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THE NON-ARCHIMEDEAN CORONA PROBLEM

Marius van der PUT

§.1. Introduction and Summary.

Let K denote a complete, non archimedean valued field. The central problem of this work is the Corona problem (see (3.1) (3.3)).

Let K be algebraically closed, $K\langle X_1, \dots, X_n\rangle$ the Banach algebra of all bounded analytic functions on the "open" polydisc $\Delta(K)^n = \{(\lambda_1, \dots, \lambda_n) \in K^n \| \lambda_i| < 1$ for all $i\}$. Suppose that $f_1, \dots, f_s \in K\langle X_1, \dots, X_n\rangle$ have the property $\inf\{\max_{1\leqslant i\leqslant s} |f_i(\lambda)| \mid \lambda \in \Delta(K)^n\} > 0$. Are there $g_1, \dots, g_s \in K\langle X_1, \dots, X_n\rangle$ such that $\sum f_i g_i = 1$?

The cases (n = 1 and all s) and (n > 1 and s = 2) are proved. The proof consists of two steps : (3.4) : A reduction of the corona statement to a problem on polynomials (2.1). (2.4) and (2.6) : Solution of this problem on polynomials for (n=1, all s) and (n > 1, s = 2).
Section 2 contains further alternative problems related to the Corona-conjecture and a discription of $\delta(I)$ in terms of complete ideals (see (2.8)).

In section 4 a detailed study of the ring $K\langle X\rangle$ (i.e. n = 1) is made. In particular a theorem of M. Lazard on zero's of analytic functions is generalized. As an application of this one gives in section 5 a complete description of the closed subspaces of $c_o(\mathbb{N}_o \longrightarrow K)$ which are invariant under the anti-shift operator : $T : c_o(\mathbb{N}_o \longrightarrow K) \longrightarrow c_o(\mathbb{N}_o \longrightarrow K)$ defined by

$$T(a_o, a_1, a_2 \dots) = (a_1, a_2, a_3, \dots)$$

In the sequel we will use the following notations

$\mathbb{N}$ = the set of positive integers ; $\mathbb{N}_o = \mathbb{N} \cup \{0\}$; for any set X, $b(X \longrightarrow K)$ is the Banach space of all bounded maps $f : X \longrightarrow K$, normed by $\|f\| = \sup|f(x)|$;

$c_o(X \longrightarrow K)$ and $c(X \longrightarrow K)$ are the closed subspaces of $b(X \longrightarrow K)$ consisting of all $f : X \longrightarrow K$ satisfying $\lim_{x \to \infty} f(x) = 0$, resp. $\lim_{x \to \infty} f(x)$ exists.

For any Banach space E, E' denotes its dual. For a bounded K-linear map $\mu : E_1 \longrightarrow E_2$, the dual map : $E_2' \longrightarrow E_1'$ is denoted by μ'. For operations on Banach spaces like direct sum ($\sum$), direct product (Π) and terms as α-orthogonal, orthonormal, weak Hahn-Banach theorem, spaces of countable type we refer to [5].

Let $X_1, \ldots, X_n$ be indeterminates, then $K\{X_1, \ldots, X_n\}$ denotes the affinoid algebra in n-indeterminates over K. That is, $K\{X_1, \ldots, X_n\}$ consists of all power series $\sum a_\alpha X^\alpha$ such that $\lim |a_\alpha| = 0$. For affinoid algebras we refer to [1,7].

§.2. - An inequality for ideals in $V[X_1, \ldots, X_n]$.

Let K be an algebraically closed field and V a (rank 1) valuation ring with quotient field K. The maximal ideal of V will be denoted by $m = m(V)$ and the residue field of V by k. For ideals $I \subset V[X_1, \ldots, X_n]$ having the property $I \cap V \neq 0$ we define : $\alpha(I) = \sup\{|\alpha| \mid \alpha \in I \cap V\}$ and

$$\delta(I) = \inf\left\{\sup_{f \in I} |f(\lambda_1, \ldots, \lambda_n)| \mid \lambda_1, \ldots, \lambda_n \in V\right\}.$$

Clearly $0 < \alpha(I) \leqslant \delta(I)$. If I is generated by $f_1, \ldots, f_s$ then $\delta(I)$ equals $\inf\left\{\max_{1 \leqslant i \leqslant s} |f_i(\lambda_1, \ldots, \lambda_n)| \mid \lambda_1, \ldots, \lambda_n \in V\right\}$. Let $c(I)$ denote the positive real number satisfying $\alpha(I) = \delta(I)^{c(I)}$. Put $c(n,s) = \sup\{c(I) \mid I$ ideal in $V[X_1, \ldots, X_n]$, generated by s elements and $I \cap V \neq 0\}$. So $c(n,s) \in \mathbb{R} \cup \{\infty\}$ and $c(n,s) \geqslant 1$.

(2.1) <u>Conjecture</u>: $c(n,s) < \infty$ <u>for all</u> n <u>and</u> s.

In this section we will show $c(1,s) = 2$ for all $s (\geqslant 2)$ and $c(n,2) = 2$ for all n. In section 3 it is shown that "$c(n,s) < \infty$ for all s and fixed n" implies the Corona statement for dimension n. We start by considering the case n = 1.

(2.2) <u>Main lemma. Let</u> I <u>be a finitely generated ideal in</u> $V[X]$ <u>such that</u> $I \cap V \neq 0$. <u>There exists a</u> $\rho \in V$, $\rho \neq 0$, <u>such that</u> $\rho^{-1} I \subset V[X]$ <u>and</u> $\rho^{-1} I \not\subset m(V)[X]$. <u>Let</u> $d = d(I)$ <u>denote the degree of a generator of the ideal</u> $\phi(\rho^{-1} I) \subset K[X]$ where ϕ <u>is the canonical map</u> $V[X] \longrightarrow K[X]$. <u>Then</u> : $\delta(I)^{2d-1} \leqslant \alpha(I)^d$ <u>or equivalenty</u>

$c(I) \leqslant 2-\frac{1}{d}$.

<u>Conversely for every</u> $d \geqslant 1$ <u>there exists an ideal</u> $J \subset V[X]$ <u>with</u> $J \cap V \neq 0$ <u>and</u> J <u>is generated by two monic polynomials of degree d such that</u> $d = d(J)$ <u>and</u> $c(J) = 2-\frac{1}{d}$.

<u>Proof</u>. The proof is done by induction on d. For convenience we introduce on $V[X]$ the valuation $\| \ \|$, extending $| \ |$ on V, and given by $\| \sum a_i X^i \| = \max |a_i|$ Since I is finitely generated, there exists an element $f \in I$ such that $\|f\| = \sup\{\|g\| \mid g \in I\}$. Take $\rho \in V$ with $|\rho| = \|f\|$, then $\rho^{-1} I \subset V[X]$ and $\rho^{-1} I \not\subset m(V)[X]$. If one has shown the inequality $c(\rho^{-1} I) \leqslant 2-\frac{1}{d}$ then it follows that $c(I) \leqslant 2-\frac{1}{d}$ since $\alpha(\rho^{-1}I) = |\rho|^{-1}\alpha(I)$ and $\delta(\rho^{-1}I) = |\rho|^{-1} \delta(I)$. So without loss of generality we may assume that $\rho = 1$. First a lemma :

2.3. <u>Lemma</u>. <u>Let</u> $f \in I$ <u>satisfy</u> $\|f\| = 1$ <u>then there exists a monic polynomial</u> $g \in I$ <u>such that</u> $f \in gV[X]$.

<u>Proof</u>. The element f can be written as $f = \mu(X-a_1)\ldots(X-a_s)(1-b_1X)\ldots(1-b_tX)$ where $|\mu| = 1$; $a_1, \ldots, a_s \in V$; $b_1, \ldots, b_t \in m(V)$. We want to show that $g = (X-a_1) \ldots (X-a_s)$ belongs to I. Put $(1-b_1X)\ldots(1-b_tX) = 1-h$ where $h \in V[X]$ satisfies $\|h\| < 1$. For some $m \geqslant 1$, $h^m \in I$ because $I \cap V \neq 0$. Hence $g = \mu^{-1}f(1+h+\ldots+h^{m-1}) + h^m g$ belongs to I.

<u>Continuation of the proof of</u> (2.2) : According to (2.3) there exists a monic polynomial $f_o \in I$ of degree $d = d(I)$. After a translation of X we may suppose that 0 is a root of f_o . Let $\{g_1, \ldots, g_s\}$ generate I. Write $g_i = q_i f_o + r_i$, where q_i , $r_i \in V[X]$ and degree $(r_i) < d$. Then $\|r_i\| > 1$ for all i, since $\phi(r_i) \in K[X]$ must be zero. Put $f_i = f_o + r_i$ for $i = 1,\ldots, s$, then $\{f_o, f_1, \ldots, f_s\}$ generates I and $\phi(f_i) = \phi(f_o)$ for all $i = 1,\ldots, s$.

In case $d = 1$ this gives that I is generated by $\{\lambda, X\}$ for some $\lambda \in m(V)$. Clearly this implies $\alpha(I) = \delta(I)$ and $c(I) = 1$. Now we proceed by induction and suppose $d = d(I) > 1$.

<u>Case</u> (1) : "$\phi(f_o) = X^d \in k[X]$". Let $\rho \in V$ satisfy $|\rho| = \max\{|a| \mid a \in V$ is root of some $f_i (i = 0,\ldots, s)\}$. By construction also $\phi(f_i) = X^d$ for all $i \geqslant 1$ and so

$|\rho| < 1$. We consider now the ideal $\tilde{I} \subset V[X]$ generated by the monic polynomials $\{\rho^{-d} f_i(\rho X) \mid i = 0,\dots, s)\}$. By definition

$\delta(\tilde{I}) = \inf \{ \max_{0\leq i\leq s} | \lambda^{-d} f_i(\lambda)| \mid \lambda \in V, |\lambda| \leq |\rho|\}$ which is also equal to

$|\rho^{-d}| \inf \{ \max_{0\leq i\leq s} | f_i(\lambda)| \mid \lambda \in V\}$ because all the roots of $f_o,\dots, f_s$ have absolute value $\leq |\rho|$. So $\delta(\tilde{I}) = |\rho^{-d}|\ \delta(I)$.

If $\alpha \in \tilde{I} \cap V$ then $\alpha = \sum_{i=0}^{s} Q_i(X)\ \rho^{-d} f_i(\rho X)$ for somme

$Q_o,\dots, Q_s \in V[X]$. After euclidean division with remainder of all $Q_i (i = 1,\dots, s)$ by the monic polynomial $\rho^{-d} f_o(\rho X)$ one finds an expression

$\alpha = \sum_{i=0}^{s} P_i(X)\ \rho^{-d} f_i(\rho X)$ such that $\deg(P_i) < d$ $(i = 0,1,\dots, s)$. Hence

$\alpha \rho^{2d-1} = \sum_{i=0}^{s} P_i(\rho^{-1}X)\rho^{d-1} f_i(X)$ and for all $i = 0, 1,\dots, s$ one has

$\rho^{d-1} P_i(\rho^{-1}X) \in V[X]$ since $\deg P_i < d$. So we have shown that

$|\rho^{2d-1}|\ \alpha(\tilde{I}) \leq \alpha(I)$.

Clearly $d(\tilde{I}) \leq d$. If $d(\tilde{I}) < d$ then by induction hypothesis $c(\tilde{I}) < 2-\frac{1}{d}$ and it follows that also $c(I) < 2-\frac{1}{d}$. If $d(\tilde{I}) = d$, then the generators $F_i(X) = \rho^{-d} f_i(\rho X)$ $(i = 0,\dots, s)$ of $\tilde{I}$ have the property $\phi(F_i) = \phi(F_o)$ for all $i = 1,\dots, s$ and $\phi(F_o) \neq X^d$. So we are reduced to

<u>Case</u> (2) : "$I = (f_o, f_1,\dots, f_s)$; $\phi(f_i) = \phi(f_o)$ for all i ; $f_o(0) = 0$ and $\phi(f_i) = \phi(f_o)$ for all i, and $\phi(f_o)$ is a polynomial of degree d, unequal to X^d".

Here we proceed as follows : write $\phi(f_o) = X^{d-}(X^{d+} +\dots+ \lambda)$ with $d^- > 0$, $d^+ > 0$, $\lambda \in k$, $\lambda \neq 0$. Put $f_i = f_i^- f_i^+$ $(i = 0,\dots, s)$ such that $\phi(f_i^-) = X^{d-}$ and $\phi(f_i^+) = (X^{d+} +\dots+ \lambda)$. Consider the ideals $I^+ = (f_o^+,\dots, f_s^+)$ and $I^- = (f_o^-,\dots, f_s^-)$. Then we have $\delta(I) = \min(\delta(I^-), \delta(I^+))$ since $\delta(I)$ equals

$$\min\left[\inf\left\{\max_{0\leq i\leq s}|f_i^-(\lambda)\, f_i^+(\lambda)|\,\Big|\,|\lambda|<1\right\},\ \inf\left\{\max_{0\leq i\leq s}|f_i^-(\lambda)\, f_i^+(\lambda)|\,\Big|\,|\lambda|=1\right\}\right]$$

and for $\lambda \in V$, $|\lambda| < 1$, we have $|f_i^+(\lambda)| = 1$ (all i) and for $\lambda \in V$, $|\lambda| = 1$, we have $|f_i^-(\lambda)| = 1$ (all i).

Also $\alpha(I) \geq \min(\alpha(I^-), \alpha(I^+))$ or in other words : if $\alpha \in I^- \cap V$ and $\alpha \in I^+ \cap V$ then $\alpha \in I \cap V$. Indeed :

$$\alpha = \sum_{i=0}^{s} P_i^- f_i^- \quad \text{and} \quad \alpha = \sum_{i=0}^{s} P_i^+ f_i^+ \quad \text{with } P_i^-, P_i^+ \in V[X].$$

Hence $\alpha f_o^+ \ldots\ldots f_s^+$ and $\alpha f_o^- \ldots f_s^-$ belong to I. The polynomials $\phi(f_o^+ \ldots f_s^+)$ and $\phi(f_o^- \ldots f_s^-)$ in $k[X]$ are relatively prime. So there are $P,Q \in V[X]$ with $1 = \phi(Pf_o^+ \ldots f_s^+ + Qf_o^- \ldots f_s^-)$. Consequently I contains the element $\alpha Pf_o^+ \ldots f_s^+ + \alpha Qf_o^- \ldots f_s^- = \alpha(1-h)$ where $\|h\| < 1$. As in the lemma (2.3) it follows that $\alpha \in I$.

Now we have $\alpha(I)^{dd^+d^-} \geq \min(\alpha(I^-)^{dd^+d^-}, \alpha(I^+)^{dd^+d^-})$ which is by induction hypothesis ($d^- = d(I^-) < d$ and $d^+ = d(I^+) < d$) greater or equal to $\min(\delta(I^-)^{dd^+(2d^--1)}, \delta(I^+)^{dd^-(2d^+-1)})$. One checks easily that $dd^+(2d^--1) \leq (2d-1)d^+d^-$ and $dd^-(2d^+-1) \leq (2d-1)d^+d^-$.

Consequently $\alpha(I)^d \geq \min(\delta(I^-)^{2d-1}, \delta(I^+)^{2d-1}) = \delta(I)^{2d-1}$.

This finishes the proof of the first part of (2.2).

To show that the bound $c(I) \leq 2-\frac{1}{d}$ is best possible we construct an example : Write $X^{2d-1} - 1 = Q.G$ where Q and G are monic polynomials of degrees d-1, resp. d. Put $f(X) = \rho^d F(\rho^{-1}X)$ and $g(X) = \rho^d G(\rho^{-1}X)$ where $\rho \in V$, and $0 < |\rho| < 1$. Then f and g are also monic polynomials belonging to $V[X]$. Take $J = (f,g)$. Using the notation of case (1) above we clearly have $\tilde{J} = (F,G)$ and $\delta(\tilde{J}) = 1$. Hence $\delta(J) = |\rho^d|$. Further let $0 \neq \alpha \in J \cap V$. Then $\alpha = p(X)f(X) + q(X)g(X)$, where $p,q \in V[X]$ and where one may suppose deg p $< d$ and deg q $< d$. Hence

$\alpha\rho^{-d} = p(\rho X)\rho^{-d} f(\rho X)g(\rho X)$. Now $F(X) = \rho^{-d} f(\rho X)$ and $G(X) = \rho^{-d} g(\rho X)$. Using the fact that the equation $1 = P_1F + Q_1G$ with $\deg P_1 < d$, $\deg Q_1 < d$ has only the solution $P_1 = X^{d-1}$ and $Q_1 = Q$, one finds that $p(\rho X) = \alpha\rho^{-d} X^{d-1}$. Hence $p(X) = \alpha\rho^{-2d+1}X^{d-1}$. But $p(X) \in V[X]$ yields $|\alpha| \leq |\rho^{2d-1}|$. So one finds $\alpha(J) = |\rho|^{2d-1}$ and $c(J) = 2-\frac{1}{d}$.

(2.4.) Corollary. Let I be a finitely generated ideal in $V[X]$ such that $I \cap V \neq 0$. Form the ideal $J = \bigcap\{I(\lambda) \mid \lambda \in V\}$, where $I(\lambda)$ denotes the image of I under the V-algebra homomorphism $V[X] \longrightarrow V$ which sends X to λ. Then :

$$J^2 \subsetneq I \cap V \subset J \text{ and } \delta(I)^2 < \alpha(I) \leq \delta(I).$$

Further $c(1,s) = 2$ for all $s (\geq 2)$.

Proof : From the definitions it follows that $\delta(I) = \sup\{|\alpha| \mid \alpha \in J\}$, $\delta(I)^2 = \sup\{|\alpha| \mid \alpha \in J^2\}$ and $\alpha(I) = \sup\{|\alpha| \mid \alpha \in I \cap V\}$. So the first two statements of (2.4) are quivalent. The second and third statement of (2.4) follow immediately from (2.2).

Remarks. Corollary (2.4) will suffice us in proving the Corona conjecture for dimension 1. In the rest of this section we discuss some more detailed results which might be useful for dimension > 1.

(2.5) Lemma. Let I be an ideal in $V[X]$ generated by $f_1, \ldots, f_s$ such that $I \cap V \neq 0$. Let Z denote the set of all roots of $f_1 \ldots f_s$ which belong to V. Then :

(i) $J = \bigcap\{I(\lambda) \mid \lambda \in V\}$ is equal to $\bigcap\{I(\lambda) \mid \lambda \in Z\}$. In particular J is a principal ideal.

(ii) $I \cap V$ is a principal ideal.

(iii) Suppose that $s = 2$, $\|f_1\| = \|f_2\| = 1$ and $\deg \phi(f_i) = d_i$. Let $p_1, p_2 \in V[X]$ be given such that $\max(\|p_1\|, \|p_2\|) = 1$; $p_1 f_1 + p_2 f_2 = \alpha \in V, \alpha \neq 0$ and $\deg \phi(p_1) < d_2$, $\deg \phi(p_2) < d_1$. Then $|\alpha| = \alpha(I)$.

Proof. (i) Put $\delta^*(I) = \min\{\max_{1\leq i\leq s} |f_i(z)| \mid z \in Z\}$. Clearly $\delta^*(I) \geqslant \delta(I)$.
The statement in (i) is equivalent to $\delta(I) = \delta^*(I)$. We prove this by induction on $\sum_{i=1}^{s} \deg(f_i)$. If $\lambda \in V$ satisfies $|\lambda - z| \geqslant \rho$ for all $z \in Z$ where $\rho = \max\{|z_1 - z_2| \mid z_1, z_2 \in Z\}$ then for any $z \in Z$ one has

$\max_{1\leq i\leq s} |f_i(\lambda)| \geqslant \max_{1\leq i\leq s} |f_i(z)|$. The set $\{\lambda \in V \mid |\lambda - z| < \rho$ for some $z \in Z\}$ is equal to a disjoint union $B_1 \cup \ldots \cup B_t$ $(t > 1)$ of "open" spheres with raddi ρ
Each f_i can be written as $\sum_{j=1}^{t} f_{ij}$ such that for all i and j, the roots of f_{ij} belonging to V also belong to B_j.

Then $\delta(I) = \min_{1\leq j\leq t} (\inf_{\lambda\in B_j} (\max_{1\leq i\leq s} |f_i(\lambda)|))$. For any $i \in \{1,\ldots, s\}$ and $j \in \{1,\ldots,t\}$ there exists a constant ρ_{ij} such that $|f_i(\lambda)| = \rho_{ij}|f_{ij}(\lambda)|$ for all $\lambda \in B_j$. Since $\sum_{i=1}^{s} \deg(f_{ij}) < \sum_{i=1}^{s} \deg(f_i)$ for all j, the induction hypothesis gives $\inf_{\lambda\in B_j} (\max_{1\leq i\leq s} |f_i(\lambda)|) = \min_{z\in Z\cap B_j} (\max_{1\leq i\leq s} |f_i(z)|)$. Hence $\delta(I) = \min_{z\in Z} (\max_{1\leq i\leq s} |f_i(z)|)$.

(ii) Let $\rho_0 \in V$ be such that $\rho_0 = \max\{\|f\| \mid f \in I\}$. (Here we use of course that I is finitely generated). Let $f_0 \in I$ denote an element in I which has minimal degree under all elements $f \in I$ with $\|f\| = |\rho_0|$. As in (2.3) one finds that $\rho_0^{-1} f_0$ is a monic polynomial of degree d. For $f \in I$ we write $\rho_0^{-1} f_0 q + R(f)$, where $q, R(f) \in V[X]$ and $\deg(R(f)) < d_0$. The ideal I_1 generated by $\{\rho_0 R(f) \mid f \in I\}$ is again finitely generated and clearly $I_1 \cap V = I \cap V$ and $d(I_1) < d_0 = d(I)$. Induction on $d(I)$ (the cases $d(I) = 0$ or 1 being trivial) completes the proof,

(iii) If $|\alpha| < \alpha(I)$ then for some $\beta \in V$, $|\beta| > |\alpha|$ and $q_1, q_2 \in V[X]$ we have $\beta = q_1 f_1 + q_2 f_2$. It follows that $p_1 = \alpha\beta^{-1} q_1 + rf_2$ and $p_2 = \alpha\beta^{-1} q_2 - rf_1$ for some $r \in V[X]$. Since $\max(\|p_1\|, \|p_2\|) = 1$ one has $\|r\| = 1$ and $\phi(p_1) = \phi(r)\phi(f_2)$, $\phi(p_2) = \phi(r)\phi(-f_1)$. This contradicts the assumption $\deg \phi(p_1) < d_2$ and $\deg \phi(p_2) < d_1$.

(2.6) Corollary. Let $I \subset V[X_1, \dots, X_n]$ be an ideal generated by two elements and satisfying $I \cap V \neq 0$. Then $c(I) < 2$. Moreover $c(n,2) = 2$ for all $n \geq 1$.

Proof. Let $\rho \in V$ satisfy $|\rho| = \max(\|f_1\|, \|f_2\|)$ where $\{f_1, f_2\}$ generates I. The inequality $c(I) < 2$ would follow from $c(\rho^{-1}I) < 2$. So without loss of generality we may assume $\rho = 1$. So we can suppose $1 = \|f_1\| \geq \|f_2\|$. If $\|f_2\| < 1$ we can replace f_2 by $f_1 + f_2$. So without loss of generality we can suppose $\|f_1\| = \|f_2\| = 1$. After a linear change of $X_1, \dots, X_n$ we have that $\phi(f_1)$ and $\phi(f_2)$ are monic polynomials in X_n with coefficients in $k[X_1, \dots, X_{n-1}]$. Using Weierstrass-preparation and division for the affinoid algebra $K\{X_1, \dots, X_n\} \supset V[X_1, \dots, X_n]$ (see [1] Satz 1,2 of Kap. I) one finds : For any $f \in V[X_1, \dots, X_n]$ and any $\pi \in V$, $0 < |\pi| < 1$ there are $q,r,s \in V[X_1, \dots, X_n]$ satisfying $f = qf_1 + r + \pi s$ and $\deg r < d_1 = \deg_{X_n}(\phi(f_1))$.

Given an expression $\beta = q_1 f_1 + q_2 f_2$, $\beta \neq 0$, $\beta \in V$. Then q_2 is not divisible by f_1 in $K\{X_1, \dots, X_n\}$. Hence for suitable $\pi \in V$, ($|\pi|$ small enough) one has

$$q_2 = qf_1 + r + \pi s \text{ with } q,r,s \in V[X_1, \dots, X_n]; \|r\| > |\pi| \quad \text{and}$$

$\deg_{X_n}(r) < d_1$.

Substituting this and possibly dividing by an element ($\neq 0$) in V one finds

$$\alpha = p_1 f_1 + p_2 f_2 \; ; \; \alpha \in V, \; \alpha \neq 0, \; \max(\|p_1\|, \|p_2\|) = 1 \quad \text{and}$$

$\deg_{X_n}\phi(p_2) < d_1$, $\deg_{X_n}\phi(p_1) < d_2 = \deg_{X_n}\phi(f_2)$. In this we substitute for $X_1, \dots, X_{n-1}$ elements $\lambda_1, \dots, \lambda_{n-1} \in V$. Put $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ then one has

$$\alpha = p_1(\lambda, X_n)f_1(\lambda, X_n) + p_2(\lambda, X_n)f_2(\lambda, X_n) \text{ and (2.5) part (iii)}$$

yields $|\alpha| = \alpha((f_1(\lambda,X_n), f_2(\lambda,X_n))) \leq \alpha(I)$. Further

$\alpha((f_1(\lambda,X_n), f_2(\lambda,X_n))) > \delta((f_1(\lambda,X_n), f_2(\lambda,X_n)))^2 \geq \delta(I)^2$ has as consequence $\alpha(I) > \delta(I)^2$. Moreover $c(n,2) \geq c(1,2) = 2$. So $c(n,2) = 2$.

Remark. In the next proposition and corollary we will give an algebraic interpretation of $\sigma(I)$ using complete ideals and integral closures of ideals. We will use tacitely the exposition on complete ideals given in [8] appendices 2, 3 and 4.

(2.7) Proposition. Let V be a (rank 1) valuation ring with field K (not necessarily algebraic closed) and I a finitely generated ideal in $V[X_1, \ldots, X_n]$ with $I \cap V \neq 0$. Let I' be the integral closure of I in $K(X_1, \ldots, X_n)$ and a an element of $V \setminus I' \cap V$. Then there exists a finite field extension L of K and a valuation-ring W with quotient field L, $W \cap K = V$ and a V-algebra homomorphism $\phi : V[X_1, \ldots, X_n] \longrightarrow W$ such that $\phi(a) \notin \phi(I)W$ (or equivalently

$|a| = |\phi(a)|_W > \sup|\phi(I)|_W$).

Proof. Since $a \notin I'$ there exists a valuationring W^* of $K(X_1, \ldots, X_n)$ such that $W^* \supset V[X_1, \ldots, X_n]$ and $a \notin IW^*$. Choose $b \in I$ with $IW^* = bW^*$. The rank of W^* is finite (in fact $\leqslant n+1$). Hence there are prime ideals $\underline{p} \supset \underline{q}$ in W^* with $\underline{p} = 1+\text{hgt}\ \underline{q}$ and $b/a \in \underline{p} \setminus \underline{q}$. Now $U = W^*_{\underline{p}}/\underline{q}W^*_{\underline{p}}$ is a valuationring of rank 1 and we have a canonical map $\Psi : V[X_1, \ldots, X_n] \longrightarrow W \longrightarrow U$ satisfying

$|\Psi(a)|_U = |a|_V > \max_{1 \leqslant i \leqslant s} |\Psi(f_u)|_U$ where $\{f_1, \ldots, f_s\}$ denotes a set of

generators for the ideal I. Let $\wedge$ denote completion with respect to the given valuation in particular $\hat{K}$ denotes the completion of K. Then Ψ extends to a $\hat{K}$-algebra homomorphism, also denoted by $\Psi : \hat{K}\{X_1, \ldots, X_n\} \longrightarrow Qt(U)^\wedge$. Here $Qt(U)$ is the quotient field of U. This map Ψ extends further to a $\hat{K}$-algebra homomorphism $\Psi_1 : \hat{K}\{X_1, \ldots, X_n, T_1, \ldots, T_s\} \longrightarrow Qt(U)^\wedge$ where $\Psi_1(X_i) = \Psi(X_i)$ and $\Psi_1(T_j) = \pi^{-1} a^{-1} \Psi(f_j)$. Here $\pi \in V$, $0 < |\pi| < 1$, is chosen such that $|\pi^{-1} a^{-1} \Psi(f_j)|_U \leqslant 1$ for all $j = 1, \ldots, t$. The kernel of Ψ_1 clearly contains the ideal J of $K\{X_1, \ldots, X_n, T_1, \ldots, T_s\}$ generated by $\{\pi a T_i - f_i\}_{i=1,\ldots,s}$

So $J \neq (1)$. Let M be a maximal ideal of $\hat{K}\{X_1, \ldots, X_n, T_1, \ldots, T_s\}$ which contains J. As is well known ([7] Theorem 4.5), M is the kernel of a map $\chi : \hat{K}\{X_1, \ldots, X_n, T_1, \ldots, T_s\} \longrightarrow F$, where F is a finite field extension of $\hat{K}$. Let W' denote the valuationring of F, then χ induces a V-algebra homomorphism $\chi : V[X_1, \ldots, X_n] \longrightarrow W'$ such that $|\chi(a)|_{W'} = |a|_V > \max_{1 \leq i \leq s} |\chi(f_i)|_{W'}$.

Choose elements $\alpha_1, \ldots, \alpha_n \in W'$ algebraic over K such that $\max_{1 \leq i \leq s} |\alpha_i - \chi(X_i)|_{W'}$ is small enough to ensure that $\phi : V[X_1, \ldots, X_n] \longrightarrow W'$ given by $\phi(X_i) = \alpha_i (i=1,\ldots,n)$ has still the property $|a|_V > \max_{1 \leq i \leq s} |\phi(f_i)|_{W'}$. Let L be the quotient field of im ϕ and $W = W' \cap L$. Then L is a finite extension of K and $\phi : V[X_1, \ldots, X_n] \longrightarrow W$ has the required properties.

<u>Definition.</u> To formulate the next corollary easily we define $\delta(I)$ for ideals $I \subset V[X_1, \ldots, X_n]$ with $V \cap I \neq 0$ and $K = Qt(V)$ not (necessarily) algebraically closed as follows : $\delta(I) = \inf\{\sup_{f \in I} |f(\lambda_1, \ldots, \lambda_n)|_W \mid W \supset V$ any valuationring such that $Qt(W)$ is a finite extension of K and $\lambda_1, \ldots, \lambda_n$ any elements $\in W\}$.

(2.8) <u>Corollary. With the notations of</u> (2.7). <u>The following ideals are equal</u> :

a) $I' \cap V$

b) $I_1 = \bigcap\{V \cap \phi^{-1}(\phi(I)W) \mid W \supset V$ <u>any rank</u> 1 <u>valuationring and</u> $\phi = V[X_1, \ldots, X_n] \longrightarrow W$ <u>any</u> V-<u>algebra homomorphism</u>$\}$

c) $I_2 : \bigcap\{V \cap \phi^{-1}(\phi(I)W) \mid W \supset V$ <u>any valuationring such that</u> $Qt(W)$ <u>is a finite extension of</u> K <u>and</u> $\phi : V[X_1, \ldots, X_n] \longrightarrow W$ <u>any</u> V-<u>algebra homomorphism</u>$\}$.

<u>In particular</u> $\delta(I) = \sup\{|\alpha| \mid \alpha \in I' \cap V\}$ <u>and for any rank</u> 1 <u>valuationring</u> $W \supset V$, $W \cap K = V$, <u>we have</u> $\delta(I) = \delta(IW[X_1, \ldots, X_n])$.

<u>Proof.</u> Clearly $I_2 \supset I_1$ and (2.7) yields $I' \cap V \supset I_2$. Take $a \in I' \cap V$. Then a is integral over I. Hence for any $W \supset V$ and any $\phi : V[X_1, \ldots, X_n] \longrightarrow W$ the element $\phi(a)$ is integral over $\phi(I)W$. Since W is a valuationring this means $\phi(a) \in \phi(I)W$. This shows $I' \cap V \subset I_1$.

Further the formula for $\delta(I)$ follows at once from the definitions and $\delta(I) = \delta(IW[X_1, \ldots, X_n])$ follows from $I_1 = I_2$.

Remarks. The conjecture $c(n,s) < \infty$ can now be restated in the following way : There exists an integer A, only depending on n and the number of generators of I such that $(I' \cap V)^A \subset I \cap V$.

In this form one does not need the condition that V is a valuationring. More general we conjecture the following :

Let R be a normal domain, I a finitely generated ideal in $R[X]$ such that $I \cap R \neq 0$. Then there exists an integer A, only depending on R and the number of generators of I, such that $(I' \cap R)^A \subset (I \cap R)^*$, where I' is the integral closure of I in $R[X]$ and $(I \cap R)^*$ is the integral closure of $I \cap R$ in R.

As we have seen this conjecture is true if R is a valuationring (then A = 2). Also if R is a Dedekind domain the conjecture is true with A = 2. Further one sees that this conjecture would imply $c(n,s) < \infty$ (all n,s) and consequently it would solve the Corona problems for any dimension.

In the following proposition we give still another formulation of the conjecture $c(n,s) < \infty$ for all n and s.

(2.9) Proposition. Let V be a rank 1 valuationring with algebraically closed quotient field K and let $f \in V[X_1, \dots, X_n]$ define a nonsingular hyperplane of $K[X_1, \dots, X_n]$. Suppose that there exists an integer A only depending on n such that the ideal $I \subset V[X_1, \dots, X_n]$ generated by f and $\frac{\partial f}{\partial X_i}$ (i=1,...,n) satisfies $(I' \cap V)^A \subset I \subset V$. Then $c(n,s) < \infty$ for all n and s.

Remark. Note that the condition $I \cap V \neq 0$ is equivalent to saying that f defines a non-singular hyperplane over K. Further both $I \cap V$ (or $\alpha(I)$) and $I' \cap V$ (or $\delta(I)$) are measures (or if one wants multiplicities) for the singularities of the hyperplane over V associated with f.

Proof of (2.9). Let an ideal $J = (g_1, \dots, g_s) \subset V[X_1, \dots, X_m]$ which satisfies $J \cap V \neq 0$ be given. Put n = m+s and consider $f = g_1 X_{m+1} + \dots + g_s X_{m+s}$. The ideal I in $V[X_1, \dots, X_n]$ generated by f and $\frac{\partial f}{\partial X_i}$ (i=1,...,m) is also generated by $g_1, \dots, g_s, h_1, \dots, h_m$ where $h_j = \sum_{i=1}^{s} \frac{\partial g_i}{\partial X_j} X_{m+i}$. Since $I \supset JV[X_1, \dots, X_n]$ it is clear that $\alpha(I) \geq \alpha(J)$ and $\delta(I) \geq \delta(J)$. The proposition will be proved if we show $\alpha(I) = \alpha(J)$. Take $\alpha \in I \cap V$. Then $\alpha = \sum p_i g_i + \sum q_j h_j$

with p_i, $q_j \in V[X_1, \ldots, X_n]$. After substituting $X_{m+1} = \ldots = X_n = \ldots = 0$ in this equation one obtains $\alpha \in J \cap V$. So $I \cap V = J \cap V$.

§.3 Bounded analytic functions on an open polydisc.

Let K be a non-archimedean valued complete field and $X_1, \ldots, X_n$ indeterminates. $K < X_1, \ldots, X_n >$ denotes the algebra of all formal power series

$f = \sum a_{\alpha_1, \ldots, \alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ with coefficients in K such that

$\sup |a_{\alpha_1, \ldots, \alpha_n}| < \infty$. It is a Banach algebra w.r.t. the multiplicative norm $\|f\| = \sup |a_{\alpha_1, \ldots, \alpha_n}|$. The "free" affinoid algebra $K\{X_1, \ldots, X_n\}$ consisting of all expressions $\sum a_{\alpha_1, \ldots, \alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ such that $\lim |a_{\alpha_1, \ldots, \alpha_n}| = 0$, is a closed subalgebra of $K < X_1, \ldots, X_n >$.

Let V denote the valuationring of K and S the multiplicative set $V \setminus \{0\}$ then $K < X_1, \ldots, X_n > = S^{-1}(V[[X_1, \ldots, X_n]])$. In particular it follows that $K < X_1, \ldots, X_n >$ is noetherean if the valuation V is discrete. (The converse is also true).

An analytic interpretation of $K < X_1, \ldots, X_n >$ is the following : If the valuation V is non-discrete then $K < X_1, \ldots, X_n >$ is the algebra of all bounded analytic functions defined on the "open" polydisc $\Delta(K)^n = \{(\lambda_1, \ldots, \lambda_n) \in K^n \mid$ all $|\lambda_i| < 1\}$. The norm as defined above, coincides with the supremumnorm on $\Delta(K)^n$. (Proofs and more details can be found in [6]). So $K < X_1, \ldots, X_n >$ is the non-archimedean analogue of the Hardy space $H^\infty(\Delta)$ of an open polydisc $\Delta \subset \mathbb{C}^n$.

The Corona conjecture is :

Let K_{alg} denote the algebraic closure of K which is given the unique valuation extending the valuation of K. Then the image of $\Delta(K_{alg})^n$ in the maximal ideal space of $K < X_1, \ldots, X_n >$ (which is given the Gelfand topology) is a dense subset.

A more explicit formulation (see [2] pg. 163, for the proof of the equivalence of the two statements) is :

(3.1) <u>The elements</u> $f_1, \dots, f_s \in K < X_1, \dots, X_n >$ <u>generate the unit ideal if and only if</u> $\delta = \inf \{ \max_{1 \leq i \leq s} | f_i(\lambda) | \mid \lambda \in \Delta(K_{alg})^n \} > 0$.

One implication in this statement is trivial, namely : if $f_1, \dots, f_s$ generate the unit ideal then $\sum_{i=1}^{s} g_i f_i = 1$ for some $g_1, \dots, g_s \in K < X_1, \dots, X_n >$. It follows that $\delta \geq (\max \| g_i \|)^{-1} > 0$. The other implication will be proved in this paper for n = 1 and for n > 1, s = 2 in a more precise form :

(3.2) <u>Theorem</u>. (<u>Corona statement for dimension</u> 1). <u>For any</u> $f_1, \dots, f_s \in K < X >$ <u>satisfying</u> $\|f_i\| \leq 1$ (i=1,...,s) <u>and</u> $\delta = \inf \{ \max_{1 \leq i \leq s} |f_i(\lambda)| \mid \lambda \in \Delta(K_{alg}) \} > 0$ <u>there are</u> $g_1, \dots, g_s \in K<X>$ <u>with</u> $\sum_{i=1}^{s} g_i f_i = 1$ <u>and</u> $\max_{1 \leq i \leq s} \|g_i\| \leq \delta^{-2}$.

(3.3) <u>Conjecture</u> $(C_{n,s})$. <u>There exists a constant</u> $A \geq 1$ <u>such that for any</u> $f_1, \dots, f_s \in K < X_1, \dots, X_n >$ <u>satisfying</u> $\|f_i\| \leq 1$ (i=1,...,s) and $\delta > 0$ there are $g_1, \dots, g_s \in K < X_1, \dots, X_n >$ with $\sum_{i=1}^{s} g_i f_i = 1$ and $\max \|g_i\| \leq \delta^{-A}$.

<u>Remarks</u>. (1) Of course (3.2) is the special case $(C_{1,s})$ of (3.3).

(2) <u>Let</u> $f_1, \dots, f_s \in K < X_1, \dots, X_n >$ <u>and</u> $L \supset K$ <u>a complete valued field</u>. <u>Then</u> δ <u>as defined in (3.1) is equal to</u> $\inf \{ \max_{1 \leq i \leq s} | f_i(\lambda) | \mid \lambda \in \Delta(L_{alg})^n \}$.

<u>In other words</u> δ <u>does not depend on the field</u> K.

<u>Proof</u>. We may of course suppose $\|f_i\| \leq 1$ for all i and $\delta > 0$. It suffices to show for any $\rho \in K_{alg}$, $0 < |\rho| < 1$, that $\delta_1 = \inf \{ \max_{1 \leq i \leq s} | f_i(\rho\lambda) | \mid \lambda \in \Delta(K_{alg})^n \}$ is equal to $\delta_2 = \inf \{ \max_{1 \leq i \leq s} | f_i(\rho\lambda) | \mid \lambda \in \Delta(L_{alg})^n \}$.

Since $\delta_1 \geq \delta > 0$ and $f_1(\rho X), \dots, f_s(\rho X) \in \hat{K}_{alg}\{X_1, \dots, X_n\}$ and every residue field of this affinoid algebra is equal to $\hat{K}_{alg}$ we find that $\{f_1(\rho X), \dots, f_s(\rho X)\}$ generate the unit ideal. Hence $\delta_1 \geq \delta_2 > 0$.

Write $f_i = \sum f_{i,\alpha} X^\alpha$ ($f_{i,\alpha} \in K$ and $i = 1,\dots,s$) and put $g_i = \sum_{|\alpha| \leq N} f_{i,\alpha} X^\alpha$; all this in the well known shorthand $X = (X_1,\dots,X_n)$; $\alpha = (\alpha_1,\dots,\alpha_n)$ and $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$. For fixed ρ there exists N such that for all i, $f_i(\rho X)-g_i(\rho X)$ considered as an element of $K\{X_1,\dots,X_n\}$ has norm $< \delta_2$. It follows that, with the notation $h_i(X) = g_i(\rho X)$ and $I = (h_1,\dots,h_s) V[X_1,\dots,X_n]$, one has

$$\delta_1 = \inf\{ \max_{1 \leq i \leq s} |h_i(\lambda)| \mid \lambda = (\lambda_1,\dots,\lambda_n) \in K^n_{alg}, \text{ all } |\lambda_i| \leq 1\} = \delta(I) \text{ and}$$

$\delta_2 = \delta(IW[X_1,\dots,X_n])$ where W denotes the valuationring of L. So the equality $\delta_1 = \delta_2$ follows from (2.8).

(3) <u>Let</u> $f_1,\dots,f_s \in K < X_1,\dots,X_n >$ <u>satisfy</u> $\|f_i\| < 1$ <u>for all</u> i <u>and let</u> $L \supset K$ <u>be a complete field. Suppose that there exists a constant</u> A <u>and</u> $h_1,\dots,h_s \in L < X_1,\dots,X_n >$ <u>satisfying</u> $\max \|h_i\| < A$ and $\sum_{i=1}^{s} h_i g_i = 1$. <u>Then there are</u> $g_1,\dots,g_s \in K < X_1,\dots,X_n >$ <u>with max</u> $\|g_i\| < A$ and $\sum_{i=1}^{s} g_i f_i = 1$.

<u>Proof.</u> Let E the closed subspace of the K-Banach space L generated by 1 and all the coefficients of all h_i. Choose an $\varepsilon > 0$ such that $(1+\varepsilon)\max\|h_i\| < A$. Since E is a Banach space over K of countable type there exists a K-linear map $l: E \to K$ with $l(1) = 1$ and $\|l\| \leq 1+\varepsilon$. Let $E < X_1,\dots,X_n >$ denote the closed subspace of $L < X_1,\dots,X_n >$ consisting of the power series with all coefficients in E. Of course $E < X_1,\dots,X_n >$ is a $K < X_1,\dots,X_n >$-module and the extension $L : E < X_1,\dots,X_n > \longrightarrow K < X_1,\dots,X_n >$ of l, defined by $L(\sum e_\alpha X^\alpha) = \sum l(e_\alpha) X^\alpha$ is $K < X_1,\dots,X_n >$-linear and $\|L\| \leq 1+\varepsilon$. Hence $g_i = L(h_i)$ $(i=1,\dots,s)$ have the required properties.

(4) The two preceeding remarks imply for the purpose of (3.2) or (3.3) we may replace K by any complete valued field $L \supset K$. In particular <u>we may suppose that</u> K <u>is algebraically closed and maximally complete</u>.

(3.4) <u>Theorem</u>. $c(n,s+1) < \infty$ <u>implies</u> $(C_{n,s})$.

Proof. We consider first the following statement :

($T_{n,s}$) : Let K be an algebraically closed, maximally complete field K. There exists a constant $A \geq 1$ such that for any $f_1, \dots, f_s \in K\{X_1, \dots, X_n\}$ with $\|f_i\| < 1$ $(i=1,\dots,s)$ and $\delta = \inf\{\max_{1\leq i\leq s} |f_i(\lambda_1, \dots, \lambda_n)| \,|\, \lambda_1, \dots, \lambda_n \in K$, all $|\lambda_i| \leq 1\} > 0$ there are $g_1, \dots, g_s \in K\{X_1, \dots, X_n\}$ such that $\sum_{i=1}^{s} g_i f_i = 1$ and $\max\|g_i\| < \delta^{-A}$.

The theorem will now follow from the following two lemmas.

(3.5) Lemma. ($T_{n,s}$) implies ($C_{n,s}$).

(3.6) Lemma. $c(n,s+1) < \infty$ implies ($T_{n,s}$).

Proof of (3.5). Choose a sequence $(\pi_t)_{t=1}^{\infty} \in K$ with $0 < |\pi_t| < 1$, $|\pi_t| < |\pi_{t+1}|$ and $\lim |\pi_t|^t = 1$. Put $f_i^t(X) = f_i(\pi_t X)$ for $i = 1,\dots,s$. Clearly $f_1^t, \dots, f_s^t \in K\{X_1, \dots, X_n\}$. Using ($T_{n,s}$) it follows that there are $g_1^t, \dots, g_s^t \in K\{X_1, \dots, X_n\}$ with $\sum g_i^t f_i^t = 1$ and $\max \|g_i^t\| < \delta^{-A}$.

Put $g_i^t = \sum_{\alpha} (g_i^t)_\alpha X^\alpha$, where $(g_i^t)_\alpha \in K$, and put $h_i^t = \sum_{|\alpha|\leq 2t} (g_i^t)_\alpha \pi_t^{-|\alpha|} X^\alpha$. Then we have $\|h_i^t\| < |\pi_t^{-2t}| \, \delta^{-A}$ and $\sum_{i=1}^{s} h_i^t f_i = 1 + \sum_{|\alpha|\geq t} a_{i,\alpha} X^\alpha$ for suitable $a_{i,\alpha} \in K$.

Unfortunately $\lim_{t\to\infty} h_i^t$ does not exist in general and we have to construct our solutions $g_1, \dots, g_s$ out of $\{h_i^t\}_{t=1}^{\infty}$ by a Banach limit process. Let $b(\mathbb{N} \longrightarrow K)$ denote the Banach space of all bounded sequences in K, provided with the supremum-norm. By $c(\mathbb{N} \longrightarrow K)$ we denote the closed subspace of $b(\mathbb{N} \longrightarrow K)$ consisting of all sequences $a = (a_n)$ for which $\lim a_n$ exists. Since K is maximally complete there exists a K-linear map $\phi : b(\mathbb{N} \longrightarrow K)$ extending "lim" on $c(\mathbb{N} \longrightarrow K)$ with $\|\phi\| = 1$.

Put $h_i^t = \sum (h_i^t)_\alpha X^\alpha$, $(h_i^t)_\alpha \in K$ and put $H_{i,\alpha} = ((h_i^t)_\alpha)_{t=1}^{\infty} \in b(\mathbb{N} \longrightarrow K)$.

Define $g_{i,\alpha} = \phi(H_{i,\alpha})$ and $g_i = \sum g_{i,\alpha} X^{\alpha}$. Clearly $\|g_i\| < \delta^{-A}$ for all i. We will have finished after showing $\sum g_i f_i = 1$. Now

$\sum g_i f_i = \sum_{\alpha}(\sum_{i=1}^{s} \sum_{\beta+\gamma=\alpha} \phi(H_{i,\beta}) f_{i,\gamma}) X^{\alpha}$. For fixed α we have

$\sum_{i=1}^{s} \sum_{\beta+\gamma=\alpha} \phi(H_{i,\beta}) f_{i,\gamma} = \phi((\sum_{i=1}^{s} \sum_{\beta+\gamma=\alpha} f_{i,\beta}(h_i^t)_{\gamma})_{t=1}^{\infty})$. From

$\sum_{i=1}^{s} h_i^t f_i = 1 + \sum_{|\gamma| \geqslant t} a_{i,\gamma} X^{\gamma}$ it follows that $\lim_{t\to\infty} \sum_{i=1}^{s} \sum_{\beta+\gamma=\alpha} f_{i,\beta}(h_i^t)_{\gamma} = 1$ or 0 according to $\alpha = 0$ or $\alpha \neq 0$.

Since ϕ extends "lim" we are done.

<u>Proof of</u> (3.6). Choose polynomials $p_1, \dots, p_s \in K[X, \dots, X_n]$ such that $\|p_i - f_i\| \leqslant \delta^{c(n,s+1)}$ for all i. Since $\delta > 0$ and $\{\lambda \in K^n | \lambda = (\lambda_1, \dots, \lambda_n)$, all $|\lambda_i| \leqslant 1\}$ is the set of all maximal ideals of $K\{X_1, \dots, X_n\}$ there are $h_1, \dots, h_s \in K\{X_1, \dots, X_n\}$ with $\max\|h_i\| \leqslant 1$ and $\sum h_i f_i = p_o$ with $p_o \in K$, $0 < |p_o| \leqslant 1$. Consider the ideal $I = (p_o, p_1, \dots, p_s)$ in $V[X_1, \dots, X_n]$. Clearly $I \cap V \neq 0$ and $\delta(I) \geqslant \delta$. Hence for some $k_o, k_1, \dots, k_s \in V[X_1, \dots, X_n]$ one has $\sum_{i=0}^{s} k_i p_i = \alpha$, $\alpha \in V$ and $|\alpha| \geqslant \delta^{c(n,s+1)}$.

Then $\sum_{i=1}^{s} \alpha^{-1} k_i f_i + \alpha^{-1} k_o \sum_{i=1}^{s} h_i f_i = 1 + \sum_{i=1}^{s} \alpha^{-1} k_i (f_i - p_i)$. By construction $\|\sum_{i=1}^{s} \alpha^{-1} k_i (f_i - p_i)\| < 1$ and consequently $u = 1 + \sum_{i=1}^{s} \alpha^{-1} k_i (f_i - p_i)$ is a unit in $K\{X_1, \dots, X_n\}$. Hence finally $\sum_{i=1}^{s} u^{-1}(\alpha^{-1} k_i + \alpha^{-1} k_o h_i) f_i = 1$ and for every i, $\|u^{-1}(\alpha^{-1} k_i + \alpha^{-1} k_o h_i)\| \leqslant \delta^{-c(n,s+1)}$.

<u>Remarks</u>. (1) Unfortunately it seems in general impossible to choose the $p_1, \dots, p_s$ in tne proof above such that $(p_1, \dots, p_s) \cap V \neq 0$. So we can not prove $c(n,s) < \infty \Longrightarrow (T_{n,s})$. However by a trick, similar to the one used in (2.6) we can prove "$c(1,2) < \infty \Longrightarrow (T_{n,2})$ for all n" :

Proof. We may suppose using Weierstrass-preparation, that f_1 and f_2 are monic polynomials in X_n of degrees d_1 and d_2. In any equation $g_1f_1 + g_2f_2 = \pi$, $\pi \in V$, $\pi \neq 0$; g_1, $g_2 \in K\{X_1, \ldots, X_n\}$, $\max(\|g_1\|, \|g_2\|) = 1$, one can, using Weierstrass-division, reduce g_1 and g_2 such that $\deg_{X_n}(g_1) < d_2$ and $\deg_{X_n}(g_2) < d_1$. Further one can assume that $\max(\|g_1\|, \|g_2\|) = 1$. Choose $\lambda \in (\lambda_1, \ldots, \lambda_{n-1}) \in V^{n-1}$ such that $\max(\|g_1(\lambda, X_n)\|, \|g_2(\lambda, X_n)\|) = 1$. Then

$g_1(\lambda, X_n)f_1(\lambda, X_n) + g_2(\lambda, X_n)f_2(\lambda, X_n) = \pi$, and $g_i(\lambda, X_n)$, $f_i(\lambda, X_n)$ are polynomials in X_n. From (2.5) part (iii) it follows that

$|\pi| = \alpha((f_1(\lambda, X_n)f_2(\lambda, X_n)) \geqslant \alpha((f_1, f_2))$. Hence $|\pi| = \alpha((f_1, f_2)) \geqslant \delta((f_1, f_2))^{-2}$ by $c(1,2) = 2$.

(2) It seems likely that Corona-conjecture for dimension n implies $c(n,s) < \infty$ for all s.

(3.7) Corollary. $(C_{n,2})$ is true for all $n \geqslant 1$ and with $A = 2$.

§4. Interpolation and zero's.

In this section we study the ring $K\langle X\rangle$ in more detail. First of all we generalize a theorem of Lazard ([3] ; théorème 2) to the case of bounded analytic functions. We use approximately the same notations as in [3] ;

A divisor D defined (or rational) over K is a map $D : \Delta(K_{alg}) \to \mathbb{Z}$ satisfying : for any ρ, $0 < \rho < 1$, there exists a rational function over K (i.e. an element of $K(X)$) which has a divisor (in classical sense) E satisfying $E(\lambda) = 0$ if $|\lambda| > \rho$, $E(\lambda) = D(\lambda)$ if $|\lambda| \leqslant \rho$. The divisor D is said to be positive if $D(\lambda) \geqslant 0$ for all λ (or $D \geqslant 0$ in the obvious ordering of divisors). Further the set $\mathcal{D}_K$ of all divisors which are rational over K is considered to be a susbet of $\mathcal{D}_L$ for every complete valued field $L \supset K$.

Let $\mathcal{A}(K)$ denote the algebra of all power series over K with radius of convergence $\geqslant 1$. For any $f \in \mathcal{A}(K)$ we denote by (f) its divisor. To show that $(f) \in \mathcal{D}_K$ we remark that for any ρ, $0 < \rho < 1$, any ideal in $K\{X, \rho\}$ is principal and generated by a polynomial $\in K[X]$. In particular thereexists a polynomial $P \in K[X]$ with $PK\{X,\rho\} = fK\{X,\rho\}$. Hence $(f) \in \mathcal{D}_K$.

There is a convenient way to represent a positive divisor over K ([3]; (4.3)): The set $\{|\lambda| \mid \lambda \in \Delta(K_{alg})\ ;\ D(\lambda) \neq 0\}$ is at most countable and can be written as $\{\mu_i\}_{i \geqslant 1}$ with $\mu_1 < \mu_2 < \dots$. Let $Q_i \in K[X]$ be a polynomial with $(Q_i)(\lambda) = D(\lambda)$ if $|\lambda| \leqslant \mu_i$ and $(Q_i)(\lambda) = 0$ if $|\lambda| > \mu_i$. Let $P_1 = Q_1$ and $P_i = Q_i(Q_{i-1})^{-1}$ for $i > 1$, and normalize the P_i's by the condition $P_i(0) = 1$ if $\mu_i > 0$ and $P_i = X^d$ if $\mu_i = 0$. Now we write (formally or with the interpretation of [3] ; (4.3)) $D = \prod P_i$.

(4.1) <u>Lemma. Let</u> $f \in \mathcal{A}(K)$ <u>and</u> $(f) = \prod P_i$. <u>Put</u> $c^{-1} = (fP_1^{-1})(0)$. <u>Let</u> $L \supset K$ <u>be any complete valued field. Then for</u> $\lambda \in L$, $|\lambda| < 1$, <u>we have</u> $|f(\lambda)| = |c| \prod |P_i(\lambda)|$.

<u>For any</u> ρ, $0 < \rho < 1$, <u>we have</u> $\|f\|_\rho = |c| \prod \|P_i\|_\rho$.

<u>Proof</u>. Take ρ, $0 < \rho < 1$. Then $f = c \prod_{i=1}^{n} P_i . u$, where n is such that for $i > n$ one has $\mu_i > \rho$. Since u has no zero's with absolute value $\leqslant \rho$, u is an invertible element of $K\{X, \rho\}$ with constant absolute value 1. Hence for $\lambda \in L$, $|\lambda| < \rho$, we have $|f(\lambda)| = |c| \prod_{i=1}^{n} |P_i(\lambda)|$ and $\|f\|_\rho = |c| \prod_{i=1}^{n} \|P_i\|_\rho$. We note further that for $i > n$, $|P_i(\lambda)| = 1$ and $\|P_i\|_\rho = 1$.

<u>Definition.</u> For a positive divisor $D = \prod P_i$ defined over K and $0 < \rho \leqslant 1$ we put $\|D\|_\rho = \prod \|P_i\|_\rho$ (which is finite if $\rho < 1$ and can be ∞ if $\rho = 1$).

(4.2) <u>Corollary. An element</u> $f \in \mathcal{A}(K)$ <u>belongs to</u> $K \langle X \rangle$ <u>if and only if</u> $\|(f)\|_1 < \infty$. <u>In particular if</u> f <u>is normalized by</u> "$f = X^d g$, $g(0) = 1$", <u>then</u> $\|f\| = \|(f)\|_1$.

(4.3) <u>Theorem. Let</u> D <u>be a positive divisor which is rational over</u> K. <u>For every</u> $\varepsilon > 0$ <u>there exists an element</u> $f \in \mathcal{A}(K)$ <u>such that</u> $(f) \geqslant D$ <u>and</u> f <u>is normalized by</u> "$f = X^d g$, $g(0) = 1$" <u>and such that for every</u> ρ, $0 < \rho \leqslant 1$:

$\|D\|_\rho \leqslant \|f\|_\rho \leqslant \|D\|_\rho (1+\varepsilon)$.

<u>If</u> $L \supset K$ <u>is a maximally complete extension of</u> K <u>then there exists</u> $g \in \mathcal{A}(L)$ <u>with</u> $(g) = D$ <u>and hence if</u> g <u>is normalized,</u> $\|g\|_\rho = \|D\|_\rho$ <u>for all</u> ρ, $0 < \rho \leqslant 1$.

<u>Proof</u>. Leaving out trivial cases, we may assume $D = \prod_{i=1}^{\infty} P_i$, $P_i(0) = 1$ for all i. Put $\sum_{i=0}^{\infty} a_{n,i} X^i = P_1 \dots P_n$ and $\sum_{i=0}^{\infty} a_{n,i}^{(j)} X^i = P_1 \dots \hat{P}_j \dots P_n$. For any ρ,

$0 < \rho < 1$, we have $|a_{n,i}|\rho^i \leq \|P_1 \dots P_n\|_\rho \leq \|D\|_\rho$ and $|a^{(j)}_{n,i}|\rho^i \leq \|D\|_\rho$

Let A_i resp. $A^{(j)}_i \in b(\mathbb{N} \to K)$ denote the bounded sequences $(a_{n,i})^{\infty}_{n=1}$ resp. $(a^{(j)}_{n,i})^{\infty}_{n=1}$. Let E be the closed subspace of $b(\mathbb{N} \to K)$ generated by $c(\mathbb{N} \to K)$, all A_i and all $A^{(j)}_i$ (We use here the notations of the proof of (3.5)). Then E is a Banach space of countable type over K and hence for every $\varepsilon > 0$ there exists a K-linear map $\phi : E \to K$ with $|\phi| \leq 1+\varepsilon$, which extends "lim" : $c(\mathbb{N} \to K) \to K$. Let $f = \sum_{i=0}^{\infty} \phi(A_i)X^i$ and $f^{(j)} = \sum_{i=0}^{\infty} \phi(A^{(j)}_i)X^i$. Clearly $f(0) = 1$ and $f \in \mathcal{A}(K)$ since for all ρ, $0 < \rho < 1$ we have $\|f\|_\rho = \max_i |\phi(A_i)|\rho^i \leq (1+\varepsilon)\max_i \|A_i\|\rho^i \leq$ $\leq (1+\varepsilon)\|D\|_\rho$. Analogous $f^{(j)} \in \mathcal{A}(K)$ for all j^i.

Let $P_j = (b_0 + b_1X + \dots + b_tX^t)$. Then $P_jf^{(j)} = \sum_{k=0}^{\infty} \phi(\sum_{i=0}^{t} b_iA^{(j)}_{k-i})X^k$.

But, using $P_j \sum_{i=0}^{\infty} a^{(j)}_{n,i} X^i = \sum_{i=0}^{\infty} a_{n,i} X^i$ for all n, one finds $\sum_{i=0}^{t} b_iA^{(j)}_{k-i} = A_k$ for all k. Consequently $P_jf^{(j)} = f$ and $(f) \geqslant D$.

Finally (4.1) shows that $\|f\|_\rho = \|(f)\|_\rho \geqslant \|D\|_\rho$ for all ρ.

Now assume that $L \supset K$ is given and L is maximally complete. We follow the construction above. Since L is maximally complete there exists an L-linear $\phi : E\hat{\otimes}_K L \to L$ with $\|\phi\| = 1$, which extends "lim" : $c(\mathbb{N} \to L) = c(\mathbb{N} \to K)\hat{\otimes}_K L \to L$. Applying this ϕ we find a $g \in \mathcal{A}(L)$ with $(g) \geqslant D$, g normalized and $\|g\|_\rho = \|D\|_\rho$ for all ρ, $0 < \rho \leq 1$.

If $(g) > D$ then $g = Pg^*$, $P \in L[X]$, $P(0) = 1$, $(g^*) \geqslant D$. For $\rho < 1$, close to 1 one has $\|P\|_\rho > 1$. This gives the contradiction $\|g\|_\rho = \|P\|_\rho \|g^*\|_\rho > \|g^*\|_\rho \geqslant \|D\|_\rho$. So $(g) = D$.

<u>Remarks</u>. (1) In the first part of (4.3) we found an element $\phi \in E'$ with $\|\phi\| \leq 1+\varepsilon$ and ϕ extends "lim". In general it is not possible to find an extension Ψ with $\|\Psi\| = 1$. The following example (due to Lazard) illustrates this :

If K is not maximally complete one can find a sequence of spheres $B(X_n, \rho_n)$ in K such that $B(X_n, \rho_n) \supset B(X_{n+1}, \rho_{n+1})$; $\rho_n > \rho_{n+1}$; $|X_n - X_{n+1}| = \rho_n$, $\lim \rho_n = 1$, $\bigcap_{n=1}^{\infty} B(X_n, \rho_n) = \emptyset$ and $\prod \rho_n < \infty$. The last condition can always be obtained by deleting out of a given sequence sufficiently many elements.

Put $y_n = (X_{n+1} - X_n)^{-1}$, then $|y_n| = \rho_n^{-1}$ and the divisor $D = \prod(1-y_n^{-1}X)$ satisfies $\|D\|_1 = \prod \rho_n < \infty$. Suppose that there exists $f \in \mathcal{A}(K)$ with $(f) = D$. Then $f \in K < X >$ and write $f = 1 + \sum_{i=1}^{\infty} a_i X^i$. For any $n \geqslant 1$ we can write :

$f = \prod_{i=1}^{n-1} (1-y_i^{-1}X)(1+h_n)$ where $h_n = \sum_{i=1}^{\infty} h_{n,i}X^i$ and $1+h_n \in K < X >$ has no zero's of absolute value $< |y_n| = \rho_n^{-1}$. It follows that $\|1+h_n\|_{\rho_n^{-1}} = 1$ and in particular $|h_{n,1}|_{n-1} \leqslant \rho_n$.

Further $a_1 = -\sum_{i=1}^{n-1} \frac{1}{y_i} + h_{n,1}$ and $\sum_{i=1}^{n-1} \frac{1}{y_i} = x_n - x_1$. So we obtain

$|(x_1-a_1)-x_n| = |h_{n,1}| \leqslant \rho_n$ for all n and $x_1 - a_1 \in \bigcap_{n=1}^{\infty} B(x_n, \rho_n) = \emptyset$. Contradiction.

(2) Let a positive divisor $D \in \mathcal{D}_K$ be given. A criterium for the existence of $f \in \mathcal{A}(K)$ with $(f) = D$ is the following:

There exists a closed subspace F of E such that $F \supsetneq c_0(\mathbb{N} \to K)$ and $F \overset{\perp}{\oplus} Ke = E$ where $e = (1,1,\ldots)$ and $\overset{\perp}{\oplus}$ denotes the direct orthogonal sum.

(3) If the valuation of K is discrete then the divisor of any $f \in K < X >$ is finite. This follows at once from [6] (2.5). But it follows also from $f = \prod P_i$. Then $\prod \|P_i\| = \|(f)\|_1 < \infty$. For $i > 1$, $\|P_i\| > 1$. Hence the divisor must be finite.

(4.4) Theorem. (Interpolation). Let $(P_i)_{i=1}^{\infty}$ be a sequence of relatively prime polynomials in $K < X >$, normalized by $\|P_i\| = 1$ and P_i has only roots with absolute value < 1. For any i and n, $1 \leqslant i \leqslant n$ we denote by $Q_{i,n}$ the unique polynomial of degree $\deg(P_i)$ satisfying $Q_{i,n} P_1 \ldots \hat{P}_i \ldots P_n = 1 \bmod (P_i)$.

Associated with this sequence we have a canonical map
$\tau : K < X > \to \prod_i K < X > /_{(P_i)}$.

(i) τ is surjective if and only if $A = \sup\{\|Q_{i,n}\| \mid n \geqslant 1, 1 \leqslant i \leqslant n\} < \infty$.

(ii) If τ is surjective then the inverse of $\tau^*: K < X > /_{\ker \tau} \to \prod_i K<X>/_{(P_i)}$ has norm A.

(iii) In particular, if D is a positive divisor, rational over K, which is decomposed as $D = \prod P_i$, then $\|D\|_1 \leqslant A \leqslant \|D\|_1^2$ and $\tau_D : K < X > \to \prod K < X > /_{(P_i)}$ is surjective if and only if $\|D\|_1 < \infty$. The kernel of τ_D is equal to $I_D^1 = \{f \in K < X > \mid (f) \geqslant D\}$. So $\ker \tau_D \neq 0$

<u>if and only if</u> $\|D\|_1 < \infty$.

(4.5) <u>Lemma</u>. <u>Let</u> $P \in K < X >$ <u>be a polynomial, normalized by</u> $\|P\| = 1$ <u>and</u> P <u>has only roots of absolute value</u> < 1. <u>Let</u> $\tau : K < X > \to K < X > /_{(P)}$ <u>be the canonical map,</u> $\alpha ; K < X > \to c_o(\mathbb{N}_o \to K)'$ <u>the bijective isometric map given by</u>

$$\alpha(\sum_{n=0}^{\infty} a_n X^n)(b_o, b_1, b_2, \ldots) = \sum_{i=0}^{\infty} a_i b_i, \text{ where } \sum_{n=0}^{\infty} a_n X^n \in K < X > \text{and}$$

$(b_o, b_1, \ldots) \in c_o(\mathbb{N}_o \to K)$.

<u>Let</u> $\beta : K < X > /_{(P)} \to (K < X > /_{(P)})''$ <u>denote the canonical bijective isometry. Then there exists a unique K-linear map</u> $\mu : (K < X > /_{(P)})' \to c_o(\mathbb{N}_o \to K)$ <u>such that the following diagram is commutative</u>

$$\begin{array}{ccc} K < X > & \xrightarrow{\tau} & K < X > /_{(P)} \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ c_o(\mathbb{N}_o \to K)' & \xrightarrow{\mu} & (K < X > /_{(P)})'' \end{array}$$

<u>Moreover</u> μ <u>is an isometry</u>.

<u>Proof</u>. Any $F \in K < X >$ can uniquely be written as $F = qP + \sum_{i=0}^{s-1} a_i(F)X^i$ where $s = \deg(P)$, $q \in K < X >$ and $a_i(F) \in K$. Moreover $\max_{0 \leq i < s} |a_i(F)| \leq \|F\|$. It follows that the images $\bar{1}, \bar{X}, \ldots, \bar{X}^{s-1}$ of $1, X, \ldots, X^{s-1}$ in $K < X > /_{(P)}$ form there an orthonormal base and that $\tau(F) = \sum_{i=0}^{s-1} a_i(F)\bar{X}^i$. Let $A : K < X > /_{(P)} \to K < X > /_{(P)}$ denote the K-linear map given by $A(f) = \bar{X}f$ for all $f \in K < X > /_{(P)}$. Then $\|A^s\| < 1$ and for all $n \geq 0$ we have

$$A^n(\bar{1}) = \sum_{i=0}^{s-1} a_i(X^n)\bar{X}^i.$$

Hence $\lim_{n\to\infty} \|\sum_{i=0}^{s-1} a_i(X^n)\bar{X}^i\| = 0$ and for $F = \sum_{n=0}^{\infty} b_n X^n$ and all $i = 0, \ldots, s-1$ we have $a_i(F) = \sum_{n=0}^{\infty} b_n a_i(X^n)$. The map μ is now defined by : if $1 \in (K < X > /_{(P)})'$ then $\mu(1) = (1(\sum_{i=0}^{s-1} a_i(X^n)\bar{X}^i))_{n=0}^{\infty} \in c_o(\mathbb{N}_o \to K)$. It is clear now that this μ is the unique map which makes the diagram commutative and that μ is isometric.

Proof of (4.4). Part (i). Let τ_i denote the canonical map $K\langle X\rangle \to K\langle X\rangle/_{(P_i)}$, β_i the canonical map $K\langle X\rangle/_{(P_i)} \to (K\langle X\rangle/_{(P_i)})''$ and $\mu_i : (K\langle X\rangle/_{(P_i)})' \to c_o(\mathbb{N}_o \to K)$ the map obtained with the help of (4.5).

Then $\tau = \Pi\tau_i$. Put $\mu = \Sigma\mu_i : \Sigma(K\langle X\rangle/_{(P_i)})' \to c_o(\mathbb{N}_o \to K)$ and $\beta = \Pi\beta_i : \Pi K\langle X\rangle/_{(P_i)} \to \Pi(K\langle X\rangle/_{(P_i)})'' = (\Sigma(K\langle X\rangle/_{(P_i)})')'$.

Then again $\beta\circ\tau = \mu'\circ\alpha$ and β,α are bijective and isometric. So we may consider μ' instead of τ. Using the weak from of Hahn-Banach which is available for the spaces $c_o(\mathbb{N}_o \to K)$ and $\Sigma(K\langle X\rangle/_{(P_i)})'$ since they are both of countable type over K, one sees that μ' is surjective if and only if

$$c = \inf\left\{\frac{\|\mu(l)\|}{\|l\|} \,\middle|\, l \in \Sigma(K\langle X\rangle/_{(P_i)})' ,\ l \neq 0\right\} > 0.$$ Moereover if $c > 0$ then $\|\tau^{*-1}\| = c^{-1}$.

Further $\|\mu(l)\| = \sup\left\{\frac{|\alpha(F)(\mu(l))|}{\|F\|} \,\middle|\, F \in K\langle X\rangle,\ F \neq 0\right\}$. After writing $l = \Sigma l_i$, $l_i \in (K\langle X\rangle/_{(P_i)})'$ one has $\alpha(F)(\mu(l)) = \sum_{i=1}^{\infty} l_i(\tau_i F)$. It suffices in the computation of c to consider finite sums $\sum_{i=1}^{N} l_i = l$.

Assume now $A < \infty$. Choose $b_o + b_1\bar{X} + \ldots + b_{s_i-1}\bar{X}^{s_i-1} \in K\langle X\rangle/_{(P_i)}$ (with $s_i = \deg P_i$) satisfying $|l_i(b_o + b_1\bar{X} + \ldots)| = \|l_i\|\ \|b_o + b_1\bar{X} + \ldots\|$. The element $F \in K\langle X\rangle$ given by $F = Q_{i,N}(b_o + b_1\bar{X} + \ldots + b_{s_i-1}\bar{X}^{s_i-1})$ satisfies :

$$|\alpha(F)(\mu(l))| = |l_i(b_o + b_1\bar{X} + \ldots)| = \|l_i\|\,\|b_o + b_1\bar{X} + \ldots\| \geq$$

$\geq A^{-1}\|F\|\ \|l_i\|$. Consequently $\|\mu(l)\| \geq A^{-1}\ \|l\|$ and $c \geq A^{-1} > 0$. Hence τ is surjective.

Assume now that τ is surjective; then $\|\tau^{*-1}\| < \infty$ according to the closed graph theorem. Hence the inverse $\rho : K\langle X\rangle/_{(P_1\,\cdots\,P_n)} \to \prod_{i=1}^{n} K\langle X\rangle/_{(P_i)}$ has norm $\leq \|\tau^{*-1}\|$. The element $\rho^{-1}(0,\ldots,0,1,0\ldots0)$ can be written as $Q_{i,n}P_1\ldots\hat{P}_i\ldots P_n$. Hence $\|Q_{i,n}\| \leq \|\tau^{*-1}\|$ for all i and n. So $A \leq \|\tau^{*-1}\| < \infty$.

(ii) $A = \|\tau^{*-1}\|$ since $A \leq \|\tau^{*-1}\|$ and $\|\tau^{*-1}\|^{-1} = c \geq A^{-1}$ are both derived in the proof of (i).

(iii) For convenience we suppose that $D(0) = 0$. In the decomposition $D = \prod_{i=1}^{\infty} P_i$ we make the normalisation : P_i is a monic polynomial of degree $s(i)$, such that all roots of P_i have absolute value μ_i. Further $\mu_1 < \mu_2 < \dots$. It follows that $\|D\|_1 = \prod_{i=1}^{\infty} |P_i(0)|^{-1}$.

The polynomial $Q_{i,n}$ is defined by $Q_{i,n} P_1 \dots \hat{P}_i \dots P_n \equiv 1 \bmod (P_1)$ and $\deg Q_{i,n} < s_i$. Equivalently $Q_{i,n} P_1 \dots \hat{P}_i \dots P_n + R_{i,n} P_i = 1$, $\deg Q_{i,n} < s_i$, $\deg R_{i,n} < (\sum_{j=1}^{n} s_j) - s_i$. According to (2.5) and the definitions of $\alpha(I)$ and $\delta(I)$ for the ideal $I = (P_i, P_1 \dots \hat{P}_i \dots P_n)$ we find $\max(\|Q_{i,n}\|, \|R_{i,n}\|) = \|Q_{i,n}\| = \alpha(I)^{-1}$ and
$\delta(I) = \min_j \min\{\max\{|P_i(z)|, |P_1 \dots \hat{P}_i \dots P_n(z)|\} \mid z \text{ zero of } P_j\}$. An easy calculation yields $|P_1(0)| \dots |P_n(0)| \leq \delta(I) \leq |P_{i+1}(0)| \dots |P_n(0)|$.
Further, using (2.4) :

$\delta(I)^2 \leq \alpha(I) \leq \delta(I)$ one finds $\|D\|_1 \leq A \leq \|D\|_1^2$. The rest of (iii) follows at once from (4.2).

(4.6) Corollary. A sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \{\lambda \in K \mid |\lambda| < 1\}$ is called an interpolation sequence if the map $\tau : K\langle X\rangle \to b(\mathbb{N} \to K)$, given by $\tau(f) = (f(\lambda_n))_{n=1}^{\infty}$, is surjective. One has τ is surjective if and only if $\inf_i \prod_{n=1}^{\infty} |\lambda_n - \lambda_i| = c > 0$.

Further if τ is surjective then the inverse of the induced map τ^*: $K\langle X\rangle/_{\ker \tau} \to b(\mathbb{N} \to K)$ has norm c^{-1}.

Proof. Apply (4.4) part(i) and (ii) with $P_i = X - \lambda_i$.

(4.7) Corollary. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an interpolation sequence and let $I \subset K\langle X\rangle$ be the ideal $\{f \in K\langle X\rangle \mid f(\lambda_i) = 0 \text{ for all } i\}$. Then the maximal ideals $M \supset I$ correspond 1-1 with ultrafilters $\mathcal{U}$ on $\mathbb{N}$, where the correspondance is given by

$$\mathcal{U} \mapsto \{f \in K\langle X\rangle \mid \lim_{\mathcal{U}} |f(\lambda_i)| = 0\} = M \subset K\langle X\rangle .$$

<u>Moreover for every maximal ideal</u> $M \supset I$ <u>the residue field</u> $K < X > /_M$ <u>provided with the quotient norm is a valued field. If</u> $\mathcal{U}$ <u>is non-trivial then</u> $K < X > /_M$ <u>is a "big" field extension of</u> K.

<u>Proof</u>. Since $K < X > /_I \cong b(\mathbb{N} - K)$, everything follows from [4] (4.1) and (4.4).

<u>Problems</u>. It is not clear and probably not true that every maximal ideal M of $K < X >$, even if K is algebraically closed and maximally complete, is obtained as in (4.7) from an interpolation sequence. However, one has a weaker result:

Let $f \in M$, $f \neq 0$ and let $(f) = \prod P_i$ be the canonical decomposition of the divisor of f. Then according to (4.4) :

$K < X > /_{(f)} \cong \prod K < X > /_{(P_i)}$ and M corresponds to a maximal ideal of $\prod K < X > /_{(P_i)}$. A study of algebras R of the type $R = \prod_{i=1}^{\infty} R_i$, where $\dim R_i < \infty$ for each i, is needed to obtain further results on maximal ideals of $K < X >$. We remark that the special case $\sup_i \dim R_i < \infty$ reduces easily to the case $b(\mathbb{N} \to K)$ (i.e. $\dim R_i = 1$ for all i) which is treated in [4]. The case $\sup \dim R_i = \infty$ seems far more complicated. Interesting questions about those algebras are

(i) Is R/M, provided with the quotient norm, a valued field for every maximal ideal M?

(ii) Does R contain closed prime ideals which are non-zero and non-maximal ?

(iii) Is the set of "trivial" maximal ideals a dense subset of the set of all maximal ideals of R ?

(iv) Can one give a filter-description for the maximal ideals of R ?

(4.8) <u>Corollary</u>. <u>Let</u> $f \in K < X >$ <u>satisfy</u> : <u>the set of all zero's</u> $\{\lambda_i\}_{i=1}^{\infty}$ <u>of</u> f <u>belongs to</u> K <u>and every zero is a simple zero of</u> f. <u>Then</u> $\{\lambda_i\}_{i=1}^{\infty}$ <u>is an interpolation sequence if and only if</u> $(f,f') = (1)$.

<u>Proof</u>. Suppose $(f,f') = (1)$. Then $\delta(f,f') > 0$ and consequently $\inf|f'(\lambda_n)| > 0$. Write $f = (X - \lambda_n)g$ with $g \in K < X >$ then it follows that $|f'(\lambda_n)| = |g(\lambda_n)| =$

$$= \prod_{\substack{i=1 \\ i \neq n}}^{\infty} |\lambda_i - \lambda_n|$$

. Hence according to (4.7) the sequence is an interpolation sequence.

Suppose that the sequence is an interpolation sequence. Then as before $\inf_{n} |f'(\lambda_n)|>0$. For every maximal ideal $M \supset fK < X >$ there exists, according to (4.7), an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $M = \{g \in K < X > | \lim_{\mathcal{U}} |g(\lambda_i)| = 0\}$. Clearly f' does not belong to any of those maximal ideals and hence $(f,f') = (1)$.

Problems. (i) Does there exist a maximal ideal M of $K < X >$ with the property : Fo every $f \in M$, also $f' \in M$?

(ii) Suppose that K is algebraically closed and maximally complete ; let $M \subset K < X >$ be a maximal ideal, $f \in M$ such that $f' \notin M$. Is M obtainable from an interpolation sequence as in (4.7) ?

(4.9) Corollary. *Let V be a non-discrete (rank 1) valuation ring. Then the Krull-dimension of $V[[X]]$ is infinite.*

Proof. If Krulldim $V[[X]] < \infty$ then also Krulldim $K < X > < \infty$ and Krulldim $b(\mathbb{N} \to K) < \infty$ since for a suitable interpolation sequence one obtains $b(\mathbb{N} \to K)$ as a residue ring of $K < X >$. The proof of (4.9) will be completed by using the next lemma, which shows that $b(\mathbb{N} \to K)$ contains infinite chains of prime ideals.

(4.10) Lemma. (i) *Let $\mathcal{U}$ be a fixed non-trivial ultrafilter on $\mathbb{N}$ and let $c = (c_1, c_2, c_3, \ldots)$ be a sequence of real numbers satisfying $0 < c_i < 1$ for all i and $\lim c_i = 0$. Then the ideal I_c of $b(\mathbb{N} \to K)$ given by*

$f \in I_c$ if for some $k \in \mathbb{N}$ and $D \in \mathbb{R}$ the set $\{n \in \mathbb{N} \mid |f(n) \leq c_n^{\frac{1}{k}} D\}$ belongs to $\mathcal{U}$, is a prime ideal.

(ii) *Let d denote the sequence $d = (c_1, c_2^2, c_3^3, c_4^4, \ldots)$ then $I_d \subsetneqq I_c$.*

Proof. (i) (a) I_c is an ideal since for $f_1, f_2 \in I_c$, $g \in b(\mathbb{N} \to K)$ we have $V_i = \{n \in \mathbb{N} \mid \|f_i(n)| \leq c_n^{\frac{1}{k_i}} D_i\} \in \mathcal{U}$ $(i=1,2)$ and with $D = \max(D_1, D_2)$, $k = \max(k_1, k_2)$ we have

$$\{n \in \mathbb{N} \mid | f_1(n) + g(n) f_2(n)| \leq c_n^{\frac{1}{k}} D \|g\|\} \supset V_1 \cap V_2 \text{ and belongs to } \mathcal{U}.$$

Hence $f_1 + gf_2 \in I_c$.

(b) I_c is a prime ideal. Indeed let $f_1, f_2 \notin I_c$ then for all

$k \in \mathbb{N}$, $D \in \mathbb{R}$ the complements W_i of $\{n \in \mathbb{N} \mid |f_i(n)| \leqslant Dc_n^{\frac{1}{k}}\}$ (i=1,2) belong to $\mathcal{U}$. Hence for all $k \in \mathbb{N}$, $D \in \mathbb{R}$, the set $\{n \in \mathbb{N} \mid |f_1(n)f_2(n)| > D^2 c_n^{\frac{2}{k}}\}$ belongs to $\mathcal{U}$. Consequently $f_1 f_2 \notin I_c$.

(ii) Take $f \in I_c$ with $\frac{1}{2} c_n \leqslant |f(n)| \leqslant c_n$ for all n. If f would belong to I_d then for some $k \in \mathbb{N}$, $D \in \mathbb{R}$ one has $\{n \in \mathbb{N} \mid |f(n)| \leqslant c_n^{\frac{n}{k}} D\} \in \mathcal{U}$. But $|f(n)| \geqslant \frac{1}{2} c_n \geqslant c_n^{\frac{n}{k}} D$ for all but finitely many integers n. Hence $f \notin I_d$.

<u>Remark</u>. The question whether Krulldim R = 1 (R non-noetherean) implies Krulldim $R[[X]] < \infty$ is recently, for more general rings than valuationrings as in (4.10), answered in the negative by J.T. Arnold (On Krulldimensions in power series rings ; to appear).

<u>Problem</u>. Although we proved that $b(\mathbb{N} \to K)$ contains infinite chains of prime ideals one can easily see that every non-zero closed prime ideal is maximal. Does the same hold for $K \langle X \rangle$?

§5. Application to invariant subspaces.

The Banach space $E = c_o(\mathbb{N}_o \to K)$ is given the orthonormal base $\{e_i\}_{i=0}^{\infty}$ where e_i denotes $(0,\dots,0,1,0,\dots)$. We consider on E the antishift operator $T : E \to E$ defined by $T(e_i) = e_{i-1}$ $(i \geqslant 1)$ and $Te_o = 0$. As shown in [6] , (3.4), the algebra of all bounded operators on E which commute with T is isomorphic to $K \langle X \rangle$; the isomorphism $\rho : K \langle X \rangle \to \mathcal{L}(E)$ is given by

$$\rho(\sum_{n=0}^{\infty} a_n X^n) e_i = \sum_{n=0}^{\infty} a_n T^n(e_i) = \sum_{n=0}^{i} a_n T^n(e_i).$$

Let $\pi_n \in E'$ $(n \geqslant 0)$ denote the map given by $\pi_n(e_i) = 0$ if $i \neq n$ and 1 if $i = n$. The composed map $\pi_o \circ \rho : K \langle X \rangle \to E'$ has obviously the property

$$\pi_o \circ \rho(\sum_{n=0}^{\infty} a_n X^n)(b_o, b_1, \dots) = \sum_{i=0}^{\infty} a_i b_i .$$ Hence $\pi_o \circ \rho = \alpha$ where α is the map considered in lemma (4.5).

In this section we investigate the set of all closed subspaces of E which are invariant under T.

(5.1) Lemma. Let $F \subset E$ be a closed subspace which is invariant under T. Then :

(i) For all $f \in K\langle X\rangle$, $\rho(f)F \subset F$.

(ii) Let $id(F) = \{f \in K\langle X\rangle \mid \rho(f)F = 0\}$. Then id(F) is a closed ideal of $K\langle X\rangle$.

The kernel of the map $K\langle X\rangle \xrightarrow{\pi_o \circ \rho} E' \xrightarrow{r} F'$, where r denotes the obvious restriction map, is also equal to id(F). Further $K\langle X\rangle/_{id(F)} \cong F'$.

(iii) Let $F_1 \subsetneqq F_2$ denote closed invariant subspaces of E. Then $id(F_1) \supsetneqq id(F_2)$.

(iv) For any ideal $I \subset K\langle X\rangle$ one defines $n(I) = \cap\{\ker \rho(f) \mid f \in I\}$. Then n(I) is a closed invariant subspace of E. Further $n(id(F)) = F$.

Proof. (i) Let $f = \sum_{n=0}^{\infty} a_n X^n \in K\langle X\rangle$ and $x \in F \subset c_o(\mathbb{N}_o \to K) = E$. Then $\rho(f)(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n T^n(x)$. Since F is closed and invariant under T, we find $\rho(f)(x) \in F$. Hence $\rho(f)F \subset F$ for all $f \in K\langle X\rangle$.

(ii) It is clear that id(F) is a closed ideal of $K\langle X\rangle$. Further let $f \in K\langle X\rangle$. Then $\pi_o \circ \rho(f)(F) = 0$ if and only if $\pi_o \circ \rho(X^n f)(F) = 0$ for all $n \geqslant 0$. But $\pi_o \circ \rho(X^n f) = \pi_n \circ \rho(f)$. Hence $\pi_o \circ \rho(f)F = 0$ if and only if $\rho(f)F = 0$. So id(F) is the kernel of $r \circ \pi_o \circ \rho$. The map $r : E' \to F'$ is surjective since a weak form of Hahn-Banach is available for the Banach spaces F and E which are of countable type over K. Hence $F' \cong K\langle X\rangle/_{id(F)}$.

(iii) The map $F_2' \to F_1'$ is surjective and has a nontrivial kernel. So using (ii) one finds $id(F_1) \supsetneqq id(F_2)$.

(iv) Apply (iii) with $F_1 = F$ and $F_2 = n(id(F))$.

(5.2) Lemma. Let $P \in K\langle X\rangle$ be a polynomial of degree s, normalized by the condition : all the roots of P have absolute value < 1. As in (4.5) there exists a commutative diagram

$$\begin{tikzcd}
K\langle X\rangle \arrow[r, dash, "\tau"] \arrow[d, "\alpha"'] & K\langle X\rangle/_{(P)} \arrow[d, "\beta"] \\
c_0(\mathbb{N}_0 \to K) \arrow[r, dash, "\mu'"] & (K\langle X\rangle/_{(P)})''
\end{tikzcd}$$

The map μ *is an isomorphism of* $(K\langle X\rangle/_{(P)})'$ *into* $n(PK\langle X\rangle)$. *Further*

(a) $id(n(PK\langle X\rangle)) = PK\langle X\rangle$.

(b) *For every closed invariant subspace* F *of* E *with* $\dim F = s < \infty$ *there exists a polynomial* $P \in K\langle X\rangle$ *of degree* s *which has only roots of absolute value* < 1 *such that* $F = n(PK\langle X\rangle)$.

Proof. We show first $\dim n(PK\langle X\rangle) = s$. Write $P = X^s + \alpha_{s-1}X^{s-1} + \ldots + \alpha_0$, all $|\alpha_i| < 1$ by assumption and write $x : \sum x_i e_i \in c_0(\mathbb{N}_0 \to K)$.

The equation $\rho(P)(x) = 0$ then reads :

$$x_{i+s} + \alpha_{s-1}x_{i+s-1} + \ldots + \alpha_0 x_i = 0 \text{ for all } i = 0,1,2,\ldots .$$

So with given $x_0, \ldots, x_{s-1}$ there exists a unique solution $(x_i)_{i=0}^{\infty}$ of this set of equations and moreover $\lim |x_i| = 0$ since all $|\alpha_i| < 1$. Hence $\dim n(PK\langle X\rangle) = s$.

In showing $\operatorname{im}\mu = n(PK\langle X\rangle)$ it suffices to prove $\operatorname{im}\mu \subset n(PK\langle X\rangle)$ since μ is already known to be isometric and $\dim(K\langle X\rangle/_P)' = s = \dim n(PK\langle X\rangle)$. Take $l \in (K\langle X\rangle/_{(P)})'$. Then for all $n \geq 0$, $\alpha(X^nP)(\mu(l)) = \mu'\circ\alpha(X^nP)(l) = \beta\circ\tau(X^nP)(l) = 0$. Hence, since $\alpha(X^nP) = \pi_n \circ \rho(P)$ for all $n \geq 0$, we find $\rho(P)(\mu(l)) = 0$. This means $\operatorname{im}\mu \subset \ker \rho(P) = n(PK\langle X\rangle)$.

a) $id(n(PK\langle X\rangle)) = QK\langle X\rangle$ where Q is a polynomial dividing P. Applying "n" again and (5.1) part (iv) one finds $n(PK\langle X\rangle) = n(QK\langle X\rangle)$. Since $\dim n(QK\langle X\rangle) = \text{degree } Q$, one obtains $P = Q$.

b) Let T^* denote the restriction of T to F. The characteristic polynomial $P \in K[X]$ of T^* satisfies $P(T^*) = 0$ or $P(T) \in id(F)$. Hence for some polynomial $Q \in K[X]$ which divides P we have $id(F) = QK\langle X\rangle$. After applying "n" one obtains $F = n(id(F)) = n(QK\langle X\rangle)$. So $\deg Q = \dim F = s$ and $Q = P$. Clearly all the roots of P have absolute value < 1, otherwise $P = uP^*$ where u is a unit in $K\langle X\rangle$ and degree $P^* < s$ which is impossible.

(5.3) *Lemma*. *Let* D *be a positive divisor over* K *and such that* $\|D\|_1 < \infty$. *Let* $D = \prod P_i$ *denote its canonical decomposition. As in the proof of (4.4) one has a commutative diagram*

$$\begin{tikzcd} K\langle X\rangle \arrow[r, "\tau"] \arrow[d, "\alpha"] & K\langle X\rangle/_{(P_i)} \arrow[d, "\beta"] \\ c_0(\mathbb{N}_0 \to K)' \arrow[r, "\mu'"] & (\Sigma(K\langle X\rangle/_{(P_i)})')'. \end{tikzcd}$$

Let I_D *denote the closed ideal* $\{f \in K\langle X\rangle \mid (f) \geqslant D\}$. *Then* I_D *is the kernel of* τ *and* im $\mu = n(I_D)$. *Further the subspaces* $\mu((K\langle X\rangle/_{(P_i)})') = n(P_i K\langle X\rangle)$ $(i=1,2,\ldots)$ *are* $\|D\|_1^{-2}$*-orthogonal and their (closed) sum* $\sum n(P_i K\langle X\rangle)$ *is equal to* im μ. *Moreover* id(im μ) $= I_D$.

Proof. As in the proof of (4.4) part(iii) one finds $\|\mu(x)\| \geqslant \|D\|_1^{-2}\|x\|$ for all $x \in \sum(K\langle X\rangle/_{(P_i)})'$. It follows immediately that the subspace $n(P_i K\langle X\rangle) = \mu((K\langle X\rangle/_{(P_i)})')$ are $\|D\|_1^{-2}$ -orthogonal and that their closed sum is equal to im μ.

Clearly $I_D = \ker\tau = \ker \beta\circ\tau = \ker \mu'\circ\alpha$. Hence $f \in I_D$ if and only if $\alpha(X^n f)(\text{im}\,\mu) = 0$ for all $n \geqslant 0$. Again $\alpha(X^n f) = \pi_n \circ \rho(f)$ yields $f \in I_D$ if and only if $\rho(f)(\text{im}\,\mu) = 0$ or equivalently $f \in \text{id}(\text{im}\,\mu)$. So $I_D = \text{id}(\text{im}\,\mu)$.

Also clearly $n(I_D) \supseteq \text{im}\,\mu$, hence $I_D \subset \text{id}(n(I_D)) \subset \text{id}(\text{im}\,\mu) = I_D$. Using (5.1) part(iv) one sees that $n(I_D) = \text{im}\,\mu$.

(5.4) *Theorem*. *Let* $\mathcal{B}\mathcal{D}_K^+$ *denote the set of all positive divisors* D *which are rational over* K *and satisfy* $\|D\|_1 < \infty$. *The map* $\phi : \mathcal{B}\mathcal{D}_K^+$ **(the set of all closed invariant** *subspaces of* $c_0(\mathbb{N}_0 \to K))$ *given by* $\phi(D) = n(I_D)$ *is bijective. Further* $\text{id}(\phi(D)) = I_D$.

Suppose in addition that K *is maximally complete, then* ϕ *induces a bijection between the set of principal ideals of* $K\langle X\rangle$ *and the set of all closed invariant subspaces of* $c_0(\mathbb{N}_0 \to K)$.

Proof. In view of (5.3) and (4.3) all we have to show is that ϕ is surjective. Let F be a closed invariant subspace of $c_0(\mathbb{N}_0 \to K)$ and $f \in K\langle X\rangle$, $f \neq 0$, $f \in \text{id}(F)$.

Let $(f) = \prod P_i$ be the canonical decomposition of the divisor of f. Then $n(fK < X >) = \sum n(P_i K < X >) \supset F$ and the set of subspaces $\{n(P_i K < X >)\}_i$ is $\|(f)\|_1^{-2}$-orthogonal.

Let $x = \sum x_i \in F$ with $x_i \in n(P_i K < X >)$ for all i. Then $\|x\| \geqslant \|(f)\|_1^{-2} \max(\|x_i\|)$ and $\lim \|x_i\| = 0$. Let $Q_{i,n}$ be the polynomial of degree $< \deg(P_i)$ satisfying $Q_{i,n} P_1 \ldots \hat{P}_i \ldots P_n \equiv 1 \mod(P_i)$.

Then $\rho(Q_{i,n} P_1 \ldots \hat{P}_i \ldots P_n)(x) = x_i + \sum_{j \geqslant n} \rho(Q_{i,n} P_1 \ldots \hat{P}_i \ldots P_n)(x_j)$. Since $\sup \|Q_{i,n} P_1 \ldots \hat{P}_i \ldots P_n\| < \infty$ according to (4.4), we obtain after taking the limit of $n \to \infty$, $x_i \in F$. So we have shown that $F = \overline{\sum F \cap n(P_i K < X >)}$ and this sum of subspaces is $\|(f)\|_1^{-2}$-orthogonal. Each $F \cap n(P_i K < X >)$ is finite dimensional and equals $n(P_i^* K < X >)$ for some P_i^* dividing P_i, according to (5.2). Let D be the divisor which has the decomposition $D = \prod P_i$. Then it is clear from (5.3) that $F = n(I_D)$.

Remarks. (1) This theorem resembles of course the following theorem in the complex case: [2] page 66, "every closed subspace S of the Hardy space $H^2(\Delta)$, invariant under multiplication by z, has the form $S = FH^2$, where F is an inner function".

However, the multiplication by z, defines a shift-operator in $H^2(\Delta)$ whereas our concern has been the anti-shift operator $T : c_o(\mathbb{N}_o \to K) \to c_o(\mathbb{N}_o \to K)$.

The non-archimedean case of a shift operator $U : c_o(\mathbb{N}_o \to K) \to c_o(\mathbb{N}_o \to K)$ is quite simple. Identify $c_o(\mathbb{N}_o \to K)$ with $K\{X\}$ by means of the map $\Upsilon : K\{X\} \to c_o(\mathbb{N}_o \to K)$ given by $\Upsilon(\sum a_n X^n) = \sum a_n U^n(e_o) = (a_o, a_1, a_2, \ldots)$. Closed invariant subspaces of $c_o(\mathbb{N}_o \to K)$ correspond then 1-1 with ideals of $K\{X\}$. As is well known every ideal in $K\{X\}$ has the form $PK\{X\}$ where P is a polynomial which has only roots of absolute value $\leqslant 1$.

(2) If the valuation of K is discrete then the non-trivial, closed subspaces of $c_o(\mathbb{N}_o \to K)$ which are invariant under the anti-shift operator T have finite dimension. This follows from (5.4) and the remark that every ideal in $K < X >$ is principal and generated by a polynomial.

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