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a higher Atiyah-Patodi-Singer index theorem**

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THE b -PSEUDODIFFERENTIAL CALCULUS ON GALOIS COVERINGS
AND A HIGHER ATIYAH-PATODI-SINGER INDEX THEOREM

ERIC LEICHTNAM[†] AND PAOLO PIAZZA[‡]

Abstract. Let $\Gamma \rightarrow \widetilde{M} \rightarrow M$ be a Galois covering with boundary. In this paper we develop a b -pseudodifferential calculus on the noncompact manifold \widetilde{M} . Our main application is the proof of a higher Atiyah-Patodi-Singer index formula, for a generalized Dirac operator \widetilde{D} on \widetilde{M} , under the assumption that the group Γ is of polynomial growth with respect to a word metric and that the L^2 -spectrum of the boundary operator \widetilde{D}_0 has a gap at zero. Our results extend work of Atiyah-Patodi-Singer, Connes-Moscovici and Lott.

Résumé. Soit $\Gamma \rightarrow \widetilde{M} \rightarrow M$ un revêtement Galoisien à bord. Dans cet article nous développons un b -calcul pseudodifférentiel sur \widetilde{M} . Ceci nous permet de prouver un théorème de l'indice supérieur d'Atiyah-Patodi-Singer, pour un opérateur de Dirac \widetilde{D} sur \widetilde{M} , sous l'hypothèse que le groupe Γ est à croissance polynomiale par rapport à une métrique des mots et que zéro est un point isolé du spectre L^2 de l'opérateur de bord \widetilde{D}_0 . Notre résultat généralise des travaux d'Atiyah-Patodi-Singer, Connes-Moscovici et Lott.

Key words: Dirac operators, Galois coverings, C^* -algebras, b -calculus, noncommutative de Rham homology, finite propagation speed estimates.

AMS Subject Classification Index: 58G12, 58G20, 46L87, 58G15.

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0. Introduction.

One of the fundamental tools in the development of index theory for elliptic operators has been the use of heat-kernel techniques. As this introduction is meant for a wide audience, we briefly recall the main point of this approach. Suppose, for simplicity, that M is an even dimensional closed *spin* compact manifold with a fixed spin structure. Let $S = S^+ \oplus S^-$ be the bundle of spinors and let D be the Dirac operator associated to the given spin structure. The operator D is formally self-adjoint and odd with respect to the \mathbb{Z}_2 -grading; thus $D^\pm : C^\infty(M, S^\pm) \rightarrow C^\infty(M, S^\mp)$ and $D^- = (D^+)^*$. The heat operator of the Dirac laplacian, $\exp(-tD^2)$, is a smoothing operator for each $t > 0$. Thus the Schwartz kernel of $\exp(-tD^2)$, the *heat kernel*, is smooth on $M \times M$ and it is therefore trace class acting on the Hilbert space of L^2 sections of S . Consider the supertrace of $\exp(-tD^2)$, $\text{STr}(\exp(-tD^2)) \equiv \text{Tr}(\exp(-tD^-D^+)) - \text{Tr}(\exp(-tD^+D^-))$. The vanishing of the trace on commutators implies that this difference does not depend on t , thus

$$\frac{d}{dt}(\text{STr}(\exp(-tD^2))) = 0 \quad (0.1).$$

Moreover, by Lidski's theorem, it is given by the difference of the integrals of the two heat kernels over the diagonal Δ of $M \times M$. It is well known that as $t \rightarrow +\infty$ the heat operator converges exponentially to the orthogonal projection onto the null space of D^2 . This implies that $\text{STr}(\exp(-tD^2))$ converges exponentially to the supertrace of the projection onto the null space of D^2 which is easily seen to be the index of D^+ . On the other hand

as $t \rightarrow 0^+$ the heat kernel restricted to the diagonal converges itself to a density on $\Delta \equiv M$ which is *explicitly computable*. We denote this density by $AS_{[n]}$, n being the dimension of the manifold M and AS being an explicit differential form constructed out of the riemannian curvature tensor. The index theorem for D^+ then follows by equating the integral over $\Delta \equiv M$ of this explicit geometric expression with the supertrace of the projection onto the null space of D^2 (which is the index of D^+). Here formula (0.1) has been used. Thus

$$\text{ind}(D^+) = \int_M AS.$$

What we have just explained is a sketch of the proof of the *local* index theorem for Dirac operators (see [ABP][G][BGV]).

The fascinating idea of using the heat equation to investigate the index of Dirac operators (due to McKean and Singer in its first formulation) opened the way to a variety of extensions of the original results of Atiyah and Singer, some of which will be now recalled.

In a fundamental series of articles, Atiyah, Patodi and Singer [APS 1,2,3] extended the results of [AS 1,3] to Dirac operators on manifolds with boundary.

Thus suppose now that M has a boundary ∂M and that the riemannian metric is of product type near the boundary. The Dirac operators D^\pm can be written, near the boundary, as $\pm \partial/\partial u + D_0$ with u equal to the normal variable to the boundary and D_0 the Dirac operator on ∂M . The operator D_0 is elliptic and essentially self-adjoint. Let Π_\geq be the spectral projection corresponding to the non-negative eigenvalues of D_0 and let

$$C^\infty(M, S^+, \Pi_\geq) = \{s \in C^\infty(M, S^+) \mid \Pi_\geq(s|_{\partial M}) = 0\}.$$

The Atiyah-Patodi-Singer theorem [APS 1] states that the operator D^+ acting on Sobolev completions of $C^\infty(M, S^+, \Pi_\geq)$ (we denote this operator by $D_{\Pi_\geq}^+$) is a Fredholm operator with index

$$\text{ind}(D_{\Pi_\geq}^+) = \int_M AS - \frac{1}{2}(\eta(D_0) + \dim \text{null } D_0).$$

Here $\eta(D_0)$ is the eta invariant of the self-adjoint operator D_0 . It is a spectral invariant that measures the *asymmetry* of the spectrum of D_0 . It

is defined as the value at $s = 0$ of the meromorphic continuation of the complex function

$$\sum_{\lambda \neq 0} \text{sign } \lambda |\lambda|^{-s} \quad \Re s \gg 0$$

with λ running over the eigenvalues of D_0 . Equivalently, using the Mellin transform,

$$\eta(D_0) = \frac{2}{\sqrt{\pi}} \int_0^\infty \text{Tr}(D_0 e^{-(tD_0)^2}) dt \equiv \int_0^\infty \eta(D_0)(t) dt. \quad (0.2)$$

The proof of the Atiyah-Patodi-Singer theorem relies heavily on the heat-kernel method.

The Atiyah-Patodi-Singer index theorem has seen a number of reformulations and alternative proofs. Among the latest contributions to the subject we mention here the b -calculus approach of Melrose [Me] (see also [P1][MeNi]). In this new approach microlocal techniques are used in order to give an elegant and conceptually simple proof of the original result of Atiyah-Patodi-Singer (in fact for metrics which are more general than those, product-like near the boundary, considered in [APS 1]). We refer the uninitiated reader to the introduction of [Me] for a very readable summary of the main ideas surrounding the b -calculus proof.

We come now to Bismut's fundamental proof of the *local* family index theorem for Dirac operators on closed manifolds [B]. Given a smooth family of Dirac operators $D = (D_z)_{z \in B}$ acting on $C^\infty(M_z, S_z)$ and parametrized by a compact manifold B , we can consider the associated (regularized) index bundle $\text{Ind}(D) = [\text{null}(D^+)] - [\text{null}(D^-)]$, an element in the K -theory $K^0(B)$ of the base B , and the associated Chern character $\text{Ch}(\text{Ind}(D))$, a cohomology class in $H^{\text{even}}(B, \mathbb{R})$. From an algebraic point of view the index bundle can be seen as the formal difference of two finitely generated projective $C^0(B)$ modules, $C^0(B)$ denoting the algebra of continuous functions on B (see [A]). Thus $\text{Ind}(D) \in K_0(C^0(B))$, with $K_0(C^0(B))$ equal to the 0th algebraic K -group of $C^0(B)$. This point of view will be exploited later

The problem is once again to give a geometric formula for $\text{Ch}(\text{Ind}(D))$, an *a priori* analytic object. The cornerstone of Bismut's treatment of the family index theorem is the use of the superconnection formalism (see also [Q]). Instead of considering the family of Dirac laplacians (D_z^2) one considers

a family of generalized laplacians *with differential form coefficients*. This new family is manufactured out of a superconnection on the infinite dimensional bundle, over B , whose fiber at $z \in B$ is equal to $C^\infty(M_z, S_z)$. The fiber supertraces of the heat-kernels associated to this new family of generalized laplacians combine to give a *smooth* differential form on B . We denote by K_t the family of heat-kernels and by $\text{STr}(K_t)$ this smooth differential form. Bismut then proves that $\text{STr}(K_t)$ satisfies the following properties

- (i) It is a closed differential form $\forall t > 0$.
- (ii) It does not depend on t modulo exact forms: $d/dt(\text{STr}(K_t)) = d_B \alpha_t$.
- (iii) It is explicitly computable as $t \rightarrow 0^+$.
- (iv) It represents $\text{Ch}(\text{Ind}(D))$ in $H^{\text{even}}(B)$.

This last property can be proved directly as in Bismut's original argument or by showing, as in [BV][BGV], that the limit as $t \rightarrow +\infty$ of $\text{STr}(K_t)$ converges as a differential form on B to the Chern character of the index bundle.

Using these properties the *local* version of the family index theorem follows. In particular

$$\text{Ch}(\text{Ind}(D)) = \int_{\text{fibre}} \text{AS} \quad \text{in } H^*(B).$$

Among the many implications of Bismut's heat-kernel treatment of the family index theorem we concentrate now on the family version of the Atiyah-Patodi-Singer index theorem. The first result in this direction is due to Bismut and Cheeger [BC 1,2,3]; (D_z) is now a family of Dirac operators on manifolds with boundary, parametrized by a compact smooth manifold B . In order for the family D_{z, Π_\geq}^+ to define a *smooth* (or even continuous) family of Fredholm operators it is necessary that the null spaces of the boundary operators $D_{0,z}$ are of constant dimension in $z \in B$. Notice that under this assumption they form a smooth vector bundle over B , $\text{null}(D_0) \rightarrow B$. Moreover the index bundle $\text{Ind}(D_{\Pi_\geq})$ is well defined and the following formula holds

$$\text{Ch}(\text{Ind}(D_{\Pi_\geq})) = \int_{\text{fibre}} \text{AS} - \frac{1}{2}(\hat{\eta} + \text{Ch}(\text{null}(D_0))) \quad \text{in } H^*(B) \quad (0.3)$$

(the formula is fully proved in the invertible case in [BC 1,2] and stated in the constant rank case in [BC 3]; see [MP 1] for a complete proof of (0.3)). In this formula $\hat{\eta}$ is the eta form of Bismut-Cheeger; it is a higher version of the eta invariant, in the sense that the 0-degree component of $\hat{\eta}$ computed

at $z \in B$ is equal to $\eta(D_{0,z})$. The Bismut-Cheeger eta form is defined in terms of the superconnection formalism by a formula similar to (0.2). The assumption that the operators of the family D_0 have null spaces of constant dimension plays a crucial role in the proof of the convergence of the integral.

The results of Bismut-Cheeger were improved in [MP 1,2]. The use of a new notion, that of *spectral section* associated to a self-adjoint family of elliptic operators (like D_0), together with the pseudodifferential b -calculus, allowed for the formulation and the proof of a *general Atiyah-Patodi-Singer family index theorem*, both in the even and in the odd dimensional case.

Suppose now, as in the beginning of this introduction, that M is a closed compact spin manifold. Let us denote by Γ the fundamental group $\pi_1(M)$ of M and by $\widetilde{M} \rightarrow M$ the universal covering of M . The Γ -manifold \widetilde{M} is again spin with a Γ -invariant Dirac operator \widetilde{D} acting on the section of a Γ -invariant spinor bundle \widetilde{S} . It is clear that \widetilde{M} will be in general non-compact. There are two sets of objects that are determined by the appearance of the fundamental group of M .

First we can consider the classifying map $\nu : M \rightarrow B\Gamma$ associated to the Γ -bundle $\Gamma \rightarrow \widetilde{M} \rightarrow M$. For each cohomology class $[\beta] \in H^*(B\Gamma, \mathbb{C})$ we can then consider $\nu^*[\beta] \in H_{dR}^*(M)$ and the complex numbers

$$\int_M \text{AS} \wedge \nu^*[\beta].$$

Recall also that there is a canonical isomorphism between $H^*(B\Gamma, \mathbb{C})$ and the group cohomology $H^*(\Gamma, \mathbb{C})$.

The second set of objects determined by the discrete group $\pi_1(M)$ is more analytic in nature. We can consider the reduced C^* -algebra $C_r^*(\Gamma)$, i.e. the closure in $B(\ell^2(\Gamma))$ of the image of $\mathbb{C}\Gamma$ by the left regular representation, and the infinite dimensional bundles

$$\mathcal{S}^\pm = S^\pm \otimes (\widetilde{M} \times_\Gamma C_r^*(\Gamma)).$$

These are bundles on M with fibres that are finitely generated projective $C_r^*(\Gamma)$ -modules. The operator \widetilde{D} defines operators $\mathcal{D}^\pm : C^\infty(M, \mathcal{S}^\pm) \rightarrow C^\infty(M, \mathcal{S}^\mp)$ which are $C_r^*(\Gamma)$ -Fredholm as maps $H^1(M, \mathcal{S}^\pm) \rightarrow L^2(M, \mathcal{S}^\mp)$, in the sense that $[\text{null}(\mathcal{D}^+)] - [\text{null}(\mathcal{D}^-)]$ (really $[\text{null}(\mathcal{D}^+ + \mathcal{R}^+)] - [\text{null}(\mathcal{D}^- + \mathcal{R}^-)] \equiv [\mathcal{L}] - [\mathcal{N}]$ for suitable compact perturbations \mathcal{R}^\pm , see [R]) is a formal difference of *finitely generated projective* $C_r^*(\Gamma)$ -modules. Thus, as

in the algebraic reinterpretation of the index bundle for families explained above, we obtain an index class $\text{Ind}(\mathcal{D}) \in K_0(C_r^*(\Gamma))$. The proof of the $C_r^*(\Gamma)$ -Fredholm property for \mathcal{D} is a consequence of the Mishenko-Fomenko $C_r^*(\Gamma)$ -pseudodifferential calculus [M-F]. Alternatively Kasparov KK -theory can be employed [BJ][K].

To obtain characteristic numbers out of this index class one must pass to cyclic (co)homology. In order to make use of the cyclic (co)homology machinery it is necessary to fix a certain dense subalgebra \mathcal{B} of $C_r^*(\Gamma)$, containing $C\Gamma$ and stable under holomorphic functional calculus of $C_r^*(\Gamma)$, and then show that \mathcal{D} defines index classes $\text{Ind}(\mathcal{D}) \in K_0(\mathcal{B}) \cong K_0(C_r^*(\Gamma))$. This step should be thought as a “smoothing” of the index class $\text{Ind}(\mathcal{D})$, quite analogous to the passage from a continuous to a smooth index bundle in the family case. The Chern character of $\text{Ind}(\mathcal{D})$ is now well defined in the topological noncommutative de Rham homology of \mathcal{B} , $\text{Ch}(\text{Ind}(\mathcal{D})) \in \overline{H}_*(\mathcal{B})$, and can be paired with topological cyclic cocycles so as to get complex numbers. The Connes-Moscovici higher index theorem on Γ -coverings can then be stated as follows. Let us fix a group cocycle $\beta \in Z^l(\Gamma, \mathbb{C})$; in a purely algebraic way β defines a cyclic cocycle τ_β in $ZC^l(C\Gamma)$; *assume* that this cyclic cocycle extends to a cyclic cocycle $\tau_\beta \in ZC^l(\mathcal{B}^\infty)$. Then

$$\langle \text{Ch}(\text{Ind}(\mathcal{D})), \tau_\beta \rangle = C_l \int_M \text{AS} \wedge \nu^*[\beta] \quad (0.4)$$

with C_l a nonzero l -dependent constant. When $l = 0$ this is the Von Neumann index theorem of Atiyah and Singer [A][S] on Γ -coverings.

A spectacular application of (0.4), given by Connes and Moscovici, is the proof of the homotopy invariance of Novikov’s higher signatures when the group Γ is hyperbolic.

Notice that index theoretic methods for establishing this homotopy invariance were pioneered by Lustzig [Lu] who established it when $\Gamma = \mathbb{Z}^k$. In this case the higher index theorem (0.4) reduces to a family index theorem with parameter space B equal to the dual group to Γ (i.e. a torus T^k).

One could regard the higher index theorem of Connes-Moscovici as a *noncommutative family index theorem*.

Recall now Bismut’s heat-kernel treatment of the genuine family index theorem. One is then led to speculate that there should exist a local heat-kernel approach to the higher index theorem. This idea is pursued by Lott in [L1], where a Bismut superconnection proof of (0.4) is given. The main

tool in [L1] is the use of a correspondence between \mathcal{B} smoothing operators on M and ordinary smoothing operators on \widetilde{M} with Schwartz kernel rapidly decreasing, in an appropriate sense, at ∞ .

Using this correspondence it is possible to define and effectively manipulate a noncommutative superconnection heat kernel K_t , an appropriate supertrace STR and the noncommutative [Ka] closed differential form $\text{STR}(K_t)$. The higher index theorem follows as in Bismut by the equality

$$\langle \text{Ch}(\text{Ind}(D)), \tau_\beta \rangle = \langle \text{STR}(K_t), \tau_\beta \rangle$$

and the short time behaviour of $\text{STR}(K_t)$:

$$\lim_{t \downarrow 0^+} \langle \text{STR}(K_t), \tau_\beta \rangle = \langle \int_M \text{AS} \wedge \omega, \tau_\beta \rangle = \int_M \text{AS} \wedge \nu^*[\beta]. \quad (0.5)$$

The form ω can be explicitly described. Notice that both equations in (0.5) need to be justified. In [L2], using the above superconnection formalism, Lott introduces the definition of the higher eta invariant $\tilde{\eta}$. It is a noncommutative \mathcal{B} -differential form essentially defined by an integral similar to (0.2). The existence of the integral, in the present context, is far from being obvious and two assumptions are needed in order to ensure the convergence of the integral of $\tilde{\eta}(t)$ at $t = 0$ and $t = +\infty$. First that the group Γ is of polynomial growth with respect to a word metric. Second that the Dirac operator on \widetilde{M} admits a bounded L^2 -inverse. These two assumptions are needed in order to use finite propagation speed estimates on the noncompact manifold \widetilde{M} . When $\Gamma = \mathbb{Z}^k$ Lott's higher eta invariant reduces to the eta form of Bismut-Cheeger.

In [L2] it is conjectured that such a higher eta invariant should enter in a natural way into a higher Atiyah-Patodi-Singer Γ -index theorem on manifolds with boundary having a product structure near the boundary.

In this paper we have two goals in mind. First we develop a b -pseudodifferential calculus on Galois coverings; second we apply this analytic machinery to the geometric problem presented above and show that Lott's conjecture holds true. The proof of the conjecture rests more precisely on such an extension of the b -calculus, on Lott's superconnection proof of the Connes-Moscovici higher index theorem and on a \mathcal{B} - b -calculus on the compact manifold with boundary M .

The same hypothesis that are needed to define the higher eta invariant must be assumed in order to formulate and prove the higher index theorem.

Actually, by using an idea of Berline and Vergne, we extend the results of [L2] and show the convergence of the integral defining the higher eta invariant only assuming that the L^2 -spectrum of the Dirac operator on the covering has a gap at zero. Consequently we prove a higher Atiyah-Patodi-Singer index theorem more general than the one conjectured by Lott. This improvement opens the way to several possible geometric applications of the higher index formula. The precise statement of our result is given at the beginning of Sect. 14.

A final comment on our assumption

$$\exists \delta > 0 \text{ such that } \text{spec}(\tilde{D}_0) \cap [-\delta, \delta] = \{0\}$$

on the boundary operator. This assumption is the precise analogue, in the noncommutative setting, of the Bismut-Cheeger hypothesis that the null spaces of the boundary family have constant rank. In the truly family case this assumption is completely removed from the picture by employing the notion of *spectral section* (see [MP 1,2]). The notion of spectral section has also been successfully used by Dai and Zhang in order to define the higher spectral flow associated to a one-dimensional deformation of a family of self-adjoint operators parametrized by a compact space B [DZ]. The higher spectral flow of [DZ] is a class in $K^0(B) = K_0(C^0(B))$. Wu, on the other hand, has extended the definition of spectral section of [MP 1,2] to the noncommutative context and, generalizing [DZ], has defined a noncommutative higher spectral flow associated to a one-parameter family $\{D_t\}$ of operators on A -Hilbert modules [W], with A a C^* -algebra. Wu's higher spectral flow is an element in $K_0(A)$. In a future publication we shall use in an essential way the analytic tools developed in this paper and the notion of noncommutative spectral section to give a *general* higher Atiyah-Patodi-Singer index theorem (i.e. when the L^2 -spectrum of the boundary operator has no gap at all).

We shall now briefly describe the contents of the paper and the structure of the proof. In the first three sections we really deal with higher index theory on *closed* manifolds, thus extending some of the results of [L1]. Since Γ is of polynomial growth we can fix, as in [L2], \mathcal{B} , the dense "smooth" subalgebra of $C_r^*(\Gamma)$, to be equal to \mathcal{B}^∞ , the convolution algebra of rapidly decreasing function on Γ .

In Sect. 1 we show how to explicitly construct "smooth" representatives of the index class associated to \mathcal{D} ; this is accomplished by developing in

a rigorous way a \mathcal{B}^∞ -Mishenko-Fomenko pseudodifferential calculus. Thus $\text{Ind}(\mathcal{D}) = [\mathcal{L}_\infty] - [\mathcal{N}_\infty]$ for suitable finitely generated projective \mathcal{B}^∞ -modules. Essential to our treatment are the finite propagation speed estimates of Cheeger-Gromov-Taylor [CGT] on the covering \widetilde{M} . The functional analytic technicalities of the proofs are gathered in Appendix A (Sect. 16). In Sect. 2 we show how to define higher eta invariants only assuming the existence of a gap at zero in the L^2 -spectrum of the Dirac operator \widetilde{D} on \widetilde{M} . Thus we assume

$$\exists \delta > 0 \text{ such that } \text{spec}(\widetilde{D}_0) \cap [-\delta, \delta] = \{0\}. \quad (0.6)$$

In Sect.3 we consider higher eta invariants for the operator $\widetilde{D} + \vartheta$, ϑ small, and study their behaviour as $\vartheta \rightarrow 0 \in \text{spec}(\widetilde{D}_0)$. It is important to point out that the correspondence between \mathcal{B}^∞ -pseudodifferential operators on the base M and rapidly decreasing operators on the covering \widetilde{M} is fundamental throughout the paper, especially when we consider \mathcal{B}^∞ -operators with $\Omega_*(\mathcal{B}^\infty)$ (i.e. noncommutative differential form) coefficients.

Sect. 4 to Sect. 10 are devoted to the extension of the b -calculus to Galois Γ -coverings with boundary, concentrating on the virtually nilpotent case.

With Sect. 11 we enter in the truly higher case, showing how a b -Dirac operator \widetilde{D} on a Γ -covering with boundary defines an index class $\text{Ind}(\mathcal{D}) \in K_0(C_r^*(\Gamma))$. The “smoothing” of the index class is dealt with in Sect. 12, where a \mathcal{B}^∞ - b -Mishenko Fomenko calculus is developed. The b -superconnection formalism is introduced in Sect. 13, where the definition of the b -version of Lott’s supertrace functional is given and its behaviour on supercommutators is investigated as in the commutative case treated in [M][MP].

In Sect. 14 we finally prove Lott’s conjecture. The structure of the proof, for simplicity in the invertible case, is as follows. Let $\overline{H}_*(\mathcal{B}^\infty)$ be the topological noncommutative de Rham homology of \mathcal{B}^∞ [Ka]. By Karoubi’s theory of characteristic classes for finitely generated projective modules, we know that the Chern character of the \mathcal{B}^∞ index class of \mathcal{D} , $\text{Ind}(\mathcal{D}) = [\mathcal{L}_\infty] - [\mathcal{N}_\infty]$, can be expressed as the STR of the exponential of the curvature ∇^2 of a connection on $\mathcal{L}_\infty \oplus \mathcal{N}_\infty$:

$$\text{Ch}(\text{Ind}(\mathcal{D})) = \text{STR}(\exp(-\nabla^2)) \in \overline{H}_*(\mathcal{B}^\infty). \quad (0.7)$$

With the help of the \mathcal{B}^∞ - b -calculus we then prove that $b\text{-STR}(\exp(-\nabla^2)) = \text{STR}(\exp(-\nabla^2))$. Using this equality, formula (0.7), various transgression

formulas as in [L1][B], the short time behaviour of the heat kernel on Γ -coverings with boundary and applying (repeatedly) the supercommutator formula for b -STR we obtain, $\forall u > 0$, the following equality

$$\text{Ch}(\text{Ind}(\mathcal{D})) = \int_M \text{AS} \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(t) dt + B(u) \text{ in } \overline{H}_*(\mathcal{B}^\infty)$$

with $B(u)$ explicit boundary terms. Employing finite propagation speed estimates on the boundary we then show that as $u \rightarrow +\infty$ the term $B(u)$ converges to zero. In these computations the assumption on the L^2 -invertibility of \tilde{D}_0 is very important. Using the convergence of the higher eta integrand at infinity the higher APS index theorem

$$\text{Ch}(\text{Ind}(\mathcal{D})) = \int_M \text{AS} \wedge \omega - \frac{1}{2} \int_0^\infty \tilde{\eta}(t) dt \equiv \int_M \text{AS} \wedge \omega - \frac{1}{2} \tilde{\eta} \text{ in } \overline{H}_*(\mathcal{B}^\infty)$$

follows in the invertible case. If the boundary operator \tilde{D}_0 only satisfies (0.6) then it is not difficult to see that $\text{null}(\mathcal{D}_0)$ is a finitely generated projective \mathcal{B}^∞ -module and the higher APS index formula becomes

$$\text{Ch}(\text{Ind}(\mathcal{D})) = \int_M \text{AS} \wedge \omega - \frac{1}{2} (\tilde{\eta} + \text{Ch}(\text{null}(\mathcal{D}_0))) \text{ in } \overline{H}_*(\mathcal{B}^\infty).$$

The passage from the invertible to the gap case is based on the limit behaviour of the higher eta invariants of $\tilde{D}_0 + \vartheta$ as $\vartheta \rightarrow 0^+$ (this was studied back in Sect.3). The precise statement of the theorem is given at the beginning of Sect.14 and can be read at this point.

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1. \mathcal{B}^∞ -Mishenko-Fomenko pseudodifferential calculus.

Let M be a compact *boundaryless* Riemannian manifold of **even** dimension n , let $\pi : \widetilde{M} \rightarrow M$ be a Galois Γ -covering of M , Γ being a finitely-generated virtually nilpotent group. This means that if $\|\cdot\|$ is a right invariant word metric on Γ then the number of points $g \in \Gamma$ such that $\|g\| < R$ is of polynomial growth with respect to R . Thus if \widetilde{M} is endowed with the lift by π of the Riemannian metric of M then the volume of the open balls $B(x, R)$ of \widetilde{M} is of polynomial growth as $R \rightarrow +\infty$. Any function in the classical Schwartz space $\mathcal{S}(\widetilde{M}, \mathbb{C})$ is integrable with respect to the Riemannian density; we will denote by $d(x, y)$ the geodesic distance between two points $x, y \in \widetilde{M}$.

Let $\Lambda = C_r^*(\Gamma)$ be the reduced C^* -algebra of Γ . Since Γ is virtually nilpotent

$$\mathcal{B}^\infty = \{f : \Gamma \rightarrow \mathbb{C} / \forall L \in \mathbb{N}, \sup_{\gamma \in \Gamma} (1 + \|\gamma\|)^L |f(\gamma)| < +\infty\}$$

is a subalgebra of Λ which is stable under holomorphic functional calculus and Γ acts on the left on \mathcal{B}^∞ . We also recall that \mathcal{B}^∞ is a Frechet unital algebra whose unit is e , the neutral element of Γ .

Let $E = E^+ \oplus E^-$ be a \mathbb{Z}_2 -graded hermitian Clifford module over M endowed with a unitary Clifford connection [BGV], let

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad (D^+)^* = D^-$$

be the associated \mathbb{Z}_2 -graded Dirac operator. Let \tilde{D} be the associated lifted Dirac operator on the covering \tilde{M} which acts on sections of the lifted vector bundle $\tilde{E} = \pi^*(E)$ endowed with the lifted hermitian metric.

The action of $\gamma \in \Gamma$ on \tilde{M} will be considered on the right and denoted R_γ . We will deal respectively with the two following \mathcal{B}^∞ and Λ bundles over M :

$$(\tilde{M} \times_\Gamma \mathcal{B}^\infty) \otimes E = \mathcal{E}^\infty, \quad (\tilde{M} \times_\Gamma \Lambda) \otimes E = \mathcal{E}$$

where for any $\gamma \in \Gamma$ the point (x, f) of $(\tilde{M} \times_\Gamma \mathcal{B}^\infty)$ is identified with $(R_\gamma(x), \gamma^{-1} \cdot f)$. Let us fix $z_0 \in \tilde{M}$; we define the usual Schwartz space $\mathcal{S}(\tilde{M}, \tilde{E})$ as the set of the so-called rapidly decreasing sections u of \tilde{E} : for any multi-index of derivation α :

$$\forall N \in \mathbb{N}, \sup_{z \in \tilde{M}} [\|\nabla^\alpha u(z)\| (1 + d(z_0, z))^N] < \infty$$

One of the main contributions of [L1] is the systematic use of a bijective correspondence between

(i) rapidly decreasing sections of \tilde{E} on \tilde{M} and smooth sections of the bundle \mathcal{E}^∞ .

(ii) smoothing Γ -invariant operators on \tilde{M} with rapidly decreasing Schwartz kernel and smoothing operators on M acting on \mathcal{E}^∞ .

More precisely we have the following

Proposition 1.1.

1] *The map:*

$$u \rightarrow s = \sum_{\gamma \in \Gamma} R_\gamma^*(u) \gamma$$

is a bijection between the Schwartz space $\mathcal{S}(\tilde{M}, \tilde{E})$ and $C^\infty(M, \mathcal{E}^\infty)$, the space of smooth sections of \mathcal{E}^∞ .

2] *Let $\text{End}_{\mathcal{B}^\infty}(M, \mathcal{E}^\infty)$ denote the algebra of smoothing linear operators $T : C^\infty(M, \mathcal{E}^\infty) \rightarrow C^\infty(M, \mathcal{E}^\infty)$ defined by a Schwartz kernel of C^∞ -class:*

$$(z, z') \in M \times M \rightarrow T(z, z') \in \text{Hom}_{\mathcal{B}^\infty}(\mathcal{E}_{z'}, \mathcal{E}_z)$$

Then there is an isomorphism between $\text{End}_{\mathcal{B}^\infty}(M, \mathcal{E}^\infty)$ and the algebra of Γ -invariant integral operators on $L^2(\tilde{M}, \tilde{E})$ with smooth kernels $\tilde{T}(x, y) \in \text{Hom}(\tilde{E}_y, \tilde{E}_x)$ such that for all $N > 0$ and all multi-indices α, β :

$$\sup_{x, y} \|((1 + d(x, y))^N \nabla_x^\alpha \nabla_y^\beta \tilde{T}(x, y))\| < +\infty \quad (1.1)$$

This isomorphism $T \rightarrow \tilde{T}$ is such that for any $u \in \mathcal{S}(\tilde{M}, \tilde{E})$:

$$T\left(\sum_{\gamma \in \Gamma} R_{\gamma}^*(u)\gamma\right) = \sum_{\gamma \in \Gamma} R_{\gamma}^*(\tilde{T}(u))\gamma.$$

We define an operator:

$$\mathcal{D}_{\infty} = \begin{pmatrix} 0 & \mathcal{D}_{\infty}^{-} \\ \mathcal{D}_{\infty}^{+} & 0 \end{pmatrix} \tag{1.2}$$

acting on $C^{\infty}(M, \mathcal{E}^{\infty})$ by setting:

$$\forall u \in \mathcal{S}(\tilde{M}, \tilde{E}^{\pm}), \quad \mathcal{D}_{\infty}^{\pm} \left(\sum_{\gamma \in \Gamma} R_{\gamma}^*(u)\gamma\right) = \sum_{\gamma \in \Gamma} R_{\gamma}^*(\tilde{D}^{\pm}(u))\gamma \tag{1.3}$$

Using the usual norm [resp. semi-norms] of Λ [resp. \mathcal{B}^{∞}] we see that the space of smooth sections $C^{\infty}(M, \mathcal{E}^{\pm})$ [resp. $C^{\infty}(M, \mathcal{E}^{\pm\infty})$] is a Frechet space. We observe that:

$$C^{\infty}(M, \mathcal{E}^{\pm}) = C^{\infty}(M, \mathcal{E}^{\pm\infty}) \otimes_{\mathcal{B}^{\infty}} \Lambda.$$

In the sequel we shall often omit the subscript \mathcal{B}^{∞} in such tensor products.

We define a Λ -hermitian product $\langle \cdot, \cdot \rangle$ by setting for any $s_j = \sum_{\gamma \in \Gamma} R_{\gamma}^*(u_j)\gamma \in C^{\infty}(M, \mathcal{E}^{\pm\infty})$ with $j = 1, 2$:

$$\langle s_1, s_2 \rangle = \sum_{\gamma \in \Gamma} \left(\int_{\tilde{M}} \langle \overline{u_1(x)}, R_{\gamma}^*(u_2)(x) \rangle_x dx \right) \gamma \in \mathcal{B}^{\infty} \tag{1.4}$$

where $\langle \cdot, \cdot \rangle_x$ in the righthandside denotes the hermitian scalar product of \tilde{E}_x . The Λ -Hilbert module $L^2(M, \mathcal{E}^{\pm})$ is defined to be the completion of $C^{\infty}(M, \mathcal{E}^{\pm\infty})$ for the norm:

$$\|s_1\| = \| \langle s_1, s_1 \rangle \|_{\Lambda}^{\frac{1}{2}}$$

where the righthandside is the C^* -norm of Λ . In a similar way [M-F] have defined $\mathcal{H}^{-1}(M, \mathcal{E}^{\pm})$, the Sobolev space of order -1 . The operator \mathcal{D}_{∞} defined by (1.2) induces an operator \mathcal{D} :

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^{-} \\ \mathcal{D}^{+} & 0 \end{pmatrix}$$

which is continuous from $L^2(M, \mathcal{E})$ to $\mathcal{H}^{-1}(M, \mathcal{E})$.

It is worthwhile to describe these spaces on a small open subset U of the base M such that over U the covering map π and the two bundles $E, \widetilde{M} \times_{\Gamma} \Lambda$ are trivial. For $i = 1, 2$ let s_i be a section in $C^\infty(U, \mathcal{E}^\infty)$ with compact support. We fix a sheet U_0 of $\pi^{-1}(U)$ and we let $\pi_0^{-1} : U \rightarrow U_0$ be the inverse of the local diffeomorphism induced by $\pi : U_0 \rightarrow U$. By definition, for each $z \in U$, $s_i(z)$ is the class fixed by an element of the following form:

$$(\pi_0^{-1}(z), \sum_{\gamma \in \Gamma} u_{i,\gamma}(z) \gamma) \quad (1.5)$$

where for each k , the C^k -norms of the $u_{i,\gamma}(z)$ are of rapid decay with respect to $\|\gamma\|$ as $\|\gamma\|$ goes to infinity. The Λ -scalar product (1.4) can be written as:

$$\langle s_1, s_2 \rangle = \sum_{\gamma, \gamma' \in \Gamma} \left(\int_U \overline{u_{1,\gamma}(z)} u_{2,\gamma\gamma'}(z) dz \right) \gamma'$$

Moreover: $\forall z \in U$, $\mathcal{D}_\infty s_i(z)$ is determined by the equivalence class of the pair

$$(\pi_0^{-1}(z), \sum_{\gamma \in \Gamma} D u_{i,\gamma}(z) \gamma)$$

We construct an orthonormal basis $(s_j)_{j \in \mathbb{N}}$ of $L^2(U, \mathcal{E})$ made of compactly supported C^∞ sections of \mathcal{E}^∞ as follows: first we fix $(u_{j,e})_{j \in \mathbb{N}}$, an orthonormal basis of $L^2(U, E)$ made of compactly supported C^∞ sections; then we define $u_{j,\gamma}$ to be identically zero for all $(j, \gamma) \neq (j, e)$, $j \in \mathbb{N}$; clearly then the $(s_j)_{j \in \mathbb{N}}$ defined by (1.5) provide the required orthonormal basis of $L^2(U, \mathcal{E})$.

So, using a finite number of open subsets U_1, \dots, U_s of M such that $M = \cup_{l=1}^s \overline{U}_l$, $U_l \cap U_k = \emptyset$ for $l \neq k$, and such that over each U_i the covering map π and the bundles \mathcal{E}^\pm, E are trivial, we can find an orthonormal basis $(e_k^\pm)_{k \geq 1}$ of $L^2(M, \mathcal{E}^\pm)$ such that for any $k \geq 1$, $e_k^\pm \in C^\infty(M, \mathcal{E}^{\pm\infty})$.

Convention 1.2. *In the sequel we fix such an orthonormal basis $(e_k^\pm)_{k \geq 1}$ and set $\mathcal{L}_m^\pm = \oplus_{k=1}^m \mathcal{B}^\infty e_k^\pm$.*

The existence of such an orthonormal basis will be useful in the proof of the main theorem of this section which we now state:

Theorem 1.3. *We can find \mathcal{L}_∞ [resp. \mathcal{N}_∞] a sub- \mathcal{B}^∞ -module projective of finite rank of $C^\infty(M, \mathcal{E}^{\infty,+})$ [resp. $C^\infty(M, \mathcal{E}^{\infty,-})$] with the following properties:*

1] \mathcal{L}_∞ is free and we have

$$\mathcal{D}_\infty^+(\mathcal{L}_\infty) \subset \mathcal{N}_\infty \tag{1.6}$$

2] As Frechet spaces

$$\mathcal{L}_\infty \oplus \mathcal{L}_\infty^\perp = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \mathcal{N}_\infty \oplus \mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) = C^\infty(M, \mathcal{E}^{\infty,-}). \tag{1.7}$$

3] *The orthogonal projection of $C^\infty(M, \mathcal{E}^{\infty,+})$ onto \mathcal{L}_∞ and the projection $P_{\mathcal{N}_\infty}$ of $C^\infty(M, \mathcal{E}^{\infty,-})$ onto \mathcal{N}_∞ along $\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)$ are smoothing operators.*

4] As Banach spaces

$$\begin{aligned} \mathcal{L}_\infty \otimes \Lambda \oplus \overline{\mathcal{L}_\infty^\perp \otimes \Lambda} &= L^2(M, \mathcal{E}^+) \\ \mathcal{N}_\infty \otimes \Lambda \oplus \overline{\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \otimes \Lambda} &= \mathcal{H}^{-1}(M, \mathcal{E}^-) \end{aligned}$$

5] *The operator*

$$\mathcal{D}_\infty^+ : \mathcal{L}_\infty^\perp \rightarrow \mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \tag{1.8}$$

is invertible for the Frechet topologies; the operator

$$\mathcal{D}^+ : \overline{\mathcal{L}_\infty^\perp \otimes \Lambda} \rightarrow \overline{\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \otimes \Lambda} \subset \mathcal{H}^{-1}(M; \mathcal{E}^-)$$

is invertible.

Remark. By definition:

$$\text{Ind } \mathcal{D}^+ = [\mathcal{L}_\infty] - [\mathcal{N}_\infty] \in K_0(\mathcal{B}^\infty) \cong K_0(\Lambda)$$

The sub-modules \mathcal{L}_∞ and \mathcal{N}_∞ should be viewed as “smooth representative” of the index class $\text{Ind } \mathcal{D}^+$. It is very likely that our proof can be adapted to the case in which Γ is no more virtually nilpotent but hyperbolic, and \mathcal{B}^∞ is equal to the *Connes-Moscovici algebra*. However our arguments do not allow to prove Theorem 1.3 for more general algebras \mathcal{B} which are “only” stable under holomorphic functional calculus.

There is of course an analogous decomposition for \mathcal{D}^- :

$$\mathcal{M}_\infty \oplus \mathcal{M}_\infty^\perp = C^\infty(M, \mathcal{E}^{\infty, -}), \quad \mathcal{I}_\infty \oplus \mathcal{D}_\infty^-(\mathcal{M}_\infty^\perp) = C^\infty(M, \mathcal{E}^{\infty, +}).$$

In contrast with the classical Hilbert-space theory we do *not* claim here the existence of a *simultaneous* \mathcal{B}^∞ -decomposition of $C^\infty(M, \mathcal{E}^{\infty, +})$ and $C^\infty(M, \mathcal{E}^{\infty, -})$ for $\mathcal{D}^+, \mathcal{D}^-$ such that $\mathcal{I}_\infty = \mathcal{L}_\infty$ and $\mathcal{M}_\infty = \mathcal{N}_\infty$.

We shall refer to Theorem 1.3 as the \mathcal{B}^∞ -Mishenko-Fomenko decomposition. Our result makes precise a statement in [L1, Sect. 6], which seems to be given there without a detailed proof. In order to pass to the boundary case we felt it was necessary to give a precise account of $\Psi_{\mathcal{B}^\infty}^*(M, \mathcal{E}^\infty)$, the space of classical \mathcal{B}^∞ -Mishenko-Fomenko pseudodifferential operators on M ; this turned out to be a rather delicate matter already in the boundaryless case. The analogous statements in the boundary case (e.g. the *specific structure* of the Schwartz kernels of the projections onto the smooth representatives $\mathcal{L}_\infty, \mathcal{N}_\infty$) will play a crucial role in the proof of the higher APS index theorem.

The following three propositions are a key tool in the development of the \mathcal{B}^∞ -Mishenko-Fomenko calculus.

Proposition 1.4.

- 1] Let $\mathcal{P} \in \Psi_{\mathcal{B}^\infty}^0(M, \mathcal{E}^\infty)$, let \tilde{P} be the associated operator on the covering so that $\mathcal{P}(\sum_\gamma R_\gamma^*(s)\gamma) = \sum_\gamma R_\gamma^*(\tilde{P}(s))\gamma$ for any $s \in \mathcal{S}(\tilde{M}, \tilde{E})$. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(z) \equiv 0$ for $z < 0$ and $\chi(z) \equiv 1$ for $z > 1$. Let $K_{\tilde{P}}$ be the Schwartz kernel of \tilde{P} . Then for any $N > 1$:

$$\sup_{R>1} \sup_{\|u\|_{L^2} < 1} R^N \left\| \int_{\tilde{M}} \chi(d(x, y) - R) K_{\tilde{P}}(x, y) u(y) dy \right\|_{L^2(\tilde{M}, \tilde{E})} < +\infty \quad (1.9)$$

- 2] Let A and B be two operators sending continuously $C^\infty(M, \mathcal{E}^\infty)$ into itself, B being moreover a smoothing operator. Then $A \circ B$ is also a smoothing operator.

Proof. We easily reduce ourselves to the case when the Schwartz kernel $K_{\mathcal{P}}(z, z')$ of \mathcal{P} has compact support in $U \times U$ where U is a small open subset of M such that over U the bundles E and $\tilde{M} \times_\Gamma \Lambda$ are trivial. By definition we can write:

$$K_{\mathcal{P}} = \sum_{\gamma \in \Gamma} K_{\gamma} \gamma$$

where the K_{γ} are the Schwartz kernel of usual pseudodifferential operators in $\Psi^0(U, E)$ defined by complete symbols of zero order $\sigma_{\gamma}(z, \xi)$ which are rapidly decreasing in the symbol topology as $\|\gamma\|$ goes to $+\infty$. Let U_0 be an open subset of \widetilde{M} such that the covering map $\pi : U_0 \rightarrow U$ induces an isomorphism which we denote by π_0 . Then:

$$\pi^{-1}(U) = \cup_{\gamma \in \Gamma} R_{\gamma}(U_0)$$

The Γ -invariant Schwartz kernel $K_{\widetilde{P}}$ of \widetilde{P} is determined by:

$$\forall (x, y) \in U_0 \times U_0, \forall \gamma \in \Gamma, K_{\widetilde{P}}(R_{\gamma}(x), y) = K_{\gamma}(\pi(x), \pi(y)). \quad (1.10)$$

The rigorous meaning of this equality is the following: let r_{γ} be the action on $M \times M$ given by $(x, y) \rightarrow (R_{\gamma}(x), y)$; then for each test function $\phi \in U_0 \times U_0$ we have

$$\langle K_{\widetilde{P}}, r_{\gamma}^* \phi \rangle = \langle K_{\gamma}, (\pi_0^{-1})^* \phi \rangle .$$

Since the covering \widetilde{M} is of polynomial growth, we get easily 1] by using the rapid decay of the symbols σ_{γ} . We omit the easy proof of 2].

Remark. Note that in 1] the Schwartz kernel of \widetilde{P} may have singularities at off-diagonal points $(R_g(x), x)$ where $g \neq e$. It is for this reason that, in contrast with (1.1), we can only give decaying estimates in terms of the L^2 -operator norm.

Proposition 1.4 gives a general result for operators on \widetilde{M} associated to \mathcal{B}^{∞} -pseudo-differential operators. For operators manufactured out of Dirac-type laplacians it is possible to improve such a result by using finite propagation speed estimates. We shall only need the following:

Proposition 1.5. *For any $\alpha \in \mathbb{R}$ the operator $(1 + \widetilde{D}^2)^{\alpha}$ on the covering \widetilde{M} has a Schwartz kernel which is smooth outside the diagonal and satisfies the same decay estimates for $d(x, y) \rightarrow +\infty$ as in Proposition 1.1, formula (1.1). Considering the associated operator on M acting on sections of \mathcal{E}^{∞} we get an invertible operator $(\text{Id} + \mathcal{D}^2)^{\alpha} \in \Psi_{B^{\infty}}^{2\alpha}(M, \mathcal{E}^{\infty})$.*

Proof. We set $f(z) = (1 + z^2)^\alpha$, for any real z . Since the Fourier transform $\hat{f}(r)$ is defined by an oscillatory integral, $\hat{f}(r)$ is smooth on $\mathbb{R} \setminus \{0\}$ and its derivatives are of rapid decay as $|r| \rightarrow +\infty$. We can write:

$$(1 + \tilde{D}^2)^\alpha = \int_0^{+\infty} \hat{f}(r) \cos(r|\tilde{D}|) dr$$

Let us recall that by the finite speed propagation property, for any $y \in \tilde{M}$ and any $r \geq 0$, the support of the distribution kernel $\cos(r|\tilde{D}|)(x, y)$ is contained in the ball $B(y, r)$. Let $\varepsilon > 0$, and $\phi(r) \in C^\infty(\mathbb{R}, [0, 1])$ be an even function such that $\phi(r) = 0$ if $|r| < \varepsilon$, $\phi(r) = 1$ if $|r| > 2\varepsilon$. So, if $d(x, y) > 2\varepsilon$ we have:

$$K_{(1+\tilde{D}^2)^\alpha}(x, y) = \int_0^{+\infty} \hat{f}(r) \phi(r) \cos(r|\tilde{D}|)(x, y) dr$$

By construction $\hat{f}(r)\phi(r)$ is the Fourier transform of a function $h(z)$ in the Schwartz space $\mathcal{S}(\mathbb{R}, \mathbb{R})$. Thanks to finite propagation speed properties it is well known that the Schwartz kernel of $h(\tilde{D})$ is of rapid decay as $d(x, y) \rightarrow +\infty$ [CGT] and since

$$K_{(1+\tilde{D}^2)^\alpha}(x, y) = K_{h(\tilde{D})}(x, y) \text{ for } d(x, y) > 2\varepsilon$$

the proposition follows.

Proposition 1.6.

- 1] Let \mathcal{P} be a pseudo-differential operator $\in \Psi_{B^\infty}^0(M, \mathcal{E}^\infty)$ such that, as a bounded operator on the Λ -Hilbert module $L^2(M, \mathcal{E})$, $\|\mathcal{P}\| < \frac{1}{2}$. Then $(\text{Id} - \mathcal{P})^{-1}$ sends continuously $C^\infty(M, \mathcal{E}^\infty)$ into itself.
- 2] Let $\mathcal{Q} \in \Psi_{B^\infty}^m(M, \mathcal{E}^\infty)$ be a pseudo-differential operator of order m which is invertible in the Λ -calculus. Then \mathcal{Q}^{-1} belongs to the space $\Psi_{B^\infty}^{-m}(M, \mathcal{E}^\infty)$.

Proof. 1]. We will use notations and results from Proposition 1.4. Since $\|\mathcal{P}\| < \frac{1}{2}$, we can claim that (see [Pa] page 449):

$$\forall u \in L^2(M, \mathcal{E}), \quad \langle \mathcal{P}(u), \mathcal{P}(u) \rangle \leq \frac{1}{2} \langle u, u \rangle$$

where this equality has meaning in $\Lambda = C_r^*(\Gamma)$. Using the trace of Λ we see therefore that for any u in the Schwartz space $\mathcal{S}(\tilde{M}, \tilde{\mathcal{E}})$:

$$\|\tilde{\mathcal{P}}(u)\|_{L^2(\tilde{M}, \tilde{\mathcal{E}})} \leq \frac{1}{2} \|u\|_{L^2(\tilde{M}, \tilde{\mathcal{E}})} \tag{1.11}$$

Let us fix $u \in \mathcal{S}(\widetilde{M}, \widetilde{E})$, we are going to prove that $\sum_{k \geq 0} \widetilde{P}^k u$ belongs to $\mathcal{S}(\widetilde{M}, \widetilde{E})$ which will prove 1].

Let A be the lift to \widetilde{M} of a differential operator of order one with diagonal principal symbol acting on $C^\infty(\widetilde{M}, \widetilde{E})$. Let us fix $N \in \mathbb{N}^*$. Then $(A^N \circ \widetilde{P}^k)u(x_0)$ is the sum of at most $(k + 2)^{(N+1)}$ terms of the following type:

$$\int_{\widetilde{M}^k} B_1(x_0, x_1) B_2(x_1, x_2) \dots B_k(x_{k-1}, x_k) (A^l u)(x_k) dx_k \dots dx_1 \quad (1.12)$$

where $l \leq N$, $(k - N) + l$ of the B_j are equal to \widetilde{P} , and the other B_j are the Schwartz kernels of iterated commutators of A with \widetilde{P} .

So we can find a constant $C_N(u)$ which does not depend on k so that:

$$\forall k \in \mathbb{N}, \quad \|(A^N \circ \widetilde{P}^k)u\|_{L^2} \leq C_N(u) 2^{N+1-k} (k + 2)^{N+1}$$

Thus $x_0 \rightarrow \sum_{k \geq 0} \widetilde{P}^k u(x_0)$ is C^∞ on \widetilde{M} . Let us fix $x \in \widetilde{M}$. Let $x_0 \in \widetilde{M}$ be such that $d(x, x_0) \geq 2$. We set $k_0 = [\sqrt{d(x, x_0)}]$. Since \widetilde{M} has bounded geometry we see, using Sobolev injection's theorem that:

$$\left| \sum_{k \geq k_0} (A^N \circ \widetilde{P}^k u)(x_0) \right| \leq C'_N(u) \left(\frac{3}{4}\right)^{k_0} \leq C''_N(u) \left(\frac{3}{4}\right)^{\sqrt{d(x, x_0)}} \quad (1.13)$$

Now let us consider $A^N \circ \widetilde{P}^k u(x_0)$ for $k \in \{0, 1, \dots, k_0 - 1\}$ and a particular term of the type (1.12). Using the fact u is of Schwartz class, we see that

$$\int_{d(x, x_0) \leq 10 d(x, x_k)} B_1(x_0, x_1) B_2(x_1, x_2) \dots B_k(x_{k-1}, x_k) (A^l u)(x_k) dx_k \dots dx_1$$

satisfies the required decay estimates. Now we recall (see (1.11)) that the L^2 -operator norm of \widetilde{P} is $< \frac{1}{2}$ and that each B_j in (1.12) satisfies the estimates (1.9) of Proposition 1.4. We are going to show that if in the following integral

$$\int_{d(x, x_0) > 10 d(x, x_k)} B_1(x_0, x_1) B_2(x_1, x_2) \dots B_k(x_{k-1}, x_k) (A^l u)(x_k) dx_k \dots dx_1$$

we replace each $B_j(x_{j-1}, x_j)$ by

$$\left[1 - \chi(d(x_{j-1}, x_j) - R) \right] B_j(x_{j-1}, x_j), \quad \text{with } R + 1 \leq \frac{\sqrt{d(x, x_0)}}{2}$$

then we will *get zero*; since all the kernels B_j satisfy estimates (1.9) of proposition 1.4 and at least $k - N - 1$ of them are equal to \tilde{P} and thus have an L^2 -operator norm $< \frac{1}{2}$, *this will prove part 1]* of the lemma. We observe that:

$$d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k) \geq d(x_0, x) - d(x, x_k)$$

So if $d(x, x_0) > 10d(x, x_k)$, we see that there exists $j \in \{1, \dots, k\}$ such that:

$$d(x_{j-1}, x_j) \geq \frac{9d(x, x_0)}{10k} \geq \frac{9d(x, x_0)}{10k_0} \geq \frac{\sqrt{d(x_0, x)}}{2}$$

Since $\chi(z) \equiv 0$ for $z \geq 1$, we see that the integral in question is indeed zero. Using inequality (1.13) we see therefore that $A^N \sum_{k \geq 0} \tilde{P}^k u(x_0)$ is of rapid decay and 1] is proven.

2] First we show that \mathcal{Q}^{-1} sends $C^\infty(M, \mathcal{E}^\infty)$ into itself. Let us consider $\mathcal{C} \in \Psi_{\mathcal{B}^\infty}^{-m}(M, \mathcal{E}^\infty)$ be such that $\mathcal{R} = \mathcal{Q}^{-1} - \mathcal{C} \in \Psi_{\Lambda}^{-m}(M, \mathcal{E})$ is very small so that $\mathcal{Q} \circ \mathcal{R} = \text{Id} - \mathcal{Q} \circ \mathcal{C}$ and $\mathcal{R} \circ \mathcal{Q} = \text{Id} - \mathcal{C} \circ \mathcal{Q}$ are bounded operators on $L^2(M, \mathcal{E})$ with small operator norms:

$$\|\mathcal{Q} \circ \mathcal{R}\| < \frac{1}{2}, \quad \|\mathcal{R} \circ \mathcal{Q}\| < \frac{1}{2}$$

Then, according to 1], $(\mathcal{Q} \circ \mathcal{C})^{-1} = (\text{Id} - \mathcal{Q} \circ \mathcal{R})^{-1}$ and $(\mathcal{C} \circ \mathcal{Q})^{-1} = (\text{Id} - \mathcal{R} \circ \mathcal{Q})^{-1}$ sends $C^\infty(M, \mathcal{E}^\infty)$ into itself. Then the same is true for $\mathcal{C} \circ (\mathcal{Q} \circ \mathcal{C})^{-1}$ and $(\mathcal{C} \circ \mathcal{Q})^{-1} \circ \mathcal{C}$ which are both equal to \mathcal{Q}^{-1} . Thus \mathcal{Q}^{-1} sends $C^\infty(M, \mathcal{E}^\infty)$ into itself. Next we show that \mathcal{Q}^{-1} belongs to $\Psi_{\mathcal{B}^\infty}^{-m}(M, \mathcal{E}^\infty)$. There exists a \mathcal{B}^∞ parametrix $\mathcal{G} \in \Psi_{\mathcal{B}^\infty}^{-m}(M, \mathcal{E}^\infty)$ such that

$$\text{Id} - \mathcal{Q} \circ \mathcal{G} = \mathcal{R}' \text{ where } \mathcal{R}' \in \text{End}_{\mathcal{B}^\infty}(\mathcal{E}^\infty, \mathcal{E}^\infty).$$

So $\mathcal{Q}^{-1} = \mathcal{G} + \mathcal{Q}^{-1} \circ \mathcal{R}'$. As we have just seen \mathcal{Q}^{-1} sends $C^\infty(M, \mathcal{E}^\infty)$ into itself; by applying Proposition 1.4 2] we see that $\mathcal{Q}^{-1} \circ \mathcal{R}'$ is smoothing in the \mathcal{B}^∞ -calculus and the Proposition follows.

2. Higher eta invariants.

In this section we consider a compact connected boundaryless Riemannian manifold N and $\pi : \tilde{N} \rightarrow N$ a Galois Γ -covering. We assume that the discrete group Γ is finitely presented and virtually nilpotent. Here, the dimension of N may be either even or odd. In Section 14 we shall use the higher-eta invariants for N equal to the *odd* dimensional *boundary* of an even dimensional manifold. In Section 13, on the other hand, we shall use the superconnection formalism recalled here in the *even* case.

Let E be a hermitian Clifford module over N endowed with a unitary Clifford connection. Let D be the associated Dirac operator. If N is even-dimensional then both E and D are assumed to be \mathbb{Z}_2 -graded. We denote by Υ the grading operator; thus $\Upsilon^2 = \text{Id}$ and $E^\pm = \ker(\Upsilon \pm \text{Id})$.

If N is odd dimensional then we consider the bundle of \mathbb{C} -vector spaces $E_\sigma = E \otimes \text{Cl}(1)$. Here $\text{Cl}(1)$ is the complex Clifford algebra of \mathbb{C} , it is generated, as a complex vector space, by 1 and σ with $\sigma^2 = 1$. There is a natural bundle isomorphism $E_\sigma \rightarrow E \oplus E$ under which σ becomes the matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let z be a base point of N ; for any endomorphism u of $(E_\sigma)_z$ we define a linear functional $\text{Str}_{\text{Cl}(1)} : \text{End}((E_\sigma)_z) \rightarrow \mathbb{C}$ by setting

$$\text{Str}_{\text{Cl}(1)} u = \frac{1}{2} \text{Str} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ u \quad (2.1)$$

If u is $\text{Cl}(1)$ -right linear, then u is of the form

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} + \sigma \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix}$$

so that $\text{Str}_{\text{Cl}(1)} u = \text{Tr } u_2$ as in [Q].

Let \tilde{D} be the associated lifted Dirac operator on the covering \tilde{N} acting on the sections of the lifted bundle $\tilde{E} = \pi^*(E) \rightarrow \tilde{N}$. We denote by \tilde{E}_σ the lift of E_σ on the covering.

We will assume that the L^2 -spectrum of \tilde{D} has a gap at zero; thus there is a $\delta > 0$ such that

$$\text{spec}(\tilde{D}) \cap]-\delta, \delta[= \{0\} \quad (2.2)$$

Our first goal is to define the higher-eta invariant associated to \tilde{D} , thus extending Lott's original construction which was only valid under the assumption \tilde{D} invertible ([L2]). That such an extension should exist was already remarked in [L2].

We first recall a few results and definitions of [L 1,2].

We set $\widehat{\Omega}_0(\mathcal{B}^\infty) = \mathcal{B}^\infty$. As already remarked the neutral element $e \in \Gamma$ is the unit element of the algebra \mathcal{B}^∞ .

For each $k \in \mathbb{N}^*$, $\widehat{\Omega}_k(\mathcal{B}^\infty)$ is the set of the functions $a : \Gamma \times (\Gamma \setminus \{e\})^k \rightarrow \mathbb{C}$:

$$(\gamma_0, \gamma_1, \dots, \gamma_k) \rightarrow a_{\gamma_0, \gamma_1, \dots, \gamma_k}$$

such that for any $M \in \mathbb{N}$:

$$\sup_{\gamma_1, \dots, \gamma_k \in \Gamma \setminus \{e\}} \sup_{\gamma_0 \in \Gamma} |a_{\gamma_0, \gamma_1, \dots, \gamma_k}| (|\gamma_0| + |\gamma_1| + \dots + |\gamma_k|)^M < \infty$$

We will view the elements of the \mathbb{C} -Frechet vector space $\widehat{\Omega}_k(\mathcal{B}^\infty)$ as homogeneous differential forms of degree k , thus identify a with:

$$\sum_{\gamma_1, \dots, \gamma_k \in \Gamma \setminus \{e\}} \sum_{\gamma_0 \in \Gamma} a_{\gamma_0, \gamma_1, \dots, \gamma_k} \gamma_0 d\gamma_1 \dots d\gamma_k \quad (2.3)$$

$\gamma_0 d\gamma_1 \dots d\gamma_k$ being an abstract symbol. Moreover we set $de = 0$, and, by convention, $d\gamma_1 \dots d\gamma_k = 0$ if at least one of the γ_j is equal to e . Let us consider the graded vector space

$$\widehat{\Omega}_*(\mathcal{B}^\infty) = \prod_{k \geq 0} \widehat{\Omega}_k(\mathcal{B}^\infty)$$

We define the product of two homogeneous forms of $\widehat{\Omega}_*(\mathcal{B}^\infty)$ by setting:

$$\begin{aligned} & (\gamma_0 d\gamma_1 \dots d\gamma_k) \cdot \gamma'_{k+1} d\gamma'_{k+2} \dots d\gamma'_{k+m} = \\ & \sum_{j=1}^{k-1} (-1)^{k-j} \gamma_0 d\gamma_1 \dots d(\gamma_j \gamma_{j+1}) \dots d\gamma'_{k+1} d\gamma'_{k+2} \dots d\gamma'_{k+m} \\ & + \gamma_0 d\gamma_1 \dots d\gamma_{k-1} d(\gamma_k \cdot \gamma'_{k+1}) d\gamma'_{k+2} \dots d\gamma'_{k+m} + \\ & (-1)^k \gamma_0 \gamma_1 d\gamma_2 \dots d\gamma'_{k+1} d\gamma'_{k+2} \dots d\gamma'_{k+m} \end{aligned}$$

Since Γ is virtually nilpotent, this product allows to define on $\widehat{\Omega}_*(\mathcal{B}^\infty) = \widehat{\Omega}_{\text{even}}(\mathcal{B}^\infty) \oplus \widehat{\Omega}_{\text{odd}}(\mathcal{B}^\infty)$ a structure of \mathbb{Z}_2 -graded Frechet algebra with unit element e . In particular, $\widehat{\Omega}_*(\mathcal{B}^\infty)$ is a \mathcal{B}^∞ -bimodule. By definition a sequence $(\omega^p)_{p \in \mathbb{N}}$ of forms tends to zero if each component of degree k of ω^p tends to zero in $\widehat{\Omega}_k(\mathcal{B}^\infty)$.

Moreover we *define* on $\widehat{\Omega}_*(\mathcal{B}^\infty)$ a graded differential d by *defining* its values on the elements e, γ_0 and $\gamma_0 d\gamma_1 d\gamma_2 \dots d\gamma_k$ which appear in (2.3) in the following way and with an obvious abuse of notations:

$$de = 0, \quad d\gamma_0 = ed\gamma_0 \quad d(\gamma_0 d\gamma_1 d\gamma_2 \dots d\gamma_k) = e d\gamma_0 d\gamma_1 \dots d\gamma_k$$

We then have $d \circ d = 0$, and for any homogeneous form ω of degree $\partial\omega$ we have $d(\omega \cdot \omega') = d\omega \cdot \omega' + (-1)^{\partial\omega} \omega \cdot d\omega'$. Now we consider the graded vector space:

$$\overline{\widehat{\Omega}}_*(\mathcal{B}^\infty) = \prod_{k \geq 0} \frac{\widehat{\Omega}_k(\mathcal{B}^\infty)}{\widehat{\Omega}_k(\mathcal{B}^\infty) \cap [\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]_t^-}$$

where $[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]_t^-$ is the closure of the spaces generated by the *graded* commutators: $\omega\omega' - (-1)^{\partial\omega\partial\omega'} \omega'\omega$. The differential d induces a differential of $\overline{\widehat{\Omega}}_*(\mathcal{B}^\infty)$ sending homogeneous forms of degree k into forms of degree $k+1$. The corresponding homology is called the topological non-commutative de Rham homology; it pairs (for positive degrees) with the topological cyclic cohomology of \mathcal{B}^∞ (see Karoubi [Ka]).

Recall that $\mathcal{S}(\widetilde{N}, \widetilde{E})$ is a right \mathcal{B}^∞ -module where the right \mathcal{B}^∞ -action on a section f is given by $f \cdot \gamma = R_{\gamma^{-1}}^*(f)$. An easy extension of Proposition 1.1 allows us to show the existence of a natural isomorphism of Frechet spaces:

$$\mathcal{S}(\widetilde{N}, \widetilde{E}) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_k(\mathcal{B}^\infty) \rightarrow C^\infty(N, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_k(\mathcal{B}^\infty))$$

$$\sum_{\gamma_1, \dots, \gamma_k \in \Gamma} f_{\gamma_1, \dots, \gamma_k} \otimes ed\gamma_1 \dots d\gamma_k \rightarrow \sum_{\gamma_0, \gamma_1, \dots, \gamma_k \in \Gamma} R_{\gamma_0}^*(f_{\gamma_1, \dots, \gamma_k}) \gamma_0 d\gamma_1 \dots d\gamma_k \tag{2.4}$$

where, for any fixed point $z_0 \in \tilde{N}$, the $f_{\gamma_1, \dots, \gamma_k}(z)$ above belong to the Schwartz space $\mathcal{S}(\tilde{N}, \tilde{E})$ and are of rapid decay, together with all their covariant derivatives, with respect to $d(z, z_0) + \|\gamma_1\| + \dots + \|\gamma_k\|$.

We recall (cf formulas (1.2), (1.3)) that \tilde{D} can also be seen as the operator associated on the covering to a differential operator \mathcal{D}_∞ in the \mathcal{B}^∞ -Mishenko-Fomenko calculus acting on $C^\infty(N, \mathcal{E}^\infty)$.

In the even case, we recall that the \mathbb{Z}_2 -grading of $C^\infty(N, \mathcal{E}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)$ is defined as follows: the (total) degree of $u \otimes \alpha$ is the sum of the degrees of u and α where u is a section of \mathcal{E}^∞ and $\alpha \in \widehat{\Omega}_*(\mathcal{B}^\infty)$.

In the odd case we shall use the \mathcal{B}^∞ -bundle $\mathcal{E}_\sigma^\infty \rightarrow N$; to any endomorphism u of \mathcal{E}^∞ we associate the endomorphism of $\mathcal{E}_\sigma^\infty$ given by $u \oplus u$.

Now we define Lott's connection and the corresponding superconnection.

Definition 2.1. *Let $h \in C_0^\infty(\tilde{N})$ be such that $\sum_{\gamma \in \Gamma} R_\gamma^*(h) \equiv 1$.*

(i) *We define a connection by setting for any $f \in S(\tilde{N}, \tilde{E})$:*

$$\begin{aligned} \nabla : C^\infty(N, \mathcal{E}^\infty) &\rightarrow C^\infty(N, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_1(\mathcal{B}^\infty)) \\ \nabla(\sum_{\gamma \in \Gamma} R_\gamma^*(f)\gamma) &= \sum_{\gamma', \gamma \in \Gamma} R_{\gamma'}^*(h) R_{\gamma'\gamma}^*(f) \gamma' \otimes_{\mathcal{B}^\infty} d\gamma \end{aligned}$$

or, in a more compact way: $\nabla f = \sum_{\gamma \in \Gamma} h R_\gamma^*(f) d\gamma$.

(ii) *If $\dim N$ is even and Υ is the grading operator, then for any real $s > 0$ the superconnection D_s is defined to be $\Upsilon \nabla + s\mathcal{D}_\infty$. Thus*

$$\forall \xi \in C^\infty(N, \mathcal{E}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty), \forall \alpha \in \widehat{\Omega}_*(\mathcal{B}^\infty),$$

$$D_s(\xi\alpha) = D_s(\xi)\alpha + (-1)^{\partial \xi} \xi d\alpha$$

where $\partial \xi$ is the (total) degree of ξ . Notice that \mathcal{D}_∞ is $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -right linear.

(iii) *If $\dim N$ is odd, the superconnection D_s is defined to be $\Upsilon \nabla + s\sigma\mathcal{D}_\infty$ where Υ is the obvious \mathbb{Z}_2 -grading of $\mathcal{E}_\sigma^\infty = \mathcal{E} \oplus \mathcal{E}$.*

Remark. As pointed out by Connes in [C] page 434, these supersign rules differ slightly from that of Quillen [Q] because we are dealing with *right*-modules.

It is important to point out that it is only with these supersign rules and with this definition of D_s that the curvature operator D_s^2 becomes $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -right-linear. In the even case for instance, for any $f \in S(\widetilde{N}, \widetilde{E})$, $D_s^2(\sum_{\gamma \in \Gamma} R_\gamma^*(f)\gamma)$ is given under the correspondence (2.4) by:

$$\nabla^2(f) + s(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla)(f) + s^2 \widetilde{D}^2(f)$$

where we recall the two following formulas:

$$(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla)f = \sum_{\gamma \in \Gamma} -\Upsilon[\widetilde{D}, h] R_\gamma^*(f) d\gamma \tag{2.5}$$

$$\nabla^2(f) = \sum_{\gamma, \gamma' \in \Gamma} h R_\gamma^*(h) R_{\gamma\gamma'}^*(f) d\gamma d\gamma' \tag{2.6}$$

Similar formulas hold in the odd case.

Now we consider the following $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -bundle over N :

$$\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty) = E \otimes_{\mathbb{C}} (\widetilde{M} \times_{\Gamma} \mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)$$

We can find a finite open cover $\mathcal{U} = \{U_j, 1 \leq j \leq q\}$ of N and associated trivializations of this bundle:

$$M_j : (\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))|_{U_j} \rightarrow U_j \times (\mathbb{C}^{\dim E} \otimes_{\mathbb{C}} \widehat{\Omega}_*(\mathcal{B}^\infty))$$

where M_j is a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear bundle isomorphism. Next we consider a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear continuous endomorphism \mathcal{K} of $C^\infty(N, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ defined by a distribution Schwartz kernel:

$$\mathcal{K}(z, z') \in \text{Hom}_{\widehat{\Omega}_*(\mathcal{B}^\infty)}[\mathcal{E}_{z'}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty), \mathcal{E}_z^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)].$$

Thus for any section u of $\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)$ we can write, with a common abuse of notation,

$$\forall z \in N, \mathcal{K}(u)(z) = \int_N \mathcal{K}(z, z') u(z') dg(z')$$

where $dg(z')$ is the riemannian density. We are going to describe locally this Schwartz kernel. Let U_i, U_j be two open subsets of \mathcal{U} . Then we can find a finite number of distributions on $U_i \times U_j : (z, z') \rightarrow \omega_l(z, z'), 1 \leq l \leq m$ with

values in $\widehat{\Omega}_*(\mathcal{B}^\infty)$ and also endomorphisms $A_l \in \text{End}(\mathbb{C}^{\dim E})$, $1 \leq l \leq m$, such that for any vector $s \in \mathbb{C}^{\dim E}$ and for any $\alpha \in C^\infty_{\text{comp}}(U_i, \widehat{\Omega}_*(\mathcal{B}^\infty))$ we have:

$$[\mathcal{K} \circ M_i^{-1}(s \otimes \alpha)]|_{U_j} = M_j^{-1} \left[\sum_{l=1}^m A_l(s) \otimes \left(\int_{U_i} \omega_l(z, z') \alpha(z') dg(z') \right) \right] \quad (2.7)$$

By definition \mathcal{K} is a smoothing operator if all the $\omega_l(z, z')$ are C^∞ functions with values in $\widehat{\Omega}_*(\mathcal{B}^\infty)$.

Now let us consider the important case where \mathcal{K} sends sections of \mathcal{E}^∞ into sections of $\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_k(\mathcal{B}^\infty)$ for a fixed k . Proceeding as in [L1] we see that \mathcal{K} corresponds to a *distribution* Schwartz kernel $(z, z') \rightarrow \widetilde{K}(z, z')$ on $\widetilde{N} \times \widetilde{N}$ of the form:

$$\widetilde{K}(z, z') = \sum_{\gamma_1, \dots, \gamma_k} \widetilde{K}_{\gamma_1, \dots, \gamma_k}(z, z') d\gamma_1 \dots d\gamma_k \quad (2.8)$$

such that $\forall f \in \mathcal{S}(\widetilde{N}, \widetilde{E})$:

$$\mathcal{K} \left(\sum_{\gamma \in \Gamma} R_\gamma^*(f) \gamma \right) = \sum_{\gamma_0, \gamma_1, \dots, \gamma_k} \int_{\widetilde{N}} \widetilde{K}_{\gamma_1, \dots, \gamma_k}(z \cdot \gamma_0, z') f(z') dz' \gamma_0 d\gamma_1 \dots d\gamma_k$$

Fundamental examples of such operators on the covering are provided by $(\Upsilon \nabla + s \widetilde{D})^2$, $\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla$. It is easy to check that for any $\gamma \in \Gamma$, $\widetilde{K}(z, z' \cdot \gamma) = \widetilde{K}(z, z') \cdot \gamma$. In contrast with the case $k = 0$ treated in Proposition 1.1, we must point out that when $k \geq 1$ the $\widetilde{K}_{\gamma_1, \dots, \gamma_k}(z, z')$ are *not individually* Γ -invariant with respect to the diagonal action of Γ on $\widetilde{N} \times \widetilde{N}$. That's why, in the next definition, the statement of the decay property for *the family of smooth kernels* $\widetilde{K}_{\gamma_1, \dots, \gamma_k}$ corresponding to the smoothing property of \mathcal{K} , is not completely obvious.

Definition 2.2. Let F be a fundamental domain for the covering \widetilde{N} .

- (i) Let \mathcal{K} be the right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator introduced above and let us assume that the Schwartz kernels $\widetilde{K}_{\gamma_1, \dots, \gamma_k}$ in equation (2.8) are all C^∞ . Then we shall say that \mathcal{K} (or \widetilde{K}) satisfies the decay property (DP) if for any $M \in \mathbb{N}$ and any multi-index of derivation α with respect to (z, z') the supremum $C(\mathcal{K}, M, \alpha)$ of the set:

$$\left\{ \left[d(z, F) + \|\gamma_1\| + \dots + \|\gamma_{k-1}\| + d(z', R_{\gamma_k}(F)) \right]^M |\nabla^\alpha \widetilde{K}_{\gamma_1, \dots, \gamma_k}(z, z')|, \right.$$

such that $\gamma_1, \dots, \gamma_k \in \Gamma, (z, z') \in \tilde{N}^2\}$

is finite.

- (ii) Let $s \rightarrow F(s)$ be a positive function on \mathbb{R}^{+*} . Let $(\mathcal{K}(s))_{s \in \mathbb{R}^{+*}}$ be a family of right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operators as in (i) above. We shall say that the $\mathcal{K}(s)$ (or $\tilde{K}(s)$) satisfy the decay property (DP) with respect to $F(s)$, s being > 0 , if for any $M \in \mathbb{N}$ and any multi-index of derivation α with respect to (z, z') we can find a constant $D(M, \alpha)$ such that:

$$\forall s > 0, \quad C(\mathcal{K}(s), M, \alpha) \leq D(M, \alpha) F(s)$$

Let $d(x, y)$ denote the geodesic distance on \tilde{N} associated with the lifted Riemannian metric. We recall that \tilde{N} and the virtually nilpotent group Γ are quasi-isometric. Using the arguments in [L 1,2] it is not difficult to prove the following proposition. We leave the cumbersome details to the interested reader.

Proposition 2.3. *A right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator sending the sections of \mathcal{E}^∞ into those of $\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_m(\mathcal{B}^\infty)$ is smoothing in the \mathcal{B}^∞ -calculus if and only if the $\tilde{K}_{\gamma_1, \dots, \gamma_k}$ of (2.8) are smooth and satisfy the decay property (DP).*

The next result is essentially a corollary of Proposition 2.3; alternatively a *direct* proof on the covering \tilde{N} can be given.

Proposition 2.4. *Let $(\mathcal{K}(s))_{s > 0}$ be a family of right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operators, as in Definition 2.2, satisfying property (DP) uniformly with respect to $F(s)$, s being > 0 . Let \mathcal{A} be a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator sending continuously the sections of \mathcal{E}^∞ into those of $\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_m(\mathcal{B}^\infty)$. Then the operators $\mathcal{K}(s) \circ \mathcal{A}, \mathcal{A} \circ \mathcal{K}(s)$ will also satisfy property (DP) uniformly with respect to $F(s)$, s being > 0 , **provided** the operator \tilde{A} associated to \mathcal{A} on the covering fulfills at least one of the following assumptions:*

- (i) \tilde{A} satisfies property (DP).
- (ii) \tilde{A} belongs to $\{\Upsilon \nabla \tilde{D} + \tilde{D} \Upsilon \nabla, \nabla^2\}$.
- (iii) \tilde{A} is a Γ -invariant pseudo-differential operator of **any order** acting on $L^2(\tilde{N}, \tilde{E})$ whose Schwartz kernel $\tilde{A}(z, z')$ is C^∞ outside of the diagonal and of rapid decay (with all its derivatives) when $d(z, z') \rightarrow +\infty$.

Now, in the odd case for instance, we set for s real > 0 , $\tilde{P}_s = -\nabla^2 - s\sigma(\Upsilon \nabla \tilde{D} + \tilde{D} \Upsilon \nabla)$. We can then define the solution of heat superconnection

equation, $\exp(-D_s^2)$, as an element of $\text{End}_{\widehat{\Omega}_*(\mathcal{B}^\infty)}(N, \mathcal{E}_\sigma^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ by defining its associated operator on the covering to be:

$$\begin{aligned} \exp(-s^2\widetilde{D}^2 + \widetilde{P}_s) &= \exp(-s\widetilde{D}^2) + \\ &\int_0^1 \exp(-u_1 s^2 \widetilde{D}^2) \widetilde{P}_s \exp(-(1-u_1)s^2 \widetilde{D}^2) du_1 + \\ &\int_0^1 \int_0^{1-u_1} (\exp(-u_1 s^2 \widetilde{D}^2) \widetilde{P}_s \exp(-u_2 s^2 \widetilde{D}^2) \widetilde{P}_s \\ &\exp(-(1-u_1-u_2)s^2 \widetilde{D}^2)) du_2 du_1 + \dots \end{aligned} \quad (2.9)$$

Using formulas (2.5) and (2.6) we see that for each $k \in \mathbb{N}$, only a finite number of terms in the expansion (2.9) will give a contribution in $\widehat{\Omega}_k(\mathcal{B}^\infty)$. Using finite propagation speed estimates for $\exp(-s\widetilde{D}^2)(z, z')$ and Proposition 2.4 we see easily that all these terms satisfy property (DP).

Now we define the supertraces in our context. First let us assume that N is even-dimensional. Let \mathcal{K} be a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator satisfying the (DP) property as in Definition 2.2. Let $x \in N$, we use the notations of equation (2.7) with $U_i = U_j$ being a neighborhood of x . Let Υ denote the grading of $\mathbb{C}^{\dim E} \simeq E_x$. We then define the supertrace $\text{Str}\mathcal{K}(x, x)$ to be:

$$\text{Str}\mathcal{K}(x, x) = \sum_{l=1}^m \text{Str}(\Upsilon^{\partial\omega_l} A_l) \omega_l(x, x) \quad (2.10)$$

$\text{Str}\mathcal{K}(x, x)$ is intrinsically defined modulo the closure $[\mathcal{B}^\infty, \widehat{\Omega}_*(\mathcal{B}^\infty)]^-$ but in fact we will view $\text{Str}\mathcal{K}(x, x)$ as an element of $\widetilde{\Omega}_*(\mathcal{B}^\infty)$ (i.e. as a differential form modulo graded commutators).

Let us now consider the case where N is odd dimensional. Let \mathcal{K} be a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator acting on $C^\infty(N, \mathcal{E}_\sigma^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ and satisfying the (DP) property. We do not assume that \mathcal{K} is σ -linear, for instance the superconnection operator which appears in Lemma 3.1 is not σ -linear because of the grading Υ .

Let $x \in N$; of course we use the notations of equation (2.7) with $U_i = U_j$ a neighborhood of x . We set:

$$\text{Str}_{\text{Cl}(1)}(\mathcal{K}(x, x)) = \sum_{l=1}^m \text{Str}_{\text{Cl}(1)}((-\Upsilon)^{\partial\omega_l} A_l) \omega_l(x, x) \quad (2.11)$$

where Υ denotes the grading of $\mathbb{C}_\sigma^{\dim E}$; this differential form is intrinsically defined in $\overline{\Omega}_*(\mathcal{B}^\infty)$. The sign minus in (2.11) will provide a definition of the higher-eta invariant so that Proposition 14.2 holds true.

In the even [resp odd] case we define the supertrace $\text{STR } \mathcal{K}$ [resp. σ -supertrace $\text{STR}_{\text{Cl}(1)} \mathcal{K}$] as:

$$\text{STR } \mathcal{K} = \int_N \text{Str } \mathcal{K}(x, x) dg(x), \quad \text{STR}_{\text{Cl}(1)} \mathcal{K} = \int_N \text{Str}_{\text{Cl}(1)} \mathcal{K}(x, x) dg(x)$$

where $dg(x)$ is the Riemannian density of N . Both $\text{STR } \mathcal{K}$ and $\text{STR}_\sigma \mathcal{K}$ belong to $\overline{\Omega}_*(\mathcal{B}^\infty)$.

Now we recall the following formula of [L1] for the supertrace.

Proposition 2.5. *Let $\Phi \in C_0^\infty(\tilde{N})$ be such that $\sum_{\gamma \in \Gamma} R_\gamma^*(\Phi) \equiv 1$. Let \mathcal{K} be a right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator satisfying the decay property (DP) with the notations of Definition 2.2. Then:*

1] *In the even case, the supertrace of \mathcal{K} is the element of $\overline{\Omega}_*(\mathcal{B}^\infty)$ given by:*

$$\text{STR } \mathcal{K} = \sum_{\gamma_0, \gamma_1, \dots, \gamma_k \in \Gamma} \int_{\tilde{M}} \Phi(z) \text{Str } \tilde{K}_{\gamma_1, \dots, \gamma_k}(z\gamma_0, z) dz \gamma_0 d\gamma_1 \dots d\gamma_k$$

Moreover the supertrace of a supercommutator is zero and $d \text{STR } \mathcal{K} = \text{STR} [D_s, \mathcal{K}]$ where D_s is the superconnection of Definition 2.1.

2] In the odd case, $\text{STR}_{\text{Cl}(1)} \mathcal{K}$ is given by replacing Str by $\text{Str}_{\text{Cl}(1)}$ in the previous formula.

Remark. It is precisely because of property (DP) that the coefficients $a_{\gamma_0, \dots, \gamma_k}$ of these supertraces or σ -supertraces are of rapid decay with respect to $\|\gamma_0\| + \dots + \|\gamma_k\|$. The proof of the equality $d \text{STR } \mathcal{K} = \text{STR} [D_s, \mathcal{K}]$ is easy and left to the reader.

We have assumed that there is $\delta > 0$ so that $L^2 - \text{spec}(\tilde{D}^2) \cap]-\delta, \delta[= \{0\}$. Let $\chi(x) \in C_0^\infty(\mathbb{R}, [0, 1])$ be such that $\chi(x) \equiv 1$ for $|x| < \frac{\delta}{2}$, $\chi(x) \equiv 0$ for $|x| > \frac{3\delta}{4}$. The orthogonal projection P_0 onto the null space of \tilde{D}^2 is given by $P_0 = \chi(\tilde{D}^2)$. Using finite propagation speed estimates we see that P_0 is a smoothing Γ -invariant operator whose Schwartz kernel is rapidly decreasing.

Now we set $\tau(x) = \frac{1-\chi(x)}{x} \in C^\infty(\mathbb{R}, \mathbb{R})$.

Proposition 2.6.

1] We define the Green operator to be $\tilde{G} = \tau(\tilde{D}^2)$. It sends the Schwartz space $S(\tilde{N}, \tilde{E})$ into itself. We have $L^2(\tilde{N}, \tilde{E}) = \text{null}(\tilde{D}^2) \oplus^\perp \text{Im } \tilde{D}^2$. Moreover $\tilde{G} : \text{Im } \tilde{D}^2 \rightarrow \text{Im } \tilde{D}^2$ is the inverse of \tilde{D}^2 acting on $\text{Im } \tilde{D}^2$.

2] $\text{null } \mathcal{D}_\infty$ is a finitely generated projective \mathcal{B}^∞ -module.

Proof. 1]. This proof is standard. Using finite propagation speed estimates we see that $\tau(\tilde{D}^2)$ is a Γ -invariant pseudo-differential operator of order -2 whose Schwartz kernel $\tau(\tilde{D}^2)(z, z')$ is C^∞ outside the diagonal of \tilde{N}^2 and of rapid decay when $d(z, z') \rightarrow +\infty$. For all $u \in S(\tilde{N}, \tilde{E})$, we can write:

$$u = P_0(u) + \tilde{D}^2 \circ \tilde{G} \circ (\text{Id} - P_0)(u)$$

Since $\tilde{G} \circ (\text{Id} - P_0) = \tilde{G}$, we get 1] immediately.

2]. Theorem 1.3 shows the existence of the following decompositions for \mathcal{D}_∞ :

$$C^\infty(N, \mathcal{E}^\infty) = \mathcal{L}_\infty \oplus^\perp \mathcal{L}_\infty^\perp \rightarrow C^\infty(N, \mathcal{E}^\infty) = \mathcal{N}_\infty \oplus \mathcal{D}_\infty(\mathcal{L}_\infty^\perp)$$

such that $\mathcal{D}_\infty(\mathcal{L}_\infty) \subset \mathcal{N}_\infty$ where \mathcal{L}_∞ and \mathcal{N}_∞ are finitely generated sub- \mathcal{B}^∞ -modules, and $\mathcal{D}_\infty : \mathcal{L}_\infty^\perp \rightarrow \mathcal{D}_\infty(\mathcal{L}_\infty^\perp)$ is invertible with inverse \mathcal{G} . Moreover the two projections $P_{\mathcal{L}_\infty}$, $P_{\mathcal{N}_\infty}$ onto \mathcal{L}_∞ , \mathcal{N}_∞ respectively are smoothing. So we have:

$$\mathcal{G} \circ (\text{Id} - P_{\mathcal{N}_\infty}) \circ \mathcal{D}_\infty = \text{Id} - P_{\mathcal{L}_\infty}$$

Applying the projection $P_{\text{null } \mathcal{D}_\infty}$ to both members of the previous equality, we get that $P_{\text{null } \mathcal{D}_\infty} = P_{\mathcal{L}_\infty} \circ P_{\text{null } \mathcal{D}_\infty}$. So $\text{null } \mathcal{D}_\infty$ is certainly finitely generated. Since the L^2 -spectrum of \tilde{D} has a gap at zero, we can argue as in the proof of 1] to check that: $\text{null } \tilde{D} \oplus^\perp \tilde{D}(S(\tilde{N}, \tilde{E})) = S(\tilde{N}, \tilde{E})$. Using Lott's correspondence between $S(\tilde{N}, \tilde{E})$ and $C^\infty(N, \mathcal{E}^\infty)$, we then get:

$$\text{null } \mathcal{D}_\infty \oplus^\perp \text{Im } \mathcal{D}_\infty = C^\infty(N, \mathcal{E}^\infty), \quad \text{null } \mathcal{D}_\infty \otimes \Lambda \oplus^\perp \overline{\text{Im } \mathcal{D}_\infty \otimes \Lambda} = L^2(N, \mathcal{E})$$

Now, Lemma 16.2 of Appendix A shows that $\text{null } \mathcal{D}_\infty$ is \mathcal{B}^∞ -projective and finitely generated. The proposition is therefore proved.

Now we state the main result of this section:

Theorem 2.7. *The following integral is absolutely convergent and defines the higher eta invariant $\tilde{\eta}$ as an element of $\widehat{\Omega}_*(\mathcal{B}^\infty)$:*

$$\text{if dim } N \text{ is even, } \tilde{\eta} = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \text{STR}[\tilde{D} \exp(-(\Upsilon \nabla + s\tilde{D})^2)] ds$$

$$\text{if dim } N \text{ is odd, } \tilde{\eta} = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \text{STR}_{\text{Cl}(1)}[\sigma \tilde{D} \exp(-(\Upsilon \nabla + s\sigma \tilde{D})^2)] ds$$

Remark. In the odd case we can pair $\tilde{\eta}$ with the trivial 0-cyclic cocycle (= evaluation at $\gamma_0 = e$). Then by Proposition 2.5 we get

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \int_{\tilde{N}} \Phi(z) \text{Tr}(\tilde{D} \exp(-t^2 \tilde{D}^2))(z, z) dz dt \equiv \frac{2}{\sqrt{\pi}} \int_0^\infty \text{Tr}_\Gamma(\tilde{D} \exp(-t^2 \tilde{D}^2)) dt.$$

This is precisely the Cheeger-Gromov Γ -eta invariant [CG] which enters in the APS Γ -index theorem of Ramachandran [R]. It should be remarked at this point that the Γ -eta invariant can be defined only assuming the group Γ finitely presented. The crucial difference between the integrand of the Γ -eta invariant and that of the *higher* eta invariant lies in the fact that in the latter case it is necessary to control the heat-kernel as $t \rightarrow +\infty$ at *arbitrarily distant points*. It is for this reason that in the higher case we assume the group virtually nilpotent.

Proof of Theorem 2.7. We will deal only with *the odd case*. The integrability near $s = 0$ is a straightforward consequence of the local index theorem as pointed out in [L2] page 219. We then study the integrability for $s \rightarrow +\infty$, extending to the present context a technique of Berline and Vergne (see [B-V]). We will work directly on the covering. Intuitively we shall show that once the use of C^k -estimates on compact manifolds is replaced by the use of the (DP) property on \tilde{N} and once finite propagation speed estimates are employed to control the heat-kernel at distant points, the large time behaviour of the *superconnection* heat-kernel can be studied by using the Berline-Vergne diagonalization lemma as in the compact (family) case. We set $P_1 = \text{Id} - P_0$; for $j = 1, 2$ we still denote by P_j be the projection acting on $S(N, \tilde{E}_\sigma)$ defined by $P_j \oplus P_j$. We obtain the decomposition:

$$S(N, \tilde{E}_\sigma) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty) = \text{Im } P_0 \otimes \widehat{\Omega}_*(\mathcal{B}^\infty) \oplus \text{Im } P_1 \otimes \widehat{\Omega}_*(\mathcal{B}^\infty) \quad (2.12)$$

In order to shorten the notations we set:

$$\tilde{C} = \Upsilon \nabla \sigma \tilde{D} + \sigma \tilde{D} \Upsilon \nabla \quad (2.13)$$

The curvature operator $\tilde{D}_s^2 = (\Upsilon \nabla + s\sigma \tilde{D})^2$ on the covering is given by the following matrix decomposition associated with (2.12):

$$\tilde{D}_s^2 = \begin{pmatrix} P_0 \nabla^2 P_0 & P_0 (\nabla^2 + s\tilde{C}) P_1 \\ P_1 (\nabla^2 + s\tilde{C}) P_0 & P_1 (\nabla^2 + s\tilde{C}) P_1 + s^2 \tilde{D}^2 P_1 \end{pmatrix}$$

Now we denote by \mathcal{A} the algebra generated by the operators:

$$\text{Id}, P_0, P_1, \tilde{C}, \nabla^2, \tilde{G}$$

Let \mathcal{M}_1 be the set of "Laurent polynomials" of degree ≤ -1 with respect to s real ≥ 1 , and with coefficient in $\mathcal{A}P_0\mathcal{A}$. Thus a generic element of \mathcal{M}_1 is if the form $\sum_{k=1}^N \frac{A_k}{s^k}$, where the A_k belong to $\mathcal{A}P_0\mathcal{A}$. Since P_0 is smoothing, Proposition 2.4 then shows that all the A_k above satisfy property (DP) of Definition 2.2.

Proceeding as in [B-V] and using the Green operator \tilde{G} , one proves easily the following lemma:

Lemma 2.8. *With respect to the decomposition (2.12) we can write:*

$$\tilde{D}_s^2 = g(s) \left(\begin{pmatrix} R & 0 \\ 0 & H(s) \end{pmatrix} + Z(s) \right) g^{-1}(s)$$

where,

$$H(s) = s^2 \tilde{D}^2 P_1 + s P_1 \tilde{C} P_1 + P_1 \nabla^2 P_1 + P_1 \tilde{C} P_0 \tilde{C} P_1 \tilde{G}$$

R is the curvature of $P_0 \nabla P_0$; $g(s)$ and $Z(s)$ are squared matrices of type (2,2) such that $g(s)$ is invertible and the three matrices $Z(s)$, $g(s) - \text{Id}$ and $g^{-1}(s) - \text{Id}$ have their coefficients in \mathcal{M}_1 .

The next theorem is crucial for the proof of Theorem 2.7

Theorem 2.9.

- 1] Let $\tilde{K}(z, z')$ be the Schwartz kernel on the covering \tilde{N} associated with a right $\hat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator \mathcal{K} satisfying property (DP) as in Definition 2.2. Then for each $k \in \mathbb{N}$, the components in $\hat{\Omega}_k(\mathcal{B}^\infty)$ of

$\exp(-H(s)) \circ \tilde{K}$ and $\tilde{K} \circ \exp(-H(s))$ all satisfy property (DP) uniformly with respect to the constant function 1, s being in $]0, 1[$.

2] For each $k \in \mathbb{N}$, we can find $\delta' > 0$ so that each component in $\hat{\Omega}_k(\mathcal{B}^\infty)$ of $\exp(s\delta') \exp(-H(s))$ satisfies property (DP) uniformly with respect to the constant function 1, s being ≥ 1 .

Proof, We begin by recalling the finite propagation speed estimates for $P_1 \circ \exp(-s^2\tilde{D}^2)$ and $\exp(-s^2\tilde{D}^2)$. Let χ be the function introduced just before Proposition 2.5 so that $\text{Id} - \chi(\tilde{D}^2)$ is the projection P_1 . Since $\chi(x) \equiv 1$ for $2|x| < \delta$, we can use the main result of [CGT] as in [L2] page 215 to show the existence of $\delta'' > 0$ so that for any $a, N \in \mathbb{N}$:

$$\forall s \geq 1, \forall x, y \in \tilde{N}, \quad |(\tilde{D}^a P_1 \exp(-s^2\tilde{D}^2))(x, y)| \leq C(a, N)(1 + d(x, y))^{-N} \exp(-s^2\delta'') \quad (2.14)$$

Now let $\varepsilon > 0$ be very small compared to the radius of injectivity of \tilde{N} . We set $R(x, y) = \max(d(x, y) - \varepsilon, 0)$. In [L2] page 215, it is shown that for any $a \in \mathbb{N}$ and any $s \in]0, 1[$:

$$\forall x, y \in \tilde{M} \text{ with } d(x, y) > 2\varepsilon, \quad |(\tilde{D}^a \exp(-s^2\tilde{D}^2))(x, y)| \leq C(a) \exp\left(-\frac{R^2(x, y)}{10s^2}\right) \quad (2.15)$$

Next we observe that since the heat kernel is Γ -invariant and almost Euclidean we have the following asymptotic expansion valid for $d(x, y) \leq 2\varepsilon$ and $0 < s < 1$:

$$\exp(-s^2\tilde{D}^2)(x, y) \sim (4\pi s^2)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{4s^2}\right) \sum_{k \geq 0} s^{2k} a_k(x, y) \quad (2.16)$$

which can be differentiated at any order.

Moreover we observe that $P_1 \exp(-s^2\tilde{D}^2) = \exp(-s^2\tilde{D}^2) - P_0$ where P_0 is a smoothing operator whose Schwartz kernel is rapidly decreasing.

Now we apply Duhamel's formula where $H(s)$ is considered as a perturbation of $s^2 P_1 \tilde{D}^2$. Defining $I(s) = (s^2 P_1 \tilde{D}^2 - H(s))$ we obtain

$$\exp(-H(s)) = \exp(-s^2 P_1 \tilde{D}^2) +$$

$$\begin{aligned}
& \int_0^1 \exp(-u_1 s^2 \tilde{D}^2 P_1) I(s) \exp(-(1-u_1) s^2 P_1 \tilde{D}^2) du_1 + \\
& \int_0^1 \int_0^{1-u_1} (\exp(-u_1 s^2 P_1 \tilde{D}^2) I(s) \exp(-u_2 s^2 P_1 \tilde{D}^2) I(s) \\
& \quad \exp(-(1-u_1-u_2) s^2 P_1 \tilde{D}^2)) du_2 du_1 + \dots \tag{2.17}
\end{aligned}$$

Now let us prove briefly 1]. The asymptotic expansion (2.16) and inequality (2.15) allow us to see that $\tilde{K} \circ \exp(-s^2 \tilde{D}^2)$ and $\exp(-s^2 \tilde{D}^2) \circ \tilde{K}$ satisfy property (DP) uniformly with respect to 1 as $s \in]0, 1]$. Using the Duhamel expansion (2.17), the definition of $H(s)$ in Lemma 2.8 and Proposition 2.4 which asserts the stability of condition (DP) under composition, we get immediately 1]. Let us prove 2]. Let us consider in expansion (2.17) *the integral over the k -simplex* associated with u_1, \dots, u_k . In *this* integral at least one of the following (nonnegative) numbers:

$$u_1, \dots, u_k, 1 - u_1 - \dots - u_k$$

will be $\geq \frac{1}{k+1}$, u_j for example. Then, estimate (2.14) allows us to see that the s -family of operators $\exp(\frac{\delta''}{k+1} s^2) \exp(-u_j s^2 \tilde{D}^2) \circ P_1$ satisfy property (DP) uniformly with respect to 1 as $s > 1$. This condition will be preserved if we compose $\exp(-u_j s^2 \tilde{D}^2)$ on the right and on the left by the operators appearing in expansion (2.17). Thus we get 2] with $\delta' = \frac{\delta''}{k+1}$.

End of the proof of Theorem 2.7

In order to shorten the next formulas we set:

$$E(s) = \begin{pmatrix} R & 0 \\ 0 & H(s) \end{pmatrix}$$

Proceeding as in the proof of lemma 14 of [B-V] we can use Lemma 2.8, Theorem 2.9 and Proposition 2.4 to write each component in $\widehat{\Omega}_k(\mathcal{B}^\infty)$ of $\exp[-E(s) - Z(s)]$ under the form:

$$\begin{pmatrix} (\exp_k R) + U_1(s) & U_3(s) \\ U_2(s) & W(s) \end{pmatrix} \tag{2.18}$$

where $(\exp_k R)$ is the component in $\widehat{\Omega}_k(\mathcal{B}^\infty)$ of $\exp R$, and the operators $U_j(s)$, $1 \leq j \leq 3$ [resp. $W(s)$] satisfy property (DP) uniformly with respect to s^{-1} [resp. s^{-2}] as $s \geq 1$.

Moreover, the uniqueness of the heat equation shows that:

$$\exp(-\tilde{D}_s^2) = g(s) \exp[-E(s) - Z(s)]g^{-1}(s)$$

Recall that according to Lemma 2.8, $g(s) - \text{Id}$ and $g^{-1}(s) - \text{Id}$ satisfy property (DP) uniformly with respect to s^{-1} as $s \geq 1$. So equation (2.18) shows that each component in $\widehat{\Omega}_k(\mathcal{B}^\infty)$ of $\exp(-\tilde{D}_s^2)$ is of the form:

$$\begin{pmatrix} (\exp_k R) + U'_1(s) & U'_3(s) \\ U'_2(s) & W'(s) \end{pmatrix}$$

where the operators $U'_j(s), 1 \leq j \leq 3$ [resp. $W'(s)$] satisfy property (DP) uniformly with respect to s^{-1} [resp s^{-2}] as $s \geq 1$. Moreover:

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma \tilde{D}P_1 \end{pmatrix} \begin{pmatrix} (\exp_k R) + U'_1(s) & U'_3(s) \\ U'_2(s) & W'(s) \end{pmatrix} = \\ \begin{pmatrix} 0 & 0 \\ \sigma \tilde{D}P_1 U'_2(s) & \sigma \tilde{D}P_1 W'(s) \end{pmatrix}$$

According to Proposition 2.4, $\sigma \tilde{D}P_1 W'(s)$ will still satisfy property (DP) uniformly with respect to s^{-2} as $s \geq 1$. Now we observe that the grading Υ and σ preserve the decomposition (2.12) and recall the definition of $\text{Str}_{\text{Cl}(1)}$ given by (2.11) and Proposition 2.5. We get therefore Theorem 2.7 by using the previous (DP) estimate for $\sigma \tilde{D}P_1 W'(s)$.

3. Modified higher eta invariants.

In this section we will assume that N is odd dimensional. We still assume (see equation (2.2)) that the L^2 -spectrum of \tilde{D} has a gap at zero. So for any $\vartheta \in (0, \delta)$, the operator $\tilde{D} + \vartheta$ admits a L^2 -bounded inverse.

Lemma 3.1. *For any $t > 0$ and $\vartheta \in]0, \delta[$, we set: $\mathbb{B}(s, \vartheta) = \Upsilon \nabla + s\sigma(\tilde{D} + \vartheta)$.*

1] *The following integral is absolutely convergent in $\overline{\tilde{\Omega}}_*(\mathcal{B}^\infty)$:*

$$\frac{2}{\sqrt{\pi}} \int_t^{+\infty} \text{STR}_{\text{Cl}(1)}(\sigma(\tilde{D} + \vartheta) \exp(-\mathbb{B}^2(s, \vartheta))) ds;$$

moreover, as $t \rightarrow 0^+$, it admits an asymptotic expansion with respect to $\log t$ and the t^k , $k \in \mathbb{Z}$. We then define $\tilde{\eta}(\vartheta)$ to be the coefficient of t^0 .

2] *For any $t > 0$ and $\vartheta \in]0, \delta[$ we have modulo graded commutators the following variation formula for the higher eta integrand:*

$$\begin{aligned} \partial_\vartheta \text{STR}_{\text{Cl}(1)}(\sigma(\tilde{D} + \vartheta) \exp(-\mathbb{B}^2(s, \vartheta))) &= \\ \partial_s \text{STR}_{\text{Cl}(1)}(s\sigma \exp(-\mathbb{B}^2(s, \vartheta))) &+ \\ +d \text{STR}_{\text{Cl}(1)}\left(\frac{d\mathbb{B}}{ds} \int_0^1 \exp(-u\mathbb{B}^2) \frac{d\mathbb{B}}{d\vartheta} \exp(-(1-u)\mathbb{B}^2) du\right). \end{aligned}$$

Proof. 1] Since $\tilde{D} + \vartheta$ is invertible the convergence of this integral is essentially a result of [L2] page 215-218. The existence of the asymptotic expansion is a consequence of the local index theorem. 2] The proof is an easy adaptation of the proof of Proposition 14 of [MP 1].

Theorem 3.2. *We have: $\lim_{\vartheta \downarrow 0} \tilde{\eta}(\vartheta) = \tilde{\eta} + \text{Ch}(\text{null } \mathcal{D}_\infty)$ modulo exact forms in $\overline{\Omega}_*(\mathcal{B}^\infty)$.*

Proof. Let us consider three real numbers $\vartheta > 0$ and $0 < t < T$. We integrate both members of the equality of Lemma 3.1 2] with respect to $s \in [t, T]$ and $\vartheta' \in [0, \vartheta]$. Modulo exact forms we get the following equality in $\overline{\Omega}_*(\mathcal{B}^\infty)$:

$$\begin{aligned} & \int_t^T \text{STR}_{\text{Cl}(1)}(\sigma(\tilde{D} + \vartheta) \exp(-\mathbb{B}^2(s, \vartheta))) ds - \\ & \int_t^T \text{STR}_{\text{Cl}(1)}(\sigma \tilde{D} \exp(-\mathbb{B}^2(s, 0))) ds \\ & = \int_0^\vartheta \text{STR}_{\text{Cl}(1)}(\sigma T \exp(-\mathbb{B}^2(T, \vartheta'))) d\vartheta' - \\ & \int_0^\vartheta \text{STR}_{\text{Cl}(1)}(\sigma t \exp(-\mathbb{B}^2(t, \vartheta'))) d\vartheta' \end{aligned} \quad (3.1)$$

We check easily that:

$$\lim_{\vartheta \rightarrow 0^+} \lim_{t \rightarrow 0^+} \int_0^\vartheta \text{STR}_{\text{Cl}(1)}(\sigma t \exp(-\mathbb{B}^2(t, \vartheta'))) d\vartheta' = 0$$

Thus, using equation (3.1), we see that the theorem is a consequence of the following assertion:

$$\lim_{\vartheta \rightarrow 0^+} \lim_{T \rightarrow +\infty} \int_0^\vartheta \text{STR}_{\text{Cl}(1)}(\sigma T \exp(-\mathbb{B}^2(T, \vartheta'))) d\vartheta' = \frac{\sqrt{\pi}}{2} \text{Ch}(\text{null } \mathcal{D}_\infty) \quad (3.2)$$

We have:

$$\mathbb{B}^2(T, \vartheta) = (\Upsilon \nabla)^2 + s\sigma \tilde{D} \Upsilon \nabla + s \Upsilon \nabla \sigma \tilde{D} + s^2(\tilde{D} + \vartheta)^2$$

We set $\tilde{D}^2(\vartheta) = \tilde{D}^2 + 2\vartheta \tilde{D}$. Since the L^2 -spectrum of \tilde{D} has a gap at zero, $\tilde{D}^2(\vartheta)$ will admit, for ϑ small enough, a Green operator $\tilde{G}(\vartheta)$ whose Schwartz kernel is smooth outside the diagonal and of rapid decay. Consider the matrix decomposition of the curvature $\mathbb{B}^2(T, \vartheta)$ with respect to the decomposition (2.12) associated with \tilde{D} . We can proceed as in [B-V] to get the following result analogous to Lemma 2.8:

Lemma 3.3. *With respect to the decomposition (2.12), we can write for ϑ small enough:*

$$\mathbb{B}^2(T, \vartheta) = g(T, \vartheta) \left(\begin{pmatrix} R + T^2 \vartheta^2 & 0 \\ 0 & H(T, \vartheta) + T^2 \vartheta^2 \end{pmatrix} + Z(T, \vartheta) \right) g^{-1}(T, \vartheta)$$

where,

$$H(T, \vartheta) = T^2 \tilde{D}^2(\vartheta) P_1 + T P_1 \tilde{C} P_1 + P_1 \nabla^2 P_1 + P_1 \tilde{C} P_0 \tilde{C} P_1 \tilde{G}(\vartheta)$$

R is the curvature of $P_0 \nabla P_0$; $g(T, \vartheta)$ and $Z(T, \vartheta)$ are squared (2,2) matrix such that $g(T, \vartheta)$ is invertible and the three matrix $Z(T, \vartheta)$, $g(T, \vartheta) - \text{Id}$ and $g^{-1}(T, \vartheta) - \text{Id}$ have their coefficients in $\mathcal{M}_1(\vartheta)$ where $\mathcal{M}_1(\vartheta)$ is defined as \mathcal{M}_1 before Lemma 2.8 but with \tilde{G} replaced by $\tilde{G}(\vartheta)$.

Now we make a few remarks. Hypothesis (2.2) shows that for ϑ small enough $\tilde{D}^2(\vartheta) \equiv \tilde{D}^2 + 2\vartheta \tilde{D}$ is a generalized self-adjoint positive laplacian, so $\exp(-t^2 \tilde{D}^2(\vartheta))(x, y)$ will satisfy expansion (2.16) for small time t and $d(x, y)$ small. Finite propagation speed techniques allow to see that $\tilde{D}^a P_1 \exp(-t^2 \tilde{D}^2(\vartheta))$ satisfy the large time estimate (2.14) for $s > 1$. Observe next that for each $\tilde{\vartheta} \in \mathbb{R}$ the value at ξ of the Fourier transform of $r \rightarrow \exp(-r^2 - 2r\tilde{\vartheta})$ is $\sqrt{\pi} \exp(-\frac{\xi^2}{4} + \tilde{\vartheta}^2 + i\tilde{\vartheta}\xi)$. Hence we can again use a finite propagation speed argument to see that $\exp(-t^2 \tilde{D}^2(\vartheta))(x, y)$ satisfy estimate (2.15) for t small and $d(x, y) > 2\varepsilon$. Now we can proceed as at the end of the proof of Theorem 2.7 to see that each component in $\widehat{\Omega}_k(\mathcal{B}^\infty)$ of $T \exp(-\mathbb{B}^2(T, \vartheta'))$ is of the form:

$$T e^{-T^2 \vartheta'^2} \left[\begin{pmatrix} \exp_k(-R) & 0 \\ 0 & 0 \end{pmatrix} + U(T, \vartheta') \right]$$

where the coefficients of the matrix $U(T, \vartheta')$ satisfy property (DP) uniformly with respect to T^{-1} as $\vartheta' \in]0, \vartheta]$ and $T > 1$.

We check then that:

$$\lim_{T \rightarrow +\infty} \int_0^\vartheta T \exp(-T^2 \vartheta'^2) d\vartheta' = \frac{\sqrt{\pi}}{2}$$

Now, using the definition of $\text{Str}_{\text{Cl}(1)}$ given at the beginning of Sect. 2, we see easily that the left-hand-side of equation (3.2) exists and is equal to:

$$\frac{\sqrt{\pi}}{2} \text{STR}_{\text{Cl}(1)}(\sigma \exp(-R)) = \frac{\sqrt{\pi}}{2} \text{Ch}(\text{null } \mathcal{D}_\infty)$$

4. Γ -manifolds with boundary and the small b -calculus.

We first give some **geometric preliminaries** that will be constantly used in the sequel.

Let M be a smooth connected compact manifold with boundary. Let Γ be a finitely generated discrete group. Let \widetilde{M} be a Galois covering of M with covering group equal to Γ . As in the previous sections we denote by R_g , $g \in \Gamma$, the action of Γ on \widetilde{M} on the right and by $\pi : \widetilde{M} \rightarrow M$ the covering map. Since Γ acts by diffeomorphisms on \widetilde{M} , there is an induced action

$$R_g : \partial\widetilde{M} \rightarrow \partial\widetilde{M}, \quad g \in \Gamma \quad (4.1)$$

with the property that $\partial\pi : \partial\widetilde{M} \rightarrow \partial M$ is a Galois Γ -covering.

We consider the Lie algebra of vector fields

$$\mathcal{V}_b(\widetilde{M}) = \{V \in C^\infty(\widetilde{M}, T\widetilde{M}) \mid V \text{ is tangent to } \partial\widetilde{M}\}$$

We can introduce as in the compact case the notion of b -tangent bundle ${}^bT\widetilde{M}$ and we have $\mathcal{V}_b(\widetilde{M}) = C^\infty(\widetilde{M}, {}^bT\widetilde{M})$. We denote by ${}^b\Omega$ the b -density bundle. Notice that by (4.1) there is an induced action of Γ on ${}^bT\widetilde{M}$. When speaking of the Clifford bundle on \widetilde{M} we shall always refer to the one associated to ${}^bT^*\widetilde{M}$. Here we follow the Clifford algebra conventions of [M], thus demanding that $\alpha\beta + \beta\alpha = 2 < \alpha, \beta >$.

We fix a boundary defining function $\tilde{x} \in C^\infty(\widetilde{M})$ by lifting from M a boundary defining function $x \in C^\infty(M)$. Let $g_M \in \Gamma^\infty(M, {}^bT^*M \otimes {}^bT^*M)$ be an *exact* b -metric on M . Thus (we can assume that)

$$g_M = (dx/x)^2 + h_M \quad (4.2)$$

with $h_M \in C^\infty(M, T^*M \otimes T^*M)$. The lift of g_M to \widetilde{M} is an exact b -metric which is furthermore Γ -invariant. We denote this lifted metric by g . The associated riemannian b -density $|dg|$ is clearly Γ -invariant. The metric g induces in a natural way a Γ -invariant riemannian metric g_0 and a Γ -invariant density $|dg_0| \in C^\infty(\partial\widetilde{M}, \Omega)$ on $\partial\widetilde{M}$. We fix once and for all the Levi-Civita connection associated to g as in [M].

We fix a Γ -invariant locally finite cover of \widetilde{M} by coordinate neighborhoods. We also fix a Γ -invariant partition of unity subordinate to this cover.

In order to investigate the analytic properties of Dirac type operators on \widetilde{M} we extend to the non-compact b -manifold \widetilde{M} the microlocal techniques developed in [M] in the compact case.

Let \widetilde{F} be a complex vector Γ -bundle over \widetilde{M} . The Γ -action on \widetilde{M} and \widetilde{F} induces in a natural way an action on $\dot{C}_c^\infty(\widetilde{M}, \widetilde{F})$, $C_c^\infty(\widetilde{M}, \widetilde{F})$, $\dot{C}^\infty(\widetilde{M}, \widetilde{F})$, $C^\infty(\widetilde{M}, \widetilde{F})$ where the dot in the first and in the third space means vanishing of infinite order at $\partial\widetilde{M}$. If we introduce an hermitian metric on \widetilde{F} then we can consider the Hilbert space $L_b^2(\widetilde{M}, \widetilde{F})$, i.e. L^2 with respect to the b -density $|dg|$ and the given metric on \widetilde{F} . The group Γ acts on $L_b^2(\widetilde{M}, \widetilde{F})$, the action being unitary if the hermitian metric on \widetilde{F} is chosen to be Γ -invariant.

Let $\widetilde{F}_0, \widetilde{F}_1$ be two Γ -bundles with Γ -invariant hermitian metrics. Certainly the notion of b -differential operator $P \in \text{Diff}_b^*(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ is meaningful on \widetilde{M} . If P is a b -differential operator, then $P : C^\infty(\widetilde{M}, \widetilde{F}_0) \rightarrow C^\infty(\widetilde{M}, \widetilde{F}_1)$ and it makes sense to define the space of Γ -invariant b -differential operator as

$$\text{Diff}_{b,\Gamma}^*(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1) = \{\widetilde{P} \in \text{Diff}_b^*(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1) \mid \widetilde{P} \circ R_\gamma = R_\gamma \circ \widetilde{P} \quad \forall \gamma \in \Gamma\}.$$

More generally a C -linear map $A : C_c^\infty(\widetilde{M}, \widetilde{F}_0) \rightarrow C^\infty(\widetilde{M}, \widetilde{F}_1)$, or $A : L_b^2(\widetilde{M}, \widetilde{F}_0) \rightarrow L_b^2(\widetilde{M}, \widetilde{F}_1)$, is Γ -invariant if $A \circ R_\gamma = R_\gamma \circ A$ for each γ in Γ .

Let now \widetilde{M} be even dimensional and let \widetilde{E} be a \mathbb{Z}_2 -graded Γ -invariant vector bundle on \widetilde{M} . Thus $\widetilde{E} = \pi^* E$, with E a \mathbb{Z}_2 -graded bundle on the compact b -riemannian manifold M . We assume that E is a unitary Clifford module endowed with a unitary connection ∇^E which is Clifford with respect to the b -Levi-Civita connection associated to the b -metric g_M . We assume that $\nabla_{x\partial_x}^E \equiv 0$ on ∂M . We denote by D the generalized Dirac operator associated to these data. The lift of D to the covering \widetilde{M} is a Γ -invariant b -differential operator \widetilde{D} ; it is precisely the Dirac operator associated to the lifted data on \widetilde{E} .

The Clifford bundle associated to $T^*(\partial\widetilde{M})$ acts in a natural way on $\widetilde{E}_{\partial\widetilde{M}}$:

$$\text{cl}_{\partial}(\eta)(e_{|\partial\widetilde{M}}) \equiv \text{cl}(i\frac{d\tilde{x}}{\tilde{x}})\text{cl}(\eta)(e_{|\partial\widetilde{M}})$$

We define \widetilde{E}_0 to be $\widetilde{E}_{\partial\widetilde{M}}^+$. It is a unitary Clifford bundle with respect to $\text{cl}_{\partial}(\cdot)$. It is endowed with the induced Clifford connection. We denote by \widetilde{D}_0 the associated Γ -invariant Dirac operator. Finally, we identify $\widetilde{E}_{\partial\widetilde{M}}^-$ with \widetilde{E}_0 through Clifford multiplication by $\text{cl}(i d\tilde{x}/\tilde{x})$. With these identifications, that will always be used in the rest of the paper, the indicial family $I(\widetilde{D}, \lambda) \in \text{Diff}_{\Gamma}(\partial\widetilde{M}; \widetilde{E}_{\partial\widetilde{M}})$ is equal to the family of Γ -invariant differential operators on $\widetilde{E}_0 \oplus \widetilde{E}_0$ given by

$$\begin{pmatrix} 0 & \widetilde{D}_0 - i\lambda \\ \widetilde{D}_0 + i\lambda & 0 \end{pmatrix}. \tag{4.3}$$

End of geometric preliminaries.

Corresponding to the four spaces of C^∞ -sections introduced after (4.2) we have four spaces of distributional sections

$$\begin{aligned} C^{-\infty}(\widetilde{M}, \widetilde{F}) &= (\dot{C}_c^\infty(\widetilde{M}, \widetilde{F}^* \otimes {}^b\widetilde{\Omega}))' & C^{-\infty}(\widetilde{M}, \widetilde{F}) &= (\dot{C}^\infty(\widetilde{M}, \widetilde{F}^* \otimes {}^b\widetilde{\Omega}))' \\ \dot{C}_c^{-\infty}(\widetilde{M}, \widetilde{F}) &= (C^\infty(\widetilde{M}, \widetilde{F}^* \otimes {}^b\widetilde{\Omega}))' & \dot{C}^{-\infty}(\widetilde{M}, \widetilde{F}) &= (C_c^\infty(\widetilde{M}, \widetilde{F}^* \otimes {}^b\widetilde{\Omega}))' \end{aligned}$$

The Schwartz kernel theorem, in this context, states the existence of a 1-1 correspondence between the space of continuous linear maps

$$\dot{C}_c^\infty(\widetilde{M}, \widetilde{F}_0) \rightarrow C^{-\infty}(\widetilde{M}, \widetilde{F}_1) \tag{4.4}$$

and the space $C^{-\infty}(\widetilde{M} \times \widetilde{M}; \text{Hom}(\widetilde{F}_0 \otimes {}^b\Omega^{-1}, \widetilde{F}_1))$ where $\text{Hom}(\widetilde{F}_0, \widetilde{F}_1)$ is the bundle over $\widetilde{M} \times \widetilde{M}$ whose fibre at (p, q) is the vector space $(\widetilde{F}_1)_q \otimes (\widetilde{F}_0)_p^*$. Following [M] we shall define a space of pseudodifferential operators, naturally extending the Γ -invariant b -differential operators, by specifying the Schwartz kernel of its elements. In order to characterize their behaviour near the corner of $\widetilde{M} \times \widetilde{M}$, we introduce as in [M] the b -stretched product \widetilde{M}_b^2 . Let $\partial M = \sqcup_{j=1, \dots, k} (\partial M)_j$ be the decomposition of ∂M in its k connected components. Then

$$\widetilde{M} = \sqcup_j (\sqcup_{\alpha \in A_j} (\partial\widetilde{M})_j^\alpha) \quad \text{with } \pi^{-1}((\partial M)_j) = \sqcup_{\alpha \in A_j} (\partial\widetilde{M})_j^\alpha$$

Definition 4.1. *The ordinary b -stretched product \widetilde{M}_b^2 is obtained by blowing up in \widetilde{M}^2 the submanifold*

$$B_\Delta = \sqcup_j (\sqcup_\alpha (\partial\widetilde{M})_j^\alpha \times (\partial\widetilde{M})_j^\alpha) \subset \partial\widetilde{M} \times \partial\widetilde{M}.$$

Thus, following the notation given in [M Ch 4],

$$\widetilde{M}_b^2 \equiv [\widetilde{M}^2; B_\Delta] = S_+(N(B_\Delta)) \sqcup (\widetilde{M} \times \widetilde{M} \setminus (B_\Delta)) \quad (4.5)$$

with blow-down map $\beta_b^2 : \widetilde{M}_b^2 \rightarrow \widetilde{M}^2$.

The notions of lifted diagonal Δ_b , of front face, as well as left and right boundary face are precisely as in [M]; we adopt the notations given there and use the symbols bf, lb, rb for these three submanifolds.

Notice that in (4.5) we only blow up the components of $\partial\widetilde{M} \times \partial\widetilde{M}$ that meet the diagonal. This space is still too small for our needs: in studying the Schwartz kernel of operators on the covering \widetilde{M} which are associated to \mathcal{B}^∞ - b -Mishenko-Fomenko pseudodifferential operators on the base M we will need to consider the following extended version of the b -stretched product.

Definition 4.2. *The extended b -stretched product \widetilde{M}_{eb}^2 is obtained by blowing up in \widetilde{M}^2 the submanifold*

$$B = \sqcup_j (\sqcup_{(\alpha,\beta) \in A_j^2} (\partial\widetilde{M})_j^\alpha \times (\partial\widetilde{M})_j^\beta). \quad (4.6)$$

We denote by β_{eb}^2 the blow-down map $\beta_{eb}^2 : \widetilde{M}_{eb}^2 \rightarrow \widetilde{M}^2$

Remark. Since $B = B_\Delta \sqcup C$ with

$$C = \sqcup_j (\sqcup_{(\alpha,\beta) \in A_j^2} (\partial\widetilde{M})_j^\alpha \times (\partial\widetilde{M})_j^\beta) \text{ with } \alpha \neq \beta.$$

we see that $\widetilde{M}_{eb}^2 = [\widetilde{M}_b^2, C]$; thus there is a partial blow-down map $\beta_e^2 : \widetilde{M}_{eb}^2 \rightarrow \widetilde{M}_b^2$ such that $\beta_b^2 \circ \beta_e^2 = \beta_{eb}^2$.

One could also consider the overblown b -stretched product

$$\widetilde{M}_{ob}^2 = [\widetilde{M}^2, \partial\widetilde{M} \times \partial\widetilde{M}]$$

where *all* the connected components of the corner $\partial\widetilde{M} \times \partial\widetilde{M}$ are blown up. We shall not need this space.

Proposition 4.3. *The diagonal action of Γ on \widetilde{M}^2 lifts in a natural way to an action of Γ on \widetilde{M}_b^2 and on \widetilde{M}_{eb}^2 . The product action of $\Gamma \times \Gamma$ of $\widetilde{M} \times \widetilde{M}$ lifts in a natural way to an action of $\Gamma \times \Gamma$ on \widetilde{M}_{eb}^2 with quotient space diffeomorphic to M_b^2 .*

Remark. The second statement in this proposition already suggests why it is natural to introduce the extended b -stretched product.

Proof. Since Γ acts on the boundary of \widetilde{M} , there is certainly an action of Γ on $(\widetilde{M} \times \widetilde{M} \setminus (B_\Delta))$. On the other hand, elements in $S_+(N(B_\Delta))$ are equivalence classes of curves $\chi : [0, 1] \rightarrow \widetilde{M}^2$ with their initial point $\chi(0) \in B_\Delta$ and $\dot{\chi}(0) \notin T_{\chi(0)}(B_\Delta)$. We obviously define $R_\gamma[\chi] = [\chi_\gamma]$ with $\chi_\gamma(t) = R_\gamma(\chi(t))$. This action is well defined and under the identification

$$\tau : S_+(N(B_\Delta)) \leftrightarrow (B_\Delta) \times [-1, 1]$$

explained in [M Lemma 4.1], it corresponds to the natural action

$$\Gamma \times (B_\Delta) \times [-1, 1] \rightarrow (B_\Delta) \times [-1, 1]$$

given by $(\gamma, (p, \lambda)) \rightarrow (p \cdot \gamma, \lambda)$ with Γ acting diagonally on B_Δ . We leave the easy proof of this fact to the reader. Exactly the same argument establishes the first statement for \widetilde{M}_{eb}^2 . Consider now the product action of $\Gamma \times \Gamma$ on \widetilde{M}^2 . We can extend this action to \widetilde{M}_{eb} by setting

$$R_{(\gamma, \gamma')}[\chi] = [\chi_{(\gamma, \gamma')}] \text{ with } [\chi] \in S_+N(B)$$

and $\chi_{(\gamma, \gamma')}(t) = R_{(\gamma, \gamma')}(\chi(t))$. This action is well defined (precisely because of the definition of B in (4.6)); it is also clear that $\widetilde{M}_{eb}/\Gamma \times \Gamma \cong M_b^2$. The proposition is proved.

In introducing b -pseudodifferential operators we first assume, for simplicity, that $\widetilde{F}_0 = \widetilde{F}_1 = {}^b\widetilde{\Omega}^{\frac{1}{2}}$. The blow-down map gives an isomorphism

$$(\beta_b^2)^* : \dot{C}_c^\infty(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \rightarrow \dot{C}_c^\infty(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$$

and, by duality, an isomorphism

$$(\beta_b^2)_* : C^{-\infty}(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \rightarrow C^{-\infty}(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}). \tag{4.7}$$

The definition of the small space of b -pseudodifferential operators is exactly as in [M] and we recall it here for the convenience of the reader:

Definition 4.4. *The (small) space of b -pseudodifferential operators of order m acting on b -half densities, $\Psi_b^m(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}})$, consists of those continuous linear operators A as in (4.4) whose Schwartz kernel $K_A \in C^{-\infty}(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$ lifts under (4.7) to an element in the space $\{K \in I^m(\widetilde{M}_b^2, \widetilde{\Delta}_b, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \mid K \equiv 0 \text{ at } \ell b, rb\}$, with \equiv meaning equality of Taylor series at the indicated set.*

We shall use the same symbol for the Schwartz kernel of $A \in \Psi_b^*$ in \widetilde{M}^2 and its lift on \widetilde{M}_b^2 .

Definition 4.5. *The extended (small) b -calculus Ψ_{eb}^* is obtained by considering the Schwartz kernels in $\{K \in I^m(\widetilde{M}_{eb}^2, \Delta_b, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \mid K \equiv 0 \text{ at } \ell b, rb\}$ with $\Delta_b, \ell b, rb$ defined in terms of β_{eb}^2 .*

Remark. Using the remark following Definition 4.2 we see that $\Psi_b^* \subset \Psi_{eb}^*$. This inclusion is simply obtained by lifting a Schwartz kernel on \widetilde{M}_b^2 to a Schwartz kernel on \widetilde{M}_{eb}^2 through the partial blow-down map β_e^2 . The lifted kernel will vanish of infinite order on the off-diagonal components of the front face of \widetilde{M}_{eb}^2 . We shall not use the extended b -calculus until Sect. 12.

We say that $A \in \Psi_b^m$ is properly supported if both the canonical projections $\pi_1, \pi_2 : \text{supp}K_A \subset \widetilde{M}^2 \rightarrow \widetilde{M}$ are proper maps. A typical example is given by ϵ -local operators (in the ordinary sense, i.e. with respect to an ordinary riemannian metric \hat{g} on \widetilde{M}).

Definition 4.4 can be extended in the usual fashion to take in account the presence of two arbitrary bundles F_0, F_1 and we denote by $\Psi_b^m(\widetilde{M}; F_0, F_1)$ the corresponding space of b -pseudodifferential operators; thus

$$\Psi_b^*(\widetilde{M}; F_0, F_1) \equiv \Psi_b^*(\widetilde{M}; {}^b\widetilde{\Omega}^{\frac{1}{2}}) \otimes_{C^\infty(\widetilde{M}_b^2)} C^\infty(\widetilde{M}_b^2, (\beta_b^2)^* \text{Hom}(F_0 \otimes {}^b\Omega^{-\frac{1}{2}}, F_1 \otimes {}^b\Omega^{-\frac{1}{2}})).$$

Using well known mapping properties of pseudodifferential operators on paracompact manifolds [Sh 1] and the results established in [M] it is easy to prove that $A \in \Psi_b^m$ maps \dot{C}_c^∞ into \dot{C}^∞ and C_c^∞ into C^∞ . If in addition A is properly supported, then A maintains the compact support property.

The usual symbolic properties of Ψ_b^* carry over to the present non-compact context. In particular there is a well defined notion of ellipticity.

The composition of two b -pseudodifferential operators of order m and m' , one of which properly supported, is again a b -pseudodifferential operator of order $m + m'$.

We now take the Γ -action into account. From now on, unless otherwise stated, we shall always work with Γ -invariant b -densities. With a small abuse of notation we keep the symbol ${}^b\tilde{\Omega}$ in order to denote the Γ -invariant b -density bundle: thus ${}^b\Omega = \pi^*({}^b\tilde{\Omega})$.

If \tilde{F}_0, \tilde{F}_1 are complex vector Γ -bundles on \tilde{M} , then we define

$$\Psi_{b,\Gamma}^m(\tilde{M}; \tilde{F}_0, \tilde{F}_1) = \{\tilde{A} \in \Psi_b^m(\tilde{M}; \tilde{F}_0, \tilde{F}_1) \mid R_\gamma \circ \tilde{A} = \tilde{A} \circ R_\gamma, \quad \forall \gamma \in \Gamma\} \quad (4.8)$$

Notice that the Γ -invariance of the operators obviously implies a Γ -invariance on the constants appearing in the symbol estimates of Definition 4.4.

Finally given a Γ -bundle \tilde{F} over \tilde{M} we can introduce the following Sobolev spaces. If $m \in \mathbb{Z}^+$ then

$$H_{b,\Gamma}^m(\tilde{M}, \tilde{F}) = \{u \in L_b^2(\tilde{M}, \tilde{F}) \mid \tilde{A}u \in L_b^2(\tilde{M}, \tilde{F}), \quad \forall \tilde{A} \in \text{Diff}_{b,\Gamma}^m(\tilde{M}; \tilde{F})\}$$

and if $m \in \mathbb{Z}^-$ then

$$H_{b,\Gamma}^m(\tilde{M}, \tilde{F}) = L_b^2(\tilde{M}, \tilde{F}) + \text{Diff}_{b,\Gamma}^{-m}(L_b^2(\tilde{M}, \tilde{F})).$$

These Sobolev spaces are in between $H_{b,c}^k$ and $H_{b,\text{loc}}^k$. Standard arguments show that $\tilde{A} \in \Psi_{b,\Gamma}^m$ properly supported defines a continuous linear operator $\tilde{A} : H_{b,\Gamma}^k \rightarrow H_{b,\Gamma}^{k-m}$ for each $k \in \mathbb{Z}$.

In the hypothesis Γ -virtually nilpotent we can also consider b -pseudodifferential operators that are not propely supported but instead rapidly decreasing outside the lifted diagonal Δ_b . The rapid decay condition refers to the action of the group Γ . In order to encode such a decaying property we introduce an auxilliary metric \hat{g} on \tilde{M} for which \tilde{M} and Γ become *quasi-isometric*. The metric \hat{g} is simply the lift to \tilde{M} of an ordinary metric on M . In the sequel we denote by $d(\cdot, \cdot)$ the distance function associated to \hat{g} .

Let $\varepsilon \in (0, 1)$ and let $\mathcal{O}(\text{bf}) = \{p \in \tilde{M}_b^2 \mid d(\beta_b^2(p), B_\Delta) < \varepsilon\}$. In $\mathcal{O}(\text{bf})$ the variables $r = \tilde{x} + \tilde{x}'$ and $\tau = (\tilde{x} - \tilde{x}')/(\tilde{x} + \tilde{x}')$ together with the boundary variables (y, y') (see [M] Ch. 4) can be used. Let $\mu \in C^\infty(\tilde{M}_b^2, {}^b\Omega^{-\frac{1}{2}})$ be the lift to \tilde{M}_b^2 of the density $|dg_M \otimes dg_M|^{-\frac{1}{2}}$ on $M \times M$.

Definition 4.6. Let Γ be virtually nilpotent and let $\tilde{A} \in \Psi_{b,\Gamma}^m(\tilde{M}, {}^b\tilde{\Omega}^{\frac{1}{2}})$.

We shall say that \tilde{A} is rapidly decreasing outside an ε -neighborhood of the lifted diagonal if

(i) for each multi-index of derivation α and any $q \in \mathbb{N}$ we can find a constant $C_{\alpha,q} > 0$ such that

$$\forall (z, z') \in \widetilde{M}_b^2 \setminus \mathcal{O}(\text{bf}) \text{ such that } d(z, z') > \varepsilon,$$

$$|\nabla^\alpha (K_{\widetilde{A}} \otimes \mu)(z, z')|(1 + d(z, z'))^q < C_{\alpha,q}$$

(ii) For any multi-index of derivation α with respect to $(r, \tau, y, y') \in \mathcal{O}(\text{bf})$ and for any $q \in \mathbb{N}$ there exists a constant $D_{\alpha,q}$ such that

$$\forall (r, \tau, y, y') \in \mathcal{O}(\text{bf}) \text{ such that } d(y, y') > \varepsilon,$$

$$|\nabla^\alpha (K_{\widetilde{A}} \otimes \mu)(\tau, r, y, y')|(1 + d(y, y'))^q < D_{\alpha,q}$$

Of course we could have used covariant differentiation on the b -density bundle (with respect to the b -Levi-Civita connection) instead of inserting the density μ .

Proposition 4.7. *If Γ is virtually nilpotent and $\widetilde{A} \in \Psi_{b,\Gamma}^m$ is rapidly decreasing, then $\widetilde{A} : H_{b,\Gamma}^k \rightarrow H_{b,\Gamma}^{k-m}$ is bounded for each $k \in \mathbb{Z}$.*

We omit the easy proof.

5. Γ -trace class operators.

Let \widetilde{F} be a complex vector Γ -bundle over \widetilde{M} endowed with a Γ -invariant hermitian metric. By considering the space $L_b^2(\widetilde{M}, \widetilde{F})$ as our reference Hilbert space, we can introduce the Von Neumann algebra $\mathcal{A}_\Gamma = \{A \in B(L_b^2(\widetilde{M}, \widetilde{F})) \mid A \circ R_\gamma = R_\gamma \circ A \quad \forall \gamma \in \Gamma\}$. We can also consider the usual Γ -trace and introduce the ideals of Γ -Hilbert-Schmidt operators and Γ -Trace class operators. We refer the reader to [A2] (see also [ES] [Sh 2]) for the necessary definitions.

Let now $\widetilde{P} \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ be an elliptic Γ -invariant b -differential operator. Then there exists a Γ -invariant ε -local parametrix \widetilde{Q}_σ

$$\widetilde{P} \circ \widetilde{Q}_\sigma = \text{Id} - \widetilde{R}_{0,\sigma} \quad \widetilde{Q}_\sigma \circ \widetilde{P} = \text{Id} - \widetilde{R}_{1,\sigma} \quad (5.1)$$

with $\widetilde{Q}_\sigma \in \Psi_{b,\Gamma}^{-m}(\widetilde{M}; \widetilde{F}_1, \widetilde{F}_0)$ and $\widetilde{R}_{i,\sigma} \in \Psi_{b,\Gamma}^{-\infty}$ and ε -local.

In fact the existence of a symbolic ε -local parametrix in $\Psi_b^{-\infty}$ is simply an application of the symbolic calculus for b -pseudodifferential operators. The use of a Γ -invariant partition of unity subordinate to a Γ -invariant cover of \widetilde{M} by coordinate charts, ensures that such a parametrix can be constructed in $\Psi_{b,\Gamma}^{-\infty}$.

Remark. In the sequel we shall often drop the tilde-notation for Γ -invariant operators on \widetilde{M} . We shall only keep it when it is necessary to make a distinction between operators on \widetilde{M} and operators on M .

Notice that if as in (4.5) we denote by β_b^2 the blow-down map and if $T \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ then the Schwartz kernel of T satisfies

$$K_T \in C^\infty(\widetilde{M}_b^2; (\beta_b^2)^* \text{Hom}(\widetilde{F}_0 \otimes {}^b\Omega^{-1}, \widetilde{F}_1))$$

and it is furthermore invariant with respect to the action induced on the latter space by the fibre action of Γ on the bundles $\widetilde{F}_0, \widetilde{F}_1$ and the (diagonal) action of Γ on \widetilde{M}_b^2 introduced in Proposition 4.3. In particular if $\widetilde{F}_0 = \widetilde{F}_1 = {}^b\widetilde{\Omega}^{\frac{1}{2}}$ and if we use $|dg|$ in order to trivialize this bundle, we obtain

$$T \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \Leftrightarrow K_T \in C^\infty(\widetilde{M}_b^2/\Gamma). \quad (5.2)$$

To understand when an operator $T \in \Psi_{b,\Gamma}^{-\infty} \cap \mathcal{A}_\Gamma$ is Γ -trace class we need to define the indicial operator and the indicial family of an arbitrary b -pseudodifferential operator. The vector field $\tilde{x}\partial/\partial\tilde{x}$ fixes a trivialization $\tilde{\nu}$ of the positive normal bundle to the boundary

$$N_+\partial\widetilde{M} \cong \partial\widetilde{M} \times [0, \infty) \quad (5.3)$$

and thus of its fibre-compactification $\overline{N_+\partial\widetilde{M}} \cong \partial\widetilde{M} \times [-1, 1]$. The action of Γ on $N_+\partial\widetilde{M}$ with respect to (5.3) is of course of product type: $R_\gamma(p, \lambda) = (R_\gamma(p), \lambda)$ for each $(p, \lambda) \in \partial\widetilde{M} \times [0, \infty)$.

Given an element $A \in \Psi_b^m(\widetilde{M}; E_0, E_1)$ we can define its indicial operator $I(A) \in \Psi_{b,\Gamma}^m(\overline{N_+\partial\widetilde{M}}; E_0, E_1)$ and its indicial family $I_{\tilde{\nu}}(A, z) \in \Psi^m(\partial\widetilde{M}; E_0, E_1), z \in \mathbb{C}$, precisely as in [M]. If $A \in \Psi_{b,\Gamma}^m$ then both $I(A)$ and $I_{\tilde{\nu}}(A, z)$ will be Γ -invariant. The indicial operator of an element $A \in \Psi_{b,\Gamma}^{-\infty} \cap \mathcal{A}_\Gamma$ represents the obstruction to the Γ -trace class property for A .

Proposition 5.1. *Let $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F}) \cap \mathcal{A}_\Gamma$. If A is Γ -trace class on $L_b^2(\widetilde{M}, \widetilde{F})$ then $I(A) = 0$.*

Proof. Let $p \in \text{bf}(\widetilde{M}_b^2)$. We want to show that $K_A(p) = 0$. Let M_0 be a fundamental domain containing p . Then $L_b^2(\widetilde{M}, \widetilde{F}) \cong \bigoplus_{\gamma \in \Gamma} L_b^2(M_0, \widetilde{F})$ and $A \in B(L_b^2(\widetilde{M}, \widetilde{F}))$ is represented by a block matrix $[A_{\gamma,\gamma'}]$ with $A_{\gamma,\gamma'} \in B(L_b^2(M_0, \widetilde{F}))$.

By assumption $A_{e,e}$ is trace class in $B(L_b^2(M_0, \widetilde{F}))$ and $\text{Tr}_\Gamma A = \text{Tr } A_{e,e}$. Moreover the Schwartz kernel of $A_{e,e}$ is equal to $K_A|_{M_0 \times M_0}$. Thus if $\phi\psi \in C_c^\infty(\overline{M_0}), p \in \text{supp}\phi \cap \text{supp}\psi, 1 = \phi(p) = \psi(p)$ then $\phi A_{e,e} \psi \in \Psi^{-\infty}(\overline{M_0}; \widetilde{F})$ and since it is trace class on $L_b^2(\overline{M_0}; \widetilde{F})$ it follows from [M] (Proposition 4.57) that $I(\phi A_{e,e} \psi) = 0$. Since $K_A(p) = K_{\phi A_{e,e} \psi}(p)$ and p was arbitrary we conclude that $I(A) = 0$ as required.

We now ask under which circumstances is the converse true. If $h \in C^\infty(\widetilde{M})^\Gamma$, we denote by $\pi_* h$ the induced smooth function on M .

Proposition 5.2. *Let $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$ and assume $I(A) = 0$. Assume one of the following conditions:*

- 1] *A defines a positive, self adjoint bounded operator on $L_b^2(\widetilde{M}, \widetilde{F})$*
- 2] *K_A is compactly supported in \widetilde{M}_b^2/Γ .*
- 3] *Γ is virtually nilpotent and K_A is rapidly decreasing on \widetilde{M}_b^2/Γ*

Then $A \in \mathcal{A}_\Gamma$, A is Γ -trace class and

$$\mathrm{Tr}_\Gamma A = \int_{M_0} \mathrm{tr} K_A|_{\Delta_b} \equiv \int_M \mathrm{tr} \pi_*(K_A|_{\Delta_b}). \quad (5.4)$$

Proof. 1] Recall that by simple functional analytic arguments it suffices to show that $\phi A \psi$ is trace class on $L_b^2(\widetilde{M}, \widetilde{F})$ for each $\phi, \psi \in C_c^\infty(\widetilde{M})$. Clearly $K_{\phi A \psi}$ is smooth and compactly supported on M_b^2 . Since by assumption the indicial operator of $\phi A \psi$ is equal to 0, it follows that

$$\phi A \psi : L_b^2(\widetilde{M}, \widetilde{F}) \rightarrow \tilde{x} H_{b,c}^\infty(\widetilde{M}, \widetilde{F}).$$

One checks that if $n = \dim \widetilde{M}$, the inclusion $\tilde{x}^\varepsilon H_{b,c}^{n+\delta}(\widetilde{M}, \widetilde{F}) \hookrightarrow L_b^2(\widetilde{M}, \widetilde{F})$ is compact $\forall \varepsilon, \delta > 0$. Thus $\phi A \psi \in B(L_b^2(\widetilde{M}, \widetilde{F}))$ is trace class as required. Formula (5.4) is standard.

2] Since, by assumption, A is properly supported, it follows that A is bounded on L_b^2 ; thus $A \in \mathcal{A}_\Gamma$. Since $I(A) = 0$ and K_A is compactly supported in \widetilde{M}_b^2/Γ , we certainly have that A is Γ -Hilbert-Schmidt (i.e. L_b^2 -integrable in \widetilde{M}_b^2/Γ). Let $B \in \mathrm{Diff}_{b,\Gamma}^k$ be an elliptic operator and let $Q_\sigma \in \Psi_{b,\Gamma}^{-k}$ be a properly supported Γ -invariant symbolic parametrix. Then $Q_\sigma \circ B = \mathrm{Id} - T$ with $T \in \Psi_{b,\Gamma}^{-\infty}$ and properly supported. We can write $A = T \circ A + Q_\sigma \circ B \circ A$. Since by assumption $I(A) = 0$ it follows that $A = \tilde{x} A'$, with $A' \in \Psi_{b,\Gamma}^{-\infty}$ and properly supported. It follows that $T \circ A = (T \tilde{x}^{\frac{1}{2}}) \circ (\tilde{x}^{\frac{1}{2}} A')$ and since on the right-hand side we have the product of two Γ -Hilbert-Schmidt operators we conclude that $T \circ A$ is Γ -trace class. Similarly $Q_\sigma \circ B \circ A = Q_\sigma \circ B'$ with $B' \in \Psi_{b,\Gamma}^{-\infty}, I(B') = 0, B'$ properly supported. Thus $Q_\sigma \circ B \circ A = (Q_\sigma \tilde{x}^{\frac{1}{2}}) \circ (\tilde{x}^{\frac{1}{2}} B'')$ with $B'' \in \Psi_{b,\Gamma}^{-\infty}$ properly supported. Choosing k large enough we see that $(Q_\sigma \tilde{x}^{\frac{1}{2}})$ is Γ -Hilbert-Schmidt. It follows that $Q_\sigma \circ B \circ A$ is Γ -trace class as the product of two Γ -Hilbert-Schmidt operators.

Since exactly the same argument establishes 3] , the proposition is proved.

If we select a function $h \in C_0^\infty(\widetilde{M})$ which near $\partial\widetilde{M}$ is constant in the normal direction and such that

$$\sum_{\gamma \in \Gamma} R_\gamma^* h \equiv 1 \quad (5.5)$$

then as in [A2] we can express the Γ -Trace in (5.4) as

$$\mathrm{Tr}_\Gamma A = \int_{\widetilde{M}} h K_A |_{\Delta_b}. \quad (5.6)$$

6. The b - Γ -Trace.

The last section has been devoted to the proof of necessary and sufficient conditions under which an element $A \in \Psi_{b,\Gamma}^{-\infty} \cap \mathcal{A}_\Gamma$ is Γ -trace class. The heat kernel of a Γ -invariant Dirac laplacian, a fundamental object in index theory, is a typical element of $\Psi_{b,\Gamma}^{-\infty}$ with non-vanishing indicial operator (we shall treat the heat kernel in Sect. 10); thus according to Proposition 5.1 it cannot be Γ -trace class on L_b^2 .

As in the compact case we shall now define an extension of the Γ -trace functional to all of $\Psi_{b,\Gamma}^{-\infty}$. Let us fix a trivialization $\nu \in C^\infty(\partial M, N_+ \partial M)$ of the normal bundle to the boundary of ∂M and let $x \in C^\infty(M)$ be a boundary defining function for ∂M with $d\nu \cdot x = 1$ on ∂M . We denote as usual by $\tilde{\nu}$ and \tilde{x} the lifted objects. Recall the b -integral of $[M]$; if $\phi \in C^\infty(M, {}^b\Omega)$ then

$$\int_M^\nu \phi = \lim_{\varepsilon \downarrow 0} \left[\int_{x>\varepsilon} \phi + \log \varepsilon \cdot \int_{\partial M} \phi|_{\partial M} \right].$$

For any element $\psi \in C_c^\infty(\widetilde{M}, {}^b\widetilde{\Omega})$, exactly the same definition can be given on \widetilde{M} . Consider now $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}})$ and $K_A|_{\Delta_b} \in (C^\infty(\widetilde{M}, {}^b\widetilde{\Omega}))^\Gamma$. We can give the following

Definition 6.1. *The b - Γ -trace of $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}})$ is equal to the b -integral of $\pi_*(K_A|_{\Delta_b})$:*

$$b\text{-Tr}_\Gamma A = \int_M^\nu \pi_*(K_A|_{\Delta_b}). \quad (6.1)$$

Equivalently, if $M_0 \subset \widetilde{M}$ is a fundamental domain for the action of Γ and if $h \in C_0^\infty(\widetilde{M})$ is a function as in (5.5), then

$$b - \text{Tr}_\Gamma A = \int_{\widetilde{M}}^\nu h K_A|_{\Delta_b} = \int_{M_0}^\nu K_A|_{\Delta_b}. \quad (6.2)$$

If, more generally, A acts between the sections of a complex vector Γ -bundle, $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$ then

$$b - \text{Tr}_\Gamma A = \int_M^\nu \text{tr} \pi_*(K_A|_{\Delta_b}).$$

Notice that the $b - \Gamma$ -trace of $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$ is, by definition, a finite number.

If $A \in \Psi_{b,\Gamma}^{-\infty}$ is a Γ -trace class element of \mathcal{A}_Γ then

$$b - \text{Tr}_\Gamma A = \text{Tr}_\Gamma A$$

since, by Proposition 5.1,

$$\int_M^\nu \pi_*(K_A|_{\Delta_b}) = \int_M \pi_*(K_A|_{\Delta_b}) = \text{Tr}_\Gamma A. \quad (6.3)$$

The first equality holds whenever the indicial operator of A vanishes.

As in the compact case the b - Γ -trace is not zero on commutators. Before giving a formula for $b - \text{Tr}_\Gamma[A, B]$, with $A, B \in \Psi_{b,\Gamma}^{-\infty}$, we need a Lemma.

Lemma 6.2. *Let $A \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$. Then $I_{\tilde{\nu}}(A, z)$ is an entire family of Γ -invariant smoothing operators, with Schwartz kernel rapidly decreasing in z on any compact subset of $\partial\widetilde{M} \times \partial\widetilde{M}$, as $|\Re z| \rightarrow \infty$, in any region where $|\Im z|$ is bounded.*

The proof of the Lemma follows at once from the properties of the Mellin transform. To simplify the notation we shall often forget about the $\tilde{\nu}$ subscript in the indicial family.

Proposition 6.3. *Let $A, B \in \Psi_{b,\Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$, B being properly supported. Then*

$$b - \text{Tr}_\Gamma[A, B] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{\partial M} \text{tr}((\partial\pi)_* K(\frac{\partial}{\partial\lambda} I(A, \lambda) \circ I(B, \lambda))|_{\partial\Delta_b}) d\lambda.$$

If moreover the operator $\partial/\partial\lambda(I(A, \lambda)) \circ I(B, \lambda)$ is Γ -trace class for each $\lambda \in \mathbb{R}$ then, more suggestively,

$$b - \text{Tr}_\Gamma[A, B] = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{\partial M} \text{Tr}_\Gamma \left(\frac{\partial}{\partial \lambda} I(A, \lambda) \circ I(B, \lambda) \right) d\lambda. \quad (6.4)$$

Lastly if B is a differential operator and $A \in \Psi_{b, \Gamma}^{-\infty}(\widetilde{M}, \widetilde{F})$ then the same is true.

Proof. We shall use the definition of $b - \text{Tr}_\Gamma$ given by the first equality in (6.2) and adapt the proof of [M] page 153. We can assume that K_A and K_B have supports contained in $\cup_{\gamma \in \Gamma} U \cdot \gamma = V$ where U is a "small" open subset of \widetilde{M}_b^2 disjoint from $\text{lb}(\widetilde{M}_b^2)$ and $\text{rb}(\widetilde{M}_b^2)$. We choose U so that we can use projective local coordinates on V : $(\tilde{x}, s = \frac{\tilde{x}}{x'}, y, y')$ and $h \equiv h(y)$ does not depend on x on $\widetilde{M}_b^2 \cap \Delta_b$. We set:

$$K_A = \alpha(\tilde{x}, s, y, y') \left| \frac{ds}{s} \frac{d\tilde{x}}{\tilde{x}} dy dy' \right|^{\frac{1}{2}}$$

$$K_B = \beta(\tilde{x}, s, y, y') \left| \frac{ds}{s} \frac{d\tilde{x}}{\tilde{x}} dy dy' \right|^{\frac{1}{2}}$$

Of course K_A and K_B are Γ -invariant. Let $R : \widetilde{M}_b^2 \rightarrow \widetilde{M}_b^2$ be the factor exchanging isomorphism: $R(z, z') = (z', z)$. As in [M] page 154 we have: (for $\varepsilon > 0$)

$$\begin{aligned} \int_{\tilde{x} > \varepsilon} [hK_{A \circ B} - hK_{B \circ A}]|_{\Delta} &= \int_{\{x > \varepsilon\} \cap \widetilde{M}_b^2} hK_A R^*(K_B) - \\ &\int_{\{x' > \varepsilon\} \cap \widetilde{M}_b^2} K_A R^*(hK_B) \end{aligned}$$

In the previous local coordinates we have $R(\tilde{x}, s, y, y') = (\frac{\tilde{x}}{s}, \frac{1}{s}, y, y')$. Since K_A, K_B are Γ -invariant and $\sum_{\gamma \in \Gamma} R_\gamma^* h \equiv 1$ we see that:

$$\begin{aligned} &\int_{\partial \widetilde{M} \times \partial \widetilde{M}} \int_0^{+\infty} \int_{\varepsilon s}^{+\infty} \alpha(\tilde{x}, s, y, y') \beta\left(\frac{\tilde{x}}{s}, \frac{1}{s}, y, y'\right) h(y') \frac{d\tilde{x}}{\tilde{x}} \frac{ds}{s} dy dy' \\ &= \int_{\partial \widetilde{M} \times \partial \widetilde{M}} \int_0^{+\infty} \int_{\varepsilon s}^{+\infty} \alpha(\tilde{x}, s, y, y') h(y) \beta\left(\frac{\tilde{x}}{s}, \frac{1}{s}, y, y'\right) \frac{d\tilde{x}}{\tilde{x}} \frac{ds}{s} dy dy' \end{aligned}$$

Therefore we get:

$$\int_{\tilde{x} > \varepsilon} [hK_{A \circ B} - hK_{B \circ A}]_{|\Delta} =$$

$$\int_{\partial \tilde{M} \times \partial \tilde{M}} \int_0^{+\infty} \int_{\varepsilon}^{s\varepsilon} \alpha(\tilde{x}, s, y, y') h(y) \beta\left(\frac{\tilde{x}}{s}, \frac{1}{s}, y', y\right) \frac{d\tilde{x}}{\tilde{x}} \frac{ds}{s} dy dy'$$

At this point the proof is completely parallel to that in [M].

7. Γ -invariant b -elliptic operators and their parametrices.

Let $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ be elliptic. We fix once and for all a trivialization ν of the positive normal bundle to the boundary of M and a boundary defining function $x \in C^\infty(M)$ such that $dx \cdot \nu = 1$. We denote by $\tilde{\nu}, \tilde{x}$ the lifted objects on \widetilde{M} . In order to investigate the properties of the null space of P we need to improve the symbolic parametrix construction of Section 5 and produce an inverse modulo operators with vanishing indicial operator. In order to accomplish this we need to impose a condition on the indicial family of P . We make the following assumption

$$\exists \delta > 0 \mid L^2 - \text{spec}(I(P, \lambda)) \cap [-i\delta, i\delta] = \emptyset \quad \forall \lambda \in \mathbb{R}. \quad (7.1)$$

Thus for each $\lambda \in \mathbb{R}$, $I(P, \lambda)^{-1}$ exists as a bounded operator on the Hilbert space $L^2(\partial\widetilde{M}; \widetilde{F}_1, \widetilde{F}_0)$.

As a fundamental example we can consider a Dirac-type operator \widetilde{D}^\pm on an even dimensional Γ -covering with boundary \widetilde{M} endowed with an exact Γ -invariant b metric. Let us denote by \widetilde{D}_0 the boundary operator. If there exists a $\delta > 0$ such that

$$L^2 - \text{spec}(\widetilde{D}_0) \cap [-\delta, \delta] = \emptyset, \quad (7.2)$$

i.e. if \widetilde{D}_0 admits a bounded L^2 -inverse, then $I(\widetilde{D}^\pm, \lambda) = \pm i\lambda + \widetilde{D}_0$ satisfies assumption (7.1).

Remark. We can introduce the set

$$\text{spec}_b(P) = \{z \in \mathbb{C} \mid I(P, \lambda) \text{ does not admit a bounded } L^2 - \text{inverse}\}.$$

We claim that, as in the compact case, this set is concentrated near the imaginary axis. In fact, if $S \in \Psi_{b,\Gamma}^{-\ell}, \ell \geq 0$ is properly supported, then we have uniform estimate for the Γ -Sobolev operator norms of the indicial family of S , i.e. on $\|I(S, z)\|_{0,\ell}$, exactly as in [M] page 148. This estimate follows ultimately from the Γ -invariant symbol estimates and allow us to infer as in [M] that if Q_σ is a Γ -invariant ε -local symbolic parametrix of $P \in \text{Diff}_{b,\Gamma}^m$ as in (5.1) and if $R_{0,\sigma}$ is the resulting ε -local remainder, then for the indicial family $I(R_{0,\sigma}, z)$ we can find a function $F : [0, \infty) \rightarrow [0, \infty)$ such that $\text{Id} - I(R_{0,\sigma}, z)$ is invertible for each z such that $|\Re z| > F(|\Im z|)$. Using

$$I(P, z) \circ I(Q_\sigma, z) = \text{Id} - I(R_{0,\sigma}, z)$$

we see that $\text{spec}_b(P)$ is contained in the complement of the set $\{z \in \mathbb{C} : |\Re z| > F(|\Im z|)\}$ and the claim follows.

Before stating the main result of this section we introduce the Γ -calculus with bounds. Let \mathcal{L} be the space of b -half densities K on \widetilde{M}_b^2 such that :

- (i) For each $\phi \in C^\infty(\widetilde{M}_b^2)$ with support disjoint from $\text{bf}(\widetilde{M}_b^2)$, $K\phi \in H_{b,\text{loc}}^\infty(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$.
- (ii) if $K = \kappa(r, \tau, y, y')|\frac{dr}{r}|^{\frac{1}{2}}$ in the neighborhood $\mathcal{O}(\text{bf})$ introduced in Definition 4.6 then

$$\kappa(r, \cdot) \in C^\infty([0, \epsilon]; H_b^\infty([-1, 1], {}^b\widetilde{\Omega}^{\frac{1}{2}}) \otimes C^\infty(B_\Delta, \widetilde{\Omega}^{\frac{1}{2}})) \quad (7.3).$$

Notice that

$$K \in \mathcal{L} \Rightarrow K \text{ is } C^\infty \text{ in the interior of } \widetilde{M}_b^2.$$

Let $\alpha, \beta \in \mathbb{R}$ and define $\widetilde{\Psi}_b^{-\infty, \alpha, \beta}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}})$ as the space of b -half densities A on \widetilde{M}_b^2 such that for some ϵ (depending on A) $\rho_{\ell b}^{-\alpha-\epsilon} \rho_{rb}^{-\beta-\epsilon} A \in \mathcal{L}$.

We define the calculus with bounds as the space of continuous operators

$$A : \dot{C}^\infty(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \rightarrow C^{-\infty}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}})$$

with Schwartz kernel in

$$\begin{aligned} \Psi_b^{m, \alpha, \beta}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) &= \Psi_b^m(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) + \widetilde{\Psi}_b^{-\infty, \alpha, \beta}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) + \\ &\rho_{\ell b}^\alpha \rho_{rb}^\beta H_{b,\text{loc}}^\infty(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}). \end{aligned}$$

We shall usually assume that $\alpha = \beta \equiv \delta > 0$ and use the shorter notation $\tilde{\Psi}_b^{-\infty, \delta}, \Psi_b^{m, \delta}$.

We denote by $\Psi_{b, \Gamma}^{m, \delta}(\tilde{M}, b\tilde{\Omega}^{\frac{1}{2}})$ the Γ -invariant calculus with bounds.

Thus

$$\Psi_{b, \Gamma}^{m, \delta}(\tilde{M}, b\tilde{\Omega}^{\frac{1}{2}}) = \Psi_{b, \Gamma}^m(\tilde{M}, b\tilde{\Omega}^{\frac{1}{2}}) + \tilde{\Psi}_{b, \Gamma}^{-\infty, \delta}(\tilde{M}, b\tilde{\Omega}^{\frac{1}{2}}) + \rho_{\ell b}^{\delta} \rho_{rb}^{\delta} (H_{b, \text{loc}}^{\infty}(\tilde{M}^2, b\tilde{\Omega}^{\frac{1}{2}}))^{\Gamma} \tag{7.4}$$

where we denote by $(H_{b, \text{loc}}^{\infty}(\tilde{M}^2, b\tilde{\Omega}^{\frac{1}{2}}))^{\Gamma}$ the Γ -invariant kernels in

$$H_{b, \text{loc}}^{\infty}(\tilde{M}^2, b\tilde{\Omega}^{\frac{1}{2}}).$$

If $A \in \Psi_{b, \Gamma}^{m, \delta}, B \in \Psi_{b, \Gamma}^{\ell, \delta}$, with B properly supported, then $A \circ B \in \Psi_{b, \Gamma}^{m+\ell, \delta}$. Finally properly supported elements in $\Psi_{b, \Gamma}^{m, \delta}(\tilde{M}, b\tilde{\Omega}^{\frac{1}{2}})$ define bounded operators $H_{b, \Gamma}^k \rightarrow H_{b, \Gamma}^{k-m}$. All these statements generalize in an obvious way to the case of operators acting between the sections of two complex vector Γ -bundles \tilde{F}_0, \tilde{F}_1 .

We can also introduce the extended b -calculus with bounds. The definition of $\tilde{\Psi}_{eb, \Gamma}^{-\infty, \delta}$ is precisely as in (i) (ii) above but with the submanifold B instead of B_{Δ} appearing in (7.3). Notice that $\Psi_{b, \Gamma}^{m, \delta} \subset \Psi_{eb, \Gamma}^{m, \delta}$ with the lift of $\tilde{A} \in \Psi_{b, \Gamma}^{m, \delta}$ vanishing of order 2δ at the off-diagonal front face of \tilde{M}_{eb}^2 .

Theorem 7.1. *Let $P \in \text{Diff}_{b, \Gamma}^m(\tilde{M}; \tilde{F}_0, \tilde{F}_1)$ be elliptic and assume (7.1). Then there exists an operator $Q \in \Psi_{b, \Gamma}^{-m, \delta}(\tilde{M}; \tilde{F}_1, \tilde{F}_0)$ such that*

$$P \circ Q = \text{Id} - R_0$$

$$Q \circ P = \text{Id} - R_1$$

with $R_i \in \rho_{bf} \tilde{\Psi}_{b, \Gamma}^{-\infty, \delta}$.

Proof. Using assumption (7.1) the proof proceeds as in the compact case, once Γ -invariant elliptic theory as in [A2][ES] is employed on $\partial\tilde{M}$. We leave the details to the reader.

Remark. It is important to note that the parametrix Q will not produce ε -local remainders. To understand this point we observe that in a neighborhood of the front face, the Schwartz kernel of Q is the sum of the Schwartz kernel of a symbolic parametrix Q_{σ} , as in (5.1), and of the Schwartz kernel given by

$$K(s, y, y') = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^{i\lambda} K(I(P, \lambda)^{-1} \circ I(R_{0, \sigma}, \lambda))(y, y') d\lambda \tag{7.5}$$

with $R_{0,\sigma}$ equal to the remainder produced by the symbolic parametrix (see (5.1)). Because of the presence of the inverse of $I(P, \lambda)$, such a parametrix is not ε -local, nor it will produce an ε -local remainder. On the other hand it is clear that we cannot simply cut the kernel given by (7.5) near the diagonal; such a cutting would destroy the property of $I(Q)$ of inverting the indicial operator of P which is in turn a mandatory requirement in order to obtain remainders which vanish on the front face as in Theorem 7.1. This lack of ε -locality should be compared with the closed case ([A2]). We point out that since the remainders R_0, R_1 are *not* properly supported, they are in general unbounded on b - Γ -Sobolev spaces; they are instead continuous as maps $H_{b,c}^k \rightarrow H_{b,\text{loc}}^\ell, \forall k, \forall \ell \in \mathbb{Z}$. In particular, although $I(R_0) = I(R_1) = 0$, nothing can be said about the Γ -trace class property of R_0, R_1

The b - Γ -trace functional can be extended to $\Psi_{b,\Gamma}^{-\infty,\delta}$, since only the restriction of the Schwartz kernel to the lifted diagonal $\Delta_b \subset \widetilde{M}_b^2$ is involved. The b -trace identity, formula (6.4) is still valid if $A \in \Psi_{b,\Gamma}^{-\infty,\delta}$ and B is differential.

We can also consider the linear functional

$$\widetilde{\text{Tr}}_\Gamma : \rho_{\text{bf}} \Psi_{b,\Gamma}^{-\infty,\delta}(\widetilde{M}, \widetilde{F}) \rightarrow \mathbb{C}$$

obtained by integrating the trace of the restriction of the Schwartz kernel of $A \in \rho_{\text{bf}} \Psi_{b,\Gamma}^{-\infty,\delta}$ over a fundamental domain $M_0 \subset \Delta_b$:

$$\widetilde{\text{Tr}}_\Gamma A = \int_{M_0} \text{tr} K_A | \Delta_b = \int_M \text{tr}(\pi_*(K_A | \Delta_b)). \quad (7.6)$$

Observe that

$$\rho_{\text{bf}} \Psi_{b,\Gamma}^{-\infty,\delta}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \subset \rho_{\text{bf}} \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma$$

(here we consider $\widetilde{F} = {}^b\widetilde{\Omega}^{\frac{1}{2}}$ for notational convenience). If $2\delta < 1$ then the latter space is contained in $\rho_{\text{bf}}^{2\delta} \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma$ which is in turn isomorphic, through the blow-down map, to the space $\rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma$.

Similarly, for $0 < \delta$ small enough, there is an injection

$$\begin{aligned} \rho_{\text{bf}} \Psi_{e_b}^{-\infty,\delta}(\widetilde{M}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) &\subset \rho_{\text{bf}} \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}_b^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma \subset \\ &\rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma. \end{aligned}$$

In fact (7.6) defines a linear functional on $\rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}))^\Gamma$. For operators with Schwartz kernel in this space the analogue of Proposition 5.2 holds:

Proposition 7.2. *If $A : \dot{C}^\infty \rightarrow C^{-\infty}$ has Schwartz kernel*

$$K_A \in \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}^2, \text{Hom}(\widetilde{F} \otimes {}^b\Omega^{-1}, \widetilde{F})))^\Gamma$$

and if one of the following three assumptions is satisfied

- (i) *A defines a bounded positive self-adjoint operator on $L_b^2(\widetilde{M}, \widetilde{F})$*
 - (ii) *K_A has compact support in \widetilde{M}^2/Γ*
 - (iii) *Γ is virtually nilpotent and K_A is rapidly decreasing on \widetilde{M}^2/Γ .*
- then $A \in \mathcal{A}_\Gamma$, A is Γ -trace class and we have $\text{Tr}_\Gamma A = \widetilde{\text{Tr}}_\Gamma A$.*

8. The Γ -index of an elliptic Γ -invariant b -differential operator.

As a consequence of the parametrix construction of Theorem 7.1 we see that if $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ is elliptic and satisfies (7.1), then the null space $\text{null}(P : H_{b,c}^k \rightarrow H_{b,\text{loc}}^{k-m})$ is contained in $x^\delta H_{b,\text{loc}}^\infty(\widetilde{M}, \widetilde{F}_0)$. Let us now consider $\Pi_{\text{null}(P)}$, the orthogonal projection, in L_b^2 , onto the null space of P acting on L_b^2 with domain $H_{b,\Gamma}^m$. Clearly $\Pi_{\text{null}(P)}$ is a positive and self-adjoint element in \mathcal{A}_Γ .

Proposition 8.1. *If $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ is elliptic and satisfies (7.1) then $\Pi_{\text{null}(P)}$ is Γ -trace class.*

Proof. Since $\Pi_{\text{null}(P)}$ is positive and self-adjoint it suffices to show that $\phi \Pi_{\text{null}(P)} \psi$ is trace class on $L_b^2(\widetilde{M}, \widetilde{F}_0)$ for each $\phi, \psi \in C_c^\infty(\widetilde{M})$. However, since by the parametrix construction we certainly have

$$\phi \Pi_{\text{null}(P)} \psi : L_b^2(\widetilde{M}, \widetilde{F}_0) \rightarrow x^\delta H_{b,c}^\infty(\widetilde{M}, \widetilde{F}_0)$$

we conclude as in the proof of Proposition 5.2 that $\phi \Pi_{\text{null}(P)} \psi$ is trace class and the Proposition follows.

We shall now investigate the structure of the Schwartz kernel of the operator $\Pi_{\text{null}(P)}$. Consider the manifold with corners \widetilde{M}^2 .

Proposition 8.2. *If $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ is elliptic and satisfies (7.1) then*

$$K(\Pi_{\text{null}(P)}) \in \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta (H_{b,\text{loc}}^\infty(\widetilde{M}^2, \text{Hom}(\widetilde{F}_0 \otimes {}^b\Omega^{-1}, \widetilde{F}_0)))^\Gamma. \quad (8.1)$$

Proof. As we shall not use this result, we only sketch the proof. We are indebted with R. Melrose for explaining to us the compact case of the following argument.

Let $\widehat{Z} = \widetilde{M} \times \widetilde{M} \equiv \widetilde{M}_X \times \widetilde{M}_Y$ with X, Y denoting the *collective* coordinates on the two copies of the manifold with boundary \widetilde{M} . Suppose, for simplicity, that the boundary $\partial\widetilde{M}$ is connected and that $\widetilde{F}_0 = \widetilde{F}_1 = {}^b\widetilde{\Omega}^{\frac{1}{2}}$. \widehat{Z} is a Γ -manifold with corners, with Γ acting diagonally. We have a corner of codimension 2, which is $\widehat{Z}_{1,1} = \partial\widetilde{M} \times \partial\widetilde{M}$, and the two boundary hypersurfaces $\widehat{Z}_{0,1} = \widetilde{M} \times \partial\widetilde{M}$ and $\widehat{Z}_{1,0} = \partial\widetilde{M} \times \widetilde{M}$. As in [atiyah] we see that our kernel $K(\Pi_{\text{null}(P)})$, denoted from now on as K_Π , belongs to the null space of the operator

$$\widehat{P} = P_X^* \circ P_X + P_Y^* \circ P_Y$$

The operator \widehat{P} is an elliptic b -differential operator on the manifold with corners \widehat{Z} . Since $K_\Pi \in \text{null}(\widehat{P})$ what we need to show is that there exists a (good) parametrix \widehat{Q} of \widehat{P} , in a space of b -pseudodifferential operators, with remainders $\widehat{R}_1, \widehat{R}_2$ in the residual space $\rho_{\text{lb}}^\varepsilon \rho_{\text{rb}}^\varepsilon H_{b,\text{loc}}^\infty(\widehat{Z}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$

This approach involves defining on a manifold with corners of codimension 2:

- (i) the small b -calculus, with its composition, symbolic and mapping properties
- (ii) the notion of indicial operator and indicial family associated to a boundary face.

Once this has been done (and we refer the reader to [M2]) we can proceed as follows. Associated to \widehat{P} there are indicial operators

$$\begin{aligned} I_{1,0}(\widehat{P}) &\in \text{Diff}_{b,\Gamma}^m(\overline{N_+ \widehat{Z}_{1,0}}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \\ I_{0,1}(\widehat{P}) &\in \text{Diff}_{b,\Gamma}^m(\overline{N_+ \widehat{Z}_{0,1}}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \\ I_{1,1}(\widehat{P}) &\in \text{Diff}_{b,\Gamma}^m(\overline{N_+ \widehat{Z}_{1,1}}, {}^b\widetilde{\Omega}^{\frac{1}{2}}) \end{aligned} \tag{8.2}$$

with indicial families

$$\begin{aligned} I_{1,0}(\widehat{P}, \lambda) &\equiv I_{\widetilde{M} \times \partial\widetilde{M}}(\widehat{P}, \lambda) = P^* \circ P + I(P, \lambda)^* \circ I(P, \lambda) \\ I_{0,1}(\widehat{P}, \mu) &\equiv I_{\partial\widetilde{M} \times \widetilde{M}}(\widehat{P}, \mu) = I(P, \mu)^* \circ I(P, \mu) + P^* \circ P. \\ I_{1,1}(\widehat{P}, \lambda, \mu) &\equiv I_{\partial\widetilde{M} \times \partial\widetilde{M}}(\widehat{P}, \lambda, \mu) = I(P, \mu)^* \circ I(P, \mu) + I(P, \lambda)^* \circ I(P, \lambda) \end{aligned}$$

Our assumption (7.1) certainly implies that the operators $I(P, \tau)^* \circ I(P, \tau)$ are invertible, with bounded L^2 -inverse, for each $\tau \in \mathbb{R}$. More precisely there exists an $\varepsilon > 0$ such that

$$L^2 - \text{spec}(I(P, \tau)^* \circ I(P, \tau)) \cap (-\infty, \varepsilon] = \emptyset \quad \forall \tau \in \mathbb{R}. \quad (8.3)$$

Using (8.3) it is not difficult to prove that there exists a positive δ such that

$$\begin{aligned} L^2 - \text{spec}(I_{1,1}(\widehat{P}, \lambda, \mu)) \cap (-\infty, \delta] &= \emptyset \\ L_b^2 - \text{spec}(I_{1,0}(\widehat{P}, \lambda)) \cap (-\infty, \delta] &= \emptyset \\ L_b^2 - \text{spec}(I_{0,1}(\widehat{P}, \mu)) \cap (-\infty, \delta] &= \emptyset \end{aligned}$$

for each $\lambda \in \mathbb{R}$ and for each $\mu \in \mathbb{R}$. We leave the easy proof of this fact to the reader. Proceeding as in the boundary case it follows that *all* three indicial operators associated to \widehat{P} are invertible on L^2 and L_b^2 , the inverse being obtained in terms of the inverse Mellin transform of (the inverse of) the respective indicial families. The inverses of the indicial operators belong to calculi with bounds (analogous to the one defined in (7.4)) on the compactified normal bundles appearing in (8.2). If \widehat{Q}_σ is a symbolic parametrix, with $\widehat{Q}_\sigma \circ \widehat{P} = \text{Id} - \widehat{R}_{1,\sigma}$, we can then find an operator \widehat{Q}' , in an appropriate calculus with bounds on \widehat{Z} , with the property that

$$I_{i,j}(\widehat{Q}') \circ I_{i,j}(\widehat{P}) = I_{i,j}(\widehat{R}_{1,\sigma})$$

for each admissible i, j . The operator $\widehat{Q} = \widehat{Q}_\sigma + \widehat{Q}'$ is such that $\widehat{Q} \circ \widehat{P} = \text{Id} - \widehat{R}_1$ with \widehat{R}_1 belonging to the same calculus with bounds as \widehat{Q}' but with all indicial operators equal to zero. The push-forward under the blow-down map

$$\beta_b^2 : \widehat{Z}_b^2 \rightarrow \widehat{Z}^2$$

of the Schwartz kernel of \widehat{R}_1 belongs to the space $\rho_{\text{lb}}^\varepsilon \rho_{\text{rb}}^\varepsilon H_{b,\text{loc}}^\infty(\widehat{Z}^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$ which is what is needed to conclude the proof.

Remark. The Proposition just established, together with the Proposition 7.2, proves once again that $\Pi_{\text{null}(P)}$ is Γ -trace class.

Definition 8.3. Let $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ be elliptic. We define the Γ -index of P as

$$\text{ind}_\Gamma P = \text{Tr}_\Gamma \Pi_{\text{null}(P)} - \text{Tr}_\Gamma \Pi_{\text{null}(P^*)} \equiv \dim_\Gamma \text{null}(P) - \dim_\Gamma \text{null}(P^*). \quad (8.4)$$

As a corollary of Proposition 8.1 we get

Proposition 8.4. *If $P \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$ is elliptic and satisfies (7.1) then $\text{ind}_\Gamma P < \infty$.*

9. Virtually nilpotent groups and spectral properties.

In this section we shall always assume that the group Γ is virtually nilpotent. Our main objective is to show that b -Dirac laplacians satisfying assumption (7.2) have a spectral function which is Γ -trace class near zero. The fundamental step is the following:

Proposition 9.1. *Let Γ be virtually nilpotent and let \tilde{D}^2 be a Dirac-type laplacian belonging to $\text{Diff}_{b,\Gamma}^2(\tilde{M}; \tilde{E}, \tilde{E})$, with $\tilde{D}_0 \in \text{Diff}_\Gamma(\partial\tilde{M}; \tilde{E}_0, \tilde{E}_0)$ satisfying assumption (7.2). Then there exists a parametrix $Q \in \Psi_{b,\Gamma}^{-2,\delta}(\tilde{M}; \tilde{E}, \tilde{E})$ with remainders R_0, R_1 which are Γ -trace class. An analogous statement holds for the Dirac operator \tilde{D} itself.*

Proof. We first analyze the behaviour near the front face. The parametrix Q of Theorem 7.1 is the sum of a symbolic ε -local parametrix Q_σ and of an operator $Q' \in \Psi_{b,\Gamma}^{-\infty,\delta}$ with indicial operator

$$K'(s, y, y') = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^{i\lambda} K((\tilde{D}_0^2 + \lambda^2)^{-1} \circ I(R_{0,\sigma}, \lambda))(y, y') d\lambda. \quad (9.1)$$

Using finite propagation speed estimates as in the proof of Proposition 1.5 we see that the Schwartz kernel of the operator $(\tilde{D}_0^2 + \lambda^2)^{-1}$ is rapidly decreasing, uniformly in λ , outside the diagonal in $\partial\tilde{M} \times \partial\tilde{M}$. Since $I(R_{0,\sigma}, \lambda)$ is smoothing and almost local, uniformly in λ , we conclude that the Schwartz kernel defined by (9.1) is smoothing and rapidly decreasing in the sense of Definition 4.6 (ii) with $r = 0$. Since Q' can be chosen to be supported near

the front face and since $R_0 = R_{0,\sigma} - P \circ Q'$ and $R_1 = R_{1,\sigma} - Q' \circ P$, we see that the remainders R_0, R_1 vanish on the front face, are rapidly decreasing in the sense of Definition 4.6 in a neighborhood of the front face and ε -local in the complement of such a neighborhood. According to Proposition 7.2 3] R_0 and R_1 belong to the von Neumann Algebra \mathcal{A}_Γ and are Γ -trace class. This proves the statement for \tilde{D}^2 . The proof of the one for \tilde{D}_0 proceeds exactly in the same way.

Theorem 9.2. *Let $E_{\tilde{D}^2}(\Delta) \in \mathcal{A}_\Gamma$ be the spectral function associated to the Borel set Δ for the generalized b -Laplacian \tilde{D}^2 . Let \tilde{D}_0 be the boundary operator and assume that $L^2 - \text{spec}(\tilde{D}_0) \cap (-\delta, \delta) = \emptyset$ for some $\delta > 0$. Then there exists $\varepsilon > 0$, which depends on δ , such that*

$$\text{Tr}_\Gamma(E_{\tilde{D}^2}((-\infty, \lambda])) < \infty \quad \forall \lambda \leq \varepsilon.$$

Proof. We shall first need the following Lemma for elements in a semifinite Von Neumann algebra \mathcal{A} endowed with a faithful, normal, semifinite trace τ . We denote by $\mathcal{S}_1(\mathcal{A}, \tau)$ the ideal of trace class elements. We refer to [Br][Sh 2] for its proof.

Lemma 9.3. *Let $P \in \mathcal{A}$ and assume $P = P^*$. Then P is invertible modulo $\mathcal{S}_1(\mathcal{A}, \tau)$ if and only if there exists an $\varepsilon > 0$ such that $\tau(E_P((-\varepsilon, \varepsilon))) < \infty$, with $E_P(\Delta)$ denoting the spectral function associated to the Borel set Δ .*

Going back to the proof of Theorem 9.2 we observe that the operator $(\text{Id} + \tilde{D}^2)^{-1}$ is an element in $\Psi_{b,\Gamma}^{-1,\delta}$ which is L_b^2 -continuous. Similarly the operator $P = (\text{Id} + \tilde{D}^2)^{-1} \circ \tilde{D}^2$ is bounded on L_b^2 and since it is obviously Γ -invariant it follows that $P \in \mathcal{A}_\Gamma$. Let $Q \in \Psi_{b,\Gamma}^{-2,\delta}$ be the rapidly decreasing parametrix constructed in Proposition 9.1. Let G be the operator $G = Q \circ (\text{Id} + \tilde{D}^2)$. Since $Q = Q_\sigma + Q'$, with Q' supported near the front face and rapidly decreasing, and since \tilde{D}^2 is differential we see that $G \in \Psi_{b,\Gamma}^{0,\delta}$ is the sum of a Γ -invariant ε -local 0th-order b -pseudodifferential operator and an element in $\Psi_{b,\Gamma}^{-\infty}$ which is supported near the front face and rapidly decreasing. As in Section 7 we have that G is bounded in L_b^2 and thus $G \in \mathcal{A}_\Gamma$. Let us show that G furnishes an inverse of P modulo Γ -trace class operators. According to proposition 9.1, $G \circ P = \text{Id} - R_1$, where R_1 belongs to the ideal $\mathcal{S}_1(\mathcal{A}_\Gamma, \tau)$ of Γ -trace class operators. A standard argument then shows that $P \circ G - \text{Id} \in \mathcal{S}_1(\mathcal{A}_\Gamma, \tau)$.

By Lemma 9.3 we infer that there exists an $\varepsilon' > 0$, $\varepsilon' \leq \delta$ such that the spectral function $E_P((-\infty, \varepsilon'])$ is Γ -trace class. It follows that $E_P((-\infty, \lambda])$ is also Γ -trace class $\forall \lambda \leq \varepsilon'$. Since $P = \tilde{D}^2 \circ (\text{Id} + \tilde{D}^2)^{-1}$ we conclude that there exists an $\varepsilon < \delta$ such that

$$\text{Tr}_\Gamma(E_{\tilde{D}^2}((-\infty, \lambda])) < \infty \quad \forall \lambda \leq \varepsilon$$

and the theorem is proved.

Remark. The result just proved allows for the definition of b -Novikov-Shubin invariants.

10. The heat equation on Galois coverings with boundary.

In this section we give a treatment of the heat equation on Γ -coverings which is suited to our particular needs. In particular, we assume the group Γ to be virtually nilpotent. The advantage in making this assumption is that it is then possible to build directly a space of kernels with the right space-variables decaying estimates to which the fundamental solution of the heat equations belongs. The decaying estimates translate as usual in the property of belonging to a \mathcal{B}^∞ -calculus which is in turn a fundamental requirement in order to define traces and b -traces (see Section 13).

First we assume that \tilde{N} is a Γ -cover without boundary with base N . The definition of the heat-space \tilde{N}_H^2 proceeds exactly as in chapter seven of [M]; thus \tilde{N}_H^2 is obtained by t -parabolically blowing up $B_H = \{(0, p, p) \mid p \in \tilde{N}\}$ in $\mathbb{R}^+ \times \tilde{N} \times \tilde{N}$. The group Γ acts on $\mathbb{R}^+ \times \tilde{N} \times \tilde{N}$ by the formula $(t, p, p') \cdot \gamma = (t, p \cdot \gamma, p' \cdot \gamma)$. Also Γ acts on the temporal front face $\text{tf}(\tilde{N}_H^2) \rightarrow B_H$ with the action on the interior equal to the Γ -action over $T\tilde{N}$. We fix Γ -invariant boundary defining functions ρ_{tf} , ρ_{tb} for the temporal front face and for the temporal boundary respectively.

Definition 10.1. *Let $-k \in \mathbb{N}$. We define the heat-calculus of order k , $\Psi_{H,\Gamma}^k(\tilde{N}; \tilde{\Omega}^{\frac{1}{2}})$, as the space of Γ -invariant kernels*

$$K \in \rho_{\text{tf}}^{-\frac{1}{2}(n+3)-k} C^\infty(\tilde{N}_H^2, \tilde{\Omega}^{\frac{1}{2}}); \quad K|_{\text{tb}} \equiv 0$$

with the following additional property : $\exists \varepsilon \in]0, 1[$ such that for any multi-index α of derivation with respect to $(t, z, w) \in \mathbb{R}^{+*} \times \tilde{N}^2$, for any $q \in \mathbb{N}$

and any $T > 0$:

$$\sup_{0 < t < T} \sup_{d(z,w) > \varepsilon} |\nabla^\alpha(K \otimes \mu)(t, z, w)|(1 + d(z, w))^q < \infty. \quad (10.1)$$

μ being the lift of a riemannian trivializing $-\frac{1}{2}$ density.

We define $\Psi_{H, \text{ev}, \Gamma}^k(\tilde{N}; \tilde{\Omega}^{\frac{1}{2}})$ by requiring K to belong to the space

$\rho_{\text{tf}}^{-\frac{1}{2}(n+3)-k} C_{\text{even}}^\infty(\tilde{N}_H^2, \tilde{\Omega}^{\frac{1}{2}})$ (see [M] for more details on the definitions used here and in the sequel). The definition with a bundle \tilde{F} instead of $\tilde{\Omega}^{\frac{1}{2}}$ can be given as usual by tensoring with the smooth sections of the appropriate homomorphism bundle. Since the following lemma may be proved as in [M] page 262, we omit its proof.

Lemma 10.2. *Let $A \in \Psi_{H, \Gamma}^k(\tilde{N}; \tilde{E})$ with $-k \in \mathbb{N}$. Then*

$$A : \mathcal{S}(\tilde{N}, \tilde{E}) \rightarrow t^{-\frac{k}{2}-1} C^\infty([0, +\infty[_{\frac{1}{2}} \times \tilde{N}, \tilde{E})$$

and $\forall t > 0, \forall u \in \mathcal{S}(\tilde{N}, \tilde{E}), Au(t, \cdot)$ defines an element in $\mathcal{S}(\tilde{M}, \tilde{E})$. If moreover $A \in \Psi_{H, \text{ev}, \Gamma}^{-2}$ then for any $t \geq 0, Au(t, \cdot)$ defines an element of $\mathcal{S}(\tilde{M}, \tilde{E})$.

Notice that if $A \in \Psi_{H, \text{ev}, \Gamma}^{-2}$ then the restriction A_0 of the operator A to $t = 0$ is well defined

$$A_0 : \mathcal{S}(\tilde{M}, \tilde{E}) \rightarrow \mathcal{S}(\tilde{M}, \tilde{E}), \quad A_0 \psi = (A\psi)|_{t=0}$$

Theorem 10.3. *Let $\tilde{P} \in \text{Diff}_\Gamma^2(\tilde{N}, \tilde{E})$ be the lift to \tilde{N} of an elliptic self-adjoint differential operator on M of order two, with non-negative principal symbol. Then there is a unique element $H_{\tilde{P}} \in \Psi_{H, \text{ev}, \Gamma}^{-2}(\tilde{N}; \tilde{E})$ such that*

$$(\partial_t + \tilde{P})H_{\tilde{P}} = 0 \text{ in } t > 0, \quad (H_{\tilde{P}})|_{t=0} = \text{Id}. \quad (10.2)$$

The kernel $H_{\tilde{P}}$ defines a semigroup of smoothing operators $\exp(-t\tilde{P})$ each of which is *rapidly decreasing* on $\tilde{N} \times \tilde{N}$.

Proof. The structure of the proof is precisely as in [M]. First we construct a parametrix $G \in \Psi_{H, \text{ev}, \Gamma}^{-2}$; thus

$$t(\partial_t + \tilde{P})G = R \in \Psi_{H, \Gamma}^{-\infty}, \quad G_0 = \text{Id}.$$

We can always arrange for G and R to be ε -local, i.e. $G(t, p, p') \equiv 0$ if $d(p, p') \geq \varepsilon$. Recall the convolution product \star of [M]. The proof of the theorem is completed by using the following

Lemma 10.4. *The kernel defined by*

$$H_{\widetilde{P}} = G + G \star \sum_{k=1}^{+\infty} \left(-\frac{R}{t}\right)^{\star k}$$

belongs to $\Psi_{H,\text{ev},\Gamma}^{-2}$ and is a solution of the heat equation as in (10.2)

Proof of the Lemma. One proves that this kernel is indeed the solution of the heat equation exactly in [M] page 271. Let T be > 0 ; then one checks that for each $k \in \mathbb{N}$, $\left(-\frac{R}{t}\right)^{\star k}$ is $k\varepsilon$ -local and that its Schwartz kernel can be estimated for all $t \in]0, T[$, by $C_T^{k+1}/k!$ where C_T is a constant which does not depend on k . Lastly, in order to prove that the heat kernel satisfies the decay properties of Definition 10.1, one uses the trick introduced in the proof of proposition 1.6 (see inequality (1.12)). Namely we assume that $d(z, w) > 100$ and set $k_0 = \lfloor \sqrt{d(z, w)} \rfloor - 5$. If $k < k_0$, then $\left(-R/t\right)^{\star k}(z, w)$ is zero. Since $\sum_{k \geq k_0} C_T^{k+1}/k!$ is rapidly decreasing with respect to k_0 as k_0 goes to $+\infty$, we have proved the lemma.

Next we consider a Γ -cover with boundary \widetilde{M} of the compact manifold with boundary M . The b -heat space \widetilde{M}_η^2 is obtained by taking the t -parabolic blow up of $\mathbb{R}^+ \times \widetilde{M}_b^2$ along $B_H = \{(0, p, p) \mid p \in \widetilde{M}\}$ in $\mathbb{R}^+ \times \widetilde{M} \times \widetilde{M}$. \widetilde{M}_η^2 is thus a manifold with corners which has five boundary hypersurfaces; namely, the two frontfaces $\text{bf}(\widetilde{M}_\eta^2)$, $\text{tf}(\widetilde{M}_\eta^2)$, and the three others $\text{lb}(\widetilde{M}_\eta^2)$, $\text{rb}(\widetilde{M}_\eta^2)$, $\text{tb}(\widetilde{M}_\eta^2)$. The group Γ acts naturally on \widetilde{M}_η^2 (via a diagonal action on the last two factors) and on the boundary hypersurfaces. We fix Γ -invariant defining functions ρ_{tf} , ρ_{tb} for the temporal front face and the temporal boundary respectively. In the next definition we will consider the neighborhood $\mathcal{O}(\text{bf})$ of $\text{bf}(\widetilde{M}_b^2)$ in \widetilde{M}_b^2 introduced in Definition 4.6.

Definition 10.5. *Let $-k \in \mathbb{N}$. We define the b -heat-calculus of order k , $\Psi_{\eta,\Gamma}^k(\widetilde{M}; {}^b\widetilde{\Omega}^{\frac{1}{2}})$, as the space of Γ -invariant kernels*

$$K \in \rho_{\text{tf}}^{-\frac{1}{2}(n+3)-k} C^\infty(\widetilde{M}_\eta^2, {}^b\widetilde{\Omega}^{\frac{1}{2}}); \quad K \equiv 0 \text{ at } \text{tb} \cup \text{lb} \cup \text{rb}$$

such that the two following additional properties are satisfied for a suitable $\varepsilon \in]0, 1[$:

- (i) *for any multi-index of derivation α with respect to (t, z, w) , for any $q \in \mathbb{N}$ and any $T > 0$: there exists a constant $C_{\alpha,q,T}$ such that*

$$\forall t \in (0, T) \forall (z, w) \in \widetilde{M}^2 \setminus \mathcal{O}(\text{bf}) \text{ such that } d(z, w) > \varepsilon$$

$$|\nabla^\alpha(K(t, z, w) \otimes \mu)|(1 + d(z, w))^q < C_{\alpha, q, T}. \quad (10.3)$$

(ii) for any multi-index β of derivation with respect to (t, r, τ, y, y') , for any $q \in \mathbb{N}$, and any real $T > 0$: there exists a constant $D_{\beta, q, T}$ such that $\forall t \in (0, T)$

$$\forall (r, \tau, y, y') \in \mathcal{O}(\text{bf}) \text{ such that } d(y, y') > \varepsilon$$

$$|\nabla^\beta(K \otimes \mu)(t, r, \tau, y, y')|(1 + d(y, y'))^q < D_{\beta, q, T}. \quad (10.4)$$

where we have as usual denoted by μ a trivializing riemannian lifted b -density.

We define $\Psi_{\eta, \text{ev}, \Gamma}^k(\widetilde{M}; {}^b\widetilde{\Omega}^{\frac{1}{2}})$ by requiring K to belong to

$\rho_{\text{tf}}^{-\frac{1}{2}(n+3)-k} C_{\text{even}}^\infty(\widetilde{M}_\eta^2, {}^b\widetilde{\Omega}^{\frac{1}{2}})$. The following theorem claims the existence and unicity of the solution of the b -heat equation in the b -heat calculus of \widetilde{M}_η^2 . We will omit the proof which uses arguments from [M] and from the boundaryless case above.

Theorem 10.6. *Let $\widetilde{P} \in \text{Diff}_{b, \Gamma}^2(\widetilde{M}, \widetilde{E})$ be the lift to \widetilde{M} of an elliptic self-adjoint b -differential operator on M of order two, with non-negative principal symbol. Then there is a unique element $H_{\widetilde{P}} \in \Psi_{\eta, \text{ev}, \Gamma}^{-2}(\widetilde{M}; \widetilde{E})$ such that*

$$(\partial_t + \widetilde{P})H_{\widetilde{P}} = 0 \text{ in } t > 0, \quad (H_{\widetilde{P}})|_{t=0} = \text{Id}. \quad (10.5)$$

Remark. Since H_P belongs to the b -heat calculus, we see that near the front face, the derivatives $\partial_r^p \partial_\tau^q K_{H_P}$ of the Schwartz kernel of H_P are smooth functions (as a consequence we can certainly define the b - Γ -Trace). Equivalently, we can control the derivatives $\partial_s^p \partial_x^q K_{H_P}$ of the heat kernel near the front face, with respect to the projective coordinates $s = (\tilde{x}/\tilde{x}')$, x . Such a property does not follow in an obvious way from the construction of the heat kernel on \widetilde{M} viewed as a complete manifold with cylindrical ends. In fact finite propagation speed estimates only allow to control the derivatives with respect to $x \partial_x = \partial_u$, $u = \log x$.

Proposition 10.7. *Let $\widetilde{D}^- \widetilde{D}^+$ be a Dirac laplacian on a Γ -covering endowed with an exact b -metric. Assume that the boundary operator of \widetilde{D}^+ admits a bounded L^2 -inverse. Then*

$$\lim_{t \rightarrow \infty} b - \text{Tr}_\Gamma e^{-t\widetilde{D}^- \widetilde{D}^+} = \text{Tr}_\Gamma(\Pi_{\text{null } \widetilde{D}^+}) = \dim_\Gamma(\text{null } \widetilde{D}^+)$$

Proof. Since the boundary operator \tilde{D}_0 is assumed to be invertible, we can define a small real $\delta > 0$ and a contour \mathcal{C}_ε in the complex domain such that for any $\lambda \in \mathcal{C}_\varepsilon$ the member of the indicial family

$$I(\tilde{D}^- \tilde{D}^+ - \lambda \text{Id}, z) \text{ is invertible } \forall z \text{ in the strip } |\Im z| < \delta. \quad (10.6)$$

The contour \mathcal{C}_ε is explicitly given by the union of C_ε and its complex conjugate $\overline{C_\varepsilon}$, with C_ε equal to be the union of a halfline \mathcal{L} with origin $\sigma + iA$ and of the three segments:

$$[-\varepsilon, -\varepsilon + i\varepsilon], [-\varepsilon + i\varepsilon, \sigma + i\varepsilon], [\sigma + i\varepsilon, \sigma + iA].$$

Here A and the slope of \mathcal{L} are $\gg 0$, and the positive reals $\varepsilon, \sigma, \delta$ are very small.

Now we analyze the resolvent of $\tilde{D}^- \tilde{D}^+$ proceeding as in [M]. Let $\phi(t) \in C_c^\infty([0, +\infty[)$ be such that $\phi(t) \equiv 1$ for $0 \leq t \leq 1$. We then set $G_s(\lambda) = \int_0^{+\infty} e^{t\lambda} e^{-t\tilde{D}^- \tilde{D}^+} \phi(t) dt$, for any $\lambda \in \mathcal{C}_\varepsilon$. So we have:

$$(\tilde{D}^- \tilde{D}^+ - \lambda \text{Id}) \circ G_s(\lambda) = \text{Id} - R_s(\lambda), \quad R_s(\lambda) = - \int_0^{+\infty} e^{t\lambda} e^{-t\tilde{D}^- \tilde{D}^+} \phi'(t) dt$$

Thanks to (10.6) we can construct as in [M] $G_B(\lambda) \in \Psi_{b,\Gamma}^{-\infty,\delta}$ so that for each $\lambda \in \mathcal{C}_\varepsilon$:

$$(\tilde{D}^- \tilde{D}^+ - \lambda \text{Id}) \circ [G_s(\lambda) + G_B(\lambda)] = \text{Id} - R_r(\lambda)$$

where $R_r(\lambda)$ belongs to $\rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta H_{b,\Gamma,\text{loc}}^\infty(\tilde{M}^2, {}^b\tilde{\Omega}^{\frac{1}{2}})$ and its Schwartz kernel is rapidly decreasing with respect to $d(z, z')$ and λ belonging to the contour \mathcal{C}_ε . Now we set $G_r(\lambda) = (\tilde{D}^- \tilde{D}^+ - \lambda \text{Id})^{-1} \circ R_r(\lambda)$. Thus we have the following decomposition of the resolvent for any $\lambda \in \mathcal{C}_\varepsilon$:

$$(\tilde{D}^- \tilde{D}^+ - \lambda \text{Id})^{-1} = G_s(\lambda) + G_B(\lambda) + G_r(\lambda)$$

Using the spectral measure representation of $\tilde{D}^- \tilde{D}^+$ we see that for any real $t > 1$:

$$b - \text{Tr}_\Gamma \exp(-t\tilde{D}^- \tilde{D}^+) = \frac{1}{2i\pi} b - \text{Tr}_\Gamma \left(\int_{\mathcal{C}_\varepsilon} \exp(-t\lambda) (\tilde{D}^- \tilde{D}^+ - \lambda \text{Id})^{-1} d\lambda \right)$$

Moreover we can let ε go to 0^+ in the previous equality. We denote by \mathcal{C}_{0^+} the limit contour. In view of theorem 9.2 we can choose σ so that $\forall \lambda \in [0, 2\sigma]$ the spectral projection $E_{\tilde{D}^-, \tilde{D}^+}(\lambda)$ is Γ -trace class. Using an integration by parts with respect to λ we can certainly show that:

$$\lim_{t \rightarrow +\infty} \int_0^\sigma \exp(-t\lambda) (b - \text{Tr}_\Gamma dE_\lambda) = b - \text{Tr}_\Gamma \Pi_{\text{null } \tilde{D}^+} = \text{Tr}_\Gamma \Pi_{\text{null } \tilde{D}^+} \tag{10.7}$$

On the other hand

$$dE_{\tilde{D}^-, \tilde{D}^+}(\lambda) = \frac{1}{2i\pi} [(\tilde{D}^- \tilde{D}^+ - (\lambda + i0^+) \text{Id})^{-1} - (\tilde{D}^- \tilde{D}^+ - (\lambda - i0^+) \text{Id})^{-1}] d\lambda$$

so that, from (10.7), we obtain

$$\lim_{t \rightarrow +\infty} \frac{1}{2i\pi} b - \text{Tr}_\Gamma \left(\int_{\mathcal{C}_{0^+} \cap \{\Re \lambda < \sigma\}} \exp(-t\lambda) [(\tilde{D}^- \tilde{D}^+ - \lambda \text{Id})^{-1}] d\lambda \right) = \text{Tr}_\Gamma (\Pi_{\text{null } \tilde{D}^+})$$

Now we observe that the L_b^2 -operator norm of $\varepsilon_1 (\tilde{D}^- \tilde{D}^+ - (\sigma \pm \varepsilon_1 i) \text{Id})^{-1}$ remains bounded as ε_1 goes to 0^+ . Thus using the structure of $G_s(\lambda)$, $G_B(\lambda)$, $G_r(\lambda)$ we see easily that:

$$\lim_{t \rightarrow +\infty} \frac{1}{2i\pi} b - \text{Tr}_\Gamma \left[\int_{\mathcal{C}_{0^+} \cap \{\Re \lambda \geq \sigma\}} \exp(-t\lambda) (\tilde{D}^- \tilde{D}^+ - \lambda \text{Id})^{-1} d\lambda \right] = 0.$$

Proposition 10.7 is now proved.

Recall now the definition of the Γ -eta invariant given in the remark following Theorem 2.7. Using the properties of the heat kernel we can prove

Proposition 10.8. *Let \tilde{M} be an even dimensional Γ -covering with boundary as in section 4. Assume Γ -virtually nilpotent. Let $\tilde{D} \in \text{Diff}_{b,\Gamma}^1(\tilde{M}; \tilde{E}, \tilde{E})$ be a Dirac-type \mathbb{Z}_2 -graded b -differential operator. Under the assumption that $0 \notin \text{spec } \tilde{D}_0$ the following formula holds :*

$$\text{ind}_\Gamma \tilde{D}^+ = \int_M \hat{A}(M) \text{Ch}'(E) - \frac{1}{2} \eta_\Gamma(0).$$

Proof. Using the short time behaviour of the b -heat kernel and the rescaled b -heat calculus, which is nothing but the b -version of Getzler calculus, we get:

$$\lim_{t \rightarrow 0^+} b - \text{STr}_\Gamma \exp(-t\tilde{D}^2) = \int_M \hat{A}(M) \text{Ch}'(E)$$

On the other hand, Proposition 10.7 shows that:

$$\lim_{t \rightarrow 0^+} b - \text{STr}_\Gamma \exp(-t\tilde{D}^2) = \text{ind}_\Gamma \tilde{D}^+.$$

Thus by the fundamental theorem of calculus the difference between the Γ -index and the Atiyah-Singer integral is given by the integral from $t = 0$ to $t = \infty$ of the derivative of the b - Γ -Supertrace of the heat kernel of \tilde{D}^2 . Proceeding as in the introduction of [M] (page 6) we can express this derivative as the b - Γ -supertrace of a commutator. By applying Proposition 6.3 we obtain the Γ -eta invariant contribution as defined in the Remark following the statement of Theorem 2.7. The Proposition is proved.

Remark. The result given above should be viewed as an Atiyah-Patodi-Singer index theorem on Galois coverings with boundary. Ramachandran [R], in the context of non-local boundary problems, has given a version of the above proposition with no assumptions on the boundary operator and only assuming the group Γ finitely presented. His result is thus much more general than Proposition 10.8 above. However in the truly higher case, which is what we are really interested in, our assumptions are crucial not only in *proving* the higher index theorem but even in *defining* the higher eta invariant appearing in the main formula (see [L2] and the remark following the statement of Theorem 2.7).

11. The Λ - b -Mishenko-Fomenko calculus.

In this section we introduce the b -Mishenko-Fomenko calculus on the manifold M . We only assume the group Γ to be finitely presented. Many of the arguments are straightforward generalizations of the concepts in [MF] [M] and are therefore only sketched. The fundamental result is the $C_r^*(\Gamma)$ -Fredholm property for elliptic b -differential $C_r^*(\Gamma)$ -operators whose indicial family is invertible.

Let $\Gamma \rightarrow \widetilde{M} \rightarrow M$ be a Galois covering with boundary as in section 4 and let F_0, F_1 be two complex vector bundles over M with lifts $\widetilde{F}_0, \widetilde{F}_1$ over \widetilde{M} . We define the C^* -algebra-bundles of $\mathcal{F}_0, \mathcal{F}_1$ as in the beginning of section 1. Let us consider the trivial $C_r^*(\Gamma)$ -bundle over \widetilde{M} , $\widetilde{C}_r^*(\Gamma) = \widetilde{M} \times C_r^*(\Gamma)$. As in [CM] we identify

$$C^\infty(M, \mathcal{F}_i) \leftrightarrow (C^\infty(\widetilde{M}, \widetilde{C}_r^*(\Gamma) \otimes \widetilde{F}_i))^\Gamma.$$

Let us write, as usual, Λ for $C_r^*(\Gamma)$. The space $\text{Diff}_{b,\Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ of Λ - b -differential operators is obtained by considering the restriction of the operators

$$\text{Id} \otimes \widetilde{P} : C^\infty(\widetilde{M}, \widetilde{C}_r^*(\Gamma) \otimes \widetilde{F}_0) \rightarrow C^\infty(\widetilde{M}, \widetilde{C}_r^*(\Gamma) \otimes \widetilde{F}_1)$$

with $\widetilde{P} \in \text{Diff}_{b,\Gamma}^m(\widetilde{M}; \widetilde{F}_0, \widetilde{F}_1)$, to the Γ -invariant elements. We shall denote the restriction of $\text{Id} \otimes \widetilde{P}$ to $C^\infty(M; \mathcal{F}_0)$ either as $\text{Id} \otimes_\Gamma \widetilde{P}$ or as \mathcal{P} . Notice that we have resumed here the tilde-notation for Γ -invariant operators on \widetilde{M} .

If $x \in C^\infty(M)$ is a boundary defining function for ∂M then, in local coordinates (x, y_1, \dots, y_n) , each Λ - b -differential operator of order m can be written as a matrix of operators of the following type:

$$\sum_{k, \alpha} a_{k, \alpha} \left(x \frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial y_n}\right)^{\alpha_n} \quad (11.1)$$

with $a_{k, \alpha} \in C^\infty(M, \Lambda)$. In fact we can also define the algebra of Λ - b -differential operators directly in terms of (11.1).

The principal symbol of $\mathcal{P} \in \text{Diff}_{b, \Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ is well defined as an element in $C^\infty({}^bT^*M, \text{Hom}_\Lambda(p^*\mathcal{F}_0, p^*\mathcal{F}_1))$, with $p : {}^bT^*M \rightarrow M$ the projection map, and there is a well defined notion of ellipticity.

Once a b -metric on M has been fixed, we can consider b -Sobolev spaces $\mathcal{H}_b^s(M, \mathcal{F})$ associated to the sections of a Λ -vector bundle \mathcal{F} over M ; the b - Λ -hermitian scalar product for $L_b^2(M, \mathcal{F})$ is defined as in (1.4) but using the b -riemannian density. These spaces are Λ -Hilbert modules (isomorphic to $\ell^2(\Lambda)$) and Banach spaces. If $\mathcal{P} \in \text{Diff}_{b, \Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ then it is easily seen that \mathcal{P} defines a bounded operator $\mathcal{P} : \mathcal{H}_b^k \rightarrow \mathcal{H}_b^{k-m}$ for each $k \in \mathbb{Z}$ which is a Λ -module homomorphism.

If $\mathcal{P} = \text{Id} \otimes_\Gamma \tilde{P}$ is a Λ - b -differential operator, then we can define its indicial family as $I(\mathcal{P}, z) = \text{Id} \otimes_\Gamma I(\tilde{P}, z)$, $z \in \mathbb{C}$. It's a family of Λ -differential operators on ∂M which is elliptic if \mathcal{P} is. The indicial family can also be defined in terms of (11.1). Notice that $I(\mathcal{P}, z)$ is holomorphic in z (in the sense that its coefficients, which belongs to $C^\infty(M, \text{Hom}_\Lambda(\mathcal{F}_0, \mathcal{F}_1))$, depend holomorphically on z).

Let $\mathcal{P} \in \text{Diff}_{b, \Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ be elliptic and such that $\exists \delta > 0$:

$$\exists s \in \mathbb{Z} \mid \forall \lambda \in \mathbb{R} \times [-i\delta, i\delta] \quad I(\mathcal{P}, \lambda) \text{ admits a bounded inverse}$$

$$\text{as a map } \mathcal{H}_b^s \rightarrow \mathcal{H}_b^{s-m}. \quad (11.2)$$

Notice that if $I(\mathcal{P}, \lambda)$ admits an inverse as an element in $B(\mathcal{H}_b^s, \mathcal{H}_b^{s-m})$ then it does so as a bounded map $\mathcal{H}_b^k \rightarrow \mathcal{H}_b^{k-m}$ for each $k \in \mathbb{Z}$. The proof is standard once we use the Mishenko-Fomenko calculus on ∂M .

Theorem 11.1. *Let $\mathcal{P} \in \text{Diff}_{b, \Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ be elliptic and assume (11.2). Then for each $k \in \mathbb{Z}$, \mathcal{P} is Λ -Fredholm as a map*

$$\mathcal{P} : \mathcal{H}_b^k(M, \mathcal{F}_0) \rightarrow \mathcal{H}_b^{k-m}(M, \mathcal{F}_1)$$

with Mishenko-Fomenko Index class

$$\text{Ind}(\mathcal{P}) = [\mathcal{L}] - [\mathcal{M}] \in K_0(\Lambda) \tag{11.3}$$

independent of $k \in \mathbb{Z}$.

Proof. Recall [MF, Th. 2.4] that a sufficient condition for a continuous Λ -homomorphism P between two Hilbert Λ -modules H_1, H_2 to be Λ -Fredholm is that there exists an inverse of $P : H_1 \rightarrow H_2$ modulo Λ -compact operators. Applying this result to $\mathcal{P} : \mathcal{H}_b^k \rightarrow \mathcal{H}_b^{k-m}$ we see that it suffices to find for each $k \in \mathbb{Z}$ a continuous Λ -homomorphism $\mathcal{Q} : \mathcal{H}_b^{k-m} \rightarrow \mathcal{H}_b^k$ with the property that

$$\mathcal{P} \circ \mathcal{Q} = \text{Id} - \mathcal{R}_1 \quad \mathcal{Q} \circ \mathcal{P} = \text{Id} - \mathcal{R}_2 \tag{11.4}$$

with $\mathcal{R}_1, \mathcal{R}_2 : \mathcal{H}_b^k \rightarrow \mathcal{H}_b^k$ Λ -compact. In order to prove Theorem 11.1 we shall need the following

Lemma 11.2. *Let $x \in C^\infty(M)$ be a boundary defining function for $\partial M \subset M$ and let \mathcal{F} be a Λ -bundle on M . For any $\varepsilon > 0$ and for any $s > t$ the inclusion*

$$x^\varepsilon \mathcal{H}_b^s(M, \mathcal{F}) \hookrightarrow \mathcal{H}_b^t(M, \mathcal{F}) \tag{11.5}$$

is Λ -compact.

Proof. We will only deal with the case $t = 0$. Since this question is local, we can replace \mathcal{E}^+ by the one-dimensional trivial bundle Λ and we have to show that the injection:

$$x^\varepsilon H_b^s(M; \Lambda) \rightarrow L_b^2(M; \Lambda)$$

is Λ -compact. Let $\delta > 0$, then we can find $C_\delta > 0$ so that:

$$\forall u \in x^\varepsilon H_b^s(M, \Lambda), \quad \|u\|_{L_b^2(M, \Lambda)} \leq \delta^\varepsilon \|u\|_{x^\varepsilon L_b^2(M \cap \{x < \delta\}; \Lambda)} +$$

$$C_\delta \|u\|_{x^\varepsilon H_b^s(M \cap \{x > \delta\}; \Lambda)}$$

Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $x^\varepsilon H_b^s(M; \Lambda)$. Since $(e_n)_{n \geq 0}$ converges weakly to 0 in $L^2(M \cap \{x > \delta\}; \Lambda)$ and the injection $H^s(M \cap \{x > \delta\}; \Lambda) \rightarrow L^2(M \cap \{x > \delta\}; \Lambda)$ is compact, we see that for all $\eta > 0$ we can find a real N_η so that:

$$\forall n \geq N_\eta, \quad \|e_n\|_{L_b^2(M; \Lambda)} \leq \delta^\varepsilon + C_\delta \eta$$

The lemma is thus proved.

The strategy for proving Theorem 11.1 is of course to develop a b -Mishenko-Fomenko pseudodifferential calculus where a parametrix \mathcal{Q} of \mathcal{P} can be found.

In order to rigorously develop such a calculus we would need several pages of rather obvious material. It is in fact clear that once the algebra of C^∞ functions on M and on M_b^2 has been replaced by the algebra of C^∞ functions with values in Λ , the development of the small calculus $\Psi_{b,\Lambda}^*(M, \mathcal{F}_0, \mathcal{F}_1)$ and of the calculus with bounds $\Psi_{b,\Lambda}^{*,\delta}(M; \mathcal{F}_0, \mathcal{F}_1)$ is simply the disjoint union of the material presented in [MF] and [M]. Thus we leave to the reader

- (1) the definition of the small b - Λ -calculus

$$\Psi_{b,\Lambda}^*(M; \mathcal{F}_0, \mathcal{F}_1) \supset \text{Diff}_{b,\Lambda}^*(M; \mathcal{F}_0, \mathcal{F}_1).$$

- (2) the construction of a symbolic parametrix for each elliptic b - Λ -differential operator.

- (3) the definition of the Λ -calculus with bounds

$$\begin{aligned} \Psi_{b,\Lambda}^{*,\delta}(M; \mathcal{F}_0, \mathcal{F}_1) &= \Psi_{b,\Lambda}^*(M; \mathcal{F}_0, \mathcal{F}_1) + \tilde{\Psi}_{b,\Lambda}^{-\infty,\delta,\delta}(M; \mathcal{F}_0, \mathcal{F}_1) \\ &\quad + \rho_{\text{lb}}^\delta \rho_{\text{rb}}^\delta \mathcal{H}_b^\infty(M^2, \text{Hom}(\mathcal{F}_0^* \otimes {}^b\Omega^{-1}, \mathcal{F}_1)) \end{aligned}$$

as in Section 7

- (4) the proof of the continuity properties of the calculus with bounds on Sobolev spaces \mathcal{H}_b^* .

- (5) the proof of the following

Proposition 11.3. *Given $\mathcal{P} \in \text{Diff}_{b,\Lambda}^m(M; \mathcal{F}_0, \mathcal{F}_1)$ elliptic and satisfying (11.2), there exists a parametrix $\mathcal{Q} \in \Psi_{b,\Lambda}^{-m,\delta}(M; \mathcal{F}_0, \mathcal{F}_1)$ such that*

$$\mathcal{P} \circ \mathcal{Q} = \text{Id} - \mathcal{R}_1, \quad \mathcal{Q} \circ \mathcal{P} = \text{Id} - \mathcal{R}_2, \quad \mathcal{R}_1, \mathcal{R}_2 \in \rho_{\text{bf}} \Psi_{b,\Lambda}^{-\infty,\delta}(M; \mathcal{F}_0, \mathcal{F}_1). \quad (11.6)$$

Using this proposition, the mapping properties of the elements in $\Psi_{b,\Lambda}^{*,\delta}$ as given in (4) above and Lemma 11.2 the proof of the first part of Theorem 11.1 follows. The fact that the index class does not depend on the particular choice of $k \in \mathbb{Z}$ also depends on the parametrix construction (the pseudoinverse on \mathcal{H}_b^{k-m} is induced by the same b - Λ -operator for each $k \in \mathbb{Z}$). The proof of Theorem 11.1 is now complete.

Notation. As in the closed case, we denote by $\text{End}_{b,\Lambda}^{\infty,\epsilon}(M, \mathcal{F})$ the space of Schwartz kernels defined by $\rho_{\text{lb}}^\epsilon \rho_{\text{rb}}^\epsilon \mathcal{H}_b^\infty(M^2, \text{Hom}(\mathcal{F}_0^* \otimes {}^b\Omega^{-1}, \mathcal{F}_1))$.

12. Virtually nilpotent groups and the $\Psi_{b, \mathcal{B}^\infty}^*$ -calculus.

We now assume that Γ is virtually nilpotent, (M, g_M) is an exact even dimensional b -manifold and $D^\pm \in \text{Diff}_b^1(M; E^\pm, E^\mp)$ is a Dirac-type operator. Our goal is to extend the results of section 1 to the b -case. Again we want to pass from the C^* -algebra $\Lambda = C_r^*(\Gamma)$ to the “smooth” subalgebra \mathcal{B}^∞ .

Observe first of all that from (11.6) it follows that the null space of $\mathcal{D}^+ : \mathcal{H}_b^1(M, \mathcal{E}^+) \rightarrow \mathcal{H}_b^0(M, \mathcal{E}^-)$ is contained in $\mathcal{H}_b^\infty(M, \mathcal{E}^+)$ which, however, is not contained in $C^\infty(M, \mathcal{E}^+)$. Thus we cannot hope for a \mathcal{B}^∞ -decomposition of $C^\infty(M, \mathcal{E}^{\infty, \pm})$ as in Section 1 and our first task is to define and characterize the space that will contain the “smooth” representatives of the index class $\text{Ind}(\mathcal{D}^+)$ which was introduced in the previous section (see Theorem 11.1).

Let $U \subset M$ a trivialisizing neighborhood for the bundles $E, \tilde{E}, \mathcal{E}, \mathcal{E}^\infty$. If $\phi \in C_c^\infty(U)$ and if $s \in L_b^2(M, \mathcal{E})$ then for each fixed $z \in U$ we can write, as in Section 1 (see (1.5))

$$(\phi s)(z) = \sum_{\gamma \in \Gamma} s_{\phi, \gamma}(z) \gamma. \quad (12.1)$$

Definition 12.1. We define $\mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ as the space of L_b^2 -sections of \mathcal{E} such that for each trivialisizing neighborhood $U \subset M$, for each smooth compactly supported function $\phi \in C_c^\infty(U)$ the following two properties are satisfied

(i) the $s_{\phi, \gamma}$ appearing in (12.1) belongs to $H_b^\infty(M, E)$ for any $\gamma \in \Gamma$.

(ii) $\forall N \in \mathbb{N}$, $\forall P \in \text{Diff}_b^*(M; E, E)$ we have:

$$\sup_{\gamma \in \Gamma} (1 + \|\gamma\|)^N \|Ps_{\phi, \gamma}\|_{L_b^2} < \infty.$$

If x is a boundary defining function for ∂M and $\varepsilon > 0$ then we define the space $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ by simply requiring the $s_{\phi, \gamma}$ in (12.1) to belong to $x^\varepsilon H_b^\infty(M, E)$ and by imposing the estimates in (ii) for the sections $x^{-\varepsilon} s_{\phi, \gamma}$.

We are interested in the space $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, \pm})$ because it will contain the smooth representatives of the index class $\text{Ind}(\mathcal{D}^+)$.

We now define the space corresponding to $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ on the cover \widetilde{M} .

Let $\varepsilon \geq 0$ and fix $p_0 \in \widetilde{M}$. Let $\tilde{x} \in C^\infty(\widetilde{M})$ be the lift of the boundary defining function $x \in C^\infty(M)$. Let $1_{B^c(p_0, R)}$ be the characteristic function of the complementary of the ball centered in p_0 and of radius R . Here the distance function is with respect to the lift of an ordinary metric on M . See the discussion leading to Definition 4.6.

Definition 12.2. We define $\mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E})$ as the space

$$\{u \in \tilde{x}^\varepsilon H_{b, \Gamma}^\infty(\widetilde{M}, \widetilde{E}) \mid \forall N \in \mathbb{N}, \forall P \in \text{Diff}_{b, \Gamma}^* \\ \sup_{R > 1} (R^N \|1_{B^c(p_0, R)} P(\tilde{x}^{-\varepsilon} u)\|_{L_b^2}) < \infty\}$$

The proof of the two following propositions is easy and will be left to the reader.

Proposition 12.3. For each $\varepsilon \geq 0$ the map

$$u \rightarrow s = \sum_{\gamma \in \Gamma} R_\gamma^*(u)\gamma$$

is a bijection between the space $\mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E})$ and $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$.

Next we define the space of “smooth endomorphisms” as the space $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M, \mathcal{E}^\infty)$ of operators T the Schwartz kernel of which belongs to

$$\rho_{\text{lb}}^\varepsilon \rho_{\text{rb}}^\varepsilon \mathcal{H}_b^\infty(M \times M, \text{Hom}(\mathcal{E}^\infty \otimes {}^b\Omega^{-1}, \mathcal{E}^\infty)).$$

When we have two different bundles $\mathcal{F}_0^\infty, \mathcal{F}_1^\infty$, a similar definition can be given, thus defining the space $\text{Hom}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M; \mathcal{F}_0^\infty, \mathcal{F}_1^\infty)$

On the cover \widetilde{M} we define the space $\mathcal{S}_{b,\Gamma}^{\infty,\varepsilon}(\widetilde{M}^2, \text{Hom}_\Omega(\widetilde{E}, \widetilde{E}))$, with $\text{Hom}_\Omega(\widetilde{E}, \widetilde{E}) = \text{Hom}(\widetilde{E} \otimes {}^b\Omega^{-1}, \widetilde{E})$, as the set of functions $u \in \tilde{x}^\varepsilon \tilde{x}'^\varepsilon H_b^\infty(\widetilde{M}^2, \text{Hom}_\Omega(\widetilde{E}, \widetilde{E}))$ which are invariant for the diagonal Γ -action and such that for any $\Gamma \times \Gamma$ -invariant b -differential operator P acting on $H_b^\infty(\widetilde{M}^2, \text{Hom}_\Omega(\widetilde{E}, \widetilde{E}))$ and any function $\phi(z) \in C_{\text{comp}}^\infty(\widetilde{M})$ we have for any nonnegative integer N :

$$\sup_{R>1} R^N |1_{B^c(p_0, R)}(z) P(\tilde{x}^{-\varepsilon} \tilde{x}'^{-\varepsilon} u(z, z') \phi(z'))|_{L_b^2(\widetilde{M} \times \widetilde{M})} < \infty$$

We then have:

Proposition 12.4. *There is a bijection $T \rightarrow \widetilde{T}$ between $\text{End}_{b,\mathcal{B}^\infty}^{\infty,\varepsilon}(M, \mathcal{E}^\infty)$ and $\mathcal{S}_{b,\Gamma}^{\infty,\varepsilon}(\widetilde{M}^2, \text{Hom}_\Omega(\widetilde{E}, \widetilde{E}))$ so that*

$$T\left(\sum_{\gamma \in \Gamma} R_\gamma^*(u)\gamma\right) = \sum_{\gamma \in \Gamma} R_\gamma^*(\widetilde{T}u)\gamma$$

for any $u \in \mathcal{S}_{b,\Gamma}^{\infty,\varepsilon}(\widetilde{M}, \widetilde{E})$, $\varepsilon \geq 0$.

The definition of the small $\mathcal{B}^\infty - b$ -pseudodifferential calculus is obtained by taking the symbols with values in \mathcal{B}^∞ . In the next proposition we will describe the correspondence between $b - \mathcal{B}^\infty$ -pseudo-differential operators on the base M and associated operators on \widetilde{M} . It is at this point that we finally use the definition of the *extended* b -stretched product introduced in Definition 4.2. First we fix a few notations. We recall (cf definition 4.5) the $\Gamma \times \Gamma$ -covering map $\pi : \widetilde{M}_{\text{eb}}^2 \rightarrow M_b^2$ and the blowdown map $\beta_b : M_b^2 \rightarrow M^2$. Let $\mathcal{O}(\text{bf})$ be an open neighborhood of the frontface of M_b^2 . Then $\pi^{-1}(\mathcal{O}(\text{bf}))$ will be a neighborhood of $\text{bf}(\widetilde{M}_{\text{eb}}^2)$ on which the usual coordinates (r, τ, y, y') may be used.

Proposition 12.5. *There is a one-to-one correspondence between operators $\mathcal{K} \in \Psi_{b,\mathcal{B}^\infty}^{-\infty}(M, \mathcal{E}^\infty)$ and Γ -invariant operators defined by Schwartz kernels \widetilde{K} belonging to $\Psi_{\text{eb},\Gamma}^{-\infty}(\widetilde{M}, \widetilde{E})$ where \widetilde{K} satisfies the two following estimates:*

- 1] For any $N \in \mathbb{N}$ and any multi-index α of derivation with respect to (z, w) we have:

$$\sup_{(z,w) \in \widetilde{M}_{\text{eb}}^2 \setminus \pi^{-1}(\mathcal{O}(\text{bf}))} \|\nabla^\alpha \widetilde{K}(z, w)\| (1 + d(z, w))^N < \infty$$

2] For any $N \in \mathbb{N}$ and any multi-index β of derivation with respect to (r, τ, y, y') we have:

$$\sup_{(r, \tau, y, y') \in \pi^{-1}(\mathcal{O}(\text{bf}))} \|\nabla^\beta \tilde{K}(r, \tau, y, y')\| (1 + d(y, y'))^N < \infty$$

The proof of the Proposition is along the lines of that given for Proposition 1.4 (see in particular formula (1.10)).

Remark. Theorem 10.6 thus shows (in view of definition 10.5) that the heat kernel $\exp(-t\tilde{D}^2)$ is the associated operator on \tilde{M}_{eb}^2 of an element of $\Psi_{b, \mathcal{B}^\infty}^{-\infty}(M, \mathcal{E}^\infty)$.

We can also define the \mathcal{B}^∞ -calculus with bounds $\Psi_{b, \mathcal{B}^\infty}^{*, \delta}(M, \mathcal{E}^\infty)$ (for $\delta > 0$) and construct a parametrix $Q \in \Psi_{b, \mathcal{B}^\infty}^{-m, \delta}(M, \mathcal{E}^\infty)$ of an elliptic element $\mathcal{P} \in \text{Diff}_{b, \mathcal{B}^\infty}^m(M, \mathcal{E}^\infty)$ whenever $I(\mathcal{P}, \lambda)$ is invertible, in the \mathcal{B}^∞ - b -calculus, for each $\lambda \in \mathbb{R} \times [-i\delta, i\delta]$. In view of the results of Section 1 (in particular Prop. 1.6), invertibility in the Λ -calculus would suffice. Summarizing :

Proposition 12.6. *Let $\mathcal{P} \in \text{Diff}_{b, \mathcal{B}^\infty}^m(M; \mathcal{E}^\infty)$ be elliptic and assume that $I(\mathcal{P}, \lambda)$ is invertible in the Λ -calculus for each $\lambda \in \mathbb{R} \times [-i\delta, i\delta]$. Then there exists a parametrix $Q \in \Psi_{b, \mathcal{B}^\infty}^{-m, \delta}(M; \mathcal{E}^\infty)$ with remainders in the space $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \epsilon}(M, \mathcal{E}^\infty)$.*

Remark. If \mathcal{P} is induced by a Dirac-type operator \tilde{D} on \tilde{M} with L^2 -invertible boundary operator \tilde{D}_0 , then Proposition 12.6 is nothing but Proposition 9.1 of Section 9.

We can now state the main theorem of this section.

Theorem 12.7. *Let \tilde{D} be a Dirac operator on \tilde{M} with L^2 -invertible boundary operator \tilde{D}_0 . Let \mathcal{D} and \mathcal{D}_∞ be the two operators induced by \tilde{D} in the Λ - and \mathcal{B}^∞ -Mishenko-Fomenko calculus. We can find $\epsilon > 0$, \mathcal{L}_∞ [resp. \mathcal{N}_∞] a sub- \mathcal{B}^∞ -module projective of finite rank of $x^\epsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, +})$ [resp. $x^\epsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, -})$] with the following properties:*

1] \mathcal{L}_∞ is free and we have

$$\mathcal{D}_\infty^+(\mathcal{L}_\infty) \subset \mathcal{N}_\infty \tag{12.2}$$

2] As Frechet spaces

$$\mathcal{L}_\infty \oplus \mathcal{L}_\infty^\perp = \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, +}), \quad \mathcal{N}_\infty \oplus \mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) = \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, -}). \tag{12.3}$$

- 3] The orthogonal projection of $\mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,+})$ onto \mathcal{L}_∞ and the projection $P_{\mathcal{N}_\infty}$ of $\mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,-})$ onto \mathcal{N}_∞ along $D_\infty^+(\mathcal{L}_\infty^\perp)$ are operators in $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M, \mathcal{E}^\infty)$.
- 4] As Banach spaces

$$\begin{aligned} \mathcal{L}_\infty \otimes \Lambda \oplus \overline{\mathcal{L}_\infty^\perp \otimes \Lambda} &= L^2(M, \mathcal{E}^+) \\ \mathcal{N}_\infty \otimes \Lambda \oplus \overline{\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \otimes \Lambda} &= \mathcal{H}_b^{-1}(M, \mathcal{E}^-) \end{aligned}$$

- 5] The operator

$$\mathcal{D}_\infty^+ : \mathcal{L}_\infty^\perp \rightarrow \mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \tag{12.4}$$

is invertible for the Frechet topologies; the operator

$$\mathcal{D}^+ : \overline{\mathcal{L}_\infty^\perp \otimes \Lambda} \rightarrow \overline{\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \otimes \Lambda} \subset \mathcal{H}_b^{-1}(M; \mathcal{E}^-)$$

is invertible.

- 6] The operator $(\mathcal{D}_\infty^+)^{-1} \circ (\text{Id} - P_{\mathcal{N}_\infty})$ belongs to the \mathcal{B}^∞ -*b*-calculus with bounds: $\Psi_{b, \mathcal{B}^\infty}^{-1, \varepsilon}$.

As a consequence of the theorem we immediately obtain

$$\text{Ind } \mathcal{D}^+ = [\mathcal{L}_\infty] - [\mathcal{N}_\infty] \in K_0(\mathcal{B}^\infty) = K_0(\Lambda). \tag{12.5}$$

Proof. See Sect. 17.

13. Noncommutative superconnections and the b -Chern character.

In this Section we use the notations of Section 12 about the exact even-dimensional b -manifold M , the graded hermitian Clifford module $E^+ \oplus E^-$, and the Dirac operators D^\pm . We will adapt to the b -setting the superconnection formalism of Section 2. For instance we will define the b -supertrace and the commutator formula (Prop 13.5) which explains why this is not a supertrace. Lastly we will prove the local higher index theorem. We adopt the identification near the boundaries explained in the geometric preliminaries of section 4.

We define $\Psi_{b, \widehat{\Omega}_*(\mathcal{B}^\infty)}^{-\infty}(M, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ to be the set \mathcal{K} of right $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operators defined by a smooth Schwartz kernel $\mathcal{K}(p)$, $p \in M_b^2$:

$$\mathcal{K}(p) \in \text{Hom}(\mathcal{E}_{q'} \otimes {}^b\Omega^{-1}; \mathcal{E}_q \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$$

where $(q, q') \in M^2$ is the image of p by the blowdown map β_b^2 . We assume that in the trivialization charts of equation (2.7) the kernel \mathcal{K} may be written locally as:

$$\sum_{l=1}^m A_l \otimes \omega_l$$

where each $A_l \in \text{End}(\mathbb{C}^{\dim E})$, and each ω_l belongs to the space $\Psi_b^{-\infty}(M, {}^b\Omega^{\frac{1}{2}}) \widehat{\otimes}_{\mathbb{C}} \widehat{\Omega}_*(\mathcal{B}^\infty)$. For each point $q \in \Delta_b$ we can define the supertrace

$$\text{Str } \mathcal{K}(q) \in \widehat{\Omega}_*(\mathcal{B}^\infty)$$

modulo the space of *graded* commutators exactly as in equation (2.10). Let us fix a trivialization $\nu \in C^\infty(\partial M; N_+ \partial M)$ of the normal bundle and $x \in C^\infty(M)$ a boundary defining function for ∂M such that $dx \cdot \nu = 1$ on ∂M . We define:

$$b - \text{STR } \mathcal{K} = \int_{\Delta_b}^\nu \text{Str } \mathcal{K}|_{\Delta_b} = \lim_{\varepsilon \rightarrow 0^+} \left[\int_{x > \varepsilon} \text{Str } \mathcal{K}|_{\Delta_b} + \log \varepsilon \int_{\partial M} \text{Str } \mathcal{K}|_{\partial \Delta_b} \right]$$

Let us assume moreover \mathcal{K} sends sections of \mathcal{E}^∞ into that of $\mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_k(\mathcal{B}^\infty)$ for some integer k . Then Lott's correspondence recalled before Definition 2.2 and the arguments used in Proposition 12.5 show that we can associate to \mathcal{K} an operator \widetilde{K} on the covering \widetilde{M} whose Schwartz kernel $\widetilde{K}(\tilde{p})$ of \widetilde{K} is defined for $\tilde{p} \in \widetilde{M}_{eb}^2$ by:

$$\widetilde{K}(\tilde{p}) = \sum_{\gamma_1, \dots, \gamma_k \in \Gamma} \widetilde{K}_{\gamma_1, \dots, \gamma_k}(\tilde{p}) d\gamma_1 \dots d\gamma_k \tag{13.1}$$

where $\widetilde{K}_{\gamma_1, \dots, \gamma_k}(p)$ defines an element of $\Psi_{eb}^{-\infty}(\widetilde{M}, \widetilde{E})$. Thus by definition of Lott's correspondence we have for any $f \in \mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E})$:

$$\mathcal{K} \left(\sum_{\gamma \in \Gamma} R_\gamma^*(f) \right) = \sum_{\gamma_0 \in \Gamma} \sum_{\gamma_1, \dots, \gamma_k \in \Gamma} \int_{\widetilde{M}} \widetilde{K}_{\gamma_1, \dots, \gamma_k}(z\gamma_0, z') f(z') \gamma_0 d\gamma_1 \dots d\gamma_k$$

We omit the proof of the following proposition which states that these operators satisfy a decay property very similar to the one of Definition 2.2

Proposition 13.1. *Let F be a fundamental domain of \widetilde{M} . Then the operator \mathcal{K} or \widetilde{K} of equation (13.1) satisfies the following decay property, which will be called property (DP). All the covariant derivatives of the operators $\widetilde{K}_{\gamma_1, \dots, \gamma_k}(p)$ satisfy the decay estimates of 1] and 2] of Proposition 12.5 with respect to:*

$$d(z; F) + \|\gamma_1\| + \dots + \|\gamma_{k-1}\| + d(w, R_{\gamma_k}(F)), \text{ if } p = (z, w) \text{ as in 1]}$$

$$d(y; F) + \|\gamma_1\| + \dots + \|\gamma_{k-1}\| + d(y', R_{\gamma_k}(F)), \text{ if } p = (r, \tau, y, y') \text{ as in 2]}$$

The following proposition may be proved as in [L1].

Proposition 13.2. *Let $\phi \in C_{\text{comp}}^\infty(\widetilde{M})$ which is constant in the normal direction near the boundary such that $\sum_{\gamma \in \Gamma} R_\gamma^* \phi \equiv 1$. For an operator \mathcal{K} as in Proposition 13.1 we have:*

$$b - \text{STR} \mathcal{K} = \sum_{\gamma_0, \dots, \gamma_k \in \Gamma} \int_{\widetilde{M}}^\nu \phi(z) \text{Str} \widetilde{K}_{\gamma_1, \dots, \gamma_k}(z\gamma_0, z)|_{\Delta_b} \gamma_0 d\gamma_1 \dots d\gamma_k$$

modulo graded commutators.

For the manifold *with boundary* M we define Lott’s connection ∇ and superconnection $D_s = \Upsilon \nabla + s\mathcal{D}$ for s real > 0 exactly as in Definition 2.1, Υ being the grading of $\widetilde{E} = \widetilde{E}^+ \oplus \widetilde{E}^-$. Then we consider $h \in C_{\text{comp}}^\infty(\widetilde{M})$ which is constant in the normal direction near the boundary and such that $\sum_{\gamma \in \Gamma} R_\gamma^*(h) \equiv 1$. If $f \in \mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E})$ represents as in Proposition 12.3 an element of $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ then:

$$\forall z \in \widetilde{M}, \nabla f(z) = \sum_{\gamma \in \Gamma} h(z) R_\gamma^*(f)(z) d\gamma \tag{13.2}$$

We state without proof the following result which is the analogue of Proposition 2.4.

Proposition 13.3. *Let \widetilde{K} be as in proposition 13.1 an operator satisfying property (DP). Let \widetilde{A} be an operator acting on $\mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E}) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)$. Then both $\widetilde{A} \circ \widetilde{K}$ and $\widetilde{K} \circ \widetilde{A}$ satisfy property (DP) provided \widetilde{A} satisfies at least one of the following three conditions:*

- (i) \widetilde{A} belongs to $\{\Upsilon \nabla \circ \widetilde{D} + \widetilde{D} \circ \Upsilon \nabla, \nabla^2\}$.
- (ii) \widetilde{A} is an element of $\Psi_{\text{eb}, \Gamma}^m(\widetilde{M}; \widetilde{E})$ whose Schwartz kernel is smooth outside the diagonal and satisfy the decay estimates 1] and 2] of Proposition 12.5.
- (iii) \widetilde{A} is the operator associated to an element of $\Psi_{b, \widehat{\Omega}_*(\mathcal{B}^\infty)}^{-\infty}(M, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ sending $\mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E})$ into $\mathcal{S}_b^{\infty, \varepsilon}(\widetilde{M}, \widetilde{E}) \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_m(\mathcal{B}^\infty)$ for some $m \in \mathbb{N}$ (so \widetilde{A} satisfies (DP)).

Now we set $\widetilde{P}_s = s(\Upsilon \nabla \circ \widetilde{D} + \widetilde{D} \circ \Upsilon \nabla) + \nabla^2$. We define the superconnection heat kernel $\exp(-\widetilde{D}_s^2)$ for s real > 0 to be the $\widehat{\Omega}_*(\mathcal{B}^\infty)$ –right linear operator whose associated operator on the covering (see equation (13.1)) is given by:

$$\begin{aligned} \exp(-s^2 \tilde{D}^2 - \tilde{P}_s) &= \exp(-s^2 \tilde{D}^2) + \\ &\int_0^1 \exp(-u_1 s^2 \tilde{D}^2) \tilde{P}_s \exp(-(1-u_1)s^2) \tilde{D}^2 du_1 + \\ &\int_0^1 \int_0^{1-u_1} (\exp(-u_1 s^2 \tilde{D}^2) \tilde{P}_s \exp(-u_2 s^2 \tilde{D}^2) \tilde{P}_s \\ &\exp(-(1-u_1-u_2)s^2 \tilde{D}^2)) du_2 du_1 + \dots \end{aligned} \tag{13.3}$$

where Proposition 13.3 and the remark following Proposition 12.5 show that in this Duhamel expansion (13.3) the component in $\widehat{\Omega}_k(\mathcal{B}^\infty)$ is a Schwartz kernel satisfying condition (DP). Thus we may state:

Definition 13.4. *The b -Chern Character is defined by: $b\text{-ch}_s \mathcal{E}^\infty = b\text{-STR}(e^{-D_s^2})$*

Since D_s^2 is even we see that: $b\text{-ch}_s \mathcal{E}^\infty$ is an even form defined modulo graded commutators. In general, the b -Chern Character is not closed and does not define an element of the topological noncommutative de Rham homology.

Now if $\mathcal{K} \in \Psi_{b, \widehat{\Omega}_*(\mathcal{B}^\infty)}^{-\infty}(M, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$ is an $\widehat{\Omega}_*(\mathcal{B}^\infty)$ -linear operator as in the beginning of this section, we can define its indicial family $I(\mathcal{K}, \lambda)$ which is an entire family (with respect to $\lambda \in \mathbb{C}$) of smoothing operators on the boundary ∂M acting on the sections of $\mathcal{E}_{|\partial M}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty)$. Proceeding as in the proof of Proposition 6.3, we can prove the following commutator formula:

Proposition 13.5. *Let $\mathcal{K}, \mathcal{K}'$ be two smoothing operators belonging to the space $\Psi_{b, \widehat{\Omega}_*(\mathcal{B}^\infty)}^{-\infty}(M, \mathcal{E}^\infty \otimes_{\mathcal{B}^\infty} \widehat{\Omega}_*(\mathcal{B}^\infty))$. Then:*

$$b\text{-STR}[\mathcal{K}, \mathcal{K}'] = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \text{STR} \left(\frac{\partial}{\partial \lambda} I(\mathcal{K}, \lambda) \circ I(\mathcal{K}', \lambda) \right) d\lambda$$

If we replace the operator \mathcal{K} by a differential operator $\in \text{Diff}_{b, \mathcal{B}^\infty}^1(M, \mathcal{E}^\infty)$ and \mathcal{K}' by the composition of \mathcal{K}' with an element of the calculus with bounds $\Psi_{b, \mathcal{B}^\infty}^{m, \delta}(M, \mathcal{E}^\infty)$ then the same commutator formula is valid.

Now we state the higher local index theorem.

Theorem 13.6.

1] We can find a bi-form $\omega \in \wedge^*(T^*M) \otimes \widetilde{\Omega}_*(\mathcal{B}^\infty)$ with the following structure:

$$\omega = \sum_{k \in \mathbb{N}} \omega_k, \quad \omega_k = \sum_{\gamma_0 \gamma_1 \dots \gamma_k = e} a_{\gamma_0, \gamma_1, \dots, \gamma_k} \otimes \gamma_0 d\gamma_1 \dots d\gamma_k$$

(thus ω_k is "concentrated in" $\gamma_0 \dots \gamma_k = e$) such that:

$$\lim_{s \rightarrow 0^+} b - \text{ch}_s \mathcal{E}^\infty = \int_M \hat{A}(M) \wedge \text{ch}' E \wedge \omega \in \widetilde{\Omega}_*(\mathcal{B}^\infty)$$

2] Let $\pi : \widetilde{M} \rightarrow M$ be the covering map, $P_k(\omega_k)$ be the projection of ω_k onto $\wedge^k(T^*M) \otimes \widetilde{\Omega}_*(\mathcal{B}^\infty)$, and let h be the function introduced in equation (13.2) then:

$$\pi^* P_k(\omega_k) = \frac{(-1)^k}{k!} \sum_{\gamma_0 \dots \gamma_k = e} R_{\gamma_0}^*(dh) \wedge \dots \wedge R_{\gamma_0 \dots \gamma_{k-1}}^*(dh) \gamma_0 d\gamma_1 \dots d\gamma_k$$

Moreover, $\omega_k - P_k(\omega_k)$ is a bi-differential form which is of degree $\leq k-1$ with respect to the M -variables and has a vanishing pairing with all the reduced cyclic cocycles τ_η of \mathcal{B}^∞ associated to the left-invariant antisymmetric cocycles η of the group-cohomology of Γ (see [L1], prop 12).

Remark. The assertion 2] is in fact part of proposition 12 of [L1].

Proof. In order to compute $b - \text{ch}_s \mathcal{E}^\infty$ we apply proposition 13.2 with \mathcal{K} = the superconnection heat kernel. Let $k \in \mathbb{N}^*$, we are going to analyse the limit as $s \rightarrow 0^+$ of the component in $\widetilde{\Omega}_k(\mathcal{B}^\infty)$ of $b - \text{ch}_s \mathcal{E}^\infty$. Using Duhamel's formula (13.3) we first consider the contribution of the term:

$$\begin{aligned} & (-1)^k \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{k-1}} \exp(-u_1 s^2 \widetilde{D}^2) s(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla) \\ & \exp(-u_2 s^2 \widetilde{D}^2) s(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla) \dots s(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla) \\ & \exp(-(1-u_1-\dots-u_k) s^2 \widetilde{D}^2) du_k \dots du_1 \end{aligned} \tag{13.4}$$

We recall that Υ denotes the \mathbb{Z}_2 -grading of \widetilde{E} and that by equation (2.5)

$$(\Upsilon \nabla \widetilde{D} + \widetilde{D} \Upsilon \nabla)(f) = \sum_{\gamma \in \Gamma} \partial h R_\gamma^*(f) \otimes d\gamma, \quad \text{where } \partial h = -\Upsilon[\widetilde{D}, h]$$

Proposition 13.2 and the Γ -invariance of the heat kernel allow us to see that the b -supertrace of the operator given by the previous formula (13.4) is given by:

$$\begin{aligned} & \sum_{\gamma_0, \dots, \gamma_k} (-1)^k \int_0^1 \int_0^{1-u_1} \dots \int_0^{1-u_1-\dots-u_{k-1}} \int_{\tilde{M}_{z_0}}^{\nu} \phi(z_0) \left(\int_{\tilde{M}_{z_1}} \dots \int_{\tilde{M}_{z_k}} \right. \\ & \quad \text{Str} \left[\exp(-u_1 s^2 \tilde{D}^2)(z_0, z_1) s \partial h(z_1 \gamma_0) \exp(-u_2 s^2 \tilde{D}^2)(z_1, z_2) \right. \\ & \quad \quad \left. s \partial h(z_2 \gamma_0 \gamma_1) \dots s \partial h(z_k \gamma_0 \gamma_1 \dots \gamma_{k-1}) \right. \\ & \quad \left. \left. \exp(-(1-u_1-\dots-u_k) s^2 \tilde{D}^2)(z_k \gamma_0 \gamma_1 \dots \gamma_k, z_0) \right] d\text{vol}(z_k) \dots d\text{vol}(z_1) \right) \Big|_{\Delta_b} \\ & \quad \quad \quad du_k \dots du_1 \gamma_0 d\gamma_1 \dots d\gamma_k \tag{13.5} \end{aligned}$$

Let us show that if the product $\gamma_0 \gamma_1 \dots \gamma_k$ is not equal to e then (13.5) goes to zero as $s \rightarrow 0^+$. We can find $\varepsilon > 0$ such that for any points z_0, z_1, \dots, z_k of \tilde{M} :

$$d(z_0, z_1) + d(z_1, z_2) + \dots + d(z_{k-1}, z_k) + d(z_k \gamma_0 \gamma_1 \dots \gamma_k, z_0) > \varepsilon$$

since the heat kernels $\exp(-u_1 s^2 \tilde{D}^2)(z_0, z_1) \dots$ etc are concentrated near the diagonal, we then see easily that (13.5) goes to zero.

Otherwise if $\gamma_0 \gamma_1 \dots \gamma_k = e$ in equation (13.5) then using the rescaled b -heat calculus (which is nothing but a b -version of Getzler's rescaling [Ge]) we see as in [L1] that the limit of (13.5) is:

$$\sum_{\gamma_0, \dots, \gamma_k} \frac{(-1)^k}{k!} \int_{\tilde{M}} \phi \hat{A}(\tilde{M}) \wedge \text{Ch}'(\tilde{E}) \wedge R_{\gamma_0}^* dh \wedge \dots \wedge R_{\gamma_0 \dots \gamma_k}^* dh \gamma_0 d\gamma_1 \dots d\gamma_k \tag{13.6}$$

As in [L1] a Γ -invariance argument shows the existence of a biform denoted $P_k(\omega_k) \in \wedge^*(T^*M) \otimes \bar{\Omega}_*(\mathcal{B}^\infty)$ such that (13.6) is equal to:

$$\int_M \hat{A}(M) \wedge \text{Ch}'(E) \wedge P_k(\omega_k)$$

Now if we examine the contribution of terms similar to (13.4) but where we have replaced at least one of the $(\Upsilon \nabla \tilde{D} + \tilde{D} \Upsilon \nabla)$ by ∇^2 then, using the same arguments as above, we will find that the limit may be written as

$$\int \hat{A}(M) \wedge \text{Ch}'(E) \wedge (\omega_k - P_k(\omega_k))$$

where the bi-form $(\omega_k - P_k(\omega_k))$ is concentrated in $\gamma_0 \dots \gamma_k = e$ and is of degree $\leq k - 1$ with respect to the M -variables. The Theorem is proved.

14. A higher Atiyah-Patodi-Singer index formula.

In this section we finally state and prove the main result of this paper. Let us recall our data. Let M be a compact even-dimensional manifold with boundary and let $\Gamma \rightarrow \widetilde{M} \rightarrow M$ be a Galois cover, the group Γ being virtually nilpotent. Let g be an exact b -metric on M . Let D be a generalized Dirac operator acting on a \mathbb{Z}_2 -graded hermitian Clifford module E ; $D \in \text{Diff}_b^1(M; E)$. We assume that D is associated to a hermitian Clifford connection. Let $\widetilde{D} \in \text{Diff}_{b,\Gamma}^1(\widetilde{M}, \widetilde{E})$ the Γ -invariant lift of D to the cover \widetilde{M} . Let $\widetilde{D}_0 \in \text{Diff}(\partial\widetilde{M}; \widetilde{E}_0)$ be the boundary operator. Let $\mathcal{D}, \mathcal{D}_0$ be the associated Mishenko-Fomenko operators, acting on the $C_r^*(\Gamma)$ -bundles $\mathcal{E}, \mathcal{E}_0$.

Theorem 14.1. *Let $\widetilde{D}, \mathcal{D}, \widetilde{D}_0, \mathcal{D}_0$ be Dirac operators as above. Let ω be the biform defined in theorem 13.6. We assume that there exists a $\delta > 0$ such that*

$$L^2 - \text{spec}(\widetilde{D}_0) \cap] - \delta, \delta[= \{0\}. \quad (14.1)$$

Then

- 1] There is a well defined index class $\text{Ind}(\mathcal{D}^+) \in \mathcal{K}_0(\mathcal{B}^\infty) \cong \mathcal{K}_0(C_r^*(\Gamma))$.
- 2] The null space of \mathcal{D}_0 acting on $C^\infty(\partial M, \mathcal{E}_0^\infty)$ is a projective, finitely generated \mathcal{B}^∞ -module.
- 3] For the Chern character of $\text{Ind}(\mathcal{D}^+)$, in the noncommutative topological de Rham homology of \mathcal{B}^∞ , the following formula holds

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \int_M \widehat{A} \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2}(\widetilde{\eta} + \text{Ch}(\text{null}(\mathcal{D}_0))) \quad \text{in } \overline{H}_*(\mathcal{B}^\infty). \quad (14.2)$$

Proof. We shall first establish the theorem under the assumption:

$$\exists \delta > 0 \mid L^2 - \text{spec}(\tilde{D}_0) \cap] - \delta, \delta[= \emptyset \quad (14.3)$$

We start by applying the higher b -trace identity, Proposition 13.5, to prove the following important transgression formula for the b -Chern character:

Proposition 14.2. *Let $u > t > 0$, let Υ be the grading of \mathcal{E} , and let $\tilde{\eta}(s)$ the higher eta integrand introduced in Section 2. The following equality holds in $\widetilde{\Omega}_*(\mathcal{B}^\infty)$:*

$$b\text{-ch}_u(\mathcal{E}^\infty) = b\text{-ch}_t(\mathcal{E}^\infty) - \frac{1}{2} \int_t^u \tilde{\eta}(s) ds - d \int_t^u b\text{-STR}(\tilde{D}e^{-(\Upsilon\nabla+s\tilde{D})^2}) ds \quad (14.4)$$

Proof. The proof is parallel to the one given in [MP 1] (Proposition 11) for the family case. Here our choice of signs, adapted to the right-module structure as in (2.11), enters in a crucial way. The details of the proof are left to the reader.

Taking the limit as $t \rightarrow 0^+$ of the right hand side of (14.4) and using Theorem 13.6 and 2.7 we obtain the following equality of elements in $\widetilde{\Omega}_*(\mathcal{B}^\infty)$:

$$b\text{-ch}_u(\mathcal{E}^\infty) = \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(s) ds - d \int_0^u b\text{-STR}(\tilde{D}e^{-(\Upsilon\nabla+s\tilde{D})^2}) ds. \quad (14.5)$$

The integrability of the last term on the right hand side near $s = 0$ follows from the local index theorem of the previous section. Using assumption (14.3) we now introduce a finite rank (in the sense of Kasparov [K]) perturbation of the operator \mathcal{D}_∞ as in [L1] [B]. Consider the projective sub- \mathcal{B}^∞ -modules of finite rank of $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, \pm})$ constructed in Section 12. Consider the \mathcal{B}^∞ -modules

$$\begin{aligned} \mathcal{H}^+ &= \mathcal{H}_b^\varepsilon(M, \mathcal{E}^{\infty, +}) \oplus \mathcal{N}_\infty = \mathcal{L}_\infty^\perp \oplus \mathcal{L}_\infty \oplus \mathcal{N}_\infty \\ \mathcal{H}^- &= \mathcal{H}_b^\varepsilon(M, \mathcal{E}^{\infty, -}) \oplus \mathcal{L}_\infty = \mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp) \oplus \mathcal{N}_\infty \oplus \mathcal{L}_\infty \\ \mathcal{H} &= \mathcal{H}^+ \oplus \mathcal{H}^-. \end{aligned}$$

As in Sect. 12 we denote by $\Pi_{\mathcal{L}_\infty}$ the orthogonal projection onto \mathcal{L}_∞ and by $P_{\mathcal{N}_\infty}$ the projection onto \mathcal{N}_∞ along $\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)$. We define the operator $\mathcal{R}_\alpha^+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$, $\alpha \geq 0$ by

$$\mathcal{R}_\alpha^+(f \oplus n) = (\mathcal{D}_\infty^+ f + \alpha n) \oplus \alpha \Pi_{\mathcal{L}_\infty} f$$

for each $(f \oplus n) \in \mathcal{H}^+$. More suggestively

$$\mathcal{R}_\alpha^+ = \begin{pmatrix} \mathcal{D}_{\mathcal{L}_\infty^\perp}^+ & 0 & 0 \\ 0 & \mathcal{D}_{\mathcal{L}_\infty}^+ & \alpha \\ 0 & \alpha & 0 \end{pmatrix}. \quad (14.6)$$

We next define $\mathcal{R}_\alpha^- : \mathcal{H}^- \rightarrow \mathcal{H}^+$ by

$$\mathcal{R}_\alpha^-(g \oplus l) = (\mathcal{D}_\infty^- g + \alpha l) \oplus \alpha P_{\mathcal{N}_\infty} g$$

Thus

$$\mathcal{R}_\alpha^- = \begin{pmatrix} \mathcal{D}_{\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)}^- & 0 & 0 \\ 0 & \mathcal{D}_{\mathcal{N}_\infty}^- & \alpha \\ 0 & \alpha & 0 \end{pmatrix}. \quad (14.7)$$

Finally we define

$$\mathcal{R}_\alpha = \begin{pmatrix} 0 & \mathcal{R}_\alpha^- \\ \mathcal{R}_\alpha^+ & 0 \end{pmatrix} \quad (14.8).$$

Remark. Since we have not claimed a simultaneous \mathcal{B}^∞ -decomposition for $\mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, \pm})$ (this decomposition *might* exist but we are not able to prove it) our definition here is slightly different from the one given in [L 1].

Denote by \mathcal{F}_∞ the finite rank module $\mathcal{L}_\infty \oplus \mathcal{N}_\infty$. Let us fix a \mathcal{B}^∞ -connection $\nabla_{\mathcal{F}}$ on \mathcal{F}_∞ by compressing Lott's connection ∇ by the orthogonal projection on \mathcal{L}_∞ and the projection onto \mathcal{N}_∞ along $\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)$. Let us denote by $\nabla' = \nabla \oplus \nabla_{\mathcal{F}}$ the sum connection on \mathcal{H} , Υ the grading of \mathcal{H} and define

$$b\text{-ch}_{u, \alpha}(\mathcal{E}^\infty) = \text{STR}(\exp(-(\Upsilon \nabla' + u \mathcal{R}_\alpha)^2)) \quad (14.9).$$

Clearly

$$b\text{-ch}_{u, 0}(\mathcal{E}^\infty) = b\text{-ch}_u(\mathcal{E}^\infty) - b\text{-STR}(\exp(-\nabla_{\mathcal{F}}^2)) \quad (14.10)$$

By [Ka] and Theorem 12.7 of Section 12 (see formula (12.5)) we have, as in [L1],

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \text{Ch}([\mathcal{L}_\infty] - [\mathcal{N}_\infty]) = [\text{STR}(\exp(-\nabla_{\mathcal{F}}^2))] \in \overline{H}_*(\mathcal{B}^\infty). \quad (14.11)$$

On the other hand

$$\Pi_{\mathcal{L}_\infty} \in \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M, \mathcal{E}^{\infty, +}) \quad P_{\mathcal{N}_\infty} \in \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M, \mathcal{E}^{\infty, -}). \quad (14.12)$$

This implies that their Schwartz kernels vanish on the front face; thus

$$\text{STR}(\exp(-\nabla_{\mathcal{F}}^2)) = b - \text{STR}(\exp(-\nabla_{\mathcal{F}}^2)).$$

Hence, by (14.10), (14.5), we obtain the following equality in $\overline{\Omega}_*(\mathcal{B}^\infty)$

$$\begin{aligned} \text{STR}(\exp(-\nabla_{\mathcal{F}}^2)) &= \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(s) ds - \\ &d \int_0^u b - \text{STR}(\widetilde{D} e^{-(\Upsilon \nabla + s \widetilde{D})^2}) ds - (b - \text{ch}_{u,0}(\mathcal{E}^\infty)) \end{aligned} \quad (14.13).$$

and thus, from (14.11), the following equality in $\overline{H}_*(\mathcal{B}^\infty)$:

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(s) ds - (b - \text{ch}_{u,0}(\mathcal{E}^\infty)). \quad (14.14)$$

Our main result will be obtained by taking the limit as $u \rightarrow +\infty$ of the right-hand side of (14.14) and showing that $b - \text{ch}_{u,0}(\mathcal{E}^\infty) \rightarrow 0 \pmod{d\overline{\Omega}_*(\mathcal{B}^\infty)}$ as $u \rightarrow +\infty$. Using the transgression formula for the b -Chern character we have, modulo $d\overline{\Omega}_*(\mathcal{B}^\infty)$,

$$b - \text{ch}_{u,\alpha}(\mathcal{E}^\infty) = b - \text{ch}_{u,0}(\mathcal{E}^\infty) + B_1(u, \alpha),$$

$B_1(u, \alpha)$ being boundary terms. In order to compute these boundary terms, as well as others that will appear later in the course of the proof, we need the following Lemma. Recall that, near the boundary, we are using the identifications explained in the geometric preliminaries of Sect.4. Also, for notational convenience, we shall drop the ∞ subscript in denoting the Dirac operator and the boundary Dirac operator in the \mathcal{B}^∞ -calculi.

Lemma 14.3. *The indicial families of the operators (14.6), (14.7) and of the superconnection $\Upsilon\nabla'$ are given by*

$$I(\mathcal{R}_\alpha^\pm, \lambda) = \pm\lambda i + \mathcal{D}_0 \quad I(\Upsilon\nabla', \lambda) = \Upsilon\nabla^\partial$$

with ∇^∂ equal to the connection induced by ∇ on the boundary ∂M .

Notice that, in particular, $I(\mathcal{R}_\alpha^2, \lambda) = \lambda^2 + \mathcal{D}_0^2$; moreover

$$I\left(\frac{d\mathcal{R}_\alpha}{d\alpha}, \lambda\right) \equiv 0 \tag{14.15}$$

The proof of Lemma follows at once from (14.12).

Now we have

$$\frac{d}{d\alpha} b - \text{ch}_{u,\alpha}(\mathcal{E}^\infty) = -b - \text{STR}[(u\mathcal{R}_\alpha + \Upsilon\nabla'), u\left(\frac{d\mathcal{R}_\alpha}{d\alpha}\right) \exp(-(\Upsilon\nabla' + u\mathcal{R}_\alpha)^2)].$$

The last expression is nothing but

$$\begin{aligned} & -d(b - \text{STR}(u\left(\frac{d\mathcal{R}_\alpha}{d\alpha}\right) \exp(-(\Upsilon\nabla' + u\mathcal{R}_\alpha)^2))) \\ & - \frac{i}{2\pi} \int_{-\infty}^{+\infty} \text{STR}\left(\frac{d}{d\lambda} I(u\mathcal{R}_\alpha, \lambda) I\left(u\frac{d\mathcal{R}_\alpha}{d\alpha}, \lambda\right) I(\exp(-(\Upsilon\nabla' + u\mathcal{R}_\alpha)^2), \lambda)\right) d\lambda \end{aligned} \tag{14.16}$$

From (14.15) it follows that the second term in (14.16) is identically equal to zero, so our $B_1(u, \alpha)$ is also identically equal to zero. Thus

$$b - \text{ch}_{u,\alpha}(\mathcal{E}^\infty) = b - \text{ch}_{u,0}(\mathcal{E}^\infty) \text{ mod } d\overline{\Omega}_*(\mathcal{B}^\infty) \tag{14.17}$$

so that (14.14) can be replaced by ($\forall u > 0$)

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(s) ds - (b - \text{ch}_{u,\alpha}(\mathcal{E}^\infty)).$$

If α is large enough then $\mathcal{R}_\alpha^+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ is invertible. By the results of Sect. 12 the inverse does belong to the $b\mathcal{B}^\infty$ -calculus with bounds. Thus for α large we can change the sum connection on \mathcal{H}^+ and consider

$$\nabla''_{\mathcal{H}^+} = (\mathcal{R}_\alpha^+)^{-1} \circ \nabla'_{\mathcal{H}^-} \circ \mathcal{R}_\alpha^+.$$

Define $\Upsilon\nabla''$ to be $\nabla''_{\mathcal{H}^+} \oplus (-\nabla'_{\mathcal{H}^-})$. The two *superconnections* $\Upsilon\nabla'$ and $\Upsilon\nabla''$ are of course homotopic through the path of connections $\Upsilon\nabla_\vartheta = \Upsilon\nabla' + \vartheta(\Upsilon\nabla'' - \Upsilon\nabla')$. Computing as usual the derivative with respect to ϑ of $b\text{-STR}(\exp(-(\Upsilon\nabla_\vartheta + u\mathcal{R}_\alpha)^2))$ applying the higher b -trace identity (proposition 13.5) and integrating in ϑ , from 0 to 1, we obtain, $\text{mod}(\widehat{d\Omega}_*(\mathcal{B}^\infty))$,

$$b\text{-ch}_{u,\alpha} = b\text{-STR}(\exp(-(\Upsilon\nabla'' + u\mathcal{R}_\alpha)^2)) + B_2(u, \alpha)$$

with $B_2(u, \alpha)$ given explicitly by

$$-\frac{i}{2\pi} \int_0^1 \int_{-\infty}^{+\infty} \text{STR}\left(\frac{d}{d\lambda} I(u\mathcal{R}_\alpha, \lambda) I(\Upsilon\nabla'' - \Upsilon\nabla', \lambda) \exp(-(uI(\mathcal{R}_\alpha, \lambda) + (1 - \vartheta)\Upsilon\nabla^\partial + \vartheta I(\Upsilon\nabla'', \lambda))^2)\right) d\lambda d\vartheta.$$

By definition

$$I(\Upsilon\nabla'', \lambda) = \begin{pmatrix} \nabla_\lambda^\partial & 0 \\ 0 & -\nabla^\partial \end{pmatrix}$$

with $\nabla_\lambda^\partial \equiv (i\lambda + \mathcal{D}_0)^{-1} \circ \nabla^\partial \circ (i\lambda + \mathcal{D}_0)$.

Now let us briefly explain why:

$$\lim_{u \rightarrow +\infty} B_2(u, \alpha) = 0 \tag{14.18}$$

To see this we apply Duhamel's formula in the expression defining $B_2(u, \alpha)$ where $(uI(\mathcal{R}_\alpha, \lambda) + (1 - \vartheta)\Upsilon\nabla^\partial + \vartheta I(\Upsilon\nabla'', \lambda))^2$ is considered as a perturbation of $(uI(\mathcal{R}_\alpha, \lambda))^2 = u^2(\lambda^2 + \mathcal{D}_0^2)$. Since \widetilde{D}_0 is invertible the heat kernel $\exp(-u^2\widetilde{D}_0^2)$ satisfies the three estimates (2.14), (2.15), (2.16) of Sect.2. Then $\exp(-u^2\widetilde{D}_0^2)$ satisfies the two assertions of Theorem 2.9. So let us consider an integral over a k -simplex with coordinates $\sigma_0, \dots, \sigma_k$ appearing in the Duhamel expansion of $B_2(u, \alpha)$. One of the σ_j will be $\geq \frac{1}{k+1}$ so, estimate (2.14) shows that, for a suitable real $\delta' > 0$, the operators $\exp(-\sigma_j u^2 \widetilde{D}_0^2)$ satisfy property (DP) uniformly with respect to $\exp(-\delta' \sigma_j u^2)$ as $u > 1$. We recall Proposition 2.4 which explains how the (DP) condition is preserved under composition. Then we can use part 1] of Theorem 2.9 to check that, in the Duhamel expansion of $B_2(u, \alpha)$, the component in each $\widehat{\Omega}_k(\mathcal{B}^\infty)$ is the supertrace of a Schwartz kernel satisfying (DP) uniformly with respect

to $\exp(-\frac{\delta' u^2}{k+1})$ as $u > 1$. Thus the assertion (14.18) follows by inspection of the expression of the supertrace as given by Proposition 2.5.

Summarizing $\forall u > 0$

$$\begin{aligned} \text{Ch}(\text{Ind}(\mathcal{D}^+)) &= \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^u \tilde{\eta}(s) ds \\ &\quad - b\text{-STR}(\exp(-(\Upsilon \nabla'' + u \mathcal{R}_\alpha)^2)) - B_2(u, \alpha) \text{ in } \overline{H}_*(\mathcal{B}^\infty) \end{aligned}$$

so that, thanks to (14.18) and the convergence of the higher eta invariant at infinity, we only need to show that

$$\lim_{u \rightarrow +\infty} b\text{-STR}(\exp(-(\Upsilon \nabla'' + u \mathcal{R}_\alpha)^2)) = 0.$$

To do so we follow an argument in [B]. By construction we have $(\Upsilon \nabla'') \cdot \mathcal{R}_\alpha^+ = 0$ so :

$$\begin{aligned} &(\Upsilon \nabla'' + u \mathcal{R}_\alpha)^2 = \\ &\left(\begin{array}{cc} (\Upsilon \nabla''_{\mathcal{H}^+})^2 + u^2 \mathcal{R}_\alpha^- \circ \mathcal{R}_\alpha^+ & u(\mathcal{R}_\alpha^- \circ \Upsilon \nabla'_{\mathcal{H}^-} + \Upsilon \nabla''_{\mathcal{H}^+} \circ \mathcal{R}_\alpha^-) \\ 0 & (\Upsilon \nabla'_{\mathcal{H}^-})^2 + u^2 \mathcal{R}_\alpha^+ \circ \mathcal{R}_\alpha^- \end{array} \right). \end{aligned}$$

Moreover

$$\mathcal{R}_\alpha^+ [(\Upsilon \nabla''_{\mathcal{H}^+})^2 + u^2 \mathcal{R}_\alpha^- \circ \mathcal{R}_\alpha^+] (\mathcal{R}_\alpha^+)^{-1} = (\Upsilon \nabla'_{\mathcal{H}^-})^2 + u^2 \mathcal{R}_\alpha^+ \mathcal{R}_\alpha^-.$$

Thus

$$\begin{aligned} &b\text{-STR}(\exp(-(\Upsilon \nabla'' + u \mathcal{R}_\alpha)^2)) \\ &= b\text{-TR}[\exp(-((\Upsilon \nabla''_{\mathcal{H}^+})^2 + u^2 \mathcal{R}_\alpha^- \circ \mathcal{R}_\alpha^+)) - \\ &\quad \mathcal{R}_\alpha^+ \exp(-((\Upsilon \nabla'_{\mathcal{H}^-})^2 + u^2 \mathcal{R}_\alpha^+ \mathcal{R}_\alpha^-)) (\mathcal{R}_\alpha^+)^{-1}] \end{aligned}$$

which by the higher commutator formula (see Proposition 13.5) is in turn equal to

$$\begin{aligned} &\pm \frac{i}{2\pi} \int_{-\infty}^{+\infty} \text{TR} \left(\frac{d}{d\lambda} I(\mathcal{R}_\alpha^+, \lambda) \circ \right. \\ &\quad \left. \exp(- (I(\Upsilon \nabla''_{\mathcal{H}^+}, \lambda)^2 + u^2(\lambda^2 + \mathcal{D}_0^2))) \circ I(\mathcal{R}_\alpha^+, \lambda)^{-1} \right) d\lambda. \end{aligned}$$

One checks that this last expression goes to zero as $u \rightarrow +\infty$, exactly as for (14.17).

Since we have showed that

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \int_M \widehat{A}(M) \wedge \text{Ch}'(E) \wedge \omega - \frac{1}{2} \int_0^{+\infty} \tilde{\eta}(s) ds$$

in the noncommutative topological de Rham homology of \mathcal{B}^∞ , the theorem is established under assumption (14.3).

Let us now relax this assumption and only require that $\exists \delta > 0 \mid L^2 - \text{spec}(\tilde{D}_0) \cap] - \delta, \delta[= \{0\}$. Let $\vartheta \in (0, \delta)$. Following the compact case we consider the weighted operators $D^\pm(\vartheta) = x^\mp \vartheta D x^\pm \vartheta \in \text{Diff}_b^1(M; E^\pm, E^\mp)$ and their lifts $\tilde{D}^\pm(\vartheta) \in \text{Diff}_{b,\Gamma}^1(\tilde{M}; \tilde{E}^\pm, \tilde{E}^\mp)$ to the Γ -cover \tilde{M} . Each operator $\tilde{D}^+(\vartheta)$ has a boundary operator $\tilde{D}_0 + \vartheta$ which is L^2 -invertible. According to the results of Sect. 12 it follows that there exists a well defined index class $\text{Ind}(\mathcal{D}_\infty^+(\vartheta)) = [\mathcal{L}_\infty(\vartheta)] - [\mathcal{N}_\infty(\vartheta)] \in \mathcal{K}_0(\mathcal{B}^\infty)$. By standard homotopy arguments this class is independent of the choice of ϑ , for $\vartheta \in (0, \delta)$. We define $\text{Ind}(\mathcal{D}^+) = \text{Ind}(\mathcal{D}_\infty^+(\vartheta))$. (In the compact case (thus with $\Gamma = e$) this is the definition of the *extended* L^2 -index, which is in turn equal to the Atiyah-Patodi-Singer index; in the higher covering case we do not have an index class in the Atiyah-Patodi-Singer framework.) Thus 1] is proved. The fact that $\text{null}(\mathcal{D}_{0,\infty})$ is a finitely generated projective \mathcal{B}^∞ -module was established in Proposition 2.6 of Sect. 2. In order to prove formula (14.2) we first observe that for each $\vartheta \in]0, \delta[$, we can define $\text{AS}_\omega(\vartheta)$ as the coefficient of t^0 in the asymptotic expansion of $b - \text{STR}(\exp(-(\Upsilon \nabla + t \tilde{D}(\vartheta))^2))$. Following the notation in [BGV]

$$\text{AS}_\omega(\vartheta) \equiv \lim_{t \downarrow 0} b - \text{STR}(\exp(-(\Upsilon \nabla + t \tilde{D}(\vartheta))^2)).$$

It is well defined as an element of $\widetilde{\Omega}_*(\mathcal{B}^\infty)$ for each $\vartheta \in \mathbb{R}$. One checks the existence of the asymptotic expansion of $b - \text{STR}(\exp(-(\Upsilon \nabla + t \tilde{D}(\vartheta))^2))$ by using the arguments of the proof of the local higher index theorem of section 13. Moreover $\text{AS}_\omega(\vartheta)$ is continuous in ϑ as a simple variational argument shows. In particular

$$\lim_{\vartheta \rightarrow 0} \text{AS}_\omega(\vartheta) = \text{AS}_\omega(0) = \int_M \widehat{A} \wedge \text{Ch}'(E) \wedge \omega \tag{14.19}$$

By Lemma 3.1 of Sect. 3 we can consider the higher eta invariant for $\tilde{D}_0 + \vartheta$ defined as:

$$\tilde{\eta}(\vartheta) = \lim_{t \downarrow 0} \int_t^{+\infty} \text{STR}_{\text{Cl}(1)} [\sigma(\tilde{D}_0 + \vartheta) \exp(-(\Upsilon \nabla + s \sigma(\tilde{D}_0 + \vartheta))^2)] ds$$

Taking the regularized limit as $t \downarrow 0$ in formula (14.4) applied to $\tilde{D}(\vartheta)$ and proceeding as in the previous proof of Theorem 14.1 (when \tilde{D}_0 is invertible) we see that $\forall \vartheta \in (0, \delta)$

$$\text{Ch}(\text{Ind}(\mathcal{D}^+)) = \text{AS}_\omega(\vartheta) - \frac{1}{2}\tilde{\eta}(\vartheta) \in \overline{H}_*(\mathcal{B}^\infty).$$

Using (14.19) and Theorem 3.2 of Sect. 3 the theorem now follows by taking $\vartheta \downarrow 0$.

15. Applications to positive scalar curvature questions.

We begin with recalling the definition of the higher rho-invariant of [L 2]. Since $\widetilde{\Omega}_*(\mathcal{B}^\infty)$ breaks up into a sum of subcomplexes labeled by the conjugacy classes of Γ , the higher eta-invariant may be written as:

$$\tilde{\eta} = \bigoplus_{\langle \gamma \rangle \in \langle \Gamma \rangle} \tilde{\eta}(\langle \gamma \rangle)$$

The higher rho-invariant is the closed form defined (modulo graded commutators) by:

$$\tilde{\rho} = \bigoplus_{\langle \gamma \rangle \neq \langle e \rangle} \tilde{\eta}(\langle \gamma \rangle)$$

We refer to [L 2] for the (stability) properties of $\tilde{\rho}$.

Lott has pointed out the following corollary of our Theorem 14.1.

Theorem 15.1. *Let M be a compact connected spin even-dimensional manifold with boundary having a product spin structure near ∂M . Let us assume that ∂M has a metric with positive scalar curvature which can be extended to the whole M as a metric with positive scalar curvature having a product structure near the boundary. If moreover, the fundamental group of M is virtually nilpotent, then the associated higher rho-invariant is zero modulo exact forms.*

Proof. We just have to apply our index theorem 14.1 with E being equal to the spin bundle. Lichnerowicz's formula on the universal covering allows to see that both \tilde{D} and the boundary operator \tilde{D}_0 are invertible. Since the bi-form ω is concentrated in $\langle e \rangle$, we get immediately the Theorem.

Remark. Thus if N is an odd-dimensional spin manifold with positive scalar curvature and a nonzero higher rho-invariant, then for *any* spin M such that $N = \partial M$ (assuming that such exist), there cannot be a positive-scalar-curvature metric on M which is a product near the boundary.

16. Appendix A: proof of the \mathcal{B}^∞ -decompositions.

The proof of Theorem 1.3 will be divided into several Lemmas and Propositions. We shall build upon the propositions of Section 1 and the arguments given by Mishenko and Fomenko [M-F] for the Λ -calculus. We have laid the stress only on those of the technical details which are specific to the \mathcal{B}^∞ -calculus and not an immediate consequence of [M-F].

Lemma 16.1. *Let \mathcal{L} be a free sub- \mathcal{B}^∞ -module of finite rank of the space $C^\infty(M, \mathcal{E}^{\infty,+})$ with the property that we can find a closed sub-module \mathcal{G} of $C^\infty(M, \mathcal{E}^{\infty,+})$ such that:*

$$\mathcal{L} \oplus \mathcal{G} = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \text{as Frechet spaces}$$

$$\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda \oplus \overline{\mathcal{G} \otimes_{\mathcal{B}^\infty} \Lambda} = L^2(M, \mathcal{E}^+), \quad \text{as Banach spaces}$$

Then \mathcal{L} admits an orthonormal basis (for the \mathcal{B}^∞ -hermitian scalar product $\langle \cdot, \cdot \rangle$ induced by that of $C^\infty(M, \mathcal{E}^{\infty,+})$) and:

$$\mathcal{L} \oplus \mathcal{L}^\perp = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \text{as Frechet spaces}$$

Moreover the orthogonal projection $\Pi_{\mathcal{L}}$ onto \mathcal{L} is a smoothing operator $\in \text{End}_{\mathcal{B}^\infty}^1(M, \mathcal{E}^{\infty,+})$.

Proof. Let (η_1, \dots, η_p) be a generating family for $\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda$, then $\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda$ is the range space of the adjointable operator $P = \sum_{j=1}^p \langle \cdot, e_j^+ \rangle \eta_j$. Since

$\text{Im } P$ is closed in $L^2(M, \mathcal{E}^+)$, P admits a polar decomposition $P = V|P|$ and $\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda = \text{Im } VV^*$, VV^* is a Λ -projection and

$$\text{Im } VV^* \oplus^\perp \text{null } VV^* = L^2(M, \mathcal{E}^+)$$

From this we see that the map:

$$\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda \rightarrow \text{Hom}_\Lambda(\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda, \Lambda)$$

$$x \rightarrow \langle \cdot, x \rangle$$

is an isomorphism. Thus if (e_1, \dots, e_p) is a \mathcal{B}^∞ -basis for \mathcal{L} then the matrix

$$A = (\langle e_i, e_j \rangle)_{1 \leq i, j \leq p}$$

is invertible in $M_p(\Lambda)$. We can adapt the proof of Lemma 1.2 of [M-F] to show that A is positive; \mathcal{B}^∞ being stable under holomorphic functional calculus, we see that $(e'_1, \dots, e'_p) = (A^{-\frac{1}{2}}e_1, \dots, A^{-\frac{1}{2}}e_p)$ is a \mathcal{B}^∞ -orthonormal basis for \mathcal{L} . Lastly the orthogonal projection onto L is defined by:

$$\Pi_{\mathcal{L}} = \sum_{j=1}^p \langle \cdot, e'_j \rangle e'_j$$

which is clearly smoothing.

Lemma 16.2. *Let \mathcal{F} and \mathcal{G} be two closed sub- \mathcal{B}^∞ modules of $C^\infty(M, \mathcal{E}^{\infty,+})$ such that \mathcal{F} is finitely generated, the projection $P_{\mathcal{F}}$ onto \mathcal{F} along \mathcal{G} is smoothing, and:*

$$\mathcal{F} \oplus \mathcal{G} = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \text{as Frechet spaces}$$

$$\mathcal{F} \otimes \Lambda \oplus \overline{\mathcal{G} \otimes \Lambda} = L^2(M, \mathcal{E}^+), \quad \text{as Banach spaces}$$

Then \mathcal{F} is \mathcal{B}^∞ -projective.

Proof. We follow [M-F] page 95. Let (f_1, \dots, f_l) be a \mathcal{B}^∞ -generating family of \mathcal{F} .

Sublemma 16.3. *There exists a constant $C > 0$ such that for any $x \in \mathcal{F}$ satisfying $\|x\| < 1$, we can find $(\lambda_1, \dots, \lambda_l) \in \Lambda^l$ so that $x = \sum_{j=1}^l \lambda_j f_j$ and $\|\lambda_j\| < C$ for any $j \in \{1, \dots, l\}$.*

Proof. Since $\mathcal{F} \otimes \Lambda$ is closed (and a Banach space) we just have to apply the open mapping theorem to the following continuous surjective linear map:

$$\begin{aligned} \bigoplus_{j=1}^l \Lambda &\rightarrow \mathcal{F} \otimes \Lambda \\ (\lambda_1, \dots, \lambda_l) &\rightarrow \sum_{j=1}^l \lambda_j f_j \end{aligned}$$

Going back to the proof of Lemma 16.2, let $m \in \mathbb{N}^*$ and Π_m be the orthogonal projection onto \mathcal{L}_m^+ (see convention 1.2). For each $k \in \{1, \dots, l\}$ we can write $\Pi_m(f_k) = f_k^1 \oplus g_k$ where $f_k^1 \in \mathcal{F}$ and $g_k \in \mathcal{G}$. Since $f_k \in \mathcal{F}$, and $P_{\mathcal{F}} \circ \Pi_m(f_k) = f_k^1$, we then have:

$$f_k - f_k^1 = P_{\mathcal{F}}(f_k - \Pi_m(f_k))$$

If m is big enough then [M-F] have proved that $(f_k^1)_{1 \leq k \leq l}$ is a Λ -generating family of $\mathcal{F} \otimes \Lambda$. Using Sublemma 16.3 we see that since $f_k - \Pi_m(f_k)$ is very small, $\Pi_m|_{\mathcal{F}}$ is injective and

$$L^2(M, \mathcal{E}^+) = \Pi_m(\mathcal{F}) \otimes \Lambda \oplus \overline{\mathcal{G} \otimes \Lambda}$$

We are going to show that $\Pi_m(\mathcal{F}) \oplus \mathcal{G} = C^\infty(M, \mathcal{E}^{\infty,+})$ which is the key point. Let us consider the map Ψ :

$$\begin{aligned} C^\infty(M, \mathcal{E}^{\infty,+}) &= \mathcal{F} \oplus \mathcal{G} \rightarrow \Pi_m(\mathcal{F}) \oplus \mathcal{G} \\ x = f \oplus g &\rightarrow \Psi(x) = \Pi_m(f) \oplus g \end{aligned}$$

We have $\Psi = \text{Id} + R_m$, where $R_m = (\Pi_m - \text{Id}) \circ P_{\mathcal{F}}$. Using Sublemma 16.3 we see that for m large enough the norm of R_m , as a bounded operator on $L^2(M, \mathcal{E}^+)$, is lower than $\frac{1}{2}$. Since R_m is smoothing, Proposition 1.6 shows that Ψ is invertible in the \mathcal{B}^∞ -calculus so that:

$$\text{Im } \Psi = \Pi_m(\mathcal{F}) \oplus \mathcal{G} = C^\infty(M, \mathcal{E}^{\infty,+})$$

Since $\text{Im } \Pi_m = \mathcal{L}_m^+$ and $\Psi(\mathcal{F}) = \Pi_m(\mathcal{F})$ we see that:

$$\mathcal{L}_m^+ = \Psi(\mathcal{F}) \oplus \mathcal{G} \cap \mathcal{L}_m^+$$

So $\Psi(\mathcal{F})$ and \mathcal{F} are \mathcal{B}^∞ -projective.

Lemma 16.4. *Let $\mathcal{K} \in \Psi_{\mathcal{B}^\infty}^{-\infty}(M, \mathcal{E}^{\infty,+})$ be a smoothing operator in the \mathcal{B}^∞ -calculus. Then we can find two free sub- \mathcal{B}^∞ -modules of finite rank $\mathcal{N}_1, \mathcal{N}_2$ of $C^\infty(M, \mathcal{E}^{\infty,+})$ and two closed sub- \mathcal{B}^∞ -modules $\mathcal{G}_1, \mathcal{G}_2$ of the space $C^\infty(M, \mathcal{E}^{\infty,+})$ such that for $1 \leq i \leq 2$:*

$$\mathcal{N}_i \oplus \mathcal{G}_i = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \mathcal{N}_i \otimes_{\mathcal{B}^\infty} \Lambda \oplus \overline{\mathcal{G}_i \otimes_{\mathcal{B}^\infty} \Lambda} = L^2(M, \mathcal{E}^+)$$

$$(\text{Id} + \mathcal{K})(\mathcal{N}_1) \subset \mathcal{N}_2, \quad \text{Id} + \mathcal{K} : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \text{ is invertible}$$

$$(\text{Id} + \mathcal{K}) \otimes \text{Id}_\Lambda : \overline{\mathcal{G}_1 \otimes_{\mathcal{B}^\infty} \Lambda} \rightarrow \overline{\mathcal{G}_2 \otimes_{\mathcal{B}^\infty} \Lambda} \text{ is invertible}$$

Moreover the projection Q of $C^\infty(M, \mathcal{E}^{\infty,+})$ onto \mathcal{N}_2 along \mathcal{G}_2 is smoothing.

Proof. The idea of the proof is as in Lemma 2.2 of [M-F] . We use the notations of Convention 1.2 and set for $m \in N$:

$$\mathcal{L}_m^+ = \oplus_{k=1}^m \mathcal{B}^\infty e_k^+ \subset C^\infty(M, \mathcal{E}^{\infty,+}) \tag{16.1}$$

Let Π_m be the orthogonal projection onto \mathcal{L}_m^+ . Since $\mathcal{K} \otimes \text{Id}_\Lambda$ is a compact operator acting on $L^2(M, \mathcal{E}^+)$, we can choose m large enough so that the operator norm of the restriction of $\mathcal{K} \otimes \text{Id}_\Lambda$ to $\overline{\mathcal{L}_m^{+\perp} \otimes \Lambda}$ is $< \frac{1}{2}$. Now we consider the matrix form of \mathcal{K} in the decomposition $\mathcal{L}_m^{+\perp} \oplus \mathcal{L}_m^+ = C^\infty(M, \mathcal{E}^{\infty,+})$:

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & \mathcal{K}_2 \\ \mathcal{K}_3 & \mathcal{K}_4 \end{pmatrix}$$

So $\text{Id} + \mathcal{K} \circ (\text{Id} - \Pi_m)$ defines an invertible operator on $\overline{\mathcal{L}_m^{+\perp} \otimes \Lambda} \oplus \mathcal{L}_m^+ \otimes \Lambda = L^2(M, \mathcal{E}^+)$ determined by:

$$\text{Id} : \mathcal{L}_m^+ \otimes \Lambda \rightarrow \mathcal{L}_m^+ \otimes \Lambda$$

$$(\text{Id} + \mathcal{K}_1) \otimes \text{Id}_\Lambda : \overline{\mathcal{L}_m^\perp \otimes \Lambda} \rightarrow \overline{\mathcal{L}_m^\perp \otimes \Lambda}$$

Using Proposition 1.6 and Lemma 16.1 we see that this operator is invertible in the \mathcal{B}^∞ -calculus so that $(\text{Id} + \mathcal{K}_1)^{-1}$ sends $\mathcal{L}_m^{+\perp}$ into itself. Now we proceed as in [M-F] and set:

$$\mathcal{U}_1 = \begin{pmatrix} \text{Id} & -(\text{Id} + \mathcal{K}_1)^{-1} \circ \mathcal{K}_2 \\ 0 & \text{Id} \end{pmatrix} \quad \mathcal{U}_2^{-1} = \begin{pmatrix} \text{Id} & O \\ \mathcal{K}_3 \circ (\text{Id} + \mathcal{K}_1)^{-1} & \text{Id} \end{pmatrix}$$

$$\mathcal{N}_1 = \mathcal{U}_1(\mathcal{L}_m^+), \quad \mathcal{G}_1 = \mathcal{U}_1(\mathcal{L}_m^{+\perp})$$

$$\mathcal{N}_2 = \mathcal{U}_2^{-1}(\mathcal{L}_m^+), \quad \mathcal{G}_2 = \mathcal{U}_2^{-1}(\mathcal{L}_m^{+\perp})$$

Let $\Pi_{\mathcal{L}_m^+}$ be the (smoothing) orthogonal projection onto \mathcal{L}_m^+ , then $Q = \mathcal{U}_2^{-1} \circ \Pi_{\mathcal{L}_M^+} \circ \mathcal{U}_2$ is also smoothing and Lemma 16.4 is proved.

Definition 16.5. Let $\mathcal{F} \in \Psi_{\mathcal{B}^\infty}^0(M, \mathcal{E}^{\infty,+}, \mathcal{E}^{\infty,-})$ be an elliptic pseudo-differential operator of order zero. We shall say that \mathcal{F} is \mathcal{B}^∞ -Fredholm if there exist $m \in \mathbb{N}^*$, \mathcal{U} an invertible pseudodifferential operator of order zero in the \mathcal{B}^∞ -calculus, Frechet spaces decompositions

$$C^\infty(M, \mathcal{E}^{\infty,+}) = \mathcal{U}(\mathcal{L}_m^+) \oplus \mathcal{U}(\mathcal{L}_m^{+\perp}), \quad C^\infty(M, \mathcal{E}^{\infty,-}) = \mathcal{N}_0 \oplus \mathcal{G}_0$$

Banach spaces decompositions

$$L^2(M, \mathcal{E}^+) = \mathcal{U}(\mathcal{L}_m^+) \otimes \Lambda \oplus \overline{\mathcal{U}(\mathcal{L}_m^{+\perp}) \otimes \Lambda}, \quad L^2(M, \mathcal{E}^-) = \mathcal{N}_0 \otimes \Lambda \oplus \overline{\mathcal{G}_0 \otimes \Lambda}$$

satisfying the following three conditions:

- (i) $\mathcal{N}_0, \mathcal{G}_0$ are closed sub- \mathcal{B}^∞ -modules
- (ii) \mathcal{N}_0 is of finite rank
- (iii) \mathcal{F} sends $\mathcal{U}(\mathcal{L}_m^+)$ into \mathcal{N}_0 , $\mathcal{F} : \mathcal{U}(\mathcal{L}_m^{+\perp}) \rightarrow \mathcal{G}_0$ is invertible and:

$$\mathcal{F} : \overline{\mathcal{U}(\mathcal{L}_m^{+\perp}) \otimes \Lambda} \rightarrow \overline{\mathcal{G}_0 \otimes \Lambda} \quad \text{is invertible.}$$

Lemma 16.6. Let $\mathcal{K} \in \text{Hom}_{\mathcal{B}^\infty}(M, \mathcal{E}^{\infty,+}, \mathcal{E}^{\infty,-})$ be a smoothing operator and let $\mathcal{F} \in \Psi_{\mathcal{B}^\infty}(M, \mathcal{E}^{\infty,+}, \mathcal{E}^{\infty,-})$ be a \mathcal{B}^∞ -Fredholm operator. Then we can find closed sub- \mathcal{B}^∞ -modules $\mathcal{N}_1^\pm, \mathcal{G}_1^\pm$ of $C^\infty(M, \mathcal{E}^{\infty,\pm})$, \mathcal{N}_1^\pm being finitely generated, \mathcal{N}_1^+ being free such that:

$$\mathcal{N}_1^\pm \oplus \mathcal{G}_1^\pm = C^\infty(M, \mathcal{E}^{\infty,\pm}), \quad \text{as Frechet spaces}$$

$$\mathcal{N}_1^\pm \otimes \Lambda \oplus \overline{\mathcal{G}_1^\pm \otimes \Lambda} = L^2(M, \mathcal{E}^\pm), \quad \text{as Banach spaces,}$$

$\mathcal{F} + \mathcal{K}$ sends \mathcal{N}_1^+ into \mathcal{N}_1^- , $\mathcal{F} + \mathcal{K} : \mathcal{G}_1^+ \rightarrow \mathcal{G}_1^-$ is (Frechet) invertible.

$$\mathcal{F} + \mathcal{K} : \overline{\mathcal{G}_1^+ \otimes \Lambda} \rightarrow \overline{\mathcal{G}_1^- \otimes \Lambda} \quad \text{is invertible.}$$

Proof. We follow the notations of Definition 16.5. We can replace \mathcal{F}, \mathcal{K} by $\mathcal{F} \circ \mathcal{U}, \mathcal{K} \circ \mathcal{U}$ and thus assume that $\mathcal{U} = \text{Id}$. Let

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 & 0 \\ 0 & \mathcal{F}_4 \end{pmatrix} \quad \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & \mathcal{K}_2 \\ \mathcal{K}_3 & \mathcal{K}_4 \end{pmatrix}$$

be the matrix decomposition of \mathcal{F} , \mathcal{K} associated with the two decompositions:

$$C^\infty(M, \mathcal{E}^{\infty,+}) = \mathcal{L}_m^{+\perp} \oplus \mathcal{L}_m^+, \quad C^\infty(M, \mathcal{E}^{\infty,-}) = \mathcal{F}(\mathcal{L}_m^{+\perp}) \oplus \mathcal{N}_0$$

Let Π_m be the orthogonal projection onto \mathcal{L}_m^+ . By assumption \mathcal{F}_1 is Frechet-invertible from $\mathcal{L}_m^{+\perp}$ to $\mathcal{F}(\mathcal{L}_m^{+\perp})$. The operator defined by zero on \mathcal{N}_0 and by \mathcal{F}_1^{-1} on $\mathcal{F}(\mathcal{L}_m^{+\perp})$ sends continuously $C^\infty(M, \mathcal{E}^{\infty,-})$ into $C^\infty(M, \mathcal{E}^{\infty,+})$. Since \mathcal{K} is B^∞ -smoothing (and thus Λ -compact) Proposition 1.4 (part 2]) then shows that $\mathcal{F}_1^{-1} \circ \mathcal{K}_1 \circ (\text{Id} - \Pi_m)$ is smoothing, moreover if m is large enough then we can assume that its operator norm on $L^2(M, \mathcal{E}^+)$ is $< \frac{1}{2}$. Using Proposition 1.6 we see that $(\text{Id} + \mathcal{F}_1^{-1} \circ \mathcal{K}_1)^{-1} \circ \mathcal{F}_1^{-1} = (\mathcal{F}_1 + \mathcal{K}_1)^{-1}$ sends $\mathcal{F}(\mathcal{L}_m^{+\perp})$ onto $\mathcal{L}_m^{+\perp}$. We set:

$$\mathcal{U}'_1 = \begin{pmatrix} \text{Id} & -(\mathcal{F}_1 + \mathcal{K}_1)^{-1} \circ \mathcal{K}_2 \\ 0 & \text{Id} \end{pmatrix}, \quad \mathcal{U}'_2 = \begin{pmatrix} & \text{Id} & 0 \\ -\mathcal{K}_3 \circ (\mathcal{F}_1 + \mathcal{K}_1)^{-1} & & \text{Id} \end{pmatrix}$$

We see that:

$$\mathcal{U}'_2 \circ (\mathcal{F} + \mathcal{K}) \circ \mathcal{U}'_1 = \begin{pmatrix} \mathcal{F}_1 + \mathcal{K}_1 & & 0 \\ 0 & -\mathcal{K}_3 \circ (\mathcal{F}_1 + \mathcal{K}_1)^{-1} \circ \mathcal{K}_2 + \mathcal{F}_4 + \mathcal{K}_4 & \end{pmatrix}$$

Therefore we get the Lemma by setting:

$$\begin{aligned} \mathcal{N}_1^+ &= \mathcal{U}'_1(\mathcal{L}_m^+) & \mathcal{G}_1^+ &= \mathcal{U}'_1(\mathcal{L}_m^{+\perp}) \\ \mathcal{N}_1^- &= \mathcal{U}'_2^{-1}(\mathcal{N}_0) & \mathcal{G}_2^- &= \mathcal{U}'_2^{-1}(\mathcal{F}(\mathcal{L}_m^{+\perp})) \end{aligned}$$

The Lemma is proved.

Remark. Using the projection of $\mathcal{N}_1^{+\perp}$ onto \mathcal{G}_1^+ along \mathcal{N}_1^+ which is invertible for the Frechet Topology, one sees easily that one can replace in the previous Lemma \mathcal{G}_1^+ [resp. \mathcal{G}_1^-] by $\mathcal{N}_1^{+\perp}$ [resp. $(\mathcal{F} + \mathcal{K})(\mathcal{N}_1^{+\perp})$].

Proof of Theorem 1.3. Since

$$\mathcal{X} = (\text{Id} + \mathcal{D}_\infty^2)^{-\frac{1}{2}} \in \Psi_{\mathcal{B}^\infty}^{-1}(M, \mathcal{E}^{\infty,+})$$

is invertible, $\mathcal{X} \circ \mathcal{D}^+$ is elliptic of zero order. So there exists a parametrix

$$\mathcal{P} \in \Psi_{\mathcal{B}^\infty}^0(M, \mathcal{E}^{\infty,-}, \mathcal{E}^{\infty,+})$$

such that: $\mathcal{X} \circ \mathcal{D}_\infty^+ \circ \mathcal{P} = \text{Id} + \mathcal{K}_1$ where $\mathcal{K}_1 \in \Psi_{\mathcal{B}^\infty}^-(M, \mathcal{E}^{\infty,-})$ is smoothing. In order to lighten the notations we will only write the \mathcal{B}^∞ -decompositions. We apply Lemma 16.4 to \mathcal{K}_1 (in place of \mathcal{K}) and get two decompositions:

$$\mathcal{N}_1 \oplus \mathcal{G}_1 = C^\infty(M, \mathcal{E}^{\infty,-}) = \mathcal{N}_2 \oplus \mathcal{G}_2$$

satisfying the properties stated in Lemma 16.4:

$$(\text{Id} + \mathcal{K}_1)(\mathcal{N}_1) \subset \mathcal{N}_2, \quad \text{Id} + \mathcal{K}_1 : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \text{ is invertible}$$

$\mathcal{N}_1, \mathcal{N}_2$ are free and finitely generated. Let Q be the projection onto \mathcal{N}_2 along \mathcal{G}_2 , thanks to Lemma 16.4 Q is smoothing and $Q \circ \mathcal{X} \circ \mathcal{D}_\infty^+$ is smoothing. We are going to show that $\mathcal{F} = (\text{Id} - Q) \circ \mathcal{X} \circ \mathcal{D}_\infty^+$ is \mathcal{B}^∞ -Fredholm in the sense of definition 16.5; then an application of Lemma 16.6 to \mathcal{F} and $\mathcal{K} = Q \circ \mathcal{X} \circ \mathcal{D}_\infty^+$ will give the Theorem. Clearly $\text{Im } \mathcal{F} = \mathcal{G}_2$. Moreover we also have: $\mathcal{P} \circ \mathcal{F} = \text{Id} + \mathcal{K}_2$, where $\mathcal{K}_2 \in \Psi_{\mathcal{B}^\infty}^-(M, \mathcal{E}^{\infty,+})$ is smoothing. We apply Lemma 16.4 to $\text{Id} + \mathcal{K}_2$ and get two decompositions:

$$\mathcal{N}'_1 \oplus \mathcal{G}'_1 = C^\infty(M, \mathcal{E}^{\infty,+}) = \mathcal{N}'_2 \oplus \mathcal{G}'_2$$

where \mathcal{N}'_1 is free of finite rank and:

$$\mathcal{P} \circ \mathcal{F}(\mathcal{N}'_1) \subset \mathcal{N}'_2, \quad \mathcal{P} \circ \mathcal{F} : \mathcal{G}'_1 \rightarrow \mathcal{G}'_2, \quad \text{is invertible.}$$

Since by construction $\mathcal{N}'_2 \cap \mathcal{G}'_2 = \{0\}$ we see that: $\text{null}(\mathcal{F}) \subset \mathcal{N}'_1$. Using \mathcal{P} we see that \mathcal{F} sends \mathcal{G}'_1 isomorphically onto a closed subspace of $\mathcal{G}_2 \subset C^\infty(M, \mathcal{E}^{\infty,+})$. Using \mathcal{P} again we see that: $\mathcal{F}(\mathcal{G}'_1) \cap \mathcal{F}(\mathcal{N}'_1) = \{0\}$. Since $\mathcal{N}'_1 \oplus \mathcal{G}'_1 = C^\infty(M, \mathcal{E}^{\infty,+})$ we get $\mathcal{G}_2 = \text{Im } \mathcal{F} = \mathcal{F}(\mathcal{G}'_1) \oplus \mathcal{F}(\mathcal{N}'_1)$

So we get a decomposition for a \mathcal{F} :

$$\mathcal{G}'_1 \oplus \mathcal{N}'_1 \rightarrow \mathcal{F}(\mathcal{G}'_1) \oplus [\mathcal{F}(\mathcal{N}'_1) \oplus \mathcal{N}_2]$$

Since $\mathcal{G}'_1 \oplus \mathcal{N}'_1$ is associated to $\text{Id} + \mathcal{K}_2$ as in the proof of Lemma 16.4 we see that \mathcal{F} is \mathcal{B}^∞ -Fredholm so that we can apply Lemma 16.6 to $\mathcal{F} = (\text{Id} - Q) \circ \mathcal{X} \circ \mathcal{D}_\infty^+$ and $\mathcal{K} = Q \circ \mathcal{X} \circ \mathcal{D}_\infty^+$. We get then the following decomposition for $\mathcal{D}_\infty^+ = \mathcal{X}^{-1} \circ (\mathcal{F} + \mathcal{K})$:

$$\mathcal{N}_1^+ \oplus \mathcal{G}_1^+ = C^\infty(M, \mathcal{E}^{\infty,+}), \quad \mathcal{N} \oplus \mathcal{D}_\infty^+(\mathcal{G}_1^+) = C^\infty(M, \mathcal{E}^{\infty,-})$$

$$\mathcal{N}_1^+ \otimes \Lambda \oplus \overline{\mathcal{G}_1^+ \otimes \Lambda} = L^2(M, \mathcal{E}^+), \quad \mathcal{N} \otimes \Lambda \oplus \overline{\mathcal{D}_\infty^+(\mathcal{G}_1^+) \otimes \Lambda} = \mathcal{H}^{-1}(M, \mathcal{E}^-)$$

where \mathcal{N} and $\mathcal{L}_\infty = \mathcal{N}_1^+$ are finitely generated sub- \mathcal{B}^∞ -modules, \mathcal{L}_∞ being free.

$\mathcal{D}_\infty^+(\mathcal{N}_1^+) \subset \mathcal{N}$. Moreover we have:

$$\mathcal{D}_\infty^+ : \mathcal{G}_1^+ \rightarrow \mathcal{D}_\infty^+(\mathcal{G}_1^+) \text{ is invertible}$$

$$\mathcal{D}^+ : \overline{\mathcal{G}_1^+ \otimes \Lambda} \rightarrow \overline{\mathcal{D}_\infty^+(\mathcal{G}_1^+) \otimes \Lambda} \subset \mathcal{H}^{-1}(M, \mathcal{E}^-) \text{ is invertible}$$

Now we see easily that we can replace \mathcal{G}_1^+ [resp. $\mathcal{D}_\infty^+(\mathcal{G}_1^+)$] by \mathcal{L}_∞^\perp [resp. $\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)$]. Let us show that the projection $P_{\mathcal{N}}$ onto \mathcal{N} along $\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)$ is smoothing. Let $\Pi_{\mathcal{L}_\infty}$ be the smoothing orthogonal projection onto \mathcal{L}_∞ . We can find a pseudo-differential operator \mathcal{H} in the \mathcal{B}^∞ -calculus and a smoothing operator \mathcal{R} so that:

$$\mathcal{D}_\infty^+ \circ (\text{Id} - \Pi_{\mathcal{L}_\infty}) \circ \mathcal{H} = \text{Id} - \mathcal{R}$$

By construction we have $P_{\mathcal{N}} \circ \mathcal{D}_\infty^+ \circ (\text{Id} - \Pi_{\mathcal{L}_\infty}) = 0$ and $P_{\mathcal{N}} = P_{\mathcal{N}} \circ \mathcal{R}$. From proposition 1.4 2] we get that $P_{\mathcal{N}}$ is smoothing. Now we see that \mathcal{N} is \mathcal{B}^∞ -projective by applying Lemma 16.2 to:

$$\mathcal{F} = \mathcal{X}(\mathcal{N}), \quad \mathcal{G} = \mathcal{X}(\mathcal{D}_\infty^+(\mathcal{L}_\infty^\perp)).$$

Theorem 1.3 is thus proved.

17. Appendix B: proof of the \mathcal{B}^∞ - b -decompositions.

We shall now establish Theorem 12.7. The structure of the proof is as the one for closed manifold, given in the previous section and we shall therefore be very brief. First of all we fix an orthonormal basis $(e_k^\pm)_{k \geq 1}$ of $L_b^2(M, \mathcal{E}^\pm)$ such that for any $k \geq 1$ $e_k^\pm \in x\mathcal{H}_b^\infty(M, \mathcal{E}^{\infty, \pm})$. We set $\mathcal{L}_m^+ = \bigoplus_{k=1}^m \mathcal{B}^\infty \varepsilon_k^+$; one has to use this space as the analogous one in Sections 1, 16.

Next we observe that for each m and for each $\varepsilon > 0$ $\Psi_{b, \mathcal{B}^\infty}^{m, \varepsilon}(M; \mathcal{E}^\infty) \circ \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M; \mathcal{E}^\infty) \subset \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M; \mathcal{E}^\infty)$ and that for each $\varepsilon' \in [0, \varepsilon]$ both $\Psi_{b, \mathcal{B}^\infty}^{m, \varepsilon}$ and $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}$ map $x^{\varepsilon'} \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ into itself. The proof of this two facts is as in [M]. We also observe that if $B : x^{\varepsilon'} \mathcal{H}_b^\infty \rightarrow x^{\varepsilon'} \mathcal{H}_b^\infty$ is continuous for the Frechet topology and $R \in \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}$ then $B \circ R \in \text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon'}$.

These mapping properties are used as in Section 1 to establish the analogue of Proposition 1.6, namely:

- (i) Let $\varepsilon > 0$ and let $\mathcal{P} \in \Psi_{b, \mathcal{B}^\infty}^{0, \varepsilon}(M; \mathcal{E}^\infty)$ be such that its operator norm $\|\mathcal{P}\|_{\mathcal{B}(L_b^2)} < \frac{1}{2}$. Then $\text{Id} - \mathcal{P}$ sends $x^\varepsilon \mathcal{H}_b^\infty(M; \mathcal{E}^\infty)$ into itself.
- (ii) Let $\mathcal{Q} \in \Psi_{b, \mathcal{B}^\infty}^m(M; \mathcal{E}^\infty)$ be invertible in the Λ - b -calculus. If \mathcal{Q} admits a parametrix in the \mathcal{B}^∞ - b -calculus then $\mathcal{Q}^{-1} \in \Psi_{b, \mathcal{B}^\infty}^{-m, \varepsilon}(M; \mathcal{E}^\infty)$.

As a consequence of (ii) we see that

$$\mathcal{X} \equiv (i\text{Id} + \mathcal{D}_\infty)^{-1} \in \Psi_{b, \mathcal{B}^\infty}^{-1, \varepsilon}(M; \mathcal{E}^\infty). \quad (17.1)$$

We will omit the easy proof of the following lemma since it relies on standard techniques of the previous section

Lemma 17.1. *Let $\varepsilon > 0$, and let \mathcal{L}, \mathcal{G} two closed sub- \mathcal{B}^∞ -modules of $\mathcal{H}_b^\infty(M; \mathcal{E}^{\infty,+})$ such that $\mathcal{L} \subset x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,+})$ is finitely generated and*

$$\mathcal{L} \oplus \mathcal{G} = \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,+}) \text{ as Frechet spaces}$$

$$\mathcal{L} \otimes_{\mathcal{B}^\infty} \Lambda \oplus \overline{\mathcal{G} \otimes_{\mathcal{B}^\infty} \Lambda} = L_b^2(M, \mathcal{E}^+) \text{ as Banach spaces.}$$

Then we have:

- 1] On $\mathcal{L}, H_b^\infty(M, \mathcal{E}^{\infty,+})$ and $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,+})$ induce the same topology.
- 2] If \mathcal{L} is moreover free then $\mathcal{L} \oplus \mathcal{L}^\perp = \mathcal{H}_b^\infty(M, \mathcal{E}^{\infty,+})$ and the orthogonal projection onto \mathcal{L} belongs to $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}$.
- 3] If the projection onto \mathcal{L} along \mathcal{G} belongs to $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}$ then \mathcal{L} is \mathcal{B}^∞ -projective.

The proof of the theorem now proceeds as that of Theorem 1.3 in the closed case once we make the following changes

- (i) the \mathcal{B}^∞ -smoothing operators are replaced by $\text{End}_{b, \mathcal{B}^\infty}^{\infty, \varepsilon}(M; \mathcal{E}^\infty)$
- (ii) the smooth sections of \mathcal{E}^∞ are replaced by $\mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$ with the finite rank sub-modules always contained in $x^\varepsilon \mathcal{H}_b^\infty(M, \mathcal{E}^\infty)$
- (iii) the \mathcal{B}^∞ -calculus is replaced by the \mathcal{B}^∞ - b -calculus with bounds and \mathcal{X} is chosen as in (17.1).
- (iv) as for the definition of $b - \mathcal{B}^\infty$ -Fredholm operator \mathcal{F} we add the following assumption, with notations parallel to that of definition 16.5, given the decomposition for \mathcal{F} :

$$H_b^\infty(M, \mathcal{E}^{\infty,+}) = \mathcal{U}(\mathcal{L}_m^+) \oplus \mathcal{U}(\mathcal{L}_m^{+\perp}) \rightarrow \mathcal{N}_0 \oplus \mathcal{G}_0 = H_b^\infty(M, \mathcal{E}^{\infty,+})$$

\mathcal{F}^{-1} sends continuously $\mathcal{G}_0 \cap x^\varepsilon H_b^\infty(M, \mathcal{E}^{\infty,+})$ into $x^\varepsilon H_b^\infty(M, \mathcal{E}^{\infty,+})$.

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