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ZERO CYCLES AND THE NUMBER OF GENERATORS OF AN IDEAL

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(to Pierre Samuel)

RESUME

Soit X une surface dans l'espace affine $\mathbb{A}^4$ sur un corps algébriquement clos k. On montre que X est ensemblistement intersection complète si $k = \overline{\mathbb{F}}_p$ ou si X n'est pas birationnelle à une surface projective de type général.

On donne aussi des exemples de variétés affines lisses de dimension n qui ne sont pas des sous-variétés fermées dans $\mathbb{A}^{2n}$. La plupart des résultats s'appuie sur les théorèmes de Mumford et de Roitman concernant le groupe de Chow $CH_0(X)$.

ABSTRACT

Let X be a local complete intersection surface in $\mathbb{A}^4$ over an algebraically closed field k. We show that X is set-theoretic complete intersection if $k = \overline{\mathbb{F}}_p$ or if X is smooth and not birational to a surface of general type.

We also give examples of smooth affine varieties of dimension n, not admitting a closed immersion in $\mathbb{A}^{2n}$. Most of the results here depend crucially on the results of Mumford and Roitman on the Chow group $CH_0(X)$.

Introduction.

Let k be a field and X a closed codimension two local complete intersection sub–scheme of the affine n–space $\mathbb{A}^n_k$. Let I be the defining ideal of X in $k[X_1,\dots,X_n]$ and suppose that there is a surjection $I/I^2 \twoheadrightarrow \omega_X$, where ω_X is the dualizing module of X. Then, the Ferrand–Szpiro Theorem ([Sz], see Cor. 0.2 below) asserts that X is a set–theoretic complete intersection. When X is a curve of dimension one, the surjection $I/I^2 \twoheadrightarrow \omega_X$ always exists and thus Ferrand–Szpiro showed that a local complete intersection curve in $\mathbb{A}^3_k$ is a set–theoretic complete intersection. The question whether any local complete intersection sub–scheme of $\mathbb{A}^n_k$ is a set–theoretic complete intersection is open.

In sections 1 and 2, we examine this question for surfaces in $\mathbb{A}^4$. It is shown that local complete intersection surfaces in $\mathbb{A}^4_{\mathbb{F}_p}$ are set–theoretic complete intersections.

For a smooth surface X in $\mathbb{A}^4_k$ (k algebraically closed), the existence of a surjection $I/I^2 \twoheadrightarrow \omega_X$ turns out to be equivalent to the vanishing of c_1^2 ($c_1 = c_1(\Omega^1_X)$) in the Chow group of zero–cycles. In view of this, it follows by looking at the classification of surfaces, that if X is not birationally equivalent to a surface of general type, then X is a set–theoretic complete intersection (Th. 2.9). We also show that for a smooth affine variety X in $\mathbb{A}^n_k$, the ideal I_X of X in $k[X_1,\dots,X_n]$ is generated by $n-1$ element if and only if Ω^1_X has a free direct summand of rank one (Th. 1.11).

In section 3, we give a partial converse to the Ferrand–Szpiro theorem. More precisely, we show that if $X \subset \mathbb{A}^4$ is a smooth surface which is an intersection of two surfaces $F_1 = F_2 = 0$ such that at each point of X either F_1 or F_2 is smooth, then $c_1^2 = 0$ (Cor. 3.7). In section 4, we prove a result about zero–cycles on the product of two curves, which enables us to produce examples of surfaces $X = C_1 \times C_2$, with C_i smooth affine curves such that X does not admit a closed immersion in $\mathbb{A}^4$. Further for this example Ω^1_X is not generated by three elements and hence X cannot be immersed in $\mathbb{A}^3$. In section 3, for all n,d with $1 \leq d \leq n \leq 2d+1$ we make examples of smooth d–dimensional affine varieties X such that X admits a closed immersion in $\mathbb{A}^n$, but not in $\mathbb{A}^{n-1}$. Further for any embedding of X in $\mathbb{A}^m$, the prime ideal $I(X)$ of X is not generated by $m-1$ element. When d=2 this also provides an example of a smooth surface in $\mathbb{A}^4$ with $c_1^2 \neq 0$. The example in sections 4 and 5 are constructed by showing that the appropriate obstructions in zero–dimensional Chow groups do not vanish.

In this paper we use extensively the results of Roitman ([Ro 1], [Ro 2], [Ro 3]) and Mumford ([Mum]) on the Chow group of zero–cycles. In section 5, we need a result about embedding of affine varieties (Th. 5.7). The simple and elegant proof of this theorem we have included here is due to M.V. Nori. Our thanks are due to him for this proof which replaces our earlier lengthly proof of Theorem 5.7. Thanks are also due to V. Srinivas for asking us a question about embedding of affine varieties. Results in section 5 were rewritten and refined recently in response to his question.

The work in this paper began in 1977. A part of this work was outlined in the survey article [Mu 3]. A major portion of this work was done in 1978 when the first and second named authors were visiting IHES and Ecole Normale Supérieure at Paris, respectively, and the third named author was at Ecole Normale Supérieure. We are grateful to these institutions for hospitality and support. The first two authors were also supported by NSF grants.

We have mentioned some of the recent work relevant to this paper in the form of "remarks".

§0. Notations and preliminaries.

We consider only commutative noetherian rings. Let A be such a ring and $I \subset A$ an ideal. We recall that I is a *complete intersection of height* r *if* I is generated by an A–regular sequence of length r. The ideal I is a *local complete intersection of height* r if for all maximal ideals M containing I, the ideal $I_M \subset A_M$ is a complete intersection of height r. The ideal I is a *set-theoretic complete intersection of height* r if there is an ideal J such that $\sqrt{J} = \sqrt{I}$ and J is a complete intersection of height r. If $I \subset A$ is a local complete intersection of height r, we write $\omega_1 = \mathrm{Ext}^r_A(A/I,A)$. It is well known that $\omega_1 \simeq \mathrm{Hom}(\Lambda^r I/I^2, A/I)$. Note that if X is a smooth affine variety and $V \subset X$ is a local complete intersection sub–scheme of codimension r and I the defining ideal of V in the coordinate ring A of V, then ω_I is the module of sections of $\omega_V \otimes \omega_X^{-1}$, where ω_V and ω_X are the canonical sheaves of V and X respectively.

We recall the following result of Ferrand–Szpiro [Sz], which is crucial for this paper.

Theorem 0 (*Ferrand-Szpiro*). *Let* A *be a commutative noetherian ring and* $I \subset A$ *local complete intersection ideal of height* 2. *Suppose there is a surjection* $I \twoheadrightarrow \omega_I$. *Then there is an exact sequence* $0 \to A \to P \to J \to 0$, *with* P *a projective* A*-module of rank* 2 (*for proof see* [Sz] *or* [Mu 2]).

For a projective R–module L of rank 1, we write $L^n = L^{\otimes n}$, $L^{-n} = \mathrm{Hom}(L^n,R)$, $n \geq 0$.

REMARK 0.1. The existence of surjection $I \twoheadrightarrow \omega_I$ is easily seen to be equivalent to the isomorphism $I/I^2 \approx \omega_I^{-2} \oplus \omega_I$, where $\omega_I^{-2} = \mathrm{Hom}(\omega_I^{\otimes 2}, A/I)$. If every projective A/I–module splits as a direct sum of a free module and a module of rank one (e.g. dim $A/I = 1$), then every projective A/I–module P of rank r is completely determined by $\Lambda^r P$ and hence in this case the surjection $I \twoheadrightarrow \omega_I$ is immediate. This remark and the fact that projective modules over polynomial rings over fields are free (Quillen–Suslin Theorem) led Ferrand–Szpiro to deduce that local complete intersection curves in $\mathbb{A}^3$ are set–theoretic complete intersections. Later, Mohan Kumar (MK1] generalized the Ferrand–Szpiro argument to show that any local complete intersection curve in $\mathbb{A}^n$ is a set–theoretic complete intersection.

We do not know the answer even when $n = 4$ and $V = V(I)$ is a smooth surface and k is algebraically closed.

LEMMA 1.2. *Let $R = k[X_1, X_2, X_3, X_4]$, where k is an algebraically closed field and $I \subset R$ a local complete intersection ideal of height 2. Let $A = R/I$ and let $\omega_I = \omega = \mathrm{Ext}^2_R(A,R)$. Then*

1) $I/I^2 \approx A \oplus \omega^{-1}$
2) *Consider the following conditions*
   a) ω *is generated by two elements.*
   b) $I/I^2 \approx \omega \oplus \overline{\omega}^2$
   c) ω^{-2} *is generated by two elements. We have* a) $\Rightarrow$ b) $\Rightarrow$ c).

PROOF : 1) Since projective R–modules are free and I has projective dimension one, we have an exact sequence $0 \to R^{\ell-1} \to R^{\ell} \to I \to 0$. Tensoring this sequence with $A = R/I$, we get an exact sequence $0 \to L \to A^{\ell-1} \to A^{\ell} \to I/I^2 \to 0$ with L a projective A–module of rank one. Thus in $K_0(A)$, we have $[I/I^2] = [A]+[L]$, and hence $L \approx \Lambda^2 I/I^2 = \omega^{-1}$. Since cancellation holds for projectives over A [Su], we have $I/I^2 \simeq A \oplus \omega^{-1}$.

2) a) $\Rightarrow$ b). By (1), $I/I^2 \oplus A \approx A^2 \oplus \omega^{-1}$. Since ω is generated by two elements, whe have $\omega \oplus \omega^{-1} \approx A^2$. So

$$I/I^2 \oplus A \approx \omega \oplus \omega^{-1} \oplus \omega^{-1} \approx \omega \oplus (\omega^{-1} \otimes A^2)$$
$$\approx \omega \oplus \omega^{-1} \otimes (\omega \oplus \omega^{-1}) \approx A \oplus \omega \oplus \omega^{-2}.$$

Now b) follows from [Su].

3) b) $\Rightarrow$ c). 1) and b) imply that $\omega^2 \oplus \omega^{-1} \approx A \oplus \omega$. Hence $\omega^2 \oplus \omega^{-2} \oplus \omega \approx \omega^2 \oplus A \oplus \omega^{-1} \approx A^2 \oplus \omega$.

Hence by cancelling ω, we have $\omega^2 \oplus \omega^{-2} \approx A^2$, i.e., ω^2 is generated by two elements.

Remarks 1.3. $SK_0(A) = \ker(\tilde{K}_0(A) \xrightarrow{\det} \operatorname{Pic} A)$ has no 2–torsion if char $k \neq 2$ [Le] or $(A/I)_{\mathrm{red}}$ is smooth (Prop. 2.1). Using this fact, it is not hard to show that, in these cases, the implication c) $\Rightarrow$ a) holds and hence a), b), c) are equivalent.

Corollary 1.4. *Let $V \subset \mathbb{A}^4_k$ $(k = \bar{k})$ be a smooth irreducible affine surface. Then V lies on a smooth hypersurface.*

Proof : If $I \subset R = k[X_1,X_2,X_3,X_4]$ is the ideal of V, then we have $I/I^2 \approx A\bar{f} \oplus \omega^{-1}$ $(A = R/I)$. Let $f \in I$ be a lift of $\bar{f}$ and let $I^2 = \Sigma_{i=1}^{\ell} Rf_i$. Then $f, f_1, \ldots, f_\ell$ have no common zeros on $\mathbb{A}^4 - V$. Hence by Bertini's theorem (as given in [Sw] applied to $\mathbb{A}^4 - V$), there exist linear polynomials $h_1, \ldots, h_\ell$ such that Spec $R/f'R$ is smooth and integral on $\mathbb{A}^4 - V$, where $f' = f + \Sigma_{i=1}^{\ell} h_i f_i$. Since V is smooth and f' is a lift of $\bar{f}$ it follows that Spec $R/f'R$ is smooth at points of V. Hence the hypersurface $f' = 0$ is smooth and integral.

Remark 1.5. Recently it has been shown that (1.4) is true for smooth n–dimensional affine varieties V in $\mathbb{A}^{2n}$ ([Mu 5]).

Proposition 1.6. *Let R, I, A and ω be as in Lemma* 1.2. *Suppose $\omega^{\otimes r}$ is generated by two elements for some $r \neq 0$. Then I is a set-theoretic complete intersection.*

Proof : We may assume $r > 0$. Let $f \in I$ such that $I/I^2 \approx A\bar{f} \oplus \omega_I^{-1}$. Set $J = I^r + Rf$. It is easy to see that J is a local complete intersection of height two. Further $J/J^2 \approx R/J.\tilde{f} \oplus \omega_J^{-1}$ (use i) of Lemma 1.2), where $\tilde{f}$ is the image of f in J/J^2. Hence $\omega_J^{-1} = J/(J^2 + Rf)$. By Corollary 0.2 and Lemma 1.2, 2), it suffices to show that ω_J^{-1} is generated by two elements. Since I/J is a nilpotent ideal in R/J it suffices to show that $\omega_J^{-1} \otimes R/I$ is generated by two elements. But

$$\omega_J^{-1} \otimes R/I = \frac{J}{IJ + Rf} = \frac{I^r + Rf}{I^{r+1} + Rf} \approx \omega_I^{-r} .$$

Hence ω_J^{-1} is generated by two elements and the proof of the proposition is complete.

Theorem 1.7. *Let $k = \bar{\mathbb{F}}_p$ and $I \subset k[X_1,X_2,X_3,X_4]$ a local complete intersection of height two. Then I is a set-theoretic complete intersection.*

Proof : Immediate from Proposition 1.6 and the following lemma.

Lemma 1.8. *Let $K/\mathbb{F}_p$ be an algebraic extension and A a d-dimensional affine ring over K $(d > 0)$. Let L be a projective A-module of rank one. Then $L^{\otimes r}$ is generated by d elements for some $r > 0$ (depending on L).*

Proof : We prove the lemma by induction on d. Suppose the lemma is proved for $d = 1$ and assume $d > 1$. Without loss, we may assume that A is reduced, Spec A is connected and $L = I \subset A$ is an invertible ideal. Then I/I^2 is a projective A/I–module of rank one and $\dim A/I = d-1$. Hence by induction hypothesis $(I/I^2)^{\otimes r} = I^r/I^{r+1}$ is generated by $d-1$ elements and hence J is generated by d elements (e.g. see [MK2]).

Thus we may assume $d = 1$. Since Pic A commutes with direct limits, we may assume K is finite. Let A' be the integral closure of A (in its total quotient ring) and F the conductor ideal from A' to A. Then A'/F is finite. Furthermore, Pic A' is finite [We ; p. 207, Th. 5–3–11]). Now the standard exact sequence [Ba, 5.6] $(A'/F)^* \longrightarrow \operatorname{Pic} A \longrightarrow \operatorname{Pic} A'$ ($(A'/F)^*$ = units in A'/F) show that Pic A is finite. This proves the lemma when $d = 1$ and the proof of the lemma is complete.

Remark 1.9. The proof of Lemma 1.8 works verbatim when $k = \mathbb{Z}$. Also recently it has been shown [MKMR] that cancellation theorem similar to [Su] holds for finitely generated rings over $\mathbb{Z}$. Hence (1.2), (1.6) and hence (1.7) hold when R is replaced by $\mathbb{Z}[X_1,X_2,X_3]$ or $\mathbb{F}_q[X_1,X_2,X_3,X_4]$.

Theorem 1.10. *Let k be an algebraically closed field and $I \subset R = k[X_1,\dots,X_n]$ a local complete intersection of height two and $\omega = \omega_1 = \operatorname{Ext}^2_R(R/I,R)$. Then the following conditions are equivalent.*

a) *I is generated by $n-1$ elements.*

b) *I/I^2 is generated by $n-1$ elements.*

c) *ω is generated by $n-2$ elements.*

Proof : a) $\Longrightarrow$ b). Obvious.
b) $\Longrightarrow$ c). Put $A = R/I$. As in the proof of 1) of Lemma 1.2, we have $[I/I^2] = [A \oplus \omega^{-1}]$ in $K_0(A)$. Hence $I/I^2 \oplus A^{\ell} \approx A^{\ell+1} \oplus \omega^{-1}$ for some $\ell \geq 0$. Now b) implies that $A^{\ell+1} \oplus \omega^{-1}$ is generated by $n-1+\ell$ elements. Let $\varphi : A^{n-1+\ell} \twoheadrightarrow A^{\ell+1} \oplus \omega^{-1}$ be a surjection with kernel M. Then

$$A^{n-2} \oplus A^{\ell+1} = A^{n-1+\ell} \approx M \oplus A^{\ell+1} \oplus \omega^{-1}.$$

Since $\dim A = n-2$, by Suslin's cancellation theorem [Su], we get $A^{n-2} \approx M \oplus \omega^{-1}$. This shows that ω^{-1} and therefore ω is generated by $n-2$ elements.
c) $\Rightarrow$ a). Since $\mathrm{Ext}^1(I,R) \approx \omega$, by [Mu2, p. 180], there exists an exact sequence $0 \to R^{n-2} \to P \to I \to 0$ with P a projective R–module. Now a) is immediate by Quillen–Suslin Theorem.

Theorem 1.11. *Let $X \subset \mathbb{A}^n_k$ be a smooth affine variety of dimension d and $I = I(X)$, the defining ideal of X in $k[X_1,\dots,X_n]$ $(k = \bar{k})$. Let A be the coordinate ring of X. Then the following conditions are equivalent.*

1) *I is generated by $n-1$ elements.*
2) *I/I^2 is generated by $n-1$ elements.*
3) *$\Omega_{A/k}$ has a free direct summand of rank one.*

Proof: 1) $\Rightarrow$ 2). Trivial.
2) $\Rightarrow$ 3). We have $I/I^2 \oplus \Omega^1_A \approx A^n$. 2) implies that $I/I^2 \oplus Q \approx A^{n-1}$ for some Q. Hence in $K_0(A)$, $[\Omega_A] = [A] + [Q]$. By [Su], $\Omega_A \approx A \oplus Q$.
3) $\Rightarrow$ 2). Let $\Omega_A \approx A \oplus Q$. Then $I/I^2 \oplus A \oplus Q \approx A^n$. Therefore by [Su], $I/I^2 \oplus Q \approx A^{n-1}$ and hence I/I^2 is generated by $n-1$ elements.
2) $\Rightarrow$ 1). Suppose $n-1 \geq d+2$ i.e., $n \geq d+3$. Then by [MK1], I is generated by $n-1$ elements. If $n = d+1$, I is principal and there is nothing to prove. Hence we may assume $n = d+2$. In this case, the result is immediate from Theorem 1.10.

Remark 1.12. By [Mu5], it follows that Condition 2 in Theorem 1.11 is equivalent to $c_d(\Omega^1_X) = 0$, where $c_d(\Omega_X)$ is the dth Chern class of Ω^1_X with values in the Chow group of zero cycles (see also [MKM, Cor. 2.6]). In particular, for example if X is rational, $I(X)$ is generated by $n-1$ elements.

Proposition 1.13. *Let X be a smooth affine variety of dimension d over a field k (not necessarily algebraically closed) with coordinate ring A. Suppoose X admits a closed immersion in $\mathbb{A}^{d+2}_k$. Then $\Omega^1_{A/k}$ is generated by $d+1$ elements.*

Proof : Let $I \subset k[X_1,\ldots,X_{d+2}]$ be the prime ideal of X. Then, as in the proof of 1) of Lemma 1.2, I/I^2 is stably isomorphic to $A \oplus \omega_I^{-1}$. Now $I/I^2 \oplus \Omega_A^1 \approx A^{d+2}$. So $A \oplus \omega_I^{-1} \oplus \Omega_A^1$ is stably isomorphic to A^{d+2}. So by Bass' cancellation theorem, we have $\Omega_A^1 \oplus \omega_I^{-1} \approx A^{d+1}$, i.e., Ω_A^1 is generated by $d+1$ elements.

§2. Smooth surfaces in $\mathbb{A}^4$.

For an algebraic scheme X over an algebraically closed field k, we denote by $A_p(X)$ the group of p–dimensional cycles modulo rational equivalence. If X is an irreducible scheme of dimension n, we write $A^p(X) = A_{n-p}(X)$. If X is complete, we denote by $A_{00}(X)$, the group of zero cycles of degree zero modulo rational equivalence. The following proposition is an easy consequence of Roitman's theorem (see [RO3] and [Mi]) on torsion in $A_0(X)$.

Proposition 2.1. *Let X be a smooth affine variety of dimension $d \geq 2$ over an algebraically closed field k. Suppose* $\dim X = 2$ *or* $\operatorname{char} k = 0$. *Then $A_0(X)$ is torsion-free.*

Proof : Because of resolution of singularities, we may choose a smooth projective completion V of X. Let $V - X = \bigcup_{i=1}^{r} C_i = C$, where C_i are irreducible sub–varieties of codimension one. We may also assume that C_i are all smooth. Since C is connected, we have an exact sequence

$$\oplus A_{00}(C_i) \xrightarrow{\varphi} A_{00}(V) \longrightarrow A_0(X) \longrightarrow 0\,.$$

Since $A_{00}(C_i)$ are divisible, $A_{00}(V) \approx \operatorname{Im} \varphi \oplus A_0(X)$. Further C generates $\operatorname{Alb}(V)$. (To see this, cutting V by hyperplane sections, we may assume V is a smooth surface. Yhen by Goodman's theorem [Go] C supports an ample divisor and hence generates $\operatorname{Alb}(V)$). Thus, we have a surjective map of abelian varieties, $\operatorname{Alb}(C_i) \twoheadrightarrow \operatorname{Alb}(V)$. This induces a surjective map

$$\psi : \bigoplus_i \operatorname{Alb}(C_i)_{torsion} \twoheadrightarrow \operatorname{Alb}(V)_{torsion}\,.$$

Thus we have the commutative diagram

$$\begin{array}{ccc} \oplus A_{00}(C_i)_{torsion} & \xrightarrow{\varphi} & A_{00}(V)_{torsion} \\ \approx\downarrow & & \downarrow\approx \\ \oplus \mathrm{Alb}(C_i)_{torsion} & \xrightarrow{\psi} & \mathrm{Alb}(V)_{torsion} \end{array} .$$

By Roitman's theorem ([Ro3] and [Mi]), the vertical maps are isomorphisms. Since ψ is surjective, it follows that $(\mathrm{Im}\ \varphi)_{torsion} = A_{00}(V)_{torsion}$ and hence $A_0(X)$ is torsion–free.

REMARK 2.2. Proposition 2.1 is valid for any smooth affine variety X over k $(k=\bar{k})$, in all characteristics (see [Sr] and [Mu5]).

Let X be a smooth affine variety of dimension d with coordinate ring A. For a projective module P, we denote by $c_p(P) \in A^p(X)$, the pth chern class of P [Fu]. Suppose now that $\dim X = 2$. It is weell known [MPS] that

$$A_0(X) = A^2(X) = SK_0(A) \overset{def}{=} \ker(\tilde{K}_0(A) \xrightarrow{det} \mathrm{Pic}(A).$$

If P is a projective A–module of rank r, it is not hard to see that $c_2(P) =$ class of $[A^{r-1}] + [\Lambda^r P] - [P]$ in $SK_0(A)$. In view of this and the cancellation theorem for projectives, we have

REMARK 2.3. Let X be a smooth affine surface over an algebraically closed field k and let A be the coordinate ring of X. Let P be a projective A–module of rank r. Then

1) $c_2(P) = 0 \Leftrightarrow P \approx A^{r-1} \oplus \Lambda^r P$.

2) If P and Q are projective A–modules, then $P \approx Q \Leftrightarrow \mathrm{rank}\ P = \mathrm{rank}\ Q$; $c_i(P) = c_i(Q)$, $i = 1,2$.

3) $L \in \mathrm{Pic}\ A$ is generated by two elements $\Leftrightarrow L \oplus L^{-1} \approx A^2 \Leftrightarrow c_1(L)^2 = 0$ in $A^2(X)$.

The following corollary is immediate from Proposition 2.1 and Remark 2.3.

COROLLARY 2.4. *With the notation as in Remark 2.3, let* $L \in \mathrm{Pic}\ A$. *Then the following conditions are equivalent.*

1) L *is generated by two elements.*

2) $c_1(L)^2 = 0$ *in* $A^2(X)$.

3) $L^{\otimes r}$ *is generated by two elements for some* $r \neq 0$.

Corollary 2.5. *Let $X \subset \mathbb{A}^4$ be a smooth affine surface. Let $I \subset R = k[X_1, X_2, X_3, X_4]$ be its prime ideal. Then the following conditions are equivalent.*

1) $I/I^2 \approx \omega_I \oplus \omega_I^{-2}$.

2) $K_X^2 = 0$ *in* $A^2(X)$ $(K_X = c_1(\omega_I)$ *is the canonical divisor of* $X)$.

3) $rK_X^2 = 0$ *for some* $r \neq 0$.

4) ω_I *is generated by two elements.*

5) I *is generated by three elements.*

If further any of these conditions is satisfied then V is a set-theoretic complete intersection in $\mathbb{A}^4$.

Proof : 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) is immediate from Corollary 2.4 and Theorem 1.10. Further, $I/I^2 \approx A \oplus \omega_I^{-1}$ by Lemma 1.3, 1). Hence 1) holds if and only if

$$c_2(\omega_I \oplus \omega_I^{-2}) = -2K_X^2 = 0 \Leftrightarrow K_X^2 = 0 .$$

The last assertion follows from Proposition 1.6.

Remark 2.6. a) When $\dim X = n \geq 3$, and $L \in \operatorname{Pic} X$, it has been proved that L is generated by n elements if and only if $c_1(L)^n = 0$. For $n = 3$ see [MKM] and for arbitrary n see [Mu5].

b) For further results about set–theoretic complete intersections see [Ly], [Bo] and [MK3].

For a smooth variety X, we write $c_i(X) = c_i(\Omega_X^1) \in A^i(X)$ and $c(X) = 1 + c_1(X) + c_2(X) + \ldots$, the total chern class of X. Following [F], let $s(X) = c(X)^{-1} = \Sigma_{p \geq 0}\, s_p(X)$, $s_p(X) \in A_p(X)$ be the total Segre class of Ω_X^1. If $X \hookrightarrow \mathbb{A}^n$ is a closed immersion with normal bundle N_X, then $s(X) = c(\check{N})_X$, where $\check{N}_X =$ dual of N_X.

Lemma 2.7. *Let* $X \xrightarrow{i} \mathbb{A}^n$ *be a smooth d-dimensional variety. Then* $s_{2d-n}(X) = 0$.

Proof : By the self–intersection formula (cf. [Fu ; Cor. 6.3]), $0 = (i^* i_*[X] = c_{n-d}(N_X))$. Hence $s_{2d-n}(X) = c_{n-d}(\check{N}_X) = 0$.

LEMMA 2.8. *Let V be a smooth projective minimal surface. Suppose there exist integers r, s such that $rc_1(U)^2 + sc_2(U) = 0$ (in $A^2(U)$) for all affine open sets U of V. Then for any smooth affine surface X birationally equivalent to V, $rc_1(X)^2 + sc_2(X) = 0$.*

PROOF : If V is ruled, then $A^2(X) = 0$ and there is nothing to prove. Otherwise, let $\tilde{V}$ be a smooth projective completion of X. Then $\tilde{V}$ dominates V birationally and therefore there exists $E \subset X$, $E = \cup_{i=1}^{\ell} E_i$, E_i rational curves such that the affine surface $U = X - E$ is an open set of V. Let $j: U \hookrightarrow X$ be the inclusion. Then we have the surjective ring homomorphism $j^*A(X) \twoheadrightarrow A(U)$. Now

$$j^*(rc_1(X)^2 + sc_2(X)) = rc_1(U)^2 + sc_2(U) = 0 .$$

Since $A_0(E) = 0$, we have $j^* : A^2(X) \underset{\sim}{\rightarrow} A^2(U)$. Hence $rc_1(X)^2 + sc_2(X) = 0$.

THEOREM 2.9. *Let $X \subset \mathbb{A}^4_k$ $(k = \bar{k})$ be a smooth affine surface. Then X is a set-theoretic complete intersection in the following cases.*

1) *X is not birationally equivalent to a surface of general type.*
2) *X is not birationally equivalent to a projective surface in $\mathbb{P}^3$.*
3) *X is not birationally equivalent to a product of two curves.*

PROOF : In view of Corollary 2.5, it suffices to check that $rc_1(X)^2 = 0$ for some $r > 0$. Let V be a smooth projective completion of X. If X is birationally equivalent to a ruled surface, then $A^2(X) = 0$, so $c_1(X)^2 = 0$ and we are done. So assume that V is not birationally equivalent to a ruled surface. First assume that V is a minimal surface. Then V is one of the following types :

a) $\kappa(V) = 0$, $12c_1(V) = 0$. Thus $12c_1(X)^2 = 0$ and $c_1(X)^2 = 0$ by Proposition 2.1.

b) $\kappa(V) = 1$; there exists r such that $r^2c_1(V)^2 = 0$. Hence again $c_1(X)^2 = 0$, by Proposition 2.1.

According to our hypothesis, if V is i) of general type, then V is a smooth surface in $\mathbb{P}^3$ or degree ≥ 5 or ii) $V = C_1 \times C_2$ where the C_i are smooth non–rational curves.

In case i), let $r = \deg C$, $C = V - X$. Let $i : X \hookrightarrow \mathbb{P}^3 - C$ be the closed immersion. If h is the restriction of a hyperplane to $\mathbb{P}^3 - C$, then $rh^2 = 0$. Hence $ri^*(h)^2 = 0$. Since $c_1(X)$ is a multiple of $i^*(h)$, it follows by Proposition 2.1, that $c_1(X)^2 = 0$.

In case ii), let K_i, $i=1,2$ be the pullback to $V=C_1\times C_2$ of the canonical divisors on C_i. Then $\Omega^1_V=\mathcal{O}(K_1)\oplus\mathcal{O}(K_2)$. Then $K_i^2=0$, $i=1,2$ and $c_2(V)=K_1K_2$, and $c_1(V)=(K_1+K_2)^2=2K_1K_2=2c_2(V)$.

Now let X be any smooth affine surface in $\mathbb{A}^4$ satisfying the hypothesis of the theorem. Then by Lemma 2.8 and the discussion above, either $c_1(X)^2=0$ or $c_1(X)^2=2c_2(X)$ (the latter holds when X is birational to product of two curves). But by Lemma 2.7, $s_0(X)=c_1(X)^2-c_2(X)=0$. Hence in any case $c_1(X)^2=0$ and the proof the theorem is complete.

Remark 2.10. If X is birational to product of two curves and is embedded in $\mathbb{A}^4$, then $2c_1(X)=c_1(X)^2=c_2(X)$. Hence $c_1(X)=c_2(X)=0$.

Remark 2.11. Mohan Kumer [MK3] has recently shown that if $X\subset\mathbb{A}^n$ $(n\geq 5)$ is a smooth affine surface birational to a product of curves, then X is a set–theoretic complete intersection.

§3. A criterion for vanishing of c_1^2.

In this section we give a partial converse to Corollary 2.5. We begin with the following well known lemma.

Lemma 3.1. *Let A be a noetherian ring and M, N finite A-modules. Let $x_1,\dots,x_r$ be a N-regular sequence which annihilates M. Then* $\mathrm{Ext}^r_A(M,N)\approx\mathrm{Hom}_A(M,N/(x_1,\dots,x_r)N)$.

Proof : We use induction on r, the case $r=0$ being trivial. Assume $r>0$ and put $\overline{N}=N/x_1N$. By induction hypothesis,

$$\mathrm{Ext}^{r-1}(M,\overline{N})\approx\mathrm{Hom}_A(M,N/(x_1,\dots,x_r)N).$$

The exact sequence

$$0\longrightarrow N\xrightarrow{x_1}N\longrightarrow\overline{N}\longrightarrow 0$$

given

$$\operatorname{Ext}_A^{r-1}(M,N) \to \operatorname{Ext}_A^{r-1}(M,\overline{N}) \to \operatorname{Ext}_A^{r-1}(M,N) \xrightarrow{0} \operatorname{Ext}_A^{r}(M,N).$$

Since $x_1,\dots,x_r$ is a regular N-sequence annihilating M, we have $\operatorname{Ext}_A^{r-1}(M,N) = 0$. Hence

$$\operatorname{Ext}_A^{r}(M,N) \approx \operatorname{Ext}_A^{r-1}(M,\overline{N}) \approx \operatorname{Hom}(M,N/(x_1,\dots,x_r)N).$$

This completes the proof of Lemma 3.1.

Corollary 3.2. *Let A be a noetherian ring and $I \subset A$ a local complete intersection of height r. Let J be a complete intersection of height r contained in I. Then*

$$\omega_I \approx \operatorname{Ext}^r(A/I,A) \approx \operatorname{Hom}(A/I,A/J) = \frac{J:I}{J}.$$

Lemma 3.3. *Let P be a projective A-module of rank $r-1$ generated by r elements. Then* $\det P = \Lambda^{r-1}P$ *is generated by r elements.*

Proof : We have the surjection $A^r \twoheadrightarrow P$. This induces the surjection $A^r \approx \Lambda^{r-1}A^r \twoheadrightarrow \Lambda^{r-1}P$.

Lemma 3.4. (Swan) *Let P be a projective A-module of rank* 1. *Suppose P is generated by r elements. Then $\otimes^n P$ is generated by r elements for all n.*

Proof : We have a surjection $A^r \overset{\varphi}{\twoheadrightarrow} P$, so that $P \oplus \ker\varphi \simeq A^r$. Taking duals, we see that $P^* = P^{-1}$ is generated by r elements. Hence we may assume $n > 0$. Let $x_1,\dots,x_r$ generated P. Set $\otimes^n x = \underbrace{x \otimes \dots \otimes x}_{n \text{ times}}$. Then $\otimes^n x_1, \dots \otimes^n x_r$ generate $\otimes^n P$: (check locally).

Theorem 3.5. *Let A be a noetherian ring and I a prime ideal which is a local complete intersection of height r. Let $J = (f_1,\dots,f_r)$ be a complete intersection of height r with $\sqrt{J} = I$. Assume that for every maximal ideal $M \supset I$, the ideal J_M contains $r-1$ elements of a minimal set of generators of I/M* (i.e., $\dim_{A/M}\operatorname{Coker}(J \to I/MI) \leq 1$). *Let $k(I)$ denote the quotient field of A/I and $n = \operatorname{length}_{k(I)}(\frac{A_I}{JA_I})$. Then*

1) $\omega_I = \operatorname{Ext}^r(A/I,A)$ *is divisible by $n-1$ in* $\operatorname{Pic} A/I$.

2) $\omega_I^{\otimes n}$ *is generated by r elements, where* $n = \operatorname{length}_{k(I)}(\frac{A_I}{JA_I})$.

PROOF : By hypothesis for every maximal ideal $M \supset I$ *there existe* $g_1,\ldots,g_r \in A_M$ such that $IA_M = (g_1,\ldots,g_r)A_M$ with $g_1,\ldots,g_{r-1} \in J$. So in $A_M/(g_1,\ldots,g_{r-1})$, $\bar{I}_M = I_M/(g_1,\ldots,g_{r-1})$ is a principal prime ideal generated by the non–zero divisor $\bar{g}_r$. Since $J_M/(g_1,\ldots,g_{r-1})$ is $\bar{I}$–primary, $J_M = (g_1,\ldots,g_{r-1},g_r^k)$. Further

$$k = \mathrm{length}_{k(I)} \frac{A_I}{(J_M)_I} = \mathrm{length}_{k(I)} \frac{A_I}{JA_I} = n\,.$$

Hence for every maximal ideal M, there exist $g_1,\ldots,g_r \in A_M$ such that $I_M = (g_1,\ldots,g_r)A_M$ and $J_M = (g_1,\ldots,g_{r-1},g_r^n)$. So for $\ell < n$, $(I^\ell + J)_M$ is generated by $g_1,\ldots,g_{r-1},g_r^\ell$. In particular, $I^\ell + J$ is a local complete intersection of height r. Now by Corollary 3.2, $\omega_I = \mathrm{Hom}(A/I,A/J) = (J{:}I)/J$. We claim that $J{:}I = I^{n-1}+J$. Since $\sqrt{I{:}J} = \sqrt{I^{n-1}+J} = I$, we have to check this locally at maximal ideals $M \supset I$. In A_M, the equality reduces to

$$(g_1,\ldots,g_{r-1},g_r^n) : (g_1,\ldots,g_r) = (g_1,\ldots,g_{r-1},g_r^{n-1}).$$

This is obvious since $g_1,\ldots,g_r$ is a regular A_M–sequence. Hence $\omega_I = I^{n-1}+J/J$. By the local description one also easily sees that I/I^2+J (in fact, all $I^k+J/I^{k+1}+J$, $1 \le k \le n-1$) are projective A/I–modules of rank 1. Set $L = I/I^2+J$. Then we have a natural surjection of

$$L^{\otimes n-1} \xrightarrow{\varphi} \frac{I^{n-1}+J}{J} = \omega_I\,.$$

φ is in fact an isomorphism, since L and ω_I are projective modules of rank 1. This establishes 1). We have the split exact sequence

$$0 \longrightarrow \frac{I^2+J}{I^2} \longrightarrow \frac{I}{I^2} \longrightarrow \frac{I}{I^2+J} = L \longrightarrow 0\,.$$

Now $\frac{I^2+J}{I^2}$ is a projective A/I–module of rank $r-1$ and is generated by r elements since J is generated by r elements. So by Lemma 3.3, $Q = \det(\frac{I^2+J}{I^2})$ is generated by r elements. Taking determinants, we see that $\Lambda^r I/I^2 = \omega_I^{-1} \approx L \otimes Q$. Hence $Q \approx L^{-1} \otimes \omega_I^{-1}$ and therefore

$$Q^{\otimes n-1} \approx (L^{\otimes n-1})^{-1} \otimes (\omega_I^{-1})^{\otimes n-1} = \omega_I^{-1} \otimes \omega_I^{1-n} = \omega_I^{\otimes -n}\,.$$

Since Q is generated by r elements, it follows by Lemma 3.4 that $\omega_I^{\otimes -n}$ and hence $\omega_I^{\otimes n}$ is generated by r elements.

COROLLARY 3.6. *Let* $V \subset \mathbb{A}^n$ *be a closed smooth variety of codimension* r. *Suppose* V *is a set-theoretic complete intersection of* r *hypersurfaces* $H_i = (f_i = 0)$, $f_i \in k[X_1,\ldots,X_n]$, $1 \le i \le r$ *with* $(f_1,\ldots,f_r)$ *containing* $r-1$ *minimal set of generators of* $I(V)_P$ *for all* $P \in V$. ($I(V)$ = prime ideal of V in $k[X_1,\ldots,X_n]$). *If* $H_1 \ldots H_r = mV$ then $m^r c_1(V)^r = 0$ *in* $A^r(V)$.

PROOF : Let A be the coordinate ring of V and L be a projective A–module of rank 1. If L is generated by r elements, we have $L \oplus P \approx A^r$, for projective A–modules P of rank $r-1$. Now $c(P) = (1+c_1(L))^{-1}$. Since rank $P = r-1$, $c_r(P) = (-1)^r c_1(L)^r = 0$. By Theorem 3.5, $\omega_V^{\otimes m}$ is generated by r elements. Hence $(mc_1(V))^r = 0$ in $A^r(V)$.

COROLLARY 3.7. *Let* $X \subset \mathbb{A}_k^{2r}$ *be a smooth affine variety of dimension* r *satisfying the hypothesis of Corollary* 3.6. *Then* $c_1(V)^r = 0$.

PROOF : Immediate from Corollary 3.6, Proposition 1.2 and Remark 2.2.

§4. Zero cycles on product of two curves.

If X is a smooth affine surface in $\mathbb{A}^4$ which is birationally equivalent to a product of curves, then we have seen that $c_1(X)^2 = c_2(X) = 0$. Here we prove a result about zero cycles on product of two curves which shows that there exists smooth affine curves C_i, $i = 1,2$ such that for $X = C_1 \times C_2$, $c_1(X)^2 \neq 0$ and $c_2(X) \neq 0$. This gives in particular an example of a surface not embeddable in $\mathbb{A}^4$.

THEOREM 4.1. *Let* $X = C_1 \times C_2$, *where* C_i *are smooth projective curves. Let* Δ *be a zero cycle of positive degree on* X. *Suppose for all* $(P_1,P_2) \in X$, *there is a positive integer* m (*depending on* (P_1,P_2) *such that* $m\Delta$ *is rationally equivalent to a zero cycle supported on* $P_1 \times C_2 \cup C_1 \times P_2$. *Let* $V = C_1' \times C_2'$, *where* $C_i' = C_i - \mathrm{Supp}\, p_{i*}(\Delta)$, *and* p_i *is the projection of* X *onto* C_i, $i = 1,2$. *Then* $A_0(V) = 0$.

Proof : We write $\sim$ for rational equivalence $\text{Fix}(P_1,P_2) \in X$. Suppose $m\,\Delta \sim D$, with D supported on $P_1 \times C_2 \cup C_1 \times P_2$. Write $D = D_1 \times P_2 + P_1 \times D_2$, where the D_i are zero cycles on C_i, $i = 1,2$. We have

$$mp_{1*}(\Delta) \sim D_1 + (\deg D_2)P_1 \text{ and } mp_{2*}(\Delta) \sim D_2 + (\deg D_1)P_1 .$$

Reading these equivalences on $C_1 \times P_2$ and $P_1 \times C_2$ respectively, we get

$$mp_{1*}(\Delta) \times P_2) \sim D_1 \times P_2 + (\deg D_2).(P_1,P_2)$$

and

$$m(P_1 \times p_{2*}(\Delta)) \sim P_1 \times D_2 + (\deg D_1).(P_1,P_2).$$

Adding these two rational equivalences and restricting to V, we get

$$m \deg \Delta.j^*(P_1,P_2) = -j^*(D) = -j^*(m\,\Delta) = 0 ,$$

where $j: V \hookrightarrow X$ is the inclusion. Since this holds for all $(P_1,P_2) \in X$, we get that $A_0(V)$ is torsion. Hence $A_0(V) = 0$ by Proposition 2.1.

Corollary 4.2. *Let $X = C_1 \times C_2$, where C_i are smooth projective curves of positive genus over $\mathbb{C}$. Let Δ be a zero cycle of positive degree. Then there exists a $(P_1,P_2) \in X$ such that $mi^*(\Delta) \neq 0$ in $A_0(V)$, for any $m > 0$, where $V = C_1' \times C_2'$, $C_i' = C_i - \{P_i\}$, $i = 1,2$, and $i: V \hookrightarrow X$ is the inclusion.*

Proof : Since $p_g(X) > 0$, by [Mum], $A_0(V) \neq 0$ for any open set V of X. Now the corollary is immediate from Theorem 4.1.

Lemma 4.3. *Let $X \subset \mathbb{A}^n$ be a smooth affine variety of dimension d. Let $I \subset k[X_1,\dots,X_n]$ be the prime ideal of X and A its coordinate ring.*

1) *If I is generated by r elements, then $\Omega^1_{A/k}$ has a free direct summand of rank $n-r$. Consequently, $c_i(X) = 0$ for $i > d+r-n$.*

2) *If* $\Omega_{A/k}$ *is generated by* s *elements, then* $s_i(X) = 0$ *for* $i < 2d-s$, (s_i = ith Segre class).

Proof : 1) If I is generated by r elements, then $I/I^2 \oplus L \approx A^r$ for some L. Hence $I/I^2 \oplus L \approx A^{n-r} \approx A^n \approx I/I^2 \oplus \Omega_A$. By [Su], $\Omega_A \approx L \oplus A^{n-r}$.

2) If Ω_A is generated by s elements, then $\Omega_A \oplus L \approx A^s$, for some L. Then I/I^2 and L are stably isomorphic.

Since rank $L = s-d$, we have $s_{d-i}(X) = c_i(I/I^2) = 0$, for $i > s-d$, i.e. $s_i(X) = 0$ for $i < 2d-s$.

Corollary 4.4. *Let* $X = C_1 \times C_2$, *where* C_i *are smooth projective curves of genus* $g_i \geq 2$, $i = 1,2$ *over* $k = \mathbb{C}$. *There exists a* $(P_1,P_2) \in X$ *such that the affine surface* $V = C_1' \times C_2'$, $C_i' = C_i - \{P_i\}$, $i = 1,2$ *has the following properties.*

1) $c_1(V)^2 \neq 0$, $c_2(V) \neq 0$, $c_1(V)^2 \neq c_2(V)$.

2) *For any closed immersion* $V \hookrightarrow \mathbb{A}^n$ *the prime ideal* $I(V) \subset \mathbb{C}[X_1,\dots,X_n]$ *of* V *is not generated by* $n-1$ *elements.*

3) Ω^1_V *is not generated by three elements. In particular, there does not exist any un ramified morphism* $V \longrightarrow \mathbb{A}^3$.

4) V *does not admit a closed immersion in* $\mathbb{A}^4$.

5) $\Lambda^2\Omega^1_V$ *is not genered by two elements.*

Proof : Fix canonical divisors K_i on C_i. Let $\overline{K}_i = p_i^*(K_i)$, $p_i : X \longrightarrow C_i$, being the projection. Then $\overline{K}_1+\overline{K}_2$ is the canonical divisor and $\Omega^1_X = \mathcal{O}(\overline{K}_1) \oplus \mathcal{O}(\overline{K}_2)$. Hence

$$c_1(X)^2 = 2\overline{K}_1.\overline{K}_2 \, , \; c_2(X) = \overline{K}_1.\overline{K}_2 = c_1(X)^2 - c_2(X) \, .$$

Further, $\deg(\overline{K}_1.\overline{K}_2 = 4(g_1-1)(g_2-1) > 0$. Now 1) follows from Corollary 4.2, with $\Delta = \overline{K}_1.\overline{K}_2$. The assertions 2) and 3) are immediate from 1) and Lemma 4.3 since, $c_2(V) \neq 0$, and $s_0(V) = c_1(V)^2 - c_2(V) \neq 0$. Again 4) is immediate from Lemma 2.7, since $s_0(V) \neq 0$. 5) follows from Corollary 2.4, since $c_1(V)^2 \neq 0$.

§5. Examples of surfaces in $\mathbb{A}^4$ with $c_1^2 \neq 0$.

Let X be a smooth affine variety of dimension d. It is well known that X can be embedded in $\mathbb{A}^{2d+1}$. In this section in the range, $d+1 \leq n \leq 2d+1$, we give examples of affine varieties X admitting a closed immersion in $\mathbb{A}^n$ but not admitting a closed immersion in $\mathbb{A}^{n-1}$. These examples will also have $c_d(X) \neq 0$ so that its ideals $I(X)$ aree not generated by $n-1$ elements. When $d = 2$, $n \geq 4$, this also provides an example of a smooth surface in $\mathbb{A}^4$ with $c_1^2 \neq 0$ (cf. Corollary 3.7).

We first collect some facts which follow easily from Roitman's methods [Ro 1], [Ro 2]. As before, for a variety X, $A_0(X)$ = group of zero cycles modulo rational equivalence.

Lemma 5.1. [B1] *Let X be a smooth projective variety of dimension d over $k = \mathbb{C}$. Let $N > 0$ be an integer, and let $\gamma : X^N \times X^N \longrightarrow A_0(X)$ denote the map $\gamma(x_1,\dots,x_N;y_1,\dots,y_N) = \Sigma\, x_i - \Sigma\, y_i$. Let Z be a non-singular variety and suppose given a morphism $f = (f_1,f_2) : Z \longrightarrow X^N \times X^N$ such that the composition $\gamma \circ f : Z \longrightarrow A_0(X)$ is the zero map. Let $\omega \in \Gamma(X,\Omega^q_X)$ be a q-form on X for some $q \geq 1$. Define a differential $\tilde{\omega} \in \Gamma(X^N,\Omega^q)$ by $\tilde{\omega} = \Sigma_{i=1}^N p_i^*(\omega)$, where $p_i : X^N \longrightarrow X$ is the projection on the ith factor. Then $f_1^*(\tilde{\omega}) = f_1^*(\tilde{\omega})$ on Z.*

Lemma 5.2. *Let X be a smooth projective variety over $\mathbb{C}$. Let Y be any complete variety (possibly reducible) of dimension q. Let $\varphi : Y \longrightarrow X$ be a morphism such that the induced map $\varphi_* : A_0(Y) \longrightarrow A_0(X)$ is surjective. Then $H^0(X,\Omega^\ell_X) = 0$ for $\ell > q$.*

Proof : By using Chow lemma first and then resolution of singularities, we may assume Y is smooth and projective. Let $Y_1,\dots,Y_r$ be irreducible components of Y. For r–tuple of non–negative integers $(\alpha_1,\dots,\alpha_r)$ we put $|\alpha| = \Sigma\,\alpha_i$ and $Y_\alpha = \Pi_{i=1}^r Y_i^{\alpha_i}$. (Here $\alpha_i = 0$ means that Y_i is omitted). For $|\alpha| = n$, the restriction to Y_α of $\varphi_n : Y^n \longrightarrow A_0(X)$, given by $\varphi_n(y_1,\dots,y_n) = \varphi(y_1) +\dots+ \varphi(y_n)$, induces a morphism of $\varphi_\alpha : Y_\alpha \longrightarrow A(X)$ in the sense of [Ro2]. Similarly $Y_\alpha \times X \longrightarrow A(X)$ induced by $Y^n \times X \longrightarrow X$, $((y_1,\dots,y_n,x) \longmapsto \varphi(y_1) +\dots+ \varphi(y_n)+x)$ is a morphism. Hence

$$Z_{\alpha,\beta,n} \subset Y_\alpha \times Y_\beta \times X,\ |\alpha| = n,\ |\beta| = n-1$$
$$Z_{\alpha,\beta,n} = \{(z_\alpha,z_\beta,x) \mid \varphi_\alpha(z_\alpha) \underset{\text{rat}}{\sim} \varphi_\beta(z_\beta)+x\}$$

is c–closed (i.e., countable union of irreducible closed sets) [Ro2 ; Lemma 3]. Let $p_{\alpha,\beta,n}$ denote the projection of $Z_{\alpha,\beta,n}$ on X. Since every $x \in X$ is in $\operatorname{Im} \varphi_*$

$$X = \bigcup_{\substack{\alpha,\beta,n \\ |\alpha| = n, |\beta| = n-1 \\ n \geq 1}} \operatorname{Im} p_{\alpha,\beta,n} \,.$$

Hence there exists an irreducible variety $Z \subset Y_\alpha \times Y_\beta \times X$ for some α,β,n, such that Z dominates X (under projection). Let $f : \bar{Z} \longrightarrow Z$ be a desiingularization. Let p_α, q_β denote projection of Z onto Y_α and $Y_\beta \times X$ respectively. $p_\alpha \circ f$ and $q_\beta \circ f$ composed with natural product maps $Y_\alpha \longrightarrow X^n$ and $Y_\beta \times X \longrightarrow X^n$ give the morphisms $f_i : \bar{Z} \longrightarrow X^n$, $i = 1,2$, such that the composite

$$\bar{Z} \xrightarrow{(f_1,f_2)} X^n \times X^n \xrightarrow{\gamma} A_0(X)$$

is zero, wheere γ as in Lemma 5.1 is the natural difference map. Let $\omega \in H^0(X,\Omega^{\ell}_X)$ with $\ell > q$ and $\tilde{\omega} = \Sigma\, p_i^*(\omega)$, $p_i : X^n \times X$ is the ith projection. Since $\dim Y_i < \ell$, $(\varphi|\, Y_i)^*(\omega) = 0$. Hence $f_1^*(\tilde{\omega}) = 0$. On the other hand $f_2^*(\tilde{\omega}) = g^*(\omega)$, where g is the composite map $\bar{Z} \xrightarrow{f} Z \xrightarrow{\text{proj.}} X$. Hence by Lemma 5.1, $f_2^*(\tilde{\omega}) = g^*(\omega) = 0$. Since we are in characteristic zero and $\bar{Z}$ dominates X, we have $\omega = 0$.

Corollary 5.3. *Let X be a smooth affine variety of dimension d over $\mathbb{C}$. Let $\tilde{X}$ be a smooth projective completion. Suppose $H^0(\tilde{X},\Omega^{\ell}_{\tilde{X}}) \neq 0$. Then $A_0(X)$ is a non-zero torsion-free divisible group.*

Proof : By Lemma 5.2, $A_0(\tilde{X}-X) \longrightarrow A_0(\tilde{X})$ is not surjective. Hence $A(X) \neq 0$. The rest follows from Proposition 2.1.

Remark 5.4. As in [MS], using Roitman's methods one can show that there is a homomorphism of an abelian variety $J \longrightarrow A(X)$ with countable kernel so that $\operatorname{rank}_{\mathbb{Q}} A(X) = \operatorname{card} \mathbb{C}$. But we do not need this here.

LEMMA 5.5. *Let K be a field of characteristic zero. Let H_r denote the vector space of homogeneous polynomials of degree r in $X_1,...,X_n$. Then H_r is spanned over K by $\{L^r \mid L$ linear polynomials in $X_1,...,X_n\}$.*

PROOF : Exercise.

COROLLARY 5.6. *Let X be a smooth affine variety of dimension d over $\mathbb{C}$. Suppose $A(X)$ is generated by the intersection products $c_1(L_1)...c_1(L_d)$, with $L_i \in \operatorname{Pic} X = A^1(X)$. If $A_0(X) \neq 0$, then there exists an $L \in \operatorname{Pic} X$ such that $c_1(L_1)^d \neq 0$ in $A_0(X)$.*

PROOF : By Lemma 5.5 (with $K = \mathbb{Q}$), some integral multiple of $c_1(L_1)...c_1(L_d)$ is an integral linea combination of $(c_1(L_1)^d \mid L \in \operatorname{Pic}(X))$. Since $A_0(X)$ is torsion–free, the corollary is immediate.

Next, we need the following result about embeddings of affine varieties. The proof we have given here is due to M.V. Nori. This proof replaces our lengthy proof.

THEOREM 5.7. *Let X be an integral variety of dimension d over an algebraically closed field. There exists a smooth affine open set V of such that V admits a closed immersion in $\mathbb{A}^{d+1}$*

PROOF : (M.V. Nori). By taking a generic projection to $\mathbb{A}^{d+1}$, we get a finite birational map $\pi : X \longrightarrow X'$, such that X' is a hypersurface in $\mathbb{A}^{d+1}$ and π induces an isomorphism $\pi^{-1}(X'_{\text{reg}}) \xrightarrow{\sim} X'_{\text{reg}}$ on regular points. Hence we may assume that X is an integral hypersurface (possibly singular) in $\mathbb{A}^{d+1}$. Let $A = k[x_1,...,x_d,x_{d+1}]$ be the coordinate ring of X. Let $F = \Sigma_{i=0}^m f_i x_{d+1}^i = 0$ be the equation of X, with $f_i \in k[x_1,...,x_d]$ and $f_0 \neq 0$. For any $h \in k[x_1,...,x_d]$, put $x'_{d+1} = x_{d+1}/(hf_0^2)$. Then we have $\Sigma_{i=0}^m f_i (hf_0^2)^i x'^{\,i}_{d+1} = 0$. Dividing this equation by hf_0^2, we see that $\frac{1}{hf_0} \in k[x_1,...,x_d,x'_{d+1}]$. It is easily seen that $A_{hf_0} = k[x_1,...,x_d,x'_{d+1}]$. Hence for any $h \in k[x_1,...,x_d]$, $h \neq 0$, Spec A_{hf_0} admits a closed immersion $\mathbb{A}^{d+1}$. Let h be any nonzero element in $J \cap k[x_1,...,x_d]$, when $J \subset A$ is the ideal defining the singular locus. Then Spec A_{hf_0} is a smooth affine hypersurface in $\mathbb{A}^{d+1}$.

THEOREM 5.8. *Let d,n be positive integers such that $d+1 \leq n \leq 2d$. Then there exists a smooth affine variety X of dimension D over $\mathbb{C}$ such that*

1) *X admits a closed immersion in $\mathbb{A}^{n+1}$, but X does not admit a closed immersion in $\mathbb{A}^n$.*

2) *$c_d(\Omega^1_X) \neq 0$ and the prime ideal $I(X)$ of X in $\mathbb{C}[X_1,\dots,X_m]$ for any closed immersion $X \hookrightarrow \mathbb{A}^m$ is not generated by $m-1$ elements.*

3) *Ω^1_X is not generated by $n-1$ elements.*

4) *$c_1(X)^d \neq 0$ and $\Lambda^d\Omega^1_X$ is not generated by d elements.*

PROOF : Let Y be a product of n elliptic curves. Clearly for any open set V of Y, $A_0(V)$ is generated by the products $c_1(L_1)\dots c_1(L_n)$, with $L_i \in \operatorname{Pic} V$. Further, by Lemma 5.2, since $H^0(Y,\Lambda^n\Omega_Y) \neq 0$, we get that $A_0(V) \neq 0$, for any open set V of Y; By Theorem 5.7, choose an affine open set V of Y such that V admits a closed immersion in $\mathbb{A}^{n+1}$. In view of Corollary 5.6, there exists an $L \in \operatorname{Pic} V$ such that $c_1(L)^n \neq 0$. Since V is affine, by Bertini's theorem, we can choose "generic" D_i, $1 \leq i \leq n-d$ such that the D_i are smooth integral divisors with $\mathcal{O}_V(D_i) \approx L$ and $X = \cap_{i=1}^{n-d} D_i$ is a smooth integral variety of dimension d. We claim that X has all the properties listed in the theorem. Let $I = I_X \subset \mathcal{O}_V$ be the defining ideal of X. Then $I = \mathcal{O}(-D_1) + \dots + \mathcal{O}(-D_{n-d})$. Hence I/I^2 is a direct sum of $n-d$ line bundles each isomorphic to $i^*(L)$, where $i : X \hookrightarrow V$ is the inclusion. Hence the total chern class $c(I/I^2) = (1-i^*c_1(L))^{n-d}$. Since Ω_Y and hence Ω_V is trivial, we have $c(\Omega_X) = (1-i^*c_1(L))^{-n-d}$. Hence

$$c_{n-d}(I/I^2) = (-1)^{n-d} i^*c_1(L)^{n-d}$$

$$c_d(X) = c_d(\Omega^1_X) = (-1)^d \binom{d-n}{d} . i^*c_1(L)^d = \binom{n-1}{d} . i^*c_1(L)^d$$

and

$$c_1(X) = c_1(\Omega^1_X) = (n-d) i^*c_1(L).$$

Since

$$i_*i^*c_1(L)^d = c_1(L)^d i_*[X] = c_1(L)^d c_1(L)^{n-d} = c_1(L)^n \neq 0 ,$$

it follows that $i^*c_1(L)^d \neq 0$. Since $n-d \leq d$ and $A_0(X)$ is torsion–free, it follows that $c_d(\Omega^1_X)$, $s_{2d-n}(X) = c_{n-d}(I/I^2)$ and $c_1(X)^d$ are all non–zero in the Chow ring of X. Now 1) follows from Lemma 2.7 and 2) and 3) are immediate from Lemma 4.2.

Since $c_1(X)^d \neq 0$ and for any $H \in \operatorname{Pic} X$, H is generated by d elements implies $c_1(H)^d = 0$, it follows that $\Lambda^d \Omega_X$ is not generated by d elements.

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