MÉMOIRES DE LA S. M. F.

H. SCHLICHTKRULL

On some series of representations related to symmetric spaces

Mémoires de la S. M. F. 2e série, tome 15 (1984), p. 277-289

<http://www.numdam.org/item?id=MSMF_1984_2_15__277_0>

Société Mathématique de France
2e série, Mémoire n° 15, 1984,p.277-289

ON SOME SERIES OF REPRESENTATIONS RELATED TO SYMMETRIC SPACES.

by

H. Schlichtkrull

In this paper, the series of representations constructed by M. Flensted-Jensen in [3] and [4] are considered. The main results of [8], on lowest K-types and Langlands parameters of the representations of [3] in the equal rank case, are generalized to the other series as well. The representations are identified with subquotients of parabolically induced representations. The parabolic subgroup we use, $P = MAN$, is cuspidal, and moreover, the symmetric space $M/M \cap H$ satisfies the equal rank condition. The inducing representation $\pi \otimes \nu \otimes 1$ of MAN is given by a Flensted-Jensen representation π of M, and thus the determination of Langlands parameters is reduced to Flensted-Jensen representations of M. Further, these results imply unitarity of the representations under certain conditions (see Theorem 4).

Since the proofs of some of our results are rather straightforward generalizations of those of [8], we do not give all the details in these cases, but refer to [8] in stead.

Our results generalize some results of G. Ólafsson [5], [6] (in fact, Theorem 1 and 3 below were obtained before we received [5] and [6]).

The author expresses his gratitude to the organizers of the conference for the invitation to participate.

1. Notation. Let G/H be a semisimple symmetric space with G and H connected and linear. Let τ be the corresponding involution, and let θ be a commuting Cartan involution. Denote by $g = h \oplus q$ and $g = k \oplus p$ the corresponding decompositions of the Lie algebra g, and let K be the maximal compact subgroup of G with Lie algebra k. Let G_0 denote the analytic subgroup of G with Lie algebra $g_0 = k \cap h + p \cap q$.

Choose a θ-invariant maximal abelian subspace a^0 of q, and put $t = a^0 \cap k$. Let $\Delta \subset a_{\mathbb{C}}^{0*}$ be the set of roots of a^0 in $g_{\mathbb{C}}$, and choose a positive system Δ^+ which is θ-compatible, i.e. $\alpha \in \Delta^+$ and $\alpha|_t \neq 0$ implies $\theta\alpha \in \Delta^+$. Put $\rho = \rho(\Delta^+) = \frac{1}{2}\sum_{\alpha\in\Delta^+}(\dim g_{\mathbb{C}}^{\alpha})\alpha \in a_{\mathbb{C}}^{0*}$.

Let $\ell = g^t$ be the centralizer of t in g, and let $\bar{\ell}$ denote the orthocomplement of t in ℓ (w.r.t. the Killing form of g). Choose t_2 maximal abelian in $\bar{\ell} \cap k \cap q$, then $\tilde{t} = t + t_2$ is maximal abelian in $k \cap q$. Let $\Delta_c = \Delta(\tilde{t}_{\mathbb{C}}, k_{\mathbb{C}})$, $\Delta_{c,1} = \{\alpha \in \Delta_c \mid \alpha|_t \neq 0\}$ and $\Delta_{c,2} = \{\alpha \in \Delta_c \mid \alpha|_t = 0\}$. Put $\Delta_{c,1}^+ = \{\alpha \in \Delta_c \mid \exists\beta \in \Delta^+ : \beta|_t = \alpha|_t\}$ and choose a positive system $\Delta_{c,2}^+$ for the root system $\Delta_{c,2}$, then $\Delta_c^+ = \Delta_{c,1}^+ \cup \Delta_{c,2}^+$ is a positive system for Δ_c. Define $\rho_c = \rho(\Delta_c^+) = \frac{1}{2}\sum_{\alpha\in\Delta_c^+}(\dim k_{\mathbb{C}}^{\alpha})\alpha \in i\tilde{t}^*$ and $\rho_{c,1} = \rho(\Delta_{c,1}^+)$ similarly. Notice that $\rho_{c,1}|_{t_2}$ does not vanish in general, but at least we have:

Lemma 1. $\langle\rho_{c,1},\alpha\rangle = 0$ *for all* $\alpha \in \Delta_{c,2}$.

Proof: Let $\alpha \in \Delta_{c,2}$, and denote by s_α reflection in α. Then $s_\alpha(\Delta_{c,1}^+) = \Delta_{c,1}^+$ and hence the lemma. □

For each $\lambda \in a_{\mathbb{C}}^{0*}$ we define $\mu_\lambda \in \tilde{t}_{\mathbb{C}}^*$ by the following equations:

$$(1) \qquad (\mu_\lambda + 2\rho_c)|_t = (\lambda+\rho)|_t \quad \text{and} \quad (\mu_\lambda + 2\rho_{c,1})|_{t_2} = 0 .$$

2. Flensted-Jensen's representations. Let $c \geq 0$ be the smallest possible constant such that [4] Theorem 1 holds, and define $\Lambda \subset a_{\mathbb{C}}^{0*}$ to be the set of those $\lambda \in a_{\mathbb{C}}^{0*}$ satisfying the following conditions (2) and (3):

(2) $\quad \mathrm{Re}\langle\lambda,\alpha\rangle > c$ for all $\alpha \in \Delta^+$ with $\alpha|_{\mathfrak{t}} = 0$

(3) $$\begin{cases} \dfrac{\langle\mu_\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}^+ & \text{for all } \alpha \in \Delta_c^+ \\ \mu_\lambda(X) \in \mathbb{Z} & \text{for } X \in \mathfrak{t},\ \exp 2\pi i X = e\,. \end{cases}$$

For each $\lambda \in \Lambda$ Flensted-Jensen [4] defines a function $\psi_\lambda \in C^\infty(G/H)$ by an integral formula (for the dual function on the dual symmetric space G^0/H^0), and the following properties hold for these functions:

a) The representation of K generated by ψ_λ is finite dimensional and irreducible. Denoting by δ_λ the contragredient of this representation of K, δ_λ is spherical for $K/K\cap H$ and has highest weight μ_λ.

(We have not included Condition (9) of [4], since it is redundant by Lemma 1).

b) ψ_λ is a joint eigenfunction for $U(\mathfrak{g})^K$ acting on $C^\infty(G/H)$ from the left. The eigenvalues are determined as follows: There is a unique homomorphism $\gamma\colon U(\mathfrak{g})^K \to U(\mathfrak{a}^0)$ such that for $u \in U(\mathfrak{g})^K$:

(4) $$u - \gamma(u) \in (\bar{\mathfrak{l}}\cap\mathfrak{k})_{\mathbb{C}}\, U(\mathfrak{g}) + U(\mathfrak{g})\,(\mathfrak{h}_{\mathbb{C}}^{\mathfrak{a}^0} + \mathfrak{n}^0)$$

where $\mathfrak{n}^0 = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Then $u\psi_\lambda = \gamma(u)(-\lambda-\rho)\psi_\lambda$.

Remark. In the sequel we use only properties a) and b) of the functions ψ_λ. If ψ_λ can be defined (e.g. by analytic continuation in λ), such that a) and b) still hold for some λ which does not satisfy (2), then our results can be extended to these parameters as well.

From a) and b) it follows by [2] Proposition 9.1.10 (iii) that the K-type $\check{\mu}_\lambda$ has multiplicity one in the $\mathfrak{g}$-module generated by ψ_λ. Consequently, this module has a unique irreducible quotient T^λ which contains $\check{\mu}_\lambda$.

If $\mathfrak{t}$ is maximal abelian in $\mathfrak{k}\cap\mathfrak{q}$, then ψ_λ is the same as the function defined in [3]. In this case $c = 0$, but (2) is not necessary for defining ψ_λ. In fact (2) is not serious since one can prove that then $\psi_{s\lambda} = \psi_\lambda$ for all elements s from the Weyl group of the root system $\{\alpha \in \Delta \mid \alpha|_{\mathfrak{t}} = 0\}$. The series of $(\mathfrak{g},K)$-

modules T^λ is in this case called the <u>fundamental series</u> for the symmetric space G/H.

If we can choose $\mathfrak{a}^0$ such that $\mathfrak{t} = \mathfrak{a}^0$, we say that G/H satisfies the <u>equal rank</u> condition. If furthermore $\langle\lambda,\alpha\rangle > 0$ for all $\alpha \in \Delta^+$, then ψ_λ is square integrable with respect to invariant measure on G/H, and hence ψ_λ generates a unitary irreducible representation π_λ^G of G, whose Harish-Chandra module is T^λ. This was proved under stronger assumptions on λ in [3], and subsequently proved in general by T. Oshima (unpublished, cf. however [10] and [13]).

<u>3. Lowest K-types</u>. Let $L = G^t$, then L is connected and has Lie algebra $\mathfrak{l}$. Put $\mathfrak{n}_1 = \sum_{\alpha\in\Delta^+,\alpha|_{\mathfrak{t}}\neq 0} \mathfrak{g}_{\mathbb{C}}^\alpha$ and $\mathfrak{n}_2 = \sum_{\alpha\in\Delta^+,\alpha|\mathfrak{t}=0} \mathfrak{g}_{\mathbb{C}}^\alpha$, and observe that $\mathfrak{l}_{\mathbb{C}} + \mathfrak{n}_1$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Choose an Iwasawa decomposition $\mathfrak{l} = \mathfrak{l}\cap\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathfrak{l}}$ such that $\mathfrak{a}^0 \cap \mathfrak{p} \subset \mathfrak{a}$ and $\mathfrak{n}_2 \subset \mathfrak{n}_{\mathfrak{l}}$. Notice that $\mathfrak{a}$ is τ-stable, and $\mathfrak{a}\cap\mathfrak{q} = \mathfrak{a}^0 \cap \mathfrak{p}$ by maximality of $\mathfrak{a}^0$ in $\mathfrak{q}$ so that $\mathfrak{a} = \mathfrak{a}^0\cap\mathfrak{p} + \mathfrak{a}\cap\mathfrak{h}$. Define $\rho_{\mathfrak{l}} \in \mathfrak{a}^*$ by $\rho_{\mathfrak{l}} = \frac{1}{2}\operatorname{Tr} \operatorname{ad}_{\mathfrak{n}_{\mathfrak{l}}}$, then it follows easily that $\rho_{\mathfrak{l}}|_{\mathfrak{a}\cap\mathfrak{q}} = \rho|_{\mathfrak{a}^0\cap\mathfrak{p}}$. Define for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^{0*}$ an element $\nu_\lambda^L \in \mathfrak{a}_{\mathbb{C}}^*$ by

(5) $$\nu_\lambda^L|_{\mathfrak{a}\cap\mathfrak{q}} = -\lambda|_{\mathfrak{a}^0\cap\mathfrak{p}} \quad\text{and}\quad \nu_\lambda^L|_{\mathfrak{a}\cap\mathfrak{h}} = \rho_{\mathfrak{l}}|_{\mathfrak{a}\cap\mathfrak{h}} .$$

<u>Theorem 1</u>. *Assume* $\lambda \in \Lambda$ *and*

(6) $$\langle(\lambda+\rho)|_{\mathfrak{t}},\ \alpha|_{\mathfrak{t}}\rangle \geq 0 \quad \textit{for all} \quad \alpha \in \Delta^+ .$$

Then $\mu_\lambda^\vee$ *is a lowest* K-*type of* T^λ, *and* T^λ *has no other lowest* K-*types*.

<u>Proof</u>: Let $\overline{V}_\lambda$ denote the spherical representation of $\overline{L}$ (the analytic subgroup with Lie algebra $\overline{\mathfrak{l}}$) with parameter $\nu_\lambda^L \in \mathfrak{a}_{\mathbb{C}}^*$, and denote by V_λ the representation of L which extends $\overline{V}_\lambda$ with the character $e^{\mu_\lambda - 2\rho(\mathfrak{n}_1\cap\mathfrak{p})}$ on $\exp i\tilde{\mathfrak{t}}$ (then V_λ is well defined, cf. [8] Lemma 5.5 and the succeeding remark).

Let $X(\mathfrak{l}_{\mathbb{C}}+\mathfrak{n}_1, V_\lambda, \mu_\lambda)$ be the $(\mathfrak{g},K)$-module induced from V_λ in the sense of [11], then one can conclude by comparing actions of $U(\mathfrak{g})^K$ on μ_λ that the module $T^{\lambda\vee}$, contragradient to T^λ, is equivalent to $X(\mathfrak{l}_{\mathbb{C}}+\mathfrak{n}_1, V_\lambda, \mu_\lambda)$, (cf. [8] Lemma 5.6 where T^λ has been interchanged with $T^{\lambda\vee}$).

When $\mathfrak{t} = \mathfrak{a}^0$ Theorem 1 is exactly [8] Theorem 5.4, and the general case follows in the same way as there, the only complication being the analogue of [8] (5.10), but at that point one can apply Lemma 1 above. □

4. Definition. The symmetric space G/H is said to satisfy Condition D, if the subgroup $\tilde{L} = G^{\tilde{\mathfrak{t}}}$ is compact or, equivalently, if

(7) $\quad \text{rank } G/H = \text{rank } G/G_0 = \text{rank } K/K \cap H .$

Notice that if G/H satisfies Condition D, then rank G = rank K, so that the discrete series of G is nonempty. In fact, by [8] Theorem 6.1, π_λ^G belongs in this case to the discrete series of G whenever $\langle\lambda,\alpha\rangle > k$ for all $\alpha \in \Delta^+$, where k is a certain nonnegative constant explicitly determined. However, for "smaller" λ it happens that π_λ^G no longer belongs to the discrete series of G (cf. [8] Example 7.5), and we do not know in general the Langlands parameter ν of π_λ^G in this case.

Examples. 1° $G\times G/d(G)$ satisfies Condition D if and only if rank G = rank K.

2° From the list of [1] exactly the following spaces with G classical satisfy Condition D:

$SU(2r,q)/SU(r,k)+SU(r,q-k)+T$, $SU(p,q)/SO(p,q)$,

$SU(2r,2s)/Sp(r,s)$, $SU(n,n)/SL(n,\mathbb{C})+\mathbb{R}$, $SO^*(2n)/SO(n,\mathbb{C})$,

$SO^*(4n)/SU^*(2n)+\mathbb{R}$, $SO(2r,q)/SO(r,k)+SO(r,q-k)$,

$SO(2r,2s)/U(r,s)$ (r and s not both odd), $Sp(n,\mathbb{R})/SL(n,\mathbb{R})+\mathbb{R}$,

$Sp(2r,q)/Sp(r,k)+Sp(r,q-k)$, $Sp(p,q)/U(p,q)$.

5. T^λ as induced representation. Let $\mathfrak{a}$ be as defined in Section 3, let $A = \exp \mathfrak{a}$ and let $P = MAN$ be a cuspidal parabolic subgroup of G with A as its split component.

Observe that M is invariant under τ, and that $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{q}$ where $\mathfrak{m}$ denotes the Lie algebra of M. Moreover, $M/(M\cap H)_e$ (where subscript e means "identity component") satisfies Condition D (which is generalized to non-connected reductive groups in the obvious fashion).

Let $\Delta_m \subset i\mathfrak{t}^*$ (resp. $\Delta_{mc} \subset i\mathfrak{t}^*$) consist of the roots of $\mathfrak{t}$ in $\mathfrak{m}_{\mathbb{C}}$ (resp. in $\mathfrak{m}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$), let $\Delta_m^+ = \Delta_m \cap \{\alpha|_{\mathfrak{t}} \mid \alpha \in \Delta^+\}$ and $\Delta_{mc}^+ = \Delta_m^+ \cap \Delta_{mc}$, and put $\rho_m = \frac{1}{2}\sum_{\alpha\in\Delta_m^+} (\dim \mathfrak{m}_{\mathbb{C}}^\alpha)\alpha$ and $\rho_{mc} = \frac{1}{2}\sum_{\alpha\in\Delta_{mc}^+} (\dim \mathfrak{m}_{\mathbb{C}}^\alpha \cap \mathfrak{k}_{\mathbb{C}})\alpha$.

For $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, $\mu_\lambda^m \in \mathfrak{t}_{\mathbb{C}}^*$ is defined by $\mu_\lambda^m = \mu + \rho_m - 2\rho_{mc}$. By the following lemma we get for $\lambda \in \mathfrak{a}_{\mathbb{C}}^{0*}$ that $\mu_\lambda^m|_{\mathfrak{t}} = \mu_\lambda|_{\mathfrak{t}}$.

Lemma 2. $\rho|_{\mathfrak{t}} - 2\rho_c|_{\mathfrak{t}} = \rho_m - 2\rho_{mc}$.

Proof: Suppose β is a weight of $i\mathfrak{t} + \mathfrak{a}$ in $\mathfrak{g}_{\mathbb{C}}$, and assume $\beta|_{\mathfrak{t}} \in \{\alpha|_{\mathfrak{t}} \mid \alpha \in \Delta^+\}$. The claim is that if $\beta|_{\mathfrak{a}} \neq 0$ then $\beta|_{\mathfrak{t}}$ contributes nothing to $(\rho-2\rho_c)|_{\mathfrak{t}}$. This follows from the fact that then $\theta\beta$ is also a weight and $\beta|_{\mathfrak{t}} \in \{\alpha|_{\mathfrak{t}} \mid \alpha \in \Delta_c^+\}$. □

Let $\lambda \in \Lambda$. Since the highest weight μ_λ of $\mathfrak{T}$ has multiplicity one in δ_λ, it follows from Lemma 1 that the multiplicity of the weight $\mu_\lambda|_{\mathfrak{t}}$ of $\mathfrak{t}$ in δ_λ is also one. Therefore, δ_λ contains a unique irreducible subrepresentation δ_λ^M of $M \cap K$ of highest weight $\mu_\lambda|_{\mathfrak{t}}$. Assuming

(8) $\qquad \langle\lambda|_{\mathfrak{t}}, \alpha\rangle > 0 \quad$ for all $\alpha \in \Delta_m^+$

it follows from the last paragraph of Section 2 above that $\lambda|_{\mathfrak{t}}$ determines a Flensted-Jensen representation π_λ^M of M in the discrete series of $M/(M\cap H)_e$ (here one should also take into account the possibility that M is not semisimple or not connected. In the latter case π_λ^M is determined by δ_λ^M rather than by $\lambda|_{\mathfrak{t}}$. See [6] Section 4.8).

Theorem 2. *Let* $\lambda \in \Lambda$ *and assume* (8). *Define* $\nu_\lambda^L \in \mathfrak{a}_{\mathbb{C}}^*$ *by* (5).

(i) $\mu_\lambda^\vee$ *is a lowest* K-*type of* $\mathrm{Ind}_P^G(\pi_\lambda^M \otimes \nu_\lambda^L \otimes 1)$ *where it occurs with multiplicity one.*

(ii) T^λ *is equivalent to the irreducible subquotient of* $\mathrm{Ind}_P^G(\pi_\lambda^M \otimes \nu_\lambda^L \otimes 1)$ *containing* $\mu_\lambda^\vee$.

We prove (i) in the next section and (ii) in Section 7.

6. Langlands parameters. For $\lambda \in \Lambda$ let $P_\lambda^G = M_\lambda^G A_\lambda^G N_\lambda^G$ and $P_\lambda^M = M_\lambda^M A_\lambda^M N_\lambda^M$ be cuspidal parabolic subgroups of G and M,

respectively, associated to the K-type $\delta_\lambda^\vee$, respectively the $M \cap K$-type $\delta_\lambda^{M\vee}$ by [12] Proposition 5.3.3, and let σ_λ^G and σ_λ^M be the associated discrete series representations of M_λ^G and M_λ^M, (cf. [12] Lemma 6.6.12). Notice that only the associate classes of P_λ^G and P_λ^M are uniquely determined.

Lemma 3. *We can choose* P_λ^G *and* P_λ^M *such that* $P_\lambda^G \subset P$ *and* $P_\lambda^M = P_\lambda^G \cap M$. *Then* $M_\lambda^M = M_\lambda^G$ *and moreover* $\sigma_\lambda^M = \sigma_\lambda^G$.

The proof is similar to the proof of [8] Lemma 6.5, and we omit it.

In particular $\mathfrak{a}_\lambda^G = \mathfrak{a}_\lambda^M \oplus \mathfrak{a}$.

Assume (8) and let $\nu_\lambda^G \in (\mathfrak{a}_\lambda^G)_{\mathbb{C}}^*$ and $\nu_\lambda^M \in (\mathfrak{a}_\lambda^M)_{\mathbb{C}}^*$ be the Langlands parameters of T^λ and π_λ^M, respectively.

Proof of Theorem 2 (i): Since by definition π_λ^M is a subquotient of $\mathrm{Ind}_{P_\lambda^M}^M(\sigma_\lambda^M \otimes \nu_\lambda^M \otimes 1)$, the composition factors of $\mathrm{Ind}_P^G(\pi_\lambda^M \otimes \nu_\lambda^L \otimes 1)$ are also composition factors of $\mathrm{Ind}_{P_\lambda^G}^G(\sigma_\lambda^M \otimes (\nu_\lambda^M + \nu_\lambda^L) \otimes 1)$ using induction by stages. Theorem 2 (i) then follows from Lemma 3. □

Though Theorem 2(ii) is still to be proved, we observe the following corollary to this and the preceding proof of Theorem 2 (i):

Corollary: $\nu_\lambda^G = \nu_\lambda^M + \nu_\lambda^L$.

Thus the determination of Langlands parameters of Flensted-Jensen's representations is reduced to the case of symmetric spaces satisfying Condition D.

For "large" values of λ, π_λ^M is itself in the discrete series of M (cf. Section 4), so $\sigma_\lambda^M = \pi_\lambda^M$ and thus Theorem 2 (ii) implies:

Theorem 3. *There is a constant* $c_1 \geq 0$ *such that if* $\lambda \in \Lambda$ *and*

$$(9) \qquad \langle \lambda|_{\mathfrak{t}}, \alpha|_{\mathfrak{t}} \rangle > c_1 \quad \text{for all } \alpha \in \Delta^+ \text{ with } \alpha|_{\mathfrak{t}} \neq 0$$

then P, π_λ^M, ν_λ^L *and* μ_λ *constitute a set of Langlands parameters for* T^λ (i.e. $T^\lambda \simeq J_G(P, \pi_\lambda^M, \nu_\lambda^L, \mu_\lambda)$ in the notation of [8] Section 3).

Since we need Theorem 3 in our proof of Theorem 2 (ii), we indicate how to prove the former without reference to the latter.

<u>Proof</u>: The proof follows that of [8] Lemma 6.7 with only minor modifications (see also [11], proof of Proposition 4.13). In short, since $T^{\lambda\vee} \simeq X(\ell_{\mathbb{C}} + n_1, V_\lambda, \mu_\lambda)$, (cf. the proof of Theorem 1), the $\mathfrak{a}$-parameters of $T^{\lambda\vee}$ and V_λ in the Langlands classification coincide when μ_λ is sufficiently "large", which is ensured by (9). V_λ however, has the same $\mathfrak{a}$-parameter as $\overline{V}_\lambda$, and since $\overline{V}_\lambda$ is spherical this is $-\nu_\lambda^L$. □

<u>Remark</u>. In particular, Theorems 1 and 3 generalize the results of [8] to the fundamental series for G/H. For these representations, the results have been obtained independently by G. Ólafsson [6], where they are also generalized to arbitrary real reductive linear groups (in the sense of [12] p. 1).

<u>7. Proof of Theorem 2 (ii)</u>. From Theorem 3 the statement of Theorem 2 (ii) immediately follows for sufficiently large values of λ. We will now prove Theorem 2 (ii) in general by explicit construction of a C^∞-vector for the induced representation $\mathrm{Ind}_{P_\lambda}^G(\pi_\lambda^M \otimes \nu_\lambda^L \otimes 1)$, generating a subrepresentation which contains T^λ as a quotient.

Consider the K-type δ_λ of highest weight μ_λ. Let U_λ be a representation space for δ_λ, and assume that δ_λ is unitary on U_λ. Let u_0 and u_λ in U_λ be a $K \cap H$-fixed vector and a vector of weight μ_λ respectively, normalized to $(u_\lambda, u_0) = 1$.

Define $\rho_P \in \mathfrak{a}^*$ by $\rho_P = \frac{1}{2}\,\mathrm{Tr}\,\mathrm{ad}_{\mathfrak{n}}$. Guided by [3] Eq. (3.18) we attempt a definition of a function φ_λ on G for $\lambda \in \Lambda$:

$$(10)\quad \varphi_\lambda(kxhan) = \int_{(M\cap K\cap H)_e} (\delta_\lambda(kl)u_\lambda, u_0)\, e^{\langle -\lambda-\rho, H(x^{-1}l)\rangle}\, dl \; e^{\langle -\nu_\lambda^L-\rho_P, \log a\rangle}$$

for $k \in K$, $x \in (M\cap G_0)_e$, $h \in (M\cap H)_e$, $a \in A$ and $n \in N$. The term $H(x^{-1}l)$ appearing in (10) is defined using the Iwasawa projection corresponding to Δ^+ of the dual group G^0 - see [3] or [4].

<u>Proposition 1</u>. *Eq.* (10) *defines a nonzero* C^∞*-function* φ_λ *on* G *which is* K*-finite of the irreducible type* $\mu_\lambda^\vee$. *When* (8) *holds the function* $m \to \varphi_\lambda(gm)$ *on* M *belongs to* $L^2(M/(M\cap H)_e)$ *for each* $g \in G$, *and is in the representation space of* π_λ^M.

Proof: For connected semisimple M it follows from [9] Example 3.5 that the formula

$$(11)\qquad \Psi_\lambda(kxh) = \int_{(M\cap K\cap H)_e} \delta_\lambda(kl)u_\lambda e^{\langle -\lambda-\rho_m, H(x^{-1}l)\rangle} dl$$

for $k \in M \cap K$, $x \in (M\cap G_0)_e$ and $h \in (M\cap H)_e$, gives a well defined U_λ-valued C^∞-function on M satisfying $\Psi_\lambda(km) = \delta_\lambda(k)\Psi_\lambda(m)$ for $k \in M \cap K$, $m \in M$. Moreover, when (8) holds the function $m \to (\Psi_\lambda(m), u_0)$ is in $L^2(M/(M\cap H)_e)$ and generates π_λ^M.

The preceding remarks are easily generalized to the general nonconnected reductive M.

From (11) we have that (10) is equivalent to

$$(12)\qquad \varphi_\lambda(k\,man) = (\delta_\lambda(k)\Psi_\lambda(m), u_0)e^{\langle -\nu_\lambda^L-\rho_P,\ \log a\rangle}$$

for $k \in K$, $m \in M$, $a \in A$ and $n \in N$. From this Proposition 1 follows.

□

From Proposition 1 we see that we may regard φ_λ as a C^∞-vector for $\mathrm{Ind}_P^G(\pi_\lambda^M \otimes \nu_\lambda^L \otimes 1)$. Since φ_λ is K-finite of type $\mu_\lambda^\vee$ which has multiplicity one, φ_λ is a joint eigenvector for $U(g)^K$.

Proposition 2. *The eigenvalues for* $U(g)^K$ *of* φ_λ *and* ψ_λ *are equal.*

Proof: Let $u \in U(g)^K$. We will first prove the existence of an element $u_1 \in U(a^0)$ such that $u\varphi_\lambda = u_1(\lambda)\varphi_\lambda$ for all $\lambda \in \Lambda$.

By symmetrization we identify the symmetric algebra $S(k+m)$ with a subspace of $U(g)$. Since $g = n \oplus a \oplus (m+k)$ we can determine elements $v_1,\ldots,v_p$ in $U(a)$ and $w_1,\ldots,w_p$ in $S(k+m)$ such that $u - \Sigma_{i=1}^p v_i w_i \in nU(g)$ (cf. [2] 2.4.14), and since a and $m \cap k$ commute we may assume that w_i is centralized by $m \cap k$ $(i=1,\ldots,p)$.

Put $\varphi_\lambda^y(g) = \varphi_\lambda(yg)$ for $y,g \in G$, then since $u \in U(g)^K$ we have that $(u\varphi_\lambda)(yg) = (u\varphi_\lambda^y)(g)$ for $y \in K$. Using the decomposition $G = KM_eAN$ we may take $g = man$, $m \in M_e$, $a \in A$, $n \in N$. Since φ_λ is invariant under N and homogeneous under A from the right we get

(13) $$(u\varphi_\lambda)(yman) = \Sigma_{i=1}^{P} v_i(-\nu_\lambda^L-\rho_P)(w_i\varphi_\lambda^y)(m)e^{\langle -\nu_\lambda^L-\rho_P,\log a\rangle}$$

To prove our claim that $u\varphi_\lambda = u_1(\lambda)\varphi_\lambda$ for some $u_1 \in U(\mathfrak{a}^0)$ it is then clearly enough to prove that for each $w \in S(\mathfrak{m}+\mathfrak{k})^{\mathfrak{m}\cap\mathfrak{k}}$ there exists $w_0 \in U(\mathfrak{t})$ such that

(14) $$(w\varphi_\lambda^y)(m) = w_0(\lambda|_{\mathfrak{t}})\varphi_\lambda^y(m)$$

for all $\lambda \in \Lambda$ and $m \in M_e$, $y \in K$.

Let $w \in S(\mathfrak{m}+\mathfrak{k})^{\mathfrak{m}\cap\mathfrak{k}}$ and write $w = \Sigma_{j=1}^{q} a_j \otimes b_j$ where $a_j \in S(\mathfrak{m}\cap\mathfrak{p})$ and $b_j \in S(\mathfrak{k})$, according to the identification $S(\mathfrak{m}+\mathfrak{k}) \simeq S(\mathfrak{m}\cap\mathfrak{p}) \otimes S(\mathfrak{k})$. Denote by $v \to v'$ the principal antiautomorphism of $U(\mathfrak{g})$. From (12) we then get for $m \in M_e$ that:

$$(w\varphi_\lambda^y)(m) = \Sigma_{j=1}^{q} (\delta_\lambda(y)\delta_\lambda(b_j')(a_j\Psi_\lambda)(m) , u_0) .$$

Let M^0 denote the group dual to M by Flensted-Jensen's duality. Put $f(x) = e^{\langle -\lambda-\rho_m,H(x)\rangle}$ for $x \in M^0$, and write $m = kxh$ where $k \in (M\cap K)_e$, $x \in (M\cap G_0)_e$ and $h \in (M\cap H)_e$, then (11) gives that

$$\Psi_\lambda(m) = \int_{(M\cap K\cap H)_e} \delta_\lambda(kl)u_\lambda f(x^{-1}l)dl$$

and therefore it follows that

$$(a_j\Psi_\lambda)(m) = \int_{(M\cap K\cap H)_e} \delta_\lambda(kl)u_\lambda([Ad(kl)^{-1}a_j]_L f)(x^{-1}l)dl$$

where $[Ad(kl)^{-1}a_j]_L$ denotes $Ad(kl)^{-1}a_j$ acting as a left invariant differential operator on $C^\infty(M^0)$ (cf. [9] Eq.'s (2.3) and (4.6)).

Now we get

$$\begin{aligned}
&\Sigma_{j=1}^{q} \delta_\lambda(b_j')(a_j\Psi_\lambda)(m)\\
&= \int_{(M\cap K\cap H)_e} \delta_\lambda(kl)\{\Sigma_{j=1}^{q} \delta_\lambda(Ad(kl)^{-1}b_j')u_\lambda([Ad(kl)^{-1}a_j]_L f)(x^{-1}l)\}dl\\
&= \int_{(M\cap K\cap H)_e} \delta_\lambda(kl)\{\Sigma_{j=1}^{q} \delta_\lambda(b_j')u_\lambda(a_{jL}f)(x^{-1}l)\}dl
\end{aligned}$$

since $w = \Sigma a_j \otimes b_j$ commutes with kl.

Using the decompositions

$$\mathfrak{m}_{\mathbb{C}} = \tau(\mathfrak{m}_{\mathbb{C}}\cap\mathfrak{n}^0) \oplus \mathfrak{m}_{\mathbb{C}}^{\mathfrak{t}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus (\mathfrak{m}_{\mathbb{C}}\cap\mathfrak{n}^0)$$

and

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{n}_{c,1} \oplus (\bar{\ell}\cap\mathfrak{k})_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \tau(\mathfrak{n}_{c,1})$$

where $n_{c,1} = \Sigma_{\alpha\in\Sigma^+_{c,1}} k^\alpha_{\mathbb{C}}$, we can define a map $\eta\colon S(m+k)^{t} \to U(t)$ uniquely by

$$w-\eta(w) \in (n_{c,1}+\bar{\ell}\cap k_{\mathbb{C}})S(m+k) + S(m+k)(m^{t}_{\mathbb{C}}\cap p_{\mathbb{C}}+m_{\mathbb{C}}\cap n^0\cap p_{\mathbb{C}}) .$$

Using Lemma 1 one can see that $\delta_\lambda(x)u_\lambda = 0$ for $X \in n_{c,1} + \bar{\ell} \cap k_{\mathbb{C}}$. Since also $X_L f = 0$ for $X \in m^{t}_{\mathbb{C}} + m_{\mathbb{C}} \cap n^0$, it follows then that

$$(w\varphi^Y_\lambda)(m) = \eta(w)(\mu_\lambda|_{t})\varphi^Y_\lambda(m)$$

as claimed in (14).

To finish the proof of Proposition 2 we prove that $u_1(\lambda) = \gamma(u)(-\lambda-\rho)$ for all $\lambda \in a^{0*}_{\mathbb{C}}$. Since φ_λ generates the K-type $\check{\mu}_\lambda$ in $\mathrm{Ind}^G_P(\pi^M_\lambda \otimes \nu^L_\lambda \otimes 1)$ this follows immediately from Theorem 3 when (9) holds. Since u_1 and $\gamma(u)$ are polynomials in λ the assertion holds for all λ.

□

Theorem 2 (ii) follows immediately from Proposition 2.

Remark. It would be interesting if one could construct a G-homomorphism from the space

$$\{f \in C^\infty(G) \mid f(gman) = f(g)e^{\langle -\nu^L_\lambda-\rho_P, \log a\rangle}$$
$$\forall m \in (M\cap H)_e,\ a \in A,\ n \in N,\ g \in G\}$$

to $C^\infty(G/H)$, taking φ_λ to ψ_λ. In the special case of $\sigma = \Theta$, ψ_λ is the spherical function, P is a minimal parabolic and φ_λ is the function $g \to e^{\langle\lambda-\rho,H(g)\rangle}$, and thus the homomorphism searched for is the Poisson transformation. In general the work of Oshima (cf. [7]) can probably be used to construct such a homomorphism.

8. Unitarity. Let $\lambda \in \Lambda$ and consider the following condition on λ

(15) $\langle\lambda|_{t}, \alpha|_{t}\rangle > 0$ for all $\alpha \in \Delta^+$ with $\alpha|_{a^0\cap p} = 0$.

Theorem 4. *Assume* (15), *and moreover that* λ *is purely imaginary on* $a^0 \cap p$. *Then* T^λ *is unitarizable.*

Proof: Choose a parabolic subgroup $\tilde{P} = \tilde{M}\tilde{A}\tilde{N}$ with Langlands decomposition as indicated, such that $\tilde{M}\tilde{A} = G^{a^0 \cap p}$ and $P \subset \tilde{P}$. Then $\tilde{a}$ is τ-invariant, and $\tilde{a} \cap q = a^0 \cap p$ since $\tilde{a}$ centralizes a^0 and a^0 is maximal in q. $\tilde{M}$ is invariant under τ and t is a maximal abelian subspace of $\tilde{m} \cap q$, and thus $\tilde{M}/(\tilde{M} \cap H)_e$ satisfies equal rank. By (15) $\lambda|_t$ determines a representation $\pi_\lambda^{\tilde{M}}$ in the discrete series of $\tilde{M}/(\tilde{M} \cap H)_e$.

Observe that $a = (a \cap \tilde{m}) \oplus \tilde{a}$. Put $\tilde{\ell} = \ell \cap \tilde{m}$, $\tilde{n}_\ell = n_\ell \cap \tilde{\ell}$ and $\tilde{\rho}_\ell = \frac{1}{2} \operatorname{Tr} ad_{\tilde{n}_\ell} \in (a \cap \tilde{m})^*$. It is then easily seen that $\tilde{\rho}_\ell = \rho_\ell|_{a \cap \tilde{m}}$. Therefore $\pi_\lambda^{\tilde{M}}$ is a subquotient of $\operatorname{Ind}_{P \cap \tilde{M}}^{\tilde{M}}(\pi_\lambda^M \otimes \nu_\lambda^L|_{a \cap \tilde{m}} \otimes 1)$ by Theorem 2, and using induction by stages and Theorem 2 once more we get that T^λ is a subquotient of $\operatorname{Ind}_{\tilde{P}}^G(\pi_\lambda^{\tilde{M}} \otimes \nu_{\lambda|\tilde{a}}^L \otimes 1)$.

Now $\tilde{a} = \tilde{a} \cap h \oplus a^0 \cap p$ and $\rho_{\ell|\tilde{a} \cap h} = 0$, therefore $\nu_{\lambda|\tilde{a}}^L$ is purely imaginary by (5), and the theorem follows. □

Remark. Theorem 4 was proved for the fundamental series for large values of λ by Ólafsson ([5]).

REFERENCES

[1] M. Berger, Les espaces symétriques non compacts, Ann. Sci. École Norm. Sup. 74 (1957), 85-177.

[2] J. Dixmier, Algèbres Enveloppantes, Gauthiers-Villars, Paris 1974.

[3] M. Flensted-Jensen, Discrete series for semisimple symmetric spaces, Ann. of Math. 111 (1980), 253-311.

[4] M. Flensted-Jensen, K-finite joint eigenfunctions of $U(g)^K$ on a non-Riemannian semisimple symmetric space G/H. Actes du Colloque d'Analyse Harmonique Non Commutative 1980, Marseille-Luminy. Lect. Notes in Math. 880 (1981), pp. 91-101.

[5] G. Ólafsson, Die Langlands-Klassifizierung, unitäre Darstellungen und die Flensted-Jensensche fundamentale Reihe, Seminar Prof. Maak, Nr. 39, Göttingen 1982.

[6] G. Ólafsson, Die Langlands-Parameter für die Flensted-Jensensche fundamentale Reihe, preprint 1983.

[7] T. Oshima, Poisson transformations on affine symmetric spaces, Proc. Japan Acad. Ser. A, 55 (1979), 323-327.

[8] H. Schlichtkrull, The Langlands Parameters of Flensted-Jensen's Discrete Series for Semisimple Symmetric Spaces, J. Func. Anal. 50 (1983), 133-150.

[9] H. Schlichtkrull, A Series of Unitary Irreducible Representations Induced from a Symmetric Subgroup of a Semisimple Lie Group, Invent. Math. 68 (1982), 497-516.

[10] H. Schlichtkrull, Applications of Hyperfunction Theory to Representations of Semisimple Lie Groups, Prize Essay, Københavns Universitet 1983.

[11] B. Speh and D. Vogan, Reducibility of generalized principal series representations, Acta Math. 145 (1980), 227-299.

[12] D. Vogan, Representations of real reductive Lie groups, Birkhäuser, Boston 1981.

[13] T. Oshima and T. Matsuki, A description of discrete series for semisimple symmetric spaces. Preprint.

Københavns Universitet
Matematisk Institut
p.t.
Institute for Advanced Study

Princeton
NJ 08540

USA