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APPROXIMATION PROPERTIES AND UNIVERSAL BANACH SPACES

by

Przemyslaw WOJTASZCZYK

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1. - In this talk we are dealing with two kinds of concepts.

The first one, approximation properties, has its orgin in the concepts of Schauder basis [10] and metric approximation property of Grothendieck [2]. By "approximation property" we mean the answer to the question in what a way we can approximate the identity operator on a Banach space X by finite dimensional operators. The second one, universality, has its orgin in the classical Banach-Mazur theorem on the universality of the space of continuous functions on the Cantor discontinuum [1]. The question is, to find for a given class of Banach spaces a space which contains (in a nice way) any element of the given class.

2. - Definitions.

DEFINITION 1. - A Banach space X has the bounded approximation property, shortly BAP (resp. unconditional bounded approximation property, shortly UBAP) iff there exists a sequence of finite dimensional operators $A_n : X \to X$, $n = 1,2,\dots$, such that for each $x \in X$ $x = \sum_{n=1}^{\infty} A_n(x)$ (and the series is unconditionally convergent).

Those concepts are modifications of the metric approximation property of Grothendieck [2].

DEFINITION 2. - A Banach space X has a (unconditional) basis of finite dimensional subspaces iff there exists a sequence $(X_n)_{n=1}^{\infty}$ of finite dimensional subspaces of X such that for each $x \in X$ we have a unique decomposition $x = \sum_{n=1}^{\infty} x_n$ where $x_n \in X_n$ (and the series is unconditionally convergent). If moreover $\dim X_n = 1$ for $n = 1,2,\dots$ the Banach space X has the Schauder basis (resp. unconditional Schauder basis).

It is easily seen that those concepts are in fact "approximation properties".

DEFINITION 3. - Let us have a class $\mathfrak{B}$ of Banach spaces. A space X is complementably universal for the class $\mathfrak{B}$ iff for each $Y \in \mathfrak{B}$ there exists in X a complemented subspace Y_1 which is isomorphic to Y .

3. - About this concepts the following results are proved.

THEOREM 1. - [9] A Banach space X has UBAP iff X is isomorphic to a complemented subspace of a Banach space with the unconditional basis of finite dimensional subspaces. Moreover there exists one space U_{fd} with the unconditional basis of finite dimensional subspaces such that any X having UBAP is isomorphic to a complemented subspace of U_{fd}. The space U_{fd} is unique up to isomorphism among spaces with UBAP.

Remark 1. - Some constructions used in the proof of theorem 1 has applications in simultaneous extensions of continuous functions (cf. [9]).

THEOREM 2. - A Banach space X has BAP iff X is isomorphic to a complemented subspace of a Banach space with the Schauder basis. Moreover there exists one space B with the Schauder basis which is complementably universal for the class of all Banach spaces with BAP. The space B is unique up to isomorphism among spaces with BAP.

This theorem was proved independently in [4] and [8] using results of [9] and [7].

Remark 2. - In [3] among others is proved the following fact.

There exists a family of separable Banach spaces C_p, $1 \leqslant p \leqslant \infty$ such that for any Banach space X such that X^* has BAP and any p the space $X \oplus C_p$ has the Schauder basis.

Remark 3. - The space B was constructed by different ways in [7], [9] and [5]. The proof that spaces constructed in this papers are exactly the same follows from [8]. The construction of Kadec [5] has interesting applications to the theory of preduals of L_1 (cf. [9] and [12]).

Remark 4. - Many interesting results concerning various approximation properties are proved in [4].

4. - To finish this talk we are going to state some unsolved problems.

Problem 1. - Is any Banach space with UBAP isomorphic to a complemented subspace of a Banach space with an unconditional basis ?

Problem 2. - Find an example of a separable Banach space with UBAP not having an unconditional basis.

It is probably that such an example can be found among spaces constructed in [6].

Problem 3. - Prove that in a reflexive Banach space X with BAP there exists a sequence of finite dimensional projections P_n such that $P_n(X) \subset P_{n+1}(X)$ for $n = 1, 2, \ldots$, and $P_n(x) \to x$ for any $x \in X$.

Problem 4. - Does any reflexive Banach space with a basis of finite dimensional subspaces have a Schauder basis ?

The positive solution of problems 3 and 4 together with results of [2] and [4] would imply that a separable, reflexive space with approximation property of Grothendieck [2] has a Schauder basis.

Problem 5. - [7]. Does there exist a separable Banach space complementably universal for the class of separable Banach spaces ?

The solution of this problem would have important consequences conected with "basis problem" and "approximation problem" (cf. [11] p. 386).

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