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On the search of genuine p-adic modular L-functions for $GL(n)$. With a correction to : on p-adic L-functions of $GL(2) \times GL(2)$ over totally real fields

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On the Search of Genuine p-adic Modular L-Functions for $GL(n)$

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Abstract — The purpose of this monograph is to state several conjectures concerning the existence and the meromorphy of many variable p-adic L-functions attached to many variable Galois representations (for example having values in $GL_n(\mathbb{Z}_p[[X_1, \ldots, X_r]])$) and to present some supporting examples for the conjectures. Our discussion in the earlier sections is therefore quite speculative, but towards the end, we gradually make things more concrete.

Résumé — Le but de cette monographie est de formuler quelques conjectures concernant l'existence et la méromorphie des fonctions L p-adiques de plusieurs variables attachées à des représentations galoisiennes de plusieurs variables (par exemple, à valeurs dans $GL_n(\mathbb{Z}_p[[X_1, \ldots, X_r]])$) et de présenter quelques exemples motivant nos conjectures. Nous commençons par une discussion assez spéculative, mais vers la fin, nous donnons des résultat plus concrets.

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1. Introduction

The purpose of this monograph is to state several conjectures concerning the existence, a precise interpolation property and the meromorphy of many variable p-adic L-functions attached to many variable (irreducible) Galois representations (for example having values in $GL_n(\mathbb{Z}_p[[X_1, \ldots, X_r]])$) and to present some supporting examples for the conjectures. Often in the non-abelian case, p-adic L-function is determined up to unit multiples. We would like to identify a precise interpolation property necessary to determine the p-adic L uniquely. This is important to have a non-abelian generalization of classical class number formulas and limit formulas which gives a direct connection between the analytic p-adic L and arithmetic objects. The theory of abelian p-adic L-functions is constructed out of a desire to better understand the class number formulas. Contrary to this, in the non-abelian case, it is ironic for us to be left in search of a p-adic L-function genuinely characterized by the Galois representation by which the generalized class number formula should be written down. Our discussion in the earlier sections is therefore quite speculative, but towards the end, we gradually make things more concrete.

Let us describe our idea. Let p be a prime, and fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let F be a finite extension of $\mathbb{Q}$. We write I for the set of all embeddings of F into $\overline{\mathbb{Q}}$. We will later fix an embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and take a field K which is a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ containing the image under $i_p\sigma$ for every $\sigma \in I$. Write $\mathbb{O}$ for the p-adic integer ring of K. We extend σ to an isomorphism: $\overline{F} \cong \overline{\mathbb{Q}}$ which we denote again by σ. For each p-adic place $\mathfrak{P}$ of F, we write $\mathcal{G}_{F_\mathfrak{P}}$ for the Galois group $\mathrm{Gal}(\overline{F}_\mathfrak{P}/F_\mathfrak{P})$, and if $\mathfrak{P}$ is induced from $i_p\sigma$, we identify it with the decomposition group at $\mathfrak{P}$ in $\mathcal{G}_F = \mathrm{Gal}(\overline{F}/F)$ via $i_p\sigma$. Let T_n be the standard diagonal torus of $\mathrm{Res}_{\mathfrak{r}/\mathbb{Z}} GL(n)$ for the integer ring $\mathfrak{r}$ of F which is split over $\mathbb{O}$. Then we consider a normal integral domain $\mathbb{I}$ finite (but not necessarily flat) over the completed group algebra $\mathbb{O}[[T_n(\mathbb{Z}_p)]]$. We assume that $\mathbb{O}$ is integrally closed in $\mathbb{I}$. Each $\mathbb{O}$-algebra homomorphism $P : \mathbb{I} \to K$ restricted to $\mathbb{O}[[T_n(\mathbb{Z}_p)]]$ induces a continuous character $\kappa(P)$ of $T_n(\mathbb{Z}_p) \to K^\times$. We call $P \in Spec(\mathbb{I})(K)$ *arithmetic* if it induces a weight (that is, an algebraic character) of the torus T_n

on an open neighborhood of the identity in $T_n(\mathbb{Z}_p)$. We want to study a continuous Galois representation $\varphi : \mathcal{G}_F \to GL_n(\mathbb{I})$ acting on $V(\cong \mathbb{I}^n)$ satisfying the following condition:

(A1) *There are arithmetic points P densely populated in $Spec(\mathbb{I})(K)$ such that the Galois representation $\varphi_P = P \circ \varphi$ is the p-adic étale realization of a rank n pure motive M_P defined over F with coefficients in a number field E_P in $\overline{\mathbb{Q}}$.*

The number field E_P depends on P, and the composite E_∞ of all E_P for P satisfying (A1) is usually an infinite extension of $\mathbb{Q}$. We call points P satisfying (A1) *motivic*. Densely populated motivic points determine an isomorphism $i_p : E_\infty \hookrightarrow \overline{\mathbb{Q}}_p$ such that $Tr(\varphi_P(Frob_\mathfrak{l}))$ for primes $\mathfrak{l}$ unramified for φ_P generates a subfield of $i_p(E_P)$. We extend i_p to $\overline{\mathbb{Q}}$. To get reasonable p-adic L-functions, we need to assume further

(A2$_\pm$) *For a dense subset of motivic points P, we have*

(i) *The Tate twist $M_P(1)$ is critical in the sense of Deligne [15], and for each p-adic place $\mathfrak{P}$ of F, the restriction of φ_P to $\mathcal{G}_{F_\mathfrak{P}}$ is of Hodge-Tate type;*

(ii) *Writing $\mathcal{F}_P^\mp$ for the middle terms of the Hodge filtration of $H_{DR}(M_P)$ as in [15], for each p-adic place $\mathfrak{P} = i_p\sigma$ of F, there exists an $\mathbb{I}$-direct summand $V_\mathfrak{P}^\pm \subset V$ (independent of P) stable under $\mathcal{G}_{F_\mathfrak{P}}$ such that the comparison isomorphism of Faltings [21] induces: $V_\mathfrak{P}^\mp \otimes_{\mathbb{I},P} B_{HT} \cong \mathcal{F}_P^\mp \otimes_{F\otimes E, i_p\sigma\otimes i_p} B_{HT}$,*

where B_{HT} is one of the Fontaine's rings [22] given as follows: Writing Ω for the p-adic completion of $\overline{\mathbb{Q}}_p$, we have $B_{HT} \cong \Omega[t, t^{-1}]$ on which $\mathcal{G}_{\mathbb{Q}_p}$ acts via the natural action on Ω and via the cyclotomic character on the indeterminate t. Since $V(\varphi_P)$ is of Hodge-Tate type, as a $\mathcal{G}_{F_\mathfrak{P}}$-module, writing $E_{P,\mathfrak{p}}$ for the topological closure of $i_p(E_P)$ in Ω, $V(\varphi_P) \otimes_{E_{P,\mathfrak{p}}} \Omega \cong \bigoplus_{1\le i\le n} \Omega(m_i)$ for integers m_i depending on $i_p\sigma$, where $\Omega(m) = \Omega t^m$ as a graded component of B_{HT}. We call $(m_1, \dots, m_n)$ the Hodge-Tate twist of $V(\varphi_P)$ at σ. We will show that the condition (ii) is equivalent to the admissibility condition of Panchishkin ([56] Section 5) for M_P if M_P is crystalline at p. A similar condition is stated and studied in [28] p.217 in terms of Selmer groups. In particular, the existence of $V_\mathfrak{P}^+$ in (A2$_-$) plays an important role in defining the Selmer group. We require by (ii) an analytic coherence of $\mathcal{F}_P^\pm$ with respect to analytically varying P. The density assumption included in (A2$_\pm$) about motivic critical points is important to guarantee the uniqueness of the p-adic L-function. We also assume

(unr) *φ is unramified at almost all places of F.*

The (conjectural) $\mathbb{I}$-adic representations arising from cohomological modular forms on $GL(n)_{/F}$ further satisfies an additional condition besides (A1-2$_\pm$). Since $T_n \cong \mathbf{G}_m^n$ over F, $X(T_n)$ is isomorphic to the product of n copies of the free module $\mathbb{Z}[I]$

generated by the set I of all embeddings of F into $\overline{\mathbb{Q}}$. Thus we can associate to each arithmetic point P, an n-tuple $(m_1(P), \dots, m_n(P))$ of elements in $\mathbb{Z}[I]$. Then the condition is

(A3) *The Hodge-Tate twist of φ_P restricted to $\mathcal{G}_{F_\mathfrak{P}}$ for the p-adic place $\mathfrak{P}$ induced by $i_p\sigma$ is given by $(m_1(P)_{c\sigma}, m_2(P)_{c\sigma}, \dots, m_n(P)_{c\sigma})$ for each $\sigma \in I$,*

where for $m \in \mathbb{Z}[I]$, we have written $m = \sum_\sigma m_\sigma \sigma$, and c stands for the complex conjugation which we specify below choosing an embedding i_∞ of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. This last condition (A3) might be always satisfied by representations satisfying (A1-2$_\pm$) after modifying the algebra structure over $\mathbb{O}[[T_n(\mathbb{Z}_p)]]$ and is equivalent to

(A3′) *The Hodge types of $H_B({}^{i_\infty\sigma}M_P)\otimes_{E,i_\infty}\mathbb{C}$ is given by $\{(m_i(P)_\sigma, m_{n+1-i}(P)_{c\sigma})\}_i$,*

where $H_B({}^{i_\infty\sigma}M_P)$ is the Betti realization of M_P at $i_\infty\sigma$. Anyway, we call the representation satisfying (A1-3) and (unr) an arithmetic Galois representation.

As conjectured by Deligne [15], for each motivic point P as in (A2$_-$), we expect to have a well defined special value

$$\frac{L(1, M_P)}{\mathbf{c}^+(M_P(1))} \in E_P \subset E_P \otimes_{\mathbb{Q}} \mathbb{C}$$

for Deligne's period $\mathbf{c}^+(M_P(1))$ in $(E \otimes_{\mathbb{Q}} \mathbb{C})^\times$. We now choose an embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and write $\frac{L_{i_\infty}(1,M_P)}{c^+_{\infty,i_\infty}(M_P(1))}$ for the i_∞-component of $\frac{L(1,M_P)}{\mathbf{c}^+(M_P(1))}$. An element L of the quotient field $\mathbb{K}$ of $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ can be regarded as a (p-adic) meromorphic function on $Spec(\mathbb{I})(\Omega)$ assigning the value $L(P) = P(L)$ to $P \in Spec(\mathbb{I})$, where $\mathbb{O}_\Omega$ is the p-adic integer ring of Ω. We call such functions "meromorphic", because it is a ratio of elements in $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ and hence a ratio of p-adic analytic functions on a non-empty Zariski open subset of $Spec(\mathbb{I})$ (or more precisely, on the formal completion of a non-empty open subscheme of $Spec(\mathbb{I})$ along its fibre over p). Since $\mathbb{I}$ is a normal integral domain, if a meromorphic function in the above sense is everywhere defined, it is actually an element in $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ (and hence is an Iwasawa function when $\mathbb{I}$ is an irreducible component of $\mathbb{O}[[T(\mathbb{Z}_p)]]$ for a quotient torus T of T_n). We expect that this is the case when φ modulo the maximal ideal of $\mathbb{I}$ is absolutely irreducible. We fix a choice of $S = \{V_\mathfrak{P}^+\}_\mathfrak{P}$. If there exists an element $L_{S,i_\infty}(\varphi) \in \mathbb{K}$ satisfying the following condition for all motivic points P as in (A1-2$_-$):

$$\text{(Int)} \qquad i_p^{-1}\left(\frac{L_{S,i_\infty}(P,\varphi)}{c^+_{p,i_\infty}(M_P(1))}\right) = *i_\infty^{-1}\left(\frac{L_{i_\infty}(1,M_P)}{c^+_{\infty,i_\infty}(M_P(1))}\right)$$

for a constant "$*$", we call $L_{S,i_\infty}(\varphi)$ a *genuine* p-adic L-function of φ of type S. The exact form of the constant "$*$" is known by Coates, Perrin-Riou and Panchishkin, when M_P is crystalline at p and φ contains all cyclotomic deformation of M_P (see, [60], [56] Section 6, and Conjecture 4.2.1 in Chapter 4 in the text). Here $c^+_{p,i_\infty}(M_P(1))$

is the p-adic period defined in exactly the same way as $c^+_{\infty,i_\infty}(M_P(1))$ replacing $\mathbb{C}$ by the Fontaine's graded ring B_{HT} (cf. [60]). We note that the period $c^+_{p,i_\infty}(M_P(1))$ actually depends on the choice of i_∞ besides its apparent dependence on i_p and on S. From definition, the p-adic period depends on the choice of S, but we do not know how its transcendency depends on S, because there might be some non-trivial relation among periods indexed by p-adic places. On the other hand, the complex L-value applied i_∞^{-1} at the right-hand side of (Int) is independent of i_∞, because, if we change i_∞ by ρi_∞ for $\rho \in Aut(\mathbb{C})$, the L-value of the right-hand side is conjectured by Deligne to be moved by ρ whose effect will be canceled by $(\rho i_\infty)^{-1}$. We will show in Chapter 2 from the admissibility $(\mathrm{A2}_+)$ (ii) that $c^+_{p,i_\infty}(M_P(1))$ lies in a single graded component Ωt^d of B_{HT} and hence can be identified with an element in the p-adic completion Ω of $\overline{\mathbb{Q}}_p$ (by trivialization, i.e., just putting $t=1$). By construction, $c^+_{p,i_\infty}(M_P(1))$ and $c^+_{\infty,i_\infty}(M_P(1))$ behave in exactly the same way under the change of basis. That is, by abusing notation, $c^+_{p,i_\infty}(M_P(1))/c^+_{\infty,i_\infty}(M_P(1))$ is constant, and the formula (Int) has an intrinsic meaning even if these periods depend on the choice of the basis of various realization of motives. This type of definition of p-adic periods has been given in various context (see [3] and [74]), but here emphasis is put on its dependence on i_∞, i_p and S. This type of formulation of p-adic L-functions is first found by Katz in his construction of p-adic Hecke L-functions [52] and in this general form, it is due to Don Blasius as indicated in [56] Conjecture II′ in Appendix. We study this type of L-functions and list some general conjectures in Chapter 4.

So far, it looks that things are going well. However there is a subtle fact (pointed out to me by Don Blasius) that we do not know whether $c^+_{p,i_\infty}(M_P(1)) \neq 0$ or not, and André [1] has given fairly general examples of the vanishing for some special choice of i_∞ (see 3.3 in the text). We expect to be able to go around this difficulty by changing i_∞. In [1], the transcendence of the field spanned by p-adic periods moving around i_p is studied, and some estimate of its transcendental degree is given for generic i_p. The embedding i_p is intrinsically determined by φ (at least on E_∞), and changing (i_∞, i_p) to $(i_\infty\sigma, i_p\sigma)$ for $\sigma \in \mathcal{G}_{\mathbb{Q}}$ does not affect anything. Thus changing i_∞ is basically the same as changing i_p. Here we study the dependence of the period relative to L-values. As already remarked, the right-hand side of (Int) is independent of i_∞ and therefore, it is all right to change i_∞. We will show that if one can choose an embedding i_∞ so that $c^+_{p,i_\infty}(M_P(1)) \neq 0$ for one P, then $c^+_{p,i_\infty}(M_P(1)) \neq 0$ for densely populated P satisfying $(\mathrm{A2}_-)$ for a fixed S (Proposition 4.1.1 in Chapter 4). Moreover we show that if the genuine p-adic L-function exists for one i_∞, it exists for all i_∞ with non-vanishing p-adic periods. Further, if p is odd and φ is self-dual up to characters, there is a canonical way of normalizing $L_{S,i_\infty}(\varphi)$ so that the normalized $L_S(\varphi)$ depends only on the isomorphism class of φ over $\mathbb{I}$ but not on i_∞ (Section 4.3 in Chapter 4). Anyway, the period $c^+_{p,i_\infty}(M_P(1))$ detects a "good" choice of

i_∞ making (Int) possible. Thus assuming the existence of i_∞ with non-vanishing $c^+_{p,i_\infty}(M_P(1))$, it would be reasonable to conjecture the existence of $L_{S,i_\infty}(\varphi)$ for such choice of i_∞ (Conjecture 4.2.1 in Chapter 4). Once we can specify the constant "$*$", the uniqueness of $L_{S,i_\infty}(\varphi)$ follows from the density requirement in $(\mathrm{A2}_\pm)$ of motivic points. A way of specifying "$*$" is known for crystalline M_P by Coates, Perrin-Riou and Panchishkin if φ contains all cyclotomic deformation (see [60], [58], [56] and Conjecture 4.2.1 in the text); in other words, $\varphi_P \otimes \mathcal{N}^m$ appears as φ_Q for some arithmetic point Q for all m with $\mathcal{N}^m \equiv 1 \bmod \mathbf{p}$ ($\mathbf{p} = 4$ or p according as $p = 2$ or not). Here we write $\mathcal{N}$ for the cyclotomic character of $\mathcal{G}_F$ with $\mathcal{N}(\phi_\mathfrak{l}) = \mathcal{N}(\mathfrak{l})$ for the geometric Frobenius element $\phi_\mathfrak{l}$.

We will show the existence of "good" i_∞ for every 2-dimensional absolutely irreducible arithmetic representation φ and for tensor products obtained from such 2-dimensional Galois representations (Corollary 3.4.2, Theorems 5.2.1 and 5.3.1 in Chapter 5). This process shows that the type S actually governs the form of complex and p-adic periods as a product of periods indexed by archimedean places of F as conjectured by Shimura [66], [67] in the complex case (see Chapter 5). In general, the existence of "good" i_∞ follows from the Tate-type conjecture asserting that the isomorphism class of a semi-simple regular motive is determined by its Galois representation (Theorem 3.4.1 in Chapter 3).

In this monograph, we have focused our study on the automorphic aspect of p-adic L-functions. However, they could have an arithmetic aspect, as typically represented by the Iwasawa theory. A precise (but conjectural) construction of analytic and arithmetic p-adic L-functions of the family of Tate twists of a given crystalline motive is studied in details by B. Perrin-Riou in [58]. Her formulation is different from ours using B_{cris} (in place of B_{HT} or Ω here). In a setting more close to ours, Greenberg [28] and [29] describes a conjectural theory of (p-ordinary) Selmer groups associated to arithmetic Galois deformations. In the automorphic theory, the genuine p-adic L-function is associated to the isomorphism class of φ over the field of fractions of $\mathbb{I}$. On the other hand, as Greenberg pointed me out, in the arithmetic theory, the Selmer group is more delicate and depends on the isomorphism class over $\mathbb{I}$. Although we expect the normalized L-function $L_S(\varphi)$ to play an important role in the Iwasawa theory for the contragredient $\check{\varphi}$ of φ, to formulate a main conjecture in our setting, some more work would be necessary to compare our p-adic L-functions and the arithmetic ones discussed in [58] and [28] (see Sections 4.3-4.7 in Chapter 4).

We define the arithmetic dual φ^* of φ by the contragredient of φ tensored by $\mathcal{N}$. If the (normalized) p-adic L-function $L_S(\varphi)$ exists, a principle of locating singularity (coming from the conjectures of Artin and Tate) would be

(AT_p) *For each prime divisor P of $Spec(\mathbb{I}\widehat{\otimes}\mathcal{O}_\Omega)$, if the normalized $L_S(\varphi)$ has a pole at P, then $\varphi_P \oplus \varphi^*_P$ contains the trivial representation as a subquotient.*

In some good cases including the case where the residual representation φ mod $\mathfrak{m}_{\mathbb{I}}$ is absolutely irreducible, we expect (AT_p) to hold, and we find some other (residually reducible) cases where (AT_p) might not hold, indicating necessity of further modification of $L_S(\varphi)$ (see Section 4.7 in Chapter 4 and Section 8.5 in Chapter 8). Thus the problem is quite subtle. In the first draft of this paper, the above principle was stated in a rough form which did not involve φ^*. The importance of counting also φ^* in addition to φ is suggested to the author by R. Greenberg. As already indicated above, the arithmetic side of the theory of p-adic L-functions of deformations of motives is discussed in [28] in terms of their Selmer groups. From his point of view ([28] Conjecture 4.1),

(G) *the order of the pole should be the exponent ℓ_P of P in the characteristic ideal of the $\mathbb{I}$-module $H_0(\mathcal{G}_F, \varphi \oplus \varphi^*)$.*

In [28], the conjecture is stated in terms of $H^0(\mathcal{G}_F, V(\varphi \oplus \varphi^*)^\vee \otimes_{\mathbb{I}} \mathbb{I}^*)$, where $\mathbb{I}^*$ is the Pontryagin dual module of $\mathbb{I}$. This cohomology group is the Pontryagin dual of $H_0(\mathcal{G}_F, \varphi \oplus \varphi^*)$. This is the reason why we have Galois homology group in place of cohomology group. Because of this, in principle, the genuine p-adic L-function should be closely related to the characteristic ideal of the Selmer group $\mathrm{Sel}(\check{\varphi})$ (defined in [28]) of the contragredient $\check{\varphi}$ of φ, although it might be a bit premature to make a main conjecture in this context. Hence (AT_p) is an assertion similar to (but weaker than) Conjecture 4.1 in [28], which is verified in various examples in [28]. In Chapters 7 and 8, we study the Katz p-adic L-functions interpolating Hecke L-values of CM fields, see that the L-function is close to be genuine and establish a stronger version of (AT_p). Actually Colmez made a precise conjecture in this case [13], and this type of fact has been shown by him [12] and Schneps [14] for abelian p-adic L-functions for either totally real F or a number field F containing an imaginary quadratic field. I should mention that our proof in the CM case is just a reinterpretation of the result of Colmez for totally real F.

When F is totally real, out of Hilbert modular forms on $GL(2)_{/F}$, we can construct examples of arithmetic Galois representations for $n = 2$. Let $\mathbf{h}$ be the universal ordinary or nearly ordinary p-adic Hecke algebra studied in [34], [35] and [33] Chapter 7 (the theory is recalled in Chapter 2). Then we have a canonical representation π of $\mathcal{G}_F$ taking values roughly in $GL_2(\mathbf{h})$ [38] (see Section 2.8 in Chapter 2). Thus for each irreducible component $Spec(\mathbb{I})$ of $Spec(\mathbf{h})$, writing the projection as $\lambda : \mathbf{h} \to \mathbb{I}$, we have an isomorphism class $\pi(\lambda) = \lambda \circ \pi$ over $\mathbb{K}$ of a Galois representation. We can describe the relation between $\pi(\lambda)$ and the Galois representations of classical Hilbert modular forms ([9], [69], [6]) in the following way. The Hecke algebra is naturally an algebra over the completed group algebra $\mathcal{O}[[T_2(\mathbb{Z}_p)]]$. By (A3) the algebra structure factors through a quotient torus T of T_2. We call T the weight group and describe it in details in Section 2.1 in

Chapter 2. We call an arithmetic point P positive if it induces locally on $T(\mathbb{Z}_p)$ a positive weight $\kappa(P)$ of T_2 with respect to the upper-triangular Borel subgroup of $GL(2)$. If P is positive arithmetic, the reduction $\pi(\lambda_P) = P \circ \pi(\lambda)$ modulo P is the Galois representation of a classical Hilbert modular form of weight $\kappa(P)$. As shown in [6], to such $\pi(\lambda_P)$, we have a rank two motive $M(\lambda_P)$ such that $L(s, M(\lambda_P)) = L(s, \pi(\lambda_P))$; thus, positive arithmetic points are motivic. Therefore, φ in the class $\pi(\lambda)$ (over $\mathbb{K}$) is arithmetic, and so are their tensor products. There are several choices of $V_{\mathfrak{P}}^+ = V_{\mathfrak{P}}^-$ in the case of $\pi(\lambda_1) \otimes \ldots \otimes \pi(\lambda_n)$ $(n > 1)$ for a given choice for each $\pi(\lambda_i)$. Each choice $S = \{V_{\mathfrak{P}}^+\}_{\mathfrak{P}|p}$ should yield a p-adic L-function satisfying (Int) for motivic points P compatible with the choice S. We make this precise in Conjectures 5.4.1 and 5.5.1 in Chapter 5. In the abelian (or rather CM) case, the choice of S gives rise to a p-adic CM type to which we attach a Katz p-adic L-function ([52] and [47]) and, even in the non-abelian case, is very much related to factorization of motivic periods into product of local periods indexed by archimedean places as discussed in [66], [67], [3], [5], [30], [31], [74], [75] and [42].

Various p-adic L-functions supported on the spectrum $Spec(\mathbf{h})$ or $Spec(\mathbf{h}\widehat{\otimes}_{\mathcal{O}}\mathbf{h})$ are studied in the works [47], [33], [37], [39], [40], [41], [53] and [32]. These p-adic L-functions include the p-adic Rankin product $D_p(\pi(\lambda) \otimes \pi(\mu)^\vee) \in \mathbb{K}\widehat{\otimes}_{\mathcal{O}}\mathbb{J}$ of $\pi(\lambda)$ and $\pi(\mu)$ for $\lambda : \mathbf{h} \to \mathbb{I}$ and $\mu : \mathbf{h} \to \mathbb{J}$, where $\pi(\mu)^\vee$ is the contragredient of $\pi(\mu)$. In Chapter 6, we study D_p through the conjectural theory described above. Although we can easily identify the class $S = \{V_{\mathfrak{P}}^+\}_{\mathfrak{P}|p}$ to which D_p belongs, there are several reasons to believe that D_p is not a single genuine p-adic L-function but a ratio of two genuine L-functions. To explain this, we pick two representations φ and ρ in the class $\pi(\lambda)$ and $\pi(\mu)$. The isomorphism class of φ over $\mathbb{I}$ may not be uniquely determined by $\pi(\lambda)$. It is known by Carayol [10] that $\pi(\lambda)$ determines uniquely φ if

(AI) $\overline{\varphi} = \varphi \mod \mathfrak{m}$ *for the maximal ideal* $\mathfrak{m}$ *of* $\mathbb{I}$ *is absolutely irreducible.*

The most significant reason among them is that $D_p(\pi(\lambda) \otimes \pi(\mu)^\vee)$ has poles at prime divisors in the support of the congruence module $C_0(\lambda; \mathbb{I})$ (see Section 2.9 in Chapter 2). Since $\varphi_P \otimes \rho_Q^\vee$ has the trivial representation as a subquotient if $\varphi_P \cong \rho_Q$, it looks all right to have a pole at the intersection of the two irreducible components λ and μ, since the support X of $C_0(\lambda; \mathbb{I})$ is the locus of P such that $\lambda_P = \mu'_P$ for irreducible components μ' other than λ (including μ). This is not correct because the two variable L-function: $(P, Q) \mapsto D_p(P, Q; \pi(\lambda) \otimes \pi(\mu)^\vee)$ has a pole on $X \times Spec(\mathbb{J})$ concentrated on the variable P, which is nothing to do with the locus W of (P, Q) satisfying $\varphi_P \cong \rho_Q$. Moreover it is easy to show that W is contained in the union of the diagonal Δ and the diagonal image W' of X in $Spec(\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathbb{J})$. The latter W' is of codimension at least 2 by definition, and the diagonal Δ is at least codimension 2 in the nearly ordinary case. Therefore if $D_p(\pi(\lambda) \otimes \pi(\mu)^\vee)$ were genuine, it would have to be holomorphic at W' and $X \times Spec(\mathbb{I})$. In [47], when λ and μ are both of

CM-type for the same CM field, we computed $D_p(\pi(\lambda)\otimes\pi(\mu)^\vee)$ in terms of p-adic Hecke L-functions of Katz and found that it is basically equal to the ratio

$$\text{(R)} \qquad D_p(P,Q;\pi(\lambda)\otimes\pi(\mu)^\vee) = \frac{L_{S,i_\infty}(P\otimes Q;\varphi\otimes\check{\rho})}{L_{p,i_\infty}(P,Ad(\varphi))},$$

where $L_{S,i_\infty}(\varphi\otimes\check{\rho})$ (resp. $L_{p,i_\infty}(Ad(\varphi))$) is the "genuine" p-adic L-functions for the Galois representations $\varphi\otimes\check{\rho}$ for a specific S (resp. $Ad(\varphi)$) of $\mathcal{G}_F$ into $GL_4(\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathbb{J})$ (resp. $GL_3(\mathbb{I})$) constructed out of p-adic Hecke L-functions of Katz [52] and [47]. Here $Ad(\varphi)$ is the 3-dimensional adjoint subrepresentation of $\varphi\otimes_{\mathbb{I}}\check{\varphi}$, that is, $\varphi\otimes_{\mathbb{I}}\check{\varphi} = Ad(\varphi)\oplus\mathbf{1}$ for the trivial representation $\mathbf{1}$. We wrote "genuine" for the above L-functions because it is not yet completely proven that the L-functions are genuine, strictly speaking (see Chapters 7 and 8). For the proof of genuineness, we need to suppose the compatibility of two comparison maps: one is that of Tate-Raynaud and the other is due to Faltings (see [50] 4.3.5). This fact is also pointed out to me by Blasius. On the other hand, if it is genuine, it is automatically normalized. Anyway this suggests us the same scenario in general. In fact, (i) the ratio (R) only depends on $\pi(\lambda)\otimes\pi(\mu)^\vee$ but not on i_∞ (see (NP) in Section 4.3 and Theorem 5.2.1 in Chapter 4), (ii) the formula (R) is a consequence of the existence of two genuine p-adic L-functions $L_{S,i_\infty}(\varphi\otimes\check{\rho})$ and $L_{p,i_\infty}(Ad(\varphi))$, and (iii) only assuming (AI), $p\geq 5$ and the existence of the normalized $L_p(Ad(\varphi))$ in $I\widehat{\otimes}_{\mathcal{O}}\Omega$, we can show that the normalized function $L_S(\varphi\otimes\check{\rho})$ defined by $L_p(Ad(\varphi))D_p(\pi(\lambda)\otimes\pi(\mu)^\vee)$ is everywhere holomorphic if one admits the following

Conjecture 1.0.1 — *Under (AI) and $p\geq 5$, we have*

$$C_0(\lambda;\mathbb{I})\otimes_{\mathcal{O}}\mathcal{O}_\Omega \cong \mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega/(L_p(Ad(\varphi)))$$

as $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$-modules.

This is a generalization of a conjecture made in [47]. We know that the conjecture holds for CM components when $F=\mathbb{Q}$ by the works [55], [71] and [47] combined, and we have everything we want when $F=\mathbb{Q}$ and λ is of CM-type (but μ can be non-abelian; see Chapter 7). Since the result of [41] supplies an evidence of the conjectures, I added at the end of this paper a list of corrections of misstatements in [41] to avoid confusion. Some other evidences supporting the conjecture are also discussed in [19]. We will use throughout the paper the notation introduced in the introduction. I would like to thank Don Blasius and R. Greenberg for their pointing me out many subtle points in the theory of motives and Selmer groups as indicated above.

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2. p-Adic Hecke algebras

We shall give a summary of the theory of p-adic Hecke algebras developed in [34], [35], [36], [41] and [47]. The formulation is a bit different from earlier treatment reflecting evolution. We assume F to be totally real in this chapter.

2.1 The weight group

First we like to specify the notion of weight for Hilbert modular forms for F. Let $T_1 = \mathrm{Res}_{\mathfrak{r}/\mathbb{Z}}\mathbf{G}_m$, where $\mathfrak{r}$ is the integer ring of F. Then T_1 is an algebraic torus over $\mathbb{Z}$ such that $T_1(A) = (\mathfrak{r}\otimes_{\mathbb{Z}} A)^\times$. We identify the character group $X(T_1)$ of T_1 with $\mathbb{Z}[I]$ so that $n = \sum_{\sigma\in I} n_\sigma\sigma$ sends $x \in T_1(\mathbb{Q}) = F^\times$ to $x^n = \prod_{\sigma\in I} x^{\sigma n_\sigma}$ in $\overline{\mathbb{Q}}^\times$. Weights are well described by identifying $T_1 \times_{\mathbb{Z}} T_1$ with the standard maximal F-split torus of the algebraic group $G = \mathrm{Res}_{\mathfrak{r}/\mathbb{Z}} GL(2)$, that is, the subgroup of diagonal matrices. Here we specify the identification of T_2 with $T_1 \times T_1$, reversing the order of diagonal entries, by $T_2 \ni \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \leftrightarrow (d, a) \in T_1 \times T_1$. The character group $X(T_1 \times T_1)$ is given by $\mathbb{Z}[I] \times \mathbb{Z}[I]$. We specify this isomorphism in a standard way so that $(x, y)^{(m,n)} = x^{-m}y^{-n}$. We take the connected component T_0 of the Zariski-closure of the diagonal image of $\mathfrak{r}^\times$ in $T_2 = T_1 \times T_1$, which is again a torus. For $T = (T_1 \times T_1)/T_0$, we have

$$X(T) = \left\{(m, n) \in X(T_1 \times T_1)\middle| m + n \in \mathbb{Z}t\right\}$$

for $t = \sum_\sigma \sigma$. We now enlarge a bit the group $T(\mathbb{Z}_p)$. Fix an open compact subgroup U of $G(\mathbb{A}^{(\infty)})$ with $U \supset GL_2(\mathfrak{r}_p)$, and write E_U for $UG_+(\mathbb{R}) \cap F^\times$ for the connected component of the identity $G_+(\mathbb{R})$ of $G(\mathbb{R})$, where $\mathbb{A}^{(\infty)}$ is the ring of finite adeles. Then we put

$$\mathbf{G} = \mathbf{G}_U = (T_1 \times T_1)(\mathbb{Z}_p)/\overline{E}_U,$$

where $\overline{E}_U$ is the closure of E_U under the p-adic topology. A typical example of U is given by

$$\widehat{\Gamma}_1(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) | c \equiv d - 1 \equiv 0 \mod N\widehat{\mathfrak{r}}\right\} \quad (\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell, \quad \widehat{\mathfrak{r}} = \prod_{\mathfrak{l}} \mathfrak{r}_{\mathfrak{l}})$$

for an ideal N prime to p. We use the p-adic character group $\mathscr{X} = \mathrm{Hom}_{conti}(\mathbf{G}, \Omega^\times)$ of $\mathbf{G}$ as the parameter space of p-adic nearly ordinary families of Hilbert modular forms. Although the full nearly ordinary family is parameterized by the space $\mathscr{X}$, the smaller ordinary family is parameterized by the closed subspace of $\mathscr{X}^{ord}$ in $\mathscr{X}$ which is made of characters factoring through

$$\mathbf{G}^{ord} = \mathbf{G}_U^{ord} = \mathbf{G}/(\{1\} \times T_1(\mathbb{Z}_p)) \cong T_1(\mathbb{Z}_p)/\overline{E}_U.$$

The subgroup $X(T_1 \times T_1)$ made of characters trivial on E_U is a subgroup of finite index in $X(T)$. We fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \Omega$. Then each $m \in X(T) = \mathrm{Hom}_{Alg\text{-}gr}(T_{/\overline{\mathbb{Q}}}, \mathbf{G}_{m/\overline{\mathbb{Q}}})$ induces $m \circ i_p$ in $\mathrm{Hom}_{gr}(T(\Omega), \Omega^\times)$. A continuous character κ in $\mathscr{X}$ is called an arithmetic weight (or character) if on a p-adic neighborhood of the identity, κ coincides with an algebraic character in $X(T)$ via i_p. This notion of being arithmetic is independent of the choice of i_p. We write $\mathscr{A}$ for the subgroup of arithmetic weights in $\mathscr{X}$. We write additively the group $\mathscr{A}$, and thus "0" is the identity character of $\mathscr{A}$. We put $\mathscr{A}^{ord} = \mathscr{A} \cap \mathscr{X}^{ord}$. We write $(m(\kappa), n(\kappa)) \in X(T)$ for the algebraic character induced by an arithmetic weight κ. An arithmetic weight κ is called positive if $m(\kappa) - n(\kappa) > 0$ (that is, $m(\kappa)_\sigma > n(\kappa)_\sigma$ for all σ). Then

$$\varepsilon_\kappa(a,d) = \kappa(a,d)(d,a)^{-(m(\kappa),n(\kappa))}$$

is a finite order character of $\mathbf{G}$, and by definition, $\kappa \in \mathscr{A}^{ord} \Leftrightarrow \kappa \in \mathscr{A}$, ε_κ is trivial on the variable a and $n(\kappa) = 0$.

In our previous works [34], [35], [38], [41] (resp. [42]), we have used weight written as (k,w) (resp. (n,v)) directly related to the automorphic factors (resp. the cohomological weight) of modular forms, while the weight (m,n) comes from the Hodge type of motives attached to modular forms. The following formulas clarify the relation among the three formulations:

$$(k,w) \leftrightarrow (m-n+t, t-n) \quad (t = \sum_{\sigma \in I} \sigma) \quad \text{and} \quad (n,v) \leftrightarrow (m-n-t, n).$$

A motivic and deformation theoretic description of the parameter spaces and conjectures predicting its general form can be found in Panchishkin [56] Appendix and in Tilouine [72].

2.2 Automorphic forms of weight κ

We choose an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then we may regard $m \in X(T) = \mathrm{Hom}_{Alg\text{-}gr}(T_{/\overline{\mathbb{Q}}}, \mathbf{G}_{m/\overline{\mathbb{Q}}})$ as an element $m\iota$ in $\mathrm{Hom}_{gr}(T(\mathbb{C}), \mathbb{C}^\times)$. Then we consider the following automorphic factor for each $\kappa \in \mathscr{A}$:

$$J_{\kappa\iota}(g,z) = det(g)^{n(\kappa)\iota - t\iota}(cz+d)^{m(\kappa)\iota - n(\kappa)\iota + t\iota} = det(g)^{-w\iota}(cz+d)^{k\iota},$$

where $z \in T_1(\mathbb{C}) = (F \otimes_{\mathbb{Q}} \mathbb{C})^\times$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})$. We assume that ε_κ factors through $(T_1 \times T_1)(\mathbb{Z}/p^\alpha\mathbb{Z})$ and put

$$\begin{aligned} U_0(p^\alpha) &= \left\{u \in U \middle| u_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ and } c \in p^\alpha \mathfrak{r}_p\right\} \text{ and} \\ U_1(p^\alpha) &= \left\{u \in U_0(p^\alpha) \middle| u_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ and } d \equiv 1 \mod p^\alpha \mathfrak{r}_p\right\}, \\ U_1^1(p^\alpha) &= \left\{u \in U_1(p^\alpha) \middle| u_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ and } a \equiv 1 \mod p^\alpha \mathfrak{r}_p\right\}, \end{aligned}$$

where $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $G_+(\mathbb{R})$ be the identity connected component of the Lie group $G(\mathbb{R})$. We write C_∞ for the standard maximal compact subgroup of $G(\mathbb{R})$, and we put $\mathfrak{Z} = G(\mathbb{R})/C_\infty Z(\mathbb{R}) = G_+(\mathbb{R})/C_{\infty+}Z(\mathbb{R})$ for the center Z of G and $C_{\infty+} = C_\infty \cap G_+(\mathbb{R})$. Then we may identify $\mathfrak{Z}$ with $\mathfrak{H}^I$ for the upper half complex plane $\mathfrak{H}$ so that $G_+(\mathbb{R}) \ni \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \mapsto x + \mathbf{i}y \in \mathfrak{H}^I$, where we write the $\mathbf{i}$ for $(\sqrt{-1}, \dots, \sqrt{-1})$ in $\mathfrak{Z}$. We write $\varepsilon_{\kappa,\iota}$ for $\iota \circ i_p^{-1} \circ \varepsilon_\kappa$. A modular form $f : G(\mathbb{A}) \to \mathbb{C}$ of weight κ with respect to an open subgroup V of $U_0(p^\alpha)$ satisfies the following automorphic condition:

(S1) $f(axu) = \varepsilon_{\kappa,\iota}(a,d) f(x) J_{\kappa\iota}(u_\infty, \mathbf{i})^{-1}$ for $u \in VC_{\infty+}Z(\mathbb{R})$ and $a \in G(\mathbb{Q})$,

where we have written $u_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By the above formula, for $z \in Z(\mathbb{R})(\cong T_1(\mathbb{R}))$, we see $f(zx) = z^{-m-n+t} f(x)$ ($m = m(\kappa)$ and $n = n(\kappa)$). Thus if f has the central character (that is, $f(xz) = \chi(z)f(x)$ for $z \in Z(\mathbb{A})$), its infinity type is given by $-w(\kappa) + t$ for $w(\kappa) = m(\kappa) + n(\kappa)$. Anyway, we define $f|\langle z\rangle(x) = |z|_{\mathbb{A}}^{-1} f(xz)$ for $z \in Z(\mathbb{A}^{(\infty)})$. In particular, for a prime ideal $\mathfrak{l}$ such that $GL_2(\mathfrak{r}_\mathfrak{l}) \subset V$, we write $\langle\mathfrak{l}\rangle$ for the operator $\langle\varpi\rangle$, where ϖ is a prime element of $\mathfrak{r}_\mathfrak{l}$. The operator $\langle\mathfrak{l}\rangle$ is well defined independent of the choice of ϖ. If $\mathfrak{a} = \prod_\mathfrak{l} \mathfrak{l}^{e(\mathfrak{l})}$ is a product of primes $\mathfrak{l}$, we define $\langle\mathfrak{a}\rangle = \prod_\mathfrak{l} \langle\mathfrak{l}\rangle^{e(\mathfrak{l})}$.

Now we interpret the condition (S1) in terms of the classical automorphy condition. Define a function f_x on $\mathfrak{Z}$ for each $x \in G(\mathbb{A}^{(\infty)})$ by $f_x(z) = f(xu_\infty)J_\kappa(u_\infty, \mathbf{i})$ for $u_\infty \in G_+(\mathbb{R})$ with $u_\infty(\mathbf{i}) = z$. This function is determined independently of the choice of u_∞ by (S1), and it is straightforward to check that

$$f_x(\gamma(z)) = f_x(z) J_\kappa(\gamma, z) \text{ for } \gamma \in \Gamma_x = xVG_+(\mathbb{R})x^{-1} \cap G(\mathbb{Q}).$$

Then we further impose

(S2) *$f_x(z)$ is a holomorphic function on $\mathfrak{Z}$ for all $x \in G(\mathbb{A}^{(\infty)})$ and is rapidly decreasing at all cusps of G.*

When we consider non-cuspidal modular forms, we instead impose

(M2) *$f_x(z)$ is a holomorphic function on $\mathfrak{Z}$ for all $x \in G(\mathbb{A}^{(\infty)})$ and is slowly increasing at all cusps of G.*

It is well known that the condition (M2) and (S2) are equivalent if $n \notin \mathbb{Z}t$ for $t = \sum_\sigma \sigma$.

2.3 Fourier-expansion and rationality

We choose a finite idele d such that $d\mathfrak{x}$ is the different of $F/\mathbb{Q}$. Write $F_{\mathbb{A}} = F \otimes_{\mathbb{Q}} \mathbb{A}$ and $F_{\mathbb{A}^{(\infty)}} = F \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$. Each modular form f satisfying (S1-2) has the following type of Fourier expansion on $G(\mathbb{A}) = GL_2(F_{\mathbb{A}})$ (see [34] Section 4 and [41] Section 2 for details):

$$f\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right) = |y|_{\mathbb{A}} \sum_{\xi \gg 0} a(\xi y d, f)(\xi y_\infty)^{-n} \mathbf{e}_F(\sqrt{-1}\xi y_\infty)\mathbf{e}_F(\xi x),$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \to \mathbb{C}^\times$ is the additive character such that

$$\mathbf{e}_F(x) = exp(2\pi\sqrt{-1}\sum_\sigma x_\sigma)$$

for $x = (x_\sigma)_{\sigma \in I} \in \mathbb{C}^I = F \otimes_{\mathbb{Q}} \mathbb{C}$, ξ runs over all totally positive elements in F, and $y \mapsto a(y, f)$ is a function on $F^\times_{\mathbb{A}^{(\infty)}}$ invariant under $\{q \in F^\times_{\mathbb{A}^{(\infty)}} | \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right) \in V\}$. When U contains $\widehat{\Gamma}_1(N)$ and $V \supset U_1^1(p^\alpha)$, the function $y \mapsto a(y, f)$ is supported on integral ideles. Here an integral idele means an element in $F^\times_{\mathbb{A}^{(\infty)}} \cap \widehat{\mathfrak{r}}$ for $\widehat{\mathfrak{r}} = \prod_{\mathfrak{l}} \mathfrak{r}_{\mathfrak{l}}$. When we consider non-cuspidal holomorphic modular form, we need to add the "constant term" to the above Fourier expansion. Namely modular form satisfying (M2) and (S1) for $n = [n]t$ with $[n] \in \mathbb{Z}$ has the following Fourier expansion:

$$f\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right) = |y|_{\mathbb{A}}\{a_0(yd, f)|y|_{\mathbb{A}}^{-[n]} + \sum_{\xi \gg 0} a(\xi y d, f)(\xi y_\infty)^{-n} \mathbf{e}_F(\sqrt{-1}\xi y_\infty)\mathbf{e}_F(\xi x)\},$$

where the constant term a_0 is a function on $F^\times_{\mathbb{A}^{(\infty)}}$ factoring through the strict ray class group $Cl_F(N)$ modulo N for sufficiently small integral ideal N.

For any $\mathbb{Q}$-subalgebra A of $\overline{\mathbb{Q}}$, we define $S_{\kappa\iota}(V; \iota(A))$ to be the space of functions satisfying (S1-2) and whose Fourier coefficients $a(y, f)$ are in $\iota(A)$ for all $y \in F^\times_{\mathbb{A}^{(\infty)}}$. If we change ι to $\iota\sigma$ for $\sigma \in \mathcal{G}_{\mathbb{Q}}$, then by a result of Shimura (see [65] Section 1 and [34] Theorem 4.4), the natural action induced by $a(y, \iota\sigma(f)) = \iota\sigma\iota^{-1}(a(y, \iota(f)))$ (for all y) takes $S_{\kappa\iota}(V; \iota(A))$ isomorphically to $S_{\kappa\sigma\iota}(V; \iota\sigma(A))$. Thus we have an intrinsically defined A-module $S_\kappa(V; A)$ (independent of the choice of ι) and an intrinsic function $a(y, f)$ having values in A such that ι induces an isomorphism $\iota : S_\kappa(V; A) \cong S_{\kappa\iota}(V; \iota(A))$ with $\iota(a(y, f)) = a(y, \iota(f))$. When U is given by $\widehat{\Gamma}_1(N)$ for an r-ideal N prime to p and $V = U_0(p^\alpha)$, we write $S_\kappa(Np^\alpha; A)$ for $S_\kappa(V; A)$. In a similar manner as above, we can define the space $M_\kappa(V; A)$ of holomorphic modular forms (given by (S1) and (M2)) requiring further $a_0(y, f)$ to have values in $\iota(A)$.

2.4 q-Expansion, integrality and Hecke operators

Let K be a p-adically complete subfield inside the p-adic completion Ω of $\overline{\mathbb{Q}}_p$ and $\mathbb{O}$ be the p-adic integer ring of K. Then we may consider $K_0 = i_p^{-1}(i_p(\overline{\mathbb{Q}}) \cap K)$

and $S_\kappa(V;K) = S_\kappa(V;K_0) \otimes_{K_0,i_p} K$. We may regard via i_p Fourier coefficients of a modular form in $S_\kappa(V;K_0)$ as a function on $F_{\mathbb{A}(\infty)}^\times$ into K. In particular, we can extend the function $f \mapsto a(y,f)$ to $S_\kappa(V;K)$ by linearlity. Then for $f \in S_\kappa(V;\Omega)$, we define the q-expansion coefficients by

$$\mathbf{a}_p(y,f) = y_p^{-n} a(y,f) \quad \text{and} \quad \mathbf{a}_{0,p}(y,f) = \mathcal{N}(yd^{-1})^{[n]} a_0(y,f),$$

where $\mathcal{N} : F_{\mathbb{A}(\infty)}^\times \to \mathbb{Z}_p^\times$ is the cyclotomic character such that $\mathcal{N}(y) = y_p^{-t}|y|_{\mathbb{A}}^{-1}$ for $y \in F_{\mathbb{A}(\infty)}^\times$ (cf. ([41] Sections 1-3)). Let F_+ be the additive monoid of totally positive elements of F. We now identify f with a function on $F_{\mathbb{A}(\infty)}^\times$ having values in the completed monoid algebra $\Omega[[q^\xi]]_{\xi\in F_+} \cong \Omega[[F_+]]$ by

$$f(y) = \mathcal{N}^{-1}(y) \sum_{\xi\in F_+} \mathbf{a}_p(\xi yd, f) q^\xi \quad \text{for cusp forms, and}$$

$$f(y) = \mathcal{N}^{-1}(y)\{\mathbf{a}_{0,p}(yd,f) + \sum_{\xi\in F_+} \mathbf{a}_p(\xi yd,f)q^\xi\} \quad \text{for non-cuspidal forms.}$$

We call this expansion the q-expansion of f. For any closed subring A of Ω, we write $S_\kappa(V;A)$ (resp. $M_\kappa(V;A)$) for the subspace made of cusp forms (resp. general modular forms) f with $\mathbf{a}_p(y,f) \in A$ (resp. $\mathbf{a}_p(y,f) \in A$ and $\mathbf{a}_{0,p}(y,f) \in A$) for all y. This definition is compatible with the one previously given when A is a $\mathbb{Q}$-subalgebra. For each prime ideal $\mathfrak{l}$ of $\mathfrak{r}$, we pick a prime element $\varpi_\mathfrak{l}$ in $F_\mathfrak{l}$. Then we can define the Hecke operator $\mathbb{T}(\varpi_\mathfrak{l})$ by

$$\text{(T1)} \qquad \mathbf{a}_p(y, f|\mathbb{T}(\varpi_\mathfrak{l})) = \begin{cases} \mathbf{a}_p(y\varpi_\mathfrak{l}, f) + \mathbf{a}_p(y\varpi_\mathfrak{l}^{-1}, f|\langle\mathfrak{l}\rangle) & \text{if } \mathfrak{l} \text{ is prime to } Np, \\ \mathbf{a}_p(y\varpi_\mathfrak{l}, f) & \text{if } \mathfrak{l}|Np. \end{cases}$$

Then this operator $\mathbb{T}(\varpi_\mathfrak{l})$ is well defined on $S_\kappa(V;A)$ as long as $U \supset \widehat{\Gamma}_1(N)$ and $U_0(p) \supset V \supset U_1^1(p^\alpha)$. The operator $\mathbb{T}(\varpi_\mathfrak{l})$ for $\mathfrak{l}$ outside Np only depends on the ideal $\mathfrak{l}$; so, we also write $\mathbb{T}(\mathfrak{l})$ for that. Here we have stated the definition of $\mathbb{T}(y)$ when $y = \varpi_\mathfrak{l}$ for a prime $\mathfrak{l}$. However, as is well known (e.g. [41] Section 3, where $\mathbb{T}(y)$ is written as $\mathbf{T}(y)$), we can define $\mathbb{T}(y)$ for all integral finite ideles y, and we have

$$\text{(T2)} \qquad \mathbf{a}_p(y,f) = \mathbf{a}_p(1, f|\mathbb{T}(y)).$$

The group $\mathbf{G}$ naturally acts on $S_\kappa(V;\mathbb{C})$ by $f(x) \mapsto f\left(x\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)\right)$. We define an action of $\mathbf{G}$ on $S_\kappa(V;\Omega)$ by

$$f|(a,d)(x) = \kappa(a,d) f\left(x\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)\right)$$

if $V \supset U_1^1(p^\alpha)$. This action can be written as $f|(a,d) = f|\mathbb{T}(a^{-1}d)\langle d^{-1}\rangle$ and hence preserves $S_\kappa(V;A)$ for any closed subalgebra A in Ω.

2.5 p-adic Hecke algebras

The Hecke algebra $h_\kappa(V;A)$ is the A-subalgebra of the A-linear endomorphism algebra of $S_\kappa(V;A)$ generated by $\mathbb{T}(y)$ for all y as above and the action of $\mathbf{G}_U$. The nearly ordinary part of these algebras is the maximal quotient (and also the maximal algebra direct summand) in which the image of $\mathbb{T}(\varpi_\mathfrak{p})$ is a unit for all $\mathfrak{p}$ dividing $p\mathfrak{r}$. We write $h_\kappa^{n.ord}$ for the nearly ordinary part, which is the maximal algebra direct factor of h_κ on which $\mathbb{T}(p)$ is a unit. We write e for the idempotent of the nearly ordinary part. By definition, we have an algebra structure of these Hecke algebras over the completed group algebra $\mathcal{O}[[\mathbf{G}]]$ induced by the action of $\mathbf{G}$. To make explicit the structure of $\mathcal{O}[[\mathbf{G}]]$ and $\mathcal{O}[[\mathbf{G}^{ord}]]$, we write $\mathbf{W}$ (resp. $\mathbf{W}^{ord}$) for the torsion-free part of $\mathbf{G}$ (resp. $\mathbf{G}^{ord}$). Thus

$$\mathbf{G} = \mathbf{W} \times \mu \quad \text{and} \quad \mathbf{G}^{ord} = \mathbf{W}^{ord} \times \mu^{ord}$$

for finite groups μ and μ^{ord}. Then we have that

$$\mathbf{W} \cong \mathbb{Z}_p^{1+[F:\mathbb{Q}]+d} \quad \text{and} \quad \mathbf{W}^{ord} \cong \mathbb{Z}_p^{1+d},$$

where d is the defect of the Leopoldt conjecture for F and p, that is, $d = \mathrm{rank}_{\mathbb{Z}} E_U - \mathrm{rank}_{\mathbb{Z}_p} \overline{E}_U$. The completed group algebra $\Lambda = \mathcal{O}[[\mathbf{W}]]$ (resp. $\Lambda^{ord} = \mathcal{O}[[\mathbf{W}^{ord}]]$) with respect to the adic topology under the augmentation ideal is non-canonically isomorphic to the power series ring of r variable with coefficients in $\mathcal{O}$ for $r = 1 + [F : \mathbb{Q}] + d$ (resp. $r = 1 + d$). We have $\mathcal{O}[[\mathbf{G}]] \cong \Lambda[\mu]$ and $\mathcal{O}[[\mathbf{G}^{ord}]] \cong \Lambda^{ord}[\mu^{ord}]$. If $V \subset V'$, there is a surjective $\mathcal{O}[[\mathbf{G}]]$-algebra homomorphism of the Hecke algebra for V' onto the Hecke algebra for V, which takes $\mathbb{T}(y)$ to $\mathbb{T}(y)$. We then put

$$h_\kappa = h_{\kappa,U}(\mathcal{O}) = \varprojlim_V h_\kappa(V;\mathcal{O}), \tag{H}$$

$$h_\kappa^{n.ord} = h_{\kappa,U}^{n.ord}(\mathcal{O}) = \varprojlim_V h_\kappa^{n.ord}(V;\mathcal{O}), \tag{N.Ord}$$

where the limit is taken with respect to open subgroups V of $U_0(p)$ satisfying $V \supset U_1^1(p^\infty) = \bigcap_\alpha U_1^1(p^\alpha)$. When $\kappa \in \mathcal{A}^{ord}$, we write h_κ^{ord} in place of $h_\kappa^{n.ord}$. Restricting the V's to those containing $U_1(p^\infty) = \bigcap_\alpha U_1(p^\alpha)$, we define for $\kappa \in \mathcal{A}^{ord}$ the ordinary p-adic Hecke algebra as

$$h_\kappa^{ord} = h_{\kappa,U}^{ord}(\mathcal{O}) = \varprojlim_V h_\kappa^{ord}(V;\mathcal{O}). \tag{Ord}$$

Then we have natural projections: $h_\kappa \to h_\kappa^{n.ord} \to h_\kappa^{ord}$ taking $\mathbb{T}(y)$ to $\mathbb{T}(y)$ and compatible with the projection: $\mathbf{G} \to \mathbf{G}^{ord}$. When $U = \widehat{\Gamma}_1(N)$, we use the symbol like $h_\kappa(Np^\infty;\mathcal{O})$ for these algebras. Then writing $\mathbf{h}_\kappa$ for any one of h_κ, $h_\kappa^{n.ord}$ and h_κ^{ord}, we have

$$\mathbf{h}_\kappa \cong \mathbf{h}_{\kappa'} \quad \text{as long as } \kappa > 0 \text{ and } \kappa' > 0. \tag{Univ}$$

These isomorphisms are $\mathcal{O}[[\mathbf{G}]]$-algebra isomorphisms and take $\mathbb{T}(y)$ to $\mathbb{T}(y)$ and $\langle \mathfrak{a} \rangle$ to $\langle \mathfrak{a} \rangle$. The assertions (Univ) is proven in [35] Theorem 2.3 and [34] Theorems 3.2 and 3.3. Because of (Univ), we omit the subscript "κ" from the notation. Now we state the control theorem:

$$\text{(Ctl)} \qquad \begin{aligned} h_U^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}]],\kappa} K &\cong h_\kappa^{n.ord}(U_0(p^\alpha); K) \quad (0 < \kappa \in \mathcal{A}), \\ h_U^{ord} \otimes_{\mathcal{O}[[\mathbf{G}^{ord}]],\kappa} K &\cong h_\kappa^{ord}(U_0(p^\alpha); K) \quad (0 < \kappa \in \mathcal{A}^{ord}). \end{aligned}$$

Here we regard K as an $\mathcal{O}[[\mathbf{G}]]$-algebra via the algebra homomorphism $\kappa : \mathcal{O}[[\mathbf{G}]] \to K$ induced by the arithmetic character κ, and we implicitly supposed that ε_κ factors through $T(\mathbb{Z}/p^\alpha\mathbb{Z})$. This has been proven in [35] Theorem 2.4 (resp. [34] Theorem 3.4) for $h^{n.ord}$ (resp. h^{ord}). The control theorem holds over $\mathcal{O}$ in place of K if $F = \mathbb{Q}$, and we conjecture that this stronger version should hold for general base field F. Under a certain circumstance, the stronger version of (Ctl) for general F follows from a recent result of Fujiwara [25].

2.6 Duality between Hecke algebras and p-adic modular forms

The projection maps $h_\kappa(W; A) \to h_\kappa(V; A)$ $(V \supset W)$ is induced by restricting the operator h in $h_\kappa(W; A)$ to the subspace $S_\kappa(V; A) \subset S_\kappa(W; A)$. Thus by definition, $\mathbf{h}$ naturally acts on $\bigcup_V S_\kappa(V; A)$. Here the union is taken, via q-expansion, inside the space of functions on $F_{\mathbb{A}(\infty)}^\times$ with values in the completed monoid ring $A[[q^\xi]] (\cong A[[F_+]])$. We even consider the sum $\sum_\kappa (\bigcup_V M_\kappa(V; A))$ over all positive arithmetic weights κ in the space of q-expansions. We then define a p-adic Banach norm on $\sum_\kappa (\bigcup_V M_\kappa(V; \Omega))$ by

$$\text{(Norm)} \qquad |f|_p = \mathrm{Sup}_y(|\mathbf{a}_{0,p}(y, f)|_p, |\mathbf{a}_p(y, f)|_p).$$

By (T1-2) in Section 2.4, each element h of the Hecke algebras gives a bounded operator under the above norm. Thus the action of Hecke algebras extends to the completions under the norm. Let $V_\infty = U_1^1(p^\infty)$ or $U_1(p^\infty)$, and for each closed $\mathcal{O}$-algebra A in Ω, we write $S_\kappa(V_\infty; A)$ for the completion of $\bigcup_{V \supset V_\infty} S_\kappa(V; A)$ under the above Banach norm. Similarly we write $M(V_\infty; A)$ (resp. $M_\kappa(V_\infty; A)$) for the completion of $\sum_\kappa (\bigcup_{V \supset V_\infty} M_\kappa(V; A))$ (resp. $\bigcup_{V \supset V_\infty} M_\kappa(V; A)$). By definition, the Hecke algebra of each space acts faithfully on the space. Write simply S for $S_\kappa(V; A)$. Here we allow V to be V_∞. Then we write $h(S)$ for the corresponding Hecke algebra embedded in $\mathrm{End}_A(S)$. Using this notation, we define a pairing

$$\langle\ ,\ \rangle : h(S) \times S \to A \quad \text{by} \quad \langle h, f \rangle = \mathbf{a}_p(1, f|h).$$

Then it is known (e.g. [41] Section 3) that

$$h(S) \cong \mathrm{Hom}_A(S, A) \quad \text{and} \quad S \cong \mathrm{Hom}_A(h(S), A).$$

Then the dual version of (Univ) is as follows: for $\kappa > 0$ and $\kappa' > 0$,

$$\text{(Univ*)} \quad S^{n.ord}_{\kappa,U} = S^{n.ord}_{\kappa',U}, \quad S^{ord}_{\kappa,U} = S^{ord}_{\kappa',U}, \quad S_\kappa(U_1^1(p^\infty);\mathbb{O}) = S_{\kappa'}(U_1^1(p^\infty);\mathbb{O}),$$

where $S^{n.ord}_{\kappa,U} = eS_\kappa(U_1^1(p^\infty);\mathbb{O})$ and $S^{ord}_{\kappa,U} = eS_\kappa(U_1(p^\infty);\mathbb{O})$, etc. The equality in (Univ*) means that the left-hand side and the right-hand side coincide in the space of functions on $F^\times_{\mathbb{A}^{(\infty)}}$ with values in $\mathbb{O}[[q^\xi]]$. For these spaces independent of weight, we omit the subscript "κ" from the notation. For each $\mathbf{G}$-module M, we write $M[\chi]$ for the subspace of M on which $\mathbf{G}$ acts via its character χ. Then the dual version of (Ctl) is given by

$$\text{(Ctl*)} \qquad \begin{aligned} S^{n.ord}_U[\kappa] &= S^{n.ord}_\kappa(U_0(p^\alpha);\mathbb{O}) \quad (0 < \kappa \in \mathscr{A}), \\ S^{ord}_U[\kappa] &= S^{ord}_\kappa(U_0(p^\alpha);\mathbb{O}) \quad (0 < \kappa \in \mathscr{A}^{ord}). \end{aligned}$$

Since (S2) is equivalent to (M2) when $n(\kappa) \notin \mathbb{Z}t$, $M_\kappa(V_\infty;\mathbb{O}) = S_\kappa(V_\infty;\mathbb{O})$ for such κ. Contrary to this, $M_\kappa(V_\infty;\mathbb{O})$ is really bigger than $S_\kappa(V_\infty;\mathbb{O})$ if $n(\kappa) \in \mathbb{Z}t$. We conjecture that $M_\kappa(V_\infty;\mathbb{O})$ is independent of κ as long as $n(\kappa) \in \mathbb{Z}t$ and coincides with $M(V_\infty;\mathbb{O})$. This fact probably follows from a detailed analysis of Eisenstein series combined with (Univ*). Anyway, we do not need the independence of weight for $M_\kappa(V_\infty;\mathbb{O})$. Since the Hecke operator $\mathbb{T}(p)$ continuously acts on $M(V_\infty;\mathbb{O})$, we can think of $M^{n.ord}(V_\infty;\mathbb{O})$ and $M^{ord}(V_\infty;\mathbb{O})$.

2.7 Modular parametrization by $\mathscr{X}$

For each linear map $\phi : h(Np^\infty;\mathbb{O}) \to \mathbb{O}$, we have by the duality $f_\phi \in S(Np^\infty;\mathbb{O})$ such that $\langle h, f_\phi\rangle = \phi(h)$. By (T2) in Section 2.4, we see $\mathbf{a}_p(y, f_\phi) = \phi(\mathbb{T}(y))$ for integral ideles y. Since $\mathbf{a}_p$ is supported on integral ideles (see Section 2.3), the value of $\mathbf{a}_p$ is completely determined by ϕ. It is easy to see that f_ϕ is a common eigenform of all Hecke operators $\mathbb{T}(y)$ if and only if ϕ is an $\mathbb{O}$-algebra homomorphism. Thus we get a parametrization of all $\mathbb{O}$-integral common eigenforms in $S(Np^\infty;\mathbb{O})$ by the $\mathbb{O}$-valued points of $Spec(h)$ for $h = h(Np^\infty;\mathbb{O})$. In particular, the closed subschemes $X = Spec(h^{n.ord})$ and $Y = Spec(h^{ord})$ parameterize eigenforms in $S^{n.ord}(Np^\infty;\mathbb{O})$ and $S^{ord}(Np^\infty;\mathbb{O})$, respectively. The structure of the schemes X and Y are fairly simple, since we know from [34] Theorem 3.3 and [35] Theorem 2.4 that

(F) *$h^{n.ord}(Np^\infty;\mathbb{O})$ (resp. $h^{ord}(Np^\infty;\mathbb{O})$) is finite and torsion-free over $\Lambda = \mathbb{O}[[\mathbf{W}]]$ (resp. $\Lambda^{ord} = \mathbb{O}[[\mathbf{W}^{ord}]]$).*

The space $\mathscr{X}(\overline{\mathbb{Q}}_p)$ of $\overline{\mathbb{Q}}_p$-valued points of $Spec(\Lambda)$ is isomorphic to the product of some copies of the open unit disk in $\overline{\mathbb{Q}}_p$, and the scheme X is a finite covering of $\mathscr{X}$.

There is another way to describe the parametrization of common eigenforms by each irreducible component of the Hecke algebra. To simplify the notation, we just

write $\mathbf{h}$ (resp. $\mathbf{\Lambda}$) for one of $h^{n.ord}(Np^\infty;\mathcal{O})$ and $h^{ord}(Np^\infty;\mathcal{O})$ (resp. Λ and Λ^{ord}). Let $\mathbb{L}$ be the quotient field of $\mathbf{\Lambda}$. We take a finite extension $\mathbb{K}$ of $\mathbb{L}$ and a $\mathbf{\Lambda}$-algebra homomorphism $\lambda : \mathbf{h} \to \mathbb{K}$. Since $\mathbf{h}$ is finite over $\mathbf{\Lambda}$, λ has values in the integral closure $\mathbb{I}$ in $\mathbb{K}$ of $\mathbf{\Lambda}$, which is finite over $\mathbf{\Lambda}$. Then enlarging $\mathcal{O}$ if necessary, we may assume that $\mathcal{O} = \overline{\mathbb{Q}}_p \cap \mathbb{I}$. We write $\mathscr{X}_\mathbb{I}$ for $Spec(\mathbb{I})_{/\mathcal{O}}$. Take $P \in \mathscr{X}_\mathbb{I}(\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}\text{-}alg}(\mathbb{I}, \mathcal{O})$, and compose it with λ getting $\lambda_P = P \circ \lambda$ which is a point of $X(\mathcal{O})$. Then we get a common eigenform $f(P) \in \mathbb{S}$ such that $\mathbf{a}_p(y, f(P)) = \lambda_P(\mathbb{T}(y))$ for integral y, where we have written $\mathbb{S}$ for one of $S^{n.ord}(Np^\infty;\mathcal{O})$ and $S^{ord}(Np^\infty;\mathcal{O})$ according to our choice of $\mathbf{h}$. We write for each closed $\mathcal{O}$-subalgebra A of Ω

$$\mathscr{A}_\mathbb{I}(A) = \left\{P : \mathbb{I} \to \mathcal{O} \in \mathscr{X}_\mathbb{I}(\mathcal{O}) \middle| P \text{ induces on } \mathbf{\Lambda} \text{ an arithmetic weight } \kappa.\right\}.$$

We write $\kappa(P)$ for the arithmetic weight induced by $P \in \mathscr{A}_\mathbb{I}(\mathcal{O})$. Then for positive $P \in \mathscr{A}_\mathbb{I}(\mathcal{O})$ (that is, $\kappa(P) > 0$), $f(P)$ is classical by (Ctl*) (see [34] Corollary 3.5 and [35] Corollary 2.5). Here we say a modular form $f \in \mathbb{S}[\kappa]$ "classical" if there exists a modular form $f^\circ \in S_\kappa(Np^\alpha;\overline{\mathbb{Q}})$ for some α such that $i_p^{-1}(\mathbf{a}_p(y,f)y_p^n) = a(y,f^\circ)$ for all integral y. Then we can think of a formal q-expansion $f \in \mathbb{I}[[q^\xi]]$ given by $\mathbf{a}_p(y,f) = \lambda(\mathbb{T}(y)) \in \mathbb{I}$ for integral y (otherwise $\mathbf{a}_p(y,f) = 0$). Thus we have

$$f(P)(y) = P(f(y)) = \sum_\xi P(\mathbf{a}_p(\xi y d, f))q^\xi.$$

We call any function $f : F^\times_{\mathbb{A}^{(\infty)}} \cap \widehat{\mathfrak{r}} \to \mathbb{I}[[q^\xi]]$ an $\mathbb{I}$-adic cusp form if it satisfies the following properties:

(i) *$f(P)$ is a p-adic modular form in $S(Np^\infty;\mathcal{O})[\kappa(P)]$;*

(ii) *For Zariski densely populated points of $\mathscr{A}_\mathbb{I}(\Omega)$ in $\mathscr{X}_\mathbb{I}$, $f(P)$ is a classical cusp form.*

We call an $\mathbb{I}$-adic form is nearly ordinary (resp. ordinary) if $\mathbf{h} = h^{n.ord}$ and $f(P) \in S^{n.ord}$ (resp. $\mathbf{h} = h^{ord}$ and $f(P) \in S^{ord}$) for all P. It is easy to check (e.g. [33] Section 7.3) that

$$\mathrm{Hom}_\Lambda(h^{n.ord}(Np^\infty;\mathcal{O}),\mathbb{I}) \cong \{\mathbb{I}\text{-adic nearly ordinary forms of level } Np^\infty\} \text{ and}$$
$$\mathrm{Hom}_{\Lambda^{ord}}(h^{ord}(Np^\infty;\mathcal{O}),\mathbb{I}) \cong \{\mathbb{I}\text{-adic ordinary forms of level } Np^\infty\},$$

where the isomorphisms are given by $\phi \mapsto (y \mapsto \sum_\xi \phi(\mathbb{T}(\xi y d))q^\xi)$.

2.8 Galois representations attached to λ

Fix an algebraic closure $\overline{F}$ of F. For a finite dimensional vector space V over $\mathbb{K}$, an $\mathbb{I}$-lattice L is an $\mathbb{I}$-submodule of finite type in V such that $L \otimes_\mathbb{I} \mathbb{K} = V$. For each

$\lambda : \mathbf{h} \to \mathbb{I}$, we have a Galois representation

$$\text{(Gal)} \qquad \pi(\lambda) : \mathcal{G}_F = \mathrm{Gal}(\overline{F}/F) \to GL(V(\lambda))$$

for a two dimensional vector space $V(\lambda)$ over $\mathbb{K}$ satisfying the following properties:

(i) *There is an $\mathbb{I}$-lattice L in $V(\lambda)$ stable under $\pi(\lambda)$, and $\pi(\lambda)$ is a continuous map into $Aut_{\mathbb{I}}(L)$ equipped with the adic topology of the maximal ideal of $\mathbb{I}$;*

(ii) *$\pi(\lambda)$ is irreducible and is unramified outside Np;*

(iii) *For each prime $\mathfrak{l}$ outside Np and for the geometric Frobenius $\phi_{\mathfrak{l}}$*

$$det(1 - \pi(\lambda)(\phi_{\mathfrak{l}})X) = 1 - \lambda(\mathbb{T}(\mathfrak{l}))X + \langle \mathfrak{l} \rangle X^2;$$

(iv) *For the place $\mathfrak{P}$ in F induced by $i_p\sigma$, writing $\mathcal{G}_{F_\mathfrak{P}} = \mathrm{Gal}(\overline{F}_\mathfrak{P}/F_\mathfrak{P})$ for the decomposition group at $\mathfrak{P}$, there is a one dimensional subspace $V_\mathfrak{P}^- = V_\mathfrak{P}^-(\lambda)$ of $V(\lambda)$ stable under $\pi(\mathcal{G}_{F_\mathfrak{P}})$ such that $\mathcal{G}_{F_\mathfrak{P}}$ acts on $V_\mathfrak{P}^-$ via the character $\delta : F_\mathfrak{P}^\times \subset \mathcal{G}_{F_\mathfrak{P}}^{ab} \to \mathbb{I}^\times$ given by $\delta(y) = \lambda(\mathbb{T}(y))$,*

where $\mathcal{G}_{F_\mathfrak{P}}^{ab}$ is the maximal abelian (continuous) quotient of $\mathcal{G}_{F_\mathfrak{P}}$ and $F_\mathfrak{P}^\times$ is viewed as a subgroup of $\mathcal{G}_{F_\mathfrak{P}}^{ab}$ via local class field theory. This fact is proven in [38] using Wiles' technique of pseudo representations. Actually we have taken the dual of the representation constructed in [38]. Thus in [38], the character δ as above appeared on a one-dimensional quotient of the representation space, but the character δ itself is unchanged because the identification $F_\mathfrak{P}^\times \subset \mathcal{G}_{F_\mathfrak{P}}^{ab}$ is given by assigning to each prime element the geometric Frobenius element instead of arithmetic Frobenius in [38]. In particular, the assertion (iv) tells us that $\pi(\lambda)(\mathcal{G}_{F_\mathfrak{P}})$ is contained in a split Borel subgroup of $GL_2(\mathbb{K})$, and λ factors through $h^{n.ord} \otimes_{\mathbb{O}[[\mathbf{G}]]} \mathbb{O}[[\mathbf{G}^{ord}]]$ if and only if δ is unramified. We have a natural projection $\rho : h^{n.ord} \otimes_{\mathbb{O}[[\mathbf{G}]]} \mathbb{O}[[\mathbf{G}^{ord}]] \to h^{ord}$ and can show that the localization of $Ker(\rho)$ at positive $P \in \mathcal{A}^{ord}$ vanishes. Thus $Ker(\rho)$ is a torsion Λ^{ord}-module. In particular, λ is ordinary (that is, factoring through h^{ord}) if and only if δ is unramified. When $F = \mathbb{Q}$, it is known [38] (iv) p.119 that ρ is an isomorphism.

For each $P \in \mathcal{X}_{\mathbb{I}}(K)$, a Galois representation $\pi(\lambda_P) : \mathcal{G}_F \to GL_2(K)$ is called a residual representation of $\pi(\lambda)$ modulo P if $\pi(\lambda_P)$ is semi-simple and $Tr(\pi(\lambda_P)(\sigma))$ is equal to $P(Tr(\pi(\lambda)(\sigma)))$ for all $\sigma \in \mathcal{G}_F$. If a residual representation exists, then it is unique up to K-isomorphisms. Again by Wiles' technique, one can show the existence of $\pi(\lambda_P)$ ([38] Theorem I). If once we know the existence of $\pi(\lambda_P)$, it is easy to show that the assertions (ii) (iii) and (iv) hold replacing $\mathbb{K}$ by K and λ by $\lambda_P = P \circ \lambda$, because the semi-simplification of the restriction of $\pi(\lambda_P)$ to an inertia group $I_\mathfrak{P}$ is the reduction modulo P of the semi-simplification of the restriction of $\pi(\lambda)$ to $I_\mathfrak{P}$. Thus we get a parametrization of 2-dimensional Galois representations

by $\mathscr{X}_{\mathbb{I}}(\Omega)$. In particular, if P is positive and arithmetic, $\pi(\lambda_P)$ is modular and hence is known to be motivic for almost all cases [6] (the exceptional cases only occur when $m(\kappa) = n(\kappa) + t$). Here the phrase "a modular Galois representation" means that it is associated to a classical common eigenform of all Hecke operators in the manner described in (iii) above. Modular Galois representations are constructed by Taylor [69] and by Blasius and Rogawski [6] independently. The weight κ of a modular Galois representation φ gives the Hodge type of the attached motive (cf. (A3')), that is, (m_σ, n_σ) and (n_σ, m_σ) for each $\sigma \in I$, and the motivic weight of φ is given by $\{\kappa\} = [m+n]$ if φ is modular of weight κ.

2.9 Congruence modules and differential modules

To describe congruence among cusp forms in terms of Hecke algebras and deformation rings of Galois representations, we here introduce a general notion of congruence modules and differential modules: Let R be an algebra over a normal integral domain A. We assume that R is an A-torsion-free module of finite type. Let $\phi : R \to A$ be an A-algebra homomorphism. We define $C_1(\phi; A) = \Omega_{R/A} \otimes_{R,\phi} \mathrm{Im}(\phi)$, which we call the differential module of ϕ. We suppose that the total quotient ring $Frac(R)$ can be decomposed uniquely into $Frac(R) = Frac(\mathrm{Im}(\phi)) \times \mathbb{X}$ as an algebra direct product. Let $\mathfrak{a} = Ker(R \to \mathbb{X})$. Then we put $C_0(\phi; A) = (R/\mathfrak{a}) \otimes_{R,\phi} \mathrm{Im}(\phi) \cong \mathrm{Im}(\phi)/(\mathrm{Im}(\phi) \cap R)$ ([39] Lemma 6.3), which is called the congruence module of ϕ but is actually a ring. Here the intersection $\mathrm{Im}(\phi) \cap R$ is taken in $Frac(R)$. Suppose now that A is a subring of a number field in $\overline{\mathbb{Q}}$. Since $Spec(C_0(\phi; A))$ is the scheme theoretic intersection of $Spec(\mathrm{Im}(\phi))$ and $Spec(R/\mathfrak{a})$ in $Spec(R)$, a prime $\mathfrak{p}$ is in the support of $C_0(\phi; A)$ if and only if there exists an A-algebra homomorphism $\phi' : R \to \overline{\mathbb{Q}}$ factoring through $R/\mathfrak{a}$ such that $\phi(a) \equiv \phi'(a) \bmod \mathfrak{p}$ for all $a \in R$. In other words, $\phi \bmod \mathfrak{p}$ factors through $R/\mathfrak{a}$ and can be lifted to ϕ'. We see easily that $Supp_A(C_0(\phi; A)) = Supp_A(C_1(\phi; A))$. By definition, we see that $C_0(\phi; A) \cong A/\eta(\phi)$ for an ideal $\eta(\phi)$ if $\mathrm{Im}(\phi) = A$. When $\mathrm{Hom}_A(R, A)$ is R-free of rank 1 (if R and A are both Gorenstein rings), $\eta(\phi)$ is principal. By abusing notation, we also write $\eta(\phi)$ for the generator of the principla ideal.

We continue to write $\mathbf{h}$ for one of $h^{n.ord}(Np^\infty; \mathbb{O})$ and $h^{ord}(Np^\infty; \mathbb{O})$. We first apply the above construction in the case where $A = \mathbb{I}$ and R is given by the Hecke algebra $\mathbf{h}$. Let $\lambda : \mathbf{h} \to \mathbb{I}$ be a $\mathbf{\Lambda}$-algebra homomorphism. We call λ *primitive* (of conductor N) if λ has minimal prime-to-p level N among $\lambda' : h^{n.ord}(Dp^\infty; \mathbb{O})$ or $h^{ord}(Dp^\infty; \mathbb{O}) \to \mathbb{I}$ with $\lambda'(\mathbb{T}(\mathfrak{q})) = \lambda(\mathbb{T}(\mathfrak{q}))$ for almost all prime ideals $\mathfrak{q}$. Since the theory of new forms is readily transmitted to $\mathbb{I}$-adic nearly ordinary and ordinary forms, if λ is primitive, the irreducible component of $Spec(\mathbf{h})$, through which λ factors, is reduced ([34] Theorem 3.6). Note that $\mathbf{h}$ is not only a $\mathbf{\Lambda}$-algebra but also

an $\mathcal{O}[[G']]$-algebra, where $G' = \mathbf{G}$ or $\mathbf{G}^{ord}$ according as $\mathbf{h} = h^{n.ord}$ or h^{ord}. Note that $G' = T_1(\mathbb{Z}_p) \times \mathbf{G}^{ord}$ if $\mathbf{h} = h^{n.ord}$. Moreover the action of $\mathbf{G}^{ord} \subset G'$ is just the action of the center of $G(\mathbb{Z}_p)$. Thus we can extend the action of $\mathbf{G}^{ord}$ to the action of the center $Z(\mathbb{A})$ of $G(\mathbb{A})$. It is known that the action of $Z(\mathbb{A}) \cong F_{\mathbb{A}}^{\times}$ factors through

$$\overline{Z} = \overline{Z}(N) = F_{\mathbb{A}}^{\times}/\overline{F^{\times}U_F(Np^{\infty})F_{\infty}^{\times}},$$

where $U_F(Np^{\alpha}) = U_1(Np^{\alpha}) \cap Z(\mathbb{A}^{(\infty)})$ and $U_F(Np^{\infty}) = \bigcap_{\alpha>0} U_F(Np^{\alpha})$. Then we can decompose $\overline{Z} = \mathbb{W}^{ord} \times \overline{Z}_{tor}$ for a torsion free subgroup $\mathbb{W}^{ord}(\supset \mathbf{W}^{ord}$ as an open subgroup) and a finite group $\overline{Z}_{tor}$. Thus we get a group $\mathbb{G}$ a bit bigger than G' such that $\mathbb{G} = T_1(\mathbb{Z}_p) \times \overline{Z} \supset G'$ if $\mathbf{h} = h^{n.ord}$ and $\mathbb{G} = \overline{Z}$ if $\mathbf{h} = h^{ord}$. Take the torsion-free part $\mathbb{W}$ of $\mathbb{G}$ and write $\mathbb{G} = \mathbb{W} \times \Delta$ and $\underline{\mathbf{\Lambda}}$ for $\mathcal{O}[[\mathbb{W}]]$. Then $\mathbf{h}$ is naturally an $\mathcal{O}[[\mathbb{G}]]$-algebra and hence is a $\underline{\mathbf{\Lambda}}$-algebra. Then λ induces a character $\psi : \Delta \to \mathcal{O}^{\times}$, and we can think of the maximal quotient $\mathbf{h}(\psi)$ of $\mathbf{h}$ on which Δ acts via ψ. By definition, λ factors through $\mathbf{h}(\psi)$ and gives a reduced irreducible component of $Spec(\mathbf{h}(\psi))$. Then taking R to be $\mathbf{h}(\psi) \otimes_{\underline{\mathbf{\Lambda}}} \mathbb{I}$, we have torsion $\mathbb{I}$-modules $C_j(\lambda; \mathbb{I})$ $(j = 0, 1)$. Note that replacing $\mathbb{I}$ by its finite extension $\mathbb{J}$ does not alter $C_0(\lambda; \mathbb{I})$ much: $C_0(\lambda; \mathbb{J}) = C_0(\lambda; \mathbb{I}) \otimes_{\mathbb{I}} \mathbb{J}$. In particular, just to study the support of $C_0(\lambda; \mathbb{I})$, we may take $\mathbb{I}$ as large as we want. Then by definition, for large enough $\mathbb{I}$, the support $Supp(C_0(\lambda; \mathbb{I}))$ in $\mathscr{X}_{\mathbb{I}}$ is the locus of P such that there exists $\mu : \mathbf{h} \to \mathbb{I} \in X(\mathbb{I})$ different from λ with $\lambda_P = \mu_P$. Form this, as already remarked, we have

$$Supp_{\mathbb{I}}(C_0(\lambda; \mathbb{I})) = Supp_{\mathbb{I}}(C_1(\lambda; \mathbb{I})).$$

Let P be an arithmetic point in $\mathscr{X}_{\mathbb{I}}(\mathcal{O})$. Then we have $\lambda_P : \mathbf{h}_P \to \mathcal{O}$, where $\mathbf{h}_P$ is the maximal $\mathcal{O}$-flat quotient of $(\mathbf{h}(\psi) \otimes_{\underline{\mathbf{\Lambda}}} \mathbb{I}) \otimes_{\mathbb{I},P} \mathcal{O}$. By the control theorem (Ctl), this algebra is isomorphic to the maximal $\mathcal{O}$-flat quotient of $h^{n.ord}_{\kappa(P)}(Np^{\alpha}; \mathcal{O})$ or $h^{ord}_{\kappa(P)}(Np^{\alpha}; \mathcal{O})$ on which Δ acts via ψ. Thus taking R to be $\mathbf{h}_P$ for $A = \mathcal{O}$, we can think of finite p-torsion modules: $C_j(\lambda_P; \mathcal{O})$. By construction, it is natural to expect that $\eta(\lambda)(P) = \eta(\lambda_P)$ up to $\mathcal{O}$-units for all arithmetic P. This is proven to be true if the local component of $\mathbf{h}$ through which λ factors is a Gorenstein ring [39].

2.10 Non-abelian class number formulas

In this section, we assume that N is prime to p. The module: $C_1(\lambda; \mathbb{I})$ introduced in the previous section is sometimes called the analytic Mazur module of λ and is studied in depth by Mazur-Tilouine [55], Taylor-Wiles ([73], [70]) and Fujiwara [25] in its relation to the Selmer group attached to the $\mathbb{I}$-adic representation $Ad(\pi(\lambda))$ (see also [49], [19] and [45]). Here we briefly describe the relation, assuming that $F = \mathbb{Q}$, $\mathbf{h} = h^{ord}$, $p \geq 5$ and that the conductor $C(\psi)$ of ψ is divisible by N.

Since classical notation is used in the papers quoted above; so, we first try to translate them into our language. We consider the classical space $S_k(\Gamma_0(Np), \psi\omega^{1-k})$ of holomorphic elliptic cusp forms of weight k and of "Neben"-type $\psi\omega^{1-k}$ for the Teichmüller character ω. This space is isomorphic to $S_\kappa(Np;\mathbb{C})[\psi]$ for $m(\kappa) = k-1$, $n(\kappa) = 0$ and $\varepsilon_\kappa(a,d) = \psi\omega^{1-k}(d)$, where "$[\psi]$" indicates the ψ-eigenspace under the action of Δ. Since $n(\kappa) = 0$, $\mathbb{T}((n))$ coincides with the classical Hecke operator $T(n)$ for the ideal (n) generated by a positive integer n. Let $\mathbb{Z}[\psi\omega^{-m(\kappa)}]$ be the subalgebra of $\overline{\mathbb{Q}}$ generated by $\psi\omega^{1-k}(n)$ for all $\psi\omega^{1-k}(n)$ regarding $\psi\omega^{-m(\kappa)}$ as a Dirichlet character. Then for each $\mathbb{Z}[\psi\omega^{-m(\kappa)}]$-subalgebra $A \subset \Omega$, we write $h_\kappa(Np, \psi; A)$ for the A-subalgebra of the linear endmorphism algebra $End_\Omega(S_\kappa(Np;\Omega)[\psi])$ generated by Hecke operators $T(n) = \mathbb{T}((n))$ for all n. Then it is known [44] that

$$h^{ord}(\psi) \otimes_{\Lambda^{ord},\kappa} \mathbb{C} \cong h_\kappa^{ord}(Np, \psi; \mathbb{C})$$

by an isomorphism taking $T(n) = \mathbb{T}((n))$ to $T(n) = \mathbb{T}((n))$. Thus for each positive arithmetic point $P \in Spec(\mathbb{I})(\mathbb{C})$ with $\kappa(P) = \kappa$, we have its specialization $\lambda_P : h_\kappa(Np, \psi; \mathbb{C}) \to \mathbb{C}$. Let f_P be a normalized Hecke eigenform in $S_k(\Gamma_0(Np), \psi\omega^{-m(\kappa)})$ associated to λ_P as described in Section 2.6. Thus $f_P = \sum_{n=1}^\infty a(n, f_P)q^n$ is a common eigenform of Hecke operators $T(n)$ for all n normalized so that $f_P|T(n) = a(n, f_P)f_P$ and $a(n, f_P) = \lambda_P(T(n))$. Since the conductor and the level match, the Hecke algebra $h_\kappa(Np, \psi; A)$ is reduced. We now write $\mathbb{Q}(\lambda_P)$ for the field generated by $\lambda_P(T(n))$ over $\mathbb{Q}$ and write A for the valuation ring $\mathbb{Q}(\lambda_P) \cap \mathbb{C}$ of $\mathbb{Q}(\lambda_P)$ with residual characteristic p. Thus we have well defined $C_j(\lambda_P; A) \cong C_j(\lambda_P; \mathbb{C})$ $(j = 0, 1)$ which is a finite module. As seen in Section 2.8, we have a Galois representation $\rho = \pi(\lambda_P) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C})$ unramified outside Np and $Tr(\rho(Frob_q)) = \lambda_P(T(q))$ for all primes q outside Np. Then we have the identity of the L-functions

$$L(s, \lambda_P) = \sum_{n=1}^\infty \lambda_P(T(n))n^{-s} = L(s, \pi(\lambda_P)).$$

Let π_P be the automorphic representation of $GL_2(\mathbb{A})$ spanned by f_P and its right translations. We write $L(s, Ad(\lambda_P) \otimes \chi)$ for the L-function of the adjoint lift $Ad(\pi_P)$ to $GL(3)$ [26] twisted by a Dirichlet character χ. Then this L-function has a meromorphic continuation to the whole complex s-plane and satisfies a functional equation of the form $1 \leftrightarrow 1 - s$ whose Γ-factor is given by

$$\Gamma(s, Ad(\lambda) \otimes \chi) = \Gamma_{\mathbb{C}}(s + \kappa - 1)\Gamma_{\mathbb{R}}(s + \frac{\chi(-1) + 1}{2}),$$

where $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$. Thus $\Gamma(s, Ad(\lambda) \otimes \chi)$ is finite at $s = 0, 1$ if $\chi(-1) = 1$, and hence $L(1, Ad(\lambda) \otimes \chi)$ is a critical value in the sense of

Deligne and Shimura for even characters χ. We shall relate the size of the module $C_1(\lambda;A)$ and the p-primary part of the critical value $\frac{\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda_P;A)\Omega(-,\lambda_P;A)}$.

To define the periods $\Omega(\pm,\lambda_P;A)$, we consider modular cohomology groups. We write simply ψ_P for $\psi\omega^{-m(\kappa(P))}$. Let $L(n,\psi_P;A)$ be the space of homogeneous polynomial in (X,Y) of degree $n=k-2=m(\kappa)-1$ with coefficients in A. We let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(Np)$ act on $P(X,Y)\in L(n,\psi;A)$ by

$$(\gamma P)(X,Y)=\psi_P(d)P((X,Y)^t\gamma^{-1}).$$

Then we can define the parabolic cohomology group $H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;A))$ as in [64] Chapter 8 and [33] Chapter 6. We have an isomorphism

$$\delta: S_k(\Gamma_0(Np),\psi_P)\oplus\overline{S}_k(\Gamma_0(Np),\psi_P)\cong H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;\mathbb{C})),$$

where $k=n+2$, $\overline{S}_k(\Gamma_0(Np),\psi_P)$ is the space of anti-holomorphic cusp forms of weight k of "Neben" type character ψ_P, and δ is specified in [33] and [64] as follows: We put

$$\omega(f)=\begin{cases}f(z)(X-zY)^n dz & \text{if } f\in S_{n+2}(\Gamma_0(Np),\psi_P),\\ f(z)(X-\overline{z}Y)^n d\overline{z} & \text{if } f\in\overline{S}_{n+2}(\Gamma_0(Np),\psi_P).\end{cases}$$

Then we associate to f_P the cohomology class of the 1-cocycle $\gamma\mapsto\int_z^{\gamma(z)}\omega(f_P)$ of $\Gamma_0(Np)$ for a fixed point z on the upper half complex plane. The map δ does not depend on the choice of z. There are two actions: one is a natural action of Hecke operators $T(n)$ on $H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;A))$, and the other is an action of complex conjugation c given by $c\omega(z)=\varepsilon\omega(-\overline{z})$ for $\varepsilon=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ and a differential form ω. In particular, δ and c commute with $T(n)$. We write $H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;A))[\pm]$ for the $\pm$-eigenspace of c. Then it is well known ([64] or [33]) that $H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;K))[\pm]$ is $h_\kappa(Np,\psi_P;K)$-free of rank 1 if K is a subfield of $\overline{\mathbb{Q}}$. Recall that A is a valuation ring of $\mathbb{Q}(\lambda_P)$. Thus

$$H^1_{cusp}(\Gamma_0(Np),L(n,\psi_P;A))[\lambda_P,\pm]=A\xi_\pm$$

for a generator $\xi_\pm$, where we indicate the λ_P-eigenspace by adding $[\lambda_P,\pm]$ to the notation. Then for the normalized eigenform $f_P\in S_k(\Gamma_0(Np),\psi_P)$ with $f_P|T(n)=\lambda_P(T(n))f$, we define $\Omega(\pm,\lambda_P;A)\in\mathbb{C}^\times$ by

$$\delta(f_P)\pm c(\delta(f_P))=\Omega(\pm,\lambda_P;A)\xi_\pm.$$

The story on the non-abelian class number formula is a bit long; so, let us give an outline first. Via the deformation theory of Galois representations (due to Mazur, Wiles and Taylor: [54], [73] and [70]), we can identify the local ring h of $h^{ord}(\psi)$ (through which λ factors) and $h_P=h\otimes_{\mathbb{I},P}\mathbb{C}$ with universal deformation rings classifying Galois representations (with appropriate properties) deforming $\overline{\rho}$ in good circumstances (see below for details). In this process, we know that

1. h and h_P are local complete intersections, yielding the identities:

$$char_{\mathbb{I}} C_0(\lambda;\mathbb{I}) = char_{\mathbb{I}} C_1(\lambda;\mathbb{I}) \quad \text{and} \quad \#C_0(\lambda_P;\mathbb{O}) = \#C_1(\lambda_P;\mathbb{O}),$$

2. $1_P H^1_{cusp}(\Gamma_0(Np), L(n,\psi_P;\mathbb{O}))[\pm] \cong h_P$ as h_P-modules,

where $char_{\mathbb{I}}(M)$ is the characteristic ideal of a torsion $\mathbb{I}$-module M of finite type, and 1_P is the idempotent of h_P in the Hecke algebra. By (2), we can compute the number $\#C_0(\lambda; A)$ cohomologically, and the outcome is that it is given by the inverse of the p-adic absolute value of the L-value $\frac{\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda_P;A)\Omega(-,\lambda_P;A)}$ [43]. By a work of Fujiwara [25], the part (1) is generalized to Hilbert modular forms under certain assumptions, although we restrict ourselves to the case where $F = \mathbb{Q}$. Moreover, again by the deformation theory [55], $C_1(\lambda_P;\mathbb{O})$ is the Pontryagin dual of the (p-adic) Selmer group $\mathrm{Sel}(Ad(\pi(\lambda_P)))$. In this way, we obtain the order formula of $\mathrm{Sel}(Ad(\pi(\lambda_P)))$ in terms of $L(1, Ad(\lambda_P))$. This is compatible with the conjecture of Bloch-Kato as explained in [73] Chapter 1 Section 2. A similar formula for $L(1, Ad(\lambda_P) \otimes \chi)$ is conjectured for a general Dirichlet character χ in [19] and [46], and some evidences for the conjecture are discussed there. As already remarked, we have a generator $\eta(\lambda)$ of the annihilator of $C_0(\lambda;\mathbb{I})$. By the above order formula, $\eta(\lambda)(P) = \eta(\lambda_P)$ is equal to $\frac{\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda_P;A)\Omega(-,\lambda_P;A)}$ up to $\mathbb{O}$-units for all arithmetic P. Thus $\eta(\lambda)$ gives a p-adic interpolation of adjoint L-values up to units in $\mathbb{I}$. Our desire in this paper is to eliminate this ambiguity of units in this p-adic interpolation.

To give some details to the above description, let us recall the deformation theory of Galois representations introduced by Mazur [54]. Let $\overline{\rho} = \pi(\lambda_{\mathfrak{m}_{\mathbb{I}}})$ be the Galois representation $\pi(\lambda)$ mod $\mathfrak{m}_{\mathbb{I}}$ for the maximal ideal $\mathfrak{m}_{\mathbb{I}}$ of $\mathbb{I}$. Thus $\overline{\rho}$ has values in $GL_2(\mathbb{F})$ for the residue field $\mathbb{F}$ of $\mathbb{O}$. The isomorphism class of the semi-simplification of $\overline{\rho}$ is uniquely determined by ρ. We consider the following condition for a number field $E \subset \overline{\mathbb{Q}}$:

(ai_E) *The restriction $\overline{\rho}_E$ of $\overline{\rho}$ to $\mathcal{G}_E$ is absolutely irreducible.*

Let $\mathcal{G}^{(Np)} = \mathrm{Gal}(\mathbb{Q}^{(Np)}/\mathbb{Q})$ for the maximal extension $\mathbb{Q}^{(Np)}/\mathbb{Q}$ unramified outside $\{\ell | Np, \infty\}$. Then $\pi(\lambda)$ factors through $\mathcal{G}^{(Np)}$. For any representation ρ of $\mathcal{G}$, we write ρ_E for the restriction of ρ to $\mathcal{G}_E$. We consider deformations of $\overline{\rho}$ over the category $CNL_{\mathbb{O}}$ of complete local noetherian $\mathbb{O}$-algebras with residue field $\mathbb{F}$. For each object B of $CNL_{\mathbb{O}}$, we write $\mathfrak{m}_B$ for the maximal ideal of B. A Galois representation $\rho : \mathcal{G}^{(Np)} \to GL_2(B)$ for $B \in CNL_{\mathbb{O}}$ is a deformation of $\overline{\rho}_E$ if ρ mod $\mathfrak{m}_B$ is identical to $\overline{\rho}_E$ as matrix representations. We look into the deformation functor $\mathcal{F} : CNL_{\mathbb{O}} \to SETS$ given by

$$\mathcal{F}(B) = \{\rho : \mathcal{G}^{(Np)} \to GL_2(B) | \rho \mod \mathfrak{m}_B = \overline{\rho}\}/\approx,$$

where "$\approx$" is the conjugation by elements in $1+M_2(\mathfrak{m}_B)$. We impose more conditions on deformations. A deformation ρ is called p-ordinary over a number field E if

$\rho|_{\mathfrak{D}_\mathfrak{p}} \cong \begin{pmatrix} \delta_\mathfrak{p} & * \\ 0 & \varepsilon_\mathfrak{p} \end{pmatrix}$ with an unramified character $\delta_\mathfrak{p}$ on every decomposition subgroup $\mathfrak{D}_\mathfrak{p}$ of $\mathcal{G}_E$ at $\mathfrak{p}|p$. We like to impose some of the following two conditions depending on the situation:

(reg) *ρ is p-ordinary over $\mathbb{Q}$ with two distinct characters δ_p and ε_p;*

By class field theory, we regard ψ as a character of $\mathcal{G}^{(Np)}$ with values in $\mathbb{C}^\times$. For primes $\mathfrak{l}|N$ outside p, we impose the following condition:

$$(\psi) \qquad \rho \cong \begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix} \text{ on the inertia subgroup } I_\ell \subset \mathcal{G}^{(Np)} \text{ for each prime } \ell|N.$$

Let $\mathcal{N}$ be the p-adic cyclotomic character. Then $det(\pi(\lambda_P)) = \mathcal{N}^{m(P)}\psi$. Hereafter we suppose that p is prime to the order of ψ. We consider the following subfunctors of $\mathcal{F}$:

$$\mathcal{F}_P(B) = \{\rho \in \mathcal{F}(B) \mid \rho \text{ satisfies } (reg), (\psi) \text{ and } det(\rho) = \mathcal{N}^{m(P)}\psi\},$$
$$\mathcal{F}^{ord}(B) = \{\rho \in \mathcal{F}(B) \mid \rho \text{ satisfies } (reg) \text{ and } (\psi)\}.$$

Under the condition that p is prime to the order of ψ, $(\mathrm{ai}_\mathbb{Q})$ and (reg) for $\overline{\rho}$, the functor $\mathcal{F}_P$ (resp. $\mathcal{F}^{ord}$) is representable by a universal couple (R_P, ϱ_P) (resp. (R^{ord}, ϱ^{ord})) (see [54], [72] and [45] Appendix).

As we have seen, under $(\mathrm{ai}_\mathbb{Q})$, there is a unique Galois representation $\rho_P : \mathcal{G}^{(Np)} \to GL_2(h_P) \in \mathcal{F}_P(h_P)$ (resp, $\rho^{ord} : \mathcal{G}^{(Np)} \to GL_2(h) \in \mathcal{F}^{ord}(h)$) up to isomorphisms, such that $Tr(\rho_P(Frob_\ell)) = T_P(\ell)$ (resp. $Tr(\rho^{ord}(Frob_\ell)) = T_h(\ell)$) for all primes ℓ outside Np (for uniqueness, see [10]), where $Frob_\ell$ is the Frobenius element in $\mathcal{G}$ at the prime ℓ, and $T_P(\ell)$ (resp. $T_h(\ell)$) is the projection of $T(\ell)$ to h_P (resp. h). It has been proven by Taylor and Wiles [73] that, under $(\mathrm{ai}_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})})$ and (reg) for $\overline{\rho}$, $\mathcal{F}_P$ (resp. $\mathcal{F}^{ord}$) is representable by the pair (h_P, ρ_P) (resp. (h, ρ^{ord})). Thus in this case, the natural morphism $\iota_P : R_P \to h_P$ (resp. $\iota : R^{ord} \to h$) with $\iota_P\varrho_P \approx \rho_P$ (resp. $\iota\varrho^{ord} \approx \rho^{ord}$) is a surjective isomorphism. When E is totally real, this result is generalized by Fujiwara [25] under certain assumptions.

In the course of the proof of the above result: $R_P \cong h_P$, it is shown that R_P is a local complete intersection. This fact is basically equivalent to

$$(C1) \qquad |C_1(\lambda_P; \mathbb{C})| = |C_0(\lambda_P; \mathbb{C})|.$$

By the control theorem (Ctl), this implies

$$(C2) \qquad char_\mathbb{I}(C_1(\lambda; \mathbb{I})) = (\eta(\lambda)),$$

where for each torsion $\mathbb{I}$-module M, its characteristic ideal $char_\mathbb{I}(M)$ is given by $\prod_P P^{\ell(M_P)}$ for the length $\ell(M_P)$ of the localization M_P of M at prime divisors P

in $Spec(\mathbb{I})$. The above fact implies that

$$\text{(mlt)} \qquad 1_P H^1_{cusp}(\Gamma_0(Np), L(n,\psi_P;\mathcal{O}))[\pm] \cong h_P \cong R_P \quad \text{as } h_P\text{-modules.}$$

By virtue of (mlt), we can compute $C_0(\lambda_P;\mathcal{O})$ using cohomology groups. To explain this, we write $L(B) = H^1_{cusp}(\Gamma_0(Np), L(n,\psi_P;B))[\varepsilon]$ for $\varepsilon = \pm$. Then $L(\mathcal{O}) = L(A)\otimes_A \mathcal{O}$. Decomposing $Frac(h_\kappa(Np,\psi;A)) = Frac(Im(\lambda_P)) \times X$, we define $L^P(A)$ be the image of $L(A)$ in $L(A)\otimes_{h_\kappa(Np,\psi;A)} Frac(Im(\lambda_P))$ and a cohomological congruence module by

$$C_0^H(\lambda_P;A) = L^P(A)/(L(A)\cap L^P(A)).$$

Then (mlt) shows that

$$C_0(\lambda_P;\mathcal{O}) \cong C_0^H(\lambda_P;A).$$

It is shown in [43] and [39] that the p-adic absolute value of $\frac{\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda_P;A)\Omega(-,\lambda_P;A)}$ is the inverse of the order of the right-hand-side module of the above equation. That is, under $(\mathrm{ai}_{\mathbb{Q}})$ and (reg) for $\overline{\rho}$, if p is outside $6N$,

$$\text{(C3)} \qquad \left|\frac{c(P)\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}\right|_p^{-r} = |C_0^H(\lambda_P;A)| = |C_1(\lambda_P;A)|,$$

$$\eta(\lambda)(P) = \frac{c(P)\Gamma(1,Ad(\lambda_P))L(1,Ad(\lambda_P))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)},$$

where $c(P) = 1$ if $\kappa(P) > t \Leftrightarrow k > 2$, $c(P) = 1 - \lambda_P(T(p))^{-2}$ if $\kappa(P) = t \Leftrightarrow k = 2$, $r = r(\mathcal{O}) = \mathrm{rank}_{\mathbb{Z}_p}\mathcal{O}$, and $|\ |_p$ is the p-adic absolute value of A normalized so that $|p|_p = p^{-1}$. Even if $p|N$, there is a similar formula (see [39] for details).

We now recall the definition of the Selmer group of $Ad(\rho : \mathcal{G}^{(Np)} \to GL_2(B))$ over $\mathbb{Q}$ related to our modules C_1. Let V be the B-free module of rank 2 on which $\mathcal{G}^{(Np)}$ acts via ρ. For each decomposition subgroup $\mathcal{D}_p \subset \mathcal{G}^{(Np)}$ at p, we write V_p for the δ_p eigenspace in V. For primes $\ell|N$ prime to p, we write V_ℓ for the subspace fixed by the inertia at ℓ. We identify $Ad(\rho)$ with trace 0 subspace W of $\mathrm{End}_B(V)$. We put for $\ell|Np$

$$W_\ell = \{\phi \in W \,|\, \phi(V_\ell) = 0\}.$$

Define, for $\ell|Np$, writing $\mathcal{I}_\ell$ for the inertia subgroup at ℓ in $\mathcal{G}^{(Np)}$,

$$L_\ell = Ker(H^1(\mathcal{D}_\ell, W^*) \to H^1(\mathcal{I}_\ell, (W/W_\ell)^*)),$$

where $X^* = X\otimes_B B^*$ for the Pontryagin dual $B^* = \mathrm{Hom}(B, \mathbb{Q}_p/\mathbb{Z}_p)$ of B. Then we put

$$\mathrm{Sel}(Ad(\rho)) = \bigcap_{\ell|Np} Ker(H^1(\mathcal{G}^{(Np)}, W^*) \to H^1(\mathcal{D}_\ell, W^*)/L_\ell).$$

It is a general fact [55] (see also [45] 3.2) that

$$\text{(C4)} \qquad \mathrm{Sel}(Ad(\rho)) \cong \begin{cases} C_1(\lambda_P \circ \iota_P : R_P \to \mathbb{O}; \mathbb{O})^* & \text{if } \rho = \pi(\lambda_P), \\ C_1(\lambda \circ \iota : R^{ord} \to \mathbb{I}; \mathbb{I})^* & \text{if } \rho = \pi(\lambda). \end{cases}$$

Thus combining what all we said, we get the following order formula of the Selmer group under $(\mathrm{ai}_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})})$ and $(p) + (6Ng) = \mathbb{Z}$ for the order g of ψ,

$$\text{(CN1)} \qquad \frac{\Gamma(1, Ad(\lambda_P))L(1, Ad(\lambda_P))}{\Omega(+, \lambda_P; A)\Omega(-, \lambda; A)} = \eta(\lambda_P) \text{ up to } A\text{-units, and}$$

$$\text{(CN2)} \qquad char_{\mathbb{I}}(\mathrm{Sel}(Ad(\pi(\lambda)))) = \eta(\lambda).$$

This is a non-abelian generalization of a classical analytic class number formula (see [73] Chapter 4 and [49]). The definition of the Selmer group can be interpreted by Fontaine's theory, and the above formula can be viewed as an example of the Tamagawa number formula of Bloch and Kato for the motive $M(Ad(\lambda_P))$ yielding $Ad(\pi(\lambda_P))$ (see [73] p.466 and [7] Section 5).

3. Periods of motives

We study p-adic periods of motives under the reducibility conditions ($\mathrm{Red}_{\pm}$) of its local Galois representation at p-adic places. After an example of vanishing of p-adic periods in Section 3.3, we will prove a general non-vanishing result in Section 3.4. At the end of this chapter, we shall show that our reducibility condition (Red_{+}) is equivalent to Panchishkin's admissibility condition when the motive is crystalline (Section 3.5).

3.1 Periods of motives

Let M be a pure motive of rank n defined over F with coefficients in E. Morphisms in the category of motives we treat are induced from absolute Hodge cycles. We always assume that E is a subfield of $\overline{\mathbb{Q}}$ and that $\mathcal{G}_F$ acts on its etale p-adic realization $H_{\mathfrak{p}}(M)$ through a compatible system φ of Galois representations. We choose an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then E can be considered to be a subfield of $\mathbb{C}$ via ι. Then the L-function of M or φ is defined by

$$\text{(L)} \qquad L_\iota(s,\varphi) = L(s,M) = \prod_{\mathfrak{l}} det\left(1 - \varphi(\phi_{\mathfrak{l}})\big|_{H_{\mathfrak{p}}(M)^{I(\mathfrak{l})}} \mathcal{N}(\phi_{\mathfrak{l}})^{-s}\right)^{-1},$$

where $\mathfrak{l}$ runs over all prime ideals of F (and $\mathfrak{p}$ is chosen so that $\mathfrak{l}$ is outside $N(\mathfrak{p})$), $I(\mathfrak{l})$ denotes an inertia group at each prime $\mathfrak{l}$ and $\phi_{\mathfrak{l}}$ denotes the geometric Frobenius class modulo $I(\mathfrak{l})$. This L-function is absolutely convergent for s with $Re(s)$ sufficiently large and is conjectured to be continued analytically to the whole s-plane with a functional equation:

$$\Gamma(s,\varphi)L(s,\varphi) = \varepsilon(s,\varphi)\Gamma(1-s,\check{\varphi})L(1-s,\check{\varphi})$$

for a suitable exponential function ε and a gamma factor $\Gamma(s,\varphi)$. When $\Gamma(0,\varphi)$ and $\Gamma(1,\check{\varphi})$ are both finite, we call φ and M critical. If M is critical, there is a conjecture of Deligne [15] describing the transcendental factor $c_\iota^+(\varphi)$ of $L_\iota(0,\varphi)$ as follows: Write the Hodge realization as $H_B({}^\sigma M) \otimes_E \mathbb{C} = \oplus_{i,j} H^{i,j}({}^\sigma M)$, where ${}^\sigma M$

is the base change of M to $\mathbb{C}$ under $\iota\sigma$ for $\sigma : \overline{F} \cong \overline{\mathbb{Q}}$ and $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then the criticality of M implies

(Cr) *At each real place σ, complex conjugation c acts on $H^{k,j}({}^\sigma M)$ via -1 (resp. $+1$) if $k = j \geq 0$ (resp. $k = j < 0$), and $H^{k,k}({}^\sigma M) = 0$ for all complex place σ.*

We now take a Tannakian full subcategory $\mathscr{C}_X$ of the category of pure absolute Hodge motives defined over F with coefficients in E. We always assume E to be a subfield of $\overline{\mathbb{Q}}$. Suppose that there exist a $\overline{\mathbb{Q}}$-algebra X with a natural transformation I_X inducing an equivalence of two fiber functors on $\mathscr{C}_X$ over X: for each $\sigma \in I$,

$$\{M \mapsto H_B({}^\sigma M) \otimes_{E,i} X_E = H_B({}^\sigma M) \otimes_{\mathbb{Q}} X\} \cong \{M \mapsto H_{DR}(M) \otimes_{F\otimes E,\sigma\otimes i} X_E = H_{DR}(M) \otimes_{F,\sigma} X\},$$

where $i : E \hookrightarrow X_E = X \otimes_{\mathbb{Q}} E$ is the natural embedding. This transformation induces the comparison isomorphism:

$$(\mathrm{C}_X) \qquad \begin{aligned} I_{X,\sigma} : H_B({}^\sigma M) \otimes_{\mathbb{Q}} X &= H_B({}^\sigma M) \otimes_{E,i} X_E \\ \cong H_{DR}(M) \otimes_{F\otimes E,\sigma\otimes i} X_E &= H_{DR}(M) \otimes_{F,\sigma} X. \end{aligned}$$

Here $I_{X,\sigma}$ may depends on σ itself not just its restriction to F. Writing $\iota_E : E \hookrightarrow \overline{\mathbb{Q}}$ for the inclusion of E, we write $\delta_{X,\sigma}(M)$ for the ι_E-component of $det(I_{X,\sigma})$ with respect to the basis of $H_{DR}(M)$ over $E \otimes F$ and $H_B({}^\sigma M)$ over E. Here we identify $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ with a product of copies of $\overline{\mathbb{Q}}$ and, the word "ι_E-component" implies the component of $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ inducing ι_E on E. We write

$$\delta_X(M) = \prod_{\sigma\in I} \delta_{X,\sigma}(M),$$

which is an element in $X^\times$ and coincides with $\delta_{X,\sigma}(Res_{F/\mathbb{Q}}M)$ up to an algebraic factor independent of X. Let $w \in \mathbb{Z}$ be the weight of M, that is, $i+j$ for $H^{i,j}(M) \neq 0$ which is independent of (i,j) and σ. Then we write $\mathscr{F}^\pm(M)$ for a member of the Hodge filtration $\{\mathscr{F}^j H_{DR}(M)\}_j$ such that for all σ,

$$\mathscr{F}^\pm(M) \otimes_{F,\sigma} \mathbb{C} = \bigoplus_{i>j} H^{i,j}({}^\sigma M) \quad \text{if } c \text{ acts on } H^{i,i}(M) \text{ by } \mp 1 \text{ and}$$
$$\mathscr{F}^\pm(M) \otimes_{F,\sigma} \mathbb{C} = \bigoplus_{i\geq j} H^{i,j}({}^\sigma M) \quad \text{if } c \text{ acts on } H^{i,i}(M) \text{ by } \pm 1.$$

We put $H^\pm_{DR}(M) = H_{DR}(M)/\mathscr{F}^\mp(M)$. When $\mathscr{F}^+(M) = \mathscr{F}^-(M)$, we simply write $\mathscr{F}(M)$ for $\mathscr{F}^\pm(M)$. Let $H^\pm_B({}^\sigma M)$ be the "$\pm$" eigenspace of $H_B({}^\sigma M) \oplus H_B({}^{c\sigma} M)$ or $H_B({}^\sigma M)$, according as σ is complex or real, under the action of complex conjugation $c = c_\iota$ induced by ι. Then under (Cr), we know from [3] or [74] (see Lemma 5.1.1 of

Chapter 5 in the text) that if F does not contain a CM field, $H^{\pm}_{DR}(M)$ is $E \otimes F$-free, and if $H^{\pm}_{DR}(M)$ is not $E \otimes F$-free, F contains a CM-field, and there exists an automorphism ρ of F inducing complex conjugation on every CM-subfield of F such that $H^{\pm}_{DR}(M) \oplus H^{\pm}_{DR}(M)^{\rho}$ is $E \otimes F$-free, where $H^{\pm}_{DR}(M)^{\rho}$ is the $E \otimes F$-module isomorphic to $H^{\pm}_{DR}(M)$ as an E-vector space on which $e \otimes f \in E \otimes F$ acts via $e \otimes \rho(f)$. Now we define the Deligne period $c^{\pm}_{X,\sigma,\iota}(\varphi)$, following [3] and [74], by the ι_E-component of the determinant of the following linear maps $I^{\pm}_{X,\sigma}$ with respect to basis of $H^{\pm}_{DR}(M)$ or $H^{\pm}_{DR}(M) \oplus H^{\pm}_{DR}(M)^{\rho}$ over $E \otimes F$ and of $H^{\pm}_{B}({}^{\sigma}M)$ over E: if $\iota\sigma$ is real

$$I^{\pm}_{X,\sigma} : H^{\pm}_{B}({}^{\sigma}M) \subset H_B({}^{\sigma}M) \xrightarrow{I_{X,\sigma}} H_{DR}(M) \otimes_{F,\sigma} X \to H^{\pm}_{DR}(M) \otimes_{F,\sigma} X,$$

and if $\iota\sigma$ is complex,

$$I^{\pm}_{X,\sigma} : H^{\pm}_{B}({}^{\sigma}M) \subset H_B({}^{\sigma}M) \oplus H_B({}^{c\sigma}M) \xrightarrow{I_{X,\sigma} \oplus I_{X,c\sigma}} H_{DR}(M) \otimes_{F,\sigma\oplus c\sigma} (X \oplus X)$$
$$\to (H^{\pm}_{DR}(M) \otimes_{F,\sigma} X) \oplus (H^{\pm}_{DR}(M) \otimes_{F,c\sigma} X).$$

We regard, once and for all, I as a complete representative set for $\mathrm{Hom}_{alg}(\overline{F}, \overline{\mathbb{Q}})/\mathcal{G}_F$ and choose a complete representative set Σ for $\langle c \rangle \backslash \mathrm{Hom}_{alg}(\overline{F}, \overline{\mathbb{Q}})/\mathcal{G}_F$ such that $\Sigma \subset I$. When $F = \mathbb{Q}$, there is a canonical choice of I made of the identity. We will make this choice always for $F = \mathbb{Q}$. We put

$$c^{\pm}_{X,\iota}(\varphi) = \prod_{\sigma \in \Sigma} c^{\pm}_{X,\sigma,\iota}(\varphi).$$

Then $c^{\pm}_{X,\iota}(\varphi)$ is equal to $c^{\pm}_{X,\iota}(\mathrm{Ind}^{\mathcal{G}_{\mathbb{Q}}}_{\mathcal{G}_F} \varphi) = c^{\pm}_{X,\iota}(\mathrm{Res}_{F/\mathbb{Q}} M)$ up to an algebraic factor independent of X (see [74] Proposition 2.2 and [42] (1.2a,b)). By definition, we have

(BC) *Base-change of the $E \otimes F$-basis of $H^{\pm}_{DR}(M)$ or $H^{\pm}_{DR}(M) \oplus H^{\pm}_{DR}(M)^{\rho}$ by a linear transformation $\alpha \in GL_{d^{\pm}}(F \otimes E)$ and base change of the E-basis of $H^{\pm}_{B}({}^{\sigma}M)$ by $\beta_\sigma \in GL_{d^{\pm}}(E)$ transform $c^{\pm}_{X,\sigma,\iota}(\varphi)$ into $\iota_E \otimes \sigma(\gamma) c^{\pm}_{X,\sigma,\iota}(\varphi)$ for $\gamma \in (E \otimes F)^{\times}$ depending only on α and β but independent of X.*

There are various choices of X with particular properties. We can take $X = \mathbb{C}$. In this case, the comparison isomorphism $I_{\mathbb{C},\sigma}$ is derived from a theorem of de Rham. The complex period $c^{\pm}_{\iota,\sigma}(\varphi) = c^{\pm}_{\mathbb{C},\sigma,\iota}(\varphi)$ does not vanish, because c interchanges $H^{k,j}({}^{\sigma}M)$ and $H^{j,k}({}^{\sigma}M)$. We write $c^{\pm}_{\iota}(\varphi)$ for $c^{\pm}_{\mathbb{C},\iota}(\varphi)$. Supposing that φ is critical, we here give a conjecture of Deligne ([15]; see also [42] Section 1) in the following form:

(D)
$$\frac{L_\iota(0,\varphi)}{c^{+}_{\iota}(\varphi)} \in \iota(\overline{\mathbb{Q}}) \quad \text{and}$$
$$\iota^{-1}\left(\frac{L_\iota(0,\varphi)}{c^{+}_{\iota}(\mathrm{Ind}^{\mathcal{G}_{\mathbb{Q}}}_{\mathcal{G}_F} \varphi)}\right) = (\iota\sigma)^{-1}\left(\frac{L_{\iota\sigma}(0,\varphi)}{c^{+}_{\iota\sigma}(\mathrm{Ind}^{\mathcal{G}_{\mathbb{Q}}}_{\mathcal{G}_F} \varphi)}\right) \quad \text{for } \sigma \in \mathcal{G}_{\mathbb{Q}}.$$

3.2 p-Adic periods of motives

We fix an embedding $i_p : \overline{\mathbb{Q}} \to \Omega$. In the p-adic case, various rings X with the property discussed in the previous section has been studied by Fontaine. Among these, the biggest is the Fontaine's field B_{DR}. The field B_{DR} is an algebraically closed and p-adically complete field extension of $\mathbb{Q}_p$ (and hence Ω) with other interesting properties (see [22] for details). Here we take as $\mathscr{C}_X$ ($X = B_{DR}$) the category of de Rham motives defined in [2]. We fix an embedding of $\overline{\mathbb{Q}}_p$ into B_{DR} so that the composite isomorphism $\overline{\mathbb{Q}}_p \hookrightarrow B_{DR}^+ \xrightarrow{\theta} \Omega$ is the identity on $\overline{\mathbb{Q}}_p$, where the subring B_{DR}^+ of B_{DR} defined in [22] is a discrete valuation ring with residue field Ω and θ is just taking residue modulo the maximal ideal of B_{DR}^+. We write the p-adic etale realization of M by $H_p(M)$. Then we assume the existence of a functorial comparison isomorphism conjectured by Fontaine, proven by Fontaine and Messing [24] under a certain condition, studied by Faltings [20] for general smooth projective varieties and further generalized to motives in $\mathscr{C}_X$ in [2] Theorem 0.3:

$$(\mathrm{C}_{DR}) \qquad I_{i_p\sigma,DR} : H_p(M) \otimes_{\mathbb{Q}} B_{DR} \cong H_{DR}(M) \otimes_{F,i_p\sigma} B_{DR}.$$

Here we take the morphism defined in [20] and [2] because there seems a problem of comparison of comparison isomorphisms [50] 4.3.5. The morphism $I_{i_p\sigma,DR}$ is an isomorphism of $\mathscr{G}_{F_{\mathfrak{P}}}$-modules if we put the following Galois module structure: If the p-adic place $\mathfrak{P}$ is induced by $i_p\sigma$, $\mathscr{G}_{F_{\mathfrak{P}}}$ acts on $H_p(M) \otimes_{\mathbb{Q}} B_{DR}$ via the diagonal action $\tau(m \otimes b) = \tau(m) \otimes \tau(b)$ and on $H_{DR}(M) \otimes_{F,i_p\sigma} B_{DR}$ through B_{DR}, that is, $\tau(f \otimes b) = f \otimes \tau(b)$. Note that we have another functorial isomorphism coming from the identification of the Betti cohomology with the etale cohomology:

$$(\mathrm{C}_{et}) \qquad I_{\iota\sigma,et} : H_B({}^{\sigma}M) \otimes_{\mathbb{Q}} \mathbb{Q}_p = H_B({}^{\sigma}M) \otimes_E (\mathbb{Q}_p \otimes E) \cong H_p(M).$$

The etale cohomology $H_p(M)$ is defined by extending scalar from F to $\overline{F}$. The motive ${}^{\sigma}M$ over $\mathbb{C}$ is the scalar extension of M to $\mathbb{C}$ via $\iota\sigma$. Thus $I_{\iota\sigma,et}$ depends on $\iota\sigma : \overline{F} \cong \overline{\mathbb{Q}} \subset \mathbb{C}$. We can thus have $I_{X,\sigma} = I_{i_p\sigma,DR} \circ I_{\iota\sigma,et}$ for $X = B_{DR}$. For $X = B_{DR}$, we write $c^{\pm}_{DR,\sigma,\iota}(\varphi)$, $c^{\pm}_{DR,\iota}(\varphi)$, $\delta_{DR,\sigma}(M)$ and $\delta_{DR}(M)$ for $c^{\pm}_{X,\sigma,\iota}(\varphi)$, $c^{\pm}_{X,\iota}(\varphi)$, $\delta_{X,\sigma}(M)$ and $\delta_X(M)$, respectively. Note here that these periods are the $i_p\iota_E$-component of the determinant of the various comparison morphisms. We further note that the above period $c^{\pm}_{DR,\sigma,\iota}(\varphi)$ may vanish depending on ι (and i_p). We study in detail such vanishing and non-vanishing of p-adic periods in the following Sections 3.3 and 3.4.

To define periods in Ω not in B_{DR}, we introduce now the local reducibility conditions. For that, we look into B_{HT}. We start recalling $\mathbb{Z}_p(1)$ which is the p-adic Tate module of $\mathbf{G}_m$. By identifying μ_N with $\mathbb{Z}/N\mathbb{Z}$ via $i_p(\iota^{-1}(\exp(2\pi i/N))) \mapsto 1$, we have a trivialization: $\mathbb{Z}_p(1) \cong \mathbb{Z}_p$. We write $\mathbb{Z}_p(-1)$ to be the $\mathbb{Z}_p$-dual of $\mathbb{Z}_p(1)$

and define as a $\mathcal{G}_{\mathbb{Q}_p}$-module, $\mathbb{Z}_p(\pm n) = \mathbb{Z}_p(\pm 1)^{\otimes n}$ for each positive integer n. We put $\mathbb{Z}_p(0) = \mathbb{Z}_p$. Then we consider $\Omega(n) = \mathbb{Z}_p(n) \otimes_{\mathbb{Z}} \Omega$ on which $\mathcal{G}_{\mathbb{Q}_p}$ acts diagonally (and semi-linearly). There is a trivialization $\Omega \cong \Omega(n)$ as specified above. Since B_{DR}^+ is a valuation ring with residue field Ω, we construct the associated graded algebra B_{HT} of B_{DR}. That is canonically isomorphic to $\oplus_n \Omega(n) = \Omega[t, t^{-1}]$. We assume the existence of the Hodge-Tate version of $I_{i_p\sigma,DR}$:

$$(\mathrm{C}_{HT}) \qquad I_{i_p\sigma,HT} : H_p(M) \otimes_{\mathbb{Q}_p} B_{HT} \cong H_{DR}(M) \otimes_{F,i_p\sigma} B_{HT}.$$

The existence of $I_{i_p\sigma,HT}$ is shown for smooth projective varieties over $\mathbb{Z}_p$ by Faltings [21]. Thus (C_{HT}) is valid at least for crystalline motives [2]. We anyway assume the existence of I_{HT}. We can define all periods as in Section 3.1 for $X = B_{HT}$ and $I_{X,\sigma} = I_{i_p\sigma,HT} \circ I_{\iota\sigma,et}$. We express these periods replacing the subscript X by HT in the notation introduced in Section 3.1. Let S_F be the set of all p-adic places of F. We suppose (C_{HT}) at every $\mathfrak{P} \in S_F$; thus, the ι_E-component $H_{\mathfrak{p}}(M)$ of $H_p(M)$ is of Hodge-Tate type, where $\mathfrak{p}$ is the p-adic place of E induced by i_p. We suppose the existence of $\mathcal{G}_{F_{\mathfrak{P}}}$-subrepresentation $V_{\mathfrak{P}}^{\pm}$ in $H_{\mathfrak{p}}(M)$ for each $\mathfrak{P} = \{i_p\sigma\} \in S_F$ such that $I_{\sigma,HT}$ induces

$$(\mathrm{Red}_{\pm}) \qquad I_{i_p\sigma,HT,\mathfrak{p}} : V_{\mathfrak{P}}^{\pm} \otimes_{E_{\mathfrak{p}},i_p} B_{HT} \cong \mathscr{F}^{\pm}(M) \otimes_{F\otimes E,\sigma\otimes i_p} B_{HT}.$$

As we will see later, the condition (Red_-) is always satisfied by admissible crystalline motives in the sense of Panchishkin [56]. However this formulation is more useful, because it does not require M to be crystalline. A similar condition asserting the existence of $V_{\mathfrak{P}}^{\pm}$ without reference to $\mathscr{F}^{\pm}(M)$ is also stated in [28] as Panchishkin's condition.

We now study how the condition $(\mathrm{Red}_{\pm})$ behaves under base change. Let F' be a subfield of F. Since the condition $(\mathrm{Red}_{\pm})$ is insensitive to changing the field of definition of M, we may assume F to be a Galois extension of F' replacing F by the Galois closure of F over F'. Pick a prime $\mathfrak{p} \in S_{F'}$ and decompose $\mathfrak{p} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e(\mathfrak{P})}$. Extending $\mathfrak{P}$ to a prime of $\overline{F}$ corresponding to $i_p\sigma$, let $D(\mathfrak{P})$ (resp. $D(\mathfrak{p})$) be the decomposition group of $\mathfrak{P} = i_p\sigma$ in $\mathcal{H} = \mathcal{G}_F$ (resp. $\mathcal{G} = \mathcal{G}_{F'}$). We embed $\bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{Ind}_{D(\mathfrak{P})}^{D(\mathfrak{p})} V_{\mathfrak{P}}^{\pm}$ into $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} V$ as follows. We have a natural bijection between the double coset $\mathcal{H}\backslash\mathcal{G}/D(\mathfrak{P})$ and primes of F dividing $\mathfrak{p}$. By fixing a complete representative set R for $\mathcal{H}\backslash\mathcal{G}$, we identify $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} V$ with the space of functions on R with values in V. Then $g \in \mathcal{G}$ acts on $v : R \to V$ by $gv(\sigma) = h_\sigma v(\tau)$, where $\sigma g = h_\sigma \tau$ for $\tau \in R$ and $h_\sigma \in \mathcal{H}$. We identify $\mathrm{Ind}_{D(\mathfrak{P})}^{D(\mathfrak{p})} V_{\mathfrak{P}}^{\pm}$ with the space of functions in $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} V$ supported on $\mathcal{H}\backslash\mathcal{H}\sigma D(\mathfrak{P})$ with values in $V_{\sigma(\mathfrak{P})}^{\pm}$. It is plain from construction that this embedding is an embedding of $D(\mathfrak{p})$-modules, and $I_{i_p\sigma,HT}$ for $\mathrm{Res}_{F/F'} M$, which

is given by $\bigoplus_{\tau|_{F'}=\sigma} I_{i_p\tau,HT}$, induces an isomorphism

$$(\bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{Ind}_{D(\mathfrak{P})}^{D(\mathfrak{p})} V_{\mathfrak{P}}^{\pm}) \otimes_{E,i_p} B_{HT} \cong \mathscr{F}^{\pm}(Res_{F/F'}M) \otimes_{F'\otimes E,i_p\sigma\otimes i_p} B_{HT}.$$

Thus if M satisfies the condition $(\mathrm{Red}_{\pm})$ over F, $\mathrm{Res}_{F/F'} M$ satisfies $(\mathrm{Red}_{\pm})$ over F'.

As observed in [2], if $c_{HT}^{\pm}(\varphi)$ is supported by a single graded component, we can naturally identify the period with a p-adic number in Ω. We now show that this is the case when $(\mathrm{Red}_{\pm})$ is satisfied. Since the multiplication by $\delta_{HT}(\varphi)$ is the comparison isomorphism of the rank 1 motive $\bigwedge^n M$ and hence is supported by a single graded component. If M satisfies $(\mathrm{Red}_{\pm})$ and $\iota\sigma$ is a real place of F, $c_{HT,\sigma,\iota}^{\pm}(\varphi)$ is the determinant of the composition of

$$I_{\iota\sigma} : H_B^{\pm}({}^{\sigma}M) \otimes_{E,i_p} E_{\mathfrak{p}} \to H_{\mathfrak{p}}(M)/V_{\mathfrak{P}}^{\mp} \quad \text{and}$$
$$I_{i_p\sigma} : (H_{\mathfrak{p}}(M)/V_{\mathfrak{P}}^{\mp}) \otimes_{E,i_p} B_{HT} \cong \left\{H_{DR}(M)/\mathscr{F}^{\mp}(M)\right\} \otimes_{F\otimes E,\sigma\otimes i_p} B_{HT}.$$

Since $I_{\iota\sigma}$ is well defined over $E_{\mathfrak{p}}$, up to numbers in $E_{\mathfrak{p}}$, $c_{HT,\sigma,\iota}^{\pm}(\varphi)$ is equal to the determinant of the isomorphism $I_{i_p\sigma}$, and the representation $V_{\mathfrak{P}}^{\pm}$ and $H_{\mathfrak{p}}(M)/V_{\mathfrak{P}}^{\pm}$ are of Hodge-Tate type. Thus we have the p-adic periods in Ω when M satisfies $(\mathrm{Red}_{\pm})$, because the periods $c_{HT,\sigma,\iota}^{\pm}(\varphi) \in B_{HT}$ is, up to the degree 0 number $det(I_{\iota\sigma})$ in $E_{\mathfrak{p}}$, the determinant of the comparison map between $\bigwedge^{d^{\pm}} H_{DR}^{\pm}(M)$ and the Hodge-Tate representation $\bigwedge^{d^{\pm}}(H_{\mathfrak{p}}(M)/V_{\mathfrak{P}}^{\mp})$ and hence is supported by a single graded component. We can similarly check that $c_{HT,\sigma,\iota}^{\pm}(\varphi)$ is supported by a single graded component also for complex place σ. Hereafter, we write $c_{p,\sigma,\iota}^{\pm}(\varphi)$ and $c_{p,\iota}^{\pm}(\varphi)$ for $c_{HT,\sigma,\iota}^{\pm}(\varphi)$ and $c_{HT,\iota}^{\pm}(\varphi)$ regarded as being in Ω. The above argument shows the principal transcendental factor comes from non-vanishing $det(I_{i_p\sigma})$. The vanishing may occur due to the first factor $I_{\iota\sigma}$.

3.3 An example of vanishing p-adic periods

Y. André [1] found an example of vanishing period, and his argument generalizes in the following case. We keep the notation in Section 3.2. Let $\mathscr{E}$ be an elliptic curve defined over $\mathbb{Q}$ without complex multiplication. We consider the p-adic Tate module $T_p(\mathscr{E})$ of $\mathscr{E}$. Choosing a base of the dual $(T_p(\mathscr{E}))^{\vee} \hookrightarrow H_p^1(\mathscr{E})$, we have a representation $\varphi : \mathscr{G}_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$. We suppose

(Fu) *The image of φ is full, that is,* $\mathrm{Im}(\varphi) \supset SL_2(\mathbb{Z}_p)$;

(Ord) *$\mathscr{E}$ has ordinary good reduction at p.*

There is a plenty of examples of $\mathscr{E}$ satisfying the above conditions. We pick two embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \Omega$. Then by (Ord), $H_p^1(\mathscr{E})$ has a one dimensional

subspace $V_+ = V_-$ stable under the decomposition group at p relative to i_p. Since c has two eigenvalues ± 1 on $H^1_p(\mathscr{E})$, we can choose a matrix $A \in SL_2(\mathbb{Z}_p)$ so that $A^{-1}\varphi(c)A$ leaves V_+ stable and acts on it through multiplication by ± 1. We choose $\tau \in \mathscr{G}_{\mathbb{Q}}$ such that $\varphi(\tau) = A$. Then the complex conjugation with respect to $\iota\tau$ leaves V_+ stable and acts on it through multiplication by ± 1, and it is clear that $c^{\pm}_{p,\iota\tau}(\varphi) = 0$. This argument shows that on a non-empty open subset of the space of $\{(i_p, \iota)\}$, the motive $H^1(\mathscr{E})$ has non-vanishing periods but also on a non-empty closed subset, the elliptic curve $\mathscr{E}$ has p-adic periods $c^+_{p,\iota\tau}(\varphi)c^-_{p,\iota\tau}(\varphi) = 0$.

3.4 Non-vanishing of p-adic periods

Suppose $(\mathrm{Red}_{\pm})$ for M and that $M_{/F}$ is pure of rank n with coefficients in E and $H_{\mathfrak{p}}(M)$ is of Hodge-Tate type at every $\mathfrak{P} \in S_F$. We write the representation of $\mathscr{G}_F$ on $H_{\mathfrak{p}}(M)$ as φ. We identify $\mathscr{G}_F$ with $\mathscr{G}_{F^\sigma}$ by σ. Then we put $\varphi_\sigma(g) = \varphi(\sigma g \sigma^{-1})$ as a representation of $\mathscr{G}_{F^\sigma}$. Then we assume

(SS_F) $\qquad\qquad$ *φ_σ is semi-simple for all $\sigma \in I$.*

As we have already remarked, $(\mathrm{Red}_{\pm})$ does not depends on F. We also know that $\prod_\sigma c^{\pm}_{p,\sigma,\iota\tau}(\varphi)$ is equal to $c^{\pm}_{p,\mathrm{id},\iota\tau}(\mathrm{Ind}_{\mathscr{G}_F}^{\mathscr{G}_{\mathbb{Q}}}\varphi)$ up to a non-zero algebraic constant. Thus we only need to prove the non-vanishing of $c^{\pm}_{p,\mathrm{id},\iota\tau}(\Phi)$ for some τ for $\Phi = \mathrm{Ind}_{\mathscr{G}_F}^{\mathscr{G}_{\mathbb{Q}}}\varphi$. We write $c^{\pm}_{p,\iota\tau}$ for $c^{\pm}_{p,\mathrm{id},\iota\tau}$. Note that $\Phi \cong \bigoplus_{\sigma\in I}\varphi_\sigma$ over the Galois closure of F over $\mathbb{Q}$, and Φ is semi-simple. Write V for the representation space of Φ, and consider the algebraic group GL_V defined over $E_{\mathfrak{p}}$ such that $GL_V(A)$ is made of A-linear automorphisms of $V \otimes_{E_{\mathfrak{p}}} A$. Let G be the Zariski closure in $GL_V(\overline{E}_{\mathfrak{p}})$ of the image of Φ (for an algebraic closure $\overline{E}_{\mathfrak{p}}$ of $E_{\mathfrak{p}}$). Then G is an algebraic subgroup of GL_V defined over $E_{\mathfrak{p}}$, and let G° be its connected component. Since the unipotent radical $R_u(G^\circ)$ acts on V by an unipotent action, the semi-simplicity of V tells us that $R_u(G^\circ)$ is trivial, that is, G° and G are reductive.

Now we consider the Tannakian subcategory $\mathscr{M}(M)$ of motives generated by the motive $Res_{F/\mathbb{Q}}M$. We write Gal_M for the motivic Galois group of $\mathscr{M}(M)$ with respect to the fiber functor H_B. Since $\mathscr{M}(M)$ is semi-simple, Gal_M is reductive. Thus Gal_M is an algebraic group acting on V defined over E. We have a natural morphism of algebraic groups of G to Gal_M defined over $E_{\mathfrak{p}}$ induced by the functor associating to each object of $\mathscr{M}(M)$ its Galois representation. The functor taking objects of $\mathscr{M}(M)$ to their Hodge structures induces a one parameter subgroup $\mu_\infty : \mathbf{G}_{m/\mathbb{C}} \to \mathrm{Gal}_{M/\mathbb{C}}$ (cf. [16] Section 3), and similarly the p-adic Hodge-Tate structure gives another $\mu_p : \mathbf{G}_{m/\Omega} \to \mathrm{Gal}_{M/\Omega}$ [62]. We fix an embedding of $E_{\mathfrak{p}}$ into $\mathbb{C}$ which coincides with

ι on E. Then we suppose

(MT) μ_∞ *has values in* $G^\circ(\mathbb{C})$;

(Reg) $Res_{F/\mathbb{Q}}M$ *is regular over* F,

that is, the image of μ_∞ is a regular 1-parameter subgroup of Gal_M, which is equivalent to asserting that all non-trivial Hodge component of $H_{DR}(M)\otimes_{F\otimes E,\sigma\otimes\iota}\mathbb{C}$ for each σ is of dimension 1. Note that the one parameter subgroup μ_p has values in $G^\circ(\Omega)$ by a result of Sen and Serre [62] Théorème 2. By (Reg), the centralizer of μ_p (resp. μ_∞) in G is a maximal torus T_p split over Ω (resp. T_∞ split over $\mathbb{C}$) of G°, and μ_p (resp. μ_∞) determines a Weyl chamber of $Y(T_p)=\mathrm{Hom}_{alg\text{-}gr}(\mathbf{G}_m,T_p)$ (resp. $Y(T_\infty)$) and hence a Borel subgroup B_p (resp. B_∞) containing T_p (resp. T_∞) in G. The Borel subgroups are stabilizer of the filtration coming from the Hodge filtration of $H_{DR}(M)$ through the comparison isomorphisms. Thus taking a big algebraically closed field U over $E_\mathfrak{p}$ containing Ω and $\mathbb{C}$ as two extensions of $E_\mathfrak{p}$, we find $g\in G^\circ(U)$ such that $gT_\infty g^{-1}=T_p$ and $gB_\infty g^{-1}=B_p$. Since $\Phi(c)$ reverses the Hodge filtration on $H_{DR}(M)$, $\Phi(c)$ brings B_∞ to its opposite. In particular, $\Phi(c)$ (resp. $g\Phi(c)g^{-1}$) is not contained in any proper parabolic subgroup containig T_∞ (resp. T_p). Let $Par(T_p)$ be the set of proper parabolic subgroups of G containing T_p, and define a subset X of G by

$$\left\{x\in G\middle| x\Phi(c)x^{-1} \text{ is in an element of } \bigcup_{P\in Par(T_p)} P\right\}.$$

Each $P\in Par(T_p)$ contains a unique Borel subgroup with torus T_p. There is finitely many such Borel subgroups. For a given Borel subgroup, there is a finitely many parabolic subgroups containing the Borel subgroup. Thus the set $Par(T_p)$ is a finite set. For each $P\in Par(T_p)$, $\{x\in G|x\Phi(c)x^{-1}\in P\}$ is obviously Zariski-closed. Then X is a union of finitely many such sets and is Zariski-closed in G, and $\mathrm{codim}_{G^\circ}(X\bigcap G^\circ)>0$ because $g\notin X$, $g\in G^\circ$ and G° is connected. Since $\mathrm{Im}(\Phi)$ is Zariski-dense in G, we can find $\tau\in\mathcal{G}_\mathbb{Q}$ such that $\Phi(\tau^{-1})\notin X$, and $\Phi(\tau^{-1}c\tau)$ is not contained in any member of $Par(T_p)$. This shows the non-vanishing of $c^\pm_{p,\iota\tau}(\Phi)$.

Theorem 3.4.1 — *Suppose* $(Red_\pm)$, (SS_F), (MT) *and* (Reg) *for the motive* M *over a number field* F. *Then there exists* $\tau\in\mathcal{G}_F$ *such that* $c^\pm_{p,\sigma,\iota\tau}(\varphi)\neq 0$ *for all* $\sigma\in I$.

If we assume the Tate-style conjecture asserting that $\mathcal{M}(M)$ is equivalent to the Tannakian category of Galois representations generated by Φ, then $G\cong\mathrm{Gal}_M$ over $E_\mathfrak{p}$, and hence the condition (MT) follows. If M is of rank 1, there is nothing to prove (see (7.2.1)). We assume that M is of rank 2 and M satisfies in addition to $(\mathrm{Red}_\pm)$:

(Irr_F) φ *is absolutely irreducible over any finite extension* F'/F.

Let V be the representation space of φ. We consider the Zariski closure H of the image of φ_σ in $GL_V(\overline{E}_\mathfrak{p})$ for σ. Note that H is independent of σ. The Hodge structure of M at $\iota\sigma$ (resp. the p-adic Hodge-Tate structure at $i_p\sigma$) induces a one parameter subgroup $\mu_{\infty,\sigma} : \mathbf{G}_m \to \mathrm{Gal}_M$ (resp. $\mu_{p,\sigma} : \mathbf{G}_m \to H$). By (Irr_F), for real σ, H is isomorphic to either $SL(2)$ or $GL(2)$ according as the weight of the Hodge structure at σ of $M \wedge M$ is zero or not. By making some Tate twist if necessary, we may assume that $H_\sigma = GL(2)$. Then $\mu_{\infty,\sigma}$ has values in $H(\mathbb{C})$ (this is an F-version of the condition (MT)). We now assume a version of the condition (Reg):

(Reg_σ) *$H_B({}^\sigma M)$ has two distinct Hodge types at each real $\sigma \in I$ and $\iota \circ \iota_E$, and $H_B({}^\sigma M) \oplus H_B({}^{c\sigma} M)$ has four distinct Hodge types at complex $\sigma \in I$ and $\iota \circ \iota_E$.*

By this, we have the maximal torus $T_{X,\sigma}$ centralizing $\mu_{X,\sigma}$ and a Borel subgroup $B_{X,\sigma}$ of H fixing the Hodge filtration. We take a universal domain U as in the proof of Theorem 3.4.1. We fix a Borel subgroup B of H defined over U. For each real place σ of F, we can choose $g_\sigma \in H^\circ(U)$ so that $g_\sigma T_{\infty,\sigma} g_\sigma^{-1} = T_{p,\sigma}$ and $g_\sigma B_{\infty,\sigma} g_\sigma^{-1} = B_{p,\sigma}$, where H° is the connected component of H. Then $g_\sigma(\varphi_\sigma)(c) g_\sigma^{-1}$ is not contained in $B_{p,\sigma}$. Thus $X_\sigma = \{x \in H | x(\varphi_\sigma)(c)x^{-1} \in \bigcup_{P \in Par(T_{p,\sigma})} P\}$ is a proper Zariski-closed subset of H, where $Par(T_{p,\sigma})$ is the set of all proper parabolic subgroups of H containing $T_{p,\sigma}$. In particular, $\mathrm{codim}_{H^\circ}(X_\sigma \bigcap H^\circ)$ is positive. We now assume σ to be a complex embedding of F. If one of $V^\pm_\mathfrak{P}$ and $V^\pm_{\mathfrak{P}^c}$ is trivial (i.e. just $\{0\}$) for the p-adic place $\mathfrak{P}$ attached to $i_p\sigma$, the period $c^\pm_{X,\sigma,\iota}(M)$ does not vanish for any choice of ι by the definition of $H^\pm_B({}^{\iota\sigma} M)$. Thus we may assume that $\dim(V^\pm_\mathfrak{P}) = \dim(V^\pm_{\mathfrak{P}^c}) = 1$. In this case, if one of $c^\pm_{X,\sigma,\iota}(M)$ vanishes, the complex conjugation $c : H_B({}^{\iota\sigma} M) \cong H_B({}^{c\iota\sigma} M)$ induces an isomorphism: $V^\pm_\mathfrak{P} \cong V^\pm_{\mathfrak{P}^c}$. This implies the p-adic Hodge types of the two $\mathcal{G}_{F_\mathfrak{P}}$-modules are the same, which contradicts to (Reg_σ). Thus under (Reg_σ), the period does not vanish for complex σ. Thus we can forget complex places. Regard naturally G° in the proof of Theorem 3.4.1 as a subgroup of the product $H^I = \prod_\sigma H_\sigma$ of I-copies of $H_\sigma = H$. Since $\mathrm{codim}_{H^\circ}(X_\sigma \bigcap H^\circ) > 0$, we have $\tau \in \mathcal{G}_F$ such that $\varphi_\sigma(\tau) \notin X_\sigma$ but $\Phi(\tau) \in G^\circ$. Thus $\Phi(\tau) = \bigoplus_\sigma \varphi_\sigma(\tau) \notin X_\sigma \times H^{I-\{\sigma\}}$; hence the codimension of $G^\circ \bigcap X_\sigma \times H^{I-\{\sigma\}}$ in G° is positive, and $G^\circ \bigcap (\bigcup_{\sigma:real} X_\sigma \times H^{I-\{\sigma\}})$ is a proper Zariski-closed subset of G°. Thus we can find $\tau \in \mathcal{G}_F$ such that $\Phi(\tau) \notin (\bigcup_{\sigma:real} X_\sigma \times H^{I-\{\sigma\}})$. By changing ι by $\iota\tau$, we have a non-vanishing p-adic period for every $\sigma \in I$. We just record the above fact:

Corollary 3.4.2 — *Suppose that M is of rank 2 over E and $(\mathrm{Red}_\pm)$, (Reg_σ) for all σ and (Irr_F). Then there exists $\tau \in \mathcal{G}_F$ such that $c^+_{p,\sigma,\iota\tau}(\varphi) c^-_{p,\sigma,\iota\tau}(\varphi) \neq 0$ for all $\sigma \in I$.*

3.5 Admissibility condition

Our condition ($\mathrm{Red}_{\pm}$) is closely related to Panchishkin's admissibility condition for the (conjectural) existence of bounded p-adic measures for motivic L-functions (Panchishkin [56] Section 5). Here we explain the relation in some detail. For simplicity, we assume F to be totally real. To define the admissibility, we need to introduce the following extra hypotheses on M:

(Cris) *$H_\ell(M)$ (resp. $H_p(M)$) is unramified for $\ell \neq p$ (resp. crystalline at each $\mathfrak{P}|p$).*

Let us recall the definition that M is crystalline at $\mathfrak{P}$ [22]: Let $B_{cris} \subset B_{DR}$ be the Fontaine's ring B_{cris} introduced in [22] Section 4, and B_{cris} is an algebra over the field $K_0 \subset \Omega$ of fractions of the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$. The Frobenius automorphism ϕ of K_0 extends to B_{cris}, and B_{cris} has a ϕ-linear map Φ_{cris} (called crystalline Frobenius map) and a decreasing filtration $F^i B_{cris}$. Then B_{HT} is defined by $gr(B_{cris}) = gr(B_{DR})$, where the filtration of B_{DR} is given by the power of the maximal ideal of B_{DR}^+, and that of B_{cris} is induced by the filtration of B_{DR}. Let $K/\mathbb{Q}_p$ be a finite extension, $\overline{K}$ be the algebraic closure of K in Ω and $K_{ur} = K_0 \bigcap K$. Then $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$ acts on B_{DR} and B_{cris} extending its action on $\overline{K}$ so that $H^0(\mathcal{G}, B_{DR}) = K$, $H^0(\mathcal{G}, B_{HT}) = K$ and $H^0(\mathcal{G}, B_{cris}) = K_{ur}$. For each continuous representation of $\mathcal{G}$ on a finite dimensional vector space V over $\mathbb{Q}_p$, we define

$$D(V) = H^0(\mathcal{G}, V \otimes_{\mathbb{Q}_p} B_{cris}), \quad D_{DR}(V) = H^0(\mathcal{G}, V \otimes_{\mathbb{Q}_p} B_{DR}) \quad \text{and}$$
$$D_{HT}(V) = H^0(\mathcal{G}, V \otimes_{\mathbb{Q}_p} B_{HT}),$$

where $\sigma \in \mathcal{G}$ acts on $V \otimes B_X$ ($X = cris$, DR or HT) by $v \otimes b \mapsto \sigma(v) \otimes \sigma(b)$. Then $D(V) \otimes_{K_{ur}} K$ is isomorphic to $H^0(\mathcal{G}, V \otimes_{\mathbb{Q}_p} (B_{cris} \otimes_{K_{ur}} K))$ and is a subspace of $D_{DR}(V)$. It has been shown in [23] Section 3 and [22] Section 5 that

(Fo1) $D(V)$ is a vector space over K_{ur},

(Fo2) $\dim_{K_{ur}} D(V) \leq \dim_K D_{DR}(V) \leq \dim_K D_{HT}(V) \leq \dim_{\mathbb{Q}_p} V$;

(Fo3) $D(V) \otimes_{K_{ur}} K$ has decreasing filtration D^i induced from that of $B_{cris} \otimes_{K_{ur}} K$,

(Fo4) $D(V)$ has ϕ-linear automorphism Φ induced by Φ_{cris}.

When the equality: $\dim_{K_{ur}} D(V) = \dim_{\mathbb{Q}_p} V$ holds, V is called crystalline. A filtered Dieudonné module D is a finite dimensional K_{ur}-vector space D with a ϕ-linear map Φ and a decreasing filtration D^i of $D \otimes_{K_{ur}} K$ satisfying the condition (Fo3-4) for D in place of $D(V)$. We call M crystalline at p if the p-adic étale realization $H_p(M)$ is crystalline for the closure $K = F_{\mathfrak{P}}$ of $i_p(\sigma(F))$ in Ω for all $\sigma \in I$. Thus we have the above functor D depending on p-adic place $\mathfrak{P}$ induced by $i_p\sigma$ of F which gives the identification of the decomposition group at $\mathfrak{P}$ and $\mathcal{G}$. Under

this setting, M has its crystalline realization $H_{cris,\mathfrak{P}}(M)$ at $\mathfrak{P}$ given by $D(H_p(M))$, which is a K_{ur}-vector space with a ϕ-linear automorphism Φ. We suppose to have a canonical identification: $H_{cris,\mathfrak{P}}(M) \otimes_{K_{ur}} K \cong H_{DR}(M) \otimes_{F,\sigma} K$ and to have a canonical isomorphism:

$$(\mathrm{C}_{cris}) \qquad I_{i_p\sigma,cris} : H_p(M) \otimes_{\mathbb{Q}_p} B_{cris} \cong H_{cris,\mathfrak{P}}(M) \otimes_{K_{ur}} B_{cris}.$$

If M is the motive attached to a smooth projective variety V over F with good reduction at $\mathfrak{P}$, by a theorem of Faltings [20] (see also [50]), M is crystalline at p, and $H_{cris,\mathfrak{P}}(M)$ is the scalar extension to K_{ur} of the crystalline cohomology group of the reduction of V modulo $\mathfrak{P}$, and (Cris) and (C_{cris}) are satisfied. We can take $\mathscr{C}_X$ to be the category of crystalline motives defined in [2]. We define the period $c^{\pm}_{cris,\sigma}(\varphi)$ taking $X = B_{cris}$ and $I_{X,\sigma} = I_{i_p\sigma,cris} \circ I_{\iota\sigma,et}$. We assume here the compatibility of $I_{i_p\sigma,cris}$ and $I_{i_p\sigma,DR}$ (see [50] 4.3.5 and [2] 5.1). Then for crystalline M, we get $c^{\pm}_{cris,\sigma}(\varphi) = c^{\pm}_{DR,\sigma}(\varphi)$ in B_{DR}. The local Euler factor at $\mathfrak{P}$ of $L(s,M)$: $L_{\mathfrak{P}}(X) = det\left(1 - \varphi(\phi^f)\big|_{H_!(M)} X\right)$, where $f = [K_{ur} : \mathbb{Q}_p]$, can be recovered as the characteristic polynomial of Φ^f acting on $H_{cris,\mathfrak{P}}(M)$. For each filtered Dieudonné module D with Frobenius Φ, we can attach two polygons. One is the Newton polygon $P_N(D)$ of $det\left(1 - \Phi^f\big|_D X\right)$, and the other is the Hodge polygon coming from the filtration. That is, the polygon $P_H(D)(t)$ passing through points

$$(0,0), \ldots, (\sum_{i \le j} \dim(D^i/D^{i+1}), \sum_{i \le j} i \cdot \dim(D^i/D^{i+1})), \ldots .$$

If V is crystalline, it is known that $P_N(D)$ is located above $P_H(D)$, and they match at the two extremes [23] Section 4.4. A filtered Dieudonné module with this property is called weakly admissible. The functor $V \mapsto D(V)$ gives an equivalence of categories from the category of crystalline representations to the category of admissible filtered Dieudonné modules ([23] Section 3). Fontaine proved that admissible Dieudonné module is weakly admissible.

Let us now describe the $\mathfrak{P}$-admissibility condition of M due to Panchishkin: Let $P_{N,\mathfrak{P}}(t)$ (resp. $P_{H,\mathfrak{P}}(t)$) be the Newton (resp. Hodge) polygon for $H_{cris,\mathfrak{P}}(M)$. We write d^+ for $\dim_{\mathbb{Q}} H^+_B({}^{\sigma}M) = \dim_F \mathscr{F}^-(M)$. The motive M is called admissible in [56] if

$$(\mathrm{Adm}) \qquad P_{H,\mathfrak{P}}(d^+) = P_{N,\mathfrak{P}}(d^+) \quad \text{for all } \mathfrak{P}.$$

Since the Newton polygon $P_{N,\mathfrak{P}}$ is defined solely using the crystalline realization, if M is admissible, we can find a filtered Dieudonné submodule $D^-_{\mathfrak{P}}$ of $H_{cris,\mathfrak{P}}(M)$ such that $D^-_{\mathfrak{P}} \otimes_{K_{ur}} B_{DR} = \mathscr{F}^-(M) \otimes_{F,i_p\sigma} B_{DR}$. Thus if M is admissible, $D^-_{\mathfrak{P}}$ is weakly admissible and is contained in the admissible module $H_{cris,\mathfrak{P}}(M)$. In

this situation, $D_{\mathfrak{P}}^-$ itself is admissible [23] 4.5.1. Thus we can find a crystalline subrepresentation $V_{\mathfrak{P}}^- \subset H_p(M)$ of $\mathcal{G}_{F_{\mathfrak{P}}}$ such that $D(V_{\mathfrak{P}}^-) = D_{\mathfrak{P}}^-$, and $I_{i_p\sigma,cris}$ induces $V_{\mathfrak{P}}^- \otimes_{\mathbb{Q}_p} B_{HT} \cong \mathcal{F}^-(M) \otimes_{F,i_p\sigma} B_{HT}$.

In order to define Euler p-factor of the genuine p-adic L-function, we recall here some of definition given in [56] Section 6. Since the Euler p-factor is the same for M and $Res_{F/\mathbb{Q}}M$, we assume here that $F = \mathbb{Q}$. Let $\mathbb{Q}_\infty$ be the unique cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and pick a finite order character χ of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We now write

$$L_p(X) = \prod_j (1 - \alpha_j X)$$

so that $\mathrm{ord}_p \alpha_1 \leq \mathrm{ord}_p \alpha_2 \leq \cdots \leq \mathrm{ord}_p \alpha_n$. Then we define

(E)

$$\mathcal{E}(M \otimes \chi; s) = \begin{cases} \prod_{d^+ < j \leq n}(1 - \alpha_j p^{-s}) \prod_{1 \leq j \leq d^+}(1 - \alpha_j^{-1} p^{s-1}) & \text{if } \chi \text{ is trivial,} \\ \prod_{1 \leq j \leq d^+}(\alpha_j^{-1} p^s)^\nu & \text{if } C(\chi) = p^\nu \text{ with } \nu > 0, \end{cases}$$

where $C(\chi)$ is the conductor of χ. We further put

$$(\Gamma) \qquad \gamma(M) = \frac{\Gamma(1, M)}{(2\pi i)^r} \quad \text{for } r = \sum_{j<0} j h^{j,k} \text{ for the Hodge number } h^{j,k} \text{ of } \check{M},$$

where $\Gamma(s, M)$ is the Gamma factor of the L-function of M as in [15].

4. Periods of arithmetic Galois representations

We study general $\mathbb{I}$-adic arithmetic Galois representations φ and its periods. We shall state conjectures on the existence of genuine p-adic L-functions and their normalization.

4.1 Variation of p-adic periods over $Spec(\mathbb{I})$

We study a bit more general representation $\varphi : \mathcal{G}_F \to GL_n(\mathbb{K})$ than in the introduction. We assume

(Cont) *There is an $\mathbb{I}$-module of finite type V in $\mathbb{K}^n$ such that*

(i) $V \otimes_{\mathbb{I}} \mathbb{K} = \mathbb{K}^n$;

(ii) *V is stable under φ;*

(iii) *$\varphi : \mathcal{G}_F \to Aut_{\mathbb{I}}(V)$ is continuous under the adic topology of the maximal ideal of $\mathbb{I}$.*

Replacing V by its reflexive closure (that is, the intersection of localizations of V at all height 1 primes of $\mathbb{I}$ in $V \otimes_{\mathbb{I}} \mathbb{K}$), we may (and will) assume that V itself is reflexive. See [8] Chapter 7 for general theory of reflexive modules. By extending the scalar field K if necessary, we assume that in $Spec(\mathbb{I})(K)$, there are densely populated arithmetic points. We also assume unramifiedness (unr) as in the introduction. For an arithmetic point P, we consider the following condition

(Fr_P) *The localization V_P of V at P is free of rank n over $\mathbb{I}_P$.*

This condition is open, i.e., it is satisfied by all P's in a non-empty Zariski-open subset of $Spec(\mathbb{I})$. Under (Fr_P), we have the residual representation $V(\varphi_P)$ given by the image of V in V_P/PV_P, which is a continuous Galois representation into $GL_n(\mathbb{C})$. We call P *motivic* if φ_P is associated to a rank n pure motive M_P defined over F with coefficients in a number field E_P via the inclusion $i_p : \overline{\mathbb{Q}} \to \Omega$. We suppose arithmeticity:

(A1) *Motivic points are densely populated in $Spec(\mathbb{I})(K)$.*

We then consider the following condition milder than $(\mathrm{A}2_\pm)$:

$(\mathrm{A}'2_\pm)$ *For a dense subset of motivic points P, we have*

(i) *The Tate twist $M_P(1)$ is critical in the sense of Deligne [15], and $H_\mathfrak{p}(M_P)$ is of Hodge-Tate type at every $\mathfrak{P} \in S_F$;*

(ii) *For each p-adic place $\mathfrak{P}$ induced by $i_p\sigma$ of F, there exists an $\mathbb{I}$-submodule $V_\mathfrak{P}^\mp \subset V$ stable under $\mathcal{G}_{F_\mathfrak{P}}$ such that the comparison isomorphism $I_{\sigma,HT}$ induces: $V_\mathfrak{P}^\pm \otimes_{\mathbb{I},P} B_{HT} \cong \mathscr{F}^\pm(M_P) \otimes_{F\otimes E, i_p\sigma\otimes i_p} B_{HT}$.*

We may assume that $V_\mathfrak{P}^\pm \otimes_\mathbb{I} \mathbb{K} \bigcap V = V_\mathfrak{P}^\pm$ and $V_\mathfrak{P}^\pm$ is reflexive. Suppose $(\mathrm{A}'2_\pm)$ and

$(\mathrm{NV}_\pm)$ *There exists an embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ such that $c^\pm_{p,i_\infty}(\varphi_P) \neq 0$ for at least one motivic P as in $(A'2_\pm)$.*

We want to prove $c^+_{p,i_\infty}(\varphi_P) \neq 0$ for densely populated P satisfying $(\mathrm{A}'2_+)$. By replacing φ by $\mathrm{Ind}_{\mathcal{G}_F}^{\mathcal{G}_\mathbb{Q}} \varphi$ (and M by $\mathrm{Res}_{F/\mathbb{Q}} M$), we may assume that $F = \mathbb{Q}$, because the condition $(\mathrm{A}'2_\pm)$ is insensitive to induction process as shown in Section 3.2 of Chapter 3. Writing $V[c \pm 1]$ for the $\mp$-eigenspace of c of the complex conjugation induced by i_∞, we have

$$V \otimes_\mathbb{Z} \mathbb{Q} = (V[c-1] \otimes_\mathbb{Z} \mathbb{Q}) \bigoplus (V[c+1] \otimes_\mathbb{Z} \mathbb{Q}).$$

Thus we have a natural linear map $I_1 : V[c-1] \otimes_\mathbb{Z} \mathbb{Q} \to (V/V_p^-) \otimes_\mathbb{Z} \mathbb{Q}$. We now localize at P. The condition (NV_+) implies that, after reducing modulo P, I_1 induces an isomorphism. Thus the cokernel $Coker(I_1)$ is a torsion $\mathbb{I} \otimes_\mathbb{Z} \mathbb{Q}$-module, and $V[c-1] \bigcap V_p^- = \{0\}$. Let $X = X(i_\infty)$ be the support of $Coker(I_1)$ in $Spec(\mathbb{I} \otimes_\mathbb{Z} \mathbb{Q})$, which is a proper Zariski-closed subset of $Spec(\mathbb{I} \otimes_\mathbb{Z} \mathbb{Q})$. Thus for critical motivic points P outside X, $c^+_{p,i_\infty}(\varphi_P) \neq 0$. Although we only discussed here about $c^+_{p,i_\infty}(\varphi_P)$, the same argument works well for $c^-_{p,i_\infty}(\varphi_P)$ under $(\mathrm{A}'2_-)$. Thus we find

Proposition 4.1.1 — *Suppose $(NV_\pm)$. Then $c^\pm_{p,i_\infty}(\varphi_P) \neq 0$ for densely populated motivic points P satisfying $(A'2_\pm)$ for a given $\{V_\mathfrak{P}^\mp\}_{P\in S_F}$.*

Now suppose that $c^\pm_{p,i_\infty\tau}(\varphi_Q) \neq 0$ for $\tau \in \mathcal{G}_\mathbb{Q}$ with another motivic point Q as in $(\mathrm{A}'2_\pm)$. We localize $\mathbb{I}$ into $\mathbb{I}'$ so that we have an open immersion $Spec(\mathbb{I}') \hookrightarrow Spec(\mathbb{I}) - Y$ for $Y = X(i_\infty) \bigcup X(i_\infty\tau)$. We write $V_\pm^\mathfrak{P}$ for $V/V_\mathfrak{P}^\mp$. Then we have a diagram:

$$\begin{array}{ccc} V[c\mp 1] \otimes_\mathbb{I} \mathbb{I}' & \cong & V_\pm^\mathfrak{P} \otimes_\mathbb{I} \mathbb{I}' \\ {\scriptstyle \varphi(\tau)^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\pm(\tau)} \\ V[\tau^{-1}c\tau \mp 1] \otimes_\mathbb{I} \mathbb{I}' & \cong & V_\pm^\mathfrak{P} \otimes_\mathbb{I} \mathbb{I}', \end{array}$$

where $\Phi_\pm(\tau)$ is defined so that the diagram becomes commutative. Then the following diagram for each motivic point P outside Y satisfying (A'2$_+$) is commutative:

$$\begin{array}{ccccc} H_B(M_P)\otimes_{E,i_p} B_{HT} & \xrightarrow{I_{i_\infty}} & H_{\mathfrak{p}}(M_P)[c\mp 1] & \longrightarrow & V_\pm^{\mathfrak{P}}\otimes_{\mathbb{I},P} B_{HT} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi(\tau)^{-1}(P)} & & \downarrow{\scriptstyle \Phi_\pm(\tau)(P)} \\ H_B(M_P)\otimes_{E,i_p} B_{HT} & \xrightarrow[I_{i_\infty\tau}]{} & H_{\mathfrak{p}}(M_P)[c_\tau\mp 1] & \longrightarrow & V_\pm^{\mathfrak{P}}\otimes_{\mathbb{I},P} B_{HT}, \end{array}$$

where $c_\tau = \tau^{-1}c\tau$. Note that $V_\pm^{\mathfrak{P}}\otimes_{\mathbb{I},P} B_{HT} \cong H^\pm_{DR}(M_P)\otimes_{F\otimes E, i_p\otimes i_p} B_{HT}$. Then we have $c^\pm_{p,i_\infty\tau}(\varphi_P) = c^\pm_{p,i_\infty}(\varphi_P)\beta_\pm(P)$ for $\beta_\pm(\tau) = det(\Phi_\pm(\tau)) \in \mathbb{I}' - \{0\}$. Here we regard β as a function on $\mathscr{X}_\mathbb{I} = Spec(\mathbb{I})$ by $\beta(P) = P(\beta)$. Writing the characteristic ideal of an $\mathbb{I}$-module M as $char_\mathbb{I}(M)$, $\beta_\pm(\tau) \in char_\mathbb{I}(Coker(I_{i_\infty}))^{-1}\mathbb{I}$ for all τ. Thus we have

Proposition 4.1.2 — *Suppose (A'2$_\pm$). Then if $c^\pm_{p,i_\infty}(\varphi_P)c^\pm_{p,i_\infty\tau}(\varphi_Q) \neq 0$ for two motivic points P and Q satisfying (A'2$_\pm$), then there exists an element $\beta_\pm = \beta_\pm(\tau) \in \mathbb{K}^\times$ such that $c^\pm_{p,i_\infty\tau}(\varphi_P) = c^\pm_{p,i_\infty}(\varphi_P)\beta_\pm(P)$ for densely populated motivic P satisfying (A'2$_\pm$). Moreover the $\mathbb{I}$-module generated by $\beta_\pm(\tau)$ for all $\tau \in \mathscr{G}_\mathbb{Q}$ is a fractional $\mathbb{I}$-ideal of $\mathbb{K}$.*

4.2 Genuine p-adic L-functions

Let φ be an arithmetic $\mathbb{I}$-adic Galois representation satisfying (Cont), (A1,3), (A'2$_-$), (NV$_-$) and (unr). We also write $\mathscr{X}_\mathbb{I}$ for $Spec(\mathbb{I})$. Thus for an $\mathbb{C}$-algebra A, $\mathscr{X}_\mathbb{I}(A) = \mathrm{Hom}_{\mathbb{C}\text{-}alg}(\mathbb{I}, A)$. We state here a refined version of a conjecture in [56] Conjecture II':

Conjecture 4.2.1 — *Under (Cont), (A1), (A'2$_-$), (A3), (NV$_-$) and (unr), there exists a genuine p-adic L-function for φ and $S = \{V_\mathfrak{P}^+\}_{P\in S_F}$, which is a meromorphic function $L_{S,i_\infty}(P,\varphi)$ defined on $\mathscr{X}_\mathbb{I}(\Omega)$ such that for all motivic points P satisfying (A'2$_-$) for S, if $c^+_{p,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1}) \neq 0$,*

(i) $$\frac{L_{S,i_\infty}(P,\varphi)}{c^+_{p,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1})} \in i_p(\overline{\mathbb{Q}});$$

(ii) $$i_p^{-1}\left(\frac{L_{S,i_\infty}(P,\varphi)}{c^+_{p,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1})}\right) = *\mathscr{E}(P)i_\infty^{-1}\left(\frac{L_{i_\infty}(1,\varphi_P)}{c^+_{\infty,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1})}\right),$$
where $\mathcal{N}$ is the cyclotomic character such that $\mathcal{N}(\phi_\mathfrak{l}) = N(\mathfrak{l})$;

(iii) *If $M_P = M\otimes\chi$ for a finite order character χ factoring through the cyclotomic $\mathbb{Z}_p$-extension of F for a motive M crystalline at p in the sense of [2], we have the following description of "$*$" and $\mathscr{E}(P)$: Replacing M and*

M_P by their restriction of scalar to $\mathbb{Q}$, we may assume that $F = \mathbb{Q}$. Then

$$\mathscr{E}(P) = \mathscr{E}(M \otimes \chi, 1) \quad \textit{for } \mathscr{E} \textit{ in (E) in Section 3.5 in Chapter 3; and}$$
$$* = \gamma(M)\varepsilon(M \otimes \chi)^{-1} \quad \textit{for } \gamma \textit{ in } (\Gamma) \textit{ in Section 3.5 of Chapter 3,}$$

where $\varepsilon(M)$ is the constant term of the functional equation of $L_{i_\infty}(s, M)$.

Here are several remarks:

(a) In Conjecture 4.2.1 (iii), the formula of the constant "$*$" looks a bit different from [56] Section 6 because our evaluation point is "1" in place of "0". By the functional equation, the value $L(1, M \otimes \chi)$ corresponds to $L(0, \check{M} \otimes \chi^{-1})$. If one applies the formula of Coates and Perrin-Riou to this dual $\check{M} \otimes \chi^{-1}$, we get the same value as in [56] Section 6. This shift of evaluation point comes from the fact that these authors use homological formulation (and the arithmetic Frobenius) instead of the cohomological one (and the geometric Frobenius) used here. Thus the Selmer group corresponding to our p-adic L is actually the one associated to $\check{\varphi}$ defined in [28]. If crystalline motivic points P of a given type is dense in $Spec(\mathbb{I})$, the condition (iii) uniquely determines the p-adic L-function, and the restriction of our p-adic L to the cyclotomic deformation of M_P should be closely related to the p-adic L-functions of M_P conjectured in [56] and [58] (see Section 4.3). The density of crystalline motivic points seems to hold in many examples of modular Galois representations φ (see Chapters 6, 7 and 8). In the examples of p-adic L-functions in the modular case, the constant "$*$" is determined completely for all motivic critical points of given type, and the expression of "$*$" confirms the formula (iii) in the conjecture for crystalline P.

(b) In the conjecture, we mean by a meromorphic function an element in the field of fractions of $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$.

(c) The identity (ii) is independent of the choice of basis of $H^+_{DR}(M_P(1))$ and $H^+_B({}^{i_\infty\sigma}M_P(1))$ because c^+_p and c^+_∞ behave exactly in the same way under the change of basis (see (BC) in Section 3.1 of Chapter 3). If we change i_∞ by $i_\infty\tau$ with non-vanishing periods, we know the existence of $\beta_- \in \mathbb{K}^\times$ such that $c^+_{p,i_\infty\tau}(\varphi_P \otimes \mathcal{N}^{-1}) = \beta_-(P)c^+_{p,i_\infty}(\varphi_P \otimes \mathcal{N}^{-1})$. Thus if the genuine p-adic L-function $L_{S,i_\infty}(\varphi)$ exists for i_∞, then it is easy to deduce from the conjecture of Deligne (D) (in Section 3.1 of Chapter 3) that $\beta_-(\tau)L_{S,i_\infty}(\varphi)$ satisfies the properties (i) and (ii) for $i_\infty\tau$ in place of i_∞. Thus the existence of $L_{S,i_\infty}(\varphi)$ is independent of the choice of i_∞.

4.3 Normalized p-adic L-functions

We assume $p > 2$ and the following conditions for an arithmetic φ:

(Fr$_{\mathbb{I}}$) *φ has values in $GL_n(\mathbb{I})$;*

(FR$_{\pm}$) *$V_{\mathfrak{P}}^{\pm}$ is an $\mathbb{I}$-direct summand of the space $V = V(\varphi) \cong \mathbb{I}^n$ for all $\mathfrak{P}$;*

(SD) *There is a Galois character $\chi : \mathcal{G}_F \to \mathbb{I}^\times$ such that $\check{\varphi} \otimes \chi \cong \varphi$ over $\mathbb{K}$.*

By self-duality (SD), the character χ is arithmetic, and the isomorphism takes $V_{\mathfrak{P}}^{\pm}(\check{\varphi} \otimes \chi) \otimes \mathbb{K}$ into $V_{\mathfrak{P}}^{\pm}(\varphi) \otimes \mathbb{K}$. Since each pure motive has a polarization, the condition (SD) should not be too restrictive. Any 2-dimensional representations and their tensor products satisfy (SD). We thus have an inner product $\langle\ ,\ \rangle$ on $V(\varphi)$ such that

$$\langle \varphi(\sigma)x, \varphi(\sigma)y\rangle = \chi(\sigma)\langle x, y\rangle.$$

We define the normalizing factor $\alpha_{\pm}(\varphi) \in \mathbb{I}$. Let $\check{V}$ be the dual of V in $V \otimes \mathbb{K}$ under the pairing. Then $\check{V}$ is Galois stable and belongs to isomorphism class of $\check{\varphi} \otimes \chi$ over $\mathbb{I}$. Although it is more legitimate to write $V((\check{\varphi} \otimes \chi)_P)$ for $\check{V}/P\check{V}$, we write it as $\check{V}(\varphi_P)$. If the place $\mathfrak{P}$ is induced by $i_p\sigma$, we write $\check{V}_{\sigma}^{\pm}$ for $\check{V}_{\mathfrak{P}}^{\pm}$. The $\mathbb{I}$-freeness of $W_{\pm} = \check{V}_{\sigma}^{\pm} \bigoplus \check{V}_{c\sigma}^{\pm} + (\check{V} \bigoplus \check{V})[c_\sigma \pm 1]$ for complex $i_\infty\sigma$ and $W_{\pm} = \check{V}_{\sigma}^{\pm} + \check{V}[c_\sigma \pm 1]$ for real $i_\infty\sigma$ follows from $(NV_{\pm})$, and we find an $\mathbb{I}$-linear endomorphism $A_{\sigma}^{\pm}$ such that $A_{\sigma}^{\pm}\check{V} = W_{\pm}$ or $A_{\sigma}^{\pm}(\check{V} \bigoplus \check{V}) = W_{\pm}$ according as $i_\infty\sigma$ is real or complex. We put $\alpha_{\pm} = a_{\pm}(\varphi) = \prod_{\sigma \in \Sigma} \alpha_{\sigma}^{\pm}$ for $\alpha_{\sigma}^{\pm} = det(A_{\sigma}^{\pm})$ computed with respect to a basis of $\check{V}$ or $\check{V} \bigoplus \check{V}$. By definition, $\alpha_{\pm}(\varphi)$, up to $\mathbb{I}$-units, only depends on the isomorphism class of $\check{\varphi}$ and hence depends only on the isomorphism class of φ, but not on the choice of the pairing. We define

(NL) $L_S(\varphi) = \alpha_+(\varphi)^{-1} L_{S,i_\infty}(\varphi) = \alpha_+(\varphi)^{-1} L_{S,i_\infty}(\check{\varphi} \otimes \chi)$,

where the last equality follows from the fact that $\varphi \cong \check{\varphi} \otimes \chi$ over $\mathbb{K}$. Since $\check{V}_{\mathfrak{P}}^{-}(\varphi \otimes \mathcal{N}^{-1}) = \check{V}_{\mathfrak{P}}^{+}(\varphi)$ and $\check{V}/\check{W}_{+} \cong (\check{V}/\check{V}_{\mathfrak{P}}^{+})/\check{V}[c_\sigma + 1]$, it is clear from the argument proving Proposition 4.1.2 that the ideal $(L_S(\varphi))$ of $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$ depends only on isomorphism class of φ but not on the choice of i_∞. We will give another proof of this fact later.

We want to relate now the values of the normalized L-function $L_S(\varphi)$ with the complex L-values. For notational simplicity, we assume that $F = \mathbb{Q}$ for this, and we leave the reader the task of working out the formula for general F. For an arithmetic point P, classically the complex period $c_{\infty,i_\infty}^{\pm}(M_P)$ is normalized choosing a suitable lattice $L_?$ of various realizations $H_?(M_P)$. Since the L-function would describe the Selmer group of $\check{\varphi}$ (see [28]), it would be natural to take $\check{V}/P\check{V}$ as $L_{\mathfrak{p}}$ of $H_{\mathfrak{p}}(M_P)$. As for L_B, writing $\mathcal{V}$ for the valuation ring of the place i_p of E, we would choose an $\mathcal{V}$-free lattice L_B of $H_B(M_P)$ so that I_{i_∞} induces an isomorphism:

$$(\infty) \qquad \begin{array}{ccc} I_{i_\infty} : H_B(M) \otimes_E E_{\mathfrak{p}} & \cong & H_{\mathfrak{p}}(M) \\ \bigcup & & \bigcup \\ L_B \otimes_{\mathcal{V}} \mathcal{O} & \cong & L_{\mathfrak{p}}. \end{array}$$

Since Faltings' comparison map preserves well the integral structure on $H_{\mathfrak{p}}(M)$ and $H_{DR}(M)$ when M is associated to a smooth projective variety over $\mathbb{Q}$ with a smooth model over $\mathbb{Z}_p$ ([20] Theorem 5.3 and [21] (c) in the introduction), it seems possible to impose a condition similar to (∞) to normalize L_{DR} under certain assumption on M. However we do not specify here how to normalize the $\mathcal{V}$-free lattice L_{DR} in $H_{DR}(M)$, because of the following reasons:

(i) the choice of L_{DR} does not really affect the final expression (see Proposition 4.3.1),

(ii) the normalization of L_{DR} could be delicate depending on M especially if M is not the total cohomology of a smooth variety over $\mathbb{Z}_p$, and

(iii) it would add further complication to the text. Instead we will give later an example, in the case where M is associated to an elliptic curve, to clarify the point.

Since the Tate motive $\mathbb{Q}(1)$ has canonical integral structure $\mathbb{Z}(1)$ whose p-adic period is 1 and complex period is $2\pi i$. By tensoring $\mathbb{Z}(1)^{\otimes m}$, we can shift $L_?$ to the integral structure $L_?(m)$ in $H_?(M(m))$. We then define the complex period $\Omega_\infty^\pm(\varphi_P \otimes \mathcal{N}^{-m})$ (resp. the p-adic period $\Omega_p^\pm(\varphi_P \otimes \mathcal{N}^{-m})$) by the i_∞-component of $det(I_{\mathbb{C}})$ (resp. the i_p-component of $det(I_{i_p})$) computed with respect to the basis of $L_B(m)[c \mp 1]$ (resp. $L_{\mathfrak{p}}(m)/(\check{V}_p^\mp(\varphi_P \otimes \mathcal{N}^{-m}))$ and $L_{DR}(m)/(L_{DR}(m) \bigcap \mathcal{F}^\mp(M_P(m)))$. Now we have

Proposition 4.3.1 — *Let the notation be as above. We assume (SD), (Fr), (FR_+) and (NV_+). Then we have*

$$\frac{L_S(P,\varphi)}{\Omega_p^+(\varphi_P \otimes \mathcal{N}^{-1})} = *i_p i_\infty^{-1}\left(\frac{L_{i_\infty}(1,\varphi_P)}{\Omega_\infty^+(\varphi_P \otimes \mathcal{N}^{-1})}\right)$$

up to p-adic units.

Proof. Note that $\check{V}_p^-(\varphi \otimes \mathcal{N}^{-1}) = \check{V}_p^+(\varphi)$ and $\check{V}(\varphi \otimes \mathcal{N}^{-1})[c-1] = \check{V}(\varphi)[c+1]$. Thus $\alpha_-(\varphi \otimes \mathcal{N}^{-1}) = \alpha_+(\varphi)$. Then, by (∞), $\alpha_+(P) = det(I_{i_\infty} : L_B[c+1] \to L_{\mathfrak{p}}/(\check{V}_p^+/P\check{V}_p^+))$ up to p-adic units. Then we know that $c_{p,i_\infty}^+(\varphi_P \otimes \mathcal{N}^{-1})$, computed with respect to $L_{DR}(1)$ and $L_B(1)$, is equal to $\alpha_+(P)\Omega_p^+(\varphi_P \otimes \mathcal{N}^{-1}))$ up to a p-adic unit. This combined with the interpolation formula (Int) in Chapter 1 for $L_{S,i_\infty}(\varphi)$ yields the result. □

Since the periods $\Omega_p^\pm$ does not depends on i_∞, Proposition 4.3.1 shows the independence of the ideal $(L_S(\varphi))$ from i_∞. Further, if L_{DR} is chosen so that $\Omega_p^+(\varphi_P \otimes \mathcal{N}^{-m}))$ is a p-adic unit, then $L_S(P,\varphi)$ is equal to the normalized complex L-value $*\frac{L_{i_\infty}(1,\varphi_P)}{\Omega_\infty^+(\varphi_P \otimes \mathcal{N}^{-1})}$ up to units. Suppose that Q is another motivic point such that $\varphi_Q = \varphi_P \otimes \psi$ for a character ψ. Then $\psi(-1) = 1$. Since we have a natural

integral structure on the motive $M(\psi)$ of ψ, similarly to the case where ψ is the cyclotomic character, we can shift the integral structure given by $L_?$ or even a basis of $L_?$ on M_P to M_Q, which again satisfies (∞). Thus, we have well defined period $\Omega_\infty^+(\varphi_P \otimes \psi\mathcal{N}^{-1})$ of M_Q for arithmetic points Q (with $\varphi_Q = \varphi_P \otimes \psi$) on the cyclotomic deformation locus in $Spec(\mathbb{I})$. On this locus,

$$\frac{L_S(\varphi)}{\Omega_p^+(\varphi_P \otimes \mathcal{N}^{-1})} \quad \text{interpolates} \quad * i_p i_\infty^{-1}\left(\frac{L_{i_\infty}(1, \varphi_P \otimes \psi)}{\Omega_\infty^+(\varphi_P \otimes \psi\mathcal{N}^{-1})}\right) \quad \text{for variable } \psi.$$

This argument combined with (iii) of Conjecture 4.2.1 shows that the restriction of $\frac{L_S(\varphi)}{\Omega_p^+(\varphi_P \otimes \mathcal{N}^{-1})}$ to the cyclotomic line passing through P is equal to the cyclotomic p-adic L-function interpolating the complex L-values normalized with respect to $\check{V}$.

We extend the definition of the normalized period $\Omega_p^\pm$ for a general base field F. We put $L_\mathfrak{p} = \check{V}/P\check{V}$ and for each σ, we choose an $\mathfrak{r}_E$-free lattice $L_{B,\sigma}$ of $H_B({}^\sigma M)$ so that the comparison isomorphism induces $L_{B,\sigma} \otimes \mathbb{C} \cong L_\mathfrak{p}$. We also choose an $\mathfrak{r}_F \otimes_\mathbb{Z} \mathfrak{r}_E$-free lattice L_{DR} of $H_{DR}(M)$. Then we define $\Omega_{p,\sigma}^\pm(\varphi_P \otimes \mathcal{N}^{-m})$ by the determinant of I_{i_p} from $L_\mathfrak{p}(m)/(L_\mathfrak{p}(m) \bigcap \check{V}_\sigma^\mp(\varphi_P \otimes \mathcal{N}^{-m})))$ into $L_{DR}(m)/(L_{DR}(m) \bigcap \mathcal{F}^\mp(M_P(m)))$. Then we have

$$\text{(NP)} \qquad c_{p,\sigma,i_\infty}^\pm(\varphi_P \otimes \mathcal{N}^{-m}) = \alpha_\sigma^{\pm(-1)^{m+1}}(P)\Omega_{p,\sigma}^\pm(\varphi_P \otimes \mathcal{N}^{-m}) \quad \text{if } i_\infty\sigma \text{ is real,}$$
$$c_{p,\sigma,i_\infty}^\pm(\varphi_P \otimes \mathcal{N}^{-m}) = \alpha_\sigma^{\pm(-1)^{m+1}}(P)\Omega_{p,\sigma}^\pm(\varphi_P \otimes \mathcal{N}^{-m})\Omega_{p,c\sigma}^\pm(\varphi_P \otimes \mathcal{N}^{-m}) \quad \text{otherwise,}$$

where the left-hand side is also computed with respect to $L_B(m)$ and $L_{DR}(m)$.

In formulating a main conjecture in the Iwasawa theory relating the characteristic ideal of the Selmer group of $\check{\varphi}$ with an analytic p-adic L-function, I expect the L-function $L_S(\varphi)$ to play an important role, although we might have to further modify the p-adic L-function in residually reducible cases (that is, the case where $\varphi \mod \mathfrak{m}_\mathbb{I}$ is not absolutely irreducible). When $\check{V}[c+1] \cong \check{V}/\check{V}_p^+$ (that is, α_+ is a unit), the ideal $(L_{S,i_\infty}(\varphi))$ and $(L_S(\varphi))$ coincide, we do not need to modify $L_{S,i_\infty}(\varphi)$ and have the precise interpolation property (Int). Thus it is natural to ask when $V[c \pm 1] \cong V/V_p^\pm$ holds. Here we list an obviously sufficient condition in the simplest case where $F = \mathbb{Q}$ and $n = 2$: Write $\overline{\varphi}$ for the representation on $V(\overline{\varphi}) = V(\varphi)/\mathfrak{m}_\mathbb{I} V(\varphi)$ induced by φ for the maximal ideal $\mathfrak{m}_\mathbb{I}$ of $\mathbb{I}$. We assume that

(Sp) $V = V_p^+ \bigoplus V'$ *for a complement* $V' \neq 0$, *and* $V = V[c-1] \bigoplus V[c+1]$.

We leave the proof of the following fact to the reader.

Proposition 4.3.2 — *Suppose* $n = 2$, $F = \mathbb{Q}$ *and (Sp). If* $\overline{\varphi}$ *is irreducible, then there exists a complex embedding* i_∞ *such that* $\check{V}[c \pm 1] \cong \check{V}/\check{V}_p^\pm$.

We now assume that $M = M_P$ is associated with a motive given by H^1 of an elliptic curve $\mathscr{E}$ defined over $\mathbb{Q}$ with ordinary good reduction at p. We use the same

symbol $\mathscr{E}_{/\mathbb{Z}}$ for the Néron model of $\mathscr{E}$. Thus $E = \mathbb{Q}$ and $\mathfrak{p} = p$. We assume the lattices $L_?(m)$ of $H_?(M(m))$ to be equal to $H^1_{et}(\mathscr{E}, \mathbb{Z}_p)$, $H^1_B(\mathscr{E}, \mathbb{Z})$ and $H^1_{DR}(\mathscr{E}, \mathbb{Z})$. Then these lattices satisfy (∞), and we assume $L_p = \check{V}/P\check{V}$, which is actually self dual. Then we have the following five related exact sequences:

(E1) $$0 \to \mu_{p^\infty} \to \mathscr{E}[p^\infty] \to p^{-\infty}\mathbb{Z}/\mathbb{Z} \to 0;$$

(E2) $$0 \to \mathbb{Z}_p \to H^1_{et}(\mathscr{E}, \mathbb{Z}_p) \to \mathbb{Z}_p(-1) \to 0;$$

(E3) $$0 \to H^0(\mathscr{E}, \Omega_{\mathscr{E}/\mathbb{Z}_p}) \to H^1_{DR}(\mathscr{E}, \mathbb{Z}_p) \to H^1(\mathscr{E}, \mathcal{O}_{\mathscr{E}/\mathbb{Z}_p}) \to 0,$$

(E4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & L_p(1) & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \| \wr & & \| \wr & & \| \wr & & \\ 0 & \longrightarrow & T_p(\mu_{p^\infty}) & \longrightarrow & T_p(\mathscr{E}) & \longrightarrow & T_p(p^{-\infty}\mathbb{Z}/\mathbb{Z}) & \longrightarrow & 0, \end{array}$$

where $\mathscr{E}[p^\infty]$ is the p-divisible group of $\mathscr{E}$, $T_p(?)$ indicate the p-adic Tate module of the object ? and the last diagram is commutative. Note that $T_p(\mathscr{E}) = \check{L}_p$ for the dual $\check{L}_p$ of L_p. This implies the complex period and p-adic period $c^\pm_*(M(1))$ can be computed by taking the generator ω of $\mathscr{F}^\pm(M(1)^\vee) = \mathscr{F}^\pm(M)$ over $\mathbb{Q}$ and choosing a generator $b^\pm$ of $H^\pm_B(\check{M}) = H^\pm_B(M(1))$, and $c^\pm_*(M(1)) = \langle b^\pm, \omega \rangle$ for the duality pairing $\langle\ ,\ \rangle$ between $M(1)^\vee$ and $M(1)$. We can normalize ω taking the Néron differential on the Néron model $\mathscr{E}_{/\mathbb{Z}}$ of $\mathscr{E}$ over $\mathbb{Z}$, that is, $H^0(\mathscr{E}_{/\mathbb{Z}}, \Omega_{\mathscr{E}/\mathbb{Z}}) = \mathbb{Z}\omega$, because $\mathscr{F}^\pm(M) \supset H^0(\mathscr{E}, \Omega_{E/\mathbb{Q}})$. We normalize $b^\pm$ taking a generator of $L_B(1)[c \mp 1]$. Under this choice, $c^\pm_\infty(M(1))$ is the canonical period $\Omega_\pm(\mathscr{E})$ of $\mathscr{E}$. Then $\Omega^+_p(\mathscr{E}) = det(I_{i_p}) = \langle e, \omega \rangle$ for a generator e of $\mathbb{Z}_p = L_p(1)/\mathbb{Z}_p(1)$ in (E4). By [20] Theorem 5.3, if $p > 2$, $\Omega^+_p(\mathscr{E})$ is a unit in $\mathcal{O}_\Omega$. Writing $I_{i_\infty}(b^-) = u(\mathscr{E})e$, we have $u(\mathscr{E}) = \alpha_+(P)$, and we see from (Int) that

$$(*) \qquad i_p^{-1}\left(\frac{L_{S,i_\infty}(P,\varphi)}{u(E)\Omega^+_p(\mathscr{E})}\right) = *i_\infty^{-1}\left(\frac{L_{i_\infty}(1,\varphi_P)}{\Omega_+(\mathscr{E})}\right) \quad \text{and}$$
$$i_p^{-1}\left(\frac{L_S(P,\varphi)}{\Omega^+_p(\mathscr{E})}\right) = *i_\infty^{-1}\left(\frac{L_{i_\infty}(1,\varphi_P)}{\Omega_+(\mathscr{E})}\right).$$

We now show that we can always choose in the isogeny class of $\mathscr{E}_{/\mathbb{Z}}$ an elliptic curve which satisfies $L_p(1)[c-1] \cong L_p(1)/\mathbb{Z}_p(1)$ for some i_∞ (that is, $u(\mathscr{E})$ is a unit). We may assume that we have chosen i_∞ such that the p-adic valuation of $u(\mathscr{E})$ is minimal. If $u(\mathscr{E})$ is a unit, there is nothing to prove. Suppose not, and write $I_{i_\infty\sigma}(b^-) = u_\sigma e$. Thus $u_\sigma \equiv 0 \mod p$ for all σ, and $\varphi(\sigma)V(\overline{\varphi})[c-1]$ coincides with $V_p^-/\mathfrak{m}_\mathbb{I}V_p^-$ for all σ. Thus $V(\overline{\varphi})[c-1]$ is stable under the global Galois group. We take maximal subgroup C of $\mathscr{E}[p^\infty][c-1]$ stable under the global Galois action. Since the Galois action on the Tate module is irreducible, C is a finite group. Then for $\mathscr{E}' = \mathscr{E}/C$, $\mathscr{E}'[p^\infty][c-1]$ does not contain subgroup stable under the global Galois action. This implies that $\mathscr{E}'$ is what we wanted. Thus $u(\mathscr{E}')$ is a unit, and $\Omega^+_p(\mathscr{E}')$ is also a unit as already seen.

4.4 Singularity

We try to locate singularity of the normalized $L_S(\varphi)$. We continue to assume $(\mathrm{Fr}_{\mathbb{I}})$, $(\mathrm{FR}_{\pm})$ and $p > 2$. A principle related to the conjectures of Artin and Tate is

(AT_∞) *the complex L-function* $\Gamma(s,\varphi_P)L(s,\varphi_P) = \Gamma(s,\varphi_P)L(0,\varphi_P \otimes \mathcal{N}^s)$ *has a pole of order* r *at* s *implies that* $s \in \mathbb{Z}$ *and* $\varphi_P \otimes \mathcal{N}^s \bigoplus (\varphi_P \otimes \mathcal{N}^s)^*$ *contains the trivial Galois character with multiplicity* r.

Thus we ask

Question 4.4.1 — *If there exists a p-adic L-function* $L_{S,i_\infty}(\varphi)$ *satisfying the requirements of Conjecture 4.2.1, can one expect the following principle for the normalized* $L_S(\varphi)$*? For each prime divisor* $\mathbf{P}$ *of Spec*$(\mathbb{I})$ *and* $\varphi^* = \check{\varphi} \otimes \mathcal{N}$,

(AT_p) $L_S(\varphi)$ has a pole at $\mathbf{P} \Rightarrow \varphi_{\mathbf{P}} \bigoplus \varphi^*_{\mathbf{P}}$ has the trivial representation as a subquotient.

One might be tempted to expect, after the analysis of the Selmer group of φ done by Greenberg [28] Section 4, a more stronger assertion

(G) *the order of the pole at each prime divisor* $\mathbf{P}$ *is equal to*

$$\ell_{\mathbf{P}}(\varphi) = \ell_{\mathbf{P}}(H_0(\mathcal{G}_F, \varphi \bigoplus \varphi^*)).$$

Here $\ell_{\mathbf{P}}(X)$ is the length of the localization at $\mathbf{P}$ of a torsion $\mathbb{I}$-module X. We say that $L_S(\varphi)$ satisfies (AT_p) in strict sense if (G) holds for all prime divisors $\mathbf{P}$ of $\mathscr{X}_{\mathbb{I}}$. When φ is residually irreducible (i.e. $\varphi \mod \mathfrak{m}_{\mathbb{I}}$ is absolutely irreducible) with dense critical crystalline points of type S, and φ contains all cyclotomic deformation of its specialization, we conjecture that $L_S(\varphi)$ satisfies (G) (or equivalently, $L_S(\varphi) \in \mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$). However, either in the residually reducible case or in the case where φ does not contain all cyclotomic deformation, this seems more subtle on the automorphic side: The occurrence of the trivial representation in $\varphi_{\mathbf{P}}$ (or $\varphi^*_{\mathbf{P}}$) does not suffice to ensure the existence of a pole at $\mathbf{P}$. For example, if we take a locus X in $\mathscr{X}_{\mathbb{I}}(\Omega)$ on which $V(\varphi_{\mathbf{P}})^{I(\mathfrak{P})} \neq \{0\}$, the L-function L_X interpolating the L-values on X has generally more Euler factors at p than the one $L_{\mathscr{X}}$ interpolating L-values all over $\mathscr{X}_{\mathbb{I}}(\Omega)$. Thus L_X may have a pole at a prime divisor $\mathbf{P}$ on X but $L_{\mathscr{X}}$ restricted to X may not have a pole at $\mathbf{P}$, that is, the pole of L_X at $\mathbf{P}$ can be canceled by the extra-Euler factor removed from L_X. If we restrict $L_{\mathscr{X}}$ to another closed subscheme Y containing $\mathbf{P}$ as a prime divisor, but generically on Y, we have $V(\varphi_Y)^{I(\mathfrak{P})} = \{0\}$, then it would be reasonable to take $L_{\mathscr{X}}$ restricted to Y to be the L-function of φ_Y. Then the L-function does not have pole at $\mathbf{P}$ and has too less singularity at $\mathbf{P}$. Contrary to this, $\alpha_+(\varphi)$ can be non-unit if φ_P is not semi-simple for some $\mathbf{P}$ (see Section 4.7 and Section 8.5 of Chapter 8). Since $\alpha_+(\varphi)(P) = \alpha_+(\varphi)(Q)$ if $\varphi_P = \varphi_Q \otimes \psi$ for a character ψ, the divisor of $\alpha_+(\varphi)$ contains cyclotomic variation.

This implies that, for a prime divisor $\mathbf{P}$ in $div(\alpha_+(\varphi))$, $\varphi_{\mathbf{P}}$ and $\varphi^*_{\mathbf{P}}$ cannot contain the trivial representation. Thus in this case, if $L_{S,i_\infty}(\varphi)$ does not have zero at $\mathbf{P}$, which is likely in some cases (see Section 8.5 of Chapter 8), $L_S(\varphi)$ has too much singularity. Thus what we hope in general is:

(G') $\mathrm{ord}_{\mathbf{P}}(L_S(\varphi)) \le \ell_{\mathbf{P}}(\varphi) = \ell_{\mathbf{P}}(H_0(\mathcal{G}_F, \varphi \bigoplus \varphi^*))$

if $V(\varphi_{\mathbf{P}})^{I(\mathfrak{P})} = V(\varphi)^{I(\mathfrak{P})} \otimes_{\mathbb{I}} \mathbb{I}_{\mathbf{P}}/\mathbf{P}$ *and* $\alpha_+(\varphi)$ *is an* $\mathbb{I}_{\mathbf{P}}$*-unit,*

where $\mathrm{ord}_{\mathbf{P}}$ indicates the order of pole at $\mathbf{P}$. If $L_S(\varphi)$ is holomorphic at $\mathbf{P}$, we just put $\mathrm{ord}_{\mathbf{P}}(L_S(\varphi)) = 0$. We note that $\mathrm{ord}_{\mathbf{P}}(L_{S,i_\infty}(\varphi)) = \mathrm{ord}_{\mathbf{P}}(L_S(\varphi))$ if $\alpha_+(\varphi)$ is an $\mathbb{I}_{\mathbf{P}}$-unit.

4.5 μ-invariant of cyclotomic restrictions of $L_S(\varphi)$

We continue to keep the assumptions of Section 4.3 and write ν for the universal cyclotomic character, (that is, the universal deformation of the trivial character of $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ into $\mathbb{Z}_p[[\Gamma]]^\times$ for $\Gamma = 1 + p\mathbb{Z}_p$). Suppose that φ contains all the cyclotomic variation, that is, for each point P of $\mathscr{X}_{\mathbb{I}}$, there exists a closed irreducible subscheme $Spec(\mathbb{I}^P)$ of $Spec(\mathbb{I})$ such that $\mathbb{I}^P$ is a finite flat $\mathbb{Z}_p[[\Gamma]]$-algebra and $\pi_P\varphi \cong \varphi_P \otimes \nu$ for the projection $\pi_P : \mathbb{I} \to \mathbb{I}^P$. To support the idea that $L_S(\varphi)$ is related to the Selmer group of $\check{\varphi}$ (and $\varphi \otimes \mathcal{N}^{-1}$) defined in [28], we compute the variation of μ-invariant of $\pi_P(L_S(\varphi))$ over the isogeny class of φ for a fixed point P satisfying $(\mathrm{A2}_\pm)$. We let φ act on $V(\varphi) = \mathbb{I}^n$, and take a sublattice $V(\varphi')$ of $V(\varphi)$ stable under φ. We compare the μ-invariants of $\pi_P(L_S(\varphi))$ and $\pi_P(L_S(\varphi'))$. We suppose $(\mathrm{Fr}_{\mathbb{I}})$ and $(\mathrm{FR}_\pm)$ also for $V(\varphi')$. Since for any character $\xi : \mathcal{G}_F \to \mathbb{C}^\times$, $P \otimes \xi(\alpha_+(\varphi)) = P(\alpha_+(\varphi))$ up to unit in $\mathbb{C}$, we have, for the p-adic valuation v with $v(p) = 1$,

$$\mu(\pi_P(L_S(\varphi))) - \mu(\pi_P(L_S(\varphi'))) = v(\alpha_+(\varphi')(P)) - v(\alpha_+(\varphi)(P)).$$

For $\mathfrak{P}$ induced by $i_p\sigma$, we write

$$W^\pm_{\mathfrak{P}}(\varphi) = \begin{cases} \check{V}^\pm_{\mathfrak{P}}(\varphi) & i_\infty\sigma \text{ is real,} \\ \check{V}^\pm_{\mathfrak{P}}(\varphi) \bigoplus \check{V}^\pm_{\mathfrak{P}^{c_\sigma}}(\varphi) & i_\infty\sigma \text{ is complex,} \end{cases}$$

$$W(\varphi) = \begin{cases} \check{V}(\varphi) & i_\infty\sigma \text{ is real,} \\ \check{V}(\varphi) \bigoplus \check{V}(\varphi) & i_\infty\sigma \text{ is complex.} \end{cases}$$

Define $\mathbb{I}$-linear maps B, $C^\pm_{\mathfrak{P}}$, $D_{\pm,\sigma}$ and $D^\pm_{\mathfrak{P}}$ so that, for $\mathfrak{P}$ induced by $i_p\sigma$

$$B(W(\varphi')) = W(\varphi), \quad C^\pm_{\mathfrak{P}}(W^\pm_{\mathfrak{P}}(\varphi') + W(\varphi')[c_\sigma \pm 1]) = W^\pm_{\mathfrak{P}}(\varphi) + W(\varphi)[c_\sigma \pm 1],$$
$$D_{\pm,\sigma}(W(\varphi')[c_\sigma \pm 1]) = W(\varphi)[c_\sigma \pm 1], \quad D^\pm_{\mathfrak{P}}(W^\pm_{\mathfrak{P}}(\varphi')) = W^\pm_{\mathfrak{P}}(\varphi).$$

Put $\beta = det(B)$, $\gamma_{\mathfrak{P}}^{\pm} = det(C_{\mathfrak{P}}^{\pm})$, $\delta_{\pm,\sigma} = det(D_{\pm,\sigma})$ and $\delta_{\mathfrak{P}}^{\pm} = det(D_{\mathfrak{P}}^{\pm})$. Then $\beta = \delta_{+,\sigma}\delta_{-,\sigma}$, $\gamma_{\mathfrak{P}}^{\pm} = \delta_{\mathfrak{P}}^{\pm}\delta_{\pm,\sigma}$ up to units, and we have

$$\alpha_{\sigma}^{\pm}(\varphi)\alpha_{\sigma}^{\pm}(\varphi')^{-1} = \beta/\gamma_{\mathfrak{P}}^{\pm} = \delta_{\mp,\sigma}/\delta_{\mathfrak{P}}^{\pm},$$
$$\alpha_{\pm}(\varphi)\alpha_{\pm}(\varphi')^{-1} = \prod_{\sigma\in\Sigma}\delta_{\mp,\sigma} \times \prod_{\mathfrak{P}}(\delta_{\mathfrak{P}}^{\pm})^{-[F_{\mathfrak{P}}:\mathbb{Q}_p]}.$$

Then we get

$$(\mu 1)\quad \mu(\pi_P(L_S(\varphi'))) - \mu(\pi_P(L_S(\varphi))) = \sum_{\sigma\in\Sigma} v(\delta_{-,\sigma}(P)) - \sum_{\mathfrak{P}}[F_{\mathfrak{P}} : \mathbb{Q}_p]v(\delta_{\mathfrak{P}}^{+}(P)).$$

Writing C for $\check{V}(\varphi')/\check{V}(\varphi)$ and $\mathscr{F}_{\mathfrak{P}}^{+}C = \check{V}_{\mathfrak{P}}^{+}(\varphi')/\check{V}_{\mathfrak{P}}^{+}(\varphi)$, this is equivalent to, when $\mathbb{O} = \mathbb{Z}_p$,

$(\mu 2)$
$$\mu(\pi_P(L_S(\varphi'))) - \mu(\pi_P(L_S(\varphi))) = \sum_{\sigma\in\Sigma} v(\#(C(F_\sigma)) - \sum_{\mathfrak{P}}[F_{\mathfrak{P}} : \mathbb{Q}_p]v(\#(\mathscr{F}_{\mathfrak{P}}^{+}C)),$$

where F_σ is the completion of F at the archimedean place $i_\infty\sigma$ and

$$C(F_\sigma) = H_0(\mathrm{Gal}(\mathbb{C}/F_\sigma), C).$$

This matches well with the formula of Perrin-Riou [59] of the variation of the μ-invariant of the cyclotomic Selmer group of $\pi_P\check{\varphi}$.

4.6 Nearly ordinary $\mathbb{I}$-adic representations

Here we show that the $\mathbb{I}$-adic representation $\pi(\lambda)$ attached to a Λ-algebra homomorphism $\lambda : h^{n.ord}(Np^\infty;\mathbb{O}) \to \mathbb{I}$ satisfies (A1) and (A'$2_\pm$). We assume that $\mathbb{I}$ is finite and torsion-free over Λ (resp. Λ^{ord}) in the nearly ordinary case (resp. the ordinary case where λ factors through h^{ord}). Since the condition (Gal) in Section 2.8 of Chapter 2 determines the representation $\pi(\lambda)$ only over $\mathbb{K}$, we take a representation φ of $\mathscr{G}_F$ into $GL_2(\mathbb{I})$ such that $Tr(\varphi) = Tr(\pi(\lambda))$. If there is no confusion is likely, we sometimes write $\pi(\lambda)$ for φ. For example, if $\overline{\varphi}$ is absolutely irreducible, there is no ambiguity of φ [10]. By local class field theory, there is a canonical inclusion of groups of $F_p^\times = T_1(\mathbb{Q}_p)$ into $\prod_{\mathfrak{P}|p}\mathscr{G}_{F_{\mathfrak{P}}}^{ab}$ with dense image. Thus $n \in \mathbb{Z}[I] = X(T_1)$ gives rise to a character of $\mathscr{G}_{F_{\mathfrak{P}}}$. We write $V_{\mathfrak{P}}(n)$ for the rank one space over $\mathbb{O}$ on which $\mathscr{G}_{F_{\mathfrak{P}}}$ acts by a character which coincides locally with n on $I_{\mathfrak{P}}$, where $I_{\mathfrak{P}}$ is the inertia subgroup of $\mathfrak{P}$ in $\mathscr{G}_{F_{\mathfrak{P}}}$. Let P be a positive arithmetic point of $\mathscr{X}_{\mathbb{I}}(\Omega)$ of weight κ. By Blasius and Rogawski [6], it is known that $V(\varphi_P)\otimes K$ is the étale realization $H_{\mathfrak{p}}(M_P)$ of a rank two motive M_P defined over

F with coefficients in a number field E_P unless P falls in an exceptional case which is very limited (in the exceptional case, $m(\kappa) = n(\kappa) + t$ for $t = \sum_{\sigma \in I} \sigma$). Here E_P is a finite extension of the field $Q(\lambda_P)$ generated by $P(\lambda(\mathbb{T}(\mathfrak{l})))$ for primes $\mathfrak{l}$ outside Np. Then as explained in Section 2.8 of Chapter 2, we have an exact sequence of $\mathcal{G}_{F_\mathfrak{P}}$-modules: $0 \to V_\mathfrak{P}(n) \to V(\varphi_P) \to V_\mathfrak{P}(m) \to 0$, where $m = m(\kappa)$ and $n = n(\kappa)$. Since $m > n$, a theorem of Tate tells us that $V(\varphi_P) \otimes K$ is of Hodge-Tate type ([68] 3.3). This fact also follows from [21] and [6] combined. Anyway we conclude from [61] Chapter III, Appendix that the Hodge-Tate type of $V(\varphi_P)$ is (m, n) and (n, m), and $V_\mathfrak{P}(n)$ plays the role of $V_\mathfrak{P}^+ = V_\mathfrak{P}^-$ in $(\mathrm{Red}_\pm)$ for $V(\varphi_P)$. By (iv) of (Gal) in Section 2.8 of Chapter 2, $V_\mathfrak{P}(n)$ is lifted to the $\mathcal{G}_{F_\mathfrak{P}}$-submodule $V_\mathfrak{P}^+(\varphi)$ of $V(\varphi)$ which is written as $V_\mathfrak{P}^+$ in (Gal). Thus φ satisfies (A1) and $(\mathrm{A}'2_\pm)$, and any subquotient of tensor products of $\pi(\lambda_i)$ over $\mathbb{O}$ and $\mathbb{I}$ satisfies (A1) and $(\mathrm{A}'2_\pm)$ for a suitable choice of S.

Let $\delta_\mathfrak{P} : \mathcal{G}_{F_\mathfrak{P}} \to \mathbb{I}^\times$ be the character given in (iv) of (Gal) in Section 2.8 of Chapter 2 for $\varphi = \pi(\lambda)$, and define $\varepsilon_\mathfrak{P}$ by $\varepsilon_\mathfrak{P}\delta_\mathfrak{P} = det(\varphi)$. Suppose that λ is nearly ordinary. By (iv) in (Gal), the maximal closed subscheme X of $Spec(\mathbb{I})$ on which one of $\delta_\mathfrak{P}$ and $\varepsilon_\mathfrak{P}$ is trivial on $I_\mathfrak{P}$ for every $\mathfrak{P}$ is of codimension $[F : \mathbb{Q}]$ and is the union of the locus of P for which φ_P is ordinary or co-ordinary (that is, contragredient to an ordinary representation). Thus, if $[F : \mathbb{Q}] > 1$, φ_P cannot contain trivial representation as a subquotient for any prime divisor P in $Spec(\mathbb{I})$. Actually the Eisenstein component of the nearly ordinary Hecke algebra is of dimension $3 + 2d$ for the defect d of the Leopoldt conjecture for F and p. Thus the dimension of the locus of P for which φ_P or φ_P^* is reducible (over K) is less than $2 + 2d$. This gives a stronger fact that the reducible locus is of codimension at least $[F : \mathbb{Q}] - d$. When $F = \mathbb{Q}$, the ordinary locus is of codimension 1, which contains positive arithmetic point P such that φ_P is irreducible. Note that $\varphi' = det(\varphi)^{-1} \otimes \varphi$ is again modular, and by this operation, the co-ordinary locus of φ is sent to the ordinary locus of φ'. This shows the locus in X on which φ has the trivial representation as a subquotient is of codimension ≥ 2 in X. Thus even if $F = \mathbb{Q}$, $\varphi_\mathbf{P}$ cannot contain trivial representation as a subquotient for any prime divisor $\mathbf{P}$ in $Spec(\mathbb{I})$. Similarly, we can show that $\varphi_\mathbf{P}^*$ cannot contain trivial representation as a subquotient for any prime divisor $\mathbf{P}$ in $Spec(\mathbb{I})$. Thus $L_{S,i_\infty}(\varphi)$ and $L_S(\varphi)$ should be both holomorphic everywhere in the nearly ordinary case of $F \neq \mathbb{Q}$.

The ordinary case is more subtle. In this case, if $\varphi_\mathbf{P}^*$ contains the trivial representation, then $(\varepsilon_\mathfrak{P})_\mathbf{P} = \mathcal{N}$ for all $\mathfrak{P}$, because unramified $(\delta_\mathfrak{P})_\mathbf{P}$ cannot be equal to $\mathcal{N}$. Thus φ_P is of weight $k = 2$ ($\Leftrightarrow \kappa = (t, 0)$), which is arithmetic and irreducible. Thus $\varphi_\mathbf{P}^*$ never contains the trivial representation for prime divisors $\mathbf{P}$. On the other hand, $\varphi_\mathbf{P}$ may contain the trivial character. We can make the following conjecture:

Conjecture 4.6.1 — *Let λ be as above and $\varphi : \mathcal{G}_F \to GL_2(\mathbb{I})$ be a Galois representation such that $Tr(\varphi) = Tr(\pi(\lambda))$. Suppose $(FR_\pm)$ for φ.*

(i) (Nearly ordinary case) *If $\mathbb{I}$ gives an irreducible component of the normalization of the reduced part of $Spec(h^{n.ord}(Np^\infty;\mathbb{C}))$, then $L_S(\varphi)$ and $L_{S,i_\infty}(\varphi)$ for $S = \{V_{\mathfrak{P}}^+(\lambda)\}$ are holomorphic outside prime divisors $\mathbf{P}$ with non-semi-simple $\varphi_{\mathbf{P}}$.*

(ii) (Ordinary case) *Suppose that λ factors through the ordinary part, giving an irreducible component of $Spec(h^{ord}(Np^\infty;\mathbb{C}))$. Then $L_S(\varphi)$ and $L_{S,i_\infty}(\varphi)$ are holomorphic outside prime divisors $\mathbf{P}$ with reducible $\varphi_{\mathbf{P}}$.*

The statement (i) above tells us that the two L-functions are *holomorphic everywhere* if $F \neq \mathbb{Q}$ because we always expect $\varphi_{\mathbf{P}}$ to be irreducible in this case. We will prove (G) in Chapter 8 if φ in (ii) is given by an induced representation of an $\mathbb{I}$-adic Galois character, assuming the compatibility of comparison isomorphisms in [50].

4.7 Residually reducible case

Here by an example, we explain what happens when φ has non-semi-simple $\varphi_{\mathbf{P}}$. Suppose that $F = \mathbb{Q}$, $\pi(\lambda)$ is p-ordinary and unramified outside p, $\mathbb{I} = \Lambda^{ord}$ and for a prime divisor $P = (\pi)$ of Λ^{ord}, we have an exact sequence of $\mathcal{G}_F$-modules:

$$0 \to \chi \to \varphi_P \to \psi \to 0, \tag{Eis}$$

where χ and ψ are global Galois characters acting on the rank one space $V(\chi)$, and $V(\psi)$. By ordinarity, on $V_p^\pm$, the inertia group I_p acts trivially. Since $det(\varphi(c)) = -1$, $\chi(c) \neq \psi(c)$. Since χ and ψ are unramified outside p, one of them has to ramify at p, and the other is trivial by ordinarity of $\pi(\lambda)$. Thus in $\pi(\lambda)$, there are at least two isomorphism classes over $\mathbb{I}$, one φ having non-semi-simple (Eis) with χ trivial, the other φ' having non-semi-simple (Eis) with trivial ψ. First suppose that χ is trivial, then $\check{V}(\chi) = \check{V}(\varphi_P)[c-1] = \check{V}_p^\pm(\varphi_P)$ and hence α_+ in (NL) of Section 4.3 is prime to P, and α_- is divisible by π. Now we suppose that χ is ramified and ψ is trivial. Since χ ramifies in this case, $\check{V}(\varphi'_P) \cong \check{V}(\chi) \bigoplus \check{V}_p^\pm(\varphi'_P)$ as $\mathcal{G}_{\mathbb{Q}_p}$-modules, and $\check{V}(\varphi'_P)[c+1] = \check{V}(\chi)$ is isomorphic to $\check{V}(\varphi'_P)/\check{V}_p^\pm(\varphi'_P)$. Thus again α_+ is prime to P. In this case, if (Eis) is not split, α_- can be a unit for a suitable i_∞. Thus $\mathrm{ord}_P(L_S(\varphi)) = \mathrm{ord}_P(L_S(\varphi')) = \mathrm{ord}_P(L_{S,i_\infty}(\varphi))$. Note that $\ell_P(\varphi) = 0$ and $\ell_P(\varphi') = 1$. Thus for φ, $L_S(\varphi)$ satisfies (G) but $L_S(\varphi')$ does not. Thus the association $\phi \mapsto L_S(\phi)$ does not distinguish the difference of φ and φ'; in other words, the normalized p-adic L-functions are invariants of classes more rough than isomorphism classes (it is more strict than isogeny classes as we will give one more example of the reducible case in Section 8.5 of Chapter 8).

5. Periods of tensor products of motives

We express the periods of tensor products of rank 2 motives as monomials of periods of the components. We then study tensor products of rank 2 arithmetic Galois representations and specialize Conjecture 4.2.1 in Chapter 4 to this case. There are several choices of $S = \{V_{\mathfrak{P}}^+\}$, and we give a classification of p-adic L-functions according to S. We fix an embedding i_∞ and often drop the subscript i_∞ from the notation.

5.1 Tensor products of motives

Let E be a number field in $\overline{\mathbb{Q}}$. We assume

(CM_E) *E is either a CM field or a totally real field.*

Let M be a pure regular motive of rank n defined over F with coefficients in E. Since every pure motive with coefficients in $\mathbb{Q}$ has a polarization [18] 6.7, the coefficient field E compatible with the polarization satisfies (CM_E). We suppose

($\pm$) $\mathscr{F}^+(M) = \mathscr{F}^-(M) = \mathscr{F}(M)$ ($\Leftrightarrow H^{p,p}({}^\sigma M) = 0$ for all p and $\sigma \in I$).

This implies $n = 2m$. The field F contains the largest totally real subfield F_+. If F contains a CM field, then F contains the largest CM subfield F_{CM}. For any $E \otimes_{\mathbb{Q}} F$-module V and $\rho \in Aut(F/\mathbb{Q})$, we write V^ρ for the $E \otimes F$-module $V \otimes_{F,\rho} F$. We quote the following lemma from [3] and [74] Lemma 2.1 (see [42] Lemma 1.1 for a simple proof):

Lemma 5.1.1 — (o) *$H_{DR}(M)$ is $E \otimes F$-free of rank n;*

(i) *If F does not contain a CM field, then $\mathscr{F}(M)$ is $E \otimes F$-free of rank m;*

(ii) *Suppose that F contains a CM field. If the generator of* $\mathrm{Gal}(F_{CM}/F_+)$ *can be extended to $\rho \in Aut(F)$, then $\mathscr{F}(M) \oplus \mathscr{F}(M)^\rho$ is $E \otimes F$-free.*

We assume that $\mathrm{rank}(M) = 2$. We write A (resp. T) for the subset of $\sigma \in I$ such that $\mathscr{F}(M) \otimes_{F,\sigma} \overline{\mathbb{Q}} = \{0\}$ (resp. $\mathrm{rank}_{E \otimes \overline{\mathbb{Q}}}(\mathscr{F}(M) \otimes_{F,\sigma} \overline{\mathbb{Q}}) = 1$). Then $I = T \bigsqcup A \bigsqcup cA$ for complex conjugation c, and $A \neq \emptyset$ implies that F contains a CM field. For a number field L, we write I_L for the set of all embeddings of L into $\overline{\mathbb{Q}}$.

Then A (resp. T) is an inflated image of a subset of $I_{F_{CM}}$ (resp. I_{F_+}). We write $c^{\pm}_{X,J}(M) = \prod_{\sigma\in J/\langle c\rangle} c^{\pm}_{X,\sigma}(M)$ and $\delta_{X,J}(M) = \prod_{\sigma\in J}\delta_{X,\sigma}(M)$ for a subset J of I. We have for M with $\mathrm{rank}_E M = 2$

$$c^{\pm}_{X,A}(M) = \delta_{X,A}(M), \quad c^{\pm}_X(M) \equiv c^{\pm}_{X,T}(M)\delta_{X,A}(M) \mod \overline{\mathbb{Q}}^{\times}. \tag{5.5.1}$$

5.2 Periods of $M_1 \otimes M_2$

Now we consider two rank 2 motives M_1 and M_2 satisfying ($\mathrm{Red}_{\pm}$) in Section 3.2 of Chapter 3 in addition to (Reg) in Section 3.4 of Chapter 3 and ($\pm$). We write $(m_i(\sigma), n_i(c\sigma))$ and $(n_i(\sigma), m_i(c\sigma))$ for the Hodge type of M_i at σ. In this section, we study the periods for $M_1\otimes M_2$ and $Ad(M)$. For each p-adic place $\mathfrak{P}$ associated to $i_p\sigma$, we write $V_{\mathfrak{P}}(M)$ for the subspace $V^+_{\mathfrak{P}} = V^-_{\mathfrak{P}}$ in $H_{\mathfrak{p}}(M)$. We describe all possible cases of $V_{\mathfrak{P}}(M_1\otimes M_2)$, up to interchange of M_1 and M_2, in the following table as long as $M_1\otimes M_2$ satisfies (Red_+) and $V_{\mathfrak{P}}(M_1\otimes M_2)$ can be described as a sum of tensor products of $H_{\mathfrak{p}}(M_i)$ and $V_{\mathfrak{P}}(M_j)$.

$V_{\mathfrak{P}}(M_1\otimes M_2)$	Conditions
(cA) $H_{\mathfrak{p}}(M_1)\otimes H_{\mathfrak{p}}(M_2)$	(i) $\sigma \in A_1\cap A_2$ (ii) $\sigma\in A_1\cap T_2$, $n_1(\sigma)-m_1(c\sigma) > m_2(c\sigma)-n_2(\sigma)$ (iii) $\sigma\in A_1\cap cA_2$, $n_1(\sigma)-m_1(c\sigma) > m_2(c\sigma)-n_2(\sigma)$
(T) $V_{\mathfrak{P}}(M_1)\otimes H_{\mathfrak{p}}(M_2)$	(i) $\sigma\in A_2\cap T_1$, $n_2(\sigma)-m_2(c\sigma) > m_1(c\sigma)-n_1(\sigma)$ (ii) $\sigma\in A_2\cap T_1$, $m_2(\sigma)-n_2(c\sigma) < m_1(c\sigma)-n_1(\sigma)$ (iii) $\sigma\in T_1\cap T_2$, $m_2(\sigma)-n_2(c\sigma) < m_1(c\sigma)-n_1(\sigma)$ $n_2(\sigma)-m_2(c\sigma) > n_1(c\sigma)-m_1(\sigma)$ (iv) $\sigma\in cA_2\cap T_1$, $n_2(\sigma)-m_2(c\sigma) > n_1(c\sigma)-m_1(\sigma)$
(C) $V_{\mathfrak{P}}(M_1)\otimes H_{\mathfrak{p}}(M_2)$ $+H_{\mathfrak{p}}(M_1)\otimes V_{\mathfrak{P}}(M_2)$	$\sigma\in T_1\cap T_2$, $m_1(\sigma)-n_1(c\sigma) > m_2(c\sigma)-n_2(\sigma)$ $n_1(\sigma)-m_1(c\sigma) > n_2(c\sigma)-m_2(\sigma)$
(cC) $V_{\mathfrak{P}}(M_1)\otimes V_{\mathfrak{P}}(M_2)$	$\sigma\in T_1\cap T_2$, $m_1(\sigma)-n_1(c\sigma) < m_2(c\sigma)-n_2(\sigma)$ $n_1(\sigma)-m_1(c\sigma) < n_2(c\sigma)-m_2(\sigma)$
(A) $\{0\}$	(i) $\sigma\in A_1\cap cA_2$, $m_1(\sigma)-n_1(c\sigma) < n_2(c\sigma)-m_2(\sigma)$ (ii) $\sigma\in cA_1\cap T_2$, $m_1(\sigma)-n_1(c\sigma) < n_2(c\sigma)-m_2(\sigma)$ (iii) $\sigma\in cA_1\cap cA_2$

In each of the above cases, one can express the periods of $M_1\otimes M_2$ as monomials of periods of M_1 and M_2. Such computation is done by Blasius and Yoshida (see [5], [74] and [42] Section 1) assuming that $X = \mathbb{C}$. Since the definition of the period is

formal as explained in Section 3.1 of Chapter 3, the result in the papers quoted above is valid for any choice of X satisfying the requirement in Section 3.1 of Chapter 3. Here is a summary of the result obtained:

Theorem 5.2.1 — *Suppose (Red$_+$), ($\pm$) and (Reg$_\sigma$) for M_1 and M_2, and write δ_σ (resp. $c^\pm_\sigma$) for $\delta_{X,\sigma}$ (resp. $c^\pm_{X,\sigma}$). These invariants only depend on σ modulo the subgroup generated by c. Then we have the following result for $M = M_1 \otimes M_2$:*

(i) *When $\sigma \neq c\sigma$, we have $c^+_\sigma(M_i) = c^-_\sigma(M_i)$ and*

Type	$\sigma \in A$	$\sigma \in T$	$\sigma \in C$
$c^+_\sigma(M)$	$\delta_\sigma(M_1)^2\delta_\sigma(M_2)^2$	$\delta_\sigma(M_2)c^+_\sigma(M_1)^2$	$c^+_\sigma(M_1)c^+_\sigma(M_2)\delta_{c\sigma}(M_1)\delta_{c\sigma}(M_2)$
$c^-_\sigma(M)$	$\delta_\sigma(M_1)^2\delta_\sigma(M_2)^2$	$\delta_\sigma(M_2)c^-_\sigma(M_1)^2$	$c^-_\sigma(M_1)c^-_\sigma(M_2)\delta_{c\sigma}(M_1)\delta_{c\sigma}(M_2)$

(ii) *When $\sigma = c\sigma$, only Case T occurs, and we have*

$$c^\pm_\sigma(M_1 \otimes M_2) = c^+_\sigma(M_1)c^-_\sigma(M_1)\delta_\sigma(M_2).$$

Proof. We start with a general motive M. First suppose that $c\sigma \neq \sigma$ for the complex conjugation c induced by i_∞. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^+_B({}^\sigma M) & \hookrightarrow & H_B({}^\sigma M) \oplus H_B({}^{c\sigma} M) & \to & \frac{H_{DR}(M)}{\mathcal{F}(M)} \otimes_{F\otimes E,(\sigma\oplus c\sigma)\otimes\iota_E} (X \oplus X) \\ \alpha_B \,\|\wr & & \alpha \,\|\wr & & \mathrm{Id}\otimes\alpha_X \,\|\wr \\ H^-_B({}^\sigma M) & \hookrightarrow & H_B({}^\sigma M) \oplus H_B({}^{c\sigma} M) & \to & \frac{H_{DR}(M)}{\mathcal{F}(M)} \otimes_{F\otimes E,(\sigma\oplus c\sigma)\otimes\iota_E} (X \oplus X), \end{array}$$

where $\alpha(v\oplus v') = v\oplus(-v')$ for $v\oplus v' \in H_B({}^\sigma M)\oplus H_B({}^{c\sigma}M)$, α_B is just induced by α, and $\alpha_X(x\oplus x') = x\oplus(-x')$ for $x\oplus x' \in X\oplus X$. This shows that $c^+_{X,\sigma}(M) = c^-_{X,\sigma}(M)$. Then we get Case T in (i) and (ii) from [74] Proposition 3.1. Case C is studied in [42] Proposition 1.4. Case A is immediate. □

We now look at the rank 3 motive $Ad(M)$ for a rank 2 motive $M = M_1$. We have a splitting: $M \otimes \check{M} = Ad(M) \oplus \mathbf{1}$, where $\mathbf{1}$ is the identity motive with trivial Galois representation. Then the Hodge type of $Ad(M)$ is given by $\pm(m(\sigma)-n(\sigma), n(c\sigma)-m(c\sigma))$ and $(0,0)$. Thus in this case, $\mathcal{F}^-(M)$ (resp. $\mathcal{F}^+(M)$) is $E\otimes F$-free of rank 1 (resp. 2). This shows that $V^+_{\mathfrak{P}}(Ad(M))$ is the image of the projection of $V_{\mathfrak{P}}(M)\otimes H_{\mathfrak{p}}(\check{M}) + H_{\mathfrak{p}}(M)\otimes V_{\mathfrak{P}}(\check{M})$ in $Ad(M)$, and $V^-_{\mathfrak{P}}(Ad(M)) = V_{\mathfrak{P}}(M)\otimes V_{\mathfrak{P}}(\check{M})$. If $Ad(M)$ is critical at some integer, F has to be totally real. We have the following result similarly to Theorem 5.2.1:

Theorem 5.2.2 — *Suppose (Red$_+$), ($\pm$) and (Reg) for rank 2 pure motive M. If $Ad(M)(1)$ is critical, then F is totally real, and we have*

$$c^+_{X,\sigma}(Ad(M)(1)) = c^+_{X,\sigma}(M)c^-_{X,\sigma}(M)\delta_{X,\sigma}(M(1)).$$

5.3 Periods of $M_1 \otimes \cdots \otimes M_r$

We assume $\mathrm{rank}(M_i) = 2$ and the following conditions

(Reg) *M_i is regular for all i,*

(Cr) *$M(1)$ is critical for $M = M_1 \otimes \cdots \otimes M_r$;*

(Fr_M) *$\mathscr{F}(M)$ is $E \otimes F$-free.*

The assumption (Fr_M) is automatic under ($\pm$) for M_i if F does not contain a CM field. When F contains a CM field, we can always bring $M' = M'_1 \otimes \cdots \otimes M'_r$ satisfying (Reg) and (Cr) into M satisfying the above three conditions by tensoring it with a rank 1 CM motive; thus, periods of M' can be written down as a monomial of $c^+_{X,\sigma}(M_i)$, $c^-_{X,\sigma}(M_i)$ and CM periods $\delta_{X,\sigma}(M_i)$. If M has a non-trivial Hodge component (q,q) at some place $\sigma \in I$, then for a Hodge type (x_i, y_i) with $x_i \neq y_i$ of M_i at σ, $\sum_i x_i = \sum_i y_i = q$ and $H^{q,q}({}^\sigma M) \supset V_1 \oplus V_2$ for $V_1 = \otimes_i H^{x_i,y_i}({}^\sigma M_i)$ and $V_2 = \otimes_i H^{y_i,x_i}({}^\sigma M_i)$. Since complex conjugation c interchanges V_1 and V_2, c does not act on $H^{q,q}(M)$ by scalar. Then the criticality of $M(1)$ prohibits this to happen, and thus M satisfies ($\pm$). We write $a_i(\sigma,+) = m_i(\sigma)$ and $a_i(\sigma,-) = n_i(\sigma)$. Following [5] and [74] Section 3, we consider the set $\Sigma = \{\{1,\dots,r\} \to \{\pm\}\}$ of functions and put

$$\Sigma^+_\sigma = \left\{ s \in \Sigma \middle| \sum_i a_i(\sigma, s(i)) > \sum_i a_i(c\sigma, -s(i)) \right\}$$
$$= \left\{ s \in \Sigma \middle| \sum_{j \in J(s)} k_j(\sigma) > \sum_{i \notin J(s)} k_i(\sigma) \right\},$$

where $J(s) = \{j | s(j) = +\}$ and $k_i(\sigma) - 1 = a_i(\sigma,+) - a_i(c\sigma,-)$. From $m_i(\sigma) > n_i(\sigma)$, we have that $k_i(\sigma) > 1$. Then $\mathscr{F}(M) \otimes_{E \otimes F, i_\infty \otimes \sigma} \mathbb{C}$ is equal to

$$\sum_{s \in \Sigma^+_\sigma} \otimes_i H^{a_i(\sigma,s(i)), a_i(c\sigma,-s(i))}({}^\sigma M_i)$$

for all $\sigma \in I$. For M to satisfy (Red_+), $V_{\mathfrak{P}}(M)$ has to be a sum of tensor products of $H_{\mathfrak{p}}(M_i)$ and $V_{\mathfrak{P}}(M_i)$. This is equivalent to assuming

($\mathrm{Red}_\pm$) *Σ^+_σ only depends on the p-adic place $\mathfrak{P}$ induced by $i_p\sigma$ for all σ.*

Under ($\mathrm{Red}_\pm$), we write $\Sigma^+_{\mathfrak{P}}$ for Σ^+_σ, and we have

$$V_{\mathfrak{P}}(M) = V^+_{\mathfrak{P}} = V^-_{\mathfrak{P}} = \sum_{s \in \Sigma^+_{\mathfrak{P}}} (\otimes_{i \in J(s)} V_{\mathfrak{P}}(M_i)) \otimes (\otimes_{j \notin J(s)} H_{\mathfrak{p}}(M_j)).$$

We define for each $\sigma \in I$ the two integers $\nu_i(\sigma)$ and $\mu_i(\sigma)$ by

$$\nu_i(\sigma) = \#\{s \in \Sigma^+_\sigma | s(i) = +\} \quad \text{and} \quad \mu_i(\sigma) = \#\{s \in \Sigma^+_\sigma | s(i) = -\}.$$

Then $\mu_i(\sigma)+\nu_i(\sigma)=2^{r-1}$, and as shown in [5] and [74] Section 3, we have

Theorem 5.3.1 — *Under the above assumption, we have*

$$c^{\pm}_{X,\sigma}(M_1\otimes M_2\otimes\cdots\otimes M_r)=\prod_i\left\{(c^+_{X,\sigma}(M_i)c^-_{X,\sigma}(M_i))^{(\nu_i(\sigma)-\mu_i(\sigma))/2}\delta_{X,\sigma}(M_i)^{\mu_i(\sigma)}\right\}.$$

Although in [74] Section 3 p.156, it is assumed that $a_i(c\sigma,\pm)=a_i(\sigma,\pm)$ for all σ to deduce Theorem 5.3.1, his proof works well in the general case as long as we assume (Fr_M).

5.4 Conjectures on $\varphi_1\otimes\cdots\otimes\varphi_r$

We suppose that $\mathbb{I}$ is an integral domain finite over $\mathcal{O}[[T_2(\mathbb{Z}_p)]]$ for T_2 given by $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}_m\times\mathbf{G}_m$. We consider arithmetic $\mathbb{I}$-adic representations $\varphi_i:\mathcal{G}_F\to GL_2(\mathbb{K})$ for $i=1,2,\dots,r$. For each arithmetic point P, we write $(m(P),n(P))$ in $\mathbb{Z}[I]^2$ for its weight. Since $M_{i,P}$ is pure, $m_\sigma(P)+n_{c\sigma}(P)$ is independent of σ and thus $m+cn$ factors through the torus T defined in Section 2.1 of Chapter 2. We write $V_{\mathfrak{P}}(\varphi_i)$ for the space $V^-_{\mathfrak{P}}$ specified in $(\mathrm{A2}_\pm)$ for φ_i. Writing $\mathbb{L}$ for the field of fractions of $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\dots\widehat{\otimes}_{\mathcal{O}}\mathbb{I}$, we have

$$\varphi=\varphi_1\otimes\cdots\otimes\varphi_r:\mathcal{G}_F\to GL_N(\mathbb{L})\quad\text{for}\quad N=2^r.$$

We write $\mathbb{J}$ for the integral closure of $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\dots\widehat{\otimes}_{\mathcal{O}}\mathbb{I}$ in $\mathbb{L}$. Then $\mathbb{J}$ is an integral domain finite over $\mathcal{O}[[T(\mathbb{Z}_p)]]$ for $T=T_2^r\cong T_N$, and φ is arithmetic in the sense of Section 4.1 in Chapter 4. We have natural projection of $\mathscr{X}_{\mathbb{J}}$ to $\mathscr{X}^r_{\mathbb{I}}$. We have several different choices of $V_{\mathfrak{P}}=V^{\pm}_{\mathfrak{P}}(\varphi)$. To describe this choice, we pick a subset $\Sigma^+_{\mathfrak{P}}$ of Σ for each $\mathfrak{P}$ such that

$$\bigcap_{s\in\Sigma^+_{\mathfrak{P}}}\left\{x\in\mathbb{R}^n_+\,\middle|\,\sum_j s(j)x_j>0\right\}\quad\text{is non-empty, and}\quad\Sigma=\Sigma^+_{\mathfrak{P}}\bigsqcup-\Sigma^+_{\mathfrak{P}}.\tag{5.5.2}$$

We write $S=\{\Sigma^+_{\mathfrak{P}}|\mathfrak{P}\in S_F\}$. For each $\mathfrak{P}$, there are 2^{r-1}-choices of $\Sigma^+_{\mathfrak{P}}$. Thus the set of S is made of $2^{(r-1)|S_F|}$ elements. For each such choice S, we expect to have a p-adic L-function. These p-adic L-functions might be related in a yet unknown way. Let $V(\varphi_j)$ be the reflexive $\mathbb{I}$-module given for φ_j in (Cont) in Section 4.1 of Chapter 4 and put $V(\varphi)$ for the $\mathbb{J}$-reflexive closure of the image of $\widehat{\otimes}_jV(\varphi_j)$ in $(\widehat{\otimes}_jV(\varphi_j))\otimes\mathbb{L}$. Then we define

$$V_{\mathfrak{P}}(\varphi;S)=V_{\mathfrak{P}}(\varphi)=V(\varphi)\bigcap\left\{\sum_{s\in\Sigma^+_{\mathfrak{P}}}\left(\widehat{\bigotimes}_{i\in J(s)}V^-_{\mathfrak{P}}(\varphi_i)\right)\widehat{\otimes}\left(\widehat{\bigotimes}_{j\notin J(s)}V(\varphi_j)\right)\otimes\mathbb{L}\right\}.$$

We write an element $a=(m,n)\in X(T_2)$ as $m=\sum_\sigma a(\sigma,+)\sigma$ and $n=\sum_\sigma a(\sigma,-)\sigma$. Then under this convention, the subset of $X(T_N)=\oplus_j X(T_2)$ given by

$$\Lambda_S=\left\{(a_j)\in\oplus_j X(T_2)\Big|\sum_i a_i(\sigma,s(i))>\sum_i a_i(c\sigma,-s(i))\quad\forall s\in\Sigma^+_{\mathfrak{P}}\quad\text{and}\quad\forall\sigma\right\}$$

gives a spanning cone in $X(T)\otimes_{\mathbb{Z}}\mathbb{R}$ by (5.5.2). Here note that $\mathfrak{P}$ is induced by $i_p\sigma$. Thus the set of motivic point $P=\otimes_i P_i\in\mathscr{X}^r_{\mathbb{I}}$ such that $\sum_i(m(P_i),n(P_i))\in\Lambda_S$ is dense in $\mathscr{X}^r_{\mathbb{I}}$, and such points are dense in $\mathscr{X}_{\mathbb{J}}$. We call such points motivic points of type S. This shows φ satisfies (A'$2_\pm$) and (A3) for the above $\mathbb{J}$-submodule $V_{\mathfrak{P}}(\varphi;S)$. The periods of $M_P=\otimes_i M_{i,P_i}$ for all motivic points P of type S have the same shape:

(Pd)
$$c^{\pm}_{X,\sigma}(M_P)=\prod_i\left\{(c^+_{X,\sigma}(M_{i,P_i})c^-_{X,\sigma}(M_{i,P_i}))^{(\nu_i(S,\sigma)-\mu_i(S,\sigma))/2}\delta_{X,\sigma}(M_{i,P_i})^{\mu_i(S,\sigma)}\right\}$$

where $\nu_i(S,\sigma)=\#\{s\in\Sigma^+_{\mathfrak{P}}|s(i)=+\}$ and $\mu_i(S,\sigma)=\#\{s\in\Sigma^+_{\mathfrak{P}}|s(i)=-\}$ for the place $\mathfrak{P}$ induced by $i_p\sigma$.

Conjecture 5.4.1 — *For each choice S, we can find a unique p-adic L-function $L_{S,i_\infty}(\varphi)$ in the field of fractions of $\mathbb{J}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ such that for all motivic points P of type S critical at 1, if the p-adic period does not vanish,*

$$i_p^{-1}\left(\frac{L_{S,i_\infty}(P,\varphi)}{c^+_{p,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1})}\right)=*i_\infty^{-1}\left(\frac{L_{i_\infty}(1,\varphi_P)}{c^+_{\infty,i_\infty}(\varphi_P\otimes\mathcal{N}^{-1})}\right),$$

where the period in the denominator is given by the formula (Pd).

This conjecture covers all critical values when F does not contain a CM field. When F contains a CM field, $\mathscr{F}(M_P)$ may not be free, and we have more cases to cover. To illustrate this, we describe the situation in a non-trivial case in the following paragraph.

5.5 The case of $\varphi_1\otimes\varphi_2$

First assume (Fr_M) in Section 5.3 for $r=2$. There are only 2 choices of $\Sigma^+_{\mathfrak{P}}$ at each $\mathfrak{P}$. Writing $s\in\Sigma^+_{\mathfrak{P}}$ as $(s(1)\ s(2))$, these two choices are $\{(++),(+-)\}$ and $\{(++),(-+)\}$. Let S_F be the set of p-adic places of F. For an S, we define a subset T of S_F by $\mathfrak{P}\in T\Leftrightarrow(+-)\in\Sigma^+_{\mathfrak{P}}$ and $\Sigma^+_{\mathfrak{P}}\in S$. For $\mathfrak{P}\in T$, we are in Case T in Theorem 5.2.1 and for $\mathfrak{P}\notin T$, we are in Case T for which the role of M_1 and M_2 is interchanged. The choice S is determined by T. Writing $J(T)$ for the set of $\sigma\in I$ such that $i_p\sigma$ induces the p-adic places $\mathfrak{P}$ in T, we call a motivic point $P\otimes Q$ of

type T if

$$\begin{aligned} k_\sigma(P) > k_{c\sigma}(Q) \quad \text{and} \quad k_{c\sigma}(P) > k_\sigma(Q) &\iff \sigma \in J(T), \quad \text{and} \\ k_\sigma(Q) > k_{c\sigma}(P) \quad \text{and} \quad k_{c\sigma}(Q) > k_\sigma(P) &\iff \sigma \notin J(T), \end{aligned} \tag{5.5.3}$$

where $k_\sigma(P) - 1 = m_\sigma(P) - n_{c\sigma}(P) > 1$. We now treat the general case without assuming (Fr_M). Thus we may assume that F contains a CM field. We choose a partition $S_{F_{CM}} = A \bigsqcup cA \bigsqcup B \bigsqcup C \bigsqcup cC$, where c is the generator of $\mathrm{Gal}(F_{CM}/F_+)$. We further choose a subset T of B. Then the above partition induces a partition of I by associating to $\sigma \in I$ a p-adic place of F_{CM} induced by $i_p\sigma|_{F_{CM}}$. We use the same symbols to write this partition of I; thus, $I = A \bigsqcup cA \bigsqcup T \bigsqcup (B-T) \bigsqcup C \bigsqcup cC$. We call a motivic point $P \otimes Q$ "*of type* (A, T, C)" if

$$\begin{aligned} \sigma \in A &\iff k_{c\sigma}(P) < -k_{c\sigma}(Q) \quad \text{and} \quad \sigma \in cA \iff k_\sigma(P) < -k_\sigma(Q) \\ \sigma \in C &\iff k_\sigma(P) > k_{c\sigma}(Q) \quad \text{and} \quad k_{c\sigma}(P) < k_\sigma(Q) \\ \sigma \in T &\iff k_\sigma(P) > k_{c\sigma}(Q) \quad \text{and} \quad k_{c\sigma}(P) > k_\sigma(Q) \\ \sigma \in B - T &\iff k_\sigma(Q) > k_{c\sigma}(P) \quad \text{and} \quad k_{c\sigma}(Q) > k_\sigma(P). \end{aligned} \tag{5.5.4}$$

This implies that

$$\begin{gathered} \mathfrak{P} \in T \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = V_{\mathfrak{P}}(\varphi_1) \otimes V(\varphi_2) \otimes \mathbb{L} \\ \mathfrak{P} \in B - T \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = V(\varphi_1) \otimes V_{\mathfrak{P}}(\varphi_2) \otimes \mathbb{L} \\ \mathfrak{P} \in A \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = V(\varphi_1) \otimes V(\varphi_2) \otimes \mathbb{L} \\ \mathfrak{P} \in cA \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = \{0\} \\ \mathfrak{P} \in C \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = \{V(\varphi_1) \otimes V_{\mathfrak{P}}(\varphi_2) + V_{\mathfrak{P}}(\varphi_1) \otimes V(\varphi_2)\} \otimes \mathbb{L} \\ \mathfrak{P} \in cC \iff V_{\mathfrak{P}}(\varphi_1 \otimes \varphi_2) \otimes \mathbb{L} = V_{\mathfrak{P}}(\varphi_1) \otimes V_{\mathfrak{P}}(\varphi_2) \otimes \mathbb{L}. \end{gathered}$$

When F does not contain any CM field, we just pretend $A = C = \emptyset$, and we call a motivic point P "*of type* $(\emptyset, T, \emptyset)$" if it is of type T.

Conjecture 5.5.1 — *For each choice of $S = (A, T, C)$, we find a p-adic L-function $L_{S,i_\infty}(\varphi_1 \otimes \varphi_2)$ in the field of fractions of $\mathbb{J}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ such that for all motivic points P of type S critical at 1, if the p-adic period does not vanish,*

$$i_p^{-1}\left(\frac{L_{S,i_\infty}(P \otimes Q, \varphi_1 \otimes \varphi_2)}{c^+_{p,i_\infty}(\varphi_{1,P} \otimes \varphi_{2,Q} \otimes \mathcal{N}^{-1})}\right) = *i_\infty^{-1}\left(\frac{L_{i_\infty}(1, \varphi_{1,P} \otimes \varphi_{2,Q})}{c^+_{\infty,i_\infty}(\varphi_{1,P} \otimes \varphi_{2,Q} \otimes \mathcal{N}^{-1})}\right),$$

where we have

$$c^+_{X,i_\infty}(\varphi_{1,P}\otimes\varphi_{2,Q}\otimes\mathcal{N}^{-1}) = \prod_{\sigma\in A}\delta_{X,\sigma}(\varphi_{1,P})^2\delta_{X,\sigma}(\varphi_{2,Q}\otimes\mathcal{N}^{-1})^2$$
$$\times\prod_{\sigma\in T/\langle c\rangle}c^+_{X,\sigma}(\varphi_{1,P})c^-_{X,\sigma}(\varphi_{1,P})\delta_{X,\sigma}(\varphi_{2,Q}\otimes\mathcal{N}^{-1})$$
$$\times\prod_{\sigma\in B-T/\langle c\rangle}c^+_{X,\sigma}(\varphi_{2,Q})c^-_{X,\sigma}(\varphi_{2,Q})\delta_{X,\sigma}(\varphi_{1,P}\otimes\mathcal{N}^{-1})$$
$$\times\prod_{\sigma\in C}c^+_{X,\sigma}(\varphi_{1,P})c^-_{X,\sigma}(\varphi_{2,Q})\delta_{X,c\sigma}(\varphi_1,Q)\delta_{X,c\sigma}(\varphi_{2,Q}\otimes\mathcal{N}^{-1}).$$

5.6 The case of $Ad(\varphi)$

We write $\varphi=\varphi_1$ and consider the tensor product $\varphi\otimes_{\mathbb{I}}\check{\varphi}$. In $V(\varphi\otimes_{\mathbb{I}}\check{\varphi}) = \mathrm{End}_{\mathbb{I}}(V(\varphi))$, there is a unique 3 dimensional subspace $V(Ad(\varphi))$ with trace 0 endomorphisms, giving the following decomposition over $\mathbb{K}$: $\varphi\otimes_{\mathbb{I}}\check{\varphi}\cong Ad(\varphi)\oplus\mathbf{1}$. As seen in Theorem 5.2.2, for $Ad(\varphi)$ being arithmetic, we need to assume that F is totally real. In this case, there is only one choice of $S=\{V^+_{\mathfrak{P}}(Ad(\varphi)), V^-_{\mathfrak{P}}(Ad(\varphi))\}_{\mathfrak{P}}$ for a given choice of $\{V_{\mathfrak{P}}(\varphi)\}_{\mathfrak{P}}$. Thus we expects only one genuine p-adic L-function $L_{p,i_\infty}(Ad(\varphi))$ for each i_∞. Thus for $S=(\emptyset,I,\emptyset)$, by Theorems 5.2.1 and 5.2.2, the ratio $\frac{L_{S,i_\infty}(\varphi_1\otimes\varphi_2)}{L_{p,i_\infty}(Ad(\varphi_1))}$, which is equal to $\frac{L_S(\varphi_1\otimes\varphi_2)}{L_p(Ad(\varphi_1))}$ by (NP) in Section 4.3 of Chapter 4 if the normalized L-functions exist, is expected to belong to $\mathbb{L}$ in $Frac(\mathbb{J}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega)$.

We now assume that $p>2$ and $(\mathrm{Fr}_{\mathbb{I}})$, $(\mathrm{FR}_\pm)$ for φ in Section 4.3. Then $Ad(\varphi)$ automatically satisfies these conditions. We further suppose the absolute irreducibility (AI) for $\overline{\varphi}$. By [54] Lemma 2 in Section 1.2, $Ad(\varphi)_P$ for any $P\in\mathscr{X}_{\mathbb{I}}$ never has the trivial subquotient. If $Ad(\varphi)^*_P = Ad(\varphi)_P\otimes\mathcal{N}$ contains the trivial subquotient, then $\overline{\varphi}\cong\overline{\varphi}\otimes\mathcal{N}$, which implies that $p=3$ and $\overline{\varphi}$ is an induced representation of a character of $\mathcal{G}_{F(\sqrt{-3})}$. Thus if $p\geq 5$ or $\overline{\varphi}$ is not an induced representation of a character of $\mathcal{G}_{F(\sqrt{-3})}$, $Ad(\varphi)^*_P$ never contains the trivial representation. Here is a refinement of Conjecture 1.0.1:

Conjecture 5.6.1 — *Suppose (AI) for $\varphi\in\pi(\lambda)$ for a primitive algebra homomorphism $\lambda:\mathbf{h}\to\mathbb{I}$. If either $p\geq 5$ or $\overline{\varphi}$ is not an induced representation of a character of $\mathcal{G}_{F(\sqrt{-3})}$, then over $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$, $C_0(\lambda;\mathbb{I})\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega$ is isomorphic to $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{O}_\Omega/(L_p(Ad(\varphi)))$, where $L_p(Ad(\varphi))$ is the normalized genuine p-adic L-function of $Ad(\varphi)$.*

5.7 The case of $\varphi_1\otimes\varphi_2\otimes\varphi_3$

We now specialize Conjecture 5.4.1 to $F=\mathbb{Q}$ and $r=3$. We write k_i for $k_i(\sigma)$ with the unique embedding σ of $\mathbb{Q}$ into $\overline{\mathbb{Q}}$. We write $Q_X(M_i)$ for $c^+_{X,\sigma}(M_i)c^-_{X,\sigma}(M_i)$. There are 4 different types S, and three out of the four types are transformed each

other by permuting M_1, M_2 and M_3 as follows [5]:

Type	Period
(i) $k_1 > k_2 + k_3$	$Q_X(M_1)^2\delta_{X,\sigma}(M_2)^2\delta_{X,\sigma}(M_3)^2$
(ii) $k_1 + k_2 > k_3$, $k_1 + k_3 > k_2$, and $k_2 + k_3 > k_1$	$\prod_{j=1}^{3} Q_X(M_j)\delta_{X,\sigma}(M_j)$

Thus there should be 3-variable p-adic L-functions of 4 different types. When $\varphi_i \in \pi(\lambda_i)$ for algebra homomorphisms $\lambda_i : h^{n.ord}(N_i p^\infty; \mathbb{O}) \to \mathbb{I}$, p-adic L-functions related to $\varphi_1 \otimes \varphi_2 \otimes \varphi_3$ has been constructed in [32] and [57]. Assuming $N_i = 1$, Panchishkin [57] deals with the case (ii). His p-adic L-function is a function sending characters χ of $\mathbb{Z}_p^\times$ to the L-value $L(1, \varphi_{P\otimes Q\otimes R} \otimes \chi)$ divided by a suitable period for a fixed a motivic point $P \otimes Q \otimes R$ of type (ii). Thus our conjecture asserts that Panchishkin's L-functions can be glued together to yield a 3-variable type (ii) L-function. I should also mention that the result in [57] is valid without assuming $(\mathrm{Red}_\pm)$. On the other hand, Harris and Tilouine treat the case (i) interpolating along $Spec(h^{n.ord}(N_1 p^\infty; \mathbb{O}))$ square roots of type (i) values. They get a p-adic L-function which sends P to $\sqrt{L(\frac{m(P)+n(P)+1}{2}, \varphi_{P\otimes Q\otimes R})}$ divided basically by the Petersson self inner product of the modular form corresponding to P for a fixed $Q \otimes R$. Their evaluation point is the center of symmetry of the functional equation of $L(s, \varphi_{P\otimes Q\otimes R})$. Thus our conjecture asserts the extensibility of the square of the L-function to a 3 variable L-function $L^{(i)}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)$. Similarly to Conjecture 5.6.1, we expect that the ratio $\frac{L^{(i)}(\varphi_1\otimes\varphi_2\otimes\varphi_3)}{L_p(Ad(\varphi_1))^2}$ falls in $\mathbb{L}$.

6. p-Adic Rankin products

We assume that F is totally real. We study the p-adic L-functions constructed in [41] in terms of the genuine p-adic L-functions of $\varphi \otimes \check{\rho}$ for two modular Galois representations.

6.1 Complex Rankin products

We write $\mathbf{h}(N)$ (resp. $\mathbf{\Lambda}$) for $h^{n.ord}(Np^\infty;\mathbb{C})$ or $h^{ord}(Np^\infty;\mathbb{C})$ (resp. Λ or Λ^{ord}) as in Section 2.9 of Chapter 2. Let $\lambda : \mathbf{h}(N) \to \mathbb{I}$ and $\mu : \mathbf{h}(J) \to \mathbb{I}$ be primitive $\mathbf{\Lambda}$-algebra homomorphisms giving irreducible components of the Hecke algebras. Take φ (resp. ρ) in the isogeny class $\pi(\lambda)$ (resp. $\pi(\mu)$). For two positive arithmetic points P and Q of $\mathscr{X}_{\mathbb{I}}$, we look at the complex L-function $L(s, \varphi_P \otimes \check{\rho}_Q)$. This L-function is convergent if $Re(s)$ is sufficiently large, is continued to the whole s-plane as a meromorphic function and satisfies the functional equation as in Section 3.1 of Chapter 3 (see [51]).

The principle (AT_∞) in Section 4.4 of Chapter 4 is known to hold for $L(s, \varphi_P \otimes \check{\rho}_Q)$. This is equivalent to saying that $L(s, \varphi_P \otimes \check{\rho}_Q)$ has a pole at $s = 1$ if and only if $\varphi_P \cong \rho_Q$ because the two representations are absolutely irreducible (see [65] Proposition 4.13). When $\varphi_P \otimes \check{\rho}_Q \otimes \mathcal{N}^{-1}$ is critical, as conjectured by Shimura and proven by Shimura, Harris and Yoshida ([65], [66], [67], [30], [31] and [74], [75]), the transcendental factor of $L(1, \varphi \otimes \check{\rho})$ has a natural factorization described in Section 5.2 of Chapter 5 depending on the two weights $\kappa(P)$ and $\kappa(Q)$ of P and Q, respectively. We look into the following special case of type $(\emptyset, I, \emptyset)$:

$$\text{(Ad)} \qquad m(P) - m(Q) > 0 \geq n(P) - n(Q),$$

where the transcendental factor $c^+_X(\varphi_P \otimes \check{\rho}_Q \otimes \mathcal{N}^{-1})$ is very close to $c^+_X(Ad(\varphi_P) \otimes \mathcal{N}^{-1})$ as shown in Theorems 5.2.1 and 5.2.2 of Chapter 5. The algebraicity theorem in this case is studied in [65] (see also [41]), and the transcendental factor is given, up to a power of π, by the self Petersson inner product (f_P, f_P) of a Hilbert cusp form f_P, where f_P is the primitive element in the automorphic representation generated

by the value $f(P)$ at P of the $\mathbb{I}$-adic form f giving rise to φ. On the other hand, as shown in [47] Theorem 7.1,

$$\text{(AV)} \qquad L_{i_\infty}(1, Ad(\varphi_P)) = *\pi^{m(P)-n(P)+2t}(f_P, f_P),$$

where $\pi^m = \prod_{\sigma\in I}\pi^{m_\sigma}$ and $*$ is a simple constant depending only on $\kappa(P)$ and the conductor of f_P. This combined with Theorem 4.2 of [65] (see also p.320 of [41]) shows that under the assumption of (Ad)

$$\text{(R1)} \qquad \frac{G(det(\rho_Q))L_{i_\infty}(1, \varphi_P \otimes \check{\rho}_Q)}{(2\pi i)^{\{\kappa(Q)\}-\{\kappa(P)\}}G(det(\varphi_P))L_{i_\infty}(1, Ad(\varphi_P))} \in \overline{\mathbb{Q}},$$

where $G(det(\rho_Q))$ is the Gauss sum of the complex avatar of the character $det(\rho_Q)$ specified in [65] (3.9). By definition, $\delta_*(M(\check{\rho}_Q \otimes \mathcal{N}^{-1})) = G(det(\rho_Q))^{-1}x^{2t+\{\kappa(Q)\}}$ for $x = 2\pi i$ or 1 according as $* = \infty$ or p. This follows from the fact that

(i) the Tate motive $T = \mathbb{Q}(1)$ is given by $H_p(T) = \mathbb{Q}_p(1)$, $H_B(T) = (2\pi i)\mathbb{Q} \subset \mathbb{C}$ and $H_{DR}(T) = \mathbb{Q}$, and

(ii) we have identified $\mathbb{Q}_p(1)$ with $\mathbb{Q}_p$.

The fact (R1) is well explained by the following result deduced from Theorems 5.2.1 and 5.2.2 in Chapter 5:

Proposition 6.1.1 — *Suppose (Ad) for $P \otimes Q$. Then we have*

$$\text{(i)} \qquad \begin{aligned}&(2\pi i)^{-\{\kappa(Q)\}}G(det(\rho_Q))c_\infty^+(\varphi_P \otimes \check{\rho}_Q \otimes \mathcal{N}^{-1})\\ &= (2\pi i)^{-\{\kappa(P)\}}G(det(\varphi_P))c_\infty^+(Ad(\varphi_P) \otimes \mathcal{N}^{-1});\end{aligned}$$

$$\text{(ii)} \qquad G(det(\rho_Q))c_p^+(\varphi_P \otimes \check{\rho}_Q \otimes \mathcal{N}^{-1}) = G(det(\varphi_P))c_p^+(Ad(\varphi_P) \otimes \mathcal{N}^{-1}).$$

This shows that $\frac{L_{S,i_\infty}(\varphi\otimes\check{\rho})}{L_{p,i_\infty}(Ad(\varphi))}$ actually falls in the field of fractions of $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}$. If there exist the normalized genuine p-adic L-functions $L_S(\varphi\otimes\check{\rho})$ of type $(\emptyset, I, \emptyset)$ and $L_p(Ad(\varphi))$, by (NP) in Section 4.3 of Chapter 4, $\frac{L_{S,i_\infty}(\varphi\otimes\check{\rho})}{L_{p,i_\infty}(Ad(\varphi))}$ is equal to $\frac{L_S(\varphi\otimes\check{\rho})}{L_p(Ad(\varphi))}$ and is *independent* of i_∞ and the choice of $\varphi \in \pi(\lambda)$ and $\rho \in \pi(\mu)$.

6.2 The p-adic Rankin product in [41]

When $\mathbf{h} = h^{n.ord}$, we constructed in [41] a p-adic (meromorphic) L-function $D_p(\pi(\lambda)\otimes\check{\pi}(\mu)) : \mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}} \to \Omega$ in $\mathbb{L}$ satisfying the following interpolation property: up to a p-Euler factor and a simple algebraic constant $*$

(Int1)

$$D_p(P, Q; \pi(\lambda)\otimes\check{\pi}(\mu)) = *i_\infty^{-1}\left(\frac{G(det(\pi(\rho_Q)))L_{i_\infty}(1, \varphi_P \otimes \check{\rho}_Q)}{G(det(\pi(\varphi_P)))(2\pi i)^{\{\kappa(Q)\}-2n(P)+2t}(f_P, f_P)}\right)$$

for motivic $(P,Q) \in \mathscr{A}_{\mathbb{I}} \times \mathscr{A}_{\mathbb{I}}$ satisfying (Ad) for $\kappa = \kappa(P)$ and $\kappa' = \kappa(Q)$. The constant $*$ can be checked to be the value $u(P \otimes Q)$ of a unit $u \in (\mathbf{\Lambda}\widehat{\otimes}\mathbf{\Lambda})^{\times}$ independent of $P \otimes Q$ after factoring out the canonical p-Euler factors which should appear in two genuine p-adic L-functions (see Conjecture 4.2.1 in Chapter 4), although the computation is a bit involved. For this fact, see [47] (8.14a). Although in [47], we have only dealt with the CM case, the computation in general is essentially the same (except possibly at places dividing 2) because at each (odd finite) place, the local automorphic representation is of CM type (that is, it is induced from a character of a quadratic extension). This combined with (AV) simply shows that $D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$ gives, up to unit factors, the ratio $\frac{L_{S,i_\infty}(\varphi \otimes \check{\rho})}{L_{p,i_\infty}(Ad(\varphi))}$ for $S = (\emptyset, I, \emptyset)$ if two genuine p-adic L-functions exist.

To support this idea, we look into the singularity of $D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$. As we will see in the following paragraph, admitting (AT_p), $L_S(\varphi \otimes \check{\rho})$ is supposed to be everywhere holomorphic in the nearly ordinary case, and in the ordinary case, $L_S(\varphi \otimes \check{\rho})$ can have singularity only at the diagonal divisor Δ of $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$ when $\varphi = \rho$. The singularity of $D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$ is not as above. Via the theory of congruence modules (cf. see Section 2.9 of Chapter 2), the locus of points P for which there exists $\nu : h^{n.ord} \to \mathbb{I}$ such that $\nu_P = \lambda_P$ is the support X of the congruence module $C(\lambda) = C_0(\lambda; \mathbb{I})$ of λ. In particular, we know from its construction in [41] that the singular locus of $D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$ is contained in $X \times \mathscr{X}_{\mathbb{I}}$. In the CM case, we have an exact information of X, and $X \times \mathscr{X}_{\mathbb{I}}$ is really different from Δ. As explained in Section 2.10 of Chapter 2 and Conjecture 5.6.1 in Chapter 5, the characteristic ideal of the congruence module $C_0(\lambda; \mathbb{I})$ is closely related to $L_p(Ad(\varphi))$. Moreover one can check in this modular case (see Theorem I and Lemma 5.3 of [41]), the removed p-Euler factor is identical to the ratio of those Euler p-factors of $L_S(\varphi \otimes \check{\rho})$ and $L_p(Ad(\varphi))$ as predicted by Coates and Perrin-Riou (see (iii) of Conjecture 4.2.1) for cyclotomic twists $P \otimes Q$ of crystalline motivic points. Thus we expect that D_p is the ratio $\frac{L_S(\varphi \otimes \check{\rho})}{L_p(Ad(\varphi))}$ of two genuine p-adic L-functions. Here the numerator $L_S(\varphi \otimes \check{\rho})$ should be legitimately called the p-adic Rankin product of type $(\emptyset, I, \emptyset)$.

6.3 Ordinary versus nearly ordinary

Take a prime divisor $\mathbf{P}$ of $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$, and write $k(\mathbf{P})$ for its residue field. It is easy to see that $V((\varphi \otimes \check{\rho})_{\mathbf{P}}) \otimes k(\mathbf{P})$ and $V((\varphi \otimes \check{\rho})^*_{\mathbf{P}} \otimes k(\mathbf{P})$ never have the trivial subquotient for prime divisors $\mathbf{P}$ of the form $P \times \mathscr{X}_{\mathbb{I}}$ or $\mathscr{X}_{\mathbb{I}} \times Q$. Thus if they have the trivial subquotient, the projection of $\mathbf{P}$ to each component is dominant, and we find densely populated prime divisors P and Q on $\mathscr{X}_{\mathbb{I}}$ such that $P \times Q$ lies on $\mathbf{P}$ and one of φ_P and ρ_Q is irreducible. Then $V(\varphi_P \otimes \check{\rho}_Q) \cong \mathrm{Hom}(V(\rho_Q), V(\varphi_P))$ has the trivial subquotient if and only if $\varphi_P \cong \rho_Q$. Thus, in this case, $\mathbf{P}$ has to be the

diagonal divisor Δ and $\dim_{\mathcal{O}}(\mathscr{X}_{\mathbb{I}})=1$. By a similar argument, if $V((\varphi\otimes\check{\rho})^*)_{\mathbf{P}}\otimes k(\mathbf{P})$ has the trivial subquotient for a prime divisor $\mathbf{P}$, $\mathbf{P}(\Omega)=\{(P,P\otimes\mathcal{N}^{-1})|P\in\mathscr{X}\}$ and $\dim_{\mathcal{O}}\mathscr{X}_{\mathbb{I}}=1$. As in Section 2.5 of Chapter 2, writing d for the defect of the Leopoldt conjecture for (p,F), we know that $\dim_{\mathcal{O}}(h^{n.ord})=[F:\mathbb{Q}]+1+d$ and $\dim_{\mathcal{O}}(h^{ord})=1+d$. By (Gal) (iv) in Section 2.8 of Chapter 2, the inertia group $I_{\mathfrak{P}}$ at $\mathfrak{P}|p$ acts on a one dimensional subspace of $V^-_{\mathfrak{P}}(\lambda)=V^-_{\mathfrak{P}}(\varphi)$ by a character δ such that $\delta(y)=\lambda(\mathbb{T}(y))$, where we identified $I_{\mathfrak{P}}$ with $\mathfrak{r}^\times_{\mathfrak{P}}$ (resp. $I_p=\prod_{\mathfrak{P}|p}I_{\mathfrak{P}}$ with $\mathfrak{r}_p^\times$) by class field theory. This shows us that δ coincides with the character: $I_p\to\mathcal{O}[[\mathbf{G}]]^\times$ given by $y\mapsto(y,1)\in\mathbf{G}$. On the other hand, on I_p, we have $det(\varphi)(z)=\langle z\rangle\in\mathcal{O}[[\mathbf{G}]]^\times$. Therefore $\varphi_P=\rho_Q$ implies $\kappa(P)=\kappa(Q)$. Similarly, $\varphi_P=\rho_Q\otimes\mathcal{N}$ implies $\kappa(P)=\kappa(Q)\otimes(t,t)$. The locus of (P,Q) with $\kappa(P)=\kappa(Q)\otimes(t,t)$ is of codimension >1. The diagonal locus of $\kappa(P)=\kappa(Q)$ is a divisor of $\mathscr{X}_{\mathbb{I}}\times\mathscr{X}_{\mathbb{I}}$ if and only if we are in the ordinary case and δ is trivial on I_p. Thus we may make the following conjectures for $\mathbb{I}$ finite and torsion-free over Λ^{ord}:

Conjecture 6.3.1 — *(Ordinary case) Take primitive Λ^{ord}-algebra homomorphisms $\lambda:h^{ord}(Np^\infty;\mathcal{O})\to\mathbb{I}$ and $\mu:h^{ord}(Jp^\infty;\mathcal{O})\to\mathbb{I}$. Let $S=(\emptyset,I,\emptyset)$. Suppose that the existence of genuine p-adic L-functions $L_{S,i_\infty}(\varphi\otimes\check{\rho})$ of type S and $L_{p,i_\infty}(Ad(\varphi))$ for the Galois representations $\varphi\in\pi(\lambda)$ and $\rho\in\pi(\mu)$. Then*

(i) *$L_{S,i_\infty}(\varphi\otimes\check{\rho})$ is finite outside the diagonal Δ in $\mathscr{X}_{\mathbb{I}}\times\mathscr{X}_{\mathbb{I}}$;*

(ii) *$L_{S,i_\infty}(\varphi\otimes\check{\rho})$ has singularity at Δ in $\mathscr{X}_{\mathbb{I}}\times\mathscr{X}_{\mathbb{I}}$, which is actually a simple pole, if and only if $\lambda=\mu$ and the Leopoldt conjecture holds for p and F;*

(iii) *$D_p^{ord}(\pi(\lambda)\otimes\check{\pi}(\mu))=\frac{L_{S,i_\infty}(\varphi\otimes\check{\rho})}{L_{p,i_\infty}(Ad(\varphi))}$ up to finite Euler factors outside p and $\mathbb{I}$-units,*

where $D_p^{ord}(\pi(\lambda)\otimes\check{\pi}(\mu))$ is the ordinary version of $D_p(\pi(\lambda)\otimes\check{\pi}(\mu))$ which is constructed in [37] when $F=\mathbb{Q}$ and will be constructed in the following sections for general F.

The complex L-values as in (Int1) may not always be the one of primitive L-functions. Because of this defect, we cannot really claim that the identity in (iii) is only up to units. By (NP) in Chapter 4 and Theorems 5.2.1 and 5.2.2 of Chapter 5, $\alpha_\pm(\varphi\otimes\check{\rho})=\alpha_+(\varphi)\alpha_-(\varphi)=\alpha_+(Ad(\varphi))$ for $S=(\emptyset,I,\emptyset)$, and hence we have

$$\mathrm{ord}_\Delta L_{S,i_\infty}(\varphi\otimes\check{\rho})=\mathrm{ord}_\Delta L_S(\varphi\otimes\check{\rho})\quad\text{and}\quad\mathrm{ord}_\Delta L_{p,i_\infty}(Ad(\varphi))=\mathrm{ord}_\Delta L_p(Ad(\varphi)).$$

We have proven in [37] when $F=\mathbb{Q}$ that the ordinary version $D_p^{ord}(\pi(\lambda)\otimes\check{\pi}(\mu))$ has a simple pole at the diagonal if and only if $\lambda=\mu$. The technique of [37] can be generalized for general totally real F as we shall give a proof later in this chapter (see Sections 6.4-6.10). Putting the proof off to these sections, here we state only the outcome. For that, we prepare some notation. Since we are in the ordinary case, we have well defined Hecke operator $\mathbb{T}(\mathfrak{P})\in h^{ord}$ for each prime factor $\mathfrak{P}$ of p and

$\lambda(\mathbb{T}(\mathfrak{P}))$ and $\mu(\mathbb{T}(\mathfrak{P}))$ are units in $\mathbb{I}$. Since $Spec(h^{ord}(N))$ is a closed subscheme of $Spec(h^{n.ord})$, we can take the irreducible component $Spec(\mathbb{J}')$ of $Spec(h^{n.ord}(N))$ through which λ factors. Similarly we take the irreducible component $Spec(\mathbb{J}'')$ of $Spec(h^{n.ord}(J))$ through which μ factors. Thus we can take a common normal finite extension $\mathbb{J}$ of $\mathbb{J}'$ and $\mathbb{J}''$ so that $Spec(\mathbb{I})$ is a closed subscheme of $Spec(\mathbb{J})$. Then we take the Λ-algebra homomorphisms $\boldsymbol{\lambda} : h^{n.ord}(N) \to \mathbb{J}$ and $\boldsymbol{\mu} : h^{n.ord}(J) \to \mathbb{J}$ which induces λ and μ on the ordinary quotient. By [41] Theorem I, we have the nearly ordinary p-adic Rankin product $D_p(\pi(\boldsymbol{\lambda}) \otimes \check{\pi}(\boldsymbol{\mu}))$ in the quotient field of $\mathbb{J}\widehat{\otimes}\mathbb{J}$. Then we can think of the restriction of $D_p(\pi(\boldsymbol{\lambda}) \otimes \check{\pi}(\boldsymbol{\mu}))$ to $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$, for which we write $D_p^{ord}(\pi(\boldsymbol{\lambda}) \otimes \check{\pi}(\boldsymbol{\mu}))$. We know that $\Lambda^{ord} = \mathbb{O}[[\mathbf{W}^{ord}]]$. If $\mathbf{W}^{ord} \cong \mathbb{Z}_p$ ($\Leftrightarrow d = 0$), we pick a generator w of $\mathbf{W}^{ord}$; otherwise, we just pick one of generators w of $\mathbf{W}^{ord}$ such that $\mathcal{N}(w)$ generates $\mathcal{N}(\mathbf{W}^{ord})$. Then, if $d = 0$, $t_w = w^{-1} \otimes w - 1$ in $\Lambda^{ord}\widehat{\otimes}\Lambda^{ord}$ is a local parameter of Δ. For every function f on $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$ with at worst a simple pole at Δ, $\mathrm{Res}_\Delta f = (t_w f / \log_p \mathcal{N}(w))\big|_{t_w=0}$ is a function on Δ independent of the choice of w.

Theorem 6.3.2 — *Let $\lambda : h^{ord}(Np^\infty; \mathbb{O}) \to \mathbb{I}$ and $\mu : h^{ord}(Jp^\infty; \mathbb{O}) \to \mathbb{I}$ be primitive Λ^{ord}-algebra homomorphisms. Then there exists $D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu))$ in the field of fractions of $\mathbb{I}\widehat{\otimes}\mathbb{I}$ satisfying the following properties:*

(i) $$D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu)) = \prod_{\mathfrak{P}|p}(1 - \lambda(\mathbb{T}(\mathfrak{P}))/\mu(\mathbb{T}(\mathfrak{P})))^{-1} D_p^{ord}(\pi(\boldsymbol{\lambda}) \otimes \check{\pi}(\boldsymbol{\mu})),$$

(ii) *$D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu))$ has at most a simple pole at Δ and*
$$\mathrm{Res}_\Delta D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu)) = \delta_{\lambda,\mu} \frac{hR_p}{w\sqrt{D}} \prod_{\mathfrak{q}|Np} \left(1 - \frac{1}{N(\mathfrak{q})}\right),$$

where $\delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise,} \end{cases}$ *R_p is the p-adic regulator of F, D is the discriminant of F, h is the class number of F and $w = 2$ is the number of roots of unity in F. As for the parity of $R_p/\sqrt{D}$, it is determined so that the p-adic Dedekind zeta function $\zeta_{F,p}(s)$ of F satisfies*

$$\mathrm{Res}_{s=1} \zeta_{F,p}(s) = \frac{2^{[F:\mathbb{Q}]} hR_p}{w\sqrt{D}} \prod_{\mathfrak{P}|p} \left(1 - \frac{1}{N(\mathfrak{P})}\right) \quad \textit{(see [12]).}$$

When $d > 0$ ($\Leftrightarrow R_p = 0$), we prove that $D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu))$ is entire, and therefore, the above assertion still remains true independent of the choice of w. To have a legitimate L-values at specialization at $P \otimes Q$, we need to add to $D_p(P, Q; \pi(\boldsymbol{\lambda}) \otimes \check{\pi}(\boldsymbol{\mu}))$ Euler factors $\prod_{\mathfrak{P}|p}(1 - \lambda_P(\mathbb{T}(\mathfrak{P}))/\mu_Q(\mathbb{T}(\mathfrak{P})))^{-1}$, which does

not show up in $D_p^{ord}(\pi(\boldsymbol{\lambda})\otimes\check{\pi}(\boldsymbol{\mu}))$ because $V(\pi(\lambda)\otimes\check{\pi}(\mu))^{I_{\mathfrak{P}}}$ is one dimensional but $V(\pi(\boldsymbol{\lambda})\otimes\check{\pi}(\boldsymbol{\mu}))^{I_{\mathfrak{P}}}=\{0\}$. Thus the residue formula (ii) actually tell us the value of the $(r-1)$-th derivative $D_p^{ord}(\pi(\boldsymbol{\lambda})\otimes\check{\pi}(\boldsymbol{\mu}))$ at $P\otimes P$, where r is the order of zero at Δ of $\prod_{\mathfrak{P}|p}(1-\lambda(\mathbb{T}(\mathfrak{P}))/\mu(\mathbb{T}(\mathfrak{P})))$. If $R_p\neq 0$, the value is non-zero. We now deal with the nearly ordinary case:

Conjecture 6.3.3 — *Let $\lambda : h^{n.ord}(Np^\infty;\mathbb{O})\to\mathbb{I}$ and $\mu : h^{n.ord}(Jp^\infty;\mathbb{O})\to\mathbb{I}$ be two primitive Λ-algebra homomorphisms. Let $S=(\emptyset,I,\emptyset)$. Suppose that there exist genuine p-adic L-functions $L_{S,i_\infty}(\varphi\otimes\check{\rho})$ of type S and $L_{p,i_\infty}(Ad(\varphi))$. Then*

(i) *$L_{S,i_\infty}(\varphi\otimes\check{\rho})$ is holomorphic on $\mathscr{X}_{\mathbb{I}}\times\mathscr{X}_{\mathbb{I}}$; in other words, $L_{S,i_\infty}(\varphi\otimes\check{\rho})\in\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\Omega$;*

(ii) *$D_p(\pi(\lambda)\otimes\check{\pi}(\mu))=\frac{L_{S,i_\infty}(\varphi\otimes\check{\rho})}{L_{p,i_\infty}(Ad(\varphi))}$ up to units in $\mathbb{I}$ and finite Euler factors outside p.*

In the following paragraphs, we shall give an outline of a proof of Theorem 6.3.2. Since a full detail of construction of such L-function is given in [41] in the nearly ordinary case, we just satisfy ourselves with a brief description of difference in the proof of the p-ordinary case from the nearly p-ordinary case.

6.4 Auxiliary L-functions

We start preparing for the proof of Theorem 6.3.2. We need to introduce several notation to describe a crude form of the p-adic L-function which we later construct. We fix two ideals N and J prime to p in $\mathfrak{r}$, and let L be the least common multiple of N and J, that is, $L=N\cap J$. We consider two primitive Λ^{ord}-algebra homomorphisms $\lambda : h^{ord}(Np^\infty;\mathbb{O})\to\mathbb{I}$ and $\mu : h^{ord}(Jp^\infty;\mathbb{O})\to\mathbb{I}$, where $\mathbb{I}$ is a normal finite extension of $\mathbb{O}[[\mathbf{G}^{ord}]]$ for $\mathbf{G}^{ord}$ with respect to $\widehat{\Gamma}_1(N)$. By extending scalar if necessary, we may assume that $\mathbb{I}\cap\overline{\mathbb{Q}}_p=\mathbb{O}$. Let f (resp. g) be the $\mathbb{I}$-adic forms corresponding to λ (resp. μ) as explained in Section 2.7 of Chapter 2, and put $\varphi\in\pi(\lambda)$ and $\rho\in\pi(\mu)$. We first define certain Dirichlet series related to $L(s,\varphi_P\otimes\check{\rho}_Q)$ for positive arithmetic points P and Q. Since $det(\pi(\mu_Q))=\chi_Q\mathcal{N}$ for the central character χ_Q of $g(Q)$, as is well known, we have

$$\check{\pi}(\mu_Q)\cong(\chi_Q\mathcal{N})^{-1}\otimes\pi(\mu_Q)\cong\mathcal{N}^{-[m(P)]}\otimes\pi(\mu_Q^c),$$

where c is complex conjugation and $m(P)=[m(P)]t$ for an integer $[m(P)]$ (because $n(P)=0$). We define $\check{\mu}_Q(\mathbb{T}(\mathfrak{n}))=\mu_Q^c(\mathbb{T}(\mathfrak{n}))N_{F/Q}(\mathfrak{n})^{-[m(Q)]}$ and

$$\begin{aligned}L(s,\lambda_P\otimes\check{\mu}_Q)&=L_{Lp}(2s,\chi_Q^{-1}\psi_P)\sum_{0\neq\mathfrak{n}\subset\mathfrak{r}}\lambda_P(\mathbb{T}(\mathfrak{n}))\check{\mu}_Q(\mathbb{T}(\mathfrak{n}))N_{F/\mathbb{Q}}(\mathfrak{n})^{-s}\\&=L_{Lp}(2s,\chi_Q^{-1}\psi_P)\sum_{0\neq\mathfrak{n}\subset\mathfrak{r}}a(\mathfrak{n},f(P))a(\mathfrak{n},g(Q)^c))N_{F/\mathbb{Q}}(\mathfrak{n})^{-[m(Q)]-s}\end{aligned}\tag{L}$$

where "c" denotes complex conjugation and χ_Q (resp. ψ_P) is the central character of $g(Q)$ (resp. $f(P)$). Here the subscript "Lp" of the first factor $L_{Lp}(2s, \chi_Q^{-1}\psi_P)$ implies that this L-function has trivial Euler factor "1" at primes dividing Lp. Since the last expression is given without referring to λ or μ, the definition works well for any classical modular forms h and h' of weight $\kappa(P)$ and $\kappa'(Q)$. We thus define a similar L-function $D(s, h, h')$ by

$$\text{(D1)} \qquad D(s,h,h') = \sum_{0 \neq \mathfrak{n} \subset \mathfrak{r}} a(\mathfrak{n}, h) a(\mathfrak{n}, h') N_{F/\mathbb{Q}}(\mathfrak{n})^{-s-([m(P)]+[m(Q)])/2}.$$

This zeta function looks a bit different from the one introduced in [41] (4.6), and these two zeta functions are related in the following manner:

$$\text{(D2)} \qquad D(s,h,h') = D(s,h,h',\mathrm{id}),$$

where the right-hand side is the zeta function defined in [41] for the identity character $\theta = \mathrm{id}$.

We now recall the Euler factorization of $L(s, \lambda_P \otimes \check{\mu}_Q)$ in terms of automorphic representations spanned by $f(P)$ and $g(Q)$. The following formula also holds in the nearly ordinary case. For each automorphic representation $\pi = \otimes_{\mathfrak{l}} \pi_{\mathfrak{l}}$ of $G(\mathbb{A})$, we define $\alpha_{\mathfrak{l}}(\pi), \beta_{\mathfrak{l}}(\pi) \in \mathbb{C}$ as follows: choosing a prime element ϖ of $F_{\mathfrak{l}}$,

$$\alpha_{\mathfrak{l}}(\pi) = \begin{cases} \eta(\varpi) & \text{if } \pi_{\mathfrak{l}} \cong \pi(\eta, \eta') \text{ with } \eta \text{ unramified,} \\ \eta(\varpi) & \text{if } \pi_{\mathfrak{l}} \cong \sigma(\eta, \eta') \text{ with } \eta \text{ unramified and } \eta\eta'^{-1} = |\ |_{\mathfrak{l}}^{-1}, \\ 0 & \text{if } \pi_{\mathfrak{l}} \text{ is super cuspidal or } \pi_{\mathfrak{l}} \cong \pi(\eta, \eta') \text{ or } \sigma(\eta, \eta') \text{ with } \eta \text{ ramified,} \end{cases}$$

$$\beta_{\mathfrak{l}}(\pi) = \begin{cases} \eta'(\varpi) & \text{if } \pi_{\mathfrak{l}} \cong \pi(\eta, \eta') \text{ with } \eta' \text{ unramified,} \\ 0 & \text{if } \pi_{\mathfrak{l}} \cong \sigma(\eta, \eta') \text{ with } \eta \text{ unramified and } \eta\eta'^{-1} = |\ |_{\mathfrak{l}}^{-1}, \\ 0 & \text{if } \pi_{\mathfrak{l}} \text{ is super cuspidal or } \pi_{\mathfrak{l}} \cong \pi(\eta, \eta') \text{ or } \sigma(\eta, \eta') \text{ with } \eta' \text{ ramified.} \end{cases}$$

Write π (resp. π') for the automorphic representation of $G(\mathbb{A}) = GL_2(F_{\mathbb{A}})$ generated by right translation of $f(P)$ (resp. $g(Q)$). We decompose $\pi = \otimes_{\mathfrak{l}} \pi_{\mathfrak{l}}$ and the contragredient $\check{\pi}' = \otimes_{\mathfrak{l}} \check{\pi}'_{\mathfrak{l}}$. We put

$$L_{\mathfrak{l}}(X) = (1 - \alpha_{\mathfrak{l}}(\pi)\alpha_{\mathfrak{l}}(\check{\pi}')X)(1 - \alpha_{\mathfrak{l}}(\pi)\beta_{\mathfrak{l}}(\check{\pi}')X)(1 - \beta_{\mathfrak{l}}(\pi)\alpha_{\mathfrak{l}}(\check{\pi}')X)(1 - \beta_{\mathfrak{l}}(\pi)\beta_{\mathfrak{l}}(\check{\pi}')X).$$

Then we know

$$\text{(E1)} \qquad L(s, \lambda_P \otimes \check{\mu}_Q) = \prod_{\mathfrak{l}: prime} L_{\mathfrak{l}}(N(\mathfrak{l})^{-s})^{-1}.$$

Then it is easy to check (see Lemma 6.6.1 below)

$$\text{(E2)} \qquad L(s, \lambda_P \otimes \check{\mu}_Q) = L_{Lp}(2s, \psi_P \chi_Q^{-1}) D(s - 1 + \frac{[m(Q) - m(P)]}{2}, f(P), g(Q)^c).$$

In some cases, in particular if $P = Q$ and $\lambda = \mu$, the above L-function may not be primitive (that is, $L(s, \lambda_P \otimes \check{\mu}_Q)$ possibly differ from $L(s, \pi(\lambda_P) \otimes \check{\pi}(\mu_Q))$ by finitely many Euler factors). It is easy to determine the missing Euler factors of $L(s, \lambda_P \otimes \check{\mu}_Q)$ comparing the definition of $L(s, \lambda_P \otimes \check{\mu}_Q)$ with that of $L(s, \varphi_P \otimes \check{\rho}_Q)$ (see (L) in Section 3.1 of Chapter 3).

For a complex modular form h of weight κ with $n(\kappa) = 0$, its unitarization h^u is defined by

$$h^u(x) = |D_F|^{-([m(\kappa)]+1)/2} h(x) |det(x)|_{\mathbb{A}}^{([m(\kappa)]-1)/2}. \tag{U}$$

The right translations of h^u under $G(\mathbb{A})$ span a unitary automorphic representation. To describe the level of $f(P)$ and $g(Q)$ exactly, we write S_F for the set of all prime factors $\mathfrak{P}$ of p in $\mathfrak{r}$. For each S_F-tuple $\alpha = (\alpha(\mathfrak{P}))_{\mathfrak{P} \in S_F}$ of integers $\alpha(\mathfrak{P})$, we write $\mathfrak{P}^\alpha = \prod_{\mathfrak{P} \in S_F} \mathfrak{P}^{\alpha(\mathfrak{P})}$. Let κ' (resp. κ) be the weight of μ (resp. λ), that is, the restriction of μ (resp. λ) to $\mathbf{G}^{ord}$. For each arithmetic point P of $\mathscr{X}_{\mathbb{I}}$, we write $\kappa(P)$ for the (arithmetic) weight of P. Then for sufficiently large α,

$$f(P) \in S^{ord}_{\kappa(P)}(N\mathfrak{P}^\alpha; \overline{\mathbb{Q}}) \quad \text{and} \quad g(Q) \in S^{ord}_{\kappa(Q)}(J\mathfrak{P}^\alpha; \overline{\mathbb{Q}}).$$

We can easily relate $g(Q)$ with the modular form $g(Q)^c$ whose Fourier coefficients are given by $a(y, g(Q))^c$. For that purpose, let $\varpi_{\mathfrak{P}}$ be the prime element in $\mathfrak{r}_{\mathfrak{P}}$ for $\mathfrak{P} \in S_F$ and let n (resp. j) be a finite idele such that $n\mathfrak{r} = N$ (resp. $j\mathfrak{r} = J$) and $n_{\mathfrak{q}} = 1$ (resp. $j_{\mathfrak{q}} = 1$) outside N (resp. J). Define an idele $n\varpi^\alpha$ so that $(n\varpi^\alpha)_{\mathfrak{q}} = 1$ for $\mathfrak{q}$ outside Np, $(n\varpi^\alpha)_{\mathfrak{q}} = n_{\mathfrak{q}}$ for prime $\mathfrak{q}$ dividing N and $(n\varpi^\alpha)_{\mathfrak{P}} = \varpi_{\mathfrak{P}}^{\alpha(\mathfrak{P})}$ for $\mathfrak{P} \in S_F$. We then define $\tau(n\varpi^\alpha)$ by

$$\tau(n\varpi^\alpha) = \begin{pmatrix} 0 & -1 \\ n\varpi^\alpha & 0 \end{pmatrix} \in G(\mathbb{A}^{(\infty)}).$$

For each modular form ϕ in $M_\kappa(N\mathfrak{P}^\alpha; \mathbb{C})$, if ϕ has a central character ξ, we define the unitarization ξ^u by $\xi^u = \xi/|\xi|$ which is of finite order. Then, we define

$$\phi|\tau(n\varpi^\alpha)(x) = \xi^u(det(x))\phi(x\tau(n\varpi^\alpha)) \in M_{\kappa^c}(N\mathfrak{P}^\alpha; \mathbb{C}),$$

where κ^c is the complex conjugation of κ. We define the complex conjugate h^c of h by $a(y, h^c) = a(y, h)^c$ for all y. Then $g(Q)^c$ is a non-zero constant multiple of $g(Q)|\tau(j\varpi^\alpha)$ if $g(Q)$ is primitive of conductor $J\mathfrak{P}^\alpha$ (that is, of exact level $J\mathfrak{P}^\alpha$).

6.5 Ordinary p-adic Rankin products

Let $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}$ be the profinite completion of $\mathbb{I} \otimes_{\mathbb{O}} \mathbb{I}$ (that is, the $\mathfrak{m}$-adic completion of $\mathbb{I} \otimes \mathbb{I}$ for the unique maximal ideal $\mathfrak{m}$ of $\mathbb{I} \otimes \mathbb{I}$). Under $\mathbb{I} \cap \overline{\mathbb{Q}}_p = \mathbb{O}$, $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}$ is an integral

domain. For each pair of points (P,Q) in $\mathcal{Y}=\mathcal{X}_{\mathbb{I}}\times\mathcal{X}_{\mathbb{I}}$, regarding P and Q as $\mathbb{O}$-algebra homomorphisms of $\mathbb{I}$ into Ω, we have an $\mathbb{O}$-algebra homomorphism $P\otimes Q:\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}\to\Omega$. Thus for any $\Phi\in\mathbb{I}\widehat{\otimes}\mathbb{I}$, we can regard Φ as a p-adic analytic function on $\mathcal{Y}$ with values in Ω by $\Phi(P,Q)=P\otimes Q(\Phi)$. Even if $\Phi=\Psi/\Psi'$ is an element in the quotient field of $\mathbb{I}\widehat{\otimes}\mathbb{I}$, we can think of Φ as a p-adic meromorphic function on $\mathcal{Y}$ whose value is given by $\Phi(P,Q)=\Psi(P,Q)/\Psi'(P,Q)$ whenever we can choose a denominator Ψ' so that $\Psi'(P,Q)\neq 0$. We first state our result in a crude form and later deduce out of it the desired p-adic L-function $D_p^{ord}=D_p^{ord}(\pi(\lambda)\otimes\check{\pi}(\mu))$.

Theorem 6.5.1 — *There exists a unique element D^{ord} in the quotient field of $\mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathbb{I}$ satisfying the following interpolation property: Let $(P,Q)\in\mathcal{A}(\mathbb{I})\times\mathcal{A}(\mathbb{I})$ satisfying (Ad) in Section 6.1 ($\Leftrightarrow m(P)>m(Q)>0$ in the ordinary case). Then D^{ord} is finite at (P,Q), and we have, for sufficiently large α,*

(a)
$$\begin{aligned}&(\chi_Q)_\infty(-1)D^{ord}(P,Q)N_{F/\mathbb{Q}}(N\mathfrak{P}^\alpha)^{([m(P)]-1)/2}N_{F/\mathbb{Q}}(J\mathfrak{P}^\alpha)^{-1}\\&=(\chi_Q\psi_P)_\infty(-1)C_0(P,Q)\frac{L_{Lp}(2,\psi_P\chi_Q^{-1})D(\frac{[m(Q)-m(P)]}{2},f(P),g_\alpha)}{((f(P)^u)^c|\tau(n\varpi^\alpha),f(P)^u)_\alpha}\end{aligned}$$

where $(\ ,\ )_\alpha$ denotes the Petersson inner product of level $N\mathfrak{P}^\alpha$ (defined in [41] Section 4), $g_\alpha=g(Q)|\tau(j\varpi^\alpha)$ and

$$C_0(P,Q)=|D|^{1+[m(Q)-m(P)]}(2\pi i)^{m(P)-m(Q)}2^{-m(P)}(2\pi)^{-m(P)-2t}\Gamma([m(Q)]+1)^{[F:\mathbb{Q}]}.$$

Let $\mathfrak{x}$ be an ideal prime to pL and $\langle\mathfrak{x}\rangle$ be the projection of the class $[\mathfrak{x}]$ of $\mathfrak{x}$ to $\mathbf{W}^{ord}$. Then, for any $H\in\mathbb{I}$ which kills the congruence module $C_0(\lambda;\mathbb{I})$ for $h^{ord}(Np^\infty;\mathbb{O})$, $(1-\delta_{\lambda,\mu}\langle\mathfrak{x}\rangle\otimes\langle\mathfrak{x}\rangle^{-1})(H\otimes 1)D^{ord}$ is integral (i.e. $(1-\delta_{\lambda,\mu}\langle\mathfrak{x}\rangle\otimes\langle\mathfrak{x}\rangle^{-1})(H\otimes 1)D^{ord}\in\mathbb{I}\widehat{\otimes}\mathbb{I}$) and we have

(b)
$$\left\{(1-\langle\mathfrak{x}\rangle\otimes\langle\mathfrak{x}\rangle^{-1})D^{ord}\right\}(P,P)=-\log_p(\mathcal{N}(\langle\mathfrak{x}\rangle)\frac{h(F)R_p}{w\sqrt{D}}\prod_{\mathfrak{q}|Np}\left(1-\frac{1}{N_{F/\mathbb{Q}}(\mathfrak{q})}\right)\delta_{\lambda,\mu},$$

where $\delta_{\lambda,\mu}=1$ or 0 according as $\lambda=\mu$ or not.

In Theorem 5.1 of [41] which is the nearly p-ordinary counterpart of the above theorem, we formulated the result in terms of the constant $C(P,Q)$ given by $(\chi_Q\psi_P)_\infty(-1)C_0(P,Q)$. This theorem is essentially the specialization of Theorem 5.1 in [41] to the p-ordinary line. However, the construction of D^{ord} is a bit different from that of $\mathfrak{D}$ in [41], because we use the ordinary Eisenstein measure instead of the nearly ordinary Eisenstein measure employed in [41]. Since the proof of Theorem 6.5.1 is exactly parallel to that of Theorem 5.1 in [41] in this sense, we recall in details the construction of the ordinary Eisenstein measure in Section 6.7

and its convolution along λ in Sections 6.8 and 6.9, but the description of the proof of Theorem 6.5.1 is brief (Section 6.10).

We now try to relate $D(s, f(P), g_\alpha)$ with $L_p(s, \lambda_P \otimes \check{\mu}_Q)$. Let us introduce further notation: Let h_0 be a common eigenform of all Hecke operators corresponding to an $\mathbb{C}$-algebra homomorphism $\phi : h_\rho(Np^\beta; \mathbb{C}) \to \mathbb{C}$ with $n(\rho) = 0$. Let h be the primitive form of conductor $N\mathfrak{P}^\alpha$ $(0 \le \alpha \le \beta)$ associated with h_0 (and ϕ). Write $h|\mathbb{T}(\mathfrak{l}) = \phi^\circ(\mathbb{T}(\mathfrak{l}))h$. Then $\phi^\circ(\mathbb{T}(\mathfrak{l})) = \phi(\mathbb{T}(\mathfrak{l}))$ for almost all primes $\mathfrak{l}$. Let $L(s, \phi) = \sum_{\mathfrak{n}} \phi^\circ(\mathfrak{n})N(\mathfrak{n})^{-s}$ be the standard L-function for ϕ. Then we know that $h^u|\tau(n\varpi^\alpha) = W(\phi)(h^u)^c$ for a constant $W(\phi)$ with absolute value 1. In fact, it is well known that (e.g. [41] (4.10)) with a suitable Γ-factor $\Gamma(s, \phi)$, we have a functional equation:

(FE1)

$$\Gamma(s, \phi)L(s, \phi) = i^{-k}W(\phi)|d^2 n\varpi^\alpha|_{\mathbb{A}}^{s-([k]/2)}\Gamma(1-s, \check{\phi})L(1-s, \check{\phi}) \quad (k = m(\rho) + t),$$

where $\check{\phi}$ corresponds to the contragredient $\check{\pi}(\phi)$ of $\pi(\phi)$, $d \in F_{\mathbb{A}(\infty)}^\times$ is a finite idele giving the absolute different dx and

$$\check{\phi}(\mathbb{T}(\mathfrak{l})) = N(\mathfrak{l})^{-[m(\rho)]}\phi(\mathbb{T}(\mathfrak{l}))^c.$$

Thus the root number of the functional equation is given by $i^{-k}W(\phi)$. Let $\pi = \otimes_{\mathfrak{q}}\pi_{\mathfrak{q}}$ be the automorphic representation generated by the right translations of h_0. As is well known, the constant term of the functional equation $i^{-k}W(\phi)$ can be decomposed into the product of local factors (e.g. [41] (4.10c)). To describe this, we assume that $\pi_{\mathfrak{P}}$ for prime factors $\mathfrak{P}$ of p is either principal or special. For each quasi-character $\xi : F_{\mathfrak{P}}^\times \to \mathbb{C}^\times$, we define

$$\varepsilon(\xi) = \xi(d\varpi^r)G(\xi)/|\xi(d\varpi^r)G(\xi)| \quad (G(\xi) = \sum_{u \bmod \varpi^r} \xi^{-1}(u)\mathbf{e}_{F_{\mathfrak{P}}}(u/\varpi^r d)),$$

where $\mathbf{e}_{F_{\mathfrak{P}}}(x) = exp(-2\pi i[Tr_{F_{\mathfrak{P}}/\mathbb{Q}_p}(x)]_p)$ for the p-fractional part $[y]_p$ of $y \in \mathbb{Q}_p$, ϖ is a prime element of $F_{\mathfrak{P}}$, $\mathfrak{P}^r$ is the conductor of ξ and $dr_{\mathfrak{P}}$ is the different of $F_{\mathfrak{P}}$ for $d \in F_{\mathfrak{P}}$. The appearance of ξ^{-1} instead of ξ in the definition of the Gauss sum $G(\xi)$ is to adjust with the classical definition. In [41], the symbol $G(\xi^{-1})$ is used for the above $G(\xi)$. Actually for $a \in F \cap \mathrm{r}_{\mathfrak{P}}^\times$ and a Hecke character η, $1 = \eta(a) = \eta_{\mathfrak{P}}(a)\eta^{(\mathfrak{P})}(a)$ and hence $\eta^{(\mathfrak{P})}(a) = \eta_{\mathfrak{P}}(a)^{-1}$. It is common to use $a \mapsto \eta^{(\mathfrak{P})}(a)$ to define the global Gauss sum. We simply put $\varepsilon(\xi) = \xi(d)/|\xi(d)|$ if ξ is unramified. Then we can decompose

$$W(\phi) = W'(\phi)\prod_{\mathfrak{P}|p} W_{\mathfrak{P}}(\phi), \tag{W1}$$

where

$$W_{\mathfrak{P}}(\phi) = \varepsilon(\xi)\varepsilon(\xi') \quad \text{if either } \pi_{\mathfrak{P}} = \sigma(\xi, \xi') \text{ with ramified } \xi \text{ or } \pi(\xi, \xi'),$$
$$W_{\mathfrak{P}}(\phi) = -(\xi(\varpi)/|\xi(\varpi)|) \quad \text{if } \pi_{\mathfrak{P}} = \sigma(\xi, \xi') \text{ with unramified } \xi.$$

Let P and Q be two positive arithmetic points of $\mathscr{X}_{\mathbb{I}}$. To state our result, write $\pi_{\mathfrak{P}}$ (resp. $\pi'_{\mathfrak{P}}$) for the $\mathfrak{P}$-component of the automorphic representation generated by $f(P)$ (resp. $g(Q)$). Since $f(P)$ and $g(Q)$ are ordinary, for $\mathfrak{P} \in S_F$, $\pi_{\mathfrak{P}}$ and $\pi'_{\mathfrak{P}}$ are either special or principal. Thus we write $\pi_{\mathfrak{P}} = \sigma(\eta_{\mathfrak{P}}, \eta'_{\mathfrak{P}})$ or $\pi(\eta_{\mathfrak{P}}, \eta'_{\mathfrak{P}})$ and $\pi'_{\mathfrak{P}} = \sigma(\xi_{\mathfrak{P}}, \xi'_{\mathfrak{P}})$ or $\pi(\xi_{\mathfrak{P}}, \xi'_{\mathfrak{P}})$. Here we may assume that $\xi_{\mathfrak{P}}$ and $\eta_{\mathfrak{P}}$ are unramified and $|\xi_{\mathfrak{P}}(p)|_p = |\eta_{\mathfrak{P}}(p)|_p = 1$. Then we introduce a Gauss sum factor:

$$\text{(W2)} \qquad W_0(P,Q) = \frac{N(J)^{([m(Q)]+1)/2} W'(\mu_Q)}{N(N)^{([m(P)]-1)/2} W'(\lambda_P)} \prod_{\mathfrak{P} \in S_F} \frac{\xi\xi'(d_{\mathfrak{P}})|\eta\eta'(d_{\mathfrak{P}})|G(\xi'_{\mathfrak{P}})}{\eta\eta'(d_{\mathfrak{P}})|\xi\xi'(d_{\mathfrak{P}})|G(\eta'_{\mathfrak{P}})},$$

where d is the idele fixed in Section 2.3 of Chapter 2 such that $d\mathfrak{r}$ is the different of $F/\mathbb{Q}$. In [41] Section 5, instead of $W_0(P,Q)$, we used the constant $W(P,Q) = (\chi_Q \psi_P)_\infty(-1) W_0(P,Q)$. We now deduce from Theorem 6.5.1 the following result, which is the ordinary version of Theorem 5.2 in [41]:

Theorem 6.5.2 — *We have*

$$D^{ord}(P,Q) = W_0(P,Q) C_0(P,Q) S(P)^{-1} E'(P,Q) \frac{L_p(1, \lambda_P \otimes \check{\mu}_Q)}{(f_P^\circ, f_P^\circ)_\delta},$$

for all pairs of arithmetic points (P,Q) satisfying (Ad), where f_P° is the primitive form of conductor $N\mathfrak{P}^\delta$ associated with $f(P)^u$, and $S(P)$ and $E'(P,Q)$ are Euler factors at p which will be described in Lemma 6.6.1 below.

6.6 Deduction of Theorem 6.5.2 from Theorem 6.5.1

We can deduce Theorem 6.5.2 from Theorem 6.5.1 in exactly the same manner as the deduction of [41] Theorem 5.2 from [41] Theorem 5.1. The only difference from the argument in [41] is that the Fourier p-coefficients of g_α in Theorem 6.5.1 does not vanish, while we removed those p-coefficients in [41] via the twist by the character θ. Thus in our ordinary case, θ is the identity (and ψ'_P and χ'_Q in [41] Section 5 are also trivial by the ordinarity of $f(P)$ and $g(Q)$). Anyway the computation in [41] p.349-351 is valid without any modification making all the values of θ and χ'_Q to be trivial, and as in (5.6c) of [41], we may assume

$$g(Q) \text{ is of exact level } J\mathfrak{P}^\alpha.$$

We now want to compute $g_\alpha|\tau(j\varpi^\alpha)$. This part is a bit different from the computation in [41] p.351 because $g(Q)$ has Fourier p-coefficients. This non-vanishing of p-coefficients affect the form of Lemma 5.3 [41] which is the key to the deduction. To formulate the p-ordinary version of Lemma 5.3 of [41], we recall some notation from [41]. Let χ be the central character of $\pi' = \otimes_{\mathfrak{q}} \pi'_{\mathfrak{q}}$ generated by right traslations of $g(Q)$. We consider its space $V = V(\pi')$ in $L_2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$, which

is the space of square integrable functions f (modulo the center) on $G(\mathbb{Q})\backslash G(\mathbb{A})$ with $f(gz)=\chi(z)f(g)$. Suppose that for each infinite place $\sigma\in I$, π'_σ is the discrete series representation $\sigma(\mu_1,\mu_2)$ with

$$\mu_1(a)=|a|^{m_\sigma/2} \quad \text{and} \quad \mu_2(a)=a^{m_\sigma-1}|a|^{m_\sigma/2},$$

where $m=m(Q)$. Let $\mathfrak{m}$ be an ideal of $\mathfrak{r}$. For any pair (ξ,ξ') of finite order character of $(\mathfrak{r}/\mathfrak{m})^\times$, regarding them as characters of $\mathfrak{r}_\mathfrak{m}^\times$, we consider the subspace $V(\xi,\xi')=V(\pi';\xi,\xi';\mathfrak{m})$ consisting of functions $\phi\in V$ satisfying the following conditions:

(i) ϕ is the lowest weight vector of weight $k=m(Q)+t$

(that is, ϕ corresponds to a holomorphic modular form of weight k);

(ii) $\phi\left(x\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)=\xi(a_\mathfrak{m})\xi'(d_\mathfrak{m})\phi(x)$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\widehat{\Gamma}_0(\mathfrak{m})$,

where $\widehat{\Gamma}_0(\mathfrak{m})=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G(\widehat{\mathbb{Z}})|c\in\mathfrak{m}\widehat{\mathfrak{r}}\}$. Since the central character of π' is given by χ, $V(\xi,\xi')$ is trivial unless $\xi\xi'=\chi$ on $\widehat{\mathfrak{r}}^\times$. Let C be the conductor of π' in the sense of [11]. Then $V(\mathrm{id},\chi;C)$ is one dimensional and spanned by a unique element f° with $a(1,f^\circ)=1$. This form f° is called the primitive form associated with the representation π'. We can describe the subspaces $V(\xi,\xi')$ locally. For that, let us write $\pi'=\otimes_\mathfrak{q}\pi'_\mathfrak{q}$ for a local representation $\pi'_\mathfrak{q}$ at each place $\mathfrak{q}$ and take a representation space $V_\mathfrak{q}$ for $\pi'_\mathfrak{q}$. When $\mathfrak{q}$ is finite, we define $V(\xi_\mathfrak{q},\xi'_\mathfrak{q};\mathfrak{m}_\mathfrak{q})$ by the subspace of $V_\mathfrak{q}$ consisting of vectors v satisfying $\pi'_\mathfrak{q}\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)v=\xi_\mathfrak{q}(a)\xi'_\mathfrak{q}(d)v$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\widehat{\Gamma}_0(\mathfrak{m})_\mathfrak{q}$. When $\mathfrak{q}$ is an infinite place σ, $V(\xi_\mathfrak{q},\xi'_\mathfrak{q};\mathfrak{m}_\mathfrak{q})$ is the space of lowest weight vectors of weight k_σ. Then $V(\xi_\mathfrak{q},\xi'_\mathfrak{q};\mathfrak{m}_\mathfrak{q})$ is one dimensional for almost all $\mathfrak{q}$ and, fixing a generator $v_\mathfrak{q}$ for such places, we can naturally identify

$$V(\xi,\xi';\mathfrak{m})=\bigotimes_\mathfrak{q} V(\xi_\mathfrak{q},\xi'_\mathfrak{q};\mathfrak{m}_\mathfrak{q})$$

as modules over the (complex) Hecke algebra with respect to $\widehat{\Gamma}_0(\mathfrak{m})$.

Let g_Q be the primitive form associated with $V=V(\pi')$. We want to express the special value:

$$D(\frac{[m(Q)-m(P)]}{2},f(P),g_\alpha)$$

by means of $L(s,\lambda_P\otimes\check{\mu}_Q)$ (which is equal to $L(s,\pi\times\check{\pi}')$ up to finitely many Euler factors). By definition, $g(Q)$ belongs to $V(\pi';\mathrm{id},\chi;J\mathfrak{P}^\alpha)$. It is known that $\pi'_\mathfrak{P}$ for $\mathfrak{P}\in S_F$ is either principal or special, and the special representation occurs only when $m_\sigma=1$ for all the p-adic places $\sigma: F\hookrightarrow\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$ (see [34] Lemma 12.2, [38] Proposition 2.1 and Corollary 2.2). Write $\pi'_\mathfrak{P}=\pi(\xi_\mathfrak{P},\xi'_\mathfrak{P})$ (resp. $\sigma(\xi_\mathfrak{P},\xi'_\mathfrak{P})$ with

$\xi_{\mathfrak{P}}{\xi'_{\mathfrak{P}}}^{-1} = |\ |_p)$ when $\pi'_{\mathfrak{P}}$ is principal (resp. special). Here, we follow the convention explained below [38] Proposition 2.1 to describe principal series representations and special ones. Writing $C(\xi'_{\mathfrak{P}}) = \mathfrak{P}^{\gamma'(\mathfrak{P})}$ for the conductor of $\xi'_{\mathfrak{P}}$, we now define subsets of S_F as follows:

$$\begin{aligned}
\Sigma_1 &= \{\mathfrak{P} \in S_F | \gamma'(\mathfrak{P}) > 0\} \\
\Sigma_3 &= \{\mathfrak{P} \in S_F | \gamma'(\mathfrak{P}) = 0 \text{ and } \pi'_{\mathfrak{P}} \text{ is principal}\} \\
\Sigma_4 &= \{\mathfrak{P} \in S_F | \gamma'(\mathfrak{P}) = 0 \text{ and } \pi'_{\mathfrak{P}} \text{ is special}\}.
\end{aligned}$$

Here we use the same index system as in [41] Lemma 5.3 in order to facilitate comparison with the lemma there. Especially, in our p-ordinary case, the set Σ_2 in [41] will not show up. By [38] Corollary 2.2, we may assume that $\xi_{\mathfrak{P}}$ (resp. $\xi'_{\mathfrak{P}}$) coincides with the trivial character on $\mathfrak{r}_{\mathfrak{P}}^{\times}$ (resp. $\chi_{\mathfrak{P}}$) and that $g(Q)|\mathbb{T}(\mathfrak{P}) = \xi_{\mathfrak{P}}(\varpi_{\mathfrak{P}})g(Q)$ for the Hecke operator $\mathbb{T}(\mathfrak{P})$.

Similarly as above, let $\pi = \otimes_{\mathfrak{q}}\pi_{\mathfrak{q}}$ be the automorphic representation spanned by $f(P)$ and write its p-component as $\pi(\eta_{\mathfrak{P}}, \eta'_{\mathfrak{P}})$ or $\sigma(\eta_{\mathfrak{P}}, \eta'_{\mathfrak{P}})$ for $\mathfrak{P} \in S_F$ so that $f(P)|\mathbb{T}(\mathfrak{P}) = \eta(\varpi_{\mathfrak{P}})f(P)$ (where $\eta_{\mathfrak{P}}$ is unramified at $\mathfrak{P} \in S_F$). We define

$$\begin{aligned}
\Sigma_1(P) &= \{\mathfrak{P} \in S_F |\ \eta'_{\mathfrak{P}} \text{ is ramified}\} \\
\Sigma_2(P) &= \{\mathfrak{P} \in S_F |\ \pi_{\mathfrak{P}} \text{ is principal}\} \\
\Sigma_3(P) &= \{\mathfrak{P} \in S_F |\ \pi_{\mathfrak{P}} \text{ is special}\}.
\end{aligned}$$

Then Theorem 6.5.2 is a direct consequence of Theorem 6.5.1 and the following result:

Lemma 6.6.1 — (i) *We have*

$$\alpha(\mathfrak{P}) = \begin{cases} 1 & \text{when } \mathfrak{P} \in \Sigma_3 \cup \Sigma_4 \\ \gamma'(\mathfrak{P}) & \text{otherwise,} \end{cases}$$

where $J\mathfrak{P}^{\alpha}$ is the exact level of $g(Q)$ and $\alpha = (\alpha(\mathfrak{P}))_{\mathfrak{P}\in S_F}$.

(ii) *Write e for $[m(Q)] - 1$ and define operators $B_{\mathfrak{P}}$ for each $\mathfrak{P} \in S_F$ acting on $V(\pi''; \mathrm{id}, \chi_Q^{-1}; J\mathfrak{P}^{\alpha})$ for the unitarization π'' of the contragredient of π' by*

$$\begin{aligned}
B_{\mathfrak{P}} &= \mathrm{Id} \quad \text{if } \mathfrak{P} \in \Sigma_1 \cup \Sigma_4, \\
B_{\mathfrak{P}} &= \mathrm{Id} - |\varpi_{\mathfrak{P}}|_p^{-(e/2)-1}{\xi'}^{-1}(\varpi_{\mathfrak{P}})[\varpi_{\mathfrak{P}}]_0 \quad \text{if } \mathfrak{P} \in \Sigma_3.
\end{aligned}$$

Then we have $\{g(Q)|\tau(j\varpi^{\alpha})\}^u = C_0(g(Q)^c)^u|B$ for $B = \prod_{\mathfrak{P}\in S_F} B_{\mathfrak{P}}$, where

$$C_0 = W(\mu_Q)|j\varpi^{\alpha}|_{\mathbb{A}}^{-e/2} \prod_{\mathfrak{P}\in\Sigma_3} (-\xi'(\varpi_{\mathfrak{P}}))|\varpi_{\mathfrak{P}}|_{\mathfrak{P}}^{(e/2)+1}$$

and $h|[\varpi_{\mathfrak{P}}^r]_0(x) = h\left(x \begin{pmatrix} \varpi_{\mathfrak{P}}^{-r} & 0 \\ 0 & 1 \end{pmatrix}\right)$ for $h \in V(\pi''; \mathrm{id}, \chi_Q^{-1}; J\mathfrak{P}^{\alpha})$.

(iii) *Define an Euler factor* $E'_{\mathfrak{P}}(P,Q)$ *at each* $\mathfrak{P} \in S_F$ *as follows:*

$$\begin{cases} \dfrac{(1-\xi'\eta^{-1}(\varpi))}{(1-\xi^{-1}\eta(\varpi)|\varpi|_{\mathfrak{P}})(1-\xi'^{-1}\eta(\varpi)|\varpi|_{\mathfrak{P}})} & \text{if } \mathfrak{P} \in \Sigma_3, \\ \xi'\eta^{-1}(\varpi^{\gamma'(\mathfrak{P})})\dfrac{1}{(1-\xi^{-1}\eta(\varpi)|\varpi|_{\mathfrak{P}})} & \text{if } \mathfrak{P} \in \Sigma_1, \\ -\xi\eta^{-1}(\varpi)\dfrac{1}{(1-\xi^{-1}\eta(\varpi)|\varpi|^2_{\mathfrak{P}})} & \text{if } \mathfrak{P} \in \Sigma_4, \end{cases}$$

where we have written ϖ *for* $\varpi_{\mathfrak{P}}$. *Then the Euler factor at* $\mathfrak{P}$ *of*

$$L_{Lp}(2,\chi_Q^{-1}\psi_P)D(\frac{[m(Q)-m(P)]}{2}, f(P), g(Q)|\tau(j\varpi^{\alpha}))$$

is given by $C_1E(P,Q)$, *where* $E'(P,Q) = \prod_{\mathfrak{P}\in S_F} E'_{\mathfrak{P}}(P,Q)$ *and*

$$C_1 = W'(\mu(Q))|j|_{\mathbb{A}}^{(1-[m(Q)])/2} \prod_{\mathfrak{P}\in S_F} \frac{\xi\xi'(d_{\mathfrak{P}})\eta(\varpi^{\alpha(\mathfrak{P})})|\varpi^{\alpha(\mathfrak{P})}|_{\mathfrak{P}}G(\xi_{\mathfrak{P}})}{|\xi\xi'(d_{\mathfrak{P}})|}.$$

(iv) *Write* f_P° *for the primitive form of the unitarization of* π *and* $C(\pi) = N\mathfrak{P}^{\delta}$ *and define for* $s = \#(\Sigma_3(P))$

$$S(P) = (-1)^s \prod_{\mathfrak{P}\in\Sigma_2(P)} (1-\eta^{-1}\eta'(\varpi_{\mathfrak{P}})|\varpi_{\mathfrak{P}}|_{\mathfrak{P}})(1-\eta^{-1}\eta'(\varpi_{\mathfrak{P}})) \times \prod_{\mathfrak{P}\in\Sigma_1(P)} \eta'\eta^{-1}(\varpi_{\mathfrak{P}}^{\delta(\mathfrak{P})})|\varpi_{\mathfrak{P}}^{\delta(\mathfrak{P})}|_{\mathfrak{P}}.$$

Then we have

$$\frac{((f(P)^c)^u|\tau(n\varpi^{\alpha}), f(P)^u)_{\alpha}}{(f_P^{\circ}, f_P^{\circ})_{\delta}} = |\varpi^{\alpha}|_{\mathbb{A}}^{([m(P)]-1)/2}\eta(\varpi^{\alpha})\psi_{P,\infty}(-1)W'(\lambda_P)S(P)\prod_{\mathfrak{P}\in\Sigma_1(P)} G(\eta')\prod_{\mathfrak{P}\in S_F}\frac{\eta\eta'(d_{\mathfrak{P}})}{|\eta\eta'(d_{\mathfrak{P}})|}.$$

The assertions (i) and (iv) follows from [41] Lemma 5.3 (ii), (vi), and the assertion (ii) follows from [41] (5.8). Since the proof (iii) is a straightforward computation similar to (and simpler than) that of Lemma 5.3 (iv) of [41], we do not repeat it here and leave it to the attentive reader. From this lemma, we conclude the assertion of Theorem 6.5.2 in the same manner as in [41] Section 5. In the sense of Conjecture 4.2.1 (iii) in Chapter 4, the Euler factor $E'(P,Q)$ as above is associated to $L_{S,i_\infty}(\pi(\lambda)\otimes\check{\pi}(\mu))$, and $S(P)$ is associated to $L_{p,i_\infty}(Ad(\pi(\lambda)))$. This fact supports the description (iii) of Conjecture 6.3.1 and (ii) of Conjecture 6.3.3.

6.7 p-Ordinary adelic Eisenstein measure

Here we give a construction of the ordinary Eisenstein measure. Let

$$U_F(Lp^\alpha) = \left\{x \in \widehat{\mathfrak{r}}^\times \middle| x \equiv 1 \mod Lp^\alpha\widehat{\mathfrak{r}}\right\} \quad \text{and} \quad U_F(Lp^\infty) = \bigcap_{\alpha>0} U_F(Lp^\alpha).$$

We consider the ray class groups Z and $\overline{Z}$ defined as follows:

$$\overline{Z} = \overline{Z}(L) = F_{\mathbb{A}}^\times/\overline{F^\times U_F(Lp^\infty)F_\infty^\times} \quad \text{and} \quad Z = Z(L) = F_{\mathbb{A}}^\times/\overline{F^\times U_F(Lp^\infty)F_{\infty+}^\times},$$

where $\overline{F^\times U_F(Lp^\infty)F_\infty^\times}$ is the topological closure of $F^\times U_F(Lp^\infty)F_\infty^\times$ in $F_{\mathbb{A}}^\times$, $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $F_{\infty+}^\times$ is the identity connected component of $F_\infty^\times$. The group $\mathbf{W}^{ord}$ can be naturally regarded as an open subgroup of $\overline{Z}$ and Z. By class field theory, we can identify Z and $\overline{Z}$ with the Galois group of a suitable abelian extension of F. Write $Cl_F^0(Lp^\alpha)$ (resp. $Cl_F(Lp^\alpha)$) for the (resp. narrow) ray class group modulo Lp^α. Then

$$\overline{Z} \cong \varprojlim_\alpha Cl_F^0(Lp^\alpha) \quad \text{and} \quad Z \cong \varprojlim_\alpha Cl_F(Lp^\alpha).$$

By considering the class of an ideal $\mathfrak{a}$ prime to Lp in every $Cl_F(Lp^\alpha)$ or $Cl_F^0(Np^\alpha)$, we can associate to $\mathfrak{a}$ an element $[\mathfrak{a}]$ of Z or $\overline{Z}$. Let ω be the Teichmüller character of Z, that is, for each ideal $\mathfrak{a}$, $\omega([\mathfrak{a}]) = \lim_{n\to\infty} N_{F/\mathbb{Q}}(\mathfrak{a})^{p^n}$, where the limit is taken in $\mathbb{Z}_p$. Let $\mathcal{N} : Z \to \mathbb{Z}_p^\times$ be the cyclotomic character, that is, $\mathcal{N}([\mathfrak{a}]) = N_{F/\mathbb{Q}}(\mathfrak{a})$. We consider ω to have values in $\overline{\mathbb{Q}}^\times$ via i_p. Thus we may think $\varepsilon\mathcal{N}^k$ as a character with values in $\overline{\mathbb{Q}}_p^\times$ for any finite order characters $\varepsilon : Z(L) \to \overline{\mathbb{Q}}^\times$. Then, by a result of Deligne and Ribet [17], for any given ideal $\mathfrak{x}$ prime to Np, there exists a unique p-adic measure $\zeta = \zeta_{\mathfrak{x},L}$ (in the sense of Mazur) such that

$$\int_{\overline{Z}} \varepsilon\mathcal{N}^m(z)d\zeta_{\mathfrak{x},L}(z) = (1 - \varepsilon\mathcal{N}^m([\mathfrak{x}]))L_{Lp}(1-m,\varepsilon)$$

for all $m > 0$ and all character $\varepsilon\mathcal{N}^m : \overline{Z}(L) \to \overline{\mathbb{Q}}_p^\times$. Here we write

$$L_{Lp}(s,\varepsilon) = \prod_{\mathfrak{q}|Lp}(1 - \varepsilon(\mathfrak{q})N_{F/\mathbb{Q}}(\mathfrak{q})^{-s})L(s,\varepsilon),$$

where $\mathfrak{P}$ runs over all prime factors of p in $\mathfrak{r}$. We write $\mathscr{C}(Cl_F^0(Lp^\alpha);\mathbb{O})$ for the space of all functions on the finite group $Cl_F^0(Lp^\alpha)$, which is naturally the $\mathbb{O}$-dual space of the group algebra $\mathbb{O}[Cl_F^0(Lp^\alpha)]$ by the pairing $\langle\ ,\ \rangle$ given by $\langle\xi,\phi\rangle = \sum_{[\mathfrak{a}]}\xi_{[\mathfrak{a}]}\phi([\mathfrak{a}]) \in \mathbb{O}$. Here $\xi_{[\mathfrak{a}]}$ is the coefficient of ξ at $[\mathfrak{a}] \in Cl_F^0(Lp^\alpha)$. Moreover these pairings for varying α are compatible with the transition maps of the natural projective system $\{Cl_F^0(Lp^\alpha)\}_\alpha$ and the injective system $\{\mathscr{C}(Cl_F^0(Lp^\alpha);\mathbb{O})\}_\alpha$. Thus the continuous group algebra $\mathbb{O}[[\overline{Z}]]$ can be considered as the space of $\mathbb{O}$-valued measures on $\overline{Z}$.

Then, writing $\mathscr{C}(T;\mathbb{O})$ for the space of continuous functions on a topological space T having values in $\mathbb{O}$, we know that $\mathbb{O}[[\overline{Z}]]$ is the $\mathbb{O}$-dual of $\mathscr{C}(\overline{Z};\mathbb{O})$. Thus $\zeta_{\mathfrak{x},L} \in \mathbb{O}[[\overline{Z}]]$. We regard $1-[\mathfrak{x}]$ as an element μ of $\mathbb{O}[[\overline{Z}]]$. We have $\int_{\overline{Z}} \phi(z)d\mu(z) = 1-\phi([\mathfrak{x}])$.

Now we define the Eisenstein measure $E_{\mathfrak{x},L} : \mathscr{C}(\overline{Z}(L);\mathbb{O}) \to M^{ord}(Lp^\infty;\mathbb{O})$ (for each $k \geq 1$) by the following formulas:

$$\text{(E)} \qquad \mathbf{a}_{0,p}(y, E_{\mathfrak{x},L}(\phi\mathcal{N})) = 2^{-[F:\mathbb{Q}]} \int_{\overline{Z}} \phi\mathcal{N}(z) d\zeta_{\mathfrak{x},L}(z),$$

$$\mathbf{a}_p(y, E_{\mathfrak{x},L}(\phi\mathcal{N})) = \sum_{\mathfrak{b}\supset y\mathfrak{r}} (\phi([\mathfrak{b}]) - N_{F/\mathbb{Q}}(\mathfrak{x})\phi([\mathfrak{x}\mathfrak{b}])).$$

To compute the q-expansion of $E_{\mathfrak{x},L}(\phi\mathcal{N})$ when $\phi\mathcal{N} : \overline{Z}(L) \to \overline{\mathbb{Q}}_p^\times$ is a product $\varepsilon\mathcal{N}^k$ for a finite order character $\varepsilon : Cl_F(Lp^\alpha) \to \overline{\mathbb{Q}}^\times$, we introduce some classical Eisenstein series. Write simply ξ for $\varepsilon\mathcal{N}^k$ ($k = [m(\kappa)]+1$). Then there exists a classical modular form $E(\xi) \in M_\kappa(Lp^\alpha;\mathbb{C})$ for $\kappa = (\varepsilon\mathcal{N}^{k-1}, 0)$ such that

$$\mathbf{a}_p(y, E(\xi)) = \sum_{\mathfrak{b}\supset y\mathfrak{r}} \xi\mathcal{N}^{-1}([\mathfrak{b}]) \quad \text{and} \quad \mathbf{a}_{0,p}(y\mathfrak{r}, E(\xi)) = 2^{-[F:\mathbb{Q}]} L_{Lp}(1-k, \varepsilon)$$

where $\varepsilon_\kappa(a,d) = \varepsilon(d)$, $n(\kappa) = 0$ and $m(\kappa) = kt - t$. Then we can easily check that $E(\xi)|\mathbb{T}(p) = E(\xi)$, and hence $E(\xi)$ is p-ordinary. Further we get by definition

$$E_{\mathfrak{x},L}(\xi) = E(\xi)|(1 - \langle\mathfrak{x}\rangle\mathcal{N}([\mathfrak{x}])^2).$$

Thus, for any locally constant function ϕ on $\overline{Z}$ with values in K, $E_{\mathfrak{x},L}(\phi\xi)$ is an element of $M_\kappa^{ord}(Lp^\infty;K)$. Since the functionals: $\phi \mapsto \mathbf{a}_p(y, E_{\mathfrak{x},L}(\phi\xi))$ and $\mathbf{a}_{0,p}(y, E_{\mathfrak{x},L}(\phi\xi))$ are bounded measures with values in $\mathbb{O}$ on $\mathscr{C}(\overline{Z};\mathbb{O})$, the linear map defined on the space $\mathscr{LC}(\overline{Z};\mathbb{O})$ of locally constant functions $E_{\mathfrak{x},L} : \mathscr{LC}(\overline{Z};\mathbb{O}) \to M_\kappa^{ord}(Np^\infty;\mathbb{O})$ extends by continuity to a p-adic measure having values in the space of p-adic modular forms $M^{ord}(Np^\infty;\mathbb{O})$. We have, by [12],

$$\text{(Res1)} \qquad E_{\mathfrak{x},L}(id) = -\log_p(\mathcal{N}(\langle\mathfrak{x}\rangle))\frac{R_p h(F)}{w} \prod_{\mathfrak{q}|pL}\left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{q})}\right).$$

6.8 Convoluted measure

Let us simply write the Eisenstein measure as $E = E_{\mathfrak{x},L}$. Note that $\mathbf{G}^{ord}$ can be naturally considered to be an open subgroup of $Z(L)$. Then through the action of the center of G, we can extend the action of $\mathbf{G}^{ord}$ on $U^* = M(Lp^\infty;\mathbb{O})$ and $S = S(Lp^\infty;\mathbb{O})$ to $Z(L)$ which actually factors through $\overline{Z}(L)$ because units in $\mathfrak{r}^\times$ acts trivially on modular forms. For each $\mathbb{O}$-module M, we write M^* for the $\mathbb{O}$-dual of M. Let $\varphi : M^* \to S(Lp^\infty;\mathbb{O})$ be an $\mathbb{O}[[\overline{Z}(L)]]$-linear map for an $\mathbb{O}[[\overline{Z}(L)]]$-module

M. In this paragraph, in order to give a proof of Theorem 6.5.1 in the following paragraph Section 6.10, we describe the theory of the convolution measure of E and φ developed in [41] Section 9. We can decompose $\overline{Z}(L) = \mathbb{W} \times \Delta(L)$ for a finite group $\Delta = \Delta(L)$ and the torsion-free subgroup $\mathbb{W}$. We write $\mathbf{\Lambda}$ for $\mathcal{O}[[\mathbb{W}]]$. We restate Lemma 9.1 of [41] which is a key to our construction:

Lemma 6.8.1 — *Suppose that M is a $\mathbf{\Lambda}$-submodule of a $\mathbf{\Lambda}$-free module of finite rank. Then M satisfies the following conditions:*

(i) $M \cong \varprojlim_\alpha M_\alpha$ *as $\mathbf{\Lambda}$-modules;*

(ii) M_α *is a $\mathbf{\Lambda}$-module and is $\mathcal{O}$-free of finite rank;*

(iii) *The transition maps: $M_\beta \to M_\alpha$ are all surjective.*

Moreover if a $\mathbf{\Lambda}$-module M satisfies (i), (ii) and (iii), we have

$$M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}) \cong \varprojlim_\alpha ((\varinjlim_\beta M_\beta^*) \otimes_{\mathcal{O}} \mathcal{O}/p^\alpha\mathcal{O}),$$
$$(M^*)^* \cong M, \quad \mathrm{Hom}_{\mathbf{\Lambda}}(M, \mathbf{\Lambda}^*) \cong M^* \quad \textit{and} \quad \mathrm{Hom}_{\mathbf{\Lambda}}(M^*, L^*) \cong M.$$

Note that $U = (U^*)^*$ and S^* satisfy the condition (i), (ii) and (iii) of Lemma 6.8.1.

We have a product $m' : U^* \times S \to |D|S$ induced by the multiplication in the ring of q-expansions (see Section 2.4 in Chapter 2). We may define a bilinear map $m : U^* \times S \to S$ by $m(f, g) = |D|^{-1} m'(f, g)$. Then by definition, we see for $z \in Z$

$$m(f|\langle z\rangle, g|\langle z\rangle) = m(f, g)|\langle z\rangle.$$

Note that $E : \mathscr{C}(\overline{Z}(L); \mathcal{O}) \to U^*$ is $\mathcal{O}[[\overline{Z}(L)]]$-linear under the action:

$$\phi \mapsto \phi\big|_2 z(z') = \phi(z'z)\mathcal{N}(z')^2 \quad \text{on} \quad \mathscr{C}(\overline{Z}(L); \mathcal{O}).$$

Consider a compact $\mathcal{O}[[\overline{Z}(J)]]$-module M satisfying the conditions (i)-(iii) of Lemma 6.8.1. We regard M as an $\mathcal{O}[[\overline{Z}(L)]]$-module via a natural projection of $\overline{Z}(L)$ to $\overline{Z}(J)$. For an $\mathcal{O}[[\overline{Z}(L)]]$-linear map $\varphi : M^* \to S$, define

$$\widehat{\varphi} : M^* \widehat{\otimes}_{\mathcal{O}} U^* \to S \quad \text{by} \quad \widehat{\varphi} = m \circ (\varphi \otimes \mathrm{Id}),$$

where $M^* \widehat{\otimes}_{\mathcal{O}} U^*$ is a p-adic completion $\varprojlim_j (M^* \otimes U^*)/p^j(M^* \otimes U^*)$. By construction, we can write $U = \varprojlim_j U_j$ for free $\mathcal{O}$-modules of finite rank U_j. For example, we could take U_j to be the $\mathcal{O}$-dual of $\{\sum_{0 \le \kappa \le (jt,0)} M_\kappa(Lp^j; K)\} \cap M(Lp^\infty; \mathcal{O})$. We write $M \widehat{\otimes}_{\mathcal{O}} U$ for the profinite completion $\varprojlim_{i,j} M_i \otimes U_j$. We say that a function $\phi : M \to U^*$ is continuous if it is continuous under the p-adic topology on U^* and under the topology of the profinite group M. Thus if ϕ is $\mathcal{O}$-linear, then ϕ is continuous if and only if there exists $i > 0$ for any $j > 0$ such that $\phi \bmod p^j : M/p^j M \to U^*/p^j U^*$ factors through $M_i/p^j M_i$, where $M = \varprojlim_i M_i$ as in

Lemma 6.8.1. Since $U = \varprojlim_k U_k$ for $\mathcal{O}$-free modules U_k of finite rank satisfying the condition of Lemma 6.8.1, the image of $\phi \mod p^j$ is actually contained in $U_k^*/p^jU_k^*$ for some k. We denote by $\mathrm{Hom}_c(M, U^*)$ the space of all continuous $\mathcal{O}$-linear maps. Then we have

$$\begin{aligned}\mathrm{Hom}_c(M,U^*) &= \varprojlim_j \varinjlim_i \varinjlim_k \mathrm{Hom}_{\mathcal{O}}(M_i/p^jM_i, U_k^*/p^jU_k^*)\\ &= \varprojlim_j \varinjlim_{i,k} \mathrm{Hom}_{\mathcal{O}}(M_i/p^jM_i \otimes_{\mathcal{O}} U_k, \mathcal{O}/p^j\mathcal{O}) = (M\widehat{\otimes}_{\mathcal{O}}U)^*,\end{aligned}$$

where $M\widehat{\otimes}_{\mathcal{O}}U$ is the profinite completion of $M \otimes_{\mathcal{O}} U$. As shown in [41] Lemma 9.2, we have

$$M^*\widehat{\otimes}_{\mathcal{O}}U^* \cong \mathrm{Hom}_c(M,U^*) \cong (M\widehat{\otimes}_{\mathcal{O}}U)^*,$$

where the identification is given by $\phi \otimes u^* \mapsto (\phi \otimes u^*)(m) = \phi(m)u^*$. For each continuous function $\Phi \in \mathscr{C}(M \times \overline{Z}; \mathcal{O})$, we define an action of $\overline{Z}(L)$ by $(\Phi|z)(m,z') = \Phi(z^{-1}m, zz')$. We define $E_* : \mathscr{C}(M \times \overline{Z}; \mathcal{O}) \to \mathscr{C}(M, U^*)$ by

$$E_*(\Phi)(m) = \int_{\mathbf{G}(L)} (\Phi|z)(m,1)dE(z).$$

Then E_* induces on $M^*\widehat{\otimes}_{\mathcal{O}}\mathscr{C}(\mathbf{G};\mathcal{O})$ ($\cong \mathrm{Hom}_c(M, \mathscr{C}(\mathbf{G};\mathcal{O}))$) a morphism into $\mathrm{Hom}_c(M,U^*)$ ($\cong M^*\widehat{\otimes}_{\mathcal{O}}U^*$). Then we have

$$E_*(\Phi)(m) = \int_{\mathbf{G}(L)} \Phi(z^{-1}m)(z)dE(z) \quad \text{for} \quad \Phi \in \mathrm{Hom}_c(M, \mathscr{C}(\overline{Z};\mathcal{O})).$$

We now define the convoluted measure $E * \varphi : M^*\widehat{\otimes}_{\mathcal{O}}\mathscr{C}(\overline{Z};\mathcal{O}) \to S$ by

$$E * \varphi(\Phi) = \widehat{\varphi}(E_*(\Phi)).$$

By our construction, if we let $z \in \overline{Z}(L)$ act on Φ via $\Phi \parallel z(m)(x) = \Phi(m)(zx)\mathcal{N}(z)^2$ regarding Φ as an element of $\mathrm{Hom}_c(M, \mathscr{C}(\overline{Z};\mathcal{O}))$, then it is easy to see

(L1) $\qquad E * \varphi$ is a morphism of $\mathcal{O}[[\overline{Z}]]$-modules.

6.9 Convolution along λ

We fix decomposition: $\overline{Z}(N) = \mathbb{W} \times \Delta(N)$ so that the projection $\overline{Z}(L) \to \overline{Z}(N)$ induces the identity on $\mathbb{W}$. We take $\lambda : h^{ord}(Np^\infty;\mathcal{O}) \to \mathbb{I}$ as in Theorem 6.3.2. Note here λ factors through $h^{ord}(\psi)$ defined in Section 2.9 of Chapter 2. It is easy to check that $\mathbb{I}$ satisfies the conditions of Lemma 6.8.1. By composing the multiplication map: $\mathbb{I} \otimes_{\mathbf{\Lambda}} \mathbb{I} \to \mathbb{I}$ with $\lambda \otimes \mathrm{Id}$, we can naturally extend λ to an $\mathbb{I}$-algebra homomorphism of $R = h^{ord}(\psi) \otimes_{\mathbf{\Lambda}} \mathbb{I}$ into $\mathbb{I}$, which we again denote by λ. We now consider the $\mathbb{I}$-algebra decomposition $R \otimes_{\mathbb{I}} \mathbb{K} = \mathbb{K} \oplus X$ introduced in Section 2.9 of Chapter 2 and write 1_λ for

the idempotent of the factor $\mathbb{K}$. We take $0 \neq H \in \mathbb{I}$ which annihilates $C_0(\lambda;\mathbb{I})$. Then $H1_\lambda \in R$. We write $S^{ord}(\psi)$ (resp. S^{ord}) for the dual of $h^{ord}(\psi)$ (resp. $h^{ord}(Np^\infty;\mathcal{O})$) in S (see Section 2.9 in Chapter 2). We also write $\check{M} = \mathrm{Hom}_{\mathbf{\Lambda}}(M,\mathbf{\Lambda})$ for any $\mathbf{\Lambda}$-module M, and we define a pairing $\langle\ ,\ \rangle$ between R and $S^{ord}(\psi) = S^{ord}(\psi)\otimes_{\mathbf{\Lambda}}\check{\mathbb{I}}$ as follows:

$$\text{(P1)} \qquad \langle h\otimes i, f\otimes\phi\rangle = \langle h\phi(i), f\rangle = \langle h, f|\phi(i)\rangle,$$

where the pairing at the right-hand side is the pairing of Section 2.6 in Chapter 2. We then define

$$\text{(L2)} \qquad \ell_\lambda : S^{ord}(\psi)\to\mathcal{O} \quad \text{by} \quad \ell_\lambda(f) = \langle H1_\lambda, f\rangle.$$

For a point $P\in\mathscr{X}_{\mathbb{I}}(\Omega)$, we have an $\mathcal{O}$-algebra homomorphism $\lambda_P = P\circ\lambda : R\to\overline{\mathbb{Q}}_p$. Suppose P is positive and arithmetic. For a suitable S_F-tuple $\alpha = (\alpha(\mathfrak{P}))_{\mathfrak{P}\in S_F}$, we find $f(P)\in S^{ord}_{\kappa(P)}(N\mathfrak{P}^\alpha;\overline{\mathbb{Q}})$ such that $\mathbf{a}_p(y,f(P)) = \lambda_P(\mathbb{T}(y))$ for $y\in\widehat{\mathfrak{r}}\cap F^\times_{\mathbb{A}^{(\infty)}}$. As seen in [41] Lemma 9.3, We have

$$\text{(L3)} \qquad \ell_\lambda(h) = H(P)\frac{(f(P)^c|\tau(n\varpi^\alpha), h|e)_\alpha}{(f(P)^c|\tau(n\varpi^\alpha), f(P))_\alpha}$$

for all $h\in S_{\kappa(P)}(N\mathfrak{P}^\alpha;\overline{\mathbb{Q}})$, where e is the p-ordinary projector, and $(\ ,\ )_\alpha = (\ ,\ )_{N\mathfrak{P}^\alpha}$ is the Petersson inner product on $S_{\kappa(P)}(N\mathfrak{P}^\alpha;\mathbb{C})$. Note that the expression on the left-hand side does not depend on the choice of $\alpha = (\alpha(\mathfrak{P}))_{\mathfrak{P}}$.

Now we take an $\mathbf{\Lambda}$-free submodule $\mathbb{X}$ in $\mathbb{K}$ containing $\mathbb{I}$. Then $\mathbb{X}/\mathbb{I}$ is a torsion $\mathbf{\Lambda}$-module. We recall our usage of symbol: M^* for the $\mathcal{O}$-dual module of M while $\check{M}$ indicates the $\mathbf{\Lambda}$-dual module of M. Define a subspace of $\mathscr{C}(\overline{Z};\mathcal{O})$ by

$$\mathscr{C}(\overline{Z}(N);\mathcal{O})[\psi] = \left\{\phi\in\mathscr{C}(\overline{Z};\mathcal{O})\middle|\phi(\zeta x) = \psi(\zeta)\phi(x) \quad \text{for all} \quad \zeta\in\Delta(N)\right\}.$$

We know, by restricting functions on $\overline{Z}(L)$ to $\mathbb{W}$, $\mathscr{C}(\overline{Z};\mathcal{O})[\psi]\cong\mathscr{C}(\mathbb{W};\mathcal{O})$ as $\mathbf{\Lambda}$-modules. Similarly, we define

$$S(L;\psi) = \left\{f\in S(Lp^\infty;\mathcal{O})\middle|\ f|\langle\zeta\rangle = \psi(\zeta)f \quad \text{for all} \quad \zeta\in\Delta(L)\right\}.$$

Then in exactly the same manner as in [41] p.381, we can show that

$$M^*\widehat{\otimes}_{\mathcal{O}}(\mathscr{C}(\overline{Z};\mathcal{O})[\psi])\otimes_{\mathbf{\Lambda}}\check{\mathbb{X}}\cong M^*\widehat{\otimes}_{\mathcal{O}}\mathbb{X}^*$$

and define, for the twisted trace operator $T_{L/N}$ in [41] p.367,

$$\Psi : M^*\widehat{\otimes}_{\mathcal{O}}(\mathscr{C}(\overline{Z};\mathcal{O})[\psi]) \xrightarrow{E*\varphi} S(L;\psi) \xrightarrow{T_{L/N}} S(N;\psi) \xrightarrow{e} S(\psi) \quad \text{and}$$
$$E*_\lambda\varphi : M^*\widehat{\otimes}_{\mathcal{O}}\mathbb{X}^* \longrightarrow \mathcal{O} \quad \text{by} \quad \ell_\lambda\circ(\Psi\otimes\mathrm{Id}).$$

The measure $E*_\lambda\varphi$ is independent of the auxiliary choice of $\mathbb{X}$, and as seen in [41] Lemma 9.4, we know $E*_\lambda\varphi\in M\widehat{\otimes}_{\mathcal{O}}\mathbb{I}$.

6.10 Proof of Theorem 6.5.1

Now the proof of Theorem 6.5.1 goes in exactly the same way as that of Theorem 5.1 in [41] Section 10. We therefore briefly recall the argument and give the proof of the residue formula in details. We take a primitive homomorphism $\mu : h^{ord}(Jp^\infty;\mathcal{O}) \to \mathbb{I}$ factoring through $h^{ord}(\chi)$. By the duality in Section 2.6 of Chapter 2, we have $\mu^* : \mathbb{I}^* \to S^{ord}(Jp^\infty;\mathcal{O})$. Set $L = N \cap J$. We take $\mathbb{I}^*$ as M^* in Section 6.8 and $[L/J] \circ \mu^*$ as φ in the previous section for the operator $[L/J]$ given in [41] Section 7B. Then $E *_\lambda \varphi$ gives an element of $(\mathbb{I}^* \widehat{\otimes}_{\mathcal{O}} \mathbb{I}^*)^* \cong \mathbb{I} \widehat{\otimes}_{\mathcal{O}} \mathbb{I}$. We define an element D^{ord} in the quotient field of $\mathbb{I} \widehat{\otimes}_{\mathcal{O}} \mathbb{I}$ by $\frac{E *_\lambda \varphi}{1 \otimes H}$. We can evaluate D^{ord} at $(P,Q) \in \mathcal{A}_{\mathbb{I}} \times \mathcal{A}_{\mathbb{I}}$ satisfying (Ad) in exactly the same manner as in [41] Section 10 and get the evaluation formula as in Theorem 6.5.1. We now briefly recall the process in order to prove the residue formula.

We can choose H so that $H(P) \neq 0$ and H kills the congruence module $C_0(\lambda;\mathbb{I})$. We first compute $\Psi = T_{L/N} \circ e \circ (E * \varphi)$. By extending scalars if necessary, we may assume that $P, Q : \mathbb{I} \to \overline{\mathbb{Q}}_p$ have values in $\mathcal{O}$. Then especially $Q \in \mathbb{I}^*$ and thus $Q \otimes \eta \in M^* \widehat{\otimes}_{\mathcal{O}} \mathcal{C}(\overline{Z};\mathcal{O})$ for any character η of $\overline{Z}(L)$ with values in $\mathcal{O}$. We first compute $\Psi(Q \otimes \psi_P) \in S^{ord}(\psi)$, for the character $\psi_P : \overline{Z}(L) \to \mathcal{O}^\times$ given by the restriction of λ_P to $\overline{Z}(L)$, that is, $\psi_P = \psi\omega^{-[m(P)]}\mathcal{N}^{[m(P)]}$. Let $g(Q) \in S^{ord}_{\kappa(Q)}(J\mathfrak{P}^\alpha;\mathcal{O})$ be the normalized cusp form with Hecke eigenvalues μ_Q. We write χ_Q for the restriction of μ_Q to $\overline{Z}(L)$. Then, regarding $Q \otimes \psi_P \in M^* \widehat{\otimes}_{\mathcal{O}} (\mathcal{C}(\overline{Z};\mathcal{O})[\psi])$, we have by definition

$$\Psi(Q \otimes \psi_P) = T_{L/N} \circ e(g(Q)|[L/J] \cdot (E(\chi_Q^{-1}\psi_P)|(1 - \delta\langle \mathfrak{x} \rangle \mathcal{N}(\mathfrak{x})^2)),$$

where $\delta = 1$ or 0 according as $\chi = \psi$ or not. This can be rewritten as

$$((1 - \delta\langle \mathfrak{x} \rangle \otimes \langle \mathfrak{x} \rangle^{-1})\Psi \otimes \mathrm{Id})(Q \otimes P) = T_{L/N} \circ e \left\{ g(Q)|[L/J] \cdot E_{\mathfrak{x},L}(\chi_Q^{-1}\psi_P) \right\},$$

which implies

$$((1 - \delta\langle \mathfrak{x} \rangle \otimes \langle \mathfrak{x} \rangle^{-1})D^{ord})(P,Q) = \ell_\lambda(T_{L/N} \circ e \left\{ g(Q)|[L/J] \cdot E_{\mathfrak{x},L}(\chi_Q^{-1}\psi_P) \right\}).$$

This is true for densely populated points (P,Q) in $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$ and hence is true everywhere over $\mathscr{X}_{\mathbb{I}} \times \mathscr{X}_{\mathbb{I}}$. When $P = Q$, we still have

$$((1 - \delta\langle \mathfrak{x} \rangle \otimes \langle \mathfrak{x} \rangle^{-1})D^{ord})(P,P) = \ell_\lambda \left(T_{L/N} \circ e \left\{ (g(P)|[L/J]) \cdot (E_{\mathfrak{x},L}(\mathrm{id})) \right\} \right).$$

It is clear from the definition that $\ell_\lambda(T_{L/N} \circ e(g(P)|[L/J])) = \delta_{\lambda,\mu}$, where $\delta_{\lambda,\mu}$ is 1 or 0 according as $\lambda = \mu$ or not. Then the residue formula follows from (Res1) in Section 6.8.

7. p-Adic Rankin products in partially CM case

We call $\lambda : \mathbf{h} = h^{n.ord}(Np^\infty; \mathbb{O}) \to \mathbb{I}$ "*of CM type*" if there exists a totally imaginary quadratic extension L of the totally real field F such that the isomorphism class $\pi(\lambda)$ contains the induced representation Φ of a character $\varphi : \mathcal{G}_L \to \mathbb{I}^\times$. In this case, using p-adic Hecke L-functions of Katz [52], we can construct a "*genuine*" p-adic Rankin product $L_S(\Phi \otimes \check{\rho})$ for $S = (\emptyset, I, \emptyset)$ and $\rho \in \pi(\mu)$ in Conjecture 5.5.1 of Chapter 5 and Conjectures 6.3.1 and 6.3.3 of Chapter 6.

7.1 CM components of Hecke algebras

Let L be as above and write $Cl_L(Cp^\alpha)$ for the ray class group modulo Cp^α, where C is an ideal of L prime to p. Then we consider $Cl_L(Cp^\infty) = \varprojlim_\alpha Cl_L(Cp^\alpha)$. By class field theory, we may identify $Cl_L(Cp^\infty)$ with the Galois group of the maximal ray class field modulo Cp^∞. Let $\mathcal{G} = \mathcal{G}_F$ and $\mathcal{H} = \mathcal{G}_L$. Let $\lambda : h^{n.ord}(Np^\infty; \mathbb{O}) \to \mathbb{I}$ be primitive and of CM type, and suppose that $\mathbb{I}$ is torsion-free over Λ. Thus $\Phi = \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \varphi \in \pi(\lambda)$ for a character $\varphi : \mathcal{H} \to \mathbb{I}^\times$. Since the ramification of $\pi(\lambda)$ outside p is known to be finite, φ factors through $Cl_L(Cp^\infty)$ for some C. We take the largest C which we call the conductor of φ. Since $\pi(\lambda)$ is known to be irreducible, $\varphi(c\sigma c^{-1}) \neq \varphi(\sigma)$ for some $\sigma \in \mathcal{G}$, where c is an element in $\mathcal{G}$ inducing a non-trivial automorphism on L/F. By (Gal) (iv) in Section 2.8 of Chapter 2, for $\mathfrak{P} \in S_F$, $\Phi|_{D_\mathfrak{P}}$ ($D_\mathfrak{P} = \mathcal{G}_{F_\mathfrak{P}}$) is reducible, and one of its characters is given by the character $\delta_\mathfrak{P} : {F_\mathfrak{P}}^\times \to \mathrm{Im}(\lambda)^\times$ taking $y \in {F_\mathfrak{P}}^\times$ to $\lambda(\mathbb{T}(y))$. There are two possibilities:

(i) $\quad D_\mathfrak{P} \subset \mathcal{H}$ or

(ii) we can find c as above in $D_\mathfrak{P}$ and $\varphi_c(\sigma) = \varphi(c\sigma c^{-1}) = \varphi(\sigma)$ for all $\sigma \in D_\mathfrak{P}$.

If Case (ii) occurs, $\varphi_\mathfrak{P} = \varphi|_{D_\mathfrak{P} \cap \mathcal{H}} = \phi_\mathfrak{P}|_{D_\mathfrak{P} \cap \mathcal{H}}$ for a character $\phi_\mathfrak{P}$ of $D_\mathfrak{P}$. Then the semi-simplification of $\pi(\lambda)|_{D_\mathfrak{P}}$ is given by $\phi_\mathfrak{P} \oplus \phi_\mathfrak{P}\chi$ for the quadratic character χ corresponding to L/F. Thus we may assume that $\phi_\mathfrak{P} = \delta_\mathfrak{P}$ and $\eta = det(\Phi) = \chi\delta_\mathfrak{P}^2$ on ${F_\mathfrak{P}}^\times$. We see that $\langle a \rangle = \eta(a) = \mathbb{T}(a)^2$ for $(a, a) \in \mathbf{G}$ sufficiently close to 1, where $\mathbf{G}$ is the weight group as in Section 2.1 of Chapter 2. Thus Case (ii) never occurs. This implies $D_\mathfrak{P} \subset \mathcal{H}$; in other words, all prime factors in S_F splits in L. We define

a set S of primes $\mathbf{P}$ of L over p such that $\varphi|_{D_{\mathbf{P}^c}} = \delta_{\mathfrak{P}}$ ($D_{\mathbf{P}} = \mathcal{G}_{L_{\mathbf{P}}}$). Then $S \bigcap S^c = \emptyset$ and $S \bigsqcup S^c$ is the set S_L of all primes of L over p. The subset S can be regarded as a (p-adic L) type $\{V_{\mathfrak{P}}^- \subset V(\Phi)\}$ as follows: for $\mathfrak{P} \in S_F$, writing $\mathbf{P}$ for the unique prime in S above $\mathfrak{P}$, we take as $V_{\mathfrak{P}}^- = V_{\mathfrak{P}}^-(\lambda)$ the subspace of $V(\lambda)$ on which $D_{\mathbf{P}}$ acts via $\varphi_c|_{D_{\mathbf{P}}}$. Thus the choice of S gives rise to the choice of $\{V_{\mathfrak{P}}^-\}_{\mathfrak{P} \in S_F}$. Let I_L be the set of all embeddings of L into $\overline{\mathbb{Q}}$, and write Σ for the set of embeddings made of $\sigma \in I_L$ such that $i_p\sigma$ induces a p-adic place in S. Then $I_L = \Sigma \bigsqcup \Sigma c$, that is, Σ is a CM type. We call S the p-adic CM type associated with Σ and also call λ of type (L, S, Σ). Then as seen in [47] Section 6 and [48] Section 4, we have $N = N_{L/F}(C)D_{L/F}$ for the relative discriminant $D_{L/F}$ of L/F and

$$\text{(CM1)} \qquad \lambda(\mathbb{T}(\mathfrak{n})) = \sum_{\mathfrak{a}\mathfrak{a}^c = \mathfrak{n}} \varphi([\mathfrak{a}]) \quad \text{if } \mathfrak{n} \text{ is prime to } p,$$

where $\mathfrak{a}$ runs over integral ideals of L with norm $\mathfrak{n}$ and $[\mathfrak{a}]$ is the class of an ideal $\mathfrak{a}$ in $Cl_L(Cp^\infty)$. Here we understand $\varphi([\mathfrak{a}]) = 0$ if $\mathfrak{a}$ is not prime to Cp. As for the value of $\lambda(\mathbb{T}(y))$ for $y \in {F_{\mathfrak{P}}}^\times$, we get from the above argument

(CM2) $\lambda(\mathbb{T}(y)) = \varphi(y)$ *identifying* $F_{\mathfrak{P}}$ *with* $L_{\mathbf{P}^c}$ *for* $\mathbf{P} \in S$ *with* $\mathbf{P} \cap F = \mathfrak{P}$.

Moreover the character φ satisfies the following properties:

(CM3) φ *is unramified at* S^c *if* λ *is* p*-ordinary;*

(CM4) φ_P *is the* p*-adic avatar of a complex Hecke character* $\boldsymbol{\varphi}_P$ *for* $P \in \mathscr{A}_{\mathbb{I}}$.

For any complex Hecke character $\omega : L^\times \backslash L_{\mathbb{A}}^\times \to \mathbb{C}^\times$ of type A_0, we define its infinity type $\infty(\omega) = m \in \mathbb{Z}[I_L]$ by $\omega(x) = x^m = \prod_\sigma x^{\sigma m_\sigma}$ for $x \in L_\infty^\times$. Writing $m = \infty(\boldsymbol{\varphi}_P)$ and $m(\Sigma) = \sum_{\sigma \in \Sigma} m_\sigma \sigma$, we have $-m - mc = m(P) + n(P)$, and by (CM2), we see that

$$(\infty 1) \qquad n(P) = -\sum_{\sigma \in \Sigma c} m_\sigma \sigma\big|_F = -m(\Sigma c)\big|_F \quad \text{and} \quad m(P) = -m(\Sigma)\big|_F.$$

7.2 Periods of $M(\lambda_P)$ of CM type

Let $M = M(\varphi_P)_{/L}$ be the rank 1 CM motive ([4]) attached to the Galois character φ_P for a positive arithmetic point P. The field E of coefficients of $M(\varphi_P)$ is given by $\mathbb{Q}(\varphi_P)$, which is generated over $\mathbb{Q}$ by the value of φ_P on finite ideles x away from p (that is, $x_p = 1$). Since $M(\varphi_P)$ is of rank 1, $\mathscr{F}(M(\varphi_P)) \otimes_{L \otimes E, \sigma \otimes i_\infty} \mathbb{C} \neq 0 \iff \sigma \in \Sigma c$. Since $\mathscr{F}(M(\varphi_P)) \otimes_{L \otimes E, \sigma \otimes i} X = H_{DR}(M) \otimes_{L \otimes E, \sigma \otimes i} X$ ($\sigma \in \Sigma c$) for $X = \mathbb{C}$ and B_{HT}, we see (e.g. [42] Lemma 1.2) that

$$(7.2.1) \qquad c_{*,\sigma}^{\pm}(\Phi_P) = \delta_{*,\sigma}(\varphi_P) \neq 0 \quad \text{and} \quad \alpha_\pm(\Phi) \in \mathbb{I}^\times.$$

Thus for any i_∞, $L_{T,i_\infty}(\Phi) = L_{S,i_\infty}(\varphi)$ (for $T = (\emptyset, I, \emptyset)$), $L_{T,i_\infty}(\Phi \otimes \check{\rho})$ and $L_{p,i_\infty}(Ad(\Phi))$ are automatically normalized (see (NP) in Chapter 4 and Theorems 5.2.1 and 5.2.2 in Chapter 5); so, we drop i_∞ from the notation (for example, we write $L_T(\Phi)$ for $L_{T,i_\infty}(\Phi)$). For $X = B_{HT}$, there could be two comparison isomorphisms I_{Fa} and I_{RT}. The isomorphism I_{Fa} is due to Faltings [21] which we used to construct $c^{\pm}_{HT,\iota}(M)$ in Section 3.2 of Chapter 3, and the other I_{RT} is due to Raynaud-Tate for abelian varieties [68]. Let $\mathscr{CM}$ be the category of absolute Hodge motives defined over $\overline{\mathbb{Q}}$ generated by Artin motives and abelian varieties of CM-type. Thus $\mathscr{CM}$ is generated by $H^1(A)$ for abelian varieties A as above and Artin motives. If a functorial extension to $\mathscr{CM}$ of I_{RT} originally defined for H^1 of A exists, then it is unique. We assume the existence of the extension:

(C_{RT}) *There exists a natural transformation between the following fibre functors on* $\mathscr{CM}$*:*

$$I_{RT} : H_p(M) \otimes_{\mathbb{Q}_p} B_{HT} \cong H_{DR}(M) \otimes_{\overline{\mathbb{Q}}, i_p\sigma} B_{HT}.$$

By [21] and [2] Theorem 0.3, I_{Fa} induces another natural transformation on $\mathscr{CM}$:

(C_{Fa}) $$I_{Fa} : H_p(M) \otimes_{\mathbb{Q}_p} B_{HT} \cong H_{DR}(M) \otimes_{\overline{\mathbb{Q}}, i_p\sigma} B_{HT}.$$

Thus $\sigma_{et} : I_{Fa}^{-1} \circ I_{RT}$ (resp. $\sigma_{DR} = I_{Fa} \circ I_{RT}^{-1}$) gives an element of the motivic Galois group $\mathscr{M}_{et}(B_{HT})$ (resp. $\mathscr{M}_{DR}(B_{HT})$) for $\mathscr{M}_{et} = Aut_{\mathscr{CM}}(H_p)$ (resp. $\mathscr{M}_{DR} = Aut_{\mathscr{CM}}(H_{DR})$). Since σ_{DR} commutes with the Galois action of $\mathscr{G}_{\mathbb{Q}_p}$, $\sigma_{DR} \in \mathscr{M}_{DR}(\mathbb{Q}_p)$. We can use I_{RT} and I_{Fa} to define the period $c^{\pm}_{p,i_\infty}(M)$, which we write $c^{\pm}_p(M; I_X)$, if necessary, to indicates its dependence on I_X for $X = Fa$ and RT. Since $I_{Fa} = \sigma_{DR} I_{RT}$, $c^{\pm}_p(M; I_{RT}) \equiv c^{\pm}_p(M) \mod \mathbb{Q}_p^\times$.

On the other hand, by using the monomial relation of CM periods due to Shimura, Deligne and Blasius [4] for complex periods and to Gillard [27] and Blasius [2] for p-adic periods, we can write $c^{\pm}_\infty(\Phi_P)$ and $c^{\pm}_p(\Phi_P; I_X)$ up to algebraic numbers in terms of the periods of an abelian variety with complex multiplication of type (L, Σ): We briefly recall the definition of the period from [52] and [47]. For that, we need some more notation. We write $\mathscr{V} = \mathbb{O}_\Omega$ for the p-adic integer ring of Ω. We then put $\mathscr{V}_0 = i_p^{-1}(\mathscr{V})$. Let $\mathscr{R}$ be the integer ring of L. We fix an element $0 \neq \delta \in \mathscr{R}$ such that

($\Delta 1$) $$\delta^c = -\delta \text{ and } \mathrm{Im}(\delta^\sigma) > 0 \text{ for all } \sigma \in \Sigma;$$

($\Delta 2$) *The alternating form* $\langle u, v \rangle = (u^c v - v^c u)/2\delta$ *induces an isomorphism* $\mathscr{R} \bigwedge_{\mathscr{R}} \mathscr{R} \cong \mathfrak{d}^{-1}\mathfrak{c}$ *for an ideal* $\mathfrak{c}$ *prime to* p*,*

where $\mathfrak{d}$ is the different of $L/\mathbb{Q}$. We can algebraize the complex torus $\mathbb{C}^\Sigma/\mathscr{R}^\Sigma$ for

$$\mathscr{R}^\Sigma = \left\{(a^\sigma)_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma \middle| a \in \mathscr{R}\right\}$$

and find an abelian scheme $A = A_\Sigma$ over $\mathcal{V}_0$ with $\mathfrak{c}$-polarization $\mathcal{P}$ induced by $\langle\ ,\ \rangle$. Let $u = (u_\sigma)_{\sigma\in\Sigma}$ be the coordinate of $\mathbb{C}^\Sigma$. We put $\omega_\infty = du = \sum_{\sigma\in\Sigma} du_\sigma$ which is a generator of $\Omega_{A/\mathbb{C}}$ over the ring $\mathbb{C}^\Sigma$. Identifying the S-adic completion $\mathcal{R}_S$ of $\mathcal{R}$ with $\mathfrak{r}_p$, we have a trivialization $i : F_p/\mathfrak{d}_p^{-1} \to A(\mathbb{C})$. This extends to a trivialization $i : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \subset A$ over $\mathcal{V}_0$. Then i induces an isomorphism of formal groups $i : \widehat{\mathbf{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \cong \widehat{A}$ for the formal completion $\widehat{A}$ of A along the origin. Writing $\mathbf{G}_m = Spec(\mathbb{Z}[t, t^{-1}])$, $\mathbf{G}_m$ has a nowhere vanishing differential $\frac{dt}{t}$. Then we put $\omega_p = i*(\frac{dt}{t})$ on A, which is a generator of $\Omega_{A/\mathcal{V}}$ over the ring $\mathcal{V}\otimes_{\mathbb{Z}}\mathcal{R}_S$. We pick a global section ω of $\Omega_{A/\mathcal{V}_0}$ which is a generator over the ring $\overline{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathcal{R}$. We can supplement ω with a section $\eta \in H^1(A, O_A)$ for the structure sheaf O_A of $A_{/\mathcal{V}_0}$ so that $H_{DR}(A)$ is generated by (ω, η) over the ring $V_0 \otimes_{\mathbb{Z}} L$. We also choose a generator α so that $H_1(A; \mathbb{Q}) = L\alpha$ and $\int_\alpha \omega_\infty = 1$. Then the two periods $\Omega_p = \Omega_p(RT; \Sigma)$ in $(\mathcal{V}\otimes_{\mathbb{Z}}\mathcal{R}_S)^\times \subset (\mathcal{V}^\times)^\Sigma$ and $\Omega_\infty = \Omega_\infty(\Sigma)$ in $(\mathbb{C}^\times)^\Sigma$ are defined by

$$\omega = \Omega_p\omega_p \quad \text{and} \quad \omega = \Omega_\infty\omega_\infty \ (\Leftrightarrow \Omega_\infty = \int_\alpha \omega). \tag{7.2.2}$$

Then we have

$$c^\pm_{\infty,\sigma}(\pi(\lambda_P)) \equiv (2\pi i)^{n(P)}\Omega_\infty^{m(P)-n(P)} \mod \overline{\mathbb{Q}}^\times, \tag{Pd_∞}$$

and we expect to have

$$c^\pm_{p,\sigma}(\pi(\lambda_P); I_{Fa}) \equiv c^\pm_{p,\sigma}(\pi(\lambda_P); I_{RT}) \equiv \Omega_p^{m(P)-n(P)} \mod \overline{\mathbb{Q}}^\times, \tag{Pd_p}$$

where $\Omega_*^d = \prod_{\sigma\in\Sigma}\Omega_{*,\sigma}^{d_\sigma}$ for $d \in \mathbb{Z}[I]$.

We now look at the specific abelian scheme $A_{\Psi/\mathcal{V}_0}$ of CM-type (L, Ψ). We choose for each Ψ, generators b_Ψ and η_Ψ so that $H^1_{et}(A_\Psi, \mathbb{Q}_p) = L\otimes_{\mathbb{Z}_p}\mathbb{Q}_p b_\Psi$ and $H^1_{DR}(A_\Psi) = L\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}\eta_\Psi$ regarding A_Ψ is defined over $\overline{\mathbb{Q}}$. Then we could have defined $\Omega_p(RT; \Psi)$ by $I_{RT}(b_\Psi) = \Omega_p(RT; \Psi)\eta_\Psi$, because the Tate module $T_p(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is the dual of $H^1_{et}(A_\Psi, \mathbb{Q}_p)$. We define $\Omega_p(Fa; \Psi)$ by $I_{Fa}(b_\Psi) = \Omega_p(Fa; \Psi)\eta_\Psi$. Note that $\Omega_p(Fa; \Psi) = c_\Psi(\sigma)\Omega_p(RT; \Psi)$, where $c_\Psi(\sigma)$ is an element of $L\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ induced by σ_{DR}. Since we can always realize M_P inside $\otimes_\Psi H^1(A_\Psi)^{\otimes k_\Psi}$ for a suitable choice of integers k_Ψ, we can write down up to algebraic numbers $c^\pm_p(M; I_X)$ as a monomial of periods $\Omega_p(X; \Psi)$. The form of expression is independent of $X = RT$ or Fa under (C_{RT}), and the ratio of two periods are given by the same monomial of $c_\Psi(\sigma)$. In particular, if $\otimes_\Psi H^1(A_\Psi)^{\otimes k_\Psi} \cong \otimes_\Psi H^1(A_\Psi)^{\otimes k'_\Psi}$, then $\frac{\prod_\Psi \Omega_p(X;\Psi)^{k_\Psi}}{\prod_\Psi \Omega_p(X;\Psi)^{k'_\Psi}}$ is independent of X. Thus

$$c^\pm_p(M_P(1); I_{RT})/\Omega_p(RT; \Sigma)^{m(P)-n(P)} = c^\pm_p(M_P(1); I_{Fa})/\Omega_p(Fa; \Sigma)^{m(P)-n(P)}. \tag{Pd_{RTF}}$$

We simply write $\Omega_p(P) = \Omega_p(RT; P)$ (resp. $\Omega_\infty(P)$) for $\Omega_p(RT;\Sigma)^{m(P)-n(P)}$ (resp. $(2\pi i)^{n(P)}\Omega_\infty^{m(P)-n(P)}$). Since

$$i_p^{-1}(c_p^\pm(M_P(1); I_{RT})/\Omega_p(RT; P)) = i_\infty^{-1}(c_\infty^\pm(M_P(1))/\Omega_\infty(P))$$

by (C_{RT}), we can replace the transcendental factor $\Omega_p(RT;\Sigma)^{m(P)-n(P)}$ by $c_p^\pm(M_P(1); I_{RT})$. It is plausible that I_{RT} coincides with I_{Fa} on H^1 of abelian varieties (and thus $I_{RT} = I_{Fa}$), but it seems that this is not yet verified ([50] 4.3.5). Anyway we assume

(U) *There exists an unit* $\varepsilon \in \mathbb{I}^\times$ *such that* $\frac{c_p^+(M_P(1);I_{RT})}{c_p^+(M_P(1);I_{Fa})} = \varepsilon(P)$ *for all motivic* P *of type* S.

If $I_{Fa} = I_{RT}$, then $\varepsilon = 1$. For the character $\varphi : \mathcal{H} \to \mathbb{I}^\times$, Katz (see [52] and [47] Theorem II) found a p-adic L-function $L_S(\varphi) \in \mathbb{I}\widehat{\otimes}_{\mathbb{O}}\mathcal{V}$ satisfying the following interpolation property for all positive arithmetic points P

(7.2.3)
$$\frac{L_S(P,\varphi)}{\Omega_p(P)} = (\mathfrak{R}^\times : \mathfrak{r}^\times)W_p(\boldsymbol{\varphi}_P)(-1)^{m(P)}|D|^{-1/2}(-2\delta)^{n(P)}E(P)\frac{\Gamma_F(m(P))L(0,\boldsymbol{\varphi}_P^{-1})}{\Omega_\infty(P)},$$

where D is the discriminant of $F/\mathbb{Q}$, $\Gamma_F(d) = \prod_{\sigma\in I}\Gamma(d_\sigma)$ for the gamma function $\Gamma(s)$, $(2\delta)^n = \prod_{\sigma\in\Sigma}(2\delta^\sigma)^{n_\sigma}$, $W_p(\boldsymbol{\varphi}_P)$ is the Gauss sum factor defined in [47] Theorem II and

$$E(P) = \prod_{\mathfrak{L}|C}(1-\boldsymbol{\varphi}_P^{-1}(\mathfrak{L}))\prod_{\mathbf{P}\in S}\left\{(1-\boldsymbol{\varphi}_P^{-1}(\mathbf{P}^c))(1-\boldsymbol{\varphi}_P(\mathbf{P})N(\mathbf{P})^{-1})\right\}.$$

Here we have written $L_S(P,\varphi) = \int \varphi_P^{-1}d\mu$ for the Katz's measure given in [47] Theorem II. Note that the ratio $\frac{\Omega_p(P)}{\Omega_\infty(P)}$ is by definition independent of the choice of the global differential ω, and hence the equality (7.2.3) makes sense. We now rewrite the value using the complex functional equation or its p-adic counter part in [47] Theorem II. The result is

(7.2.4)
$$L_S(P,\varphi) = (\mathfrak{R}^\times : \mathfrak{r}^\times)N(\mathfrak{c})^{-1}\boldsymbol{\varphi}_P(\mathfrak{c})W(\boldsymbol{\varphi}_P)|D|^{-1/2}(2\delta)^{n(P)}E(P)\frac{\Gamma_F(m(P))L(1,\boldsymbol{\varphi}_P)}{\Omega_\infty(P)},$$

where $W(\varphi_P) = \boldsymbol{\varphi}_{P,\infty}(-1)W'(\boldsymbol{\varphi}_P^{-1})W_p((\boldsymbol{\varphi}_P)_c)$ for the Gauss sums W_p and W' defined in [47] Theorem II. If $I_{RT} = I_{Fa}$, as we have seen, $\Omega_p(P)/c_p^+(\varphi_P)$ and $\Omega_\infty(P)/c_\infty^+(\varphi_P)$ are both algebraic and equal in $\overline{\mathbb{Q}}$. Under the milder assumption (U), $L_S(\varphi)$ is a genuine p-adic L-function up to units if $p > 2$. Assuming the complete splitting of p in F, the extensibility (C_{RT}) is basically proven by Gillard [27]. The full splitting of p in F guarantees that any complex CM type is actually ordinary;

thus, A_Σ falls in the ordinary case treated by [27]. In this case, we could say that $L_S(\varphi)$ is "genuine" with respect to I_{RT} replacing I_{Fa} by I_{RT} in the definition of $c_p^+(\varphi_P)$.

7.3 Genuine p-adic Rankin products

We find, by computing Euler factors of $L_{i_\infty}(s, Ad(\Phi_P))$ for an arithmetic point P,

$$L_{i_\infty}(1, Ad(\Phi_P)) = L_{i_\infty}(1, \varphi_P^{-1}(\varphi_P)_c) L_{i_\infty}(1, \chi_L), \tag{7.3.1}$$

where $(\varphi_P)_c(\sigma) = \varphi_P(c\sigma c^{-1})$, and χ_L is the quadratic character corresponding to L/F. We suppose

(Prim) *Every prime factor of C outside the conductor of $\varphi^{-1}\varphi_c$ is either ramified or inert over F.*

Under this condition, the imprimitive adjoint L-function obtained from λ_P is just a non-zero constant multiple of the primitive one; thus, under (Prim), we need not worry about the vanishing of the extra Euler factors. We thus define

$$L_p(Ad(\Phi)) = L_S(\varphi^{-1}\varphi_c) c_p^+(\chi_L \mathcal{N}^{-1}) \frac{L(1, \chi_L)}{c_\infty^+(\chi_L \mathcal{N}^{-1})}, \tag{7.3.2}$$

where $L_S(\varphi^{-1}\varphi_c)$ is the Katz p-adic L-function of type S studied in the previous section. In the CM case, the restriction of $\pi(\lambda)$ to $\mathcal{G}_{F_\mathfrak{P}}$ is reducible, and hence we have two choices of $V_\mathfrak{P}^-(\lambda_P)$. Our choice is given by S as in Section 7.1 and gives a unique choice of $V_\mathfrak{P}^-(Ad(\Phi_P))$ as explained in Section 5.6 of Chapter 5. Let $\mu : h^{n.ord}(Jp^\infty; \mathcal{O}) \to \mathbb{I}$ be another primitive Λ-algebra homomorphism. We have a p-adic L-function $D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$ in the field of fractions of $\mathbb{I}\widehat{\otimes}_\mathcal{O}\mathbb{I}$. Then we simply define for $T = (\emptyset, I, \emptyset)$

$$L_T(\Phi \otimes \check{\pi}(\mu)) = L_p(Ad(\Phi)) D_p(\pi(\lambda) \otimes \check{\pi}(\mu)). \tag{7.3.3}$$

Here we could have written $L_T(\Phi \otimes \check{\rho})$ for $L_T(\Phi \otimes \check{\pi}(\mu))$ taking $\rho \in \pi(\mu)$, but it is actually independent of the choice of ρ by Theorem 5.2.1 and (NP) in Section 4.3 of Chapter 4.

Conjecture 7.3.1 — *$L_T(\Phi \otimes \check{\pi}(\mu))$ is entire ($\iff L_T(\Phi \otimes \check{\pi}(\mu)) \in \mathbb{I}\widehat{\otimes}_\mathcal{O}\mathbb{I}\widehat{\otimes}_\mathcal{O}\mathcal{O}_\Omega$).*

This conjecture implies Conjecture 6.3.3 in Chapter 6 in our situation if $I_{Fa} = I_{RT}$. Conjecture 7.3.1 follows from the conjecture in [47] p.192 (see also Conjecture 5.6.1 in Chapter 5 in the text). This conjecture is known to be true by [55] and [71] (see also [48]) when $F = \mathbb{Q}$ and hence, Conjecture 7.3.1 hold in this case. We will show in the following paragraph that Conjecture 7.3.1 holds when μ is also of CM-type for general totally real F.

7.4 Totally CM Case

Let L' be another CM quadratic extension of F. We assume that μ is of CM type with data (L', S', Σ', ψ) like (L, S, Σ, φ). Thus $\pi(\mu)$ has an induced representation Ψ of a character $\psi : \mathcal{G}_{L'} \to \mathbb{I}^\times$. We define the 2×2 Hecke matrix $A_\mathfrak{l}$ for λ_P so that $det(1 - A_\mathfrak{l} X) = 1 - \lambda_P(\mathbb{T}(\mathfrak{l}))X + \lambda_P(\langle\mathfrak{l}\rangle)X^2$ for each prime $\mathfrak{l}$ outside p. Thus the above polynomial can be of degree less than 2 only when $\mathfrak{l}$ divides the conductor of λ_P. We define $B_\mathfrak{l}$ similarly for μ_Q. Then we define imprimitive Euler products

$$\mathscr{L}(s, \lambda_P \otimes \check{\mu}_Q) = \prod_{\mathfrak{l}} det\left(1 - A_\mathfrak{l} \otimes B_\mathfrak{l} N(\mathfrak{l})^{-s}\right)^{-1},$$

$$\mathscr{L}(s, Ad(\lambda_P)) = \prod_{\mathfrak{l}} det\left(1 - Ad(A_\mathfrak{l}) N(\mathfrak{l})^{-s}\right)^{-1}.$$

Then the p-adic L-function $D_p = D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$ basically interpolates the ratio: $\frac{\mathscr{L}(1, \lambda_P \otimes \check{\mu}_Q)}{\mathscr{L}(1, Ad(\lambda_P))}$. It is obvious that $\mathscr{L}(s, \lambda_P \otimes \check{\mu}_Q)$ (resp. $\mathscr{L}(s, Ad(\lambda_P))$) is equal to $L(s, \Phi_P \otimes \check{\Psi}_Q)$ (resp. $L(s, Ad(\Phi_P))$ up to finitely many Euler factors. By computing Euler factors of $L(s, \lambda_P \otimes \check{\mu}_Q)$ for two arithmetic points P and Q, we find

$$\mathscr{L}(s, \lambda_P \otimes \check{\mu}_Q) = E(s) \times \begin{cases} L(s, \varphi_P \psi_Q^{-1}) L(s, \varphi_P(\psi_Q)_c^{-1}) & \text{if } L' = L, \\ L(s, \widehat{\varphi}_P \widehat{\psi}_Q^{-1}) & \text{if } L' \neq L, \end{cases}$$

where $E(s)$ is a finite Euler factor, and $\widehat{\varphi}_P$ (resp. $\widehat{\psi}_Q$) is the restriction of φ_P (resp. ψ_Q) to $\mathcal{G}_{LL'}$ for the composite LL'. Let $E_p(s)$ be the Euler factor of $E(s)$ outside p. If $E_p : (P, Q) \mapsto E(1)$ is a function in $\mathbb{I}\widehat{\otimes}\mathbb{I}$, we just put

$$L_T(\Phi \otimes \check{\Psi}) = E_p \times \begin{cases} L_S(\varphi\psi^{-1}) L_S(\varphi\psi_c^{-1}) & \text{if } L' = L, \\ L_S(\widehat{\varphi}\widehat{\psi}^{-1}) & \text{if } L' \neq L, \end{cases} \tag{N}$$

where the L-function appearing on right-hand side is the corresponding Katz p-adic L-functions of type S, and $\widehat{\varphi}$ (resp. $\widehat{\psi}$) is the restriction of φ (resp. ψ) to $\mathcal{G}_{LL'}$. Here the subset S in $S_{LL'}$ is made of p-adic places given by the extensions of places in S to LL'. Then the above $L_T = L_T(\Phi \otimes \check{\Psi})$ gives the desired "genuine" p-adic Rankin product. As was done in [47] p. 250-51 when $L = L'$, one can check that the p-Euler factors of D_p and L_T/L_p for $L_p = L_p(Ad(\Phi))$ match in general. Thus we only need to prove $E_p \in \mathbb{I}\widehat{\otimes}\mathbb{I}$ as follows. Let Ξ be the set of prime ideals outside of p satisfying the following condition:

$$U_\mathfrak{l} = (V(\Phi_P) \otimes V(\check{\Psi}_Q))^{I_\mathfrak{l}} / (V(\Phi_P)^{I_\mathfrak{l}}) \otimes (V(\check{\Psi}_Q)^{I_\mathfrak{l}}) \neq \{0\}. \tag{7.4.1}$$

Then E_p is the product of Euler factors at primes in Ξ. We can also check that Ξ is independent of P, because the Euler factor depends only on the restriction of Φ and Ψ to $I_\mathfrak{l}$ whose image is finite and hence independent of P and Q. Since E_p is just the polynomial of the eigenvalue of the Frobenius on $U_\mathfrak{l}$ ($\mathfrak{l} \in \Xi$), it resides in $\mathbb{I}\widehat{\otimes}\mathbb{I}$.

8. p-Ordinary Katz p-adic L-functions

Suppose that $\lambda : h^{ord}(Np^\infty;\mathbb{O}) \to \mathbb{I}$ is of CM type, and $\Phi = \mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}(\varphi) \in \pi(\lambda)$ for a character $\varphi : \mathscr{H} \to \mathbb{I}^\times$. We sketch a proof of the fact that the ordinary Katz p-adic L-function $L_p(\Phi) = L_S(\varphi)$, under the Leopoldt conjecture for F and p, satisfies (G) in Section 4.4 of Chapter 4.

8.1 p-Ordinary Eisenstein measures

We use the notation introduced in Chapter 7. Consider the maximal unramified extension K_0 in Ω over $\mathbb{Q}_p$. Let $\mathscr{V}$ be a finite extension of the p-adic integer ring of K_0. Let $\mathfrak{f}$ be an integral ideal and $\mathfrak{c}$ be a fractional ideal prime to $p\mathfrak{f}$. For each ideal $\mathfrak{a}$, we write $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{b}^{-1}$. We write $V(\mathfrak{c},\mathfrak{f}^2;R)$ for the space of Katz's p-adic modular functions of prime-to-p-level $\mathfrak{f}$ with coefficients in R (see [47] 1.5 for precise definition). It is defined geometrically as an affine p-adic ring whose formal spectrum is the solution of a p-adic ordinary formal moduli problem classifying test objects made of a Hilbert-Blumenthal abelian acheme A, its $\mathfrak{c}$-polarization $\mathscr{P}$ and $p^\infty\mathfrak{f}^2$-level structure i, everything being defined over a ring R. For any choice of fractional ideals $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$, evaluation of such modular function at the Tate abelian variety $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \mathscr{P}_{can}, i_{can})$ (described in [52]) gives an isomorphism of $V(\mathfrak{c},\mathfrak{f}^2;R)$ onto the p-adic completion of the union (with respect to the level $\mathfrak{f}^2p^\alpha$ $(\alpha = 1,2,\dots,\infty)$ and weight k) of classical holomorphic modular forms with i_p-integral coefficients.

Over $\mathbb{C}$, the category of the test objects is equivalent to the category of triples $(\mathscr{L},\lambda,i)$, where $\mathscr{L}$ is an $\mathfrak{r}$-lattice in $F_{\mathbb{C}} = F\otimes_{\mathbb{Q}}\mathbb{C}$, i is a level $p^\infty\mathfrak{f}^2$ structure (that is, an injective homomorphism $i : F_p/\mathfrak{d}_p^{-1}\times(\mathfrak{f}^2)^*/\mathfrak{d}^{-1} \to p^{-\infty}\mathscr{L}/\mathscr{L}\times\mathfrak{f}^{-2}\mathscr{L}/\mathscr{L}$), and λ is an isomorphism $\mathscr{L}\bigwedge_{\mathfrak{r}}\mathscr{L} \cong \mathfrak{d}^{-1}\mathfrak{c}^{-1}$. The link between A and $\mathscr{L}$ is given by $A(\mathbb{C}) \cong F_{\mathbb{C}}/\mathscr{L}$. The Eisenstein measure is a bounded measure with values in $V = V(\mathfrak{c},\mathfrak{f}^2;\mathscr{V})$ supported on an open subset of $T = \{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\}\times\{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{s})\}$ $(\mathfrak{s}\supset\mathfrak{f})$. We consider the space of locally constant functions

$$\mathscr{C}_k^{\pm} = \left\{\phi : T\to\mathbb{C}\middle|\phi(\varepsilon^{\pm1}x,\varepsilon^{\pm1}x',\varepsilon y,\varepsilon y') = N(\varepsilon)^k\phi(x,x',y,y')\right\}.$$

Each $\phi \in \mathscr{C}_k^-$, we associate an Eisenstein series of weight k by

$$\text{(E)} \quad E_k((\mathscr{L},\lambda,i);\phi,\mathfrak{c}) = \frac{(-1)^{kt}\Gamma_F(kt+st)}{\sqrt{|D|}} \sum_{w\in\mathfrak{f}^{-1}p^{-\infty}\mathscr{L}/\mathfrak{r}^\times} \frac{P\phi(w)}{N(w)^k|N(w)|^{2s}}\Big|_{s=0},$$

where P is the partial Fourier transform with respect to (x,x') which sends ϕ to a function on $\mathfrak{f}^{-1}p^{-\infty}\mathscr{L}$ supported by an r-lattice (see [47] (2.2)). Then we get, if $k \geq 2$,

$$\text{(Q)} \quad \begin{aligned} E_k(\phi,\mathfrak{c})(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\mathscr{P}_{can},i_{can}) &= N(\mathfrak{a})\{2^{-[F:\mathbb{Q}]}L(1-k;\phi,\mathfrak{a}) \\ &+ \sum_{0\ll\xi\in\mathfrak{a}\mathfrak{b}} \sum_{(a,b)\in\{\mathfrak{a}\times\mathfrak{b}\}/\mathfrak{r}^\times, ab=\xi} \phi(a,b)sgn(N(a))N(a)^{k-1}q^\xi\}, \end{aligned}$$

where $L(s,\phi;\mathfrak{a}) = \sum_{\xi\in(\mathfrak{a}-\{0\})/\mathfrak{r}^\times} \phi(\xi,0)sgn(N(\xi))^k|N(\xi)|^{-s}$. The construction of the Eisenstein measure is done in the following way: To each $\phi \in \mathscr{C}_k^+$, we associate linearly $\phi^\circ \in \mathscr{C}_k^-$ as in [47] (3.1). Let $N : T \to \mathbb{Q}_p$ be $N(x,x',y,y') = N_{F_p/\mathbb{Q}_p}(x)$. Then $\mathbb{E}^{n.ord} : N^{-k}\phi \mapsto E_k(\phi^\circ) \in V$ gives a linear map into V defined on functions on $T/\overline{\mathfrak{r}^\times}$ which is locally a negative power of N. If ϕ is supported on $T^\times/\overline{\mathfrak{r}^\times}$, by (Q), the constant term of the q-expansion disappears, and hence $\mathbb{E}^{n.ord}$ extends to a unique bounded measure with values in V on $T^\times/\overline{\mathfrak{r}^\times}$. This measure is studied in [52] and [47]. Here instead of considering functions on $T^\times$, we consider functions ϕ on

$$\mathscr{T} = (\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f}) \times (\mathfrak{r}/\mathfrak{s}))/\overline{\mathfrak{r}^\times} \quad \text{for} \quad \mathfrak{s} \supset \mathfrak{f}.$$

The space $\mathscr{T}$ is a quotient of $T = \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{s})^\times\}$ via $\pi : (x,x',y,y') \mapsto (x,x',y')$. Then we still get a linear map $\mathbb{E} : N^{-k}\phi \mapsto E_k((\phi\circ\pi)^\circ) \in V$. However we can no longer expect the vanishing of the constant term of $\mathbb{E}(\phi)$. Actually the functional of constant term of $\mathbb{E}$ given by $N^{-k}\phi \mapsto$ (the constant term of $E_k(\phi^\circ)$) is the Deligne-Ribet pseudo measure [17] interpolating Hecke L-values of F (see Section 6.7 in Chapter 6), which can have singularity. Thus, for each choice of an ideal $\mathfrak{x}$ prime to $p\mathfrak{f}$, we can define the ordinary Eisenstein measure $\mathbb{E}_{\mathfrak{c},\mathfrak{x}}$ and the Eisenstein pseudo-measure $\mathbb{E}_\mathfrak{c}$ on $\mathscr{T}^\times$ with values in $V(\mathfrak{c},\mathfrak{f}^2;\mathscr{V})$ as follows:

$$\mathbb{E}_{\mathfrak{c},\mathfrak{x}}(N^{-k}\phi) = E_k(\phi^\circ;\mathfrak{c})\big|(1-\langle\mathfrak{x}\rangle N(\mathfrak{x})^2) \quad \text{and} \quad \mathbb{E}_\mathfrak{c}(N^{-k}\phi) = E_k(\phi^\circ;\mathfrak{c}).$$

Here the word "pseudo-measure" just means that $\mathbb{E}_{\mathfrak{c},\mathfrak{x}}(N^{-k}\phi)$ is a bounded measure.

8.2 The p-ordinary Katz p-adic L-functions

Let (φ,Σ,S) be the triple studied in Chapter 7 for the CM quadratic extension L/F. Let $\mathbf{p} = \prod_{\mathbf{P}\in S}\mathbf{P}$. We define the p-ordinary Katz measure on $\mathscr{G}(C) =$

$Cl_L(C\mathbf{p}^\infty)$ for a fixed ideal C prime to p and evaluate the integral of the p-adic avatars of S-ramified (but S^c-unramified) Hecke characters. We identify $F_{\mathbb{C}}$ with $\mathbb{C}^\Sigma$ via $a \mapsto (a^\sigma)_{\sigma\in\Sigma}$ for $a \in F$. For each fractional ideal $\mathfrak{A}$ prime to Cp in L, we take a test object $(A(\mathfrak{A}), \mathcal{P}(\mathfrak{A}), \omega(\mathfrak{A}), i(\mathfrak{A}))$ defined as follows: Over $\mathbb{C}$, $A(\mathfrak{A})(\mathbb{C}) = F_{\mathbb{C}}/\Sigma(\mathfrak{A})$, where $\Sigma(\mathfrak{A}) = \{(a^\sigma)_{s\in\Sigma} | a \in \mathfrak{A}\}$. We pick an element $\delta \in L$ as in (Δ1-2) in Section 7.2. Then $\langle\ ,\ \rangle$ induces $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$-polarization $\mathcal{P}(\mathfrak{A})$ on $A(\mathfrak{A})$. We define the level structure $i(\mathfrak{A}) : F_p/\mathfrak{d}_p^{-1} \times (\mathfrak{f}^2)^*/\mathfrak{d}^{-1} \to A(\mathfrak{A})$ in exactly the same way as in [47] p.211. We recall the definition of $i(\mathfrak{A})$. We decompose $C = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ so that

(Ia) $\mathfrak{F} + \mathfrak{F}_c = \mathfrak{R}$, $\mathfrak{F} + \mathfrak{F}^c = \mathfrak{R}$, $\mathfrak{F}_c + \mathfrak{F}_c^c = \mathfrak{R}$ and $\mathfrak{F}_c \supset \mathfrak{F}^c$,

(Ib) $\mathfrak{I}$ consists of ideals inert or ramified in L/F.

Put $\mathfrak{f} = \mathfrak{F}\mathfrak{I} \cap F$. We write $\mathfrak{f} = \prod_{\mathfrak{l}} \mathfrak{l}^{\varepsilon(\mathfrak{l})}$. Thus we know that $\mathfrak{F} = \prod_{\mathfrak{L}|\mathfrak{F}} \mathfrak{L}^{\varepsilon(\mathfrak{L})}$ for primes $\mathfrak{L}$ in L for $\varepsilon(\mathfrak{L}) = \varepsilon(\mathfrak{L} \cap F)$. We choose a prime element $\varpi_{\mathfrak{l}}$ for each prime $\mathfrak{l}$ dividing $p\mathfrak{f}$ in F so that $\varpi_{\mathfrak{l}} \equiv 1 \mod \mathfrak{f}\mathfrak{l}^{-\varepsilon(\mathfrak{l})}$ (in this formula, we agree to put $\varepsilon(\mathfrak{P}) = 0$ if $\mathfrak{P}|p$) and $\varpi_{\mathfrak{l}}$ is prime to other prime $\mathfrak{l}'$ dividing $p\mathfrak{f}$. We choose a differental idele d of F such that $d = d_{p\mathfrak{f}\mathfrak{d}}$ and $d_{\mathfrak{q}} = (2\delta)_{\mathfrak{Q}}$ for prime ideal $\mathfrak{Q}|\mathfrak{F}$ ($\mathfrak{q} = \mathfrak{Q} \cap F$). Then we have

$$i(\mathfrak{A}) : F_p/\mathfrak{d}_p^{-1} \times (\mathfrak{f}^2)^*/\mathfrak{d}^{-1} \to \mathbf{p}^{-\infty}(\mathfrak{F}\mathfrak{I})^{-2}\mathfrak{A}/\mathfrak{A} \quad \text{by} \quad x \mapsto dx,$$

which induces $i(\mathfrak{A}) : F_p/\mathfrak{d}_p^{-1} \times (\mathfrak{f}^2)^*/\mathfrak{d}^{-1} \to A(\mathfrak{A})$. We can find a model

$$(A(\mathfrak{A}), \mathcal{P}(\mathfrak{A}), i(\mathfrak{A})\big|_{(\mathfrak{f}^2)^*/\mathfrak{d}^{-1}})_{/L'}$$

for a finite extension $L'/\mathbb{Q}$ which extends to an abelian scheme over the valuation ring $\mathcal{V}'$ of L' corresponding to the embedding $i_p : L' \hookrightarrow \overline{\mathbb{Q}}_p$, and there exists a nowhere vanishing differential $\omega(\mathfrak{A})_{/\mathcal{V}'}$ on $A(\mathfrak{A})_{/\mathcal{V}'}$. Moreover, defining $\mathcal{V}$ as the composite of $\mathcal{O}_{K_0}$ and $i_p(\mathcal{V}')$, $(A(\mathfrak{A}), \mathcal{P}(\mathfrak{A}), \omega(\mathfrak{A}), i(\mathfrak{A}))$ extends to an abelian scheme over $\mathcal{V}_0 = i_p^{-1}(i_p(\overline{\mathbb{Q}}) \cap \mathcal{V})$. We put $\mathfrak{s} = \mathfrak{F}_c \cap F = \mathfrak{F}_c^c \cap F$ and $\mathfrak{i} = \mathfrak{I} \cap F$. Then we have homomorphisms

$$\iota = \iota_C : \left\{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times \times (\mathfrak{r}/\mathfrak{s})^\times\right\}/\overline{\mathfrak{r}^\times} \to \mathcal{G}(C) \tag{8.2.1}$$

induced by the natural inclusion of F into L, where by ι, the factor $\mathfrak{r}_p^\times$ (resp. $(\mathfrak{r}/\mathfrak{f})^\times$, $(\mathfrak{r}/\mathfrak{s})^\times$) is identified with $\mathcal{R}_S = \prod_{\mathbf{P}\in S} \mathcal{R}_{\mathbf{P}}^\times$ (resp. $(\mathcal{R}/\mathfrak{F})^\times \times (\mathfrak{r}/\mathfrak{i})^\times$, $(\mathcal{R}/\mathfrak{F}_c)^\times$). The morphism ι has finite kernel, and its cokernel is isomorphic to $Cl^-(\mathfrak{I})$ which is the quotient of $Cl_L(\mathfrak{I})$ by the natural image of $(\mathfrak{r}/\mathfrak{i})^\times$. We now choose a complete representative set $\{\mathfrak{A}_j\}$ for $Cl^-(\mathfrak{I})$ consisting of fractional ideals prime to pCC^c. Let $[\mathfrak{A}]$ denote the class of $\mathfrak{A}$ in $\mathcal{G}(C)$. Then $\mathcal{G}(C) = \bigsqcup_{j\in Cl^-(\mathfrak{I})} \mathrm{Im}(\iota)[\mathfrak{A}_j]^{-1}$. To define measures on $\mathcal{G}(C)$, we give a recipe of extending functions defined on $\mathcal{G}(C)$ to $\mathcal{T}$ which supports the Eisenstein measure. For each function ϕ on $\mathcal{G}(C)$ and

index $j \in Cl^-(\mathfrak{I})$, we define functions ϕ_j on $\mathcal{T}$ as follows: First on $\mathcal{T}^\times$, we put $\phi_j(x) = \phi(x[\mathfrak{A}_j]^{-1})$. We extend this function to $\mathcal{T}$ by 0 outside $\mathcal{T}^\times$. We write the ideal of the polarization $\mathcal{P}(\mathfrak{A}_j)$ as $\mathfrak{c}_j$ and E_j (resp. $E_{j,\mathfrak{x}}$) for the specialization of $\mathbb{E}_{\mathfrak{c}_j}$ (resp. $\mathbb{E}_{\mathfrak{c}_j,\mathfrak{x}}$) at the test object $(A(\mathfrak{A}_j), \mathcal{P}(\mathfrak{A}_j), i(\mathfrak{A}_j))$. Then we define a measure $\mu_{\mathfrak{x}}$ on $\mathcal{G}(C)$ by

$$(8.2.2) \qquad (\mathcal{R}^\times : \mathfrak{r}^\times) \int_{\mathcal{G}(C)} \phi d\mu_{\mathfrak{x}} = \phi\left(\prod_{\mathfrak{Q}|\mathfrak{F}} \varpi_{\mathfrak{Q}}^{\varepsilon(\mathfrak{Q})} \prod_{\mathfrak{l}|\mathfrak{i}} \varpi_{\mathfrak{l}}^{\varepsilon(\mathfrak{l})}\right) \sum_j \int_{\mathcal{T}^\times} \phi_j dE_{j,\mathfrak{x}}.$$

Let $\chi : L_{\mathbb{A}}^\times / L^\times \to \mathbb{C}^\times$ be a Hecke character of conductor $C\mathbf{p}^e$ for $\mathbf{p}^e = \prod_{\mathbf{P}\in S} \mathbf{P}^{e(\mathbf{P})}$ such that $\chi(x_\infty) = x_\infty^\eta$ with

$$(\infty) \qquad \eta = m_0\Sigma \quad \text{for} \quad m_0 > 0.$$

Then, in the same manner as in [47] Sections 4 and 5, we can compute $\int \widehat{\chi} d\mu_{\mathfrak{x}}$, where $\widehat{\chi}$ is the p-adic avatar of χ defined on $\mathcal{G}(C)$. We just state the outcome:

Theorem 8.2.1 — *Let the notation and the assumption be as above. Let C be an integral ideal prime to p in L. We put $W_p(\chi) = \prod_{\mathbf{P}\in S} N_{L/\mathbb{Q}}(\mathbf{P}^{-e(\mathbf{P})}) G(2\delta; \chi_{\mathbf{P}})$ for the local Gauss sum $G(2\delta;\chi_{\mathbf{P}})$ as in [47] (0.8). Then there exist a (unique) measure $\mu_{\mathfrak{x}}$ on $\mathcal{G}(C)$ for each ideal $\mathfrak{x}$ prime to pC with values in $\mathcal{V}$ such that*

$$\frac{\int_{\mathcal{G}(C)} \widehat{\chi} d\mu_{\mathfrak{x}}}{\Omega_p^{m_0\Sigma}(1-\chi(\mathfrak{x}))} = W_p(\chi) \frac{(-1)^{m_0 t}\Gamma_F(m_0 t)}{\sqrt{|D|}\Omega_\infty^{m_0\Sigma}} \prod_{\mathfrak{Q}|C}(1-\chi(\mathfrak{Q})) \prod_{\mathbf{P}\in S}\left(1 - \frac{\chi(\mathbf{P})^{-1}}{N(\mathbf{P})}\right) L(0,\chi)$$

for all Hecke characters χ of conductor $C\mathbf{p}^e$ such that $\chi_\infty(x_\infty) = x_\infty^{m_0\Sigma}$ for $m_0 > 0$.

Since $1 - \langle \mathfrak{x} \rangle$ has zero at the trivial character, we expect a pole of the p-adic L-function at the trivial character. By the construction of the Eisenstein measure, the value $\mathbb{E}_{\mathfrak{c},\mathfrak{x}}(\mathbf{1})$ for the characteristic function $\mathbf{1}$ of $\mathfrak{r}_p^\times$ is a constant, and we see from [12] Lemme 5.4 and its corollary that

$$(\text{Res}) \qquad \mathbb{E}_{\mathfrak{c},\mathfrak{x}}(\mathbf{1}) = \log_p(\mathcal{N}(\langle \mathfrak{x} \rangle)) \frac{R_p}{w\sqrt{D}} \prod_{\mathbf{P}\in S}\left(1 - \frac{1}{N_{L/\mathbb{Q}}(\mathbf{P})}\right).$$

8.3 p-Ordinary L-function $L_p(\Phi)$

Here we use the notation introduced in Chapter 7. Let $\lambda : h^{ord}(Np^\infty; \mathbb{O}) \to \mathbb{I}$ be a Λ^{ord}-algebra homomorphism of CM type for a finite torsion-free Λ^{ord}-algebra $\mathbb{I}$. Since λ factors through h^{ord}, we see from $(\infty 1)$ in Section 7.1 of Chapter 7 that for a positive arithmetic point P, writing $\infty(\varphi_P) = m$, we have $m = -m_0\Sigma$ for $0 < m_0 \in \mathbb{Z}$. Thus φ_P^{-1} satisfies the condition (∞) in Section 8.2. We then just put

$$L_p(P, \Phi) = \int_{\mathcal{G}(C)} \varphi_P^{-1} d\mu \quad \text{for} \quad \mu = (1 - [\mathfrak{x}])^{-1} \mu_{\mathfrak{x}}.$$

Then assuming $I_{Fa} = I_{RT}$ and using the complex functional equation, we see in the same manner as in (7.2.4) that

$$\frac{L_p(P,\Phi)}{c_p^+(\Phi_P)} = (\mathfrak{R}^\times : \mathrm{r}^\times)N(\mathfrak{c})^{-1}\varphi_P(\mathfrak{c})W(\varphi_P)\sqrt{D}E^{ord}(P)\frac{\Gamma_F(m(P))L(1,\varphi_P)}{c_\infty^+(\pi(\lambda_P))},$$

where $E^{ord}(P) = \prod_{\mathfrak{Q}|C}(1-\varphi_P^{-1}(\mathfrak{Q})) \times \prod_{\mathbf{P}\in S}(1-\varphi_P(\mathbf{P})N(\mathbf{P})^{-1})$. Thus we know from (Res) and (8.2.2) the following result:

Theorem 8.3.1 — *Let the notation and the assumption be as above. We assume that $I_{Fa} = I_{RT}$ and the Leopoldt conjecture for F and p. Then $L_p(\Phi)$ defined as above satisfies Conjecture 4.6.1 (ii) in Chapter 4. Moreover $L_p(\Phi)$ is holomorphic everywhere (i.e. belongs to $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}_\Omega$) if φ is non-trivial on the torsion part of $\mathcal{G}(C)$. If φ is trivial on the torsion-part, it is holomorphic outside the unique point P such that $\varphi_P = \mathrm{id}$, and it has a simple pole at P.*

We can deduce from (Res) a precise residue formula for $L_p(\Phi)$. We will deduce the formula in the following paragraph in a bit different way from Theorem 6.3.2 in Chapter 6.

8.4 Another proof of Theorem 8.3.1

There is another way of getting the residue formula in Theorem 8.3.1 from Theorem 6.3.2 in Chapter 6. We shall give an outline of the proof here supposing that φ induces the identity character on the torsion part. To simplify the exposition, we assume that $C = 1$ and write $\mathcal{G}$ for $\mathcal{G}(1)$. The idea of the proof is the comparison of the residues of two p-adic L-functions: one is the p-ordinary Katz p-adic L-function given in Theorem 8.3.1 and the other is the p-ordinary p-adic Rankin product L-function constructed in Theorem 6.3.2. We use the notation introduced in the previous sections. We decompose $\mathcal{G} = \Delta \times W_L$ for a finite group Δ and a torsion free subgroup W_L. Take a character $\psi : \Delta \to \overline{\mathbb{Q}}^\times$. Thus we have the associated projection $\psi_* : \mathcal{O}[[\mathcal{G}]] \to \Lambda_L = \mathcal{O}[[W_L]]$. We assume that $\varphi = \psi_*$ for the identity character ψ. Let $\mathcal{X}_L = \mathrm{Hom}_{\mathcal{O}\text{-}alg}(\Lambda_L, \overline{\mathbb{Q}}_p) = Spec(\Lambda_L)(\overline{\mathbb{Q}}_p)$. Then for each $P \in \mathcal{X}_L$, we have a continuous character $\varphi_P = P \circ \psi_* : \mathcal{H} \to \overline{\mathbb{Q}}_p^\times$. We call φ_P^{-1} (or P) *critical* (of type Σ or S) if, regarding $\mathcal{G}$ as a quotient of $L_{\mathbb{A}}^\times$, we have $\varphi_P(x_p) = x_p^{-\eta}$ for $\eta \in \mathbb{Z}[I_L]$ such that

$$\text{the } p\text{-type } \eta \text{ of } \varphi_P^{-1} \text{ as above is } m_0\Sigma \text{ for an integer } m_0 > 0.$$

Then, there exists a complex avatar $\boldsymbol{\varphi}_P : L_{\mathbb{A}}^\times \to \mathbb{C}^\times$ with ∞-type $-\eta$ such that $i_\infty^{-1}(\boldsymbol{\varphi}_P(\mathfrak{a})) = i_p^{-1}(\varphi_P(\mathfrak{a}))$ for all ideals $\mathfrak{a}$ prime to $\mathbf{p}$. Let

$$L(s,\boldsymbol{\varphi}_P) = \sum_{\mathfrak{n}} \boldsymbol{\varphi}_P(\mathfrak{n})N_{L/\mathbb{Q}}(\mathfrak{n})^{-s}$$

be the complex L-function. Write $L_p(\varphi_P)$ (resp. $L_\infty(\varphi_P)$) for $(1-\varphi_P(\mathfrak{x}))^{-1}\int \varphi_P d\mu_{\mathfrak{x}}$ for the Katz measure $\mu_{\mathfrak{x}}$ in Theorem 8.2.1 (resp. $L(0,\varphi_P)$). Then we have

$$\frac{L_p(\varphi_P^{-1})}{\Omega_p^{m_0\Sigma}} = W_p(\varphi_P^{-1})\prod_{\mathbf{P}\in S}(1-\varphi_P^{-1}(\mathbf{P})N(\mathbf{P})^{-1})\frac{(-1)^{m_0[F:\mathbb{Q}]}\Gamma_F(m_0 t)L_\infty(\varphi_P^{-1})}{\sqrt{|D|}\Omega_\infty^{m_0\Sigma}}.$$

We now want to compare L_p and $D_p^{ord}(\pi(\lambda)\otimes\check{\pi}(\lambda))$ in Theorem 6.3.2 of Chapter 6. Since $\mathbf{W}^{ord}$ is naturally a subgroup of finite index of W_L, Λ_L is a finite flat algebra extension of Λ^{ord}. In the same manner as in [47] Section 6, we have the θ-measure $\theta : \mathscr{C}(\mathcal{G};\mathbb{O}_\Omega) \to S^{ord}(D_{L/F}p^\infty;\mathbb{O})$ given by

$$\mathbf{a}(\mathfrak{n},\theta(\phi)) = \sum_{\substack{N_{L/F}(\mathfrak{m})=\mathfrak{n}\\ \mathfrak{m}+\mathbf{p}=\mathfrak{R}}}\phi(\mathfrak{m}).$$

Then θ induces an algebra homomorphism $\theta^* : h^{ord}(D_{L/F}p^\infty;\mathbb{O}) \to \mathbb{O}[[\mathcal{G}]]$ satisfying [47] (6.7b). By our assumption, $\lambda = \psi_*\circ\theta^* : h^{ord}(Dp^\infty;\mathbb{O}) \to \Lambda_L$. Then λ is primitive in the sense of [34] Theorem 3.6. We pick two points $P, Q \in \mathscr{A}_{\Lambda_L}$. Let $m_0\Sigma$ and $m_0'\Sigma$ be the infinity type of φ_P^{-1} and φ_Q^{-1}, respectively. Then writing $\kappa(P)$ for the weight of the point P, we have

$$m_0 = [m(\kappa(P))] \quad \text{and} \quad n(\kappa(P)) = 0.$$

Then $\theta(\varphi_P) = f(P)$ for the Λ_L-adic form f with Hecke eigenvalues given by λ; thus, $\theta(\varphi_P) \in S_{\kappa(P)}(D_{L/F}p^\alpha;\mathbb{O})$ for suitable $\alpha > 0$. Now we write $\mathcal{H}$ for the Katz measure on $\mathbb{G} = Cl_L(p^\infty)$ given in [52]. By the functional equation [52] Theorem (5.3.0), we have

$$\mathcal{H}(\xi) = \mathcal{N}(\mathfrak{c})^{-1}\varphi(\mathfrak{c})^{-1}(\xi)\mathcal{H}(\xi^*) \quad (\Leftrightarrow N_{L/\mathbb{Q}}(\mathfrak{c})^{-1}[\mathfrak{c}]^{-1}\mathcal{H}^* = \mathcal{H}),$$

where $\xi^*(\mathfrak{n}) = \xi(\mathfrak{n}^c)^{-1}\mathcal{N}(\mathfrak{n})^{-1}$ for characters $\xi : \mathcal{G} \to \overline{\mathbb{Q}}_p^\times$ and $a \mapsto a^*$ is the involution of $\mathbb{O}[[\mathbb{G}]]$ induced by $[\mathfrak{n}]^* = \mathcal{N}(\mathfrak{n})^{-1}[\mathfrak{n}^{-c}]$. Thus comparing the evaluation formula of Theorem 8.2.1 and that of the Katz measure (see [52] and [47] Theorem II), we get

$$\mathcal{H}(\xi^*) = \mathcal{N}(\mathfrak{c})\xi(\mathfrak{c})K(\xi) = (\mathcal{R}^\times : \mathfrak{r}^\times)\mathcal{N}(\mathfrak{c})\xi(\mathfrak{c})\prod_{\mathbf{P}\in S}(1-\xi(\mathbf{P}^c))\mu(\xi)$$

for all character $\xi : \mathcal{G}/\Delta(= W_L) \to \overline{\mathbb{Q}}_p^\times$. In other words, for the projection $\rho : \mathbb{G} \to W_L$, we have

$$\rho_*\mathcal{H}^* = (\mathcal{R}^\times : \mathfrak{r}^\times)\mathcal{N}(\mathfrak{c})[\mathfrak{c}]\prod_{\mathbf{P}\in S}(1-[\mathbf{P}^c])\mu.$$

Let L_1, L_1^*, $L_2^* \in \mathcal{O}[[W_L \times W_L]]$ and L^-, $(L^-)^* \in \mathcal{O}[[W_L]]$ be such that

$$L_1(P,Q) = \mu_{\mathfrak{x}}(\varepsilon_{P,Q}), \quad L_1^*(P,Q) = \mathcal{H}(\varepsilon_{P,Q}^*),\ L_2(P,Q) = \mathcal{H}(\varepsilon'_{P,Q}),$$
$$L_2^*(P,Q) = \mathcal{H}((\varepsilon'_{P,Q})^*) \quad \text{and} \quad L^-(P) = \mathcal{H}(\varepsilon_P),\ (L^-)^*(P) = \mathcal{H}(\varepsilon_P^*)$$

for all arithmetic points (P,Q), where $\varepsilon_{P,Q} = \varphi_P^{-1}\varphi_Q$, $\varepsilon'_{P,Q} = \varphi_P^{-1}(\varphi_Q)_c$ and $\varepsilon_P = \varphi_P^{-1}(\varphi_P)_c = \varepsilon'_{P,P}$. As seen in [47] Section 8, these measures exist and are uniquely determined. Then in exactly the same manner as in [47] Section 8, we have

Theorem 8.4.1 — *Suppose that $\varphi|_\Delta$ is the identity character. Then we have*

$$D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\lambda)) = (\mathcal{R}^\times : \mathrm{r}^\times)(1 - \langle \mathfrak{x} \rangle \otimes \langle \mathfrak{x} \rangle^{-1})^{-1} \frac{UL_1^*L_2^*}{\Delta(L/F)(L^-)^*}$$

for a unit U in $\mathcal{O}_\Omega[[W_L \times W_L]]$, where $U(P,P) = 1$ for all P and $\Delta(L/F) = \prod_{\mathfrak{q}|D_{L/F}}(1 - N_{F/\mathbb{Q}}(\mathfrak{q})^{-1})h(L)/h(F)$ for the class numbers $h(L)$ and $h(F)$ of L and F.

Since $L_2^*(P,P) = (L^-)^*(P)$ and $L_1^*(P,Q) = (\mathcal{R}^\times : \mathrm{r}^\times)\mathcal{N}(\mathfrak{c})[\mathfrak{c}]\prod_{\mathbf{P}\in S}(1 - [\mathbf{P}^c])\mu_{\mathfrak{x}}(\varepsilon_{P,Q})$, from the definition of L_p in Section 8.3 and the residue formula of D_p^{ord} in Theorem 6.3.2 of Chapter 6, again we conclude Theorem 8.3.1 and the following residue formula:

$$Res_{P=\mathrm{id}}(L_p(\Phi))$$
$$= (\mathcal{R}^\times : \mathrm{r}^\times)^{-1}\frac{h(L)R_p}{2\sqrt{D}}\prod_{\mathfrak{P}|p}(1 - N_{F/\mathbb{Q}}(\mathfrak{P})^{-1}) = \frac{h(L)R_S}{w\sqrt{D}}\prod_{\mathbf{P}\in S}(1 - N_{L/\mathbb{Q}}(\mathbf{P})^{-1}),$$

where w is the number of roots of unity in L, and the relation of the regulator R_p with the S-adic regulator R_S is given by the following formula:

$$\frac{2R_S(\mathcal{R}^\times : \mathrm{r}^\times)}{w} = R_p,$$

while $\frac{2R_{L,\infty}(\mathcal{R}^\times:\mathrm{r}^\times)}{2^{[F:\mathbb{Q}]}w} = R_\infty$ for the archimedean regulators. There is a canonical choice of sign of $\frac{R_p}{\sqrt{D}}$ (see [12] Section 5) given by ordering the columns of the regulator matrix and the discriminant matrix according to any fixed order of embeddings $i_p\sigma$ ($\sigma \in I$).

8.5 Representations isogenous to Φ

Here we analyze the singularity of $L_p(\phi)$ for ϕ isogenous to Φ. Suppose that $F = \mathbb{Q}$ and $p > 2$ for simplicity. Let $W_L = \mathbf{W}^{ord} \cong \mathbb{Z}_p$ and $W \cong \mathbb{Z}_p^2$ be the torsion-free part of $\mathcal{G}(C)$ and $\mathbb{G} = Cl_L(Cp^\infty)$. There are three elements Φ and $\Phi_\pm$

in the isomorphism class of $\pi(\lambda)$ when φ is trivial on the torsion-part of $\mathcal{G}(C)$. To describe these, we note that $\mathbb{I}$ in this case is given by $\mathcal{O}[[W_L]]$, and φ is the inclusion map of W_L into $\mathbb{I}$ composed with the projection: $\mathcal{G}(C) \to W_L$. Thus $\mathbb{I}$ is a unique factorization domain, and for a prime divisor $P = (\pi)$, $\Phi_P = \mathrm{Ind}_L^{\mathbb{Q}} \mathrm{id} \cong \mathrm{id} \oplus \chi$ for the quadratic character χ corresponding to L/F and the identity character id. Let $W_\pm = \{x \in W | cxc = x^{\pm 1}\}$. Then for a generator w of W_-, its projection to W_L generates W_L, and $\pi = \varphi(w) - \varphi(cwc)$. We let these representations act on $V = \mathbb{I}^2$ so that

$$V[c-1] = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in \mathbb{I} \right\} \quad \text{and} \quad V[c+1] = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \middle| x \in \mathbb{I} \right\}.$$

These representations have an exact sequence of $\mathcal{G}_{\mathbb{Q}}$-modules:

$$0 \to V(\xi) \to V(\Phi_P) \to V(\psi) \to 0. \tag{8.8.1}$$

We can characterize $\phi \in \{\Phi_+, \Phi, \Phi_-\}$ as follows:

	Φ_+	Φ	Φ_-
Sequence(8.8.1)	non-split	split	non-split
(ξ, ψ)	(χ, id)	(id, χ)	(id, χ)

Then on V_p^+, $\mathcal{H} = \mathcal{G}_L$ acts via φ_c, and we have $V_p^+(\phi)$ and $\alpha_\pm(\phi)$ as follows:

ϕ	$V_p^+(\phi)$	$\alpha_+(\phi)$	$\alpha_-(\phi)$	$\mathrm{ord}_P(L_p(\phi))$	$\ell_P(\phi)$
Φ_+	$\{\begin{pmatrix} \pi x \\ -x \end{pmatrix} \mid x \in \mathbb{I}\}$	non-unit: π	unit	2	2
Φ	$\{\begin{pmatrix} -x \\ x \end{pmatrix} \mid x \in \mathbb{I}\}$	unit	unit	1	1
Φ_-	$\{\begin{pmatrix} -x \\ \pi x \end{pmatrix} \mid x \in \mathbb{I}\}$	unit	non-unit: π	1	0

In particular, for Φ and Φ_+, the principle (AT_p) holds in strict sense (i.e. (G) in Section 4.4 of Chapter 4 holds), but for Φ_-, only weaker (AT_p) holds.

Let $\nu : \mathcal{G}_{\mathbb{Q}} \to \mathcal{O}[[\Gamma]]^\times$ be the universal cyclotomic character as in Section 4.5 of Chapter 4. Then we can think about $\Phi_? \otimes \nu$. Since $W \cong W_L \times \Gamma$ identifying Γ with W_+, again using the (non-ordinary) Katz p-adic L-function, we can construct $L_p(\Phi \otimes \nu)$. Since $\alpha_\pm(\Phi_? \otimes \nu) = \alpha_\pm(\Phi_?) \in \mathcal{O}[[W_L]] \subset \mathcal{O}[[W_L \times \Gamma]]$, by the effect of Euler factor at p, $\mathrm{ord}_P(L_p(\Phi \otimes \nu)) = 0$ at the prime divisor $\mathbf{P} : \mathcal{O}[[W_L \times \Gamma]] \to \mathcal{O}[[\Gamma]]$ induced by the projection to Γ. Thus we have $\mathrm{ord}_{\mathbf{P}}(L_p(\Phi_+ \otimes \nu)) = 1$ and $\mathrm{ord}_{\mathbf{P}}(L_p(\Phi_- \otimes \nu)) = 0$. On the other hand, $(\Phi_+ \otimes \nu)_{\mathbf{P}}$ has a non-split exact sequence

$$0 \to \chi\nu \to (\Phi_+ \otimes \nu)_{\mathbf{P}} \to \nu \to 0.$$

This shows, on the contrary to the ordinary case, for $\Phi_- \otimes \nu$ and $\Phi \otimes \nu$, the assertion (G) holds, but for $\Phi_+ \otimes \nu$, the principle (AT_p) fails to hold.

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A. Correction to [41]

page 317 (0.4b): Φ'_P	should read	ψ'_P
page 317 line 8 from the bottom:		
$G(\xi'^{-1}_{\mathfrak{p}}\psi'_{\mathfrak{p}} - 1)$	should read	$G(\xi'^{-1}_{\mathfrak{p}}\psi'^{-1}_{\mathfrak{p}})$
page 318 $(*)$: $L(0, M(\mathbf{f}_P) \otimes M(\mathbf{g}_Q)^\vee)$	should read	$L(1, M(\mathbf{f}_P) \otimes M(\mathbf{g}_Q)^\vee)$
page 319 lines 2, 15 from the top:		
$M(\mathbf{f}_P)$	should read	$M(\mathbf{f}_P)(1)$ (the Tate twist)
page 320 lines 3, 4, 8 from the bottom:		
$\Lambda(0, M(P,Q))$	should read	$\Lambda(1, M(P,Q))$
page 330 line 11 from the bottom:		
$Z(N) \times F^\times_{\mathbf{A}}/\overline{F^\times U_F(Np^\infty)F^\times_{\infty+}}$	should read	$Z(N) = F^\times_{\mathbf{A}}/\overline{F^\times U_F(Np^\infty)F^\times_{\infty+}}$
page 330 line 2 from the bottom:		
$\mathfrak{q}$ outside N	should read	$\mathfrak{q}$ dividing N
page 331 line 2 from the bottom:		
$t_p(\mathscr{V})$	should read	$\iota_p(\mathscr{V})$
page 333 line 9 from the top: $\mathbf{M}_{k,w}$	should read	$\overline{\mathbf{M}}_{k,w}$
page 333 line 8 from the bottom:		
$f \notin \widehat{\mathfrak{r}} \cap F^\times_{\mathbf{A}_f}$	should read	$f \in \widehat{\mathfrak{r}} \cap F^\times_{\mathbf{A}_f}$
page 340 line 3 from the top:		
$\mathbf{e}_F(-\xi x)$	should read	$\mathbf{e}_F(\xi x)$
page 341 line 4 from the top:		
$D^{(1+2s)/}$	should read	$D^{(1+2s)/2}$

page 348 line 16 from the top

$\lvert D\rvert^{1+m(Q)-(P)}$	should read	$\lvert D\rvert^{1+m(Q)-m(P)}$
page 363 (6.8): $\sum_{\substack{\gamma\in\mathfrak{r}^\times\backslash F^\times \\ \xi\in F}}$	should read	$\sum_{\substack{\gamma\in\mathfrak{r}^\times\backslash F^\times \\ \delta\in F}}$
page 366 line 3 from the top: $x, \tau(m)$	should read	$x\tau(m)$

page 374 (8.1a):

$\mathbf{a}_p(y, \int_{G(L)} \phi(w^{-1}, z) w^{-t} \mathcal{N}(z) dE)$	should read	$\mathbf{a}_p(y, \int_{G(L)} \phi(w^{-1}, z) \mathcal{N}(z) dE)$
$\mathbf{a}_{0,p}(y, \int_{G(L)} w^{-t-r} \phi(z) \mathcal{N}(z) dE)$	should read	$\mathbf{a}_{0,p}(y, \int_{G(L)} \phi(z) \mathcal{N}(z) dE)$

B. List of Symbols

Chapter 1

T_n: above (A1)
$\mathcal{G}_F$, $\mathcal{G}_{F_\mathfrak{P}}$: above (A1)
$\mathbb{I}$: above (A1)
M_P, E_P : in (A1)
$V_\mathfrak{P}^\pm$: in (A2$_\pm$)

Chapter 2

T: Section 2.1
$\mathbf{G}$, $\mathbf{G}^{ord}$: Section 2.1
$\widehat{\Gamma}_1(N)$: Section 2.1
$\mathscr{X}$, $\mathscr{X}^{ord}$: Section 2.1
$\mathscr{A}$, $\mathscr{A}^{ord}$: Section 2.1
$m(\kappa), n(\kappa)$: Section 2.1
κ, ε_κ: Section 2.1
$U_0(p^\alpha)$,$U_0(p^\alpha)$, $U_1^1(p^\alpha)$: Section 2.2
$\langle\mathfrak{a}\rangle$: Section 2.2
$S_\kappa(V;A)$, $M_\kappa(V;A)$: Sections 2.3-2.4
$\mathbf{a}_p(y,f)$: Section 2.3
$\mathbb{T}(y)$, $\mathbb{T}(\mathfrak{l})$: Section 2.3
h_κ^{ord}, $h_\kappa^{n.ord}$, $h_\kappa(Np^\infty;\mathbb{O})$: Section 2.5
$\mathbf{W}$, $\mathbf{W}^{ord}$: Section 2.5
$\langle\ ,\ \rangle$: Section 2.6
$S_{\kappa,U}^{n.ord}$, $S_{\kappa,U}^{ord}$: Section 2.6
$M_\kappa(V_\infty;\mathbb{O})$, $S_\kappa(V_\infty;\mathbb{O})$: Section 2.6
$f(P)$: Section 2.7
$\mathscr{A}_\mathbb{I}$, $\mathscr{X}_\mathbb{I}$: Section 2.7
$\pi(\lambda)$, $\pi(\lambda_P)$: Section 2.8
$\overline{Z}, \overline{Z}(N)$: Section 2.9
$h^{ord}(\psi)$, $h^{n.ord}(\psi)$: Section 2.9
$C_0(\lambda,\mathbb{I})$, $C_1(\lambda;\mathbb{I})$: Section 2.9
$S_\kappa(Np;\mathbb{C})[\psi]$: Section 2.10
$\Omega(\pm,\lambda_P;A)$, $L(n,\psi_P;A)$: Section 2.10
Sel$(Ad(\rho))$: Section 2.10

Chapter 3

$L_\iota(s,\varphi)$, $L(s,M)$: Section 3.1
$H_\mathfrak{p}(M)$: Section 3.1
$\Gamma(s,\varphi)$: Section 3.1
$I_{X,\sigma}$, $I_{X,\sigma}^\pm$: Section 3.1
$\mathscr{F}^\pm(M)$, $H_B^\pm({}^\sigma M)$: Section 3.1
$H_{DR}^\pm$: Section 3.1
$\delta_{X,\sigma}(\varphi)$: Section 3.1
$c_{X,\iota}^\pm(\varphi)$, $c_{X,\iota,\sigma}^\pm(\varphi)$: Section 3.1
$c_\iota^\pm(\varphi)$, $c_{\iota,\sigma}^\pm(\varphi)$: Section 3.1
$I_{i_p\sigma,DR}$, $I_{\iota\sigma,et}$, $I_{i_p\sigma,HT}$: Section 3.2
S_F: Section 3.2
$c_{p,\sigma,\iota}^\pm(\varphi)$, $c_{p,\iota}^\pm(\varphi)$: Section 3.2

Chapter 4

$V[c\pm 1]$: Section 4.1
$\Phi_\pm(\tau)$, $\beta_\pm(\tau)$: Section 4.1
$L_{S,i_\infty}(P,\varphi)$, $\mathcal{N}$: Section 4.2
$L_S(\varphi)$,$\alpha_\pm(\varphi)$: Section 4.3
$\Omega_\infty^\pm$, $\Omega_p^\pm$: Section 4.3
φ^*, $\ell_P(\varphi)$: Section 4.4
ord$_P$: Section 4.4

Chapter 5

$c_{X,J}^\pm(M)$, $\delta_{X,J}(M)$: Section 5.1
$Ad(M)$: Section 5.2

$m_i(\sigma)$, $n_i(\sigma)$: Section 5.3
type (A, T, C): Section 5.4

Chapter 6
$L(\varphi_P \otimes \check{\rho}_Q)$: Section 6.1
$D_p(\pi(\lambda) \otimes \check{\pi}(\mu))$: Section 6.2
$\kappa(P)$: Section 6.3
$D_p^{ord}(\pi(\lambda) \otimes \check{\pi}(\mu))$: Section 6.3
χ_Q, ψ_P: Section 6.4
$L(s, \lambda_P \otimes \check{\mu}_Q)$: Section 6.4
$h^u(x)$ (unitarization): Section 6.4 (U)
$\mathfrak{P}^\alpha$, $\tau(n\varpi^\alpha)$: Section 6.4
$C_0(P, Q)$: Section 6.5 Theorem 6.5.1
$(\ ,\ )_\alpha$: Section 6.5 Theorem 6.5.1
$W(\phi)$: Section 6.5 (FE1), (W1)
$W_0(P, Q)$: Section 6.5 (W2)
$\widehat{\Gamma}_0(\mathfrak{m})$: Section 6.6
$E'(P, Q)$, $S(P)$: Section 6.6, Lemma 6.6.1
$U_F(Lp^\alpha)$: Section 6.7
$Z(L)$, $Cl_F(N)$, $Cl_F^0(N)$: Section 6.7
$\mathscr{C}(T; \mathbb{O})$, $\zeta_{\mathfrak{x},L}$: Section 6.7
$E_{\mathfrak{x},L}$: Section 6.7
$E(\xi)$: Section 6.7
$E * \varphi$: Section 6.8
$E *_\lambda \varphi$: Section 6.9

Chapter 7
$L_S(\Phi \otimes \check{\rho})$: Section 7.1
Σ: Section 7.1
$\mathscr{C}\mathscr{M}$, I_{Fa}, I_{RT}: Section 7.2
$c_p^\pm(\Phi_P; I_X)$: Section 7.2
Ω_p, Ω_∞: Section 7.2
$L_p(Ad(\Phi))$: Section 7.3

Chapter 8
$\mathbb{E}_{\mathfrak{c},\mathfrak{x}}$, $\mathbb{E}_{\mathfrak{c}}$: Section 8.1
$\mathscr{G}(C)$, $\mathbf{p}$: Section 8.2
$\mu_{\mathfrak{x}}$, $Cl^-(\Im)$: Section 8.2
$W_p(\chi)$: Section 8.2,Theorem 8.2.1
μ: Section 8.3