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ON THE CLASSIFICATION OF QUADRATIC FORMS OVER SEMI LOCAL RINGS

by

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1. Notations and definitions

Let A be a semi local ring. In this note we shall only consider non singular quadratic spaces over A. Let $W_q(A)$ be the Witt-group of quadratic spaces over A and $W(A)$ be the Witt-ring of bilinear spaces over A (see [1], [2] for the definitions). It is well-known, that $W_q(A)$ is a $W(A)$-algebra. We shall use the notation $q \sim 0$ to express the fact that the class $[q]$ is 0 in $W_q(A)$. More generally, if q_1, q_2 are two quadratic spaces with $[q_1] = [q_2]$ in $W_q(A)$, we shall write $q_1 \sim q_2$. Let $\Delta(A)$ be the group of isomorphism classes of quadratic separable algebras over A, i.e. of algebras $A(\wp^{-1}(b)) = A \oplus Az$ with $z^2=z+b$, $1+4b \in A^*$ = groups of unities of A. The product in $\Delta(A)$ is defined by

$$A(\wp^{-1}(a)) \circ A(\wp^{-1}(b)) = A(\wp^{-1}(a+b+4ab))$$

For example if $z \in A^*$, then $\Delta(A) \cong A^*/A^{*2}$, and if $4 = 0$, then $\Delta(A) \cong A/\wp(A)$, where $\wp(A) = \{a^2 - a \mid a \in A\}$.

Let $Br(A)$ be the Brauer group of A. Then we have the following usual invariants for quadratic forms (see [1], [2])

$$\begin{aligned} d &: W_q(A) \to \mathbb{Z}/z\mathbb{Z} && \text{(dimension)} \\ a &: W_q(A) \to \Delta(A) && \text{(Arf-invariant)} \\ w &: W_q(A) \to Br(A) && \text{(Witt-invariant)} \end{aligned}$$

The Arf-and Witt-invariants of a quadratic space (E,q) are defined as follows : let $C(E)$ be the Clifford algebra of (E,q) and $D(E)$ be the centralizer of the sub-algebra $C(E)^+$ of elements of degree 0 in $C(E)$. It is easy to see that $D(E)$ is a quadratic separable algebra over A. Thus we define $a(q) = [D(E)] \in \Delta(A)$. a is a group homomorphism on the subgroup $W_q(A)_o$ of $W_q(A)$, which consist of the elements of even dimension, i.e. $W_q(A)_o = \mathrm{Ker}\,(d)$. If $\dim E$ is even, then $C(E)$ is an Azumaya algebra over A, and we define in this case $w(q) = [C(E)] \in Br(A)$. If $\dim E$ is odd, then $C(E)^+$ is an Azumaya algebra over A, and we set $w(q) = [C(E)^+]$. For example let us consider the quadratic space $\langle d\rangle \otimes [1,b]$ with $d \in A^*$, $1 - 4b \in A^*$. Here $\langle d\rangle$ is the one dimensional bilinear space defined by d

and $[1,b]$ is the quadratic space $(Ae \oplus Af,\ q)$ with $q(e) = 1$, $q(f) = b$, $b_q(e,f) = 1$. Then we have

$$a(\langle d\rangle \otimes [1,b]) = [A(\mathfrak{p}^{-1}(-b))]$$
$$w(\langle d\rangle \otimes [1,b]) = [(-d,\ -b]],$$

where $(-d,\ -b]$ is the quaternion algebra $A \oplus Az \oplus Ae \oplus Aze$ with $z^2 = z-b$, $e^2=-d$ $ze + ez = e$.

Let I_A be the maximal ideal of $W(A)$ of bilinear spaces of even dimension. If $2 \in A^{*A}$ we may identify I_A with $W_q(A)_o$, but if $2 \notin A^*$ we have $W_q(A)_o = W_q(A)$. Then it is easy to show that

$$I_A\, W_q(A)_o = \text{Ker}\ (a\ |W_q(A)_o)$$
$$I_A^2\, W_q(A)_o \subseteq \text{Ker}\ (w\ |I_A W_q(A)_o)$$

A long standing question of Pfister is wether the equality $I_A^2\, W_q(A)_o =$ $\text{Ker}\ (w|_{I_A W_q(A)_o})$ is true (if A is a field of characteristic 2 the answer is yes(see [5]). In the next section we shall prove a weak version of the equality above for semi local rings.

Now we introduce another type of invariants, namely the signatures of quadratic forms. A signatur of the ring A is a ring homomorphism $\sigma : W(A) \to \mathbb{Z}$ (= ring of integers). Let Sig (A) be the set of all signatures of A. The canonical homomorphism $\beta : W_q(A) \to W(A)$, which assigns to every quadratic form q its associated bilinear form b_q, induces a ring homomorphism $\bar\sigma = \sigma \circ \beta : W_q(A) \to \mathbb{Z}$ for every $\sigma \in \text{Sig}(A)$, such that Ker $(\bar\sigma)$ is a $W(A)$-submodule of $W_q(A)$. Let us denote the set of such ring homomorphisms $\bar\sigma : W_q(A) \to \mathbb{Z}$ by $\overline{\text{Sig}}\ (A)$. The correspondence $\sigma \leftrightarrow \bar\sigma = \sigma \circ \beta$ defines a bijection $\text{Sig}(A) \to \overline{\text{Sig}}\ (A)$. Then if can be shown that

$$W(A)_t = \bigcap_{\sigma\in\text{Sig}(A)} \text{Ker}\ (\sigma)$$
$$W_q(A)_t = \bigcap_{\bar\sigma\in\overline{\text{Sig}}(A)} \text{Ker}\ (\bar\sigma)$$

(see [2]). Now we define the total signature map

$$s : W_q(A) \to \prod_{\bar\sigma\in\overline{\text{Sig}}(A)} \mathbb{Z}_{\bar\sigma}$$

$(\mathbb{Z}_{\bar\sigma} = \mathbb{Z})$ by $$s(q) = (\bar\sigma\ (q)) \in \prod_{\bar\sigma\in\overline{\text{Sig}}(A)} \mathbb{Z}_{\bar\sigma}$$

2. <u>A classification theorem for quadratic forms</u>

Combining the maps of section 1 we can define

$$\Phi : W_q(A) \to \mathbb{Z}/2\mathbb{Z} \times \Delta(A) \times Br(A) \times \prod_{\bar{\sigma}\in Sig(A)} \mathbb{Z}_{\bar{\sigma}}$$

$$\Phi = (d,a,w,s)$$

Then the following theorem is a generalization of some results of Elman and Lam (see [3]).

<u>Theorem.</u> <u>Assume</u> $|A/m| \geq 3$ <u>for all maximal ideals of</u> A. <u>Then</u> Φ <u>is injective if and only if</u> $I_A^2 W_q(A)_o$ <u>is torsion free</u>.

This result means, that if $I_A^2 W_q(A)_o$ is torsion free, quadratic forms over A are classified by its dimension, Arf-invariant, Witt-invariant and total signature. The main step in the proof of the theorem is the following.

<u>Lemma.</u> <u>Assume</u> $|A/m| \geq 3$ <u>for all</u> $m \in \max(A)$. <u>Let</u> $\Sigma(A)$ <u>be the set of elements of</u> A <u>of the form</u> $b = d+d^2 + \sum_i c_i^2$ <u>with</u> $1+4b \in A^*$. <u>Define</u> $B = A(\mathfrak{p}^{-1}(b))$ <u>for some</u> $b \in \Sigma(A)$. <u>Then if</u> $I_A^2 W_q(A)_o$ <u>is torsion free</u>, <u>it follows that</u> $I_B^2 W_q(B)_o$ <u>is torsion free</u>, <u>too</u>.

Let us now use this lemma to prove the theorem.

Let q be an anisotropic quadratic form over A with $\Phi(q) = 0$. We want to show $q = 0$. Thus let us assume $q \neq 0$. Since $s(q) = 0$, it follows that $q \in W_q(A)_t$, and since $a(q) = 0$, we have $q \in (I_A W_q(A)_o)_t$. Using (7.13), V in [1] or (8.10), V in [2], we obtain

$$q \sim \perp_{i=1}^{r} \langle a_i\rangle \otimes [1, -b_i]$$

with $b_i \in \Sigma(A)$. If $r \leq 2$, then comparing invariants on both sides, we conclude $q \sim 0$, which is a contradiction. Assume now $r > 2$. Taking $B = A(\mathfrak{p}^{-1}(b))$ with $b = b_1$, it follows that

$$q \otimes B \sim \perp_{i=2}^{r} \langle a_i\rangle \otimes [1, -b_i]$$

On the other hand $I_B^2 W_q(B)_o$ is still torsion free (see the lemma) and $\Phi(q \otimes B)=0$, thus we obtain $q \otimes B \sim 0$ by induction on r. Now q was assumed to be anisotropic, thus we get from this last relation

$$q \cong \varphi \otimes [1, -b]$$

for a suitable $\varphi = \langle c_1,\ldots,c_r\rangle$, $c_i \in A^*$ (see (4.9), V in [1] or (4.10), V in [B]). Hence

$$q \sim \langle c_1\rangle \otimes \langle 1, c_1c_2\rangle \otimes \langle 1, c_1c_3\rangle \otimes [1,-b] + \varphi_1 \otimes [1, -b]$$

with $\varphi_1 = \langle d_1,\ldots,d_{r-2}\rangle$ for some $d_i \in A^*$. But

$$\langle 1, c_1 c_2 \rangle \otimes \langle 1, c_1 c_3 \rangle \otimes [1, -b] \in (I^2_A W_q(A)_o)_t = 0, \text{ thus}$$

$$q \sim \varphi_1 \otimes [1, -b].$$

Now we apply again induction to the right side and obtain $q \sim 0$. This is a contradiction. Hence $q = 0$, proving the theorem.

Corollary. If A is a semi local ring with $I^2_A W_q(A)_o = 0$, then

$$w : I_A W_q(A)_o \to B_r(A)$$

is a monomorphism.

This follows from the fact, that $I^2_A W_q(A)_o$ implies $Sig(A) = \emptyset$, and from the theorem above. This corollary was proved by Mandelberg in [4].

Remark. Let $u(A)$ be the u-invariant of A, i.e. the maximal dimension of anisotropic quadratic forms over A, which are torsion elements in $W_q(A)$. Then if $u(A) < 8$, it follows that $I^2_A W_q(A)_o$ is torsion free. This fact can be seen as follows. Take $[q] \in (I^2_A W_q(A)_o)_t$ and assume that q is anisotropic. Since $u(A) < 8$ implies $u(A) \leq 6$ (see Appendix B in [2]), we have $\dim q \leq 6$. But $[q] \in I^2_A W_q(A)_o$ implies $\dim q \equiv 0(2)$, $a(q) = 1$, $w(q) = 1$, so that we can apply (4.13), V in [1] or (4.14), V in [2], to conclude that $q \sim 0$.

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