

Since $\gamma_k(t,t)$ commutes with $D(g_{T_k} \otimes e_{ij})q_k(t)$, (14) implies that

(15) $\|\gamma_k(t,t)D(z_A \otimes e_{ij})(t) - D(z_A \otimes e_{ij})(t)\gamma_k(t,t)\| \leq (18(L+N)+24)\kappa,$

for all $t \in [a,s]$ and all i,j. Over the support of g_{T_k} each G_l varies by no more than δ_0 times the diameter of the support of g_{T_k}, i.e. by more than $\delta_0^{-1}(3(L+N)+4)\kappa$. Set $\lambda(k,l) = G_l(c)$. Using again that we are in case (8) so that the support of g_{T_k} contains no x_i we see that

$$\|D(g_{T_k}G_l \otimes p_{ij}^l)(t) - \lambda(k,l)D(g_{T_k} \otimes p_{ij}^l)(t)\| \leq \delta_0^{-1}(3(L+N)+4)\kappa, \quad t \in J,$$

and hence that

(16)
$$\|\gamma_k(t,t)D(G_l \otimes p_{ij}^l)(t) - D(G_l \otimes p_{ij}^l)(t)\gamma_k(t,t)\| \leq \delta_0^{-1}(6(L+N)+8)\kappa,$$

for all $t \in J$ and all l,i,j.

Define $V\colon [a,s] \to M_m$ by $V(t) = \sum_{k=1}^{Q} \gamma_k(t,t)u_i$. Then V is a path of unitaries, $V(a) = u_i$ and by combining (12),(13),(15) and (16) we find that

$$\begin{aligned}
&\|V(t)\varphi(f)(t)V(t)^* - D(f)(t)\| \\
\leq\ & \|\sum_{k=1}^{Q} \gamma_k(t,t)D(f)(t) - D(f)(t)\sum_{k=1}^{Q}\gamma_k(t,t)\| + \kappa \quad \text{(by (8))} \\
\leq\ & (L+N)\max_k \|\gamma_k(t,t)D(f)(t) - D(f)(t)\gamma_k(t,t)\| + \kappa \\
\leq\ & \kappa + (L+N)\max\left\{\delta_0^{-1}(6(L+N)+8)\kappa, (18(L+N)+24)\kappa, 4\delta_0^{-1}\delta, 4\delta\right\} \\
<\ & \frac{\varepsilon}{2}, \qquad \text{(by (6))},
\end{aligned}$$

for all $t \in [a,s]$, $f \in F$. Furthermore $\|V(s) - u_{i+1}\| = \|w(s)u_i - u_{i+1}\| \leq \xi(4C\kappa)$. We extend V to a continuous path of unitaries $V\colon [a,b] \to M_m$ such that $V(b) = u_{i+1}$ and $\|V(t) - u_{i+1}\| \leq \xi(4C\kappa)$ for all $t \in [s,b]$.

Then

$$\begin{aligned}
&\|V(t)\varphi(f)(t)V(t)^* - D(f)(t)\| \\
\leq\ & 2\xi(4C\kappa) + \|u_{i+1}\varphi(f)(t)u_{i+1}^* - D(f)(t)\| \\
\leq\ & 2\xi(4C\kappa) + \kappa \quad \text{(by (8))} \\
<\ & \frac{\varepsilon}{2}, \quad \text{(by (5))},
\end{aligned}$$

for all $t \in [s,b], f \in F$, and hence

$$\|V(t)\varphi(f)(t)V(t)^* - D(f)(t)\| < \varepsilon/2,$$

for all $t \in J$ and all $f \in F$. □

COROLLARY 1.5. — *Let A and B be building blocks of any type. Let $\varphi, \psi \colon A \to B$ be unital $*$-homomorphism such that $\varphi^* = \psi^*$ on $T(B)$.*

It follows that φ and ψ are approximately inner equivalent.

Proof. — Consider the case where A and B are building blocks of type 3. Set

$$\alpha(t) = e^{\pi i t}, t \in [0,1],$$

and

$$\kappa(e^{2\pi i t}) = 2t,\ t \in [0, 1/2], \quad \kappa(e^{2\pi i t}) = 2 - 2t,\ t \in [1/2, 1].$$

Then $\kappa \circ \alpha(t) = t$. It is obvious how to define building blocks A_1 and B_1 of type 1 such that $f \mapsto f \circ \kappa$ defines unital $*$-homomorphisms $\lambda_A \colon A \to A_1$ and $\lambda_B \colon B \to B_1$. Then $f \mapsto f \circ \alpha$ defines $*$-homomorphisms $\pi_A \colon A_1 \to A$ and $\pi_B \colon B_1 \to B$ such that $\pi_A \circ \lambda_A = id_A, \pi_B \circ \lambda_B = id_B$. Since $\varphi^* = \psi^*$ on $T(B)$ we have that $(\lambda_B \circ \psi \circ \pi_A)^* = (\lambda_B \circ \varphi \circ \pi_A)^*$. By Theorem 1.4 this implies that $\lambda_B \circ \psi \circ \pi_A$ and $\lambda_B \circ \varphi \circ \pi_A$ are approximately inner equivalent. By applying π_B on the left and λ_A on the right, we see that ψ and φ are approximately inner equivalent. The other cases are handled in a similar way. □

We can now give the following description of the unital $*$-homomorphisms between building blocks of type 1,

$$A = A(n, d_1, d_2, \ldots, d_N) \quad \text{and} \quad B = A(m, e_1, e_2, \ldots, e_M),$$

with exceptional points $x_j \in \mathbb{T}, j = 1, 2, \ldots, N$, and $t_1, t_2, \ldots, t_M \in [0,1]$, respectively. By Theorem 1.4 and Lemma 1.2, any unital $*$-homomorphism $\varphi \colon A \to B$ is approximately inner equivalent to one of the following *standard form*: Let $r_k^\varphi \in \mathbb{N}$ and let

$$\mu_i^\varphi \colon [0,1] \to \mathbb{T}, \quad i = 1, 2, \ldots, \frac{m - \sum_{i=1}^N r_i^\varphi d_i}{n} \equiv L_\varphi,$$

be continuous functions such that

$$\#\{i : \mu_i^\varphi(t_k) = x_j\} \frac{n}{d_j} + r_j^\varphi \in \mathbb{N}\frac{m}{e_k}, \tag{17}$$

$j = 1, 2, \ldots, N$, and

$$\#\{i : \mu_i^\varphi(t_k) = t\} \in \mathbb{N}\frac{m}{e_k}, \quad t \in \mathbb{T} \setminus \{x_1, x_2, \ldots, x_N\} \tag{18}$$

for all $k = 1, 2, \ldots, M$, and such that

$$(\mu_1^\varphi(0), \mu_2^\varphi(0), \ldots, \mu_{L_\varphi}^\varphi(0)) = (\mu_1^\varphi(1), \mu_2^\varphi(1), \ldots, \mu_{L_\varphi}^\varphi(1)) \tag{19}$$

as unordered L_φ-tuples. There is then a unitary $u \in C[0,1] \otimes M_m$ such that

$$\varphi(f)(t) =$$
$$u(t)\operatorname{diag}\big(f(\mu_1^\varphi(t)),\dots,f(\mu_{L_\varphi}^\varphi(t)),\Lambda_1^{r_1^\varphi}(f),\Lambda_2^{r_2^\varphi}(f),\dots,\Lambda_N^{r_N^\varphi}(f)\big)u(t)^*,$$

$f \in A, t \in [0,1]$, defines a unital $*$-homomorphism $\varphi\colon A \to B$.

A similar notion of standard homomorphims exists for maps between building blocks of other types too. They will also be important for us later, so let us describe them. Consider a building block $A = A(n,d_1,d_2,\dots,d_N)$ of type 3 and $B = A(m,e_1,e_2,\dots,e_M)$ a building block of type 1 or 3 with exceptional points $x_j \in [0,1], j = 1,2,\dots,N$, and $t_1,t_2,\dots,t_M \in [0,1]$, respectively.

LEMMA 1.6. — *There are integers $r_k \in \{0,1,2,\dots,n/d_k - 1\}$, $k = 1,2,\dots,N$, and continuous functions $\mu_1,\mu_2,\dots,\mu_{N_\varphi}\colon [0,1] \to [0,1]$ with the following properties:*

- *$\mu_1(t) \le \mu_2(t) \le \cdots \le \mu_{N_\varphi}(t)$, $t \in [0,1]$.*
- *For every $t \in [0,1]$ there is a unitary $u_t \in M_m$ such that*

$$u_t\varphi(f)(t)u_t^*$$
$$= \operatorname{diag}\big(f(\mu_1(t)),f(\mu_2(t)),\dots,f(\mu_{N_\varphi}(t)),\Lambda_1^{r_1}(f),\Lambda_2^{r_2}(f),\dots,\Lambda_N^{r_N}(f)\big),$$

for all $f \in A$.

Proof. — Fix first a $t \in [0,1]$. Take

$$\kappa_1(t),\kappa_2(t),\dots,\kappa_L(t) \in [0,1]\setminus\{x_1,x_2,\dots,x_N\}$$

and $s_1,s_2,\dots,s_N \in \mathbb{N}$ such that $A \ni f \mapsto \varphi(f)(t)$ is unitarily equivalent to

$$A \ni f \mapsto \operatorname{diag}\big(f(\kappa_1(t)),\dots,f(\kappa_L(t)),\Lambda_1^{s_1}(f),\dots,\Lambda_N^{s_N}(f)\big).$$

We write $s_k = m_k n/d_k + r_k, m_k \in \mathbb{N}, r_k \in \{0,1,2,\dots,n/d_k - 1\}$. Exactly as in the proof of Lemma 1.1 we see that the r_k's do not depend on $t \in [0,1]$. Set $m_0 = 0$ and define $\kappa_i(t) = x_k$, when

$$i = L + \sum_{j=0}^{k-1} m_j + 1, L + \sum_{j=0}^{k-1} m_j + 2,\dots,L + \sum_{j=0}^{k-1} m_j + m_k,$$
$$k = 1,2,\dots,N.$$

As before we denote $L + \sum_{j=1}^N m_k$ by N_φ. Then $A \ni f \mapsto \varphi(f)(t)$ is unitarily equivalent to

$$A \ni f \mapsto \operatorname{diag}\big(f(\kappa_1(t)),\dots,f(\kappa_{N_\varphi}(t)),\Lambda_1^{r_1}(f),\dots,\Lambda_N^{r_N}(f)\big).$$

As in the proof of Lemma 1.2 we see that the map $t \mapsto (\kappa_1(t), \dots, \kappa_{N_\varphi}(t))$ is continuous into the unordered N_φ-tuples from $[0,1]$. We define the functions $\mu_i\colon [0,1] \to [0,1]$, $i = 1,2,\dots,N_\varphi$, as the unique set of functions such that

$$\mu_1(t) \le \mu_2(t) \le \cdots \le \mu_{N_\varphi}(t)$$

and

$$\big(\mu_1(t), \mu_2(t), \dots, \mu_{N_\varphi}(t)\big) = (\kappa_1(t), \dots, \kappa_{N_\varphi}(t))$$

as unordered tuples for all t. Then each μ_i is automatically continuous, cf. [**CE**], proof of Theorem 10, or use the min-max principle. $\square$

Let $r_k^\varphi \in \mathbb{N}$ and let

$$\mu_i^\varphi\colon [0,1] \to [0,1], \quad i = 1,2,\dots, \frac{m - \sum_{i=1}^N r_i^\varphi d_i}{n} \equiv L_\varphi,$$

be continuous functions such that

$$\#\{i : \mu_i^\varphi(t_k) = x_j\}\,\frac{n}{d_j} + r_j^\varphi \in \mathbb{N}\frac{m}{e_k}, \tag{20}$$

$j = 1,2,\dots,N$, and

$$\#\{i : \mu_i^\varphi(t_k) = t\} \in \mathbb{N}\frac{m}{e_k}, \quad t \in [0,1]\setminus\{x_1, x_2, \dots, x_N\} \tag{21}$$

for all $k = 1,2,\dots,M$. When B is of type 1 we also need to have

$$\big(\mu_1^\varphi(0), \mu_2^\varphi(0), \dots, \mu_{L_\varphi}^\varphi(0)\big) = \big(\mu_1^\varphi(1), \mu_2^\varphi(1), \dots, \mu_{L_\varphi}^\varphi(1)\big) \tag{22}$$

as unordered L_φ-tuples. There is then a unitary $u \in C[0,1] \otimes M_m$ such that

$$\varphi(f)(x) =$$
$$u(x)\operatorname{diag}\big(f(\mu_1^\varphi(x)), \dots, f(\mu_{L_\varphi}^\varphi(x)), \Lambda_1^{r_1^\varphi}(f), \Lambda_2^{r_2^\varphi}(f), \dots, \Lambda_N^{r_N^\varphi}(f)\big)u(x)^*,$$

$f \in A, x \in [0,1]$, defines a $*$-homomorphism $\varphi\colon A \to B$, and by combining Corollary 1.5 and Lemma 1.6 we see that any unital $*$-homomorphism from A to B is approximately inner equivalent to one of this form. By Lemma 1.6 we can assume, in addition, that (21) holds.

Note that we can always take

$$r_k^\varphi \in \left\{0, 1, 2, \dots, \frac{n}{d_k} - 1\right\}, \quad k = 1,2,\dots,N, \tag{23}$$

regardless of which type of building blocks we are considering. A standard homomorphism as described above will be said to have *minimal multiplicity* when (23) holds and

$$\frac{\#\{i : \mu_i^\varphi(y_k) = x_j\}\frac{n}{d_j} + r_j^\varphi}{\frac{m}{e_k}} < \frac{n}{d_j}, \quad j = 1,2,\ldots,N, k = 1,2,\ldots,M. \tag{24}$$

The functions $\mu_i^\varphi, i = 1,2,\ldots,L_\varphi$, will be called the *characteristic functions* of φ and the numbers $r_1^\varphi,\ldots,r_N^\varphi$ will be referred to as the *remainders* of φ. The numbers

$$\frac{\#\{i : \mu_i^\varphi(y_k) = x_j\}\frac{n}{d_j} + r_j^\varphi}{\frac{m}{e_k}}$$

will be called *the small remainders* of φ and denoted by $s^\varphi(k,j)$, $k = 1,2,\ldots,M$, $j = 1,2,\ldots,N$. Observe that

$$s^\varphi(k,j)\frac{m}{e_k} = r_j^\varphi \quad \text{modulo } \frac{n}{d_k}, \tag{25}$$

and that $s^\varphi(k,j)$ is the multiplicity of the representation Λ_j^A in $\Lambda_k^B \circ \varphi$.

By Corollary 1.5 two $*$-homomorphisms $\varphi,\psi\colon A \to B$ between building blocks, A and B, of standard form and minimal multiplicity, are approximately inner equivalent if and only if $r_j^\varphi = r_j^\psi, j = 1,2,\ldots,N$, $L_\varphi = L_\psi$, and $(\mu_1^\varphi(t),\mu_2^\varphi(t),\ldots,\mu_{L_\varphi}^\varphi(t)) = (\mu_1^\psi(t),\mu_2^\psi(t),\ldots,\mu_{L_\psi}^\psi(t))$ as unordered tuples for all $t \in [0,1]$.

LEMMA 1.7. — *Let $\varphi\colon A \to B$ be a unital $*$-homomorphism between two building blocks A and B, of any type. For any finite subset $F \subset A$ and any $\varepsilon > 0$ there is a unital $*$-homomorphism $\psi\colon A \to B$ of standard form and minimal multiplicity, and a unitary $w \in B$ such that $\|\operatorname{Ad} w \circ \varphi(a) - \psi(a)\| < \varepsilon$, $a \in F$, and $s^\varphi(j,k) = s^\psi(j,k)$ modulo n/d_j, $k = 1,2,\ldots,M$, $j = 1,2,\ldots,N$.*

Proof. — We present the proof in the case where $A = A(n,d_1,\ldots,d_N)$ and $B = A(m,e_1,\ldots,e_M)$ are both of type 1. The proof in the other cases are the same (except for notation). Let $y_1,y_2,\ldots,y_M \in [0,1]$ be the exceptional points of B. We may assume that φ is of standard form, i.e. is given by

$$r_k \in \{0,1,2,\ldots,n/d_k - 1\}, \quad k = 1,2,\ldots,N,$$

and

$$\mu_i\colon [0,1] \to \mathbb{T}, \quad i = 1,2,\ldots,(m - \sum_{i=1}^N r_i d_i)/n \equiv L,$$

through the formula

$$\varphi(f)(t) = u(t)\operatorname{diag}\big(f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_L(t)), \Lambda_1^{r_1}(f), \Lambda_2^{r_2}(f), \dots, \Lambda_N^{r_N}(f)\big)u(t)^*.$$

If we freeze $\mu_1, \dots, \mu_L$ in very small intervals $I_r = [a_r, b_r]$ around each $y_r, r = 1, 2, \dots, M$, we can achieve that $\mu_i(t) = \mu_i(y_r)$, $t \in I_r$, $i = 1, 2, \dots, L$. By freezing u in the same way we can achieve that

$$\varphi(f)(t) = \varphi(f)(y_r),$$

for all $t \in I_r$, $r = 1, 2, \dots, N$. The resulting perturbation of φ needed to obtain these things can be made arbitrarily small (on F) by choosing each I_r small enough, so we can simply assume that we have such "frozen" intervals to begin with and that they are mutually disjoint, with y_r in the interior of I_r.

Fix an r and let $\iota_r \colon M_{e_r} \to M_m$ be the given embedding. There is then a unitary $v \in M_{e_r}$ such that

$$\varphi(f)(t) = \iota_r\big(v\operatorname{diag}\big(f(t_1), f(t_2), \dots, f(t_K), \Lambda_1^{s_1}(f), \Lambda_2^{s_2}(f), \dots, \Lambda_N^{s_N}(f)\big)v^*\big),$$

$f \in A$, $t \in I_r$, for some $s_i \in \{0, 1, \dots, n/d_i - 1\}$, $i = 1, 2, \dots, N$, and some $t_1, t_2, \dots, t_K \in \mathbb{T}$. Note that $s_i = s^{\varphi}(i, r)$ modulo n/d_i. Choose continuous functions $g_i \colon I_r \to \mathbb{T}$ such that

$$|g_i(s) - t_i| < \delta, \quad s \in I_r, \quad i = 1, 2, \dots, K,$$

$g_i = t_i$ on the boundary of I_r, and

$$\{x_1, x_2, \dots, x_N\} \cap \{g_1(y_r), g_2(y_r), \dots, g_K(y_r)\} = \varnothing.$$

We can then define a perturbation φ_r of φ by $\varphi_r(f)(t) = \varphi(f)(t)$, $f \in A, t \notin I_r$, and

$$\varphi_r(f)(t) = \iota_r\big(v\operatorname{diag}\big(f(g_1(t)), f(g_2(t)), \dots, f(g_K(t)), \Lambda_1^{s_1}(f), \Lambda_2^{s_2}(f), \dots, \Lambda_N^{s_N}(f)\big)v^*\big),$$

$f \in A$, $t \in I_r$. By making such a change over each I_r and by choosing $\delta > 0$ sufficiently small, we get the desired perturbation of φ. $\square$

LEMMA 1.8. — *Let p be a projection $A = A(n, d_1, d_2, \dots, d_N)$, where A is a building block of type* 1. *Set $r = \operatorname{Tr}(p(t)), t \in \mathbb{T}$, the rank of p. Then $n/d_i | r$ for all $i = 1, 2, \dots, N$, and*

$$pA(n, d_1, \dots, d_N)p \simeq A(r, d_1', \dots, d_N')$$

where $A(r, d'_1, \ldots, d'_N)$ *is a building block of type* 1 *and* $d'_i = rd_i/n$, $i = 1, 2, \ldots, N$.

Proof. — The proof is a standard exercise. □

Of course, Lemma 1.8 holds for building blocks of other types too. Lemma 1.8 will be used repeatedly in the following, often without comment. However, one particular application deserves to be mentioned because it will be used over and over. If

$$A(n, d, N) = A(n, \underbrace{d, d, \ldots, d}_{N \text{ times}})$$

is a building block of type 2 and $x \in \{1, 2, \ldots, d\}$, then there is an imbedding $A(nx/d, x, N) \subset A(n, d, N)$, making $A(nx/d, x, N)$ a full corner in $A(n, d, N)$, and if $x_1, x_2, \ldots, x_M \in \mathbb{N}$ are natural numbers such that $\sum_{k=1}^{M} x_k = d$, then there is a unital imbedding $\bigoplus_{k=1}^{M} A(nx_k/d, x_k, N) \subset A(n, d, N)$.

CHAPTER 2

THE KK - THEORY OF BUILDING BLOCKS OF TYPE 2

In this chapter we will only consider building blocks of type 2. The KK-group $KK(A,B)$, where $A = A(n,d,N)$ and $B = A(m,e,M)$ are building blocks of type 2, is easily calculated by use of the universal coefficient theorem, [**RS**]; the result being that

$$KK(A,B) \simeq \mathrm{Hom}(K_0(A), K_0(B)) \oplus \mathrm{Hom}(K_1(A), K_1(B)) \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1}.$$

Nonetheless, it is not clear what information is coded into $KK(A,B)$. Of particular importance for us here, is it to determine the signifigance of the direct summand $\left(\mathbb{Z}_{n/d}\right)^{N-1}$ and to decide which elements of $KK(A,B)$ are represented by unital $*$-homomorphisms $A \to B$. We will answer this in the case where e is larger that $(2N+1)nd$, and this will suffice for our purposes.

When $A = A(n,d,N)$ is a building block of type 2 with exceptional points $x_1, \dots, x_N$, we set

$$A_0 = A_0(n,d,N) = \{f \in A : f(x_N) = 0\}.$$

There is then a split-exact sequence $0 \to A_0 \to A \to M_d \to 0$ from which we deduce that

$$KK(A,B) \simeq K_0(B) \oplus KK(A_0,B),$$

for any separable, nuclear C^*-algebra B, under the map

$$KK(A,B) \ni \alpha \mapsto (\alpha_*([e_{11}]), \iota^*(\alpha)),$$

where $\iota \colon A_0 \to A$ is the inclusion and e_{11} is a minimal non-zero projection in $M_d \subset A$. The following lemma and its proof was pointed out to the author by Terry Loring.

LEMMA 2.1. — *For any nuclear C^*-algebra B one has*

$$KK(A_0,B) = \varinjlim [A_0, M_k(B)].$$

Proof. — By **[L2]** and **[DL1]** it suffices to show that A_0 is homotopic symmetric. In other words we must show that the identity map id_{A_0} has an inverse in $\varinjlim[A_0, M_k(A_0)]$. To this end we may assume that $x_N = 1$ and that $\{x_1, x_2, \dots, x_{N-1}\}$ is left globally invariant by the map $\mathbb{T} \ni z \mapsto z^{-1}$. We can then define $\overline{id_{A_0}}\colon A_0 \to A_0$ by

$$\overline{id_{A_0}}(f)(z) = f(z^{-1}),$$

$z \in \mathbb{T}$. We leave the reader to check that

$$\underbrace{id_{A_0} \oplus id_{A_0} \oplus \cdots \oplus id_{A_0}}_{\frac{n}{d} \text{ times}} \oplus \underbrace{\overline{id_{A_0}} \oplus \overline{id_{A_0}} \oplus \cdots \oplus \overline{id_{A_0}}}_{\frac{n}{d} \text{ times}}$$

is homotopic to the zero map so that

$$\underbrace{id_{A_0} \oplus id_{A_0} \oplus \cdots \oplus id_{A_0}}_{\frac{n}{d} - 1 \text{ times}} \oplus \underbrace{\overline{id_{A_0}} \oplus \overline{id_{A_0}} \oplus \cdots \oplus \overline{id_{A_0}}}_{\frac{n}{d} \text{ times}}$$

represents the inverse of id_{A_0} in $\varinjlim[A_0, M_k(A_0)]$. □

Let $\Lambda_i^A\colon A(n,d,N) \to M_d$, $i = 1, 2, \dots, N$, be the exceptional representations of A. When $\mu\colon A_0 \to M_r(B) = A(rm, re, M)$ is a $*$-homomorphism we let $s_j^i \in \mathbb{Z}_{n/d}$ be the multiplicity of $\Lambda_j^A|_{A_0}$ in $\Lambda_i^{M_r(B)} \circ \mu$, taken modulo n/d, $j = 1, 2, \dots, N-1$, $i = 1, 2, \dots, M$. The arguments from Lemma 1.1 show that $s_j^i \in \mathbb{Z}_{n/d}$ only depend on μ up to homotopy, see also **[DL2]**, proof of Lemma 3.1. Hence

$$KK(A_0, B) \ni [\mu] \mapsto (s_1^i, s_2^i, \dots, s_{N-1}^i)$$

defines a group homomorphism

$$\kappa_A^i\colon KK(A_0, B) \longrightarrow \left(\mathbb{Z}_{n/d}\right)^{N-1}$$

for each $i = 1, 2, \dots, M$. We get immediately the following conclusion.

LEMMA 2.2. — *Let $\varphi, \psi\colon A \to B$ be unital $*$-homomorphisms such that $[\varphi] = [\psi]$ in $KK(A, B)$. Then $s^\varphi(k,j) = s^\psi(k,j)$ modulo n/d for all k, j. In other words, φ and ψ have the same small remainders modulo n/d.*

Proof. — It follows from the preceding that

$$s^\varphi(k,j) = s^\psi(k,j), \quad j = 1, 2, \dots, N-1.$$

The last small remainder, $s^\varphi(k, N)$, is determined, modulo n/d, from the fact that φ is unital; indeed $s^\varphi(k, N)$ is the remainder obtained by dividing

$$e/d - \sum_{j=1}^{N-1} s^\varphi(k,j)$$

with n/d. □

Assume now that $n \leq e$. For each

$$i \in \{1, 2, \ldots, N-1\}, \quad t_i \in \{0, 1, 2, \ldots, n/d - 1\},$$

define $\varphi_i^{t_i} \colon A_0 \to B = A(m, e, M)$ by

$$\varphi_i^{t_i}(f) = \operatorname{diag}(\underbrace{\Lambda_i^A(f), \Lambda_i^A(f), \ldots, \Lambda_i^A(f)}_{t_i \text{ times}}, 0, 0, \ldots, 0) \in M_e \subset B.$$

Then

$$(t_1, t_2, \ldots, t_{N-1}) \mapsto [\varphi_1^{t_1} \oplus \varphi_2^{t_2} \oplus \cdots \oplus \varphi_{N-1}^{t_{N-1}}]$$

defines a group homomorphism $\lambda_A \colon (\mathbb{Z}_{n/d})^{N-1} \to \varinjlim[A_0, M_n(B)]$ such that $\kappa_A^i \circ \lambda_A = id$, $i = 1, 2, \ldots, N-1$. It is clear that

$$\operatorname{im} \lambda_A \subset \ker\bigl(KK(A_0, B) \mapsto \operatorname{Hom}(K_1(A), K_1(B))\bigr).$$

By the universal coefficient theorem, **[RS]**,

$$\ker\bigl(KK(A_0, B) \to \operatorname{Hom}(K_1(A), K_1(B))\bigr)$$

is the image of a homomorphism $(\mathbb{Z}_{n/d})^{N-1} \to KK(A_0, B)$. So we see, just by counting, that

$$\operatorname{im} \lambda_A = \ker\bigl(KK(A_0, B) \to \operatorname{Hom}(K_1(A), K_1(B))\bigr)$$

when $n \leq e$. We conclude that the direct summand $(\mathbb{Z}_{n/d})^{N-1}$ of $KK(A, B)$ keeps track of the small remainders.

Note that if $e \geq Nn$, we have that every element of $\operatorname{im} \lambda_A$ is represented by a $*$-homomorphism $A_0 \to B$. If, in addition, $d|e$, every element of

$$\operatorname{im} \lambda_A = \ker\bigl(KK(A_0, B) \to \operatorname{Hom}(K_1(A), K_1(B))\bigr)$$

is of the form $[\varphi|A_0]$ for some unital $*$-homomorphism $\varphi \colon A \to B$. Indeed, if $(t_1, t_2, \ldots, t_{N-1}) \in (\mathbb{Z}_{n/d})^{N-1}$, we can set $r = e/d - \sum_{i=1}^{N-1} t_i$. Then

$$\varphi_1^{t_1} \oplus \varphi_2^{t_2} \oplus \cdots \oplus \varphi_{N-1}^{t_{N-1}} \oplus {\Lambda_N^A}^r$$

can be realized as unital $*$-homomorphism $\varphi \colon A \to M_e \subset B$ such that

$$[\varphi|A_0] = \lambda_A(t_1, t_2, \ldots, t_{N-1})$$

in $KK(A_0, B)$. Thus, if we identify $\operatorname{Hom}(K_0(A), K_0(B)) = K_0(B) = \mathbb{Z}$, we have that

(1) when $Nn \leq e$ and $d|e$, every element of the form
$$(e/d, 0, x)$$
in $\mathbb{Z} \oplus \mathrm{Hom}(K_1(A), K_1(B)) \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1} = KK(A,B)$ is represented by a unital $*$-homomorphism $A \to B$.

To proceed further into the investigation of which elements of $KK(A,B)$ are represented by unital $*$-homomorphisms from A to B, we need to take a closer look at the K_1-group of a building block of type 2. Let A be a building block of type 2. For simplicity we realize it as a subalgebra of an interval algebra, say
$$\begin{aligned} &A = A(n,d,N) = \\ &\{f \in C[0,1] \otimes M_n : f(x_j) \in M_d, j = 1,2,\ldots,N, f(0) = f(1)\}. \end{aligned}$$
In this case $K_1(A) \simeq \mathbb{Z} \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1}$, or closer to the unitaries:
$$K_1(A) \simeq \pi_1(U_n) \oplus \left(\pi_1(U_n/U_d)\right)^{N-1}.$$
Let us describe how we get from a unitary $U \in A$ to an element of $\mathbb{Z} \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1}$. By Theorem 1.4 there is a sequence W_n of unitaries such that $\lim_{n\to\infty} W_n U W_n^*$ exists and is a unitary of the following form: There are continuous functions $\mu_i \colon [0,1] \to \mathbb{T}, i = 1,2,\ldots,n$, such that
$$\#\{i : \mu_i(x_j) = t\} \in n/d\mathbb{Z}, \quad t \in \mathbb{T}, \quad j = 1,2,\ldots,N. \tag{2}$$
and a unitary $V \in C[0,1] \otimes M_n$ such that
$$\lim_{n\to\infty} W_n U W_n^*(t) = V(t)\,\mathrm{diag}\big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big) V(t), \quad t \in [0,1].$$
The element
$$[U] = (z_0, (z_1, z_2, \ldots, z_{N-1})) \in K_1(A) = \mathbb{Z} \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1}$$
can be determined from the μ_i's in the following way. Choose continuous functions $F_k \colon [0,1] \to \mathbb{R}$ such that
$$e^{2\pi i F_k(t)} = \mu_k(t),$$
$t \in [0,1]$, $k = 1,2,\ldots,n$. By condition (2) there are d continuous function $\lambda_i \colon [x_j, x_{j+1}] \to \mathbb{T}$, $i = 1,2,\ldots,d$, such that $\{\lambda_i(x_j)\}$ and $\{\lambda_i(x_{j+1})\}$ are the eigenvalues (counting multiplicities) of $U(x_j) \in M_d$ and $U(x_{j+1}) \in M_d$, respectively. Choose continuous functions $r_k \colon [x_j, x_{j+1}] \to \mathbb{R}$ such that
$$\lambda_k(x) = e^{2\pi i r_k(x)}, \ x \in [x_j, x_{j+1}].$$
Then
$$z_0 = \sum_{k=1}^{n} (F_k(1) - F_k(0)) \tag{3}$$

and

$$(4) \quad z_j = \sum_{k=1}^{n}(F_k(x_{j+1}) - F_k(x_j)) + \frac{n}{d}\sum_{k=1}^{d}(r_k(x_j) - r_k(x_{j+1})) \quad \text{modulo } \frac{n}{d}\mathbb{Z},$$

$j = 1, 2, \ldots, N-1$. Let us give the arguments for this. Firstly, the formula for z_0 follows from the fact that the class of U in $K_1(C(\mathbb{T}) \otimes M_n)$ is the winding number of the loop $t \mapsto \operatorname{Det} U(t)$. The formula for z_j is obtained as follows. Since

$$U(x_j), U(x_{j+1}) \in U_d,$$

$U|_{[x_j, x_{j+1}]}$ determines a loop in U_n/U_d and z_j is the element of $\mathbb{Z}_{n/d} = \pi_1(U_n/U_d)$ represented by this loop. Choose a unitary $S \in C[x_j, x_{j+1}] \otimes M_d$ such that

$$S(x_j) = U(x_j), S(x_{j+1}) = U(x_{j+1}).$$

Then $[U] = [US^*]$ in $\pi_1(U_n/U_d)$. But US^* is a loop in U_n and hence $[US^*] \in \pi_1(U_n/U_d) = \mathbb{Z}_{n/d}$ is the image of the winding number of the loop $\operatorname{Det} US^*(t)$ under the canonical surjection $\mathbb{Z} \to \mathbb{Z}_{n/d}$. Set

$$\gamma(x) = \exp(2\pi i n/d \sum_{k=1}^{d} r_k(x)),$$

$x \in [x_j, x_{j+1}]$. Then $\operatorname{Det} US^* = \operatorname{Det} U\gamma^{-1} \operatorname{Det} S^*\gamma$ and $[\operatorname{Det} S^*\gamma] \in n/d\mathbb{Z}$. Hence z_j is the winding number of $t \to \operatorname{Det} U(t)\gamma^{-1}(t)$ taken modulo $n/d\mathbb{Z}$, yielding (4).

Another, perhaps more transparent way to describe z_j is obtained if we first perturb U a little so that $U(x_j)$ and $U(x_{j+1})$ have d distinct eigenvalues and $U(x)$ has n distinct eigenvalues for all $x \in]x_j, x_{j+1}[$. That this is possible follows from the fact that U admits an arbitrarily close unitary approximant in $C[0,1] \otimes M_n$ with n distinct eigenvalues at every point of $[0,1]$, cf. **[E1]**, proof of Theorem 4.4. Then we can choose the F_i's such that

$$F_i(x_j) \in [0,1[\text{ for all } i,$$

$$F_1(x) < F_2(x) < \cdots < F_n(x), \text{ and}$$

$$e^{2\pi i F_k(x)} \neq e^{2\pi i F_j(x)}, \ j \neq k, \text{ for all } x \in]x_j, x_{j+1}[.$$

It follows that

$$F_{(k-1)n/d+j}(x_j) = F_{(k-1)n/d+1}(x_j), \ j = 1, 2, \ldots, n/d, \ k = 1, 2, \ldots, d.$$

Set $p = \max\{l : F_l(x_{j+1}) = F_1(x_{j+1})\}$. Then

$$(5) \qquad z_j = -p \text{ modulo } n/d\mathbb{Z}.$$

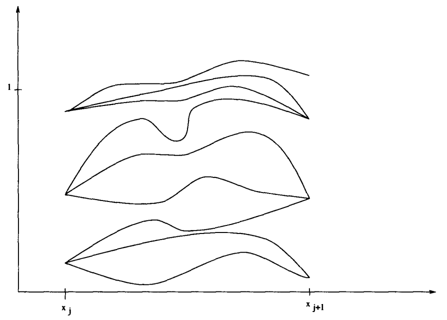

FIGURE 1. Illustration of the case $n = 9$, $d = 3$.

Indeed, we can choose the r_i's such that

$$r_k(x_j) = F_{(k-1)n/d+1}(x_j), \quad r_k(x_{j+1}) = F_{(k-1)\frac{n}{d}+1}(x_{j+1})$$

for all $k = 1, 2, \ldots, d$. Then

$$\sum_{k=1}^{n}(F_k(x_{j+1}) - F_k(x_j)) + \frac{n}{d}\sum_{k=1}^{d}(r_k(x_j) - r_k(x_{j+1})) = \frac{n}{d} - p$$

so (5) follows from (4).

LEMMA 2.3. — *Let $n, m, e \in \mathbb{N}$ such that $e|m$ and let $A = A(2mn/e, 2n, M)$ be a building block of type 2. For every homomorphism*

$$f\colon K_1(C(\mathbb{T}) \otimes M_n) \to K_1(A),$$

there is a unital $*$-homomorphism $\psi\colon C(\mathbb{T}) \otimes M_n \to A$ such that $\psi_* = f$.

Proof. — $K_1(C(\mathbb{T}) \otimes M_n) \simeq \mathbb{Z}$, generated by the class of the unitary $z_0 = \mathrm{diag}(z, 1, 1, \ldots, 1)$, and $K_1(A) \simeq \mathbb{Z} \oplus (\mathbb{Z}_{m/e})^{M-1}$. Let $\big(a_0, (a_1, a_2, \ldots, a_{M-1})\big) \in \mathbb{Z} \oplus \left(\mathbb{Z}_{m/e}\right)^{M-1} = K_1(A)$. We must exhibit a unital $*$-homomorphism

$$\psi\colon C(\mathbb{T}) \otimes M_n \to A,$$

on standard form, such that

$$[\psi(z_0)] = \big(a_0, (a_1, a_2, \ldots, a_{M-1})\big).$$

We will describe a set $\mu_1, \mu_2, \ldots, \mu_{2m/e}$ of characteristic functions for ψ. Let $0 = y_1 < y_2 < \cdots < y_M < 1$ be the exceptional points of $A(2mn/e, 2n, M)$. Choose continuous functions $F_i\colon [0,1] \to \mathbb{R}$, $i = 1, 2, \ldots, 2m/e$, such that

$$\#\left\{j : e^{2\pi i F_j(y_r)} = 1\right\} = m/e, \ \#\left\{j : e^{2\pi i F_j(y_r)} = -1\right\} = m/e,$$

for $r = 1, 2, \ldots, M$,

$$\#\left\{e^{2\pi i F_j(t)} : j = 1, 2, \ldots, 2m/e\right\} = 2m/e,$$

when $t \notin \{y_1, y_2, \ldots, y_M\} \cap [0, y_M]$, and

$$\#\{j : F_j(y_r) \in \mathbb{Z}, F_j(y_{r+1}) \notin \mathbb{Z}\} = a_r,$$

$r = 1, 2, \ldots, M-1$. These conditions can be met in many ways and are sufficient to ensure that the torsion part of $[\psi(z_0)] \in K_1(A(2mn/e, 2n, M))$ is $(a_1, a_2, \ldots, a_{M-1})$. To get the total winding number of $t \mapsto \operatorname{Det} \psi(z_0)(t)$ to become a_0 we choose the F_j's such that

$$F_j(t) = F_j(y_M), \ t \in [y_M, 1],$$

$j = 2, 3, \ldots, 2m/e$, and let $F_1 \colon [y_M, 1] \to \mathbb{R}$ be continuous such that

$$F_1(1) - F_1(y_M) = b \in \mathbb{Z}.$$

Here $b \in \mathbb{Z}$ is free to choose and, since the total winding number of $[0,1] \ni t \mapsto \operatorname{Det} \psi(z_0)(t)$ is b plus the total winding number of the loop

$$[0, y_M] \ni t \mapsto \prod_{j=1}^{\frac{2m}{e}} e^{2\pi i F_j(t)},$$

we can clearly choose b such that the total winding number of $\operatorname{Det} \psi(z_0)$ becomes a_0. □

LEMMA 2.4. — *Let $A(n,d,N)$ and $A(m,e,M)$ be building blocks of type 2 such that $d|e$ and $(N+1)n \leq e$. For any group homomorphism*

$$\chi \colon K_1(A(n,d,N)) \to K_1(A(m,e,M))$$

there is a unital $*$-homomorphism $\varphi \colon A(n,d,N) \to A(m,e,M)$ such that $\varphi_* = \chi$.

Proof. — Following the notation used by Dadarlat and Loring, **[DL2]**, we denote the unital dimension drop C^*-algebra

$$\{f \in C[0,1] \otimes M_n : f(0), f(1) \in \mathbb{C}1\}$$

by $\tilde{\mathbb{I}}_n$. Note that $K_1(M_d(\tilde{\mathbb{I}}_{n/d})) = \mathbb{Z}_{n/d}$. For any group homomorphism $\pi \colon \mathbb{Z}_{n/d} \to (\mathbb{Z}_{m/e})^{M-1}$ there is a unital $*$-homomorhism

$$\psi \colon M_d(\tilde{\mathbb{I}}_{n/d}) \to A(m/en, n, M)$$

such that $\psi_* = \pi$. To see this, choose first $m_i \in \{0, 1, 2, \ldots, m/e - 1\}$ such that $p_2(m_i z) = q_i \circ \pi(p_1(z))$, $z \in \mathbb{Z}$, $i = 1, 2, \ldots, M-1$, where $p_1 \colon \mathbb{Z} \to \mathbb{Z}_{n/d}$, $p_2 \colon \mathbb{Z} \to \mathbb{Z}_{m/e}$ and $q_i \colon (\mathbb{Z}_{m/e})^{M-1} \to \mathbb{Z}_{m/e}$ are the natural surjections; the last one to the i'th coordinate. If we consider $M_d(\tilde{\mathbb{I}}_{n/d})$ as a building block of type 4 in the natural way, (so that the exceptional points are $x_1 = 0$ and $x_2 = 1$), we may define a standard homomorphism

$$\psi\colon M_d(\tilde{\mathbb{I}}_{n/d}) \to A((m/e)n, n, M)$$

in the following way. Let $0 < y_1 < y_2 < \cdots < y_M \in]0,1[$ be the exceptional points of $A((m/e)n, n, M)$ and set $y_0 = 0, y_{M+1} = 1$. For each $i \in \{1, 2, \ldots, M\}$, we let $h_i\colon [0,1] \to [0,1]$ be the function whose graph is drawn in Figure 2.

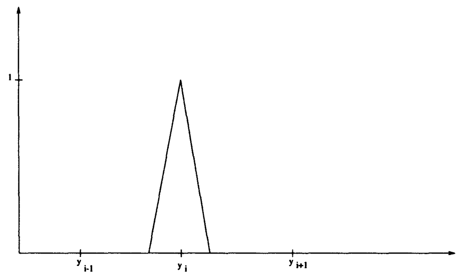

FIGURE 2

Set $a_1 = 0$ and for $j = 2, 3, \ldots, M$, let $a_j \in \{0, 1, 2, \ldots, m/e - 1\}$ be $m_1 + m_2 + \cdots + m_{j-1}$, taken modulo m/e. Then $m_j = (a_{j+1} - a_j)$ modulo m/e. Furthermore, since

$$\frac{m}{e} \mid \frac{m_j n}{d}$$

for all j, we see that

$$\frac{m}{e} \mid \frac{a_j n}{d}$$

for all j. Let σ be a permutation of $\{1, 2, \ldots, M\}$ such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(M)}$. Let ψ be the standard homomorphism whose characteristic functions consists of $a_{\sigma(1)}$ copies of $\sum_{i=1}^{M} h_{\sigma(i)}$ and $a_{\sigma(j)} - a_{\sigma(j-1)}$ copies of $\sum_{i=j}^{M} h_{\sigma(i)}, j = 2, 3, \ldots, M$, and the remainders $r_0 = (1/d)(mn/e - a_{\sigma(M)}n)$ and $r_1 = 0$. Since

$$\frac{m}{e} \mid \frac{a_{\sigma(j)} n}{d}$$

for all j (and $a_{\sigma(M)}n \leq mn/e$), these data will satisfy (20)-(22) in Chapter 1 and define a unital $*$-homomorphism $\psi\colon M_d(\tilde{\mathbb{I}}_{n/d}) \to A(m/en, n, M)$. It is straightforward to check that $\psi_* = \pi$ on $K_1(M_d(\tilde{\mathbb{I}}_{n/d}))$.

Let $0 < x_1 < x_2 < \cdots < x_N < 1$ such that

$$A(n,d,N) = \{f \in C[0,1] \otimes M_n : f(x_i) \in M_d, i = 1,2,\dots N, f(0) = f(1)\}.$$

By identifying

$$\{f \in C[x_i, x_{i+1}] \otimes M_n : f(x_i), f(x_{i+1}) \in M_d\}$$

with $M_d(\tilde{\mathbb{I}}_{n/d})$ for all $i = 1,2,\dots,N-1$, we can define a $*$-homomorphism $\gamma\colon A(n,d,N) \to \left(M_d(\tilde{\mathbb{I}}_{n/d})\right)^{N-1}$ by

$$\gamma(f) = \left(f|_{[x_1,x_2]}, \dots, f|_{[x_{N-1},x_N]}\right).$$

As shown above, we can choose, for each $j \in \{1,\dots,N-1\}$, a unital $*$-homomorphism $\lambda_j\colon M_d(\tilde{\mathbb{I}}_{n/d}) \to A((m/e)n, n, M)$ such that $\lambda_{j*} = \chi \circ \iota_j$ on $K_1(M_d(\tilde{\mathbb{I}}_{n/d}))$, where $\iota_j\colon \mathbb{Z}_{n/d} \to \left(\mathbb{Z}_{n/d}\right)^{N-1}$ is the inclusion on the j'th coordinate. Let

$$\begin{aligned}\xi\colon \left(M_d(\tilde{\mathbb{I}}_{n/d})\right)^{N-1} &\to (A((m/e)n,n,M))^{N-1}\\ &\subset A((N-1)(m/e)n,(N-1)n,M)\end{aligned}$$

be the direct sum of the λ_j's. Then

$$\xi \circ \gamma\colon A(n,d,N) \to A((N-1)(m/e)n,(N-1)n,M)$$

is a unital $*$-homomorphism such that

$$\xi_* \circ \gamma_* = \chi|_{\left(\mathbb{Z}_{n/d}\right)^{N-1}}.$$

By Lemma 2.3 there is a unital $*$-homomorphism

$$\rho\colon C(\mathbb{T}) \otimes M_n \to A(2mn/e, 2n, M)$$

such that

$$\rho_* = (\chi - \xi_* \circ \gamma_*)|_{\mathbb{Z}}.$$

Set $m_1 = m - (N+1)(m/e)n, e_1 = e - (N+1)n$ so that

$$\begin{aligned}&A((N-1)(m/e)n,(N-1)n,M) \oplus A(2(m/e)n,2n,M) \oplus A(m_1,e_1,M)\\ &\subset\ A(m,e,M)\end{aligned}$$

and define

$$\begin{aligned}&\varphi\colon A(n,d,N) \to\\ &A((N-1)(m/e)n,(N-1)n,M) \oplus A(2(m/e)n,2n,M) \oplus A(m_1,e_1,M)\\ &\subset A(m,e,M)\end{aligned}$$

by

$$\varphi = \xi \circ \gamma \oplus \rho|_A \oplus \Lambda_N^{e_1/d},$$

where $\Lambda_N^{e_1/d}\colon A \to M_{e_1} \subset A(m_1, e_1, M)$. Then $\varphi_* = \chi$ on $K_1(A)$. □

PROPOSITION 2.5. — *Let $A = A(n,d,N)$ and $B = A(m,e,M)$ be building blocks of type 2. Assume that $(2N+1)nd \leq e$.*

For any element $\alpha \in KK(A,B)$ such that $\alpha_*\colon K_0(A) \to K_0(B)$ is positive and order-unit preserving, there is a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\alpha = [\varphi]$ in $KK(A,B)$.

Proof. — Since $\alpha_*\colon K_0(A) \to K_0(B)$ is positive and order-unit preserving, it follows that $d|e$. Write $e/d = x_1 + x_2$ where $(N+1)n \leq x_1$ and $Nn \leq x_2$. Then

$$A(\frac{mx_1 d}{e}, x_1 d, M) \oplus A(\frac{mx_2 d}{e}, x_2 d, M) \subset A(m,e,M)$$

as a unital subalgebra. Let

$$i_j\colon A(mx_j d/e, x_j d, M) \longrightarrow A(m,e,M), \quad j = 1,2,$$

be the corresponding embeddings and note that $[i_j]$ is invertible in

$$KK(A(mx_j d/e, x_j d, M), B), \quad j = 1,2.$$

By Lemma 2.4 there is a unital $*$-homomorphism

$$\varphi_1\colon A \longrightarrow A(mx_1 d/e, x_1 d, M)$$

such that $\varphi_{1*} = i_{1*}^{-1} \circ \alpha_*$ on $K_1(A)$. Since $x_2 d \geq Nn$ we know from (1) that there is unital $*$-homomorphism

$$\varphi_2\colon A \longrightarrow A(mx_2 d/e, x_2 d, M)$$

such that $[i_2 \circ \varphi_2] = \alpha - [i_1 \circ \varphi_1] \in KK(A,B)$. Then

$$\varphi = \varphi_1 \oplus \varphi_2\colon A \longrightarrow B$$

is a unital $*$-homomorphism such that $\alpha = [\varphi]$ in $KK(A,B)$. □

CHAPTER 3

AN APPROPRIATE UNIQUENESS RESULT

Theorem 1.4 says that all we need to know about a unital $*$-homomorphism between building blocks can be obtained from the affine function between the tracial state spaces induced by the map. In the proof of our main result, however, we will only know the map on the level of traces approximately and, although we only ask for an approximate conclusion, Theorem 1.4 will not suffice. This is to be expected, of course, since the tracial state space, with its pairing with K_0, can not be a complete invariant. The purpose of this chapter is to obtain the substitute for Theorem 1.4, which "gives an approximate conclusion for approximate assumptions", rather than a precise conclusion for precise assumptions, and which can be made to work in the course of the proof of the main results. Thus, what we are seeking here is, in Elliotts terminology, the "uniqueness theorem".

Let

$$A(n,d,N) = \{f \in C[0,1] \otimes M_n : f(x_1), f(x_2), \dots, f(x_N) \in M_d, f(0) = f(1)\}$$

be a building block of type 2. For once it is convenient to assume that $x_1 = 0$. A unitary $U \in A(n,d,N)$ will said to be *of minimal multiplicity* when there are continuous functions $F_i \colon [0,1] \to \mathbb{R}$ such that

$$\begin{aligned}
&F_i(0) \in [0,1[, \ i = 1,2,\dots,n,\\
&F_1(t) < F_2(t) < \dots < F_n(t), \ t \notin \{x_1, x_2, \dots, x_N\},\\
&e^{2\pi i F_j(t)} \neq e^{2\pi i F_k(t)}, \ t \notin \{x_1, x_2, \dots, x_N\}, \ j \neq k,\\
&\# \operatorname{Sp} U(x_j) = d, \ j = 1,2,\dots,N,
\end{aligned}$$

and orthogonal projections

$$q_1, q_2, \dots, q_n \in C[0,1] \otimes M_n$$

such that

$$U(t) = \sum_{k=1}^{n} e^{2\pi i F_k(t)} q_k(t), \quad t \in [0,1].$$

The projections $q_1, q_2, \ldots, q_n$ are called *continuous eigenprojections* for U.

By using the fact that the unitaries with minimal multiplicity in each fiber are dense in the unitary group of $C(\mathbb{T}) \otimes M_n$, see **[E1]**, proof of Theorem 4.4, it follows easily that the unitaries of minimal multiplicity are dense in the unitary group of $A(n,d,N)$.

For each $r \in \mathbb{N}, j \in \{1,2,\ldots,r\}$, let

$$I_j^r = \{e^{2\pi i t} : t \in [(j-1)/r, j/r[\}$$

and choose a non-zero continuous function $\xi_j^r \colon \mathbb{T} \cup \{0\} \to [0,1]$ with support in I_j^r.

LEMMA 3.1. — *For each pair $k, l \in \mathbb{N}$ such that $l > 12$, there is a finite set $F_0 \subset C(\mathbb{T},[0,1])$ of non-zero elements with the following property: When U, V are unitaries in a building block, $A = A(n,d,N)$, of type 2, and $\delta > 0$ such that*

- $\theta(\xi_j^k(U)) > 1/l$, $j = 1,2,\ldots,k, \theta \in T(A)$,
- $\theta(\xi_j^{3l}(U)) > 2\delta$, $j = 1,2,\ldots,3l, \theta \in T(A)$,
- $|\theta(f(U)) - \theta(f(V))| \leq \delta$, $f \in F_0, \theta \in T(A)$,
- *there is a continuous function* $\alpha \colon \mathbb{T} \to]-n/l, n/l[$ *and a constant* $\mu \in \mathbb{T}$ *such that* $\operatorname{Det} U(t) = \mu e^{2\pi i \alpha(t)} \operatorname{Det} V(t), t \in \mathbb{T}$, *and*
- $[U] = [V]$ *in* $K_1(A)$,

then, for any finite subset $F \subset C(\mathbb{T})$ and any $\varepsilon > 0$, there is a unitary $W \in A$ such that

$$\|Wf(U)W^* - f(V)\| \leq \sup\left\{\frac{|f(s)-f(t)|}{|s-t|} : s,t \in \mathbb{T}, s \neq t\right\} \left(\frac{28}{k} + \frac{6}{l}\right)\pi + \varepsilon$$

for all $f \in F$.

Proof. — Let $F_0 \subset C(\mathbb{T},[0,1])$ be the finite subset of Lemma 2.3 of **[NT]** corresponding to $m = k$, $n = l$ and let $U, V \in A$ be unitaries meeting the five conditions of the lemma. After an initial arbitrarily small perturbation of U and V we may assume that they are both of minimal multiplicity. Let $\{q_i\}$ and $\{q'_i\}$ be the continuous eigenprojections of U and V, respectively. Thus

$$U(t) = \sum_{i=1}^{n} h_i(t) q_i(t), \ V(t) = \sum_{i=1}^{n} g_i(t) q'_i(t), \ t \in [0,1],$$

where $h_i, g_i \colon [0,1] \to \mathbb{T}$ are continuous functions such that $h_i(t) \neq h_j(t)$ and $g_i(t) \neq g_j(t)$ when $t \notin \{x_1, x_2, \dots, x_N\}$ and $i \neq j$. Since $[U] = [V]$ in $K_1(C(\mathbb{T}) \otimes M_n)$ there is a common permutation $\sigma \in \Sigma_n$ such that

$$q_i(1) = q_{\sigma(i)}(0), \ q_i'(1) = q_{\sigma(i)}'(0)$$

for all $i = 1, 2, \dots, n$. We can therefore find a unitary $S \in C[0,1] \otimes M_n$ with $S(0) = S(1)$ such that $Sq_iS^* = q_i'$, $i = 1, 2, \dots, n$. The second part of the proof of Lemma 2.3 in **[NT]** now applies to show that

$$|g_i(t) - h_i(t)| \leq (28/k + 6/l)\pi$$

for all i and all $t \in [0,1]$. For each $j \in \{1, 2, \dots, N\}$ there are partitions

$$\{1, 2, \dots, n\} = P_1^U(j) \cup P_2^U(j) \cup \dots \cup P_d^U(j),$$

and

$$\{1, 2, \dots, n\} = P_1^V(j) \cup P_2^V(j) \cup \dots \cup P_d^V(j),$$

such that

$$g_i(x_j) = g_k(x_j), \ i, k \in P_l^U(j), \ h_i(x_j) = h_k(x_j), \ i, k \in P_l^V(j),$$

$l = 1, 2, \dots, d$. It follows from the description in Chapter 2 of the class in $K_1(A(n,d,N))$ represented by U and V, that the two partitions are identical for each j. Set $P_l(j) = P_l^U(j) = P_l^V(j)$, $l = 1, 2, \dots, d$, $j = 1, 2, \dots, N$. For each j we choose a small interval Ω_j around x_j such that

$$\Omega_j \cap \Omega_i = \varnothing, i \neq j,$$

and such that

$$\begin{aligned}
&\sum_i |f \circ h_i(x_j)| \|q_i(t) - q_i(x_j)\| \leq \varepsilon/4, \\
&\sum_i |f \circ g_i(x_j)| \|q_i'(t) - q_i'(x_j)\| \leq \varepsilon/4, \\
&\|f(U)(t) - f(U)(x_j)\| \leq \varepsilon/4, \text{ and} \\
&\|f(V)(t) - f(V)(x_j)\| \leq \varepsilon/4, \ t \in \Omega_j,
\end{aligned}$$

for all j and all $f \in F$. Since

$$\sum_{i \in P_l(j)} q_i(x_j), \ \sum_{i \in P_l(j)} q_i'(x_j) \in M_d,$$

there are unitaries $T_j \in M_d$ such that

$$T_j \sum_{i \in P_l(j)} q_i(x_j) T_j^* = \sum_{i \in P_l(j)} q_i'(x_j)$$

for all $l = 1, 2, \ldots, d$, and all j. Then $T_j^* S(x_j)$ commutes with each $\sum_{i \in P_l(j)} q_i(x_j)$. Let $V \in C(\mathbb{T}) \otimes M_n$ be a unitary such that

$$
\begin{aligned}
&V(t) = 1, \; t \notin \cup_j \Omega_j, \\
&[V(t), \sum_{i \in P_l(j)} q_i(x_j)] = 0, \; t \in \Omega_j, l = 1, 2, \ldots, d, \text{ and} \\
&V(x_j) = S(x_j)^* T_j
\end{aligned}
$$

for all j. Set $W = SV$ and note that $W \in A(n, d, N)$. We have that

$$
\begin{aligned}
&\|Wf(U)W^*(t) - f(V)(t)\| \\
&\le \|W(t)f(U)(x_j)W(t)^* - f(V)(x_j)\| + \frac{2\varepsilon}{4} \\
&= \|S(t)f(U)(x_j)S(t)^* - f(V)(x_j)\| + \frac{2\varepsilon}{4} \\
&\le \|S(t)\sum_{i=1}^{n} f \circ h_i(x_j) q_i(t) S(t)^* - \sum_{i=1}^{n} f \circ g_i(x_j) q_i'(t)\| + \varepsilon \\
&\le \sup \left\{ \frac{|f(s) - f(t)|}{|s - t|} : s, t \in \mathbb{T}, s \neq t \right\} (\frac{28}{k} + \frac{6}{l})\pi + \varepsilon
\end{aligned}
$$

for all $t \in \Omega_j$ and all j. Since

$$
\begin{aligned}
&\|Wf(U)W^*(t) - f(V)(t)\| = \|Sf(U)S^*(t) - f(V)(t)\| \le \\
&\sup \{|f(s) - f(t)|/|s - t| : s, t \in \mathbb{T}, s \neq t\} (28/k + 6/l)\pi
\end{aligned}
$$

for $t \notin \cup_j \Omega_j$, the proof is complete. $\square$

The next issue will be 'eigenvalue crossovers'; a notion introduced by George Elliott in **[E3]**. Here, of course, we shall use a version of the procedure for maps between building blocks of type 2. But first we need to introduce a collection of generators for such building blocks which we can consider to be canonical. So let $A = A(n, d, N)$ be a building block of type 2 with exceptional points at $x_1, x_2, \ldots, x_N \in \mathbb{T}$. Set

$$\delta_A = 1/2 \min \{|x_i - x_j| : i \neq j\}$$

when $N \geq 2$, and $\delta_A = 1$ when $N = 1$. (It must be remarked here that we can always take $N \neq 0$. Indeed, $C(\mathbb{T}) \otimes M_n = A(n, n, N)$ for all $N \in \mathbb{N}$.) Set

$$g_i(t) = \max \{0, 1 - (1/\delta_A)|t - x_i|\}.$$

Then g_i vanishes at x_j for all $j \neq i$ and takes the value 1 at x_i. Furthermore, we have control of the variation of g_i;

$$|g_i(s) - g_i(t)| \le (1/\delta_A)|t - s|, \; t, s \in \mathbb{T}.$$

As in the proof of Theorem 1.4 we shall also use the function

$$z_A(t) = t \operatorname{dist}(t, \{x_1, x_2, \dots, x_N\}),\ t \in \mathbb{T}.$$

Let $\{e_{ij}\}$ and $\{p_{ij}\}$ be the canonical matrix units in M_n and $M_d \subset M_n$, respectively. The set

$$\cup_{k=1}^{N} \{g_k \otimes p_{ij}\} \cup \{z_A \otimes e_{ij}\}$$

generates A as a C^*-algebra and hence could serve as the canonical set of generators. However, it is convenient to include the following additional elements.

Let $0 \le y_1 < y_2 < \cdots < y_N$ be points in $[0,1[$ such that $e^{2\pi i y_j} = x_j$, $j = 1, 2, \dots, N$. We take y_A to be the function $y_A(e^{2\pi i t}) = e^{2\pi i H(t)}$, where $H\colon [0,1] \to [0,1]$ is 0 on $[0, y_1]$, grows linearly from 0 to 1 on $[y_1, y_2]$ and is constant equal to 1 on $[y_2, 1]$. If $N = 1$, we can take y_A to be the identity function z on $\mathbb{T}$. As *the convenient set of generators* for A we take

$$cg(A) = \cup_{k=1}^{N} \{g_k \otimes p_{ij}\} \cup \{z_A \otimes e_{ij}\} \cup \{z \otimes 1\} \cup \left\{ y_A \otimes e_{11} + \sum_{i \ge 2} 1 \otimes e_{ii} \right\}.$$

Note that $[y_A \otimes e_{11} + \sum_{i\ge 2} 1 \otimes e_{ii}]$ generates the direct summand $\mathbb{Z}$ of $K_1(A) = \mathbb{Z} \oplus \left(\mathbb{Z}_{n/d}\right)^{N-1}$. We observe that we have the estimate

$$\|f(s) - f(t)\| \le \frac{2}{\delta_A}|s - t| \quad s, t \in \mathbb{T},$$

for all $f \in cg(A)$.

We shall also need some additional notation. When $\varphi\colon A \to B$ is a unital $*$-homomorphism between C^*-algebras, we let $\hat{\varphi}$ denote the map $\operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ induced by φ, viz. $\hat{\varphi}(f)(\omega) = f(\omega \circ \varphi), f \in \operatorname{Aff} T(A), \omega \in T(B)$. When a is a selfadjoint element of A we denote the corresponding element of $\operatorname{Aff} T(A)$ by $\hat{a}$, i.e. $\hat{a}(\omega) = \omega(a)$, $\omega \in T(A)$. Note that when $A = A(n,d,N)$ is a building block of type 2, we can identify $\operatorname{Aff} T(A)$ with the selfadjoint part of the center of A, i.e. with $C_{\mathbb{R}}(\mathbb{T})$. When $g \in C_{\mathbb{R}}(\mathbb{T})$, we will not distinguish between g considered as a central element of A and g considered as an element of $\operatorname{Aff} T(A)$.

LEMMA 3.2 (A single eigenvalue crossover). — *Let*

$$\varphi\colon A(n,d,N) \to A(m,e,M)$$

be a unital $*$-homomorphism between building blocks, $A = A(n,d,N)$ and $B = A(m,e,M)$, of type 2. Let $y_1, y_2, \dots, y_M$ be the exceptional points of B. Assume that φ is of standard form and let $\mu_1, \mu_2, \dots, \mu_L\colon [0,1] \to \mathbb{T}$ be characteristic functions for φ.

If $t \in]0,1[\setminus \{y_1, y_2, \dots, y_M\}$ *and* $i, j \in \{1, 2, \dots, L\}$ *are such that* $|\mu_i(t) - \mu_j(t)| < \varepsilon$, *then, for all sufficiently small* $\kappa > 0$, *there is a unital* $*$*-homomorphism* $\psi\colon A \to B$ *with the same small remainders as* φ *such that*

- $\operatorname{Det} \varphi(z \otimes 1)(t) = \operatorname{Det} \psi(z \otimes 1)(t)$, $t \in \mathbb{T}$,
- $\|\hat{\varphi} - \hat{\psi}\| \leq 2n/m$,
- $\|\varphi(x) - \psi(x)\| \leq 2\varepsilon/\delta_A$, $x \in cg(A)$,
- *there are characteristic functions* ν_k, $k = 1, 2, \dots, L$, *for* ψ, *such that* $\mu_k = \nu_k$, $k \notin \{i, j\}$, $\nu_i(s) = \mu_i(s)$, $\nu_j(s) = \mu_j(s)$ *when* $s \leq t - \kappa$, $\nu_i(s) = \mu_j(s)$, $\nu_j(s) = \mu_i(s)$ *when* $s \geq t + \kappa$, *and* $|\nu_i(s) - \mu_i(s)| < \varepsilon$, $|\nu_j(s) - \mu_j(s)| < \varepsilon$ *when* $s \in [t - \kappa, t + \kappa]$,
- $\varphi(f)(s) = \psi(f)(s)$, $s \notin [t - \kappa, t + \kappa]$, $f \in A$.

Proof. — Without loss of generality we may assume that $i = 1$, $j = 2$. We have that

$$\begin{aligned}\varphi(f)(s) = u(s) \operatorname{diag}\big(&f(\mu_1(s)), f(\mu_2(s)), \dots, f(\mu_L(s)),\\ &\Lambda_1^{r_1}(f), \Lambda_2^{r_2}(f), \dots, \Lambda_N^{r_N}(f)\big) u(s)^*,\end{aligned}$$

$s \in [0,1], f \in A$, for some unitary $u \in C[0,1] \otimes M_m$. Let $\kappa > 0$ be so small that $[t - \kappa, t + \kappa] \cap \{y_1, y_2, \dots, y_M\} = \varnothing$ and $|\mu_i(s) - \mu_j(s)| < \varepsilon$ when $|t - s| < \kappa$. Choose continuous functions $\nu_k\colon [0,1] \to \mathbb{T}, k = 1, 2, \dots, L$, such that the fourth requirement of the lemma is satisfied and, at the same time,

$$\nu_1(s)\nu_2(s) = \mu_1(s)\mu_2(s), \quad s \in [0,1].$$

Let v be the permutation unitary in $U_{2n} \subset U_m$ which exchanges the first and second n-block on the diagonal, specifically

$$v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1 \in M_2 \otimes M_n.$$

There is then a path $w_0\colon [0,1] \to U_2$ such that

$$w_0(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s \leq t - \kappa,$$

$$w_0(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s \geq t + \kappa.$$

Set $w(s) = w_0(s) \otimes 1 \in U_{2n} \subset U_m$ and

$$\begin{aligned}&\psi(f)(s) =\\ &u(s)w(s) \operatorname{diag}\big(f(\nu_1(s)), \dots, f(\nu_L(s)), \Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f)\big) w(s)^* u(s)^*,\end{aligned}$$

$s \in [0,1], f \in A$. Then ψ maps into B and it is straightforward to check that ψ meets the requirements. $\square$

LEMMA 3.3 (Multiple eigenvalue crossovers). — *Let*

$$\varphi : A(n,d,N) \to A(m,e,M)$$

be a unital $*$-homomorphism between building blocks, $A = A(n,d,N)$ and $B = A(m,e,M)$, of type 2. Let $y_1, y_2, \dots, y_M \in [0,1]$ be the exceptional points of B. Assume that φ is on standard form and let $\mu_1, \mu_2, \dots, \mu_L \colon [0,1] \to \mathbb{T}$ be characteristic functions for φ.

Let $\kappa_1, \kappa_2, \dots, \kappa_R \colon [0,1] \to \mathbb{T}$ be continuous functions and $\varepsilon > 0$ such that

(A) *for each $s \in \{y_1, y_2, \dots, y_M\} \cup \{0,1\}$, there are mutually distinct elements $i_1, i_2, \dots, i_R \in \{1,2,\dots,L\}$ such that $\mu_{i_j}(s) = \kappa_j(s)$, $j = 1,2,\dots,R$, and*

(B) *for each $t \in [0,1]$, there are mutually distinct elements $m_1, m_2, \dots, m_R \in \{1,2,\dots,L\}$ such that $|\kappa_j(t) - \mu_{m_j}(t)| < \varepsilon$, $j = 1,2,\dots,R$.*

It follows that there is a unital $*$-homomorphism $\psi \colon A \to B$ such that

- *φ and ψ have the same small remainders,*
- $\operatorname{Det} \varphi(z \otimes 1)(t) = \operatorname{Det} \psi(z \otimes 1)(t)$, *$t \in \mathbb{T}$,*
- $\|\hat{\varphi} - \hat{\psi}\| \le 2n/m$,
- $\|\varphi(x) - \psi(x)\| \le 4\varepsilon/\delta_A$, *$x \in cg(A)$,*
- *There are characteristic functions, $\nu_1, \nu_2, \dots, \nu_L$, for ψ such that*

$$|\kappa_i(t) - \nu_i(t)| \le 5\varepsilon, \quad t \in [0,1],$$

and

$$\kappa_i(x) = \nu_i(x), \ x \in \{y_1, y_2, \dots, y_M\} \cup \{0,1\}\,, \ for all i = 1,2,\dots,R,$$

- $\big(\nu_1(x), \nu_2(x), \dots, \nu_L(x)\big) = \big(\mu_1(x), \mu_2(x), \dots, \mu_L(x)\big)$ *as unordered L-tuples for all $x \in \{y_1, y_2, \dots, y_M\} \cup \{0,1\}$.*

Proof. — Choose $0 = s_0 < s_1 < \dots < s_T = 1$ such that

$$x, y \in [s_l, s_{l+1}] \Rightarrow |\mu_i(x) - \mu_i(y)| < \varepsilon, \quad |\kappa_j(x) - \kappa_j(y)| < \varepsilon,$$

$i = 1,2,\dots,L$, $j = 1,2,\dots,R$, $l = 0,1,2,\dots T-1$. We may arrange that $\{y_1, y_2, \dots, y_M\} \subset \{s_0, s_1, \dots, s_T\}$. For each $k \in \{0,1,2,\dots,T\}$ we choose distinct elements $m_1^k, m_2^k, \dots, m_R^k \in \{1,2,\dots,L\}$ such that

$$|\mu_{m_j^k}(s_k) - \kappa_j(s_k)| < \varepsilon, \ j = 1,2,\dots,R.$$

If $s_k \in \{y_1, y_2, \dots, y_M\} \cup \{0,1\}$, we ensure that

$$\mu_{m_j^k}(s_k) = \kappa_j(s_k), \ j = 1,2,\dots,R.$$

Perform a single eigenvalue crossover in a small interval in the interior of $]s_0, s_1[$ such that the resulting $*$-homomorphism has the same characteristic functions as φ, except that $\mu_{m_1^0}$ and $\mu_{m_1^1}$ have been interchanged. In a second small interval, disjoint from the first, we perform another eigenvalue crossover, now with the new $*$-homomorphism, in order to interchange $\mu_{m_2^0}$ with $\mu_{m_2^1}$. By continuing through R single eigenvalue crossovers in this way, performed over mutually disjoint subintervals in $]s_0, s_1[$, we get a unital $*$-homomorphism $\varphi_1 \colon A \to B$ such that the first four requirements of the lemma hold with $\psi = \varphi_1$ and

(1) there are characteristic functions $\mu_1^1, \dots, \mu_L^1$ for φ_1 such that $\mu_i^1(t) = \mu_i(t)$, $t \geq s_1$, $i = 1, 2, \dots, L$, and $|\mu^1_{m_j^1}(t) - \kappa_j(t)| \leq 5\varepsilon$, $t \in [0, s_1]$, $j = 1, 2, \dots, R$, and
(2) $\mu^1_{m_j^1}(x) = \kappa_j(x)$, $x \in [0, s_1] \cap \big(\{0,1\} \cup \{y_1, y_2, \dots, y_M\}\big)$, $j = 1, 2, \dots, R$.

In the interval $]s_1, s_2[$ we perform a series of single eigenvalue crossovers with φ_1 in the same way, in order to exchange $\mu_{m_j^1}$ with $\mu_{m_j^2}$, $j = 1, 2, \dots, R$. The result is a unital $*$-homomorphism $\varphi_2 \colon A \to B$ such that the first four requirements of the lemma hold with $\psi = \varphi_2$ and

(3) there are characteristic functions $\mu_1^2, \dots, \mu_L^2$ for φ_2 such that $\mu_i^2(t) = \mu_i(t)$, $t \geq s_2$, $i = 1, 2, \dots, L$, $|\mu^2_{m_j^2}(t) - \kappa_j(t)| \leq 5\varepsilon$, $t \in [0, s_2]$, $j = 1, 2, \dots, R$, and
(4) $\mu^2_{m_j^2}(x) = \kappa_j(x)$, $x \in [0, s_2] \cap \big(\{0,1\} \cup \{y_1, y_2, \dots, y_M\}\big)$, $j = 1, 2, \dots, R$.

After T steps of this kind, we reach a unital $*$-homomorphism ψ with the stated properties. $\square$

LEMMA 3.4. — *Let $A = A(n, d, N)$ be a building block of type 2. For every pair $k, l \in \mathbb{N}$ with $l > 12$, $24\pi/(\delta_A k) < 1$, there is a finite subset*

$$H \subset C(\mathbb{T}, [0,1]) \subset A$$

of non-zero elements with the following property: When $\varphi, \psi \colon A \to B$ are unital $*$-homomorphisms into another building block, $B = A(m, e, M)$, of type 2, satisfying the following requirements:

1. *φ and ψ have the same small remainders (modulo n/d),*
2. *$\hat{\varphi}(\xi_j^k) > 2/l$, $j = 1, 2, \dots, k$,*
3. *$\hat{\varphi}(g) > 3\kappa$, $g \in H$,*
4. *$\|\hat{\varphi}(g) - \hat{\psi}(g)\| < \kappa/2$, $g \in H$,*
5. *there is a continuous function $\alpha \colon \mathbb{T} \to]-(\kappa/2)m, (\kappa/2)m[$ and a $\mu \in \mathbb{T}$ such that* $\operatorname{Det} \varphi(z \otimes 1)(t) = \mu e^{2\pi i \alpha(t)} \operatorname{Det} \psi(z \otimes 1)(t), t \in \mathbb{T}$,
6. *$16Nn/e < \kappa$,*
7. *$\varphi_* = \psi_*$ on $K_1(A)$,*

for some $\kappa < 1/(2l)$*, then there is a unitary* $w \in B$ *such that*

$$\|\operatorname{Ad} w \circ \varphi(a) - \psi(a)\| < (\frac{72}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad a \in cg(A).$$

Proof. — Set $H = \left\{\xi_j^k\right\} \cup \left\{\xi_j^{3l}\right\} \cup F_0$ where F_0 is the finite set from Lemma 3.1 corresponding to the present k and l. Let

$$x_1, x_2, \ldots, x_N \in \mathbb{T} \quad \text{and} \quad t_1, t_2, \ldots, t_M \in]0,1[$$

be the exceptional points of A and B, respectively. We may assume, by Lemma 1.7, that φ and ψ are on standard form and of minimal multiplicity. Since φ and ψ have the same small remainders, they also have the same remainders $r_1, r_2, \ldots, r_N$, cf. (25) in Chapter 1. Let

$$\mu_i, \nu_i \colon [0,1] \to \mathbb{T}, \quad i = 1, 2, \ldots, L,$$

be characteristic functions for φ and ψ, respectively. From the fact that φ and ψ have the same small remainders we deduce that

$$\#\{i : \mu_i(t_r) = x_j\} = \#\{i : \nu_i(t_r) = x_j\} \equiv N_{rj}$$

for all $r = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$. Set $L_0 = \max_r \sum_j N_{rj}$ and note that $L_0/m \leq N/e$ because φ (and ψ) is of minimal multiplicity, cf. (24) in Chapter 1. We choose continuous functions $\kappa_i \colon [0,1] \to \mathbb{T}$, $i = 1, 2, \ldots, L_0$, such that

$$\#\{i : \kappa_i(t_r) = x_j\} = N_{rj}$$

for all r, j, $\kappa_i(0) = \kappa_i(1)$ for all i and the $\kappa_i(0)$'s are mutually distinct. Additionally, we want that m/e divides $\#\{i : \kappa_i(t_r) = s\}$ for all r and all $s \in \mathbb{T} \setminus \{x_1, x_2, \ldots, x_N\}$. This can be achieved because $L - \sum_j N_{rj}$ is m/e-divisible for all r. Let L' be a subset of $\{1, 2, \ldots, L\}$ obtained by removing $\leq L_0$ elements. Then, for any $g \in C([0,1], \mathbb{T})$,

$$\begin{aligned}
\|\hat{\varphi}(g) - \frac{n}{m} \sum_{i \in L'} g \circ \mu_i\| &\leq \|\hat{\varphi}(g) - \frac{n}{m} \sum_{i=1}^{L} g \circ \mu_i\| + \frac{n}{m} L_0 \\
&\leq \frac{\sum_{i=1}^N r_i d}{m} + \frac{n}{m} L_0 \leq \frac{Nn}{m} + \frac{n}{m} L_0 \leq \frac{Nn}{e} + \frac{nN}{e} = \frac{2Nn}{e}.
\end{aligned}$$

Similarly,

$$\|\hat{\psi}(g) - \frac{n}{m} \sum_{i \in L'} g \circ \nu_i\| \leq \frac{2Nn}{e}.$$

It follows from 6. that $2Nn/e \leq \kappa/8$, so 3. implies that

$$\frac{n}{m}\sum_{i\in L'} \xi_j^k \circ \mu_i(t) \ > \ 2\kappa \ > \ 0$$

for all j, t. Using 4. we find that

$$\|\frac{n}{m}\sum_{i\in L'} g\circ\mu_i - \frac{n}{m}\sum_{i\in L'} g\circ\nu_i\| \ \leq \ \frac{3\kappa}{4}$$

so that

$$\frac{n}{m}\sum_{i\in L'} \xi_j^k \circ \nu_i(t) \ \geq \ 2\kappa - \kappa = \kappa \ > \ 0$$

for all j, t. It follows that for each $t \in [0,1]$, the sets $\{\mu_i(t) : i \in L'\}$ and $\{\nu_i(t) : i \in L'\}$ are $2\pi/k$-dense in $\mathbb{T}$. In particular, by perturbing each κ_i in neighbourhoods of 0 and 1 we can obtain continuous functions $\kappa_i^\varphi, \kappa_i^\psi : [0,1] \to \mathbb{T}$, $i = 1, 2, \ldots, L_0$, which have the same properties as $\{\kappa_i\}$ and, in addition, satisfy that

$$\begin{gathered}\|\kappa_i - \kappa_i^\varphi\| \leq 2\pi/k, \quad \|\kappa_i - \kappa_i^\psi\| \leq 2\pi/k \text{ for all } i,\\ \{\kappa_i^\varphi(0)\} \subset \{\mu_i(0)\} \quad \text{and} \quad \left\{\kappa_i^\psi(0)\right\} \subset \{\nu_i(0)\}.\end{gathered}$$

We can therefore meet the two conditions of Lemma 3.3 with $R = L_0$ and $\varepsilon = 2\pi/k$, both for φ and ψ, and hence perform eigenvalue crossovers to perturb φ and ψ to φ_1 and ψ_1, respectively, such that the characteristic functions, $\{\mu_i'\}$ of φ_1 and $\{\nu_i'\}$ of ψ_1, satisfy

(5) $\quad \mu_i'(0) = \mu_i'(1), \quad \nu_i'(0) = \nu_i'(1), \quad i = L - L_0 + 1, L - L_0 + 2, \ldots, L,$

(6) $\quad \mu_i'(t_r) = \nu_i'(t_r) = \kappa_{i-(L-L_0)}(t_r), \quad i = L - L_0 + 1, L - L_0 + 2, \ldots, L,$

for all r, and

$$\begin{aligned}|\mu'_{(L-L_0)+i}(t) - \kappa_i^\varphi(t)| \leq 10\pi/k,\\ |\nu'_{(L-L_0)+i}(t) - \kappa_i^\psi(t)| \leq 10\pi/k\end{aligned}$$

for all $t \in [0,1]$ and all $i = 1, 2, \ldots, L_0$. The last two conditons imply that

(7)

$$|\mu_i'(t) - \kappa_{i-(L-L_0)}(t_r)| \leq \frac{12\pi}{k}, \quad t \in [0,1], \ i = L - L_0 + 1, L - L_0 + 2, \ldots, L,$$

and

(8)

$$|\nu_i'(t) - \kappa_{i-(L-L_0)}(t_r)| \leq \frac{12\pi}{k}, \quad t \in [0,1], \ i = L - L_0 + 1, L - L_0 + 2, \ldots, L.$$

By combining 6. with condition 6. of Lemma 3.3 we find that

$$(9)\qquad \begin{aligned} &\#\left\{i \in \{1,2,\dots,L-L_0\} : \mu_i'(t_r) = x_j\right\} \\ &= \#\left\{i \in \{1,2,\dots,L-L_0\} : \nu_i'(t_r) = x_j\right\} = 0, \end{aligned}$$

for all r, j. Combining (5) with Lemma 3.3 we conclude that

$$\left(\mu_1'(0),\dots,\mu_{L-L_0}'(0)\right) = \left(\mu_1'(1),\dots,\mu_{L-L_0}'(1)\right)$$

as unordered tuples. Similarly, $\nu_i'(0) = \nu_i'(1)$ for all $i = L-L_0+1, L-L_0+2,\dots,L$, and

$$\left(\nu_1'(0),\dots,\nu_{L-L_0}'(0)\right) = \left(\nu_1'(1),\dots,\nu_{L-L_0}'(1)\right),$$

as unordered tuples. By Lemma 3.3 we have

$$\begin{aligned} &\|\hat{\varphi} - \hat{\varphi}_1\| \le 2n/m, \\ &\|\hat{\psi} - \hat{\psi}_1\| \le 2n/m, \\ &\|\varphi(x) - \varphi_1(x)\| \le 8\pi/(k\delta_A), \\ &\|\psi(x) - \psi_1(x)\| \le 8\pi/(k\delta_A), \ x \in cg(A), \end{aligned}$$

and

$$\begin{aligned} &\operatorname{Det}\varphi_1(z\otimes 1)(t) = \operatorname{Det}\varphi(z\otimes 1)(t), \\ &\operatorname{Det}\psi_1(z\otimes 1)(t) = \operatorname{Det}\psi(z\otimes 1)(t), \ t \in \mathbb{T}. \end{aligned}$$

Set $m' = m - (L-L_0)n$, $e' = m'e/m$, $m'' = (L-L_0)n$, $e'' = m''e/m$ and $B_2 = A(m', e', M)$, $B_3 = A(m'', e'', M)$. Then $B_2 \oplus B_3 \subset B$ as a unital C^*-subalgebra. Up to approximate inner equivalence φ_1 and ψ_1 are direct sums of two $*$-homomorphisms of standard form, $\varphi_2 \colon A \to B_2$, $\varphi_3 \colon A \to B_3$ and $\psi_2 \colon A \to B_2$, $\psi_3 \colon A \to B_3$, respectively, such that φ_2 and ψ_2 are given by the remainders

$$r_j^{\varphi_2} = r_j^{\varphi} = r_j^{\psi} = r_j^{\psi_2}, \ j = 1,2,\dots,N,$$

and the characteristic functions

$$\mu_i', \ i = L-L_0+1,\dots,L,$$

and

$$\nu_i', \ i = L-L_0+1,\dots,L,$$

respectively, and φ_3 and ψ_3 are given by the remainders

$$r_j^{\varphi_3} = r_j^{\psi_3} = 0, \ j = 1,2,\dots,N,$$

and the characteristic functions

$$\mu_i', \ i = 1,2,\dots,L-L_0,$$

and

$$\nu'_i,\ i = 1, 2, \dots, L - L_0,$$

respectively. It follows from (6) - (8) that φ_2 and ψ_2 may be taken such that

$$\|\varphi_2(x) - \psi_2(x)\| \le \frac{48\pi}{\delta_A k}, \quad x \in cg(A). \tag{10}$$

In particular, since $48\pi/(\delta_A k) < 2$ and $y = y_A \otimes e_{11} + \sum_{i\geq 2} 1 \otimes e_{ii} \in cg(A)$, it follows that

$$[\varphi_3(y)] = [\psi_3(y)]$$

in $K_1(B_3)$. φ_3 and ψ_3 satisfy that

$$\|\hat{\varphi}_3 - \hat{\varphi}\| \le \frac{2n + L_0 n + Nn}{m} \le \frac{4Nn}{e},$$

$$\|\hat{\psi}_3 - \hat{\psi}\| \le \frac{2n + L_0 n + Nn}{m} \le \frac{4Nn}{e}.$$

Since $\dfrac{24\pi}{k} < 1$, it follows from (5), (7) and (8) that

$$\prod_{i=L-L_0+1}^{L} \mu'_i(t) = e^{2\pi i \beta'(t)} \prod_{i=L-L_0+1}^{L} \nu'_i(t)$$

for some function $\beta' \colon [0,1] \to \mathbb{R}$ such that $\beta'(0) = \beta'(1)$ and

$$|\beta'(t)| \le \frac{12 L_0 \pi}{k}, \quad t \in [0,1].$$

Since

$$\frac{\kappa}{2} + \frac{12 L_0 \pi}{mk} < \frac{\kappa}{2} + \frac{12 N\pi}{ek} \le \frac{\kappa}{2} + \frac{N}{e} < \kappa,$$

it follows from (5) that

$$\operatorname{Det} \varphi_3(z \otimes 1)(t) = \mu e^{2\pi i \gamma(t)} \operatorname{Det} \psi_3(z \otimes 1)(t), \quad t \in \mathbb{T}, \tag{11}$$

for some continuous $\gamma \colon \mathbb{T} \to]-\kappa m, \kappa m[$ and some $\mu \in \mathbb{T}$. Furthermore,

(12) $\|\hat{\varphi}_3(g) - \hat{\psi}_3(g)\| \ \le\ 8\dfrac{Nn}{e} + \|\hat{\varphi}(g) - \hat{\psi}(g)\| = \kappa, \quad g \in H,$
(by 4. and 6.),

(13) $\hat{\varphi}_3(g) \ \ge\ \hat{\varphi}(g) - 4\dfrac{Nn}{e} \ >\ 2\kappa, \quad g \in H,$ (by 3. and 6.),

and

(14) $\hat{\varphi}_3(\xi_j^k) \ge \hat{\varphi}(\xi_j^k) - 3\dfrac{Nn}{e} \ >\ \dfrac{1}{l}, \quad j = 1, 2, \dots, k,$ (by 2. and 6.).

Note that (5) and (9) imply the existence of unital $*$-homomorphisms $\varphi_4, \psi_4 \colon C(\mathbb{T}) \otimes M_n \to B_3$ extending φ_3 and ψ_3, respectively. The above estimates, (12) - (14), hold for φ_4 and ψ_4 also. Take a unitary $w_1 \in B_3$ such that $\operatorname{Ad} w_1 \circ \varphi_4(1 \otimes e_{ij}) = \psi_4(1 \otimes e_{ij})$ for all i,j. Set $B_4 = B_3 \cap \{\psi_4(1 \otimes e_{ij})\}'$ and note that B_4 is also a building block of type 2. It follows from (11) that

$$\operatorname{Det}' \psi_4(z \otimes 1)(t) = \mu' e^{2\pi i \frac{\gamma(t)}{n}} \operatorname{Det}' \operatorname{Ad} w_1 \circ \varphi_4(z \otimes 1)(t),$$

$t \in \mathbb{T}$, for some $\mu' \in \mathbb{T}$, where Det' denotes the determinant in M_{L-L_0}. In order to apply Lemma 3.1 to the unitaries $\operatorname{Ad} w_1 \circ \varphi_4(z \otimes 1)$ and $\psi_4(z \otimes 1)$, we need to know that $|\gamma(t)/n| < (L - L_0)/l$, $t \in \mathbb{T}$. We have that

$$|\frac{\gamma(t)}{n}| \leq \frac{\kappa m}{n}, \ t \in \mathbb{T}.$$

Since $m \leq Ln + Nn$ we see that

$$\frac{(L-L_0)n}{lm} \geq \frac{m - Nn - L_0 n}{lm} \geq \frac{1}{l} - 2\frac{Nn}{el} \geq \frac{1}{l} - \kappa \geq \frac{1}{2l} > \kappa,$$

from which we get the desired bound, $|\gamma(t)/n| < (L - L_0)/l$, $t \in \mathbb{T}$. In order to meet the last condition of Lemma 3.1 we observe that y_A is homotopic to z in $U(C(\mathbb{T}))$ so that

$$\begin{aligned}[\operatorname{Ad} w_1 \circ \varphi_4(z \otimes e_{11} + \sum_{i \geq 2} 1 \otimes e_{ii})] &= [\varphi_4(y)] \\ = [\varphi_3(y)] = [\psi_3(y)] &= [\psi_4(z \otimes e_{11} + \sum_{i \geq 2} 1 \otimes e_{ii})]\end{aligned}$$

in $K_1(B_3)$. It follows that

$$[\operatorname{Ad} w_1 \circ \varphi_4(z \otimes 1)] = [\psi_4(z \otimes 1)]$$

in $K_1(B_4)$. We can now conclude from Lemma 3.1 that there is a unitary $w_2 \in B_4$ such that

$$\begin{aligned}&\| \operatorname{Ad} w_2 w_1 \circ \varphi_4(f \otimes 1) - \psi_4(f \otimes 1) \| \\ &\leq \sup \left\{ \frac{|f(s) - f(t)|}{|t-s|} : s,t \in \mathbb{T},\ s \neq t \right\} \left(\frac{28}{k} + \frac{6}{l}\right)\pi + \frac{\pi}{\delta_A l}\end{aligned}$$

for all $f \in \{g_k : k = 1, 2, \ldots, N\} \cup \{z_A\} \cup \{z\} \cup \{y_A\}$, cf. the definition of $cg(A)$. Then $w_3 = w_2 w_1 \in B_3$ is a unitary such that

$$\| \operatorname{Ad} w_3 \circ \varphi_4(x) - \psi_4(x) \| < \left(\frac{56}{\delta_A k} + \frac{13}{\delta_A l}\right)\pi, \quad x \in cg(A).$$

Combined with the previous perturbations we get a unitary $w \in B$ such that

$$\| \operatorname{Ad} w \circ \varphi(x) - \psi(x)\| < (\frac{72}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad x \in cg(A).$$

□

Let

$$A = \bigoplus_{i=1}^{L} A_i$$

be the direct sum of the L building blocks of type 2, $A_i = A(n_i, d_i, N_i), i = 1, 2, \ldots, L$. We consider each $A_i = A(n_i, d_i, N_i)$ as a (non-unital) C^*-subalgebra of A and as the convenient set of generators for A we take $cg(A) = \bigcup_{i=1}^{L} cg(A_i)$, and we define $\delta_A = \min_i \delta_{A_i}$. The center of A is $\bigoplus_{i=1}^{L} C(\mathbb{T})$ and we will let $cu(A)$ denote the L unitaries

$$(z, 1, 1, \ldots, 1), (1, z, 1, \ldots, 1), \ldots, (1, 1, \ldots, 1, z)$$

in $\bigoplus_{i=1}^{L} C(\mathbb{T}) \subset A$. The corresponding set of partial unitaries

$$(z, 0, 0, \ldots, 0), (0, z, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, z)$$

in $\bigoplus_{i=1}^{L} C(\mathbb{T}) \subset A$ will be denoted by $cu_0(A)$.

Consider $\mathbb{Z}$ with its natural ordering and let A_i, $i = 1, 2, \ldots, I$ and B_j, $j = 1, 2, \ldots, J$, be unital C^*-algebras (building blocks for example) such that

$$K_0(A_i) \simeq K_0(B_j) \simeq \mathbb{Z}$$

as partially ordered groups for all i, j, k. Set

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_I, \; B = B_1 \oplus B_2 \oplus \cdots \oplus B_J.$$

The multiplicity matrix of a unital $*$-homomorphism $\varphi\colon A \to B$ is the $J \times I$ integer matrix $\big(S_{ji}\big)$ such that the composition

$$\mathbb{Z} \simeq K_0(A_i) \to K_0(A) \xrightarrow{\varphi_*} K_0(B) \to K_0(B_j) \simeq \mathbb{Z}$$

is multiplication by S_{ji}. We set $\operatorname{mult}(\varphi) = \min_{i,j} S_{ij}$. Later, in Chapter 6, we shall also need to consider $\operatorname{mult}_0(\varphi) = \min\{S_{ij} : S_{ij} \neq 0\}$.

To formulate the next proposition we remind the reader that $DU(B)$ denotes the commutator subgroup of the unitary group $U(B)$ of a unital C^*-algebra B.

PROPOSITION 3.5. — *Let $A = \bigoplus_{i=1}^{L} A(n_i, d_i, N_i)$ be a finite direct sum of building blocks of type 2. For every pair $k, l \in \mathbb{N}$ with $l > 12$, $24\pi/(k\delta_A) < 1$, there is a finite subset $G \subset C(\mathbb{T} \cup \{0\}, [0,1])$ of non-zero elements with the following property: When $\varphi, \psi\colon A \to B$ are unital $*$-homomorphisms into the same finite direct sum of building blocks of type 2, B, such that*

1. $[\varphi] = [\psi]$ *in* $KK(A, B)$,
2. $\theta(\varphi(\xi_j^k(u_0))) > 2/l$, $j = 1, 2, \ldots, k, \theta \in T(B), u_0 \in cu_0(A)$,
3. $\theta(\varphi(g(u_0))) > 3\kappa$, $g \in G, \theta \in T(B), u_0 \in cu_0(A)$,
4. $|\theta(\psi(g(u_0)) - \varphi(g(u_0)))| < \kappa^2$, $g \in G, \theta \in T(B), u_0 \in cu_0(A)$,
5. $\operatorname{dist}(\varphi(u)\psi(u)^*, DU(B)) < \kappa^2$, $u \in cu(A)$,
6. $\max_i 16N_i n_i < \kappa \operatorname{mult}(\varphi)$,

for some $\kappa < 1/(2l)$, *then there is a unitary* $w \in B$ *such that*

$$\| \operatorname{Ad} w \circ \varphi(a) - \psi(a)\| \leq (\frac{72}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad a \in cg(A).$$

Proof. — For each $i \in \{1, 2, \ldots, L\}$, let $H_i \subset C(\mathbb{T}, [0,1])$ be the finite subset of Lemma 3.4 corresponding to $A(n_i, d_i, N_i)$ and the present choice of k, l. We may assume that each H_i contains the constant function 1. Each $f \in H_i$ extends to a continuous function $\tilde{f} : \mathbb{T} \cup \{0\} \to [0,1]$ with $\tilde{f}(0) = 0$. Set

$$G = \bigcup_{i=1}^{L} \left\{ \tilde{f} : f \in H_i \right\}.$$

Assume that we are given $\varphi, \psi \colon A \to B$ and a $\kappa \in]0, 1/2l[$ such that 1. - 6. hold. To produce the desired unitary in B we can assume that B is a building block of type 2, rather than a direct sum of such algebras. Let $p_i, i = 1, 2, \ldots, L$, be the minimal non-zero central projections in A. After a standard argument, using that φ and ψ agree on $K_0(A)$ by 1., we can assume that $\varphi(p_i) = \psi(p_i)$, $i = 1, 2, \ldots, L$.

Fix $i \in \{1, 2, \ldots, L\}$ and set $q_i = \varphi(p_i) = \psi(p_i)$. φ and ψ restrict to unital $*$-homomorphisms $A(n_i, d_i, N_i) \to q_i B q_i$ which we denote by φ_i and ψ_i, respectively. We may assume that $q_i \neq 0$ and then q_i must be a full projection, so that $q_i B q_i \subset B$ is a KK-equivalence and $[\varphi_i] = [\psi_i]$ in $KK(A_i, q_i B q_i)$. In particular, $\varphi_{i*} = \psi_{i*}$ on $K_1(A_i)$. Note that $B_i = q_i B q_i$ can be identified with the building block, $A(m', e', M)$, of type 2, where $m' = \operatorname{Tr}(q_i(t)), t \in \mathbb{T}$, and $e' = m'e/m$, cf. Lemma 1.8. We conclude from Lemma 2.2 that φ_i and ψ_i have the same small remainders. Thus conditions 1. and 7. of Lemma 3.4 are met.

Every trace state of $q_i B q_i$ is of the form $x \mapsto \omega(q_i)^{-1}\omega(x)$ for some trace state ω of B. Since $\omega(q_i)^{-1} \geq 1$, $\omega \in T(B)$, 2. and 3. imply that

$$\hat{\varphi}_i(\xi_j^k) > \frac{2}{l}, \; j = 1, 2, \ldots, k, \tag{15}$$

$$\hat{\varphi}_i(g) > 3\kappa, \; g \in H_i. \tag{16}$$

But 3. implies that $\omega(q_i) > 3\kappa$, $\omega \in T(B)$, so 4. yields that

$$\|\hat{\varphi}_i(g) - \hat{\psi}_i(g)\| < \frac{\kappa}{3}, \ g \in H_i. \tag{17}$$

There is an element $u \in cu(A)$ such that $\varphi_i(z \otimes 1) = \varphi(u)q_i$ and $\psi_i(z \otimes 1) = \psi(u)q_i$. 5. implies that there is a selfadjoint element $b \in B$ with $\|b\| < \kappa^2$ and a $c \in DU(B)$ such that

$$\varphi(u) = ce^{2\pi i b}\psi(u).$$

Thus, if we take determinants in M_m, we have that

$$\operatorname{Det}\varphi(u)(t) = e^{2\pi i \operatorname{Tr}(b(t))} \operatorname{Det}\psi(u)(t), \ t \in \mathbb{T}.$$

Hence

$$\operatorname{Det}\varphi_i(z \otimes 1)(t) = e^{2\pi i \operatorname{Tr}(b(t))} \operatorname{Det}\psi_i(z \otimes 1)(t), \quad t \in \mathbb{T}, \tag{18}$$

where the determinants are now calculated in $M_{m'}$. Note that

$$|\operatorname{Tr}(b(t))| \leq m\kappa^2 = m'\kappa^2\frac{m}{m'} = m'\kappa^2\frac{1}{3\kappa} = m'\frac{\kappa}{2}$$

for all $t \in \mathbb{T}$, so that we have condition 5. of Lemma 3.4 satisfied. Finally, $e' \geq \operatorname{mult}(\varphi)$, so that the present assumption 6. gives condition 6. of Lemma 3.4. We can now apply Lemma 3.4 to obtain a unitary $w_i \in B_i$ such that

$$\|\operatorname{Ad} w_i \circ \varphi_i(a) - \psi_i(a)\| \leq (\frac{72}{\delta_{A_i}k} + \frac{13}{\delta_{A_i}l})\pi, \quad a \in cg(A(n_i, d_i, N_i)).$$

Then $w = \sum_{i=1}^{L} w_i$ does the job. $\square$

CHAPTER 4

INJECTIVE CONNECTING MAPS

The purpose of this chapter is to establish the following

THEOREM 4.1. — *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks of type 2. Assume that A is infinite dimensional and simple. There is then a sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$ such that each A_n is a finite direct sum of building blocks of type 2, each φ_n is unital and injective and $A \simeq \varinjlim(A_n, \varphi_n)$.* $\square$

We shall need a set of generators for a building block of type 4 which we can consider to be canonical. So let $A = A(n, d, N)$ be a building block of type 4 with exceptional points at $x_1, x_2, \ldots, x_N \in [0,1]$. Set

$$\delta_A = 1/2 \min \{|x_i - x_j| : i \neq j\},$$

(and $\delta_A = 1$ when $N = 1$), and

$$g_i(t) = \max\left\{0, 1 - \frac{1}{\delta_A}|t - x_i|\right\}.$$

Let as before $\{p_{ij}\}$ and $\{e_{ij}\}$ be the standard matrix units in M_d and M_n, respectively. Define $g_A \colon [0,1] \to \mathbb{C}$ by

$$g_A(t) = e^{it/2} \operatorname{dist}(t, \{x_1, x_2, \cdots, x_N\}).$$

The set $\cup_{k=1}^N \{g_k \otimes p_{ij}\} \cup \{g_A \otimes e_{ij}\}$ will be called *the canonical set of generators* for A and denoted $cg(A)$. Observe that

$$\|f(t) - f(s)\| \leq \frac{1}{\delta_A}|t - s|, \; f \in cg(A), \; t, s \in [0,1].$$

For the proof of Theorem 4.1 we choose, for each $m \in \mathbb{N}$, non-zero continuous functions $\chi_j^m \colon [0,1] \to [0,1]$ with support in $]\frac{j-1}{m}, \frac{j}{m}[$, $j = 1, 2, \ldots, m$.

LEMMA 4.2. — *Let A and B be direct sums of building blocks of type 2 or 4, $A = \bigoplus_{i=1}^{n_2} A_i \oplus \bigoplus_{i=n_2+1}^{n_4} A_i$, where A_i is of type 2 when $i \leq n_2$ and of type 4 when $n_2 < i \leq n_4$. Assume that $\varphi\colon A \to B$ is a unital $*$-homomorphism such that*

$$\varphi|_{A_i}(\chi_j^k) \neq 0, \ j = 1, 2, \ldots, k,$$

where $2/k < \delta_{A_i}$, when $n_2 < i \leq n_4$, and

$$\varphi|_{A_i}(\xi_j^k) \neq 0, \ j = 1, 2, \ldots, k,$$

where $4\pi/k < \delta_{A_i}$, when $1 \leq i \leq n_2$.

It follows that there is an injective unital $*$-homomorphism $\psi\colon A \to B$ such that

$$\|\psi|_{A_i}(x) - \varphi|_{A_i}(x)\| \leq \frac{8\pi}{\delta_{A_i} k}, \ x \in cg(A_i)$$

for all $i = 1, 2, \ldots, n_4$.

Proof. — It suffices to prove this when A is a building block of type 2 or 4. We give the proof only in the case when A is of type 4. When A is of type 2, the proof is the same, except for notation. Write

$$B = \bigoplus_{i=1}^{m_2} B_i \oplus \bigoplus_{i=m_2+1}^{m_4} B_i,$$

where B_i is of type 2 when $i \leq m_2$ and of type 4 when $m_2 < i \leq m_4$. Then φ decomposes as a direct sum, $\varphi = \bigoplus_{i=1}^{m_4} \varphi_i$, where each $\varphi_i\colon A \to B_i$ is a unital $*$-homomorphism.

Let $x_1, x_2, \ldots, x_N \in [0,1]$ and $y_1^i, y_2^i, \ldots, y_{M_i}^i \in [0,1]$ be the exceptional points of A and B_i, respectively. We may assume that φ_i is of standard form and minimal multiplicity, i.e. is given by continuous functions

$$\mu_j^i\colon [0,1] \to [0,1], \quad j = 1, 2, \ldots, L_i,$$

and remainders

$$r_j^i \in \{0, 1, 2, \ldots, n/d - 1\}, \quad j = 1, 2, \ldots, N,$$

such that

$$\varphi_i(f)(t) = u_i(t) \operatorname{diag}\big(f \circ \mu_1^i(t), \ldots, f \circ \mu_L^i(t), \Lambda_1^{r_1^i}(f) \ldots, \Lambda_N^{r_N^i}(f)\big) u_i(t)^*,$$

$t \in [0,1]$, $f \in A$, for some unitary u_i. The assumption on φ implies that

$$\bigcup_{i=1}^{m_4} \bigcup_{j=1}^{L_i} \mu_j^i([0,1])$$

is $3/k$-dense in $[0,1]$. Set $\mu_j^i([0,1]) = [a_j^i, b_j^i]$ and perturb μ_j^i to ν_j^i such that ν_j^i and μ_j^i agree on $\{y_1^i, y_2^i, \dots, y_{M_i}^i\}$,

$$|\mu_j^i(t) - \nu_j^i(t)| \le \frac{4}{k},\ t \in [0,1],$$

and

$$\nu_j^i([0,1]) = [a_j^i - \frac{3}{k}, b_j^i + \frac{3}{k}] \cap [0,1].$$

Then

$$\psi_i(f)(t) = u_i(t)\operatorname{diag}\big(f \circ \nu_1^i(t), , \dots, f \circ \nu_{L_i}^i(t), \Lambda_1^{r_1^i}(f), \dots, \Lambda_N^{r_N^i}(f)\big)u_i(t)^*$$

defines a unital $*$-homomorphism $\psi_i\colon A \to B_i$ such that

$$\|\psi_i(x) - \varphi_i(x)\| \ \le\ \frac{4}{\delta_{A_i}k} \ \le\ \frac{8\pi}{\delta_{A_i}k},\ x \in cg(A).$$

Set $\psi = \bigoplus_{i=1}^{m_4} \psi_i$. Since

$$\bigcup_{i=1}^{m_4}\bigcup_{j=1}^{L_i} \nu_j^i([0,1]) = [0,1],$$

ψ is injective. □

LEMMA 4.3. — *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks of type 2. Then there is a sequence $B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots$ such that*

- *each B_n is a finite direct sum of building blocks of type 2, building blocks of type 4 and matrix algebras,*
- *each φ_n is unital and injective,*
- *$A \simeq \varinjlim(B_n, \varphi_n)$.*

Proof. — Assume that A is the inductive limit of the sequence $A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi} \cdots$ of finite direct sums of building blocks of type 2 and let $\rho_k\colon A_k \to A$ be the canonical $*$-homomorphism. If C is a quotient of a building block of type 2, then there is a closed subset $F \subset \mathbb{T}$ and points $x_1, x_2, \dots, x_N \in F$ such that

$$C \simeq \{f \in C(F) \otimes M_n : f(x_i) \in M_d,\ i = 1, 2, \dots, N\}$$

for some $n, d \in \mathbb{N}$, $d|n$. For every $\varepsilon > 0$ there is a subset $R \subset F$, such that R is either a circle or the disjoint union of closed intervals and points, and a continuous map $\alpha\colon F \to R$ with the properties that $\alpha(t) = t$, $t \in R$, and $|\alpha(s) - s| \le \varepsilon$, $s \in F$. Using these facts inductively, in combination with Lemma 4.2, we obtain a sequence $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$, a sequence

B_k of finite direct sums of building blocks of type 2, building blocks of type 4 and matrix algebras, together with unital and injective $*$-homomorphisms $\psi_k \colon B_k \to B_{k+1}$ making the diagram

$$\begin{tikzcd}
A_{n_1}/\ker \rho_{n_1} \arrow[r] \arrow[d] & A_{n_2}/\ker \rho_{n_2} \arrow[r] \arrow[d] & A_{n_3}/\ker \rho_{n_3} \arrow[r] \arrow[d] & \cdots \\
B_1 \arrow[r, "\psi_1"] \arrow[ur] & B_2 \arrow[r, "\psi_2"] \arrow[ur] & B_3 \arrow[r, "\psi_3"] \arrow[ur] & \cdots
\end{tikzcd}$$

into an approximate intertwining in the sense of Elliott, cf. [**E1**, Theorem 2.2]. Then $A \simeq \varinjlim(B_k, \psi_k)$ and the proof is complete. □

LEMMA 4.4. — *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks of type* 2. *Assume that A is simple and infinite dimensional. There is then a unitary $u \in A$ with full spectrum, i.e. with* $\mathrm{Sp}(u) = \mathbb{T}$.

Proof. — By Lemma 4.3 we can realize A as the inductive limit of a sequence $B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots$ such that each B_n is a finite direct sum of building blocks of type 2, building blocks of type 4 and matrix algebras, and each φ_n is unital and injective. Since a building block of type 2 or 4 contains a unitary with full spectrum, the conclusion follows from this, unless each B_n is finite dimensional. But then A is an AF-algebra and it is wellknown fact that a simple unital and infinite dimensional AF-algebra contains a unitary with full spectrum. □

LEMMA 4.5. — *Let B be a separable unital C^*-algebra. Then the following conditions are equivalent.*

- *B is $*$-isomorphic to the inductive limit of a sequence of finite direct sums of building blocks of type* 2 *and* 4 *with injective unital connecting $*$-homomorphisms.*
- *Given a finite subset $F \subset B$ and an $\varepsilon > 0$, there exists a unital C^*-subalgebra $C \subset B$ such that C is a finite direct sum of building blocks of type* 2 *and* 4 *and $F \subset_\varepsilon C$.*

Proof. — It is trivial that the first condition implies the second. To prove the reversed implication, we use that building blocks of type 2 and 4 have stable relations by [**L2**]. We can then proceed as in the proof of [**L1**], Theorem 3.8, except that we use Lemma 4.2 to choose the γ_k's injective. □

LEMMA 4.6. — *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks of type* 2. *Assume that A is simple and infinite dimensional. Then there is a sequence $B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \ldots$ such that each B_i*

is a finite direct sum of building blocks of type 2 *and 4, each* φ_i *is unital and injective and* $A \simeq \varinjlim(B_n, \varphi_n)$.

Proof. — By combining Lemma 4.3 and Lemma 4.5 we see that it suffices to prove that every unital C^*-subalgebra B of A which is a finite direct sum of building blocks of type 2, building blocks of type 4 and matrix algebras, is contained in a unital C^*-subalgebra B_1 of A which is a finite direct sum of building blocks of type 2 and 4. Since the cutdown pAp by a central projection $p \in B$ is also a simple unital infinite dimensional inductive limit of finite direct sums of building blocks of type 2, it suffices to consider the case where B is a full matrix algebra. But then $A \simeq B \otimes (A \cap B')$, where $A \cap B'$ is also a simple unital infinite dimensional inductive limit of finite direct sums of building blocks of type 2. By Lemma 4.4, $A \cap B'$ contains a unitary u with full spectrum. Set $B_1 = C^*(B, u)$. □

LEMMA 4.7. — *Let* $A = A(n, d, N)$ *be a building block of type* 4 *and let* $k \in \mathbb{N}$ *such that* $2/k < \delta_A$. *When* $\varphi\colon A \to B = A(m, e, M)$ *is a unital and injective* $*$*-homomorphism into a building block of type* 2 *or 4 such that*

$$\hat{\varphi}(\chi_j^k) \;>\; \frac{2(N+1)n}{e},\ j = 1, 2, \dots, k,$$

then there are non-zero building blocks, $B_i = A(m_i, e_i, M), i = 1, 2$, *of the same type as* B, *such that* $B_1 \oplus B_2 \subset B$ *(as a unital subalgebra) and unital* $*$*-homomorphisms* $\psi_1\colon A \to B_1, \psi_2\colon A \to B_2$ *and* $\psi_3\colon B_1 \oplus B_2 \to B$, *such that* ψ_1 *and* ψ_3 *are injective and*

$$\|\varphi(x) - \psi_3(\psi_1(x), \psi_2(x))\| \le \frac{9}{\delta_A k}, \quad x \in cg(A).$$

Proof. — We can assume that φ is on standard form and of minimal multiplicity with characteristic functions $\mu_1^\varphi, \dots, \mu_L^\varphi$. Let $x_1, x_2, \dots, x_N \in [0,1]$ and $y_1, y_2, \dots, y_M \in [0,1]$ be the exceptional points of A and B, respectively. Set $N_{rj} = \#\{i : \mu_i^\varphi(y_r) = x_j\}$ and $L_0 = \max_r \sum_j N_{rj}$. The same estimates as in the proof of Lemma 3.4 give that

$$\sum_{i=1}^{L} \chi_j^k \circ \mu_i^\varphi > 0$$

for all j. We can therefore perform eigenvalue crossovers as in that proof and in this way perturb φ to φ' such that

$$\|\varphi(x) - \varphi'(x)\| \le \frac{5}{\delta_A k},\ x \in cg(A),$$

and the characteristic functions of φ' are partitioned into the following two sets:

$$\{\mu_i : i = 1, 2, \ldots, L - L_0\} \text{ and } \{\nu_i : i = 1, 2, \ldots, L_0\}$$

in such a way that the second set satisfies conditions (20), (21) from Chapter 1 plus condition (22) when B is of type 2, while the first set satisfies the same three conditions, but with $r_j^{\varphi} = 0$ for all j. Set

$$B_1 = A(m_1, e_1, M), \quad B_2 = A(m_2, e_2, M)$$

with

$$m_1 = (L - L_0)n, \; e_1 = em_1/m, \; m_2 = m - m_1, \; e_2 = em_2/m$$

and note that $B_1 \oplus B_2 \subset B$ as a unital C^*-subalgebra. Let $\lambda \colon A \to B$ be the standard homomorphism whose characteristic functions are

$$\{\mu_i : i = 1, 2, \ldots, L - L_0\}$$

and whose remainders are 0. Let $\psi_2 \colon A \to B_2$ be the standard homomorphism with remainders $r_1^{\varphi}, r_2^{\varphi}, \ldots, r_N^{\varphi}$ and characteristic functions $\nu_i : i = 1, 2, \ldots, L_0$. By Corollary 1.5, φ' is approximate inner equivalent to the map $f \mapsto (\lambda(f), \psi_2(f)) \in B$. We have that

$$\hat{\lambda}(\chi_j^k) \;\geq\; \frac{2n}{e} \;>\; 0$$

for all j, so we can apply Lemma 4.2 to get a unital injective $*$-homomorphism $\psi_1 \colon A \to B_1$ such that

$$\|\lambda(x) - \psi_1(x)\| \;\leq\; \frac{4}{\delta_A k}, \; x \in cg(A).$$

If we let ψ be the inclusion $B_1 \oplus B_2 \subset B$, there is an inner automorphism $\operatorname{Ad} u$ of B such that ψ_1, ψ_2 and $\psi_3 = \operatorname{Ad} u \circ \psi$ have the desired properties. $\square$

Let A be a building block of type 4,

$$A = \{f \in C[0,1] \otimes M_n : f(x_i) \in M_d, i = 1, 2, \ldots, N\}.$$

Define $\kappa \colon \mathbb{T} \to [0,1]$ by

$$\kappa(e^{2\pi i t}) = 2t, \; t \in [0, 1/2], \quad \kappa(e^{2\pi i t}) = 2 - 2t, \; t \in [1/2, 1].$$

Then $\kappa^{-1}(\{x_1, x_2, \ldots, x_N\})$ consists of $2N$ points, $\{y_1, y_2, \ldots, y_{2N}\}$, with N points on the upper semi-circle and another N points on the lower semi-circle. Set

$$A^{\mathbb{T}} = \{f \in C(\mathbb{T}) \otimes M_n : f(y_i) \in M_d, i = 1, 2, \ldots, 2N\}$$

Define $\iota_1 \colon [0,1] \to \mathbb{T}$ and $\iota_2 \colon [0,1] \to \mathbb{T}$ by

$$\iota_1(t) = e^{\pi i t}$$

and

$$\iota_2(t) = e^{-\pi i t},$$

respectively. We can then define $\lambda_A \colon A \to A^{\mathbb{T}}$ and $i_j \colon A^{\mathbb{T}} \to A, j = 1,2$, by $\lambda_A(f) = f \circ \kappa$ and $i_j(g) = g \circ \iota_j$, respectively. Then $i_j \circ \lambda_A = id_A, j = 1,2$, λ_A is injective and i_1 and i_2 are jointly injective in the sense that

$$i_1(g) = 0, i_2(g) = 0 \implies g = 0.$$

Proof of Theorem 4.1. — By Lemma 4.6 we can assume that A is the inductive limit of a sequence $A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \dots$ of finite direct sums of building blocks of type 2 and 4 with unital and injective connecting maps. Consider $m \in \mathbb{N}$ and let H be any finite subset of non-zero positive elements of A_m. Since A is simple and the connecting maps injective, there is an $m_0 > m$ and a $\kappa > 0$ such that

$$\widehat{\psi_{n,m}}(h) \; > \; \kappa, \; h \in H,$$

for all $n \geq m_0$. By using this in combination with Lemma 4.2, we can find a sequence $m_1 < m_2 < \cdots$ in $\mathbb{N}$ and unital $*$-homomorphism $\varphi_n \colon A_{m_n} \to A_{m_{n+1}}$ such that the partial maps of φ_n are all injective and

$$\|\varphi_n(x) - \psi_{m_{n+1},m_n}(x)\| \leq \varepsilon_n, \quad x \in cg(A_{m_n}),$$

for any sequence $\{\varepsilon_n\} \subset]0,1[$. With an appropriate choice, Theorem 2.2 of **[E1]** shows that $A \simeq \varinjlim(A_{m_n}, \varphi_n)$. So we may assume to begin with that all the partial maps of the connecting $*$-homomorphisms are injective (and not only the maps themselves). By using Lemma 4.7 in a similar approximate intertwining argument, we may next arrange that each A_n has more than one direct summand. The partial maps (of the connecting $*$-homomorphisms) may then no longer all be injective, but that is then corrected by repeating the first approximate intertwining argument. So all in all we may assume to begin with that each A_n has more than one direct summand and that all the partial maps of the connecting $*$-homomorphisms, the ψ_n's, are injective.

The next, and final step, is to substitute the direct summands of type 4 with others of type 2 as follows. Write

$$A_n = \bigoplus_{j=1}^{m_n} X_j^n$$

where $X_1^n, X_2^n, \ldots, X_{a_n}^n$ are building blocks of type 4 and $X_{a_n+1}^n, \ldots, X_{m_n}^n$ are building blocks of type 2. Set

$$D_n = \bigoplus_{j=1}^{a_n} X_j^{n\mathbb{T}} \bigoplus_{i=a_n+1}^{m_n} X_i^n$$

and define $\lambda_n \colon A_n \to D_n$ by

$$\begin{aligned}&\lambda_n(x_1, x_2, \ldots, x_{a_n}, x_{a_n+1}, \ldots, x_{m_n})\\&= \big(\lambda_{X_1^n}(x_1), \ldots, \lambda_{X_{a_n}^n}(x_{a_n}), x_{a_n+1}, \ldots, x_{m_n}\big).\end{aligned}$$

Since all the partial maps defined by ψ_n are injective and A_{n+1} contains at least two direct summands, we can write $\psi_n = \psi_n^1 \oplus \psi_n^2$ where ψ_n^1 and ψ_n^2 are both injective. Then

$$\alpha_j(x_1, x_2, \ldots, x_{m_n}) = (i_j(x_1), \ldots, i_j(x_{a_n}), x_{a_n+1}, \ldots, x_{m_n}), \quad j = 1, 2,$$

define unital $*$-homomorphisms $\alpha_j \colon D_n \to A_n$ such that $\alpha_1(x) = \alpha_2(x) = 0 \Rightarrow x = 0$ and $\alpha_j \circ \lambda_n = id_{C_n}$, $j = 1, 2$. Define $\pi_n \colon D_n \to A_{n+1}$ by

$$\pi_n(x) = \psi_n^1(\alpha_1(x)) \oplus \psi_n^2(\alpha_2(x)).$$

Then $\pi_n \circ \lambda_n = \psi_n$. Therefore the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\psi_1"] \arrow[d, "\lambda_1"'] & A_2 \arrow[r, "\psi_2"] \arrow[d, "\lambda_2"] & A_3 \arrow[r, "\psi_3"] \arrow[d, "\lambda_3"] & \cdots \\
B_1 \arrow[r, "\lambda_2\circ\pi_1"'] \arrow[ur, "\pi_1"] & B_2 \arrow[r, "\lambda_3\circ\pi_2"'] \arrow[ur, "\pi_2"] & B_3 \arrow[r, "\lambda_4\circ\pi_3"'] \arrow[ur, "\pi_3"] & \cdots
\end{tikzcd}$$

is something so unusual as a (truly) commuting diagram. It follows that A is the inductive limit of the lower sequence. Since each $\lambda_{n+1} \circ \pi_n$ is injective, the proof is complete. $\square$

CHAPTER 5

APPROXIMATE DIVISIBILITY

The purpose with this chapter is to prove the following result which is applied in the proof of our main result. It plays exactly the same role here as in [**E3**] and [**NT**].

THEOREM 5.1. — *Let A be a unital and infinite dimensional inductive limit of a sequence of finite direct sums of building blocks of type 2. Assume that A is simple. Then A is approximately divisible.* □

A unital $*$-homomorphism $\varphi\colon A(n,d,N) \to A(m,e,M)$, between building blocks of type 2, is called *extendible* when all remainders $r_i^{\varphi}, i = 1,2,\dots,N$, are 0, modulo n/d, and

$$\#\{i : \mu_i(y_k) = \lambda\} \in \mathbb{N}m/e, \quad \lambda \in \mathbb{T},\ k = 1,2,\dots,M,$$

for some (and hence any) set of characteristic functions $\mu_1, \mu_2, \dots, \mu_L$ (here $y_1, y_2, \dots, y_M$ are the exceptional points of $A(m,e,M)$). By Theorem 1.4, φ is extendible if and only if φ is approximately inner equivalent to the restriction of a unital $*$-homomorphism $C(\mathbb{T}) \otimes M_n \to A(m,e,M)$.

LEMMA 5.2. — *Let $A = A(n,d,N)$ be a building block of type 2, $k,l \in \mathbb{N}$ such that $l > 12$, $24\pi/(\delta_A k) < 1$ and let $H \subset C(\mathbb{T},[0,1]) \subset A$ be the finite subset of Lemma 3.4 corresponding to k and l. Assume that*

$$\varphi\colon A = A(n,d,N) \to B = A(m,e,M)$$

is a unital $*$-homomorphism into the building block B of type 2, such that

(A) $\hat{\varphi}(\xi_j^k) > 3/l,\ j = 1,2,\dots,k,$

(B) $\hat{\varphi}(g) > 4\kappa,\ g \in H,$

(C) $\dfrac{4Xn}{m} < \kappa,\ \dfrac{16Nn}{e} < \kappa,$

for some $\kappa < 1/(2l)$. *Let* $g_i\colon [0,1] \to \mathbb{T}$, $i = 1,2,\dots,X$, *be any set of continuous functions such that*

$$(g_1(0), g_2(0), \dots, g_X(0)) = (g_1(1), g_2(1), \dots, g_X(1))$$

as unordered X*-tuples and* $\#\{i : g_i(y_r) = t\} \in \mathbb{N}m/e$ *for all* $t \in \mathbb{T}$ *and all* $r = 1,2,\dots,M$, *where* $y_1, y_2, \dots, y_M \in [0,1]$ *are the exceptional points of* B.

It follows that there are unital $*$*-homomorphisms*

$$\varphi_1\colon A(n,d,N) \longrightarrow A(m_1, e_1, M),$$
$$\varphi_2\colon A(n,d,N) \longrightarrow A(m_2, e_2, M),$$

where $m_1 = Xn$, $e_1 = Xen/m$, $m_2 = m - m_1$, $e_2 = e - e_1$, *such that*

- φ_1 *is extendible with characteristic functions* $g_1, g_2, \dots, g_X$,
- φ, φ_2 *and* $\varphi_1 \oplus \varphi_2$ *have the same small remainders,*
- $\|\hat{\varphi} - \widehat{\varphi_1 \oplus \varphi_2}\| \leq 2(X+1)n/m$, *and*
- *there is a unitary* $u \in A(m,e,M)$ *such that*

$$\| \operatorname{Ad} u \circ \varphi(x) - (\varphi_1 \oplus \varphi_2)(x) \| < \Big(\frac{80}{\delta_A k} + \frac{13}{\delta_A l}\Big)\pi, \quad x \in cg(A).$$

Proof. — We may assume that φ is on standard form and of minimal multiplicity. Let $\mu_1, \mu_2, \dots, \mu_L$ be characteristic functions and $r_1, r_2, \dots, r_N$ the remainders for φ. Let $x_1, x_2, \dots, x_N \in \mathbb{T}$ be the exceptional points of A. Take $\lambda \in \mathbb{T}$ such that $\operatorname{dist}(\lambda, \{x_1, x_2, \dots, x_N\}) \geq \delta_A/2$. Since

$$\frac{n}{m} \# \left\{ i : |\mu_i(t) - \lambda| \leq \frac{2\pi}{k} \right\} \;\geq\; \hat{\varphi}(\xi^k_{j_0})(t)$$

for some j_0, it follows from (A) and (C) that

$$\# \left\{ i : |\mu_i(t) - \lambda| \leq \frac{2\pi}{k} \right\} \;\geq\; 2X$$

for all $t \in [0,1]$. We can then perform eigenvalue crossovers to obtain a perturbation φ' of φ with characteristic functions $k_i, i = 1,2,\dots,2X$, and $h_i, i = 2X+1, \dots, L$, such that

$$k_i(0) = k_i(1), \ i = 1,2,\dots,2X,$$

$$\#\{i : k_i(y_r) = t\} \in \mathbb{N}m/e, \ t \in \mathbb{T},$$

for all r, and

$$|k_i(t) - \lambda| \;\leq\; \frac{10\pi}{k}, \ i = 1,2,\dots,2X.$$

By Lemma 3.3, φ and φ' have the same small remainders,

$$\operatorname{Det}\varphi'(z\otimes 1)(t) = \operatorname{Det}\varphi(z\otimes 1)(t),\ t\in\mathbb{T},$$

$$\|\hat{\varphi}-\hat{\varphi'}\| \le 2n/m$$

and

$$\|\varphi'(x)-\varphi(x)\| \le \frac{8\pi}{\delta_A k},\ x\in cg(A).$$

Set

$$m' = 2Xn,\ e' = 2Xne/m,\ m'' = m-m' \text{ and } e'' = e-e'.$$

It follows from Theorem 1.4 that there are unital $*$-homomorphisms

$$\psi_1\colon A\to A(m',e',M),\quad \psi_2\colon A\to A(m'',e'',M)$$

such that ψ_1 is the extendible $*$-homomorphism with characteristic functions $k_1,k_2,\dots,k_{2X}$, ψ_2 is the standard homomorphism with the same small remainders as φ and characteristic functions h_i, $i=2X+1,\dots,L$, and $\psi_1\oplus\psi_2$ is approximately inner equivalent to φ'. We assert that $\psi_{1*}=0$ on $K_1(A)$. Since ψ_1 is extendible, ψ_{1*} must vanish on the torsion part of $K_1(A)$. So it suffices to check that ψ_{1*} vanish on the $\mathbb{Z}$-summand. Let $\psi\colon C(\mathbb{T})\otimes M_n\to A(m',e',M)$ be a unital $*$-homomorphism such that ψ_1 is approximately inner equivalent to $\psi|_A$. It suffices to show that

$$[\psi_1(y_A\otimes e_{11}+\sum_{i\ge 2}1\otimes e_{ii})] = [\psi(z\otimes e_{11}+\sum_{i\ge 2}1\otimes e_{ii})] = 0$$

in $K_1(A(m',e',M))$. As the continuous eigenvalue functions of

$$\psi(z\otimes e_{11}+\sum_{i\ge 2}1\otimes e_{ii})$$

we can take $k_1,\dots,k_{2X}$ and $2X(n-1)$ copies of the constant function 1. There is a unitary $u\in C[0,1]\otimes M_{m'}$ such that

$$W(t) = u(t)\operatorname{diag}(k_1(t),k_2(t),\dots,k_{2X}(t),1,1,\dots,1)u(t)^*$$

and

$$S_\mu(t) = u(t)\operatorname{diag}(\underbrace{\mu,\mu,\dots,\mu}_{2X\text{ times}},1,1,\dots,1)u(t)^*$$

define unitaries in $A(m',e',M)$ for any $\mu\in\mathbb{T}$. (It is important that it is the same unitary u.) By Theorem 1.4 there is a sequence $\{T_n\}$ of unitaries in

$A(m', e', M)$ such that

$$\lim_{n\to\infty} T_n\psi(z \otimes e_{11} + \sum_{i\geq 2} 1 \otimes e_{ii})T_n^* = W.$$

Since

$$\|W - S_\lambda\| \leq \frac{10\pi}{k} \leq 2$$

and S_λ is homotopic to 1, this proves that $[\psi(z \otimes e_{11} + \sum_{i\geq 2} 1 \otimes e_{ii})] = 0$ in $K_1(A(m', e', M))$.

Now we consider the extendible standard homomorphism

$$\psi_3 \colon A \longrightarrow A(m', e', M)$$

whose characteristic functions are $g_1, \ldots, g_X$ and $\overline{g_1}, \ldots, \overline{g_X}$. We want to apply Lemma 3.4 with $\varphi = \psi_1 \oplus \psi_2$ and $\psi = \psi_3 \oplus \psi_2$. So we check the conditions of that lemma one by one. First note that $\psi_1 \oplus \psi_2, \psi_2, \varphi$ and $\psi_3 \oplus \psi_2$ all have the same small remainders, modulo n/d. Next observe that

$$\|\widehat{\psi_1 \oplus \psi_2} - \hat{\varphi}\| \leq 2n/m \leq \kappa < \frac{1}{l},$$

so that

$$\widehat{\psi_1 \oplus \psi_2}(\xi_j^k) \geq \hat{\varphi}(\xi_j^k) - \frac{1}{l} > \frac{2}{l}$$

for all j, and

$$\widehat{\psi_1 \oplus \psi_2}(g) \geq \hat{\varphi}(g) - \kappa > 3\kappa$$

for all $g \in H$. Since

$$\|\widehat{\psi_1 \oplus \psi_2} - \widehat{\psi_3 \oplus \psi_2}\| \leq \frac{2Xn}{m} < \frac{\kappa}{2},$$

we have verified conditions 1. – 4. of Lemma 3.4. Since

$$|k_i(t) - \lambda| \leq \frac{10\pi}{k} \leq \frac{10\pi}{\delta_A k} < 2$$

for all t, i, we conclude that there is a continuous function

$$\alpha \colon \mathbb{T} \to [-3\pi Xn/k, 3\pi Xn/k]$$

and a constant $\mu \in \mathbb{T}$ such that

$$\operatorname{Det}(\psi_1 \oplus \psi_2)(z \otimes 1)(t) = \mu e^{2\pi i\alpha(t)} \operatorname{Det}(\psi_3 \oplus \psi_2)(z \otimes 1)(t), \ t \in \mathbb{T}.$$

Since

$$\frac{6Xn\pi}{km} \leq \frac{3}{2}\frac{\pi}{k}\frac{4Xn}{m} < \kappa,$$

we have condition 5. fulfilled. Condition 6. of Lemma 3.4 follows from (C). Finally, we have checked that $\psi_{1*} = 0$ on $K_1(A)$. It is clear that also $\psi_{3*} = 0$ on $K_1(A)$, so we have that $(\psi_1 \oplus \psi_2)_* = (\psi_3 \oplus \psi_2)_*$ on $K_1(A)$.

It follows now from Lemma 3.4 that there is a unitary $v \in B$ such that

$$\|v(\psi_1 \oplus \psi_2)(x)v^* - (\psi_3 \oplus \psi_2)(x)\| < (\frac{72}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad x \in cg(A).$$

Since

$$\|\varphi(x) - \varphi'(x)\| \leq \frac{8\pi}{\delta_A k}, \ x \in cg(A),$$

it follows that there is unitary $u \in B$ such that

$$\|u\varphi(x)u^* - (\psi_3 \oplus \psi_2)(x)\| < (\frac{80}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad x \in cg(A).$$

Clearly,

$$\|\hat{\varphi} - \widehat{\psi_3 \oplus \psi_2}\| \ \leq \ \frac{2(X+1)n}{m}.$$

Finally, it is clear that $\psi_3 \oplus \psi_2$ is approximately inner equivalent to a direct sum $\varphi_1 \oplus \varphi_2$ as in the statement of the lemma: Remove the characterisitic functions $\overline{g_1}, \dots, \overline{g_X}$ from ψ_3 to get φ_1 and add them to those of ψ_2 to get φ_2. By Theorem 1.4, $\psi_3 \oplus \psi_2$ is approximately inner equivalent to $\varphi_1 \oplus \varphi_2$. $\square$

LEMMA 5.3. — *Let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$ be a sequence of finite direct sums of building blocks of type 2 with unital connecting $*$-homomorphisms. If $A = \varinjlim(A_n, \varphi_n)$ is infinite dimensional and simple, then*

$$\lim_{k \to \infty} \operatorname{mult}(\varphi_{k,n}) = \infty$$

for all $n \in \mathbb{N}$.

Proof. — $K_0(A)$ is a simple dimension group, so if the conclusion fails, we must have $K_0(A) = \mathbb{Z}$. But then we may assume that $A_n = A(m_n, e, N_n)$, $n \in \mathbb{N}$, for the same $e \in \mathbb{N}$. It follows in this case that $A = M_e(B)$ where B is the limit of building blocks of the form $A(m_n/e, 1, N_n)$. However, it is easily seen that such a B must have $\mathbb{C}$ as a quotient, and this is not possible when A is simple and infinite dimensional. $\square$

We can now begin the

Proof of Theorem 5.1. — Let A be the inductive limit of the sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$ where each A_i is a finite direct sum of building blocks of type 2. By Theorem 4.1 we may assume that each φ_i is unital and injective. Let $N_0 \in \mathbb{N}$ and $0 < \varepsilon < 1$ be given. It suffices to show that for any $t \in \mathbb{N}$

there is an $s > t$ and a unital finite-dimensional C^*-subalgebra $F \subset A_s$, $F \simeq M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_K}$ with $\min\{n_1, n_2, \ldots, n_K\} \geq N_0$, such that

$$\operatorname{dist}\big(\varphi_{s,t}(a), A_s \cap F'\big) < \varepsilon, \quad a \in cg(A).$$

Write

$$A_t = \bigoplus_{i=1}^{L_1} A(n_i, d_i, N_i), \quad A_s = \bigoplus_{j=1}^{L_2} A(m_j, e_j, M_j).$$

Choose $k, l \in \mathbb{N}$ such that $l > 12$, $24\pi/(\delta_A k) < 1$ and

$$\left(\frac{248}{\delta_{A_t} k} + \frac{39}{\delta_{A_t} l}\right)\pi \ < \varepsilon.$$

Since A is simple and the connecting maps injective, there is for any non-zero positive element $h \in A_t$ an integer $n_0 \in \mathbb{N}$ and a $\kappa > 0$ such that

$$\omega(\varphi_{s,t}(h)) \ > \ \kappa, \ \omega \in T(A_s),$$

for all $s > n_0$. By choosing k first and then l subsequently, we can therefore assume that

$$(1) \qquad \omega(\varphi_{s,t}(\xi_j^k(u_0))) > \frac{5}{l}, \quad \omega \in T(A_s), \ j = 1, 2, \ldots, k, \ u_0 \in cu_0(A_t).$$

Let $H \subset C(\mathbb{T}, [0,1])$ be the finite subset of Lemma 3.4 corresponding to k, l. We can ensure that

$$(2) \qquad \omega(\varphi_{s,t}(g(u_0))) > 5\kappa, \quad \omega \in T(A_s), \ g \in H, \ u_0 \in cu_0(A_t),$$

for some $\kappa \in \,]0, 1/(2l)[$. Note that we can increase s further without spoiling (1) and (2). Since $\lim_{s\to\infty} \operatorname{mult}(\varphi_{s,t}) = \infty$ by Lemma 5.3, we may assume $\operatorname{mult}(\varphi_{s,t})$ to be as large as we want. Let $p_i, i = 1, 2, \ldots, L_1$, be the minimal non-zero projections of the center of A_t. Let $\pi_j \colon A_s \to A(m_j, e_j, M_j)$ be the projection and set

$$\varphi = \pi_j \circ \varphi_{s,t}|_{A(n_i,d_i,N_i)} \colon A(n_i, d_i, N_i) \to \pi_j \circ \varphi_{s,t}(p_i) A(m_j, e_j, M_j) \pi_j \circ \varphi_{s,t}(p_i).$$

To simplify notation, set $A = A(n_i, d_i, N_i) = A(n, d, N)$ and

$$B = \pi_j \circ \varphi_{s,t}(p_i) A(m_j, e_j, M_j) \pi_j \circ \varphi_{s,t}(p_i) = A(m, e, M).$$

It will suffice for us to find a unital finite-dimensional C^*-subalgebra $F \subset B = A(m, e, M)$ such that $F \simeq M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_K}$ with $\min\{n_1, n_2, \ldots, n_K\} \geq N_0$, and

$$\operatorname{dist}\big(\varphi(a), B \cap F'\big) < \varepsilon, \quad a \in cg(A).$$

The fact that we may assume $\mathrm{mult}(\varphi_{s,t})$ to be as large as we want, implies that we can take e as large as we want. How large e should be, will be specified as we go along. Set

$$X = m/e[(2N+1)(4N_0+2)d + N_0].$$

If e is large enough we have that $4Xn/m \ < \kappa$, $16Nn/e \ < \kappa$. Set

$$m_1 = Xn, \ e_1 = \frac{Xen}{m}, \ m_2 = m - m_1, \ e_2 = e - e_1.$$

Let $\varphi_1 \colon A \to A(m_1, e_1, M)$ be the standard $*$-homomorphism whose remainders are all 0 and which have X copies of the constant function 1 as characteristic functions. By Lemma 5.2 there is a unital $*$-homomorphism $\varphi_2 \colon A \to A(m_2, e_2, M)$ such that

$$\|\hat{\varphi} - \widehat{\varphi_1 \oplus \varphi_2}\| \ \leq \ \frac{2(X+1)n}{m}$$

and

$$\| \operatorname{Ad} u \circ \varphi(x) - (\varphi_1 \oplus \varphi_2)(x)\| < (\frac{80}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad x \in cg(A).$$

Set $e_0 = (2N+1)nd$ and $m_0 = (2N+1)ndm/e$. Note that $A(m_0, e_0, M) \subset B$ as a full corner. By Proposition 2.5 there are unital $*$-homomorphisms $\chi^+, \chi^- \colon A \to A(m_0, e_0, M)$ such that

$$[i \circ \chi^{\pm}|_{A_0}] = \pm[\varphi_1 \oplus \varphi_2|_{A_0}] \in KK(A_0, B)$$

when $i \colon A(m_0, e_0, M) \to B$ is the imbedding. Define

$$\psi_1 \colon A \to \bigoplus_{i=1}^{2N_0+1} A(m_0, e_0, M)$$

by

$$\psi_1(a) = (\underbrace{\chi^+(a), \dots, \chi^+(a)}_{N_0+1 \text{ times}}, \underbrace{\chi^-(a), \dots, \chi^-(a)}_{N_0 \text{ times}})$$

and $\psi_2 \colon A \to \bigoplus_{i=1}^{2N_0+1} A(m_0, e_0, M)$ by

$$\psi_2(a) = (\underbrace{\chi^+(a), \dots, \chi^+(a)}_{N_0 \text{ times}}, \underbrace{\chi^-(a), \dots, \chi^-(a)}_{N_0+1 \text{ times}})$$

Set $m_3 = 2(2N_0+1)m_0$, $e_3 = 2(2N_0+1)e_0$ and consider $\bigoplus_{i=1}^{4N_0+2} A(m_0, e_0, M)$ as a unital C^*-subalgebra of $A(m_3, e_3, M)$ such that $\psi_1 \oplus \psi_2 \colon A \to A(m_3, e_3, M)$

is a unital $*$-homomorphism. Then $(\psi_1 \oplus \psi_2)_* = 0$ on $K_1(A)$, so we have in particular that the loop

$$t \mapsto \operatorname{Det}(\psi_1 \oplus \psi_2)(z \otimes 1)(t)$$

is homotopically trivial. There is therefore a continuous function $g\colon \mathbb{T} \to \mathbb{R}$ such that

$$\operatorname{Det}(\psi_1 \oplus \psi_2)(z \otimes 1)(t) = e^{ig(t)}, \ t \in \mathbb{T}.$$

Let $\psi\colon A \to A((m/e)N_0n, N_0n, M)$ be the standard homomorphism whose remainders are all zero and which has $(m/e)N_0$ copies of the function

$$[0,1] \ni t \mapsto \exp(-i\frac{e}{N_0mn}g(e^{2\pi it}))$$

as characteristic functions. Then $\psi_1 \oplus \psi_2 \oplus \psi\colon A \to A(m_1, e_1, M)$ is a unital $*$-homomorphism such that

$$\operatorname{Det}(\psi_1 \oplus \psi_2 \oplus \psi)(z \otimes 1)(t) = 1, \ t \in \mathbb{T}.$$

Set $\varphi_3 = \psi_1 \oplus \psi_2 \oplus \psi$ and note that

$$\|\widehat{\varphi_3 \oplus \varphi_2} - \widehat{\varphi_1 \oplus \varphi_2}\| \ \leq \ \frac{m_1}{m} \ \leq \ \frac{\kappa}{4}.$$

Since

$$\|\hat{\varphi} - \widehat{\varphi_1 \oplus \varphi_2}\| \leq \frac{2(X+1)n}{m}$$

can be made arbitrarily small by increasing e, we may assume that e is so large that conditions 2.-4. of Lemma 3.4 are satisfied, with $\varphi = \varphi_1 \oplus \varphi_2$ and $\psi = \varphi_3 \oplus \varphi_2$. Note that condition 5. is trivially satisfied since

$$\operatorname{Det}(\varphi_3 \oplus \varphi_2)(z \otimes 1)(t) \ = \ \operatorname{Det}(\varphi_1 \oplus \varphi_2)(z \otimes 1)(t), \ t \in \mathbb{T},$$

by construction. As $[\varphi_3 \oplus \varphi_2] = [\varphi_1 \oplus \varphi_2]$ in $KK(A,B)$, we have conditions 1. and 7. of Lemma 3.4 satisfied by Lemma 2.2. Finally, condition 6. holds if e is large enough. It follows that there is a unitary $v \in B$ such that

$$\|\operatorname{Ad} v \circ (\varphi_1 \oplus \varphi_2)(x) - (\varphi_3 \oplus \varphi_2)(x)\| < (\frac{72}{\delta_A k} + \frac{13}{\delta_A l})\pi, \quad x \in cg(A).$$

Set

$$e_4 = (2N_0+1)e_0, \ m_4 = (2N_0+1)m_0, \ e_5 = e - e_4, \ m_5 = m - m_4,$$

and

$$\varphi_5 = \psi_2 \oplus \psi \oplus \varphi_2\colon A \to A(m_5, e_5, M).$$

Then $\varphi_3 \oplus \varphi_2 = \psi_1 \oplus \varphi_5$ and $[\varphi_5|_{A_0}] = 0$ in $KK(A_0, A(m_5, e_5, M))$. Note that $\psi_1(A)' \cap A(m_4, e_4, M)$ contains a copy of $M_{N_0} \oplus M_{N_0+1}$ as a unital C^*-algebra and that

$$\begin{aligned}
\|\hat{\varphi} - \hat{\varphi}_5\| &\leq \|\hat{\varphi} - \widehat{\varphi_3 \oplus \varphi_2}\| + \frac{m_4}{m} \\
&\leq \|\hat{\varphi} - \widehat{\varphi_1 \oplus \varphi_2}\| + \frac{m_1 + m_4}{m} \\
&\leq \frac{2(X+1)n + m_1 + m_4}{m} \leq \frac{(3X+2)n + (2N_0+1)(2N+1)nd}{e}.
\end{aligned}$$

Thus, if just e is large enough, we have that

$$\hat{\varphi}_5(\xi_j^k) > \frac{4}{l}, \; j = 1, 2, \ldots, k,$$

and

$$\hat{\varphi}_5(g) > 4\kappa, \; g \in H.$$

We may assume that φ_5 is on standard form and of minimal multiplicity. Since $[\varphi_5|_{A_0}] = 0$ in $KK(A_0, A(m_5, e_5, M))$, we know that most of the small remainders of φ_5 vanish, specifically that $s^{\varphi_5}(i,j) = 0$, $j = 1, 2, \ldots, N-1$, for all i, by Lemma 2.2. Let $\mu_1, \mu_2, \ldots, \mu_L$ be characteristic functions for φ_5. Then we have that

$$\# \{ r : \mu_r(y_i) = x_j \} = 0, \; j = 1, 2, \ldots, N-1,$$

for all i. Set

$$L_i = \# \{ r : \mu_r(y_i) = x_N \}, \; i = 1, 2, \ldots, M,$$

and $L_0 = \max_i L_i$. Since φ_5 is of minimal multiplicity, $L_0 \leq m/e$. Note that $m/e | L_0 - L_i$ for all i. Since φ_5 has minimal multiplicity, we must therefore have that $L_i = L_0$ for all i. By Lemma 5.2 there is a unital $*$-homomorphism $\varphi_5' \colon A \to A(m_5, e_5, M)$ such that φ_5 and φ_5' have the same small remainders (in particular, also the same remainders),

$$\|\widehat{\varphi_5} - \widehat{\varphi_5'}\| \leq \frac{2((m/e)N_0 + L_0 + 1)n}{m_5},$$

$$\|\operatorname{Ad} s \circ \varphi_5(x) - \varphi_5'(x)\| < \left(\frac{80}{\delta_A k} + \frac{13}{\delta_A l}\right)\pi, \quad x \in cg(A), \tag{3}$$

for some unitary $s \in A(m_5, e_5, M)$ and such that there is a set of characteristic functions for φ_5' containing (at least) $(m/e)N_0 + L_0$ copies of the constant function x_N. Set

$$m' = r_N^{\varphi_5} d + (L_0 + N_0 m/e)n, \quad e' = \frac{m'e}{m}.$$

Then, by Theorem 1.4, φ_5' is approximately inner equivalent to the direct sum $\psi_3 \oplus \varphi_6$, where $\psi_3 \colon A \to A(m', e', M)$ is a standard homomorphism with remainders

$$r_1^{\psi_3} = r_2^{\psi_3} = \cdots = r_{N-1}^{\psi_3} = 0, \quad r_N^{\psi_3} = r_N^{\varphi_5}$$

and whose characteristic functions are $L_0 + (m/e)N_0$ copies of x_N, and $\varphi_6 \colon A \to A(m_6, e_6, M)$, $m_6 = m_5 - m'$, $e_6 = e_5 - e'$, is a unital $*$-homomorphism with all small remainders equal to zero, modulo n/d. Observe that

$$\frac{m'}{m} \le \frac{n}{m} + \frac{n}{e} + \frac{N_0 n}{e},$$

which may be as small as we want. Furthermore, observe that the relative commutant

$$\psi_4(A)' \cap A(m', e', M)$$

contains a copy of M_D, where $D = (L_0 n/d + r_N^{\varphi_5})e/m + N_0 n/d \ge N_0$, as a unital C^*-subalgebra.

Since a standard homomorphism of minimal multiplicity and with all small remainders equal to zero must be extendible, we can use Lemma 1.7 to approximate φ_6 arbitrarily well with an extendible $*$-homomorphism. Hence we may assume that φ_6 is extendible, i.e. we may assume that there is a unital $*$-homomorphism $C(\mathbb{T}) \otimes M_n \to A(m_6, e_6, M)$ extending it. We denote also the extension by φ_6. Furthermore, since $\varphi_{5*} = 0$ and $\psi_{3*} = 0$ on $K_1(A)$, it follows from (3) that $\varphi_{6*} = 0$ on $K_1(C(\mathbb{T}) \otimes M_n)$. (We use here that $y_A \otimes e_{11} + \sum_{i \ge 2} 1 \otimes e_{ii} \in cg(A)$ and that $(80/(\delta_A k) + 13/(\delta_A l))\pi < 2$). Since

$$\|\hat{\varphi}_6 - \hat{\varphi}_5\| \le \frac{m'}{m_5},$$

we may assume that e is so large that we have

$$\hat{\varphi}_6(\xi_j^k) > \frac{2}{l}$$

for all $j = 1, 2, \ldots, k$. We factorize

$$A(m_6, e_6, M) = A(\frac{m_6}{n}, \frac{e_6}{n}, M) \otimes M_n$$

in such a way that

$$\operatorname{Ad} w_1 \circ \varphi_6(f \otimes e_{ij}) = \varphi_7(f) \otimes e_{ij}, \ f \in C(\mathbb{T}), \ i, j = 1, 2, \ldots, n,$$

for some unitary $w_1 \in A(m_6, e_6, M)$ and some unital $*$-homomorphism

$$\varphi_7 \colon C(\mathbb{T}) \longrightarrow A(m_6/n, e_6/n, M).$$

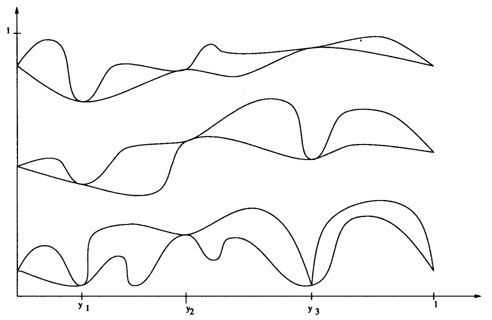

FIGURE 1. Illustration of the case $S = 6$, $m/e = 2$, $M = 3$.

After an arbitrarily small perturbation of φ_7, which we can safely ignore, we may assume that $\varphi_7(z)$ has minimal multiplicity in $A(m_6/n, e_6/n, M)$. There are then continuous functions $F_i\colon [0,1] \to \mathbb{R}, i = 1,2,\ldots,S$, such that $F_i(0) \in [0,1[$ for all i,

$$F_1(t) < F_2(t) < \cdots < F_S(t), \ \ t \notin \{y_1, y_2, \ldots, y_M\},$$

$$e^{2\pi i F_k(t)} \neq e^{2\pi i F_j(t)}, \ \ t \notin \{y_1, y_2, \ldots, y_M\}, \ \ k \neq j,$$

$$\#\{F_i(y_r) : i = 1,2,\ldots,S\} = \frac{e_6}{n}, \ \ r = 1,2,\ldots,M,$$

and orthogonal projections $q_1, q_2, \ldots, q_S \in C[0,1] \otimes M_{m_6/n}$ such that

$$\varphi_7(f)(t) = \sum_{j=1}^{S} f(e^{2\pi i F_j(t)}) q_j(t), \quad t \in [0,1], \quad f \in C(\mathbb{T}).$$

Since $\varphi_{6*} = 0$ on $K_1(A)$, we find that

$$[\varphi_7(z) \otimes e_{11} + \sum_{i \geq 2} 1 \otimes e_{ii}] = [\varphi_6(z \otimes e_{11} + \sum_{i \geq 2} 1 \otimes e_{ii})] = 0$$

in $K_1(A(m_6, e_6, M))$. Hence $[\varphi_7(z)] = 0$ in $K_1(A(m_6/n, e_6/n, M))$. This fact is equivalent to the following two additional properties of the F_i's:

$$F_j(0) = F_j(1), \ \ j = 1,2,\ldots,S,$$

$$F_{(k-1)m/e+i}(y_r) = F_{(k-1)m/e+1}(y_r),$$

$i = 1,2,\ldots,m/e, \ \ k = 1,2,\ldots,(Se)/m, \ \ r = 1,2,\ldots,M.$

The fact that φ_7 maps into $A(m_6/n, e_6/n, M)$, not just into $C(\mathbb{T}) \otimes M_{m_6/n}$, implies that

$$\sum_{i=1}^{\frac{m}{e}} q_{(k-1)\frac{m}{e}+i}(y_r) \in M_{\frac{e_6}{n}}, \tag{9}$$

for all $k = 1, 2, \ldots, Se/m$, $r = 1, 2, \ldots, M$. Write $Se/m = XN_0 + Y$ where $X \in \mathbb{N}$, $Y \in \{N_0, N_0 + 1, \ldots, 2N_0 - 1\}$. Now define $G_j \colon [0,1] \to \mathbb{R}, j = 1, 2, \ldots, S$, by

$$G_{(k-1)N_0\frac{m}{e}+j} = F_{(k-1)N_0\frac{m}{e}+1}, \quad j = 1, 2, \ldots, N_0\frac{m}{e}, \; k = 1, 2, \ldots, X,$$

and

$$G_j = F_{XN_0\frac{m}{e}+1}, \quad j \geq XN_0\frac{m}{e} + 1.$$

Since

$$\frac{1}{m_6}\left(\#\left\{r : e^{2\pi i F_r(t)} \in I_j^k\right\}\right) \;\geq\; \widehat{\varphi_7}(\xi_j^k)(t) = \widehat{\varphi_6}(\xi_j^k)(t) \;>\; \frac{2}{l},$$

and we may assume that e is so large that $2/l \;>\; N_0 m/(em_6)$, it follows that

$$\#\left\{r : e^{2\pi i F_r(t)} \in I_j^k\right\} \;>\; \frac{N_0 m}{e},$$

for all $j = 1, 2, \ldots, k$ and all $t \in [0,1]$. Hence

$$|e^{2\pi i G_j(t)} - e^{2\pi i F_j(t)}| \;\leq\; \frac{8\pi}{k},$$

for all j, t. Thus, if we define $\varphi_8 \colon C(\mathbb{T}) \otimes M_n \to A(m_6, e_6, M)$ by

$$\varphi_8(f \otimes e_{ij}) = \Big(\sum_{j=1}^{S} f(e^{2\pi i G_j(t)}) q_j(t)\Big) \otimes e_{ij}, \quad f \in C(\mathbb{T}), \; i, j = 1, 2, \ldots, n,$$

then

$$\|\operatorname{Ad} w_1 \circ \varphi_6(x) - \varphi_8(x)\| \leq \frac{16\pi}{\delta_A k}, \quad x \in cg(A).$$

(Note that φ_8 maps into $A(m_6, e_6, M)$ because of (9) and the choice of the G_j's.) Since $\varphi_8(A)' \cap A(m_6, e_6, M)$ contains a copy of $M_{N_0} \oplus M_Y$ as a unital C^*-subalgebra, we can now put everything together and conclude that there is a unitary $w_2 \in B$ and a unital $*$-homomorphism $\mu \colon A \to B$ such that

$$\|\operatorname{Ad} w_2 \circ \varphi(x) - \mu(x)\| \leq \left(\frac{248}{\delta_A k} + \frac{39}{\delta_A l}\right)\pi < \varepsilon, \quad x \in cg(A),$$

and such that $\mu(A)' \cap B$ contains $M_{N_0} \oplus M_{N_0+1} \oplus M_D \oplus M_{N_0} \oplus M_Y$ as a unital C^*-subalgebra. Since $D, Y \geq N_0$, we are done. $\square$

CHAPTER 6

THE FINAL PREPARATIONS

In this chapter we collect a series of lemmas which will used in the proof of our main results. They are centered around the problem of controlling the determinant function for certain unitaries, via the distance to the commutator subgroup. We will adopt the notation used in [**NT**], Section 3. See also [**Th4**].

LEMMA 6.1. — *Let $A = A(n,d,N)$ be a building block of type* 2 *and $u \in U_0(A)$ a unitary such that* $\operatorname{Det} u(t) = 1$, $t \in \mathbb{T}$. *It follows that there is a $\lambda \in \mathbb{T}$ and a $w \in \overline{DU_0(A)}$ such that $\lambda^n = 1$ and $u = \lambda w$.*

Proof. — Since $u \in U_0(A)$ there are selfadjoints $a_1, \dots, a_N$ in A such that

$$u = e^{2\pi i a_1} e^{2\pi i a_2} \cdots e^{2\pi i a_N}.$$

Since

$$\exp(2\pi i a_1)\exp(2\pi i a_2)\dots\exp(2\pi i a_N) = \exp(2\pi i \sum_{j=1}^{N} a_j)$$

modulo $\overline{DU_0(A)}$, it suffices to show that

$$\exp(2\pi i \sum_{j=1}^{N} a_j) \in \lambda\overline{DU_0(A)}$$

for some $\lambda \in \mathbb{T}$ with $\lambda^n = 1$. To this end, set $b = \sum_{j=1}^{N} a_j$ and note that $\operatorname{Tr}(b(t)) \in \mathbb{Z}$ since $\operatorname{Det} u(t) = 1$, $t \in \mathbb{T}$. Let $z \in \mathbb{Z}$ be the constant value of $\operatorname{Tr}(b(t))$ and set $\lambda = e^{2\pi i z/n}$. Then

$$e^{2\pi i b} = \lambda e^{2\pi i a}$$

where $a = b - z/n \in A$ satisfies that $\operatorname{Tr}(a(t)) = 0$ for all $t \in \mathbb{T}$. Now the proof of Lemma 1.4 in [**Th3**] can be used to show that for every $\varepsilon > 0$ there are two elements $v_1, v_2 \in A$ such that

$$\|a - (v_1 v_1^* - v_1^* v_1 + v_2 v_2^* - v_2^* v_2)\| < \varepsilon.$$

Since

$$e^{2\pi i(v_1 v_1^* - v_1^* v_1 + v_2 v_2^* - v_2^* v_2)} \in \overline{DU_0(A)},$$

this shows that $e^{2\pi i a} \in \overline{DU_0(A)}$. □

LEMMA 6.2. — *Let $A = A(n,d,N)$ be a building block of type 2 and p a non-zero projection in A. For any unitary $u \in A$, there is a unitary $v \in pAp + \mathbb{C}(1-p)$ such that $u = v$ modulo $\overline{DU_0(A)}$.*

Proof. — Since p automatically is a full projection and the natural map

$$\pi_0(U(pAp)) \longrightarrow K_1(pAp)$$

is an isomorphism, there is a unitary $w \in pAp + \mathbb{C}(1-p)$ such that $[w] = [v]$ in $K_1(A)$. Thus $wu^* \in U_0(A)$ and hence

$$\operatorname{Det} wu^*(t) = e^{i\alpha(t)}, \ t \in \mathbb{T},$$

for some continuous function $\alpha \colon \mathbb{T} \to \mathbb{R}$. Take a selfadjoint element $x \in pAp + \mathbb{C}(1-p)$ such that $\operatorname{Tr}(x(t)) = \alpha(t)$, $t \in \mathbb{T}$. Then $e^{-ix}wu^* \in \lambda\overline{DU_0(A)}$ for some $\lambda \in \mathbb{T}$ by Lemma 6.1. Set $v = \overline{\lambda} e^{-ix} w$. □

LEMMA 6.3. — *Let $A(n,d,N)$ be a building block of type 2 and let $U, V \in A(n,d,N)$ be unitaries such that $[U] = [V]$ in $K_1(A(n,d,N))$ and $\operatorname{Det} U(t)$ and $\operatorname{Det} V(t)$ are both constant in $t \in \mathbb{T}$. It follows that*

$$\operatorname{dist}(UV^*, DU_0(A)) \leq \frac{\pi}{d}.$$

Proof. — There is a number $\lambda \in \mathbb{T}$ such that $\operatorname{Det} \lambda UV^*(t) = 1$, $t \in \mathbb{T}$. By Lemma 6.1 there is then another number $\mu \in \mathbb{T}$ such that $\mu UV^* \in \overline{DU_0(A)}$. But if τ is any d'th root of unity, we have that $\tau 1 \in DU_0(A)$. Hence $\operatorname{dist}(\mu 1, DU_0(A)) \leq \frac{\pi}{d}$. □

Let A be a unital C^*-algebra. We use the notation

$$\rho \colon K_0(A) \to \operatorname{Aff} T(A)$$

for the canonical map. Recall that $U(A)/\overline{DU(A)}$ comes equipped with the quotient metric,

$$D_A(q'(u), q'(v)) = inf \left\{ \|uv^* - c\| : c \in \overline{DU(A)} \right\},$$

where $q'\colon U(A) \to U(A)/\overline{DU(A)}$ is the quotient map. $\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}$, on the other hand, is a metric space with the metric

$$d_A(f,g) = \begin{cases} 2 & \text{when } d'(f,g) \geq 1/2 \\ |e^{2\pi i d'(f,g)} - 1| & \text{when } d'(f,g) < 1/2, \end{cases}$$

where d' is the quotient metric of $\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}$, cf. [**NT**].

The following two lemmas were stated in [**NT**] for unital inductive limits of finite direct sums of circle algebras. However, the proofs only used that the canonical maps $\pi_1(U(A)) \to K_0(A)$ and $\pi_0(U(A)) \to K_1(A)$ are isomorphisms.

LEMMA 6.4. — *Let A be a unital C^*-algebra such that the canonical maps $\pi_1(U(A)) \to K_0(A)$ and $\pi_0(U(A)) \to K_1(A)$ are isomorphisms.*

– *There is a split exact sequence*

$$0 \to \operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \xrightarrow{\pi_A} K_1(A) \to 0.$$

– *λ_A is an isometry when $\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}$ is given the metric d_A.* □

LEMMA 6.5. — *Let A be a unital C^*-algebra such that the canonical maps $\pi_1(U(A)) \to K_0(A)$ and $\pi_0(U(A)) \to K_1(A)$ are isomorphisms. Assume that $\psi_1\colon K_1(A) \to K_1(B)$ and $\psi_0\colon \operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho(K_0(B))}$ are group homomorphisms such that ψ_0 is a contraction with respect to d_A and d_B.*

There is then a group homomorphism $\psi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$, which is contractive with respect to D_A and D_B, such that

$$\begin{array}{ccccc}
\operatorname{Aff} T(A)/\overline{\rho(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle \psi_0} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi_1} \\
\operatorname{Aff} T(B)/\overline{\rho(K_0(B))} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B)
\end{array}$$

commutes. □

LEMMA 6.6. — *Let $A = \bigoplus_{i=1}^{S} A(n_i, d_i, N_i)$ and $B = \bigoplus_{j=1}^{V} A(m_j, e_j, M_j)$ be finite direct sums of building blocks of type 2. Let $F \subset \operatorname{Aff} T(A)$ be a finite subset and $\delta > 0$. Let $M\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ be a Markov operator and $h\colon K_0(A) \to K_0(B)$ a group homomorphism such that*

$$\begin{array}{ccc}
K_0(A) & \xrightarrow{\rho} & \operatorname{Aff} T(A) \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle M} \\
K_0(B) & \xrightarrow{\rho} & \operatorname{Aff} T(B)
\end{array}$$

commutes. There is then an integer $T \in \mathbb{N}$ so large that whenever

$$H = M_{l_1} \oplus M_{l_2} \oplus \cdots \oplus M_{l_R}$$

is a finite dimensional C^*-algebra with $\min_j l_j \geq T$, then there is a unital $*$-homomorphism $\psi\colon A \to B \otimes H$ such that $\psi_* = d_* \circ h$ on $K_0(A)$, and

$$\|\widehat{\psi}(f) - \widehat{d} \circ M(f)\| < \delta, \ f \in F,$$

where $d\colon B \to B \otimes H$ is the $*$-homomorphism $d(a) = a \otimes 1_H$.

Proof. — Set $A_0 = \bigoplus_{i=1}^S C(\mathbb{T}) \otimes M_{d_i}$ and $B_0 = \bigoplus_{j=1}^V C(\mathbb{T}) \otimes M_{e_j}$. We will use the identifications $T(A_0) = T(A)$, $K_0(A_0) = K_0(A)$, and $T(B_0) = T(B)$, $K_0(B_0) = K_0(B)$. By Corollary 4.3 of **[NT]** there is a matrix algebra M_K and a unital $*$-homomorphism $\psi_0\colon A_0 \to B_0 \otimes M_K$ such that $\psi_{0*} = d_{0*} \circ h$ on $K_0(A_0)$ and

$$\|\widehat{\psi_0}(f) - \widehat{d_0} \circ M(f)\| \leq \frac{\delta}{2}, \quad f \in F,$$

where $d_0(b) = b \otimes 1_{M_K}$, $b \in B$. Let L be a common multiple of $n_1/d_1, n_2/d_2, \ldots$, n_S/d_S. We can then consider A as a unital C^*-subalgebra of $A_0 \otimes M_L$. Set

$$\psi_1 = \psi_0 \otimes id_{M_L}|_A\colon A \to A \otimes M_K \otimes M_L$$

and

$$d_1(b) = b \otimes 1_{M_K \otimes M_L}.$$

Then $\psi_{1*} = d_{1*} \circ h$ on $K_0(A)$ and

$$\|\widehat{\psi_1}(f) - \widehat{d_1} \circ M(f)\| \ \leq \ \frac{\delta}{2}, \ f \in F.$$

Choose $T \in \mathbb{N}$ so large that

$$\sup_{f \in F} \|f\| \frac{KL}{T} = \frac{\delta}{2}.$$

Consider a finite dimensional C^*-algebra $H = M_{l_1} \oplus M_{l_2} \oplus \cdots \oplus M_{l_R}$ with $\min_j l_j \geq T$. To define $\psi\colon A \to B \otimes H$ we shall use a unital $*$-homomorphism $\rho\colon A \to B$ which satisfies that $\rho_* = h$ on $K_0(A)$. The existence of ρ follows from the fact that evaluation at exceptional points, one for each direct summand, gives rise to two split surjections

$$A \ \to \ \bigoplus_{i=1}^{S} M_{d_i} \text{ and } B \ \to \ \bigoplus_{i=1}^{V} M_{e_i}$$

which induce isomorphisms on K_0. Since h defines a positive order unit preserving group homomorphism $K_0(\bigoplus_{i=1}^S M_{d_i}) \to K_0(\bigoplus_{i=1}^V M_{e_i})$, we know that there is a unital $*$-homomorphism $\bigoplus_{i=1}^{S} M_{d_i} \to \bigoplus_{i=1}^{V} M_{e_i}$ inducing h. By composition with the first split surjection and a splitting map for the second surjection, we get ρ. For each j, we write $l_j = X_j KL + R_j$, where $X_j \in \mathbb{N}$ and $R_j \in \{0, 1, 2, \ldots, KL-1\}$, and define $\lambda_j \colon A \to B \otimes M_{l_j}$ by

$$\lambda_j(a) = \mathrm{diag}\big(\underbrace{\psi_1(a), \ldots, \psi_1(a)}_{X_j\text{-times}}, \underbrace{\rho(a), \ldots, \rho(a)}_{R_j\text{-times}}\big).$$

Then $\psi(a) = \big(\lambda_1(a), \lambda_2(a), \ldots, \lambda_R(a)\big)$ defines a unital $*$-homomorphism with the desired properties. □

LEMMA 6.7. — *Let $A = A(n, d, N)$ be a building block of type 2. There is then a set $u, v_1, v_2, \ldots, v_{N-1}$ of unitaries in A such that*

1. *$[u]$ generates the direct summand $\mathbb{Z}$ in $K_1(A) = \mathbb{Z} \oplus \big(\mathbb{Z}_{n/d}\big)^{N-1}$.*
2. *$[v_i], i = 1, \ldots, N-1$, generate the direct summand $\big(\mathbb{Z}_{n/d}\big)^{N-1}$ in $K_1(A) = \mathbb{Z} \oplus \big(\mathbb{Z}_{n/d}\big)^{N-1}$.*
3. $\mathrm{Det}\, u(t) = t$, $t \in \mathbb{T}$.
4. $v_i^{n/d} \in \overline{DU(A)}$.

Proof. — The existence of the v_i's follows from the fact that the canonical surjection $U(A)/\overline{DU(A)} \to K_1(A)$ splits, cf. Lemma 6.4. The element

$$y = y_A \otimes e_{11} + \sum_{i \geq 2} 1 \otimes e_{ii} = \mathrm{diag}\big(y_A, 1, 1, \ldots, 1\big),$$

which we took as an element of $cg(A)$, does generate the direct summand $\mathbb{Z}$ of $K_1(A)$, but does not have the right determinant function. However, the loop $t \mapsto \mathrm{Det}\, y(t)$ is homotopic to the identity loop, so there is a continuous function $\alpha \colon \mathbb{T} \to \mathbb{R}$ such that $e^{i\alpha(t)}\, \mathrm{Det}\, y(t) = t$, $t \in \mathbb{T}$. Take

$$u(t) = \mathrm{diag}\big(y_A(t) e^{i\alpha(t)/n}, e^{i\alpha(t)/n}, e^{i\alpha(t)/n}, \ldots, e^{i\alpha(t)/n}\big),$$

$t \in \mathbb{T}$. □

A set $u, v_1, \ldots, v_{N-1}$ of unitaries in A satisfying conditions 1.-4. of Lemma 6.7 will be called *a set of unitary K_1-generators* in A.

LEMMA 6.8. — *Let $A = A(n, d, N)$ and $B = A(m, e, M)$ be building blocks of type 2, u a unitary in A such that $\mathrm{Det}\, u(t) = t$, $t \in \mathbb{T}$, $\alpha \in KK(A, B)$ an element of $KK(A, B)$ such that $\alpha_* \colon K_0(A) \to K_0(B)$ is positive and order unit preserving, and $v \in B$ a unitary such that $[v] = \alpha_*([u])$ in $K_1(B)$.*

Let $\varphi \colon A \to B$ be a unital $*$-homomorphism satisfying the following conditions:

1. $\hat{\varphi}(\xi_j^k) > 3/l$, $j = 1, 2, \ldots, k$,
2. $\hat{\varphi}(g) > 4\kappa, g \in H$,

where $k, l \in \mathbb{N}$ are natural numbers such that $l > 12$ and $24\pi/(\delta_A k) < 1$, $H \subset C(\mathbb{T}, [0,1])$ is the finite set of Lemma 3.4 corresponding to k, l, and

$$\frac{((8N+4)nd+4)n}{e} < \kappa < \frac{1}{2l}, \quad \frac{16Nn}{e} < \kappa.$$

It follows that there is a unital $*$-homomorphism $\psi\colon A \to B$ such that $[\psi] = \alpha$ in $KK(A, B)$, $\operatorname{Det}\psi(u)(t) = \operatorname{Det}v(t)$, $t \in \mathbb{T}$, and

$$\|\hat{\varphi} - \hat{\psi}\| \leq \frac{(6N+3)n^2d + 5n}{e}.$$

Proof. — We use Lemma 5.2 to perturb φ to φ' such that φ' is a standard homomorphism with $((2N+1)nd+1)m/e$ copies of the constant function 1 among its characteristic functions and

$$\|\hat{\varphi} - \hat{\varphi}'\| \leq \frac{(4n+2)n^2d + 4n}{e}.$$

Then φ' is approximately inner equivalent to $\psi_1 \oplus \psi_2 \oplus \psi_3$ where $\psi_1\colon A \to M_n \subset A(mn/e, n, M)$ is given by

$$\psi_1(f) = \operatorname{diag}(\underbrace{f(1), \ldots, f(1)}_{m/e \text{ times}}),$$

$\psi_2\colon A \to M_{(2N+1)nd} \subset A((2N+1)mnd/e, (2N+1)nd, M)$, is given by

$$\psi_2(f) = \operatorname{diag}(\underbrace{f(1), \ldots, f(1)}_{\frac{(2N+1)mnd}{e} \text{ times}}),$$

and $\psi_3\colon A \to A(m_1, e_1, M)$, $e_1 = e - (2N+1)nd - n$, $m_1 = me_1/e$, is a unital $*$-homomorphism on standard form, whose specific data are irrelevant for the present purposes. We may suppose that $\varphi' = \psi_1 \oplus \psi_2 \oplus \psi_3$. By Proposition 2.5 there is a unital $*$-homomorphism

$$\psi_2'\colon A \to A\left(\frac{(2N+1)mnd}{e}, (2N+1)nd, M\right)$$

such that

$$[\psi_1 \oplus \psi_2' \oplus \psi_3] = \alpha$$

in $KK(A, B)$. Then $[(\psi_1 \oplus \psi_2' \oplus \psi_3)(u)] = [v]$ in $K_1(B)$ and there is therefore a homotopically trivial loop $\beta\colon \mathbb{T} \to \mathbb{T}$ such that

$$\beta(t)\operatorname{Det}(\psi_1 \oplus \psi_2' \oplus \psi_3)(u)(t) = \operatorname{Det}v(t), \ t \in \mathbb{T}.$$

There is then also a homotopically trivial loop $\gamma\colon \mathbb{T} \to \mathbb{T}$ such that

$$\gamma(t)^{m/e} = \beta(t), \ t \in \mathbb{T}.$$

Define $\psi_1'\colon A \to A(mn/e, n, M)$ by

$$\psi_1'(f) = \operatorname{diag}\underbrace{\big(f(\gamma(t)), \ldots, f(\gamma(t))\big)}_{m/e \text{ times}},$$

and note that ψ_1' is homotopic to ψ_1. Thus $\psi = \psi_1' \oplus \psi_2' \oplus \psi_3$ is a unital $*$-homomorphism which represents α in $KK(A,B)$. Since $\operatorname{Det} u(t)$ is the identity map on $\mathbb{T}$ it follows that

$$\operatorname{Det}\psi(u)(t) = \gamma(t)^{m/e} \operatorname{Det}(\psi_1 \oplus \psi_2' \oplus \psi_3)(u)(t) = \operatorname{Det} v(t), \quad t \in \mathbb{T}.$$

Since

$$\|\hat{\psi} - \hat{\varphi}'\| \le \frac{(2N+1)nd}{e} + \frac{n}{e},$$

the proof is complete. □

We need an appropriate version of Lemma 6.8 which handles finite direct sums of building blocks of type 2. To ease the formulation of this lemma, which the reader will find messy enough as it is, we introduce some additional notation. When $A = \bigoplus_{i=1}^R A(n_i, d_i, N_i)$ is a finite direct sum of building blocks of type 2 and $u \in A(n_i, d_i, N_i)$ is a unitary, we write $\tilde{u}$ for the unitary $(1, 1, \ldots, 1, u, 1, \ldots, 1) \in A$, where u (of course) occurs as the i'th entry.

LEMMA 6.9. — *Let $A = \bigoplus_{i=1}^R A(n_i, d_i, N_i)$ and $B = \bigoplus_{i=1}^S A(m_i, e_i, M_i)$ be finite direct sums of building blocks of type 2, $u^i, v_1^i, v_2^i, \ldots, v_{N-1}^i$ a set of unitary K_1-generators for $A(n_i, d_i, N_i)$, $i = 1, 2, \ldots, R$, α an element of $KK(A,B)$, and $S^i, T_1^i, T_2^i, \ldots, T_{N-1}^i$ unitaries in B such that $T_j^{i n_j/d_j} \in \overline{DU(B)}$, $\alpha_*([\widetilde{u^i}]) = [S^i]$, $\alpha_*([\widetilde{v_j^i}]) = [T_j^i]$ in $K_1(B)$ for all i, j.*

Let $\varphi\colon A \to B$ be a unital $*$-homomorphism such that

- *$\varphi_* = \alpha_*$ on $K_0(A)$,*
- *$\theta(\varphi(\xi_j^k(u_0))) > \frac{3}{l}$, $j = 1, 2, \ldots, k$, $u_0 \in cu_0(A)$, $\theta \in T(B)$,*
- *$\theta(\varphi(g(u_0))) > 4\kappa$, $g \in G$, $u_0 \in cu_0(A)$, $\theta \in T(B)$,*

where $k, l \in \mathbb{N}$ are natural numbers such that $l > 12$, and $24\pi/(\delta_A k) < 1$, $G \subset C(\mathbb{T} \cup \{0\}, [0,1])$ is the finite set of Proposition 3.5 corresponding to the present choice of k, l, and

$$\max_j \frac{(8N_j + 4)n_j^2 d_j + 4n_j}{\operatorname{mult}(\varphi)} < \kappa < \frac{1}{2l}, \quad \max_j \frac{16 N_j n_j}{\operatorname{mult}(\varphi)} < \kappa.$$

It follows that there is a unital $*$-homomorphism $\psi\colon A \to B$ such that

$$[\psi] = \alpha \text{ in } KK(A,B),$$

$$\|\hat{\varphi} - \hat{\psi}\| \le \max_j \frac{(6N_j+3)n_j^2 d_j + 5n_j}{\operatorname{mult}(\varphi)},$$

and

$$D_B(q'(\psi(\tilde{u^i})), q'(S^i)) \le \max_r \frac{\pi}{e_r}, \quad D_B(q'(\psi(\tilde{v^i_j})), q'(T^i_j)) \le \max_r \frac{\pi}{e_r},$$

for all $j = 1,2,\ldots,N_i - 1$, $i = 1,2,\ldots,R$.

Proof. — It is straightforward to reduce the proof to the case where there is only one direct summand in B. We may therefore assume that $B = A(m,e,M)$. Let $p_1, p_2, \ldots, p_R$ be the minimal non-zero central projections in A. Let

$$\varphi_i\colon A(n_i,d_i,N_i) \to \varphi(p_i)B\varphi(p_i) = A(m_i,e_i,M), \ i = 1,2,\ldots,R,$$

be the partial $*$-homomorphisms of φ. Every trace state of $\varphi(p_i)B\varphi(p_i)$ is of the form $\omega(\varphi(p_i))^{-1}\omega(\cdot)$ for some $\omega \in T(B)$. By the choice of G, cf. the proof of Proposition 3.5, this means that second and third condition on φ turn into

$$\hat{\varphi}_i(\xi^k_j) > \frac{3}{l}, \ j = 1,2,\ldots,k,$$

and

$$\hat{\varphi}_i(g) > 4\kappa, \ g \in H,$$

respectively, where $H \subset C(\mathbb{T},[0,1])$ is the finite subset of Lemma 3.4, corresponding to the present choice of k,l. By Lemma 6.2 there is a unitary $S^i_0 \in A(m_i,e_i,M_i)$ such that $S^i_0 + (1 - \varphi(p_i)) = \lambda S^i$ modulo $\overline{DU(B)}$ for some $\lambda \in \mathbb{T}$. By Lemma 6.8 there is a unital $*$-homomorphism $\psi_i\colon A \to A(m_i,e_i,M)$ such that

$$[\iota_i \circ \psi_i] = \iota_{i*}(\alpha)$$

in $KK(A,B)$, where $\iota_i\colon \varphi(p_i)B\varphi(p_i) \to B$ is the inclusion,

$$\operatorname{Det}\psi_i(u^j)(t) = \operatorname{Det} S^i_0(t), \ t \in \mathbb{T},$$

where the determinant is calculated in M_{m_i}, and

$$\|\hat{\varphi}_i - \hat{\psi}_i\| \le \frac{(6N_i+3)n_i^2 d_i + 5n_i}{e} \le \frac{(6N_i+3)n_i^2 d_i + 5n_i}{\operatorname{mult}(\varphi)}.$$

Define $\psi\colon A \to B$ by

$$\psi(a_1, \dots, a_R) = \sum_{i=1}^{R} \iota_i \circ \psi_i(a_i).$$

Then

$$\|\hat{\varphi} - \hat{\psi}\| \;\leq\; \max_j \frac{(6N_j + 3)n_j^2 d_j + 5n_j}{\operatorname{mult}(\varphi)},$$

$$[\psi] = \alpha \text{ in } KK(A, B)$$

and

$$t \mapsto \operatorname{Det} \psi(\tilde{u^i})(t) \operatorname{Det} S^i(t)^{-1}$$

is constant for each $i = 1, 2, \dots, R$. When W is a unitary in B such that

$$W^{n/d} \in \overline{DU(B)},$$

then $t \mapsto \operatorname{Det} W(t)$ must be constant. It follows from this that also

$$\operatorname{Det} \psi(\tilde{v_j^i})(t) \operatorname{Det} T_j^i(t)^{-1}$$

is constant in t, for all $j = 1, 2, \dots, N_i - 1$, $i = 1, 2, \dots, R$. We can therefore conclude from Lemma 6.3 that

$$D_B(q'(\psi(\tilde{u^i})), q'(S^i)) \leq \frac{\pi}{e}, \quad D_B(q'(\psi(\tilde{v_j^i})), q'(T_j^i)) \leq \frac{\pi}{e},$$

for all $j = 1, 2, \dots, N_i - 1$, $i = 1, 2, \dots, R$. □

LEMMA 6.10. — *Let* $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$ *be a sequence of finite direct sums of building blocks of type* 2 *with unital connecting* $*$*-homomorphisms and set* $A = \varinjlim(A_n, \varphi_n)$. *If* A *is approximately divisible,* $\lim_{k\to\infty} \operatorname{mult}_0(\varphi_{k,n}) = \infty$ *for all* $n \in \mathbb{N}$.

Proof. — As in the proof of [**NT**], Lemma 4.4, it follows from the approximate divisibility that $K_0(A)$ has large denominators in the sense of Nistor [**N**]. By applying [**Th1**], Lemma 4.4, to the AF-algebra whose K_0-group is the limit of

$$K_0(A_1) \xrightarrow{\varphi_{1*}} K_0(A_2) \xrightarrow{\varphi_{2*}} K_0(A_3) \xrightarrow{\varphi_{3*}} \cdots,$$

we conclude that $\lim_{k\to\infty} \operatorname{mult}_0(\varphi_{k,n}) = \infty$. □

CHAPTER 7

THE MAIN RESULTS

In this chapter A and B will be unital inductive limits of sequences of finite direct sums of building blocks of type 2. To formulate the results, observe that a unital $*$-homomorphism $\psi\colon A \to B$ induces a contractive group homomorphism $\psi^\natural\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ in the obvious way.

THEOREM A. — *Assume that A is simple and that B is approximately divisible. Let α be an element of $KK(A,B)$ such that $\alpha_*[1] = [1]$ in $K_0(B)$ and $\varphi_T\colon T(B) \to T(A)$ an affine continuous map such that*

$$r_B(\omega)(\alpha_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\mathbf{\Phi}\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that

$$\begin{array}{ccccc}
\operatorname{Aff} T(A)/\overline{\rho(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle \widetilde{\varphi}} & & \Big\downarrow{\scriptstyle \mathbf{\Phi}} & & \Big\downarrow{\scriptstyle \alpha_*} \\
\operatorname{Aff} T(B)/\overline{\rho(K_0(B))} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B)
\end{array}$$

commutes, where $\widetilde{\varphi}\colon \operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho(K_0(B))}$ is the map induced by $\varphi_{T*}\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$.

It follows that there is a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\varphi^*|_{T(B)} = \varphi_T$, $\varphi^\natural = \mathbf{\Phi}$ and $[\varphi \circ \mu] = \mu^*(\alpha)$ in $KK(D,B)$, whenever D is a finite direct sum of building blocks of type 2 and $\mu\colon D \to A$ is a unital $*$-homomorphism.

This result has the following corollaries.

COROLLARY A1. — *Assume that A is simple and that B is approximately divisible. Let α be an element of $KK(A,B)$ such that $\alpha_*[1] = [1]$ in $K_0(B)$*

and $\varphi_T\colon T(B) \to T(A)$ *an affine continuous map such that*

$$r_B(\omega)(\alpha_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

It follows that there is a unital $*$*-homomorphism* $\varphi\colon A \to B$ *such that* $\varphi^* = \varphi_T$ *on* $T(B)$ *and* $[\varphi \circ \mu] = \mu^*(\alpha)$ *in* $KK(D, B)$*, whenever* D *is a finite direct sum of building blocks of type* 2 *and* $\mu\colon D \to A$ *is a unital* $*$*-homomorphism.*

COROLLARY A2. — *Assume that* A *is simple and that* B *is approximately divisible. Let* $\varphi_0\colon K_0(A) \to K_0(B)$*,* $\varphi_1\colon K_1(A) \to K_1(B)$ *be group homomorphisms such that* $\varphi_0([1]) = [1]$ *in* $K_0(B)$ *and* $\varphi_T\colon T(B) \to T(A)$ *a continuous affine map such that*

$$r_B(\varphi_0(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

It follows that there is a unital $*$*-homomorphism* $\varphi\colon A \to B$ *such that* $\varphi_* = \varphi_0$ *on* $K_0(A)$*,* $\varphi_* = \varphi_1$ *on* $K_1(A)$ *and* $\varphi^* = \varphi_T$ *on* $T(B)$*.*

Examples in [**NT**] show that Theorem A is a stronger result than Corollary A1, in the sense that $*$-homomorphisms (or even automorphisms) which agree on the Elliott invariant and satisfy the KK-condition, may not agree on $U(A)/\overline{DU(A)}$.

THEOREM B. — *Assume that* A *is simple. Let* $\varphi, \psi\colon A \to B$ *be unital* $*$*-homomorphisms such that* $\varphi^* = \psi^*$ *on* $T(B)$*,* $\varphi^\natural = \psi^\natural$ *on* $U(A)/\overline{DU(A)}$ *and* $[\varphi \circ \mu] = [\psi \circ \mu]$ *in* $KK(D, B)$*, whenever* D *is a finite direct sum of building blocks of type* 2 *and* $\mu\colon D \to A$ *a unital* $*$*-homomorphism.*

It follows that φ *and* ψ *are approximately inner equivalent.*

THEOREM C. — *Assume that* A *and* B *are simple. Let* $\varphi_1\colon K_1(A) \to K_1(B)$ *be an isomorphism,* $\varphi_0\colon K_0(A) \to K_0(B)$ *an isomorphism of partially ordered abelian groups with order units and* $\varphi_T\colon T(B) \to T(A)$ *an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

It follows that there is a $*$*-isomorphism* $\varphi\colon A \to B$ *such that* $\varphi_* = \varphi_1$ *on* $K_1(A)$*,* $\varphi_* = \varphi_0$ *on* $K_0(A)$ *and* $\varphi^* = \varphi_T$ *on* $T(B)$*.*

Proof of Theorem A. — The conclusion is trivial when A is finite dimensional so we assume that A is infinite dimensional. We set $\varphi_0 = \alpha_*\colon K_0(A) \to K_0(B)$ and $\varphi_1 = \alpha_*\colon K_1(A) \to K_1(B)$. Note that the compatibility condition on α_* and φ_T implies that φ_0 is positive.

We shall adopt the notation already established, and introduce the following additional notation. A unital $*$-homomorphism $\varphi\colon A \to B$ between C^*-algebras induces maps

$$\operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho(K_0(B))}$$

and $U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ in the obvious way, and these maps will be denoted by $\widetilde{\varphi}$ and $\varphi^\natural$, respectively. Write $A = \varinjlim A_n$ and $B = \varinjlim B_n$ where $A_1 \xrightarrow{\mu_1} A_2 \xrightarrow{\mu_2} A_3 \xrightarrow{\mu_3} \dots$ and $B_1 \xrightarrow{\rho_1} B_2 \xrightarrow{\rho_2} B_3 \xrightarrow{\rho_3} \cdots$. Each A_n and B_n is a finite direct sum of building blocks of type 2 and the connecting maps are unital. By Theorem 4.1 we may assume that μ_n is injective for all n and we will therefore, occasionally, suppress the connecting maps of this sequence in the notation. Let $\mu_{\infty,n}\colon A_n \to A$, $\rho_{\infty,n}\colon B_n \to B$, denote the canonical maps. Then $\operatorname{Aff} T(A)$ and $\operatorname{Aff} T(B)$ are the inductive limits of

$$\operatorname{Aff} T(A_1) \xrightarrow{\widehat{\mu_1}} \operatorname{Aff} T(A_2) \xrightarrow{\widehat{\mu_2}} \operatorname{Aff} T(A_3) \xrightarrow{\widehat{\mu_3}} \dots$$

and

$$\operatorname{Aff} T(B_1) \xrightarrow{\widehat{\rho_1}} \operatorname{Aff} T(B_2) \xrightarrow{\widehat{\rho_2}} \operatorname{Aff} T(B_3) \xrightarrow{\widehat{\rho_3}} \dots,$$

respectively, and the canonical maps

$$\operatorname{Aff} T(A_n) \to \operatorname{Aff} T(A), \quad \operatorname{Aff} T(B_n) \to \operatorname{Aff} T(B)$$

are $\widehat{\mu_{\infty,n}}$ and $\widehat{\rho_{\infty,n}}$, respectively. Similarly, $U(A)/\overline{DU(A)}$ and $U(B)/\overline{DU(B)}$ are the inductive limits, in the category of complete metric groups, of the sequences

$$U(A_1)/\overline{DU(A_1)} \xrightarrow{\mu_1^\natural} U(A_2)/\overline{DU(A_2)} \xrightarrow{\mu_2^\natural} U(A_3)/\overline{DU(A_3)} \xrightarrow{\mu_3^\natural} \dots$$

and

$$U(B_1)/\overline{DU(B_1)} \xrightarrow{\rho_1^\natural} U(B_2)/\overline{DU(B_2)} \xrightarrow{\rho_2^\natural} U(B_3)/\overline{DU(B_3)} \xrightarrow{\rho_3^\natural} \dots,$$

respectively. The canonical maps

$$U(A_n)/\overline{DU(A_n)} \to U(A)/\overline{DU(A)}$$

and

$$U(B_n)/\overline{DU(B_n)} \to U(B)/\overline{DU(B)}$$

are then $\mu_{\infty,n}^\natural$ and $\rho_{\infty,n}^\natural$, respectively.

As in **[NT]** we have the following fact.

ASSERTION 7.1. — *For every $n \in \mathbb{N}$, any finite subset $F \subset \operatorname{Aff} T(A_n)$ and any $\varepsilon > 0$ there is a $m \in \mathbb{N}$ and a Markov operator $M\colon \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_m)$ such that*

$$\|\widehat{\rho_{\infty,m}} \circ M(f) - \varphi_{T*} \circ \widehat{\mu_{\infty,n}}(f)\| < \varepsilon, \ f \in F,$$

and a group homomorphism $h\colon K_0(A_n) \to K_0(B_m)$ such that $\rho_{\infty,m_*} \circ h = \varphi_0 \circ \mu_{\infty,n_*}$ and such that h and M are compatible in the sense that

$$\begin{array}{ccc} \operatorname{Aff} T(A_n) & \xrightarrow{M} & \operatorname{Aff} T(B_m) \\ \rho \big\uparrow & & \rho \big\uparrow \\ K_0(A_n) & \xrightarrow{h} & K_0(B_m) \end{array}$$

commutes.

This assertion can be proved exactly as Assertion 1 in the proof of Theorem A in [**NT**]. This is because a building block of type 2 contains a finite direct sum of circle algebras with the same tracial state space and the same K_0-group, cf. the proof of Lemma 6.6.

A major step in the proof is to establish the following

ASSERTION 7.2. — *Let $F_1 \subset \operatorname{Aff} T(A_n)$ and $F_2 \subset U(A_n)/\overline{DU(A_n)}$ be finite subsets and $\varepsilon > 0$. There is then a $k \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon A_n \to B_k$ such that*

(1) $\|\widehat{\rho_{\infty,k}} \circ \widehat{\psi}(f) - \varphi_{T*} \circ \widehat{\mu_{\infty,n}}(f)\| < \varepsilon$ *for all* $f \in F_1$,
(2) $D_B(\rho^{\natural}_{\infty,k} \circ \psi^{\natural}(u), \mathbf{\Phi} \circ \mu^{\natural}_{\infty,n}(u)) < \varepsilon$ *for all* $u \in F_2$.
(3) $[\rho_{\infty,k} \circ \psi] = \mu^{*}_{\infty,n}(\alpha)$ *in* $KK(A_n, B)$,
(4) $\operatorname{mult}(\psi) > 0$.

So let us first prove Assertion 7.2. Write

$$A_n = \bigoplus_{i=1}^{R} A(n_i, d_i, N_i),$$

where each $A(n_i, d_i, N_i)$ is a building block of type 2. Let $k \in \mathbb{N}$ such that $20\pi/(\delta_{A_n} k) < 1$. For each i we choose a unitary set, $u^i, v^i_j, j = 1, 2, \dots, N_i - 1$, of K_1-generators for $A(n_i, d_i, N_i)$. Every element $x \in U(A_n)/\overline{DU(A_n)}$ has, by Lemma 6.4, a representation

$$x = \prod_{i=1}^{R} \prod_{j=1}^{N_i - 1} \lambda_{A_n}(a^x_{ij}) q'(\tilde{u^i})^{k^x_i} q'(\tilde{v^i_j})^{n^x_{ij}}$$

where

$$a_{ij}^x \in \operatorname{Aff} T(A_n)/\overline{\rho(K_0(A_n))}, \quad k_i^x,\, n_{ij}^x \in \mathbb{Z}.$$

For each $x \in F_2$ and each $j = 1, 2, \ldots, N_i - 1$, $i \in \{1, 2, \ldots, R\}$, choose $b_{ij}^x \in \operatorname{Aff} T(A_n)$ such that $q(b_{ij}^x) = a_{ij}^x$, where $q\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(A)/\overline{\rho(K_0(A))}$ is the quotient map. Since A is simple and the connecting maps injective, there are numbers $l_0, l \in \mathbb{N}, l_0 > n$, $l > 12$, such that

$$\theta(\mu_{l_0,n}(\xi_j^k(u_0))) > \frac{4}{l}, \quad j = 1, 2, \ldots, k,\ \theta \in T(A_{l_0}),\ u_0 \in cu_0(A_n).$$

Note that we can take l as large as we want; the appropriate condition is that

$$\Big(\sum_{i=1}^{R} N_i\Big)|e^{2\pi i/l} - 1| + \frac{1}{l}\big(1 + \sup_{x \in F_2} \sum_{i,j} |k_i^x| + |n_{ij}^x|\big) < \varepsilon.$$

Let $G \subset C(\mathbb{T} \cup \{0\}, [0,1])$ be the finite set of Proposition 3.5 corresponding to k, l. Let $\kappa \in]0, 1/(2l)[$ such that

$$\theta(\mu_{l_0,n}(g(u_0))) \;>\; 5\kappa, \quad g \in G,\ u_0 \in cu_0(A_n),\ \theta \in T(A_{l_0}).$$

Again we may take κ arbitrarily small; we shall require that

$$(\sup_{f \in F_3} \|f\| + 1)\kappa < \frac{1}{2l},$$

where

$$F_3 = F_1 \cup \left\{b_{ij}^x : x \in F_2, j = 1, 2, \ldots, N_i - 1, i = 1, 2, \ldots, R\right\}$$
$$\cup \left\{\widehat{\xi_j^k(u_0)}, \widehat{g(u_0)} : j = 1, 2, \ldots, k,\ g \in G,\ u \in cu_0(A_n)\right\}.$$

We remark that if $r > l_0$, then

$$\theta(\mu_{r,n}(\xi_j^k(u_0))) \;>\; \frac{4}{l},\ j = 1, 2, \ldots, k,\ \theta \in T(A_r),\ u \in cu_0(A_n),$$

and

$$\theta(\mu_{r,n}(g(u_0))) \;>\; 5\kappa,\ g \in G,\ u_0 \in cu_0(A_n),\ \theta \in T(A_r).$$

Since $\lim\limits_{r\to\infty} \operatorname{mult}(\mu_{r,n}) = \infty$ by Lemma 5.3, we can increase l_0 to get $\operatorname{mult}(\mu_{l_0,n})$ as large as we want; we will insist that

$$\max_j \frac{(8N_j + 4)n_j^2 d_j + 5n_j}{\operatorname{mult}(\mu_{l_0,n})} < \kappa \quad \text{and} \quad \max_j \frac{16N_j n_j}{\operatorname{mult}(\mu_{l_0,n})} < \kappa.$$

Take now a $\delta > 0$ such that

$$\pi\delta < \min\left\{\kappa, \frac{1}{2l}\right\} \text{ and } \delta \max_j \frac{n_j}{d_j} < 1.$$

From Assertion 7.1 we get an $m \in \mathbb{N}$, a Markov operator $M\colon \operatorname{Aff} T(A_{l_0}) \to \operatorname{Aff} T(B_m)$ such that

$$\|\widehat{\rho_{\infty,m}} \circ M(f) - \varphi_{T*} \circ \widehat{\mu_{\infty,l_0}}(f)\| \ < \ \delta, \ f \in \widehat{\mu_{l_0,n}}(F_3),$$

and a group homomorphism $h\colon K_0(A_{l_0}) \to K_0(B_m)$ such that

$$\rho_{\infty,m_*} \circ h = \varphi_0 \circ \mu_{\infty,l_0*}$$

and such that h and M are compatible. Choose a finite set V of selfadjoints in B_m such that $\{\widehat{a} : a \in V\} = M \circ \widehat{\mu_{l_0,n}}(F_3)$. Let P be a finite set of projections in B_m which generate $K_0(B_m)$. Let $T \in \mathbb{N}$ be the integer from Lemma 6.6, corresponding to $A = A_{l_0}$, $B = B_m$, $F = \widehat{\mu_{l_0,n}}(F_3), M, h$, and the present choice of $\delta > 0$. By the approximate divisibility of B, and a standard perturbation argument, which uses that B_m has stable relations by **[L2]**, we can find $k_1 > m$, a finite dimensional C^*-subalgebra

$$H = M_{l_1} \oplus M_{l_2} \oplus \cdots \oplus M_{l_R}$$

of B_{k_1} with $\min_j l_j \geq T$ and a $*$-homomorphism $\mu\colon B_m \to B_{k_1} \cap H'$ such that

$$\|\mu(x) - \rho_{k_1,m}(x)\| \ < \ \delta, \ x \in V \cup P.$$

By Lemma 6.6 there is then a unital $*$-homomorphism $\psi_0\colon A_{l_0} \to \mu(B_m) \otimes H$ such that $\psi_{0*} = d_* \circ \mu_* \circ h$ on $K_0(A_{l_0})$ and

$$\|\widehat{\psi_0}(f) - \widehat{d} \circ \widehat{\mu} \circ M(f)\| < \delta, \quad f \in \widehat{\mu_{l_0,n}}(F_3),$$

where $d(a) = a \otimes 1_H, a \in \mu(B_m)$. Let $\kappa\colon \mu(B_m) \otimes H \to B_{k_1}$ be a unital $*$-homomorphism mapping onto $C^*(H, \mu(B_m))$ such that $\kappa \circ d(x) = x$, $x \in \mu(B_m)$ and set $\psi_1 = \kappa \circ \psi_0$. Then $\psi_{1*} = \rho_{k_1,m_*} \circ h$ on $K_0(A_{l_0})$; the last equality requires only that $\delta < 1$. Furthermore,

$$\begin{aligned}\|\widehat{\psi_1}(x) - \widehat{\rho_{k_1,m}} \circ M(x)\| \ &\leq \ \|\widehat{\psi_1}(x) - \widehat{\mu} \circ M(x)\| + \delta \\ \leq \ \|\widehat{\psi_0}(x) - \widehat{d} \circ \widehat{\mu} \circ M(x)\| &+ \|\widehat{\kappa} \circ \widehat{d} \circ \widehat{\mu} \circ M(x) - \widehat{\mu} \circ M(x)\| + \delta \ \leq \ 2\delta\end{aligned}$$

for all $x \in \widehat{\mu_{l_0,n}}(F_3)$. Set $\psi_2 = \psi_1 \circ \mu_{l_0,n}\colon A_n \to B_{k_1}$. Then

$$\|\widehat{\rho_{\infty,k_1}} \circ \widehat{\psi_2}(f) - \varphi_{T*} \circ \widehat{\mu_{\infty,n}}(f)\| < 3\delta, \ f \in F_3,$$

and

$$\rho_{\infty,k_1*} \circ \psi_{2*} = \rho_{\infty,k_1*} \circ \rho_{k_1,m_*} \circ h \circ \mu_{l_0,n_*} = \varphi_0 \circ \mu_{\infty,n_*}$$

on $K_0(A_n)$. In particular,

$$\widehat{\rho_{\infty,k_1}} \circ \ > \ \widehat{\psi_2}(\widehat{\xi_j^k(u_0)}) \ > \ \frac{4}{l} - 3\delta \ > \ \frac{3}{l}, \quad j = 1, 2, \ldots, k,$$

and

$$\widehat{\rho_{\infty,k_1}} \circ \widehat{\psi_2}(\widehat{g(u_0)}) > 5\kappa - 3\delta > 4\kappa, \quad g \in G.$$

We can therefore choose $k_2 > k_1$ such that

$$\theta(\rho_{k_2,k_1} \circ \psi_2(\xi_j^k(u_0))) > \frac{3}{l}, \ j = 1, 2, \dots, k, \ u_0 \in cu_0(A_n), \ \theta \in T(B_{k_2}),$$

and

$$\theta(\rho_{k_2,k_1} \circ \psi_2(g(u_0))) > 4\kappa, \ g \in G, \ u_0 \in cu_0(A_n), \ \theta \in T(B_{k_2}).$$

Furthermore, by increasing k_2 if necessary, we may assume that B_{k_2} contains unitaries, S^i, T_j^i, $j = 1, 2, \dots, N_i - 1$, $i = 1, 2, \dots, R$, with

$$(5) \qquad D_B\big(\rho^{\natural}_{\infty,k_2}(q'(S^i)), \mathbf{\Phi} \circ \mu^{\natural}_{\infty,n}(q'(\tilde{u^i}))\big) < \delta,$$

and

$$D_B\big(\rho^{\natural}_{\infty,k_2}(q'(T_j^i)), \mathbf{\Phi} \circ \mu^{\natural}_{\infty,n}(q'(\tilde{v_j^i}))\big) = \delta$$

for all i, j. For each i, j we have that

$$D_B(\rho^{\natural}_{\infty,k_2}(q'(T_j^{i\,n_j/d_j})), q'(1)) < \frac{n_j}{d_j}\delta,$$

so by increasing k_2, we may assume that

$$\operatorname{dist}(T_j^{i\,n_j/d_j}, \overline{DU(B_{k_2})}) \leq \frac{n_j}{d_j}\delta.$$

Since $\delta \max_j n_j/d_j = 1$, we have that

$$T_j^{i\,n_j/d_j} e^a \in \overline{DU(B_{k_2})}$$

for some $a = -a^*$ (depending on i, j) with

$$\|a\| \leq \delta\pi \frac{n_j}{d_j}.$$

By exchanging each T_j^i with $T_j^i e^{a d_j/n_j}$, we may suppose that

$$T_j^{i\,n_j/d_j} \in \overline{DU(B_{k_2})}.$$

The price we pay is that we only have that

$$(6) \qquad D_B\big(\rho^{\natural}_{\infty,k_2}(q'(T_j^i)), \mathbf{\Phi} \circ \mu^{\natural}_{\infty,n}(q'(\tilde{v_j^i}))\big) < \delta\pi$$

for all i, j. Since $K_*(A_n)$ is finitely generated, the functor $KK(A_n, \cdot)$ is continuous, **[RS]**, so by increasing k_2 again we may assume that $\mu^*_{\infty,n}(\alpha) = \rho_{\infty,k_2*}(\beta)$ for some $\beta \in KK(A_n, B_{k_2})$. And, since $\alpha_* = \varphi_0$ on $K_0(A)$ and $\rho_{\infty,k_1*} \circ \psi_{2*} =$

$\varphi_0 \circ \mu_{\infty,n*}$ on $K_0(A_n)$, we may assume that $\beta_* = \rho_{k_2,k_1*} \circ \psi_{2*}$ on $K_0(A_n)$. Since $\pi\delta < 2$, it follows from (5) and (6) that

$$\rho_{\infty,k_2*}([T^i_j]) = \varphi_1([\mu_{\infty,n}(\tilde{v^i_j})])$$

and

$$\rho_{\infty,k_2*}([S^i]) = \varphi_1([\mu_{\infty,n}(\tilde{u^i})])$$

in $K_1(B)$ for all i,j. But $\varphi_1 = \alpha_*$ on $K_1(A)$, so we can also assume that

$$\beta_*([\tilde{v^i_j}]) = [T^i_j],\ \beta_*([\tilde{u^i}]) = [S^i]$$

in $K_1(B_{k_2})$ for all $j = 1,2,\ldots,N_i - 1$, $i = 1,2,\ldots,R$. Since $\mathrm{mult}(\mu_{l_0,n}) \leq \mathrm{mult}(\rho_{k_2,k_1} \circ \psi_2)$, our choice of l_0 guarentees that

$$\max_j \frac{(8N_j+4)n_j^2 d_j + 5n_j}{\mathrm{mult}(\rho_{k_2,k_1}\circ\psi_2)} < \kappa \quad \text{and} \quad \max_j \frac{16N_j n_j}{\mathrm{mult}(\rho_{k_2,k_1}\circ\psi_2)} < \kappa.$$

Thus Lemma 6.9 gives us a unital $*$-homomorphism $\psi\colon A_n \to B_{k_2}$ such that $[\psi] = \beta$ in $KK(A_n, B_{k_2})$,

$$\|\hat{\psi} - \widehat{\rho_{k_2,k_1}\circ\psi_2}\| \leq \kappa,$$

and

$$D_{B_{k_2}}(q'(\psi(\tilde{u^i})), q'(S^i)) \leq \kappa,\quad D_{B_{k_2}}(q'(\psi(\tilde{v^i_j})), q'(T^i_j)) \leq \kappa, \tag{7}$$

for all i,j. Note that $\mathrm{mult}(\psi) > 0$ since

$$\psi_* = \beta_* = \rho_{k_2,k_1*}\circ\psi_{2*} = \rho_{k_2,k_1*}\circ\psi_{1*}\circ\mu_{l_0,n*},$$

and $\mathrm{mult}(\mu_{l_0,n}) > 0$. Hence (4) holds. Observe that

$$\begin{aligned} &\|\widehat{\rho_{\infty,k_2}}\circ\hat{\psi}(f) - \varphi_{T*}\circ\widehat{\mu_{\infty,n}}(f)\| \\ &\leq \|\widehat{\rho_{\infty,k_2}}\circ\hat{\psi}(f) - \widehat{\rho_{\infty,k_1}}\circ\widehat{\psi_2}(f)\| + \|\widehat{\rho_{\infty,k_1}}\circ\widehat{\psi_2}(f) - \varphi_{T*}\circ\widehat{\mu_{\infty,n}}(f)\| \\ &\leq \sup_{f\in F_3}\|f\|\kappa + 3\delta < \frac{1}{l} \end{aligned} \tag{8}$$

for all $f \in F_3$, by the choice of δ and κ. In addition,

$$[\rho_{\infty,k_2}\circ\psi] = \rho_{\infty,k_2*}(\beta) = \mu^*_{\infty,n}(\alpha)$$

in $KK(A_n, B)$. In particular, this gives (1) and (3). (8) implies that

$$d'\big(\widetilde{\rho_{\infty,k_2}}\circ\tilde{\psi}(a^x_{ij}),\ \ \tilde{\varphi}\circ\widetilde{\mu_{\infty,n}}(a^x_{ij})\big) = \frac{1}{l}$$

for all $x \in F_2$ and all i, j. Hence the D_B-distance between

$$(\rho_{\infty,k_2} \circ \psi)^\natural(x)$$

$$= \prod_{i=1}^{R} \prod_{j=1}^{N_i-1} \lambda_B(\widetilde{\rho_{\infty,k_2}} \circ \tilde{\psi}(a_{ij}^x))q'(\rho_{\infty,k_2} \circ \psi(\tilde{u^i}))^{k_i^x} q'(\rho_{\infty,k_2} \circ \psi(\tilde{v_j^i}))^{n_{ij}^x}$$

and

$$\prod_{i=1}^{R} \prod_{j=1}^{N_i-1} \lambda_B(\tilde{\varphi} \circ \widetilde{\mu_{\infty,n}}(a_{ij}^x))q'(\rho_{\infty,k_2} \circ \psi(\tilde{u^i}))^{k_i^x} q'(\rho_{\infty,k_2} \circ \psi(\tilde{v_j^i}))^{n_{ij}^x}$$

is less than $(\sum_{i=1}^R N_i)|e^{2\pi i/l} - 1|$. By combining (5) and (7), we see that

$$D_B\big(\rho_{\infty,k_2}^\natural(q'(\psi(\tilde{u^i}))), \mathbf{\Phi} \circ \mu_{\infty,n}^\natural(q'(\tilde{u^i}))\big) < \delta + \kappa < \frac{1}{l},$$

for all i, and by combining (6) with (7) we get,

$$D_B\big(\rho_{\infty,k_2}^\natural(q'(\psi(\tilde{v_j^i}))), \mathbf{\Phi} \circ \mu_{\infty,n}^\natural(q'(\tilde{v_j^i}))\big) < \pi\delta + \kappa < \frac{1}{l}$$

for all i, j. It follows that the D_B-distance between

$$\prod_{i=1}^{R} \prod_{j=1}^{N_i-1} \lambda_B(\tilde{\varphi} \circ \widetilde{\mu_{\infty,n}}(a_{ij}^x))q'(\rho_{\infty,k_2} \circ \psi(\tilde{u^i}))^{k_i^x} q'(\rho_{\infty,k_2} \circ \psi(\tilde{v_j^i}))^{n_{ij}^x}$$

and

$$\prod_{i=1}^{R} \prod_{j=1}^{N_i-1} \lambda_B(\tilde{\varphi} \circ \widetilde{\mu_{\infty,n}}(a_{ij}^x))\mathbf{\Phi} \circ \mu_{\infty,n}^\natural(q'(\tilde{u^i}))^{k_i^x} \mathbf{\Phi} \circ \mu_{\infty,n}^\natural(q'(\tilde{v_j^i}))^{n_{ij}^x} = \mathbf{\Phi} \circ \mu_{\infty,n}{}^\natural(x)$$

is less than $1/l(\sup_{y \in F_2} \sum_{i,j} |k_i^y| + |n_{ij}^y|)$. Combined with the previous estimate, this shows that

$$\begin{aligned} &D_B(\rho_{\infty,k_2}^\natural \circ \psi^\natural(u), \mathbf{\Phi} \circ \mu_{\infty,n}^\natural(u)) \\ &< \frac{1}{l} \sup_{y \in F_2} \sum_{i,j} |k_i^y| + |n_{ij}^y| + (\sum_{i=1}^R N_i)|e^{2\pi \frac{i}{l}} - 1| \ < \ \varepsilon \end{aligned}$$

for all $u \in F_2$. We have proved Assertion 7.2.

The construction of $\varphi\colon A \to B$ is now similar to the corresponding step in the proof of Theorem A in [**NT**]. Choose finite subsets

$$F_n \subset \operatorname{Aff} T(A_n), \ K_n \subset U(A_n)/\overline{DU(A_n)} \text{ and } cg(A_n) \subset H_n \subset A_n$$

such that

$$\widehat{\mu_n}(F_n) \subset F_{n+1}, \ \mu_n^\natural(K_n) \subset K_{n+1}, \ \mu_n(H_n) \subset H_{n+1},$$

and $\bigcup_n \widehat{\mu_{\infty,n}}(F_n)$, $\bigcup_n \mu^{\natural}_{\infty,n}(K_n)$ are dense in $\operatorname{Aff} T(A)$ and $U(A)/\overline{DU(A)}$, respectively. Let $\delta(A_n)$ be a sequence in $]0,1[$ such that

$$\|\lambda(y) - \eta(y)\| \leqslant 2^{-n}, \ y \in H_n,$$

whenever $\lambda, \eta\colon A_n \to D$ are unital $*$-homomorphisms into the same C^*-algebra D satisfying that

$$\|\lambda(a) - \eta(a)\| < \delta(A_n), \ a \in cg(A_n).$$

We will construct sequences $n_1 < n_2 < n_3 < \cdots$ and $m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ and unital $*$-homomorphisms $\psi_k\colon A_{n_k} \to B_{m_k}$ such that

$$\|\rho_{m_{k+1},m_k} \circ \psi_k(x) - \psi_{k+1}(x)\| < \delta(A_{n_k}), \quad x \in cg(A_{n_k}), \tag{9}$$

$$\|\widehat{\rho_{\infty,m_k}} \circ \widehat{\psi_k}(a) - \varphi_{T*} \circ \widehat{\mu_{\infty,n_k}}(a)\| < 2^{-k}, \quad a \in F_{n_k}, \tag{10}$$

$$D_B\big(\rho^{\natural}_{\infty,m_k} \circ \psi^{\natural}_k(x), \quad \mathbf{\Phi} \circ \mu^{\natural}_{\infty,n_k}(x)\big) < 2^{-k}, \ x \in K_{n_k}, \tag{11}$$

and

$$[\rho_{\infty,m_k} \circ \psi_k] = \mu^*_{\infty,n_k}(\alpha) \tag{12}$$

in $KK(A_{n_k}, B)$ for all k. Let us check that such sequences will give us what we want. First, it is standard to define $\varphi\colon A \to B$ by

$$\varphi(\mu_{\infty,m}(x)) = \lim_{l\to\infty} \rho_{\infty,m_l} \circ \psi_l \circ \mu_{n_l,m}(x), \ x \in A_m,$$

for all $m \in \mathbb{N}$. Then

$$\hat{\varphi}(\widehat{\mu_{\infty,m}}(a)) = \lim_{l\to\infty} \widehat{\rho_{\infty,m_l}} \circ \widehat{\psi_l} \circ \widehat{\mu_{n_l,m}}(a), \ a \in \operatorname{Aff} T(A_m),$$

and

$$\varphi^{\natural}(\mu^{\natural}_{\infty,m}(x)) \ = \ \lim_{l\to\infty} \rho^{\natural}_{\infty,m_l} \circ \psi^{\natural}_l \circ \mu^{\natural}_{n_l,m}(x), \ x \in U(A_m)/\overline{DU(A_m)},$$

for all m, so (10) and (11) imply that

$$\hat{\varphi}(\widehat{\mu_{\infty,m}}(a)) \ = \ \varphi_{T*}(\widehat{\mu_{\infty,m}}(a)), \ a \in F_m,$$

and

$$\varphi^{\natural}(\mu_{\infty,m}(x)) = \mathbf{\Phi}(\mu^{\natural}_{\infty,m}(x)), \ x \in K_m,$$

respectively. The density of $\bigcup_n \widehat{\mu_{\infty,n}}(F_n)$ in $\operatorname{Aff} T(A)$, and $\bigcup_n \mu^{\natural}_{\infty,n}(K_n)$ in $U(A)/\overline{DU(A)}$, imply that $\hat{\varphi} = \varphi_{T*}$ and $\varphi^{\natural} = \mathbf{\Phi}$, respectively. Furthermore, since each A_m has exactly stable relations, **[L1]**, **[L2]**, we have that $\varphi \circ \mu_{\infty,m}$ is homotopic to $\rho_{\infty,m_l} \circ \psi_l \circ \mu_{n_l,m}$ for all sufficiently large l, and hence

$$\begin{aligned}[\varphi \circ \mu_{\infty,m}] = [\rho_{\infty,m_l} \circ \psi_l \circ \mu_{n_l,m}] &= \mu^*_{n_l,m}([\rho_{\infty,m_l} \circ \psi_l]) \\ = \mu^*_{n_l,m}(\mu^*_{\infty,n_l}(\alpha)) = \mu^*_{\infty,m}(\alpha)\end{aligned}$$

in $KK(A_m, B)$ by (12). Now, if D is a finite direct sums of building blocks of type 2 and $\mu \colon D \to A$ a unital $*$-homomorphism, then, by **[L2]** and **[L1]**, there is a unital $*$-homomorphism $\lambda \colon D \to A_m$, for some m, such that $\mu_{\infty,m} \circ \lambda$ is homotopic to μ. Hence

$$[\varphi \circ \mu] = [\varphi \circ \mu_{\infty,m} \circ \lambda] = \lambda^*([\varphi \circ \mu_{\infty,m}]) = \lambda^*(\mu^*_{\infty,m}(\alpha)) = \mu^*(\alpha)$$

in $KK(D, B)$. It now suffices to construct the sequences. This will be done by induction, of course, but to make the induction work we have to impose the following additional conditions: There are integers $r_k, t_k \in \mathbb{N}$, $t_k > 12$, $24\pi/(\delta_{A_{n_k}} r_k) < 1$ and numbers $\kappa_k \in]0, 1/(2t_k)[$ such that

$$\left(\frac{72}{\delta_{A_{n_k}} r_k} + \frac{13}{\delta_{A_{n_k}} t_k}\right)\pi < \delta(A_{n_k}),$$

$$\theta(\xi_j^{r_k}(u_0)) > \frac{2}{t_k}, \; j = 1, 2, \ldots, r_k, \; \theta \in T(A), \; u_0 \in cu_0(A_{n_k}),$$

$$\theta(g(u_0)) > 3\kappa_k, \; g \in G_k, \; \theta \in T(A), \; u_0 \in cu_0(A_{n_k}),$$

$$D_B\big(\rho^\natural_{\infty,m_k} \circ \psi_k^\natural(q'(u)), \mathbf{\Phi} \circ \mu^\natural_{\infty,n_k}(q'(u))\big) \leqslant \kappa_k^2, \; u \in cu(A_{n_k}),$$

$$\|\widehat{\rho_{\infty,m_k}} \circ \widehat{\psi_k}(\widehat{f(u_0)}) - \varphi_{T*} \circ \widehat{\mu_{\infty,n_k}}(\widehat{f(u_0)})\| \leqslant \kappa_k^2, \; f \in G_k, \; u_0 \in cu_0(A_{n_k}),$$

and

$$\operatorname{mult}(\psi_k) > 0,$$

where $G_k \subset C(\mathbb{T} \cup \{0\}, [0,1])$ is the subset of Proposition 3.5 corresponding to r_k and t_k. Let us assume that $n_1 < n_2 < \cdots < n_k$, $m_1 < m_2 < \cdots < m_k$, $r_1 < r_2 < \cdots < r_k$, $t_1 < t_2 < \cdots < t_k$, $\kappa_i, 1 \le i \le k$, and $\psi_i, 1 \le i \le k$, have been constructed. We shall construct $m_{k+1}, n_{k+1}, r_{k+1}, t_{k+1}, \kappa_{k+1}$ and ψ_{k+1}. By (14) and (15) we can choose $n_{k+1} > n_k$ so large that

$$\theta(\xi_j^{r_k}(u_0)) > \frac{2}{t_k}, \; j = 1, 2, \ldots, r_k, \; \theta \in T(A_{n_{k+1}}), \; u_0 \in cu_0(A_{n_k})$$

and

$$\theta(g(u_0)) > 3\kappa_k, \; g \in G_k, \; \theta \in T(A_{n_{k+1}}), \; u \in cu_0(A_{n_k}).$$

Choose $r_{k+1} \in \mathbb{N}$ such that

$$\frac{72\pi}{\delta_{A_{n_{k+1}}} r_{k+1}} < \frac{1}{2}\delta(A_{n_{k+1}}),$$

and subsequently $t_{k+1} \in \mathbb{N}$ such that

$$\frac{13\pi}{\delta_{A_{n_{k+1}}} t_{k+1}} < \frac{1}{2}\delta(A_{n_{k+1}})$$

and

$$\theta(\xi_j^{r_{k+1}}(u_0)) \;>\; \frac{2}{t_{k+1}},\; j = 1, 2, \ldots, r_{k+1},\; \theta \in T(A),\; u_0 \in cu_0(A_{n_{k+1}}).$$

Then take $\kappa_{k+1} \in \,]0, 1/(2t_{k+1})[$ such that

$$\theta(g(u_0)) \;>\; 3\kappa_{k+1},\; g \in G_{k+1},\; \theta \in T(A),\; u_0 \in cu_0(A_{n_{k+1}}).$$

By Assertion 7.2 there is an $m_{k+1} > m_k$ and a $*$-homomorphism

$$\lambda\colon A_{n_{k+1}} \longrightarrow B_{m_{k+1}}$$

such that

$$[\rho_{\infty, m_{k+1}} \circ \lambda] \;=\; \mu^*_{\infty, n_{k+1}}(\alpha)$$

in $KK(A_{n_{k+1}}, B)$, $\operatorname{mult}(\lambda) > 0$,

$$\|\widehat{\rho_{\infty,m_{k+1}}} \circ \hat{\lambda}(f) - \varphi_{T*} \circ \widehat{\mu_{\infty,n_{k+1}}}(f)\| \leqslant \varepsilon,\; f \in \mathcal{F}_1,$$

and

$$D_B\big(\rho^\natural_{\infty,m_{k+1}} \circ \lambda^\natural(u), \mathbf{\Phi} \circ \mu^\natural_{\infty,n_{k+1}}(u)\big) \leqslant \varepsilon,\; u \in \mathcal{F}_2,$$

where $\varepsilon > 0$ and the finite subsets

$$\mathcal{F}_1 \subset \operatorname{Aff} T(A_{n_{k+1}}), \quad \mathcal{F}_2 \subset U(A_{n_{k+1}})/\overline{DU(A_{n_{k+1}})}$$

are free to choose. We take

$$\varepsilon = \min\left\{\kappa_{k+1}^2, 2^{-k-1}\right\}$$

and $\mathcal{F}_1$ to contain $F_{n_{k+1}}$ and the images in $\operatorname{Aff} T(A_{n_{k+1}})$ of

$$\{\mu_{n_{k+1},n_k}(f(u_0)) : f \in G_k, u_0 \in cu_0(A_{n_k})\} \\ \cup \{g(u_0) : g \in G_{k+1}, u_0 \in cu_0(A_{n_{k+1}})\}$$

and $\mathcal{F}_2$ to contain

$$q'\big(cu(A_{n_{k+1}}) \cup \mu_{n_{k+1},n_k}(cu(A_{n_k}))\big) \cup K_{n_{k+1}}.$$

With these choices, the $k+1$-versions of (10)-(18) hold (with $\psi_{k+1} = \lambda$). By choosing ε even smaller and m_{k+1} even larger, if necessary, we can also assume, by using (16) and (17), that

$$D_{B_{m_{k+1}}}\big(\lambda \circ \mu_{n_{k+1},n_k}(u), \rho_{m_{k+1},m_k} \circ \psi_k(u)\big) \leqslant \kappa_k^2, \quad u \in cu(A_{n_k}),$$

and

$$\|\widehat{\rho_{m_{k+1},m_k}} \circ \widehat{\psi_k}(f(u_0)) - \widehat{\psi_{k+1}} \circ \widehat{\mu_{n_{k+1},n_k}}(f(u_0))\| < \kappa_k^2,$$

$f \in G_k, u_0 \in cu_0(A_{n_k})$. Since

$$\begin{aligned}[\rho_{\infty,m_{k+1}} \circ \lambda \circ \mu_{n_{k+1},n_k}] &= \mu^*_{n_{k+1},n_k}([\rho_{\infty,m_{k+1}} \circ \lambda]) \\ &= \mu^*_{n_{k+1},n_k}(\mu^*_{\infty,n_{k+1}}(\alpha)) = \mu^*_{\infty,n_k}(\alpha) = [\rho_{\infty,m_k} \circ \psi_k]\end{aligned}$$

in $KK(A_{n_k}, B)$, the continuity of the functor $KK(A_{n_k}, \cdot)$ implies that we may assume, if neccesary by increasing m_{k+1} again, that $[\lambda] = [\rho_{m_{k+1},m_k} \circ \psi_k]$ in $KK(A_{n_k}, B_{m_{k+1}})$. Finally, since $\lim_{l\to\infty} \operatorname{mult}(\rho_{l,m_k} \circ \psi_k) = \infty$ by Lemma 6.10, because $\operatorname{mult}(\psi_k) > 0$, we can also increase m_{k+1} to get the last condition in Proposition 3.5 satisfied. Then that proposition gives us a unitary $w \in B_{m_{k+1}}$ such that

$$\|\operatorname{Ad} w \circ \lambda(x) - \rho_{m_{k+1},m_k} \circ \psi_k(x)\| \leq \frac{72\pi}{\delta_A r_k} + \frac{13\pi}{\delta_{A_{n_k}} t_k} < \delta(A_{n_k}), \quad x \in cg(A_{n_k}).$$

By choosing $\psi_{k+1} = \operatorname{Ad} w \circ \lambda$, we will have the $k+1$-versions of (9)-(18) satisfied. This completes the induction step and the proof. $\square$

Proof of the Corollary A1. — The compatibility between φ_T and α_* implies that we get a contractive map

$$\widetilde{\varphi}\colon \operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho(K_0(B))}$$

induced by $\varphi_{T*}\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$. By Lemma 6.5 there is then a contractive group homomorphism $\Phi\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that

$$\begin{array}{ccccc}\operatorname{Aff} T(A)/\overline{\rho(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ {\scriptstyle\widetilde{\varphi}}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle\alpha_*}\big\downarrow \\ \operatorname{Aff} T(B)/\overline{\rho(K_0(B))} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B)\end{array}$$

commutes. The corollary then follows immediately from Theorem A. $\square$

Proof of Corollary A2. — By the UCT theorem, **[RS]**, there is an element $\alpha \in KK(A, B)$ such that $\alpha_* = \varphi_0 \oplus \varphi_1$ on $K_*(A)$. Apply Corollary A1. $\square$

Proof of Theorem B. — We adopt the notation and general set up from the proof of Theorem A. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. It clearly suffices to prove that there is a unitary $u \in B$ such that

$$\| \operatorname{Ad} u \circ \psi \circ \mu_{\infty,n}(x) - \varphi \circ \mu_{\infty,n}(x) \| < \varepsilon, \quad x \in cg(A_n).$$

To this end we shall apply Proposition 3.5. So let $k \in \mathbb{N}$ such that $24\pi/(\delta_{A_n} k) < 1$, and $72\pi/(\delta_{A_n} k) < \varepsilon/6$. Since A is simple and the connecting $*$-homomorphisms injective, there is an $m > n$ and an $l > 12$ in $\mathbb{N}$, such that $13\pi/(\delta_{A_n} l) < \varepsilon/6$ and

$$\theta(\mu_{m,n}(\xi_j^k(u_0))) \ > \ \frac{2}{l},\ j = 1, 2, \dots, k,\ u_0 \in cu_0(A_n),\ \theta \in T(A_m).$$

Choose $\kappa \in]0, 1/(2l)[$ such that

$$\theta(\mu_{m,n}(g(u_0))) \ > \ 3\kappa,\ g \in G,\ \theta \in T(A_m),\ u_0 \in cu_0(A_n),$$

where G is the finite set of Proposition 3.5 corresponding to the present choice of k, l. Since $\lim_{m\to\infty} \operatorname{mult}(\mu_{m,n}) = \infty$ by Lemma 5.3, we can take m so large that $\max_i 16 N_i n_i < \kappa \operatorname{mult}(\mu_{m,n})$, where N_i and n_i are the numbers occuring in the decomposition $A_n = \bigoplus_{i=1}^R A(n_i, d_i, N_i)$ of A_n as a sum of building blocks of type 2. From the fact that A_m is generated by a set of exactly stable relations, **[L2]**, we conclude that there is a $r \in \mathbb{N}$ and $\varphi_1, \psi_1 \colon A_m \to B_r$ such that

$$\|\rho_{\infty,r} \circ \varphi_1(x) - \varphi \circ \mu_{\infty,m}(x)\| \leqslant \delta,\ x \in F,$$

and

$$\|\rho_{\infty,r} \circ \psi_1(x) - \psi \circ \mu_{\infty,m}(x)\| \leqslant \delta,\ x \in F,$$

where $\delta > 0$ and the finite set $F \subset A_m$ are free to choose. In particular, we shall require that $\delta < \varepsilon/3$ and that $\mu_{m,n}(cg(A_n)) \subset F$. By assumption we have that $\hat{\varphi} = \hat{\psi} \colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ and that $\psi^\natural = \varphi^\natural$ on $U(A)/\overline{DU(A)}$. By increasing r and taking a sufficiently small δ we can therefore arrange that

$$|\theta(\psi_1 \circ \mu_{m,n}(g(u_0)) - \varphi_1 \circ \mu_{m,n}(g(u_0)))| < \kappa^2, \quad \theta \in T(B_r)\ g \in G,$$

and that

$$\operatorname{dist}(\psi_1 \circ \mu_{m,n}(u)\varphi_1 \circ \mu_{m,n}(u^*), DU(B_r)) < \kappa^2, \quad u \in cu(A_n).$$

Furthermore, we can assume that $\psi \circ \mu_{\infty,n}$ and $\varphi \circ \mu_{\infty,n}$ are homotopic to $\rho_{\infty,r} \circ \psi_1 \circ \mu_{m,n}$ and $\rho_{\infty,r} \circ \varphi_1 \circ \mu_{m,n}$, respectively, cf. **[L1]**. Our assumption on $[\varphi], [\psi] \in KK(A, B)$ therefore shows that

$$[\rho_{\infty,r} \circ \varphi_1 \circ \mu_{m,n}] = [\varphi \circ \mu_{\infty,n}] = [\psi \circ \mu_{\infty,n}] = [\rho_{\infty,r} \circ \psi_1 \circ \mu_{m,n}]$$

in $KK(A_m, B)$. By the continuity of the functor $KK(A_n, \cdot)$, we can therefore assume, by increasing r if necessary, that $[\varphi_1 \circ \mu_{m,n}] = [\psi_1 \circ \mu_{m,n}]$ in $KK(A_n, B_r)$. We now get a unitary $w \in B_r$ from Proposition 3.5 such that

$$\| \operatorname{Ad} w \circ \varphi_1 \circ \mu_{m,n}(x) - \psi_1 \circ \mu_{m,n}(x) \| < (\frac{72}{\delta_{A_n} k} + \frac{13}{\delta_{A_n} l})\pi < \frac{\varepsilon}{3}, \quad x \in cg(A_n).$$

Hence the unitary $u = \rho_{\infty,r}(w) \in B$ will do the job. □

Proof of Theorem C. — If one of A and B is finite dimensional they must both have K_0 group $\mathbb{Z}$. By the argument of Lemma 5.3 they are then both matrix algebras and the conclusion is trivial. We may therefore assume that A and B are both infinite dimensional. By **[RS]** there is a KK-equivalence $\alpha \in KK(A, B)$ such that $\alpha_* = \varphi_0$ on $K_0(A)$ and $\alpha_* = \varphi_1$ on $K_1(A)$. Let

$$\mathbf{\Phi}\colon U(A)/\overline{DU(A)} \longrightarrow U(B)/\overline{DU(B)}$$

be a homomorphism compatible with φ_T, φ_0 and φ_1, in the sense that the diagram of Theorem A commutes. Such a $\mathbf{\Phi}$ exists by Lemma 6.5. Note that $\mathbf{\Phi}$ is an isometric isomorphism and that $\mathbf{\Phi}^{-1}$ is compatible with $\varphi_T^{-1}, \varphi_0^{-1}$ and φ_1^{-1}. By Theorem A, which can be applied thanks to Theorem 5.1, there are unital $*$-homomorphisms $\lambda\colon A \to B$ and $\psi\colon B \to A$ such that $\lambda^* = \varphi_T$ on $T(B)$, $\psi^* = \varphi_T^{-1}$ on $T(A)$, $\lambda^\natural = \mathbf{\Phi}$, $\psi^\natural = \mathbf{\Phi}^{-1}$, and $[\lambda \circ \mu] = \mu^*(\alpha)$ in $KK(D, B)$, when D is a building block of type 2 and $\mu\colon D \to A$ a unital $*$-homomorphism, and $[\psi \circ \nu] = \nu^*(\alpha^{-1})$ in $KK(C, A)$, when C is a building block of type 2 and $\nu\colon C \to B$ a unital $*$-homomorphism. Thus $(\psi \circ \lambda)^*$ is the identity map on $T(A)$, $(\psi \circ \lambda)^\natural$ is the identity on $U(A)/\overline{DU(A)}$ and, when D is a building block of type 2 and $\mu\colon D \to A$ a unital $*$-homomorphism,

$$\begin{aligned}[\psi \circ \lambda \circ \mu] = (\lambda \circ \mu)^*(\alpha^{-1}) &= [\lambda \circ \mu] \cdot [\alpha^{-1}] \\ &= \mu^*(\alpha) \cdot \alpha^{-1} = \mu^*(\alpha \cdot \alpha^{-1}) = \mu^*([id_A]) = [\mu]\end{aligned}$$

in $KK(D, A)$. ($\cdot$ denotes here the Kasparov product.) By Theorem B we see that $\psi \circ \lambda$ is approximately inner equivalent to the identity map of A. In the same way we see that $\lambda \circ \psi$ is approximately inner equivalent to the identity map of B. It then follows from a standard approximate intertwining argument, cf. e.g. **[R1]**, Proposition A, that there is a $*$-isomorphism $\varphi\colon A \to B$ with the same action on $K_*(A)$ and $T(B)$ as λ. □

CHAPTER 8

ON THE AUTOMORPHISM GROUP

Let A and B be unital inductive limits of sequences of finite direct sum of building blocks of type 2.

The purpose of this chapter is to describe the automorphism group $\mathrm{Aut}(A)$ of A modulo the normal subgroup $\overline{\mathrm{Inn}(A)}$ of approximately inner automorphisms when A is simple. Besides the results of Chapter 7, we shall use ideas and results from **[R2]** and **[DL3]**. In **[R2]** Rørdam introduced a quotient of the Kasparov group $KK(C,D)$, called $KL(C,D)$, into the classification program. The advantage of $KL(C,D)$ over $KK(C,D)$ lies in the fact that two approximately inner $*$-homomorphisms $C \to D$ define the same element of $KL(C,D)$, cf. **[R2]**, Proposition 5.4, and that the contravariant functor $KL(\cdot,D)$ is continuous on the Bootstrap category $\mathcal{N}$ for which the UCT is known to hold, **[RS]**, provided only that D is σ-unital, cf. **[DL3]**, UCMT and Lemma 2.2 (ii). Thanks to this we can use KL in place of KK to improve the formulation of some of the results in Chapter 7.

THEOREM 8.1. — *Assume that A is simple and that B is approximately divisible. Let α be an element of $KL(A,B)$ such that $\alpha_*[1] = [1]$ in $K_0(B)$ and $\varphi_T\colon T(B) \to T(A)$ an affine continuous map such that*

$$r_B(\omega)(\alpha_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\mathbf{\Phi}\colon U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that

$$\begin{array}{ccccc}
\mathrm{Aff}\,T(A)/\overline{\rho(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle \tilde{\varphi}} & & \Big\downarrow{\scriptstyle \mathbf{\Phi}} & & \Big\downarrow{\scriptstyle \alpha_*} \\
\mathrm{Aff}\,T(B)/\overline{\rho(K_0(B))} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B)
\end{array}$$

commutes, where $\widetilde{\varphi}\colon \operatorname{Aff} T(A)/\overline{\rho(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho(K_0(B))}$ *is the map induced by* $\varphi_{T*}\colon \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$.

It follows that there is a unital $*$-homomorphism $\varphi\colon A \to B$ *such that* $\varphi^*|_{T(B)} = \varphi_T$, $\varphi^\natural = \mathbf{\Phi}$ *and* $[\varphi] = \alpha$ *in* $KL(A, B)$. □

COROLLARY 8.2. — *Assume that A is simple and that B is approximately divisible. Let α be an element of $KL(A, B)$ such that $\alpha_*[1] = [1]$ in $K_0(B)$ and $\varphi_T\colon T(B) \to T(A)$ an affine continuous map such that*

$$r_B(\omega)(\alpha_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

It follows that there is a unital $*$-homomorphism $\varphi\colon A \to B$ such that $\varphi^* = \varphi_T$ on $T(B)$ and $[\varphi] = \alpha$ in $KL(A, B)$. □

THEOREM 8.3. — *Assume that A is simple. Let $\varphi, \psi\colon A \to B$ be unital $*$-homomorphisms such that $\varphi^* = \psi^*$ on $T(B)$, $\varphi^\natural = \psi^\natural$ on $U(A)/\overline{DU(A)}$ and $[\varphi] = [\psi]$ in $KL(A, B)$.*

It follows that φ and ψ are approximately inner equivalent. □

It should be noted that 8.1-8.3 take a particular simple form when $K_1(A)$ is torsion-free since $KL(A, B) = \operatorname{Hom}(K_0(A), K_0(A)) \oplus \operatorname{Hom}(K_1(A), K_1(A))$ in this case. Also it should be noted that we have that $KL(A, B) = KK(A, B)$ when $K_*(A)$ is finitely generated.

Theorem 8.1 and Corollary 8.2 follow straightforwardly from Theorem A and Corollary A1 in Chapter 7. Theorem 8.3 does not follow directly from the statement of Theorem B in Chapter 7. Rather, it follows from the following slight change of the proof: Instead of the conclusion that $[\rho_{\infty,r}\circ\varphi_1] = [\rho_{\infty,r}\circ\psi_1]$ in $KK(A_m, B)$, we get (a priori) only this conclusion in $KL(A_m, B)$. However, $KK(A_m, B) = KL(A_m, B)$ by **[DL3]**, Proposition 2.9, since $K_*(A_m)$ is finitely generated. So we do actually get identity in $KK(A_m, B)$ also. The rest of the proof is unchanged.

With these KL-reformulations of the results from Chapter 7 we can now proceed to the desired description of $\operatorname{Aut}(A)/\overline{\operatorname{Inn}(A)}$. This group is put together by three components, the first of which is the group $\operatorname{Aut}(\mathcal{E}_A)$ of automorphisms of the Elliott invariant $\mathcal{E}_A$ of A, i.e. $\operatorname{Aut}(\mathcal{E}_A)$ is group of triples $(\alpha_0, \alpha_1, \alpha_T)$ where α_0 is an order unit preserving ordered-group automorphism of $K_0(A)$, α_1 is a group automorphism of $K_1(A)$ and $\alpha_T\colon T(A) \to T(A)$ is an affine homeomorphism such that

$$r_A \circ \alpha_T^{-1}(\omega) = r_A(\omega) \circ \alpha_0 \text{ on } K_0(A)$$

for all $\omega \in T(A)$. The second component is $\operatorname{Hom}\big(K_1(A), \operatorname{Aff} T(A)/\overline{\rho(K_0(A))}\big)$, the group of homomorphisms from $K_1(A)$ into $\operatorname{Aff} T(A)$ modulo the closure of the canonical image of $K_0(A)$. The third component, $\operatorname{ext}(K_1(A), K_0(A))$, was discovered by Dadarlat and Loring in [**DL3**]. $\mathrm{ext}(K_1(A), K_0(A))$ is a quotient of the group $\mathrm{Ext}(K_1(A), K_0(A))$ of abelian group extensions of $K_1(A)$ by $K_0(A)$; namely that group modulo the subgroup of such extensions which splits over every finitely generated subgroup of $K_1(A)$.

To explain how

$$\mathrm{Aut}(\mathcal{E}_A), \quad \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$$

and

$$\mathrm{ext}(K_1(A), K_0(A))$$

fit together to form $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$, we let $KL(A,A)^{-1}$ denote the group of invertible elements α in $KL(A,A)$ (invertible with respect to the ring structure coming from the Kasparov product) such that $\alpha_*[1] = [1]$ and $\alpha_*(K_0(A)^+) = K_0(A)^+$ in $K_0(A)$. Let $\mathrm{Aut}(T(A))$ denote the group of continuous affine homeomorphisms of $T(A)$ and set

$$\Gamma(A) = \\ \left\{(\psi, \chi) \in \mathrm{Aut}(T(A)) \oplus KL(A,A)^{-1} : r_A(\psi^{-1}(\omega)) = r_A(\omega) \circ \chi_*,\ \omega \in T(A)\right\}.$$

The map $\tilde{\pi}\colon \mathrm{Aut}(A) \to \Gamma(A)$ given by $\tilde{\pi}(\alpha) = (\alpha^{-1^*}, [\alpha])$ is then a group homomorphism which annihilates $\overline{\mathrm{Inn}(A)}$ by Proposition 5.4 of [**R2**] and gives rise to a homomorphism $\pi\colon \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \to \Gamma(A)$. We want to show that π is a split surjection and determine its kernel. To this end we first observe that the group $\mathrm{Aut}\big(U(A)/\overline{DU(A)}\big)$ of isometric group automorphisms of $U(A)/\overline{DU(A)}$ is isomorphic to the semi-direct product

$$\mathrm{Hom}(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}) \rtimes \big(\mathrm{Aut}(K_1(A)) \oplus \mathrm{Aut}(\mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))})\big)$$

where $\mathrm{Aut}(\mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))})$ denotes the group of isometric group automorphisms of $\mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}$ and the action of

$$(\alpha, \beta) \in \mathrm{Aut}(K_1(A)) \oplus \mathrm{Aut}\big(\mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$$

on $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$ is given by

$$(\alpha, \beta)(\varphi) = \beta \circ \varphi \circ \alpha^{-1}.$$

This follows straightforwardly from Lemma 6.4. An illuminating way of visualizing this semi-direct product is by using matrix notation:

$$\begin{pmatrix} \alpha & 0 \\ \varphi & \beta \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \varphi' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha \circ \alpha' & 0 \\ \varphi \circ \alpha' + \beta \circ \varphi' & \beta \circ \beta' \end{pmatrix},$$

where

$$\alpha, \alpha' \in \operatorname{Aut}(K_1(A)), \quad \beta, \beta' \in \operatorname{Aut}\big(\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}\big),$$

and

$$\varphi, \varphi' \in \operatorname{Hom}(K_1(A), \operatorname{Aff} T(A)/\overline{\rho(K_0(A))}).$$

In particular, there is a group homomorphism

$$\begin{aligned} &\operatorname{Aut}(K_1(A)) \oplus \operatorname{Aut}\big(\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}\big) \\ &\ni (\alpha, \beta) \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \operatorname{Aut}\big(U(A)/\overline{DU(A)}\big). \end{aligned}$$

Let $\psi \in \operatorname{Aut}(T(A))$ such that $f \circ \psi^{-1} \in \overline{\rho(K_0(A))}$ for all $f \in \overline{\rho(K_0(A))} \subset \operatorname{Aff} T(A)$. Then ψ determines an element

$$\widetilde{\psi} \in \operatorname{Aut}\big(\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}\big)$$

given by

$$\widetilde{\psi}\left(f + \overline{\rho(K_0(A))}\right) = f \circ \psi^{-1} + \overline{\rho(K_0(A))}, \quad f \in \operatorname{Aff} T(A).$$

In particular, $(\psi, \chi) \mapsto (\chi_*, \widetilde{\psi})$ defines a homomorphism

$$\Gamma(A) \to \operatorname{Aut}(K_1(A)) \oplus \operatorname{Aut}\big(\operatorname{Aff} T(A)/\overline{\rho(K_0(A))}\big).$$

Now, by using Theorem 8.1 and Theorem 8.3 as Theorem A and B were used in the proof of Theorem C of Chapter 7, we get an automorphism $\alpha_{\psi,\chi} \in \operatorname{Aut}(A)$ such that

$$\alpha^*_{\psi,\chi} = \psi^{-1}$$

on $T(A)$,

$$[\alpha_{\psi,\chi}] = \chi$$

in $KL(A, A)$ and

$$\alpha^{\natural}_{\psi,\chi} = \begin{pmatrix} \chi_* & 0 \\ 0 & \widetilde{\psi} \end{pmatrix}$$

in $\operatorname{Aut}(U(A)/\overline{DU(A)})$. In particular, $\tilde{\pi}(\alpha_{\psi,\chi}) = (\psi, \chi)$. Using Theorem 8.3 we see that

$$\alpha_{\psi,\chi} \circ \alpha_{\psi',\chi'} = \alpha_{\psi\circ\psi',\chi\cdot\chi'}$$

modulo $\overline{\operatorname{Inn}(A)}$, proving that π is a split surjection. To identify the kernel of π, note that Theorem 8.1 and Theorem 8.3 tell us that it consists of the elements

$\boldsymbol{\Phi}$ of $\mathrm{Aut}(U(A)/\overline{DU(A)})$ which fit into a commuting diagram of the form

$$\begin{tikzcd} & & U(A)/\overline{DU(A)} \arrow[d, "\Phi"] \arrow[dr, "\pi_A"] & & \\ 0 \arrow[r] & \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))} \arrow[ur, "\lambda_A"] \arrow[r, "\lambda_A"] & U(A)/\overline{DU(A)} \arrow[r, "\pi_A"] & K_1(A) \arrow[r] & 0 \end{tikzcd}$$

But this is exactly the group $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$, considered as a subgroup of $\mathrm{Aut}(U(A)/\overline{DU(A)})$. It follows that $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is a semi-direct product

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big) \rtimes \Gamma(A)$$

of $\Gamma(A)$ and $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$. To identify the corresponding action of $\Gamma(A)$ on $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$, note that when $(\psi, \chi) \in \Gamma(A)$ and φ is a homomorphism in $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$, we find that

$$\begin{aligned} \alpha^{\natural}_{\psi,\chi} \circ \begin{pmatrix} id & 0 \\ \varphi & id \end{pmatrix} \circ {\alpha^{\natural}_{\psi,\chi}}^{-1} &= \begin{pmatrix} \chi_* & 0 \\ 0 & \widetilde{\psi} \end{pmatrix} \begin{pmatrix} id & 0 \\ \varphi & id \end{pmatrix} \begin{pmatrix} \chi_*^{-1} & 0 \\ 0 & \widetilde{\psi}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} id & 0 \\ \widetilde{\psi} \circ \varphi \circ \chi_*^{-1} & id \end{pmatrix} \end{aligned}$$

in $\mathrm{Aut}(U(A)/\overline{DU(A)})$. So the action is

$$(\psi, \chi)(\varphi) = \widetilde{\psi} \circ \varphi \circ \chi_*^{-1}.$$

To decipher the group structure of $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ further, we use the description of $KL(A,A)^{-1}$ given in **[DL3]**. Following the notation of Dadarlat and Loring we let

$$\mathrm{Aut}(K_0(A), [1])^+$$

denote the group of order and order-unit preserving automorphisms of $K_0(A)$. The direct sum $\mathrm{Aut}(K_0(A), [1])^+ \oplus \mathrm{Aut}(K_1(A))$ acts on $\mathrm{Ext}(K_1(A), K_0(A))$ in the natural way; in standard notation the action is

$$(\alpha, \beta)(e) = \alpha_* \circ {\beta^{-1}}^*(e),$$

where $e \in \mathrm{Ext}(K_1(A), K_0(A))$ and $(\alpha, \beta) \in \mathrm{Aut}(K_0(A), [1])^+ \oplus \mathrm{Aut}(K_1(A))$. This action passes to an action on $\mathrm{ext}(K_1(A), K_0(A))$ for which we use the same notation. It follows from **[DL3]** that $KL(A,A)^{-1}$ is the semi-direct product $\mathrm{ext}(K_1(A), K_0(A)) \rtimes [\mathrm{Aut}(K_0(A), [1])^+ \oplus \mathrm{Aut}(K_1(A))]$ corresponding to this action. Hence

$$\Gamma(A) = \mathrm{ext}(K_1(A), K_0(A)) \rtimes \mathrm{Aut}(\mathcal{E}_A)$$

where the action of $\mathrm{Aut}(\mathcal{E}_A)$ on $\mathrm{ext}(K_1(A), K_0(A))$ is given by

$$(\alpha_0, \alpha_1, \alpha_T)(e) = \alpha_{0*} \circ \alpha_1^{-1*}(e).$$

To combine this semi-direct product decomposition of $\Gamma(A)$ with the one we have obtained for $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$, observe that $\mathrm{Aut}(\mathcal{E}_A)$ also acts on

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$$

by

$$(\alpha_0, \alpha_1, \alpha_T)(\varphi) = \widetilde{\alpha_T} \circ \varphi \circ \alpha_1^{-1}.$$

We have now proved the following

THEOREM 8.4. — *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks of type* 2. *Then* $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ *is isomorphic to the semidirect-product*

$$[\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big) \oplus \mathrm{ext}(K_1(A), K_0(A))] \rtimes \mathrm{Aut}(\mathcal{E}_A),$$

where the action of $\mathrm{Aut}(\mathcal{E}_A)$ *on*

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big) \oplus \mathrm{ext}(K_1(A), K_0(A))$$

is given by

$$(\alpha_0, \alpha_1, \alpha_T)(\varphi, e) = (\widetilde{\alpha_T} \circ \varphi \circ \alpha_1^{-1}, \alpha_{0*} \circ \alpha_1^{-1*}(e)). \quad \square$$

When A is a simple unital inductive limit of a sequence of finite direct sums of circle algebras, the structure of $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ reduces a little. In this case $K_1(A)$ is an inductive limit of finitely generated torsionfree abelian groups, so $\mathrm{ext}(K_1(A), K_0(A)) = 0$ in this case, cf. [**DL3**]. We therefore have the following corollary.

COROLLARY 8.5. — *Assume that A is a simple unital inductive limit of a sequence of finite direct sums of circle algebras. Then*

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \simeq \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big) \rtimes \mathrm{Aut}(\mathcal{E}_A),$$

where the action of $\mathrm{Aut}(\mathcal{E}_A)$ *on* $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))}\big)$ *is given*

$$(\alpha_0, \alpha_1, \alpha_T)(\varphi) = \widetilde{\alpha_T} \circ \varphi \circ \alpha_1^{-1}. \quad \square$$

If in addition A has real rank 0 (or equivalently, is the closed linear span of its projections), then $\mathrm{Aff}\, T(A)/\overline{\rho(K_0(A))} = 0$ and $\mathrm{Aut}(\mathcal{E}_A) = \mathrm{Aut}(K_1(A)) \oplus \mathrm{Aut}(K_0(A), [1])^+$, so Corollary 8.5 reduces to Theorem 2.1 of [**ER**] (with A simple.)

CHAPTER 9

THE RANGE OF THE ELLIOTT INVARIANT

It is a very interesting and challenging problem to determine the range of the Elliott invariant for simple (separable, unital, nuclear) C^*-algebras. The research towards this goal has in recent years enlarged our stock of examples of simple C^*-algebras quite dramatically, see [**V1**], [**V2**] and [**Th6**]. In the context of this paper, and in particular in view of Theorem C of Chapter 7, the problem is most relevant for simple unital inductive limits of sequences of finite direct sums of building blocks of type 2, and it can also be answered thanks to the results of Villadsen [**V1**]. Note first that $K_0(B)$ is a simple (countable) dimension group when B is a simple unital inductive limit of a sequence of finite direct sums of building blocks of type 2 and that $K_0(B)$ is not cyclic unless B is a matrix algebra. Furthermore, for any finite direct sum A of building blocks of type 2 the restriction map $r_A \colon T(A) \to SK_0(A)$ is extreme-point preserving, so it follows from Corollaries 1.6 and 1.7 of [**V1**] that the same must be the case for B. Except for the general condition that $K_1(B)$ must be a countable abelian group and r_A surjective, these are the only restrictions. More precisely, we have the following

THEOREM 9.1. — *Let G be a countable non-cyclic dimension group with order unit, H a countable abelian group, Δ a compact metrizable Choquet simplex and $r \colon \Delta \to SG$ an affine continuous extreme-point preserving surjection. There is then a simple unital inductive limit of a sequence of finite direct sums of building blocks of type 2, A, such that*

$$(T(A), r_A, K_0(A), K_1(A)) \simeq (\Delta, r, G, H).$$

Recall that $(T(A), r_A, K_0(A), K_1(A)) \simeq (\Delta, r, G, H)$ means that there is a group isomorphism $\varphi_1 \colon K_1(A) \to H$, an affine homeomorphism $\varphi_T \colon \Delta \to T(A)$ and an isomorphism $\varphi_0 \colon K_0(A) \to G$ of partially ordered groups with order unit such that

$$r_A \circ \varphi_T(\omega)(x) = r(\omega)(\varphi_0(x)), \quad \omega \in \Delta,\ x \in K_0(A).$$

Actually, in order to relate this work to the work of others, where the dimension drop C^*-algebras are used to include the possibility of having torsion in K_1, we shall prove the following.

THEOREM 9.2. — *Let G be a countable non-cyclic dimension group with order unit, H a countable abelian group, Δ a compact metrizable Choquet simplex and $r\colon \Delta \to SG$ an affine continuous extreme-point preserving surjection. There is then a simple unital inductive limit of a sequence of finite direct sums of circle algebras and matrices over dimension-drop C^*-algebras, A, such that*

$$(T(A), r_A, K_0(A), K_1(A)) \simeq (\Delta, r, G, H). \quad \square$$

As explained in the introduction, Theorem 9.1 follows from Theorem 9.2. But it also gives the following

COROLLARY 9.3. — *Let A be a simple unital inductive limit of finite direct sums of building blocks of type 2. Then A is $*$-isomorphic to a unital inductive limit of a sequence of finite direct sums of circle algebras and matrices over dimension-drop C^*-algebras.*

Proof. — The conclusion is trivial when A is finite dimensional, so we can assume that $K_0(A)$ is not cyclic. By Theorem 9.2 the Elliott-invariant of A is also realized by a simple unital inductive limit of sequences of finite direct sums of circle algebras and matrices over dimension-drop C^*-algebras. By Theorem C of Chapter 7, the two algebras are $*$-isomorphic. $\square$

For the proof of Theorem 9.2 we need a couple of lemmas. In the following we will consider a matrix algebra $M_n(\tilde{\mathbb{I}}_k)$ over the dimensiondrop C^*-algebra $\tilde{\mathbb{I}}_k$ as a building block of type 4 in the natural way, i.e. as

$$\{f \in C[0,1] \otimes M_{kn} : f(0), f(1) \in M_n\}.$$

We will let $\iota\colon C[0,1] \otimes M_n \to M_n(\tilde{\mathbb{I}}_k)$ denote the natural embedding.

LEMMA 9.4. — *Let $k \in \mathbb{N}$. For every finite set $F \subset C_{\mathbb{R}}[0,1]$ and any $\varepsilon > 0$ there is a $N \in \mathbb{N}$ with the following property: When $\varphi\colon C[0,1]\otimes M_n \to C[0,1]\otimes M_m$ is a unital $*$-homomorphism such that*

$$\hat{\varphi}(\chi_j^N) > 0, \; j = 1, 2, \ldots, N,$$

then there is a unital $*$-homomorphism $\psi\colon M_n(\tilde{\mathbb{I}}_k) \to C[0,1] \otimes M_m$ such that

$$\|\widehat{\psi \circ \iota}(f) - \hat{\varphi}(f)\| \leq \varepsilon + k\frac{n}{m}\|f\|, \; f \in F.$$

Proof. — Choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in F$ when $|x - y| < \delta$. Let N be so large that $k/N < \delta$. We assert that this N has the required property, so assume that φ is a $*$-homomorphism with the properties of the lemma. By **[Th9]** there are continuous functions

$$g_1, g_2, \ldots, g_{m/n} \colon [0,1] \longrightarrow [0,1]$$

such that $g_1 \leq g_2 \leq \cdots \leq g_{m/n}$ and φ is approximately inner equivalent to the map

$$f \mapsto \operatorname{diag}(f \circ g_1, f \circ g_2, \ldots, f \circ g_{\frac{m}{n}}).$$

So for the present purpose we have assume that φ is this map. Note that

$$|g_{i+j}(t) - g_i(t)| \leq \frac{k}{N} < \delta,\ i \leq \frac{m}{n} - k,\ j \leq k,$$

and

$$|g_j(t) - 1| \leq \frac{k}{N},\ j = k - r, k - r + 1, \ldots, k$$

for all $t \in [0,1]$. Write $m/n = lk + r$ where $r, l \in \mathbb{N}$ and $r < k$. Define $h_j = g_{(j-1)k+1}, j = 1, 2, \ldots, l$, and define $\psi \colon M_n(\tilde{\mathbb{I}}_k) \to C[0,1] \otimes M_m$ by

$$\psi(f) = \operatorname{diag}(f \circ h_1, f \circ h_2, \ldots, f \circ h_l, \underbrace{\Lambda_1^r(f), \Lambda_1^r(f), \ldots, \Lambda_1^r(f)}_{r \text{ times}})$$

It is straightforward to check that ψ meets the requirements. $\square$

As in Chapter 4 we shall consider the functions $\kappa \colon \mathbb{T} \to [0,1]$ and $\iota_1 \colon [0,1] \to \mathbb{T}$ given by $\kappa(e^{2\pi i t}) = 2t$, $t \in [0, 1/2]$, $\kappa(e^{2\pi i t}) = 2 - 2t$, $t \in [1/2, 1]$, and $\iota_1(t) = e^{\pi i t}$, respectively. They give rise to $*$-homomorphisms

$$\mu \colon C[0,1] \otimes M_n \to C(\mathbb{T}) \otimes M_n$$

and

$$\nu \colon C(\mathbb{T}) \otimes M_n \to C[0,1] \otimes M_n$$

given by $\mu(f) = f \circ \kappa$ and $\nu(g) = g \circ \iota_1$, respectively. Note that $\nu \circ \mu$ is the identity map on $C[0,1] \otimes M_n$. These homomorphisms are considered in the following lemma.

LEMMA 9.5. — *Let $\varphi \colon C[0,1] \otimes M_n \to C[0,1] \otimes M_m$ be a unital $*$-homomorphism, where $m > n$. There is then a unital $*$-homomorphism $\psi \colon C(\mathbb{T}) \otimes M_n \to C[0,1] \otimes M_m$ such that*

$$\hat{\psi} \circ \hat{\mu} = \hat{\varphi}$$

on $C_{\mathbb{R}}[0,1]$.

Proof. — By **[Th9]** there are continuous functions

$$g_i\colon [0,1] \longrightarrow [0,1], \quad i = 1,2,\dots,m/n,$$

such that φ is approximately inner equivalent to the map

$$f \mapsto \operatorname{diag}(f \circ g_1, f \circ g_2, \dots, f \circ g_{\frac{m}{n}}).$$

So for the present purpose we may assume that φ is this map. Set $\psi = \varphi \circ \nu$. Then $\psi \circ \mu = \varphi \circ \nu \circ \mu = \varphi$. □

Proof of Theorem 9.2. — Assume first that H is finitely generated, i.e. that

$$H \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_N}$$

for some $n, k_1, k_2, \dots, k_N \in \mathbb{N}$. By Theorem 3.2 of **[V1]** there is a sequence $B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots$ of finite direct sums of interval algebras such that $B = \varinjlim(B_n, \varphi_n)$ is simple and has $(T(B), r_B, K_0(B), K_1(B)) \simeq (\Delta, r, G, 0)$. Set $B_i = \bigoplus_{j=1}^{m_i} C[0,1] \otimes M_{d_j}$. As pointed out in the proof of Theorem 4.2 in **[V1]**, we may assume that $\lim_{j\to\infty} m_j = \infty$. In particular, we can assume that $m_j \geq N + n$ for all j. Furthermore, by construction each of the partial $*$-homomorphisms of the connecting maps, the φ_j's, are injective. By simplicity of B we have that $\lim_{l\to\infty} \operatorname{mult}(\varphi_{l,j}) = \infty$ for all $j \in \mathbb{N}$ and, for an arbitrary finite subset G of non-zero positive elements in $\operatorname{Aff} T(B_j)$, we can choose $l > j$ such that

$$\widehat{\varphi_{l,j}}(f) \; > \; 0, \; f \in G.$$

Let $\varepsilon > 0$ and fix a finite set $F_0 \subset B_j$. For any $\delta > 0$ and any finite set $F \subset \operatorname{Aff} T(B_j)$ of positive non-zero elements we can apply Lemma 9.4 and Lemma 9.5 to get $*$-homomorphisms

$$\varphi\colon B_j \to \bigoplus_{i=1}^{N} M_{n_i}(\tilde{\mathbb{I}}_{k_i}) \bigoplus_{i=N+1}^{m_j} C(\mathbb{T}) \otimes M_{d_i}$$

and

$$\psi\colon \bigoplus_{i=1}^{N} M_{n_i}(\tilde{\mathbb{I}}_{k_i}) \bigoplus_{i=N+1}^{m_j} C(\mathbb{T}) \otimes M_{d_i} \to B_l$$

such that $\varphi_{l,j_*} = \psi_* \circ \varphi_*$ on $K_0(B_j)$,

$$\|\widehat{\varphi_{l,j}}(f) - \widehat{\psi \circ \varphi}(f)\| \; < \; \varepsilon, \; f \in F.$$

With the appropriate choice of F and $\delta > 0$, we may now conclude from Theorem 6 of **[E2]** that there is a unitary $u \in B_l$ such that

$$\| \operatorname{Ad} u \circ \psi \circ \varphi(x) - \varphi_{l,j}(x)\| < \varepsilon, \; x \in F_0.$$

In this way we can proceed inductively to obtain an infinite diagram

$$\begin{tikzcd}
D_1 \arrow[r, "\lambda_1"] \arrow[dr, "\mu_1"'] & D_2 \arrow[r, "\lambda_2"] \arrow[dr, "\mu_2"'] & D_3 \arrow[r, "\lambda_3"] \arrow[dr] & \cdots \\
B_{l_1} \arrow[u, "\pi_1"] \arrow[r, "\varphi_{l_2,l_1}"'] & B_{l_2} \arrow[u, "\pi_2"'] \arrow[r, "\varphi_{l_3,l_2}"'] & B_{l_3} \arrow[u, "\pi_3"'] \arrow[r] & \cdots
\end{tikzcd}$$

which is an approximate intertwining in the sense of [**E1**], such that the first infinite row is a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras. By Theorem 2.2 of [**E1**] we can conclude that $D = \varinjlim(D_j, \lambda_j)$ is $*$-isomorphic to B. Next we want to change the connecting maps, the λ_j's, to other maps, say λ'_j's, to get the right K_1-group for the limit algebra. Note that all the partial $*$-homomorphisms of the λ_j's are 0 on K_1 since they factor through an interval algebra by construction. By construction

$$K_1(D_j) \simeq \mathbb{Z}^a \oplus \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_N}$$

and

$$K_1(D_{j+1}) \simeq \mathbb{Z}^b \oplus \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_N}$$

for some $a, b \geq n$. We want $\lambda'_{j_*} \colon K_1(D_j) \to K_1(D_{j+1})$ to be the map

$$\lambda'_{j_*}(z_1, \ldots, z_a, x_1, \ldots, x_N) = (z_1, \ldots, z_n, 0, 0, \ldots, 0, x_1, \ldots, x_N)$$

under these identifications. Thus we need only change the partial maps between direct summands of the same type, and we need only consider maps between matrix algebras over the same dimension drop C^*-algebras. But we must take care to make the changes so that the limit algebra remains simple. For the last purpose we take dense sequences $\{t_i\}$ and $\{s_i\}$ on the circle and the interval, respectively. Consider two of the relevant partial maps,

$$\varphi\colon C(\mathbb{T}) \otimes M_n \to C(\mathbb{T}) \otimes M_m$$

and

$$\psi\colon M_n(\tilde{\mathbb{I}}_k) \to M_m(\tilde{\mathbb{I}}_k).$$

Since the total number of direct summands tends to ∞ and the K_0-group is a simple dimension group, we may suppose, after a compression of the given sequence, that there are projections $p \in C(\mathbb{T}) \otimes M_m$ and $q \in M_m(\tilde{\mathbb{I}}_k)$ such that

$$\operatorname{rank}(p) > 2n, \ 2^j \operatorname{rank}(p) \leq m$$

and

$$\operatorname{rank}(q) > 2n, \ 2^j \operatorname{rank}(q) \leq m$$

and such that

$$\varphi(a)p = p\varphi(a), \ a \in C(\mathbb{T}) \otimes M_n$$

and

$$\psi(a)q = q\psi(a), \ a \in M_n(\tilde{\mathbb{I}}_k).$$

Note that $qM_m(\tilde{\mathbb{I}}_k)q$ contains a copy of

$$\underbrace{M_n(\tilde{\mathbb{I}}_k) \oplus M_n(\tilde{\mathbb{I}}_k) \oplus \cdots \oplus M_n(\tilde{\mathbb{I}}_k)}_{\frac{rank(q)}{n} \text{ times}}$$

as a unital C^*-subalgebra. Define $\psi_1 \colon M_n(\tilde{\mathbb{I}}_k) \to M_n(\tilde{\mathbb{I}}_k) \oplus M_n(\tilde{\mathbb{I}}_k) \oplus \cdots \oplus M_n(\tilde{\mathbb{I}}_k)$ by

$$\psi_1(f)(t) = (f(t), f(s_j), \ldots, f(s_j))$$

and consider ψ_1 as a unital $*$-homomorphism into $qM_m(\tilde{\mathbb{I}}_k)q$. The new partial map ψ', replacing ψ in λ_j, is then given by

$$\psi'(f) = \psi(f)(1-q) + \psi_1(f).$$

To change φ we proceed in essential the same way. $pC(\mathbb{T}) \otimes M_m p$ contains a copy of

$$\underbrace{C(\mathbb{T}) \otimes M_n \oplus C(\mathbb{T}) \otimes M_n \oplus \cdots \oplus C(\mathbb{T}) \otimes M_n}_{\frac{\operatorname{rank}(p)}{n} \text{ times}}$$

as a unital C^*-subalgebra, and we define $\varphi_1 \colon C(\mathbb{T}) \otimes M_n \to C(\mathbb{T}) \otimes M_n \oplus C(\mathbb{T}) \otimes M_n \oplus \cdots \oplus C(\mathbb{T}) \otimes M_n$ by

$$\varphi_1(f)(z) \ = \ (f(z), f(t_j), \ldots, f(t_j)).$$

The new partial map φ', replacing φ in λ_j, is then given by

$$\psi'(f) = \psi(f)(1-q) + \varphi_1(f).$$

It is now straightforward to see that $\varinjlim(D_j, \lambda_j')$ is simple and that the Elliott invariant of this algebra is (Δ, r, G, H).

To handle the case of a general H we use that $H = \bigcup_n H_n$ where each H_n is finitely generated. By the first part of the proof we may choose simple unital inductive limits of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras, A_n, such that $(T(A_n), r_{A_n}, K_0(A_n), K_1(A_n)) \simeq (\Delta, r, G, H_n)$. From Corollary A2 of Chapter 7 we conclude that there are unital $*$-homomorphisms $\rho_n\colon A_n \to A_{n+1}$ such that $\rho_n^*\colon T(A_{n+1}) \to T(A_n)$, $\rho_{n*}\colon K_0(A_n) \to K_0(A_{n+1})$ are both identity maps when $T(A_{n+1})$ and $T(A_n)$ are identified with Δ, and $K_0(A_{n+1})$ and $K_0(A_n)$ are identified with G. In addition, we can arrange that $\rho_{n*}\colon K_1(A_n) \to K_1(A_{n+1})$ is the inclusion $H_n \subset H_{n+1}$ (under the identifications $K_1(A_n) = H_n$ and $K_1(A_{n+1}) = H_{n+1}$). Set $A = \varinjlim(A_n, \rho_n)$. By continuity of the Elliott invariant we have that

$$(T(A), r_A, K_0(A), K_1(A)) \simeq (\Delta, r, G, H),$$

as desired. A is simple because each A_n is. The fact that A itself is the inductive limit of a sequence of finite direct sums of circle algebras and matrix algebras over dimension-drop C^*-algebras follows from **[L1]**, Theorem 6.2 and Theorem 3.8. □

CHAPTER 10

THE NON-UNITAL CASE

In this chapter we show how our main result can be adopted to cover the non-unital simple inductive limits of finite direct sums of building blocks of type 2. The main idea behind the approach appeared in [**Th8**]. For any C^*-algebra A with an approximate unit of projections we denote by $\mathcal{T}_A$ the set of lower semi-continuous densely defined traces on A. We endow $\mathcal{T}_A$ with weakest topology such that the functional $\mathcal{T}_A \ni \tau \mapsto \tau(a)$ is continuous for every positive element a of A which is dominated by a projection. This topology is Hausdorff because A has an approximate unit consisting of projections.

LEMMA 10.1. — *Assume that A is simple and that $e \in A$ is a non-zero projection. Then $\{\tau \in \mathcal{T}_A : \tau(e) = 1\}$ is compact in $\mathcal{T}_A$ and the restriction map $R^e(\tau) = \tau|_{eAe}$ is an affine homeomorphism from $\{\tau \in \mathcal{T}_A : \tau(e) = 1\}$ onto $T(eAe)$.*

Proof. — See Lemma 3 of [**Th8**]. □

Since every element $\tau \in \mathcal{T}_A$ extends canonically to $M_n(A)$ for all n, we can define a map

$$r_A \colon \mathcal{T}_A \to \mathrm{Hom}_+(K_0(A), \mathbb{R}) = \{\rho \in \mathrm{Hom}(K_0(A), \mathbb{R}) : \rho(K_0(A)^+) \subset [0, \infty[\}$$

by

$$r_A(\omega)([p] - [q]) = \omega(p) - \omega(q),$$

where p, q are projections in $\bigcup_n M_n(A)$.

LEMMA 10.2. — *Let A be a unital simple inductive limit of finite direct sums of building blocks of type 2. Let $p \in A$ be a non-zero projection in A. The* *inclusion $pA_{sa}p \subset A_{sa}$ and the map $U(pAp) \to U(A)$ given by $u \mapsto u + (1-p)$ induce isomorphisms*

$$\begin{aligned}
&\operatorname{Aff} T(pAp)/K_0(pAp) \to \operatorname{Aff} T(A)/K_0(A),\\
&U(pAp)/\overline{DU(pAp)} \to U(A)/\overline{DU(A)} \quad \textit{and}\\
&K_1(pAp) \to K_1(A)
\end{aligned}$$

such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Aff} T(pAp)/K_0(pAp) & \xrightarrow{\lambda_{pAp}} & U(pAp)/\overline{DU(pAp)} & \xrightarrow{\pi_{pAp}} & K_1(pAp) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Aff} T(A)/K_0(A) & \xrightarrow[\lambda_A]{} & U(A)/\overline{DU(A)} & \xrightarrow[\pi_A]{} & K_1(A) & \longrightarrow & 0
\end{array}$$

commutes.

Proof. — It is straightforward to check that the diagram commutes and it is wellknown that $K_1(pAp) \to K_1(A)$ is an isomorphism. Furthermore, it is easy to see that $\operatorname{Aff} T(pAp)/K_0(pAp) \to \operatorname{Aff} T(A)/K_0(A)$ is an isomorphism, e.g. by using Lemma 10.1, and the wellknown fact that $pAp \subset A$ induces an isomorphism $K_0(pAp) \simeq K_0(A)$. Then Lemma 6.4 (or the three lemma) implies that $U(pAp)/\overline{DU(pAp)} \to U(A)/\overline{DU(A)}$ is an isomorphism. □

It should be observed that the isomorphism

$$U(pAp)/\overline{DU(pAp)} \to U(A)/\overline{DU(A)}$$

is not isometric (with respect to the natural metric).

THEOREM 10.3. — *Let A and B be simple inductive limits of finite direct sums of building blocks of type 2. Assume that $\varphi_0\colon K_0(A) \to K_0(B)$ is an isomorphism of scaled dimension groups, $\varphi_1\colon K_1(A) \to K_1(B)$ an isomorphism of groups and $\varphi_T\colon \mathcal{T}_B \to \mathcal{T}_A$ an affine homeomorphism such that*

$$r_A \circ \varphi_T(\omega)(x) = r_B(\omega)(\varphi_0(x)), \quad x \in K_0(A),\ \omega \in \mathcal{T}_B.$$

It follows that there is a $*$-isomorphism $\varphi\colon A \to B$ such that

$$\begin{aligned}
&\varphi_* = \varphi_0 \textit{ on } K_0(A),\\
&\varphi_* = \varphi_1 \textit{ on } K_1(A) \quad \textit{and}\\
&\tau \circ \varphi = \varphi_T(\tau),\ \tau \in \mathcal{T}_B.
\end{aligned}$$

Proof. — Except for considerations regarding KK and $U/\overline{DU}$ the proof is identical to the proof of Theorem 4 in **[Th8]**. By **[RS]** we may choose a KK-equivalence $\alpha \in KK(A,B)$ such that $\alpha_* = \varphi_0 \oplus \varphi_1$ on $K_0(A) \oplus K_1(A)$. Since A and B have cancellation of projections it is easy to construct sequences $p_1 \leq p_2 \leq p_3 \leq \cdots$ and $q_1 \leq q_2 \leq q_3 \leq \cdots$ of projections which form approximate units in A and B, respectively, such that

$$\varphi_0([p_i]) = [q_i]$$

for all i. Let $\alpha_i \in KK(p_iAp_i, q_iBq_i)$ be the image of α under the isomorphism $KK(A,B) \simeq KK(p_iAp_i, q_iBq_i)$ induced by the inclusions $p_iAp_i \subset A$ and $q_iBq_i \subset B$. Let

$$S_i \colon U(p_{i-1}Ap_{i-1})/\overline{DU(p_{i-1}Ap_{i-1})} \to U(p_iAp_i)/\overline{DU(p_iAp_i)}$$

and

$$T_i \colon U(q_{i-1}Bq_{i-1})/\overline{DU(q_{i-1}Bq_{i-1})} \to U(q_iBq_i)/\overline{DU(q_iBq_i)}$$

be the isomorphisms given by Lemma 10.2. By recursive application of Theorem A in Chapter 7 we can construct unital $*$-isomorphisms $\psi_i \colon p_iAp_i \to q_iBq_i$ such that

$$\psi_i^* = R_{p_i} \circ \varphi_T \circ R_{q_i}^{-1},$$

$$\psi_i^\natural \circ S_i = T_i \circ \psi_{i-1}^\natural$$

on $U(p_{i-1}Ap_{i-1})/\overline{DU(p_{i-1}Ap_{i-1})}$ and

$$[\psi_i \circ \mu] = \mu^*(\alpha_i)$$

in $KK(D, q_iBq_i)$ for any unital $*$-homomorphism $\mu \colon D \to p_iAp_i$ defined on a finite direct sum of building blocks of type 2. After conjugation with unitaries from q_iBq_i we may assume that $\psi_i(p_{i-1}) = q_{i-1}$ for all i. Consider the infinite diagram

$$\begin{tikzcd}
p_1Ap_1 \arrow[r, phantom, "\subset"] \arrow[d, "\psi_1"'] & p_2Ap_2 \arrow[r, phantom, "\subset"] \arrow[d, "\psi_2"'] & p_3Ap_3 \arrow[r, phantom, "\subset"] \arrow[d, "\psi_3"'] & \cdots \\
q_1Bq_1 \arrow[r, phantom, "\subset"] \arrow[ur, "\lambda_1"] & q_2Bq_2 \arrow[r, phantom, "\subset"] \arrow[ur, "\lambda_2"] & q_3Bq_3 \arrow[r, phantom, "\subset"] \arrow[ur, "\lambda_3"] & \cdots
\end{tikzcd}$$

where λ_i is the inverse of $\psi_{i+1}|_{q_iBq_i}$ for all i. It follows from Theorem B of Chapter 7 that $\lambda_i \circ \psi_i$ is approximately inner equivalent to the inclusion $p_iAp_i \subset p_{i+1}Ap_{i+1}$ and that $\psi_{i+1} \circ \lambda_i$ is approximately inner equivalent to the inclusion $q_iBq_i \subset q_{i+1}Bq_{i+1}$ for all i. So by conjugating the ψ_i's and the λ_i's by suitable unitaries from their target algebras we can make the above diagram into an approximate intertwining in the sense of [**E1**]. Hence this diagram gives rise to an isomorphism $\varphi\colon A \to B$ with the stated properties. □

It is also possible to extend Theorem 9.1 to the non-unital case and we refer the reader to [**Vi**] for this. Visoiu only handles the case of circle algebras, but her methods carry over to building blocks of type 2 without trouble. As in the unital case the only new feature, when compared to the circle algebra case, is the possibility of having torsion in the K_1-group.

It is less obvious how the other results from Chapter 7, Theorem A and Theorem B, should be generalized to the non-unital case.

CHAPTER 11

QUALITATIVE CONCLUSIONS

THEOREM 11.1. — *Let A be a unital and simple inductive limit of a sequence of finite direct sums of building blocks of type 2. Then*

1. *A is the inductive limit of a sequence of finite direct sums of circle algebras if and only if $K_1(A)$ is torsionfree.*
2. *A is the inductive limit of a sequence of finite direct sums of interval algebras if and only if $K_1(A)$ is zero.*
3. *A has real rank 0 if and only if $r_A \colon T(A) \to SK_0(A)$ is injective.*
4. *A is an AF-algebra if and only if A has real rank zero and $K_1(A) = 0$.*
5. *A is the inductive limit of a sequence of finite direct sums of interval algebras and matrix algebras over dimension-drop C^*-algebras if and only if $K_1(A)$ is a torsion group.*

Proof

1. Since a circle algebra has torsionfree K_1, the necessity of the condition is obvious. On the other hand, if $K_1(A)$ is torsionfree, then the Elliott invariant of A is also exhibited by a simple unital inductive limit of a sequence of direct sums of circle algebras by Theorem 4.2 of [**V1**]. By Theorem C of Chapter 7, A is then isomorphic to that algebra.

2. Again the necessity of the condition is obvious and the reversed implication follows in a similar way by using Theorem 3.2 of [**V1**].

3. If A has real rank zero, A is the closed linear span of its projections and hence r_A is clearly injective. The reversed implication can be proved in two ways (at least). The first is almost identical to the previous reasoning; one simply combines Theorem C of Chapter 7 with Theorem 8.3 of [**E1**]. The second way is to combine Theorem 5.1 here with Theorem 1.4 of [**BKR**].

4. The necessity is clear and the reversed implication can again be obtained in different ways. One is to combine Theorem C of Chapter 7 with the theorem of Effros, Handelman and Shen [**EHS**]. The other is to use 1. or 2. in combination with [**E1**].

5. Since the K_1-group of an interval algebra is 0 and the K_1-group of a dimension-drop C^*-algebra is finite, the condition is clearly necessary. For the converse, observe that the proof of Theorem 9.2 can easily be modified to show that the Elliott invariant of a given A whose K_1-group is a torsion group can also be realized by a simple unital inductive limit of a sequence of finite direct sums of interval algebras and matrix algebras over dimension-drop C^*-algebras. Apply Theorem C of Chapter 7. □

In [**Th7**] it was shown that a unital inductive limit of a sequence of circle algebras is an inductive limit of interval algebras if (and only if) K_1 is zero, also in the non-simple case. It is therefore natural ask if the conclusion '$K_1(A)$ torsionfree $\Rightarrow A$ is the inductive limit of a sequence of finite direct sums of circle algebras' also holds for a non-simple unital inductive limit of a sequence of finite direct sums of building blocks of type 2. That this is not the case can be seen from the following example.

EXAMPLE 11.2. — For each $n \in \mathbb{N}$, $n \geq 2$, set

$$A_n = \{f \in C(\mathbb{T}) \otimes M_n : f(1) \in \mathbb{C}\}.$$

Let Λ_n be the unique one-dimensional irreducible representation of A_n and define $\varphi_n \colon A_n \to A_{n+1}$ by $\varphi_n(f) = \operatorname{diag}\bigl(f, \Lambda_n(f)\bigr)$. Then $A = \varinjlim(A_n, \varphi_n)$ has $K_1(A) \simeq \mathbb{Z}$ and $(K_0(A), [1]) \simeq (\mathbb{Z}, 1)$ as partially ordered groups with order-unit. It is easily see that if A was an inductive limit of a sequence of finite direct sums of circle algebras, A would have to be the limit of a sequence of the form $C(\mathbb{T}) \to C(\mathbb{T}) \to C(\mathbb{T}) \to \cdots$ which is of course not possible since A is not abelian. If we instead set $\varphi_n(f) = \operatorname{diag}\bigl(f \circ g, \Lambda_n(f)\bigr)$, where $g \colon \mathbb{T} \to \mathbb{T}$ is some homotopically trivial continuous and surjective map which takes 1 to 1, then the inductive limit will have trivial K_1-group, but can not be the inductive limit of a sequence of finite direct sums of interval algebras. Hence the conclusion '$K_1(A) = 0 \Rightarrow A$ is the inductive limit of a sequence of finite direct sums of interval algebras', does not extend to the class of (non-simple) inductive limits of a sequence of building blocks of type 2.

Despite the preceding example we have the following result.

THEOREM 11.3. — *Let $A = \varinjlim(A_n, \varphi_n)$ be a unital inductive limit of a sequence of finite direct sums of building blocks of type* 1 *and let Q denote the* *universal UHF-algebra (the one with $K_0(Q) = \mathbb{Q}$). Then $A \otimes Q$ is a the inductive limit of a sequence of finite direct sums of circle algebras.*

For the proof we need the following

LEMMA 11.4. — *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks of type 1 and let $\varphi\colon A \to B$ be a unital $*$-homomorphism. There is then a natural number D and a unital $*$-homomorphism $\psi\colon C(\mathbb{T}) \otimes M_n \to B \otimes M_D$ such that $\psi|_A$ is approximately unitarily equivalent to $\varphi \otimes 1_{M_D}$.*

Proof. — We may assume that φ is of the standard form described in Chapter 1, i.e. is given by $r_1, r_2, \dots, r_N \in \mathbb{N}$ and $\mu_i\colon [0,1] \to \mathbb{T}, i = 1, 2, \dots, L$ with the stated restrictions. Choose $D \in \mathbb{N}$ so large that

$$m/e_i \mid D, \quad n/d_k \mid D \text{ and } m/e_i \mid r_k d_k D/n,$$

$k = 1, 2, \dots, N$, $i = 1, 2, \dots, M$. Let

$$\kappa_i\colon [0,1] \longrightarrow \mathbb{T}, \quad i = 1, 2, \dots, Dm/n = DL + \sum_{k=1}^{N} r_k d_k D/n,$$

be a tuple a continuous functions containing D copies of μ_i, $i = 1, 2, \dots, L$, and $r_k d_k D/n$ copies of the constant function x_k, $k = 1, 2, \dots, N$. Because the multiplicities are m/e_i-divisible for all i there is a unitary $u \in C(\mathbb{T}) \otimes M_{Dm}$ such that

$$u(t) \operatorname{diag}\big(f(\kappa_1(t)), f(\kappa_2(t)), \dots, f(\kappa_{Dm/n}(t))\big) u(t)^*, t \in [0,1],$$

defines an element of $M_D(A(m, e_1, \dots, e_M))$ for all $f \in C(\mathbb{T}) \otimes M_n$. If we let $\psi\colon C(\mathbb{T}) \otimes M_n \to M_D(A(M, e_1, \dots, e_M))$ be the corresponding $*$-homomorphism, then $\psi|_A$ is approximately unitarily equivalent to $\varphi \otimes 1_{M_D}$ by Theorem 1.4. $\square$

Proof of Theorem 11.3. — Let A be the inductive limit of the sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$ of finite direct sums of building blocks of type 1 with unital connecting $*$-homomorphisms. By applying Lemma 11.4 inductively, we construct a sequence D_i, $i \in \mathbb{N}$, in $\mathbb{N}$ such that $D_i | D_{i+1}$ for all i and an infinite diagram

$$\begin{tikzcd}
M_{D_1}(A_1) \arrow[r, "\varphi_1 \otimes \tau_1"] \arrow[d, "\lambda_1"] & M_{D_2}(A_2) \arrow[r, "\varphi_2 \otimes \tau_2"] \arrow[d, "\lambda_2"] & M_{D_3}(A_3) \arrow[r, "\varphi_3 \otimes \tau_3"] \arrow[d, "\lambda_3"] & \cdots \\
B_1 \arrow[r, "\psi_1"'] \arrow[ur, "\mu_1"] & B_2 \arrow[r, "\psi_2"'] \arrow[ur, "\mu_2"] & B_3 \arrow[r, "\psi_3"'] \arrow[ur, "\mu_3"] & \cdots
\end{tikzcd}$$

where the B_i's are finite direct sums of circle algebras, the λ_i's, the μ_i's and ψ_i's are unital $*$-homomorphisms and $\tau_i\colon M_{D_i} \to M_{D_{i+1}}$ is the standard unital homomorphism. The diagram commutes in the sense that $\psi_i = \lambda_{i+1} \circ \mu_i$ and $\mu_i \circ \lambda_i$ is approximately inner equivalent to $\varphi_i \otimes \tau_i$ for all i. It follows that the inductive limits of the two horisontal sequences are isomorphic, see Theorem 2.2 of **[E1]**. The limit of the upper sequence is of the form $A \otimes (UHF)$. Since $A \otimes Q = A \otimes (UHF) \otimes Q$, it follows that $A \otimes Q$ is $*$-isomorphic to the limit of the sequence $(B_i \otimes Q, \psi_i \otimes id_Q)$ which is clearly the inductive limit of a sequence of finite direct sums of circle algebras. □

We now turn to the non-stable K-theory, in the sense of **[Th1]**, i.e. to the calculation of the homotopy groups of the unitary group $U(A)$. We want to show that the approximate divisibility, which now plays a prominent role in the classification program, also has important consequences for the non-stable K-theory. If A is a finite direct sum of building blocks of type 2, then the natural map $\pi_0(U(A)) \to K_1(U(A))$ is an isomorphism. Therefore the same conclusion holds when A is a unital inductive limit of building blocks of type 2. We show that a similar conclusion holds for the higher homotopy groups, $\pi_k(U(A))$, $k \geq 1$, whenever A is approximately divisible.

LEMMA 11.5. — *Let A and B be unital C^*-algebras with A approximately divisible. Then the minimal C^*-tensor product $A \otimes B$ is approximately divisible.*

Proof. — Let $F \subset A \otimes B$ be a finite subset, $n \in \mathbb{N}$ a natural number and $\varepsilon > 0$. For each $f \in F$ there is a finite sum

$$\sum_{i=1}^{m_f} a_i^f \otimes b_i^f$$

of simple tensors such that

$$\|f - \sum_{i=1}^{m_f} a_i^f \otimes b_i^f\| < \frac{\varepsilon}{2}.$$

By the approximate divisibility of A there is a finite-dimensional C^*-subalgebra $A_0 \simeq \bigoplus_{j=1}^k M_{n_j}$ of A such that $n_j \geq n$ for all j and

$$\sum_{j=1}^{m_f} \|b_j^f\| \operatorname{dist}(a_i^f, A_0' \cap A) \;<\; \frac{\varepsilon}{2}$$

for all f, i. Set $B_0 = A_0 \otimes 1$ and note that

$$\operatorname{dist}(\sum_{i=1}^{m_f} a_i^f \otimes b_i^f, B_0' \cap A \otimes B) < \frac{\varepsilon}{2}.$$

□

THEOREM 11.6. — *Let A be a unital approximately divisible C^*-algebra. Then $A \otimes B$ is K-stable in the sense of* **[Th1]** *for all C^*-algebras B. ($\otimes$ denotes here the minimal C^*-tensor product.) In particular,*

$$\pi_k(U(A)) \simeq \begin{cases} K_0(A) & \textit{when } k \textit{ is odd,} \\ K_1(A) & \textit{when } k \textit{ is even.} \end{cases}$$

Proof. — The proof is modelled on the proof of Theorem 4.5 of **[Th1]**. By Lemma 3.2 of **[Th1]** it suffices to prove that the canonical $*$-homomorphism $A \otimes B \to \mathcal{K} \otimes A \otimes B$ induces a group isomorphism

$$k_{-1}(A \otimes B) \longrightarrow k_{-1}(\mathcal{K} \otimes A \otimes B) = K_1(A \otimes B)$$

for all B. Let B^+ be the C^*-algebra obtained by adding a unit to B. There is then a split-exact sequence

$$0 \to A \otimes B \to A \otimes B^+ \to A \to 0.$$

By applying the half-exactness of k_{-1} and K_1 to this extension, it follows that we need only consider the case where B is unital. Furthermore, by Lemma 11.5, $A \otimes B$ is approximately divisible when A is, so we need only show that the canonical $*$-homomorphism $A \to \mathcal{K} \otimes A$ induces an isomorphism $k_{-1}(A) \simeq K_1(A)$. Fix a $k \in \mathbb{N}$. We must show that the map

$$U(A) \ni u \mapsto \operatorname{diag}(u, 1, 1, \ldots, 1) \in U(M_k(A))$$

induces an isomorphism $\pi_0(U(A)) \to \pi_0(U(M_k(A)))$. Surjectivity: Let w be a unitary in $M_k(A)$. Since A is approximately divisible there is a finite dimensional unital C^*-subalgebra $\bigoplus_{i=1}^N M_{n_i} \simeq F \subset A$ with $n_i \geq k$ for all i, and a unitary $w_1 \in M_k(A \cap F')$ such that

$$\|w - w_1\| < 1.$$

In particular, w_1 is homotopic to w. Let e_i, $i = 1, 2, \ldots, N$, be the minimal non-zero central projections in F. Then

$$A \cap F' \simeq \bigoplus_{i=1}^{N} M_{n_i}(B_i),$$

where $B_i = e_i(A \cap F')$. Thus it suffices, as far as the surjectivity is concerned, to show that for any unital C^*-algebra B and natural number $n \geq k$, a unitary

$u \in M_k \otimes M_n \otimes B$, which commutes with $1 \otimes M_n \otimes 1 \subset M_k \otimes M_n \otimes B$ is homotopic to a unitary of the form

$$\operatorname{diag}(v, 1, 1, \ldots, 1) \in M_k(M_n \otimes B)$$

for some unitary $v \in M_n(B)$. Since u commutes with $1 \otimes M_n \otimes 1$, it has the form

$$u = \operatorname{diag}(s, s, \ldots, s) \in M_n(M_k \otimes B)$$

for some unitary $s \in M_k \otimes B$. By standard arguments u is therefore homotopic to

$$\operatorname{diag}(s^n, 1, 1, \ldots, 1) \in M_n(M_k \otimes B).$$

This shows that if we consider $M_k \otimes M_k \otimes B \oplus \mathbb{C}$ as a unital C^*-subalgebra of $M_k \otimes M_n \otimes B$ in the natural way, by using that $n \geq k$, then u is homotopic to a unitary of the form

$$(u_1, 1) \in M_k \otimes M_k \otimes B \oplus \mathbb{C}.$$

In fact, if we let e be a minimal non-zero projection in M_k, then

$$u_1 = u_2 + 1 - 1 \otimes e \otimes 1,$$

where $u_2 u_2^* = u_2^* u_2 = 1 \otimes e \otimes 1$. The "flip" $*$-automorphism of $M_k \otimes M_k \otimes B$ which exchanges the two copies of M_k is homotopic to the identity so we see that u_1 is homotopic in the unitary group of $M_k \otimes M_k \otimes B$ to a unitary u_3 of the form

$$u_3 = u_4 + 1 - e \otimes 1 \otimes 1$$

where $u_4 u_4^* = u_4^* u_4 = e \otimes 1 \otimes 1$. Since u_3 is homotopic to (in fact equal to, if the projection e is chosen right) a unitary of the form

$$\operatorname{diag}(v, 1, 1, \ldots, 1) \in M_k(M_n \otimes B)$$

for some unitary $v \in M_n(B)$, we have established the surjectivity.

Injectivity: Let u, v be unitaries in A such that $\operatorname{diag}(u, 1, 1, \ldots, 1)$ and $\operatorname{diag}(v, 1, 1, \ldots, 1)$ are homotopic in the unitary group of $M_k(A)$ for some $k \in \mathbb{N}$. We must show that u and v are homotopic in $U(A)$. Let

$$\gamma \colon [0,1] \longrightarrow U(M_k(A))$$

be a path of unitaries connecting $\operatorname{diag}(u, 1, 1, \ldots, 1)$ to $\operatorname{diag}(v, 1, 1, \ldots, 1)$. By using that $C[0,1] \otimes A$ is approximately divisible by Lemma 11.5, we can find a finite dimensional unital C^*-subalgebra

$$\bigoplus_{i=1}^{N} M_{n_i} \simeq F \subset A$$

with $n_i \geq k$ for all i and a path γ' of unitaries in $M_k(A \cap F')$ such that

$$\sup_t \|\gamma'(t) - \gamma(t)\|$$

is as small as we want. After a subsequent perturbation we may arrange that $\gamma'(0)$ and $\gamma'(1)$ are of the form

$$\mathrm{diag}(u_1, 1, 1, \ldots, 1) \in M_k(A \cap F')$$

and

$$\mathrm{diag}(v_1, 1, 1, \ldots, 1) \in M_k(A \cap F'),$$

respectively, where $\|u - u_1\| < 1$ and $\|v - v_1\| < 1$. Since u and v are homotopic in $U(A)$ to u_1 and v_1, respectively, it suffices to show that u_1 and v_1 are homotopic in $U(A)$. With the same notation as above we have that

$$M_k(F' \cap A) \simeq \bigoplus_{i=1}^{N} M_k(M_{n_i}(B_i)).$$

Thus, for the present purpose, it suffices to consider a unital C^*-algebra B, a natural number $n \geq k$ and unitaries $u, v \in B$, such that

$$\mathrm{diag}(u, 1, 1, \ldots, 1)$$

is homotopic to

$$\mathrm{diag}(v, 1, 1, \ldots, 1)$$

within the unitary group of $M_k \otimes B$, and show that $1 \otimes u \in M_n \otimes B$ is homotopic to $1 \otimes v$ in $U(M_n \otimes B)$. But $1 \otimes u$ and $1 \otimes v$ are homotopic in the unitary group of $M_n(B)$ to

$$\mathrm{diag}(u^n, 1, 1, \ldots, 1) = \mathrm{diag}(u, 1, 1, \ldots, 1)^n$$

and

$$\mathrm{diag}(v^n, 1, 1, \ldots, 1) = \mathrm{diag}(v, 1, 1, \ldots, 1)^n,$$

respectively, and these two unitaries are homotopic since $\mathrm{diag}(u, 1, 1, \ldots, 1)$ and $\mathrm{diag}(v, 1, 1, \ldots, 1)$ are homotopic in the unitary group of $M_k(B)$. □

COROLLARY 11.7. — *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks of type 2. Then*

$$\pi_k(U(A \otimes B)) \simeq \begin{cases} K_0(A \otimes B) & \text{when } k \text{ is odd,} \\ K_1(A \otimes B) & \text{when } k \text{ is even} \end{cases}$$

for every unital C^*-algebra B.

Proof. — A is approximately divisible by Theorem 5.1, so Theorem 11.6 applies. □

By using approximate divisibility we also obtain an alternative calculation of the homotopy groups of the unitary group of a nonrational noncommuative torus.

COROLLARY 11.8 ([**Rf**, Theorem 3.4]). — *Let A be a nonrational noncommutative torus. Then*

$$\pi_k(U(A \otimes B)) \simeq \begin{cases} K_0(A \otimes B) & \text{when } k \text{ is odd,} \\ K_1(A \otimes B) & \text{when } k \text{ is even,} \end{cases}$$

for every unital C^*-algebra B.

Proof. — A is approximately divisible by [**BKR**], Theorem 1.5, so Theorem 11.6 applies. □

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