

IRREGULARITY OF AN ANALOGUE OF THE GAUSS-MANIN SYSTEMS

BY CÉLINE ROUCAIROL

ABSTRACT. — In \mathcal{D} -modules theory, Gauss-Manin systems are defined by the direct image of the structure sheaf \mathcal{O} by a morphism. A major theorem says that these systems have only regular singularities. This paper examines the irregularity of an analogue of the Gauss-Manin systems. It consists in the direct image complex $f_+(\mathcal{O}e^g)$ of a \mathcal{D} -module twisted by the exponential of a polynomial g by another polynomial f , where f and g are two polynomials in two variables. The analogue of the Gauss-Manin systems can have irregular singularities (at finite distance and at infinity). We express an invariant associated with the irregularity of these systems at $c \in \mathbb{P}^1$ by the geometry of the map (f, g) .

RÉSUMÉ (*Irrégularité d'un analogue des systèmes de Gauss-Manin*)

Dans la théorie des \mathcal{D} -modules, on définit les systèmes de Gauss-Manin par l'image directe par un morphisme du faisceau structural \mathcal{O} . Un résultat essentiel est leur régularité. On s'intéresse à l'irrégularité d'un analogue des systèmes de Gauss-Manin. Il s'agit de l'image directe $f_+(\mathcal{O}e^g)$ par un polynôme f d'un \mathcal{D} -module tordu par une exponentielle d'un second polynôme g , où f et g sont des polynômes à deux variables. Les analogues des systèmes de Gauss-Manin peuvent avoir des singularités irrégulières. On exprimera alors un invariant attaché à l'irrégularité en $c \in \mathbb{P}^1$ de ces systèmes à l'aide de la géométrie de l'application (f, g) .

Texte reçu le 8 novembre 2004, révisé le 9 mai 2005 et le 4 octobre 2006, accepté le 24 janvier 2006.

CÉLINE ROUCAIROL, Laboratoire Angevin de Recherche en Mathématiques, Université d'Angers • *E-mail* : celine.roucairol@univ-angers.fr • *Url* : <http://math.univ-angers.fr/~roucairol/index.html>

2000 Mathematics Subject Classification. — 32S40, 32C38.

Key words and phrases. — Gauss-Manin connection, irregularity complex, direct image, elementary \mathcal{D} -modules.

Contents

1. Introduction	270
2. Irregularity complex along an hypersurface	272
3. Regular holonomic \mathcal{D} -modules twisted by an exponential	274
4. Topological interpretation of the irregularity of $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$	276
5. When f and g are algebraically independant	281
6. When f and g are algebraically dependant	285
Bibliography	286

1. Introduction

1.1. We denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of regular functions on \mathbb{C}^n and by $\mathcal{D}_{\mathbb{C}^n}$ the sheaf of algebraic differential operators on \mathbb{C}^n .

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial. In \mathcal{D} -module theory, we define the Gauss-Manin systems as the cohomology modules of a complex of $\mathcal{D}_{\mathbb{C}}$ -modules, it being the direct image complex $f_+(\mathcal{O}_{\mathbb{C}^n})$. They are holonomic and regular. These systems coincide with the Gauss-Manin connections outside a finite subset Σ of \mathbb{C} such that $f : f^{-1}(\mathbb{C} \setminus \Sigma) \rightarrow \mathbb{C} \setminus \Sigma$ is a locally trivial fibration.

Now, let $g : \mathbb{C}^n \rightarrow \mathbb{C}$ be another polynomial. We denote by $\mathcal{O}_{\mathbb{C}^n} e^g$ the $\mathcal{D}_{\mathbb{C}^n}$ -module obtained from $\mathcal{O}_{\mathbb{C}^n}$ by twisting by e^g . We are interested in an analogue of the Gauss-Manin systems, it being the direct image complex $f_+(\mathcal{O}_{\mathbb{C}^n} e^g)$.

In [7], F. Maaref calculates the generic fibre of the sheaf of horizontal analytic sections of the systems $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n} e^g))$. It consists in a relative version of a result of C. Sabbah in [12]. Indeed, the generic fiber of the sheaf of horizontal analytic sections of $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n} e^g))$ is canonically isomorphic to the cohomology group with closed support $H_{\Phi_t}^{k+n-1}(f^{-1}(t)^{\text{an}}, \mathbb{C})$, where Φ_t is a family of closed subsets of $f^{-1}(t)$, on which e^{-g} is rapidly decreasing. More precisely, this family is defined as follow. Let $\pi : \widetilde{\mathbb{P}^1} \rightarrow \mathbb{P}^1$ be the oriented real blow-up of \mathbb{P}^1 at infinity. $\widetilde{\mathbb{P}^1}$ is diffeomorphic to $\mathbb{C} \cup S^1$, where S^1 is the circle of directions at infinity. A is in Φ_t if A is a closed subset of $f^{-1}(t)$ and the closure of $g(A)$ in $\mathbb{C} \cup S^1$ intersects S^1 in $] -\frac{1}{2}\pi, \frac{1}{2}\pi[$.

This isomorphism can be better understood using relative cohomology group. F. Maaref shows that for all $t \notin \Sigma$ and for all ρ , such that $\text{Re}(-\rho)$ is sufficiently large, the fibre at t of the sheaf of horizontal analytic sections of $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^n} e^g))$ is isomorphic to the relative cohomology group $H^{k+n-1}(f^{-1}(t)^{\text{an}}, (f^{-1}(t) \cap g^{-1}(\rho))^{\text{an}}, \mathbb{C})$.

Finally, he proves the quasi-unipotence of the corresponding local monodromy.

1.2. The Gauss-Manin systems have only regular singularities. In our case, the complex $f_+(\mathcal{O}_{\mathbb{C}^n} e^g)$ can have irregular singularities. The aim of this paper

is to characterize this irregularity in terms of the geometry of the map (f, g) , when f and g are two polynomials in two variables.

f and g are algebraically independent, we will prove that the complex $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$ is essentially concentrated in degree zero. Then, we can associate to this complex a system of differential equations in one variable. We want to calculate the irregularity number of this system at a point at finite distance and at infinity.

Let \mathbb{X} be a smooth projective compactification of \mathbb{C}^2 such that there exists $F, G : \mathbb{X} \rightarrow \mathbb{P}^1$, two meromorphic maps, which extend f and g . Let us denote by D the divisor $\mathbb{X} \setminus \mathbb{C}^2$. In the following, we identify \mathbb{P}^1 with $\mathbb{C} \cup \{\infty\}$.

Let Γ be the critical locus of (F, G) . We denote by Δ_1 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(\Gamma) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$ where the image is counted with multiplicity and by Δ_2 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(D) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$ where the image is counted with multiplicity.

For all $c \in \mathbb{P}^1$, the germs at (c, ∞) of the support of Δ_1 and Δ_2 are some germs of curves or are empty. Then, we denote by $I_{(c, \infty)}(\Delta_i, \mathbb{P}^1 \times \{\infty\})$ the intersection number of the cycles Δ_i and $\mathbb{P}^1 \times \{\infty\}$. If the germ at (c, ∞) of Δ_i is empty, this number is equal to 0.

THEOREM 1. — *Let $f, g \in \mathbb{C}[x, y]$ be algebraically independent. Let $c \in \mathbb{P}^1$. Then, the irregularity number at c of the system $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}e^g))$ is equal to*

$$I_{(c, \infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}) + I_{(c, \infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}).$$

When $c \in \mathbb{C}$, we can prove that the germ at (c, ∞) of Δ_2 is empty. Moreover, the germ at (c, ∞) of Δ_1 coincide with the one of the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(f, g)(\tilde{\Gamma}) \setminus \{c\} \times \mathbb{C}$, where $\tilde{\Gamma}$ is the critical locus of (f, g) .

1.3. In general, we do not know how to calculate directly the irregularity number of a system associated with $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$. The notion of irregularity complex along an hypersurface defined by Z. Mebkhout (see [9] and [10]) is the appropriate tool to express the irregularity of $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$ (see §2). Indeed, this irregularity complex along an hypersurface is a generalization of the irregularity number at a point of a system of differential equations in one variable. Moreover, Z. Mebkhout proves a theorem of commutation between the direct image functor and the irregularity functor (see Theorem 2.4). Then, the irregularity number at $c \in \mathbb{P}^1$ of the system of differential equations associated with $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$ can be expressed with the help of an irregularity complex of a \mathcal{D} -module in two variables along a curve.

In the general case where f and g are not necessarily algebraically independent, the complex $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$ is not necessarily concentrated in degree 0. Then, we want to calculate the alternative sum of the irregularity number at $c \in \mathbb{P}^1$ of the systems $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^2}e^g))$. This irregularity number IR_c is equal

to the Euler characteristic of a complex of vector spaces over \mathbb{C} , it being the irregularity complex of $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$ at $c \in \mathbb{P}^1$. When f and g are algebraically independent, this number coincide with the irregularity number of the system $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}e^g))$. Then, we can prove that the irregularity number IR_c is equal to

$$-\chi(\mathbb{R}\Gamma(F^{-1}(c) \cap G^{-1}(\infty), \text{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G))),$$

where $\text{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)$ is the irregularity complex of $\mathcal{O}_{\mathbb{X}}[*D]e^G$ along $F^{-1}(c)$.

Then, according to C. Sabbah [12], we know that, for $x \in F^{-1}(c) \cap G^{-1}(\infty)$, the Euler characteristic of $(\text{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G))_x$ is equal to the Euler characteristic of the fiber $f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho) \cap B(x, \epsilon)$, where ϵ and η are small enough and $|\rho|$ is big enough. This result is stated in Theorem 3.4, §3 in terms of complex of nearby cycles.

Then, we have to globalize the situation (see §4). First of all, we prove that for η small enough and R big enough,

$$g : f^{-1}(D^*(c, \eta)) \cap g^{-1}(\{|\rho| > R\}) \longrightarrow \{|\rho| > R\}$$

is a locally trivial fibration. Then, the irregularity number IR_c is equal to the opposite of the Euler characteristic of its fiber $f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)$. This result hold in the general case where f and g are not necessarily algebraically independent.

Then, we have to study the topology of this fiber. We have to distinguished the case where f and g are algebraically independent (see §5) and the one where they are algebraically dependant (see §6).

2. Irregularity complex along an hypersurface

We will use the definition of regularity given by Z. Mebkhout [9], [10]. First of all, we recall the definition of irregularity complex of analytic \mathcal{D} -modules. Then, we define the notion of irregularity complex for algebraic \mathcal{D} -modules. Here, we have to take into account the behaviour of these modules at infinity. Moreover, we state major theorems on irregularity: the positivity theorem, the stability of the category of complex of regular holonomic \mathcal{D} -modules (analytic) by direct image by a proper map and the comparison theorem of Grothendieck.

2.1. The analytic case. — Let X be a smooth analytic variety over \mathbb{C} . In this section, \mathcal{D}_X denotes the sheaf of analytic differential operators on X .

Let Z be an analytic closed subset of X . Denote by i the canonical inclusion of $X \setminus Z$ in X . Let \mathcal{M}^\bullet be a bounded complex of analytic \mathcal{D}_X -modules with holonomic cohomology.

DEFINITION 2.1. — We define the *irregularity complex* of \mathcal{M}^\bullet along Z as the complex

$$\begin{aligned} \mathrm{IR}_Z(\mathcal{M}^\bullet) &:= R\Gamma_Z(DR(\mathcal{M}^\bullet[*Z]))[+1] \\ &:= \mathrm{cone}(DR(\mathcal{M}[*Z]) \rightarrow Ri_*i^{-1}(DR(\mathcal{M}^\bullet[*Z]))). \end{aligned}$$

According to the constructibility theorem (cf. [6] and [11]), this complex is a bounded complex of constructible sheaves on X with support in Z . Then, we can define the covariant exact functor IR_Z between the category of bounded complexes of \mathcal{D}_X -modules with holonomic cohomology and the category of bounded complexes of constructible sheaves on X with support in Z .

DEFINITION 2.2. — \mathcal{M}^\bullet is said to be *regular* if its irregularity complex along all hypersurfaces of X is zero.

In one variable, the previous definition of regularity generalises the notion of regular singular point of a differential equation which is characterized by the annulation of the irregularity number (Fuchs Theorem). Indeed, irregularity complex along an hypersurface generalizes irregularity number in the case of one variable. According to Z. Mebkhout [9], [10], the characteristic cycle of the irregularity complex of a holonomic \mathcal{D} -module along an hypersurface is positive.

THEOREM 2.3 (Positivity Theorem). — *If Z is an hypersurface of X and \mathcal{M} is a holonomic \mathcal{D}_X -module, the complex $\mathrm{IR}_Z(\mathcal{M})$ is perverse on Z .*

The category of complexes of \mathcal{D} -modules with regular holonomic cohomology is stable by proper direct image. Let us state the theorem which proves this stability (see [9] and Proposition 3.6-4 of [10]). It will be a major tool in this paper.

Let $\pi : X \rightarrow Y$ be a proper morphism of smooth analytic varieties over \mathbb{C} . Let T be a hypersurface of Y .

THEOREM 2.4. — *Let \mathcal{M}^\bullet be a bounded complex of analytic \mathcal{D}_X -modules with holonomic cohomology. We have an isomorphism*

$$\mathrm{IR}_T(\pi_+(\mathcal{M}^\bullet))[\dim Y] \simeq R\pi_*(\mathrm{IR}_{\pi^{-1}(T)}(\mathcal{M}^\bullet))[\dim X].$$

2.2. The algebraic case. — Let X be a smooth affine variety over \mathbb{C} . In this section, \mathcal{D}_X denotes the sheaf of algebraic differential operators on X . Denote by $j : X \rightarrow \mathbb{P}^n$ an immersion of X in a projective space. Let Z be a locally closed subvariety of \mathbb{P}^n and \mathcal{M}^\bullet a bounded complex of algebraic \mathcal{D}_X -modules with holonomic cohomology.

DEFINITION 2.5. — We define the *irregularity complex* of \mathcal{M}^\bullet along Z as

$$\mathrm{IR}_Z(j_+(\mathcal{M}^\bullet)) := \mathrm{IR}_{Z^{\mathrm{an}}}(j_+(\mathcal{M}^\bullet)^{\mathrm{an}}),$$

where Z^{an} denotes the analytic variety associated with Z and $j_+(\mathcal{M}^\bullet)^{\text{an}}$ denotes the complex of analytic \mathcal{D} -modules associated with $j_+\mathcal{M}$.

DEFINITION 2.6. — \mathcal{M}^\bullet is said to be *regular* if its irregularity complex along all subvariety of \mathbb{P}^n is zero.

This condition of regularity does not depend on the choice of the immersion j (see Proposition 9.0-4 in [10]).

The Definition 2.6 of regular holonomic complex was motivated by Grothendieck's Comparison Theorem [4] and Deligne's Comparison Theorem [2]. As shown in [8], the Comparison Theorem of Grothendieck is a consequence of the following theorem:

THEOREM 2.7. — *The structure sheaf \mathcal{O}_X is regular in the sense of Definition 2.6.*

Concerning to the stability of regularity under direct image, Theorem 2.4 allows to prove the following theorem (see Theorem 9.0-7 of [10]):

THEOREM 2.8. — *The category of complexes of \mathcal{D} -modules with regular holonomic cohomology is stable under direct image.*

NOTATION 2.9. — We denote by $\text{IR}_Z^k(j_+(\mathcal{M}^\bullet))$ the k -th space of cohomology of the complex $\text{IR}_Z(j_+(\mathcal{M}^\bullet))$.

REMARK 2.10. — We are interested in the irregularity of the direct image complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$. This is a complex of $\mathcal{D}_{\mathbb{C}}$ -modules in one variable, with holonomic cohomology. According to the Definition 2.5 of irregularity complex, we have to consider an immersion $j : \mathbb{C} \rightarrow \mathbb{P}^1$. Let $c \in \mathbb{P}^1$. We want to examine the complex $\text{IR}_c(j_+f_+(\mathcal{O}_{\mathbb{C}^n} e^g))$. As this complex has its support in c , we want to compute its Euler characteristic

$$\chi(\text{IR}_c(j_+f_+(\mathcal{O}_{\mathbb{C}^n} e^g))_c).$$

In the following, we will denote this number by IR_c .

3. Regular holonomic \mathcal{D} -modules twisted by an exponential

3.1. Definitions. — Let X be an algebraic variety over \mathbb{C} . We denote by \mathcal{O}_X the sheaf of regular functions on X . We identify \mathbb{P}^1 to $\mathbb{C} \cup \{\infty\}$. Let $g : X \rightarrow \mathbb{P}^1$ be a meromorphic function on X .

DEFINITION 3.1. — We define the \mathcal{D}_X -module

$$\mathcal{O}_X[*g^{-1}(\infty)]e^g$$

as a \mathcal{D}_X -module which is isomorphic to $\mathcal{O}_X[*g^{-1}(\infty)]$ as \mathcal{O}_X -module; the action of ξ , vector field on an open subset of X , on a section he^g of $\mathcal{O}_X[*g^{-1}(\infty)]e^g$ is defined by

$$\xi(he^g) = \xi(h)e^g + h\xi(g)e^g.$$

Let \mathcal{M} be a holonomic \mathcal{D}_X -module.

DEFINITION 3.2. — We define the \mathcal{D}_X -module $\mathcal{M}[*g^{-1}(\infty)]e^g$ as the \mathcal{D}_X -module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[*g^{-1}(\infty)]e^g$.

REMARK 3.3. — The \mathcal{D}_X -module $\mathcal{O}_X[*g^{-1}(\infty)]e^g$ is the direct image by an open immersion of a vector bundle with integrable connection. Then, it is a holonomic \mathcal{D}_X -module as algebraic direct image of a holonomic \mathcal{D} -module. The \mathcal{D}_X -module $\mathcal{M}[*g^{-1}(\infty)]e^g$ is a holonomic left \mathcal{D}_X -module as tensor product of two holonomic left \mathcal{D}_X -modules.

We have analogous definitions in the analytic case. We just have to transpose in the analytic setting.

3.2. On irregularity of regular holonomic \mathcal{D} -modules twisted by an exponential. — Let X be a complex analytic manifold and let $f, g : X \rightarrow \mathbb{C}$ be two analytic functions. Assume that \mathcal{M} is a regular holonomic \mathcal{D}_X -module (analytic). Generally, $\mathcal{M}[1/g]e^{1/g}$ is an irregular \mathcal{D}_X -module.

We want to relate the irregularity complex of this module along $f^{-1}(0)$ with some topological data.

LEMMA 3.4. — *The complex $\mathrm{IR}_{f=0}(\mathcal{M}[1/g]e^{1/g})$ and the complex of nearby cycles $\Psi_g(\mathrm{DR}(\mathcal{M}[1/f]))$ have the same characteristic function on $f^{-1}(0) \cap g^{-1}(0)$.*

Proof of Lemma 3.4. — According to Corollary 5.2 of [12], this lemma is true in the case where f and g are the same function. Assume that f and g are not equal. Then, using the case where the two functions are equal, we remark that it is sufficient to prove that the complex $\mathrm{IR}_{f=0}(\mathcal{M}[1/g]e^{1/g})$ and the complex $\mathrm{IR}_{g=0}(\mathcal{M}[1/fg]e^{1/g})$ have the same characteristic function on $f^{-1}(0) \cap g^{-1}(0)$.

- Let us first prove that

$$\mathrm{IR}_{f=0}(\mathcal{M}[1/g]e^{1/g}) = R\Gamma_{f=0}(\mathrm{IR}_{g=0}(\mathcal{M}[1/fg]e^{1/g})).$$

Let X^* denote $X \setminus g^{-1}(0)$ and η be the inclusion of X^* in X . By definition, we have

$$\begin{aligned} \mathrm{IR}_{g=0}(\mathcal{M}[1/fg]e^{1/g}) &= \mathrm{cone}(\mathrm{DR}(\mathcal{M}[1/fg]e^{1/g}) \rightarrow R\eta_*\eta^{-1}\mathrm{DR}(\mathcal{M}[1/fg]e^{1/g})) \\ &= \mathrm{cone}(\mathrm{DR}(\mathcal{M}[1/fg]e^{1/g}) \rightarrow R\eta_*(\mathrm{DR}(\mathcal{M}[1/f])|_{X^*})). \end{aligned}$$

Now, consider the following diagram :

$$\begin{array}{ccc}
 X^* & \xrightarrow{\eta} & X \\
 \uparrow j & & \uparrow j' \\
 X^* \setminus f^{-1}(0) & \xrightarrow{\eta'} & X \setminus f^{-1}(0).
 \end{array}$$

Then, since \mathcal{M} is regular, according to the Definition 2.6 of regularity, we have

$$\begin{aligned}
 R\eta_*(DR(\mathcal{M}[1/f])|_{X^*}) &= R\eta_*Rj_*(DR(\mathcal{M})|_{X^* \setminus f^{-1}(0)}) \\
 &= Rj'_*R\eta'_*(DR(\mathcal{M})|_{X^* \setminus f^{-1}(0)}).
 \end{aligned}$$

As $R\Gamma_{f=0}Rj'_* = 0$, we obtain

$$\begin{aligned}
 R\Gamma_{f=0}(\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g})) &= R\Gamma_{f=0}(DR(\mathcal{M}[1/fg]e^{1/g}))[+1] \\
 &= IR_{f=0}(\mathcal{M}[1/g]e^{1/g}).
 \end{aligned}$$

• Then, we are led to show that the complex $R\Gamma_{f=0}(\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g}))$ and the complex $\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g})$ have the same characteristic function on $f^{-1}(0) \cap g^{-1}(0)$. Using the following distinguished triangle,

$$\begin{array}{ccc}
 & Rj'_*j'^{-1}(\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g})) & \\
 \swarrow [+1] & & \nwarrow \\
 R\Gamma_{f=0}(\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g})) & \longrightarrow & \mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g}),
 \end{array}$$

it is sufficient to show that the characteristic function of the complex $Rj'_*j'^{-1}(\mathbb{R}_{g=0}(\mathcal{M}[1/fg]e^{1/g}))$ is zero on $f^{-1}(0) \cap g^{-1}(0)$. Now, if \mathcal{F} is a constructible sheaf on X and $x \in f^{-1}(0)$,

$$\chi((Rj'_*j'^{-1}\mathcal{F})_x) = \chi((\mathbb{D}(j'_*j'^{-1}\mathbb{D}\mathcal{F}))_x) = \chi((j'_*j'^{-1}\mathbb{D}\mathcal{F})_x) = 0$$

(\mathbb{D} is the Verdier duality, see [1]). □

4. Topological interpretation of the irregularity of $f_+(\mathcal{O}_{\mathbb{C}^2}e^g)$

4.1. Notations. — Let $f, g : \mathbb{C}^2 \rightarrow \mathbb{C}$ be two polynomials. Let \mathbb{X} be a smooth projective compactification of \mathbb{C}^2 such that there exists $F, G : \mathbb{X} \rightarrow \mathbb{P}^1$ two meromorphic maps which extend f and g . In view to construct \mathbb{X} , F and G , we consider an immersion of \mathbb{C}^2 in \mathbb{P}^2 and we define a rational map (\tilde{f}, \tilde{g}) on \mathbb{P}^2 which extends the map (f, g) . Then, after a finite number of blowing ups, we

lift the indeterminacies of the rational map (\tilde{f}, \tilde{g}) . In the following, we fix such a compactification and use the following notations:

$$\begin{array}{ccc}
 \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C} \\
 \downarrow i & & \downarrow j \\
 \mathbb{X} & \xrightarrow{F} & \mathbb{P}^1
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{C}^2 & \xrightarrow{g} & \mathbb{C} \\
 \downarrow i & & \downarrow j \\
 \mathbb{X} & \xrightarrow{G} & \mathbb{P}^1.
 \end{array}$$

4.2. Two fibration theorems

LEMMA 4.1. — *Let $c \in \mathbb{C}$. There exists $R > 0$ big enough such that*

$$g : g^{-1}(\{|\rho| > R\}) \setminus (f^{-1}(c) \cap g^{-1}(\{|\rho| > R\})) \longrightarrow \{|\rho| > R\}$$

is a locally trivial fibration.

Proof. — Let \mathcal{S} be an algebraic Whitney stratification of \mathbb{X} such that D and $F^{-1}(c)$ are union of strata. According to the Sard’s theorem, there exists U , a dense Zariski open subset of \mathbb{P}^1 , such that $G : G^{-1}(U) \rightarrow U$ is transverse to the Whitney stratification \mathcal{S}' of $G^{-1}(U)$ induced by \mathcal{S} .

According to the first isotopy lemma of Thom-Mather, $G : G^{-1}(U) \rightarrow U$ is a locally trivial fibration with respect to \mathcal{S}' . Then, we choose $R > 0$ big enough such that $\{|\rho| > R\} \subset U$. □

Using the inclusion $j : \mathbb{C} \rightarrow \mathbb{P}^1$, we identify \mathbb{P}^1 to $\mathbb{C} \cup \{\infty\}$. If $c \in \mathbb{C}$, we denote by $D(c, \eta)$ the open disc in \mathbb{C} centered at c of radius η . Let

$$D(\infty, \eta) = \{z \in \mathbb{C}; |z| > 1/\eta\} \cup \{\infty\}.$$

If $c \in \mathbb{P}^1$, we denote by $D^*(c, \eta) \subset \mathbb{C}$ the set $D(c, \eta) \setminus \{c\}$.

LEMMA 4.2. — *Let $c \in \mathbb{P}^1$. There exists η small enough and R big enough such that*

$$g : f^{-1}(D^*(c, \eta)) \cap g^{-1}(\{|\rho| > R\}) \longrightarrow \{|\rho| > R\}$$

is a locally trivial fibration.

Proof. — According to the first isotopy lemma of Thom-Mather, we want to find a Whitney stratification \mathcal{S}' of $F^{-1}(\overline{D(c, \eta)})$ such that $D \cap F^{-1}(\overline{D(c, \eta)})$, $F^{-1}(c)$ and $F^{-1}(S(c, \eta))$ are union of strata and such that the morphism

$$G : F^{-1}(\overline{D(c, \eta)}) \cap G^{-1}(\{|\rho| > R\}) \longrightarrow \{|\rho| > R\}$$

is transverse to \mathcal{S}' .

Let \mathcal{S} be an algebraic Whitney stratification of \mathbb{X} such that $F^{-1}(c)$ and D are union of strata. For $\eta > 0$, we denote by \mathcal{T} the real analytic stratification $\{F^{-1}(S(c, \eta)), F^{-1}(D(c, \eta))\}$. For $\eta > 0$ small enough, \mathcal{S} and \mathcal{T} are transverse. Let \mathcal{S}' be the real analytic stratification $\mathcal{S} \cap \mathcal{T}$.

Now, let us prove that for η small enough and R big enough, the map

$$G : F^{-1}(\overline{D(c, \eta)}) \cap G^{-1}(\{|\rho| > R\}) \longrightarrow \{|\rho| > R\}$$

is transverse to S' .

- If $S' = S \cap F^{-1}(S(c, \eta))$, for a $S \in \mathcal{S}$, we have to prove that for η small enough and R big enough, $G|_{S \cap F^{-1}(S(c, \eta))}$ is a submersion. It is sufficient to prove that for η small enough and R big enough, $F^{-1}(c')$ is transverse to $G|_{S'}^{-1}(\rho)$, where $|\rho| > R$ and $c' \in S(c, \eta)$.

Let Γ_S be the critical locus of $(F, G)|_S$ and $\Delta_S = (F, G)(\Gamma_S)$ be the discriminant variety of $F|_S$ and $G|_S$. We denote by Δ'_S the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $\Delta_S \cap \mathbb{C}^2$. In our case, the dimension of Δ'_S is always less than 1. Then we argue by the absurd.

- If $S' = S \cap F^{-1}(D(c, \eta))$, for a $S \in \mathcal{S}$, as for R big enough, the map $G : S \cap G^{-1}(\{|\rho| > R\}) \rightarrow \{|\rho| > R\}$ is a submersion, the map

$$G : S' \cap G^{-1}(\{|\rho| > R\}) \longrightarrow \{|\rho| > R\}$$

is also a submersion. □

4.3. Topological interpretation of the irregularity of $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$

In this section, we describe the relation between the irregularity of the complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ and the fibre $f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)$ given by Lemma 4.2.

THEOREM 4.3. — *Let $c \in \mathbb{P}^1$. For η small enough and $|\rho|$ big enough,*

$$\text{IR}_c = -\chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)).$$

This theorem can be proved in two steps.

LEMMA 4.4. — *One has*

$$\text{IR}_c = -\chi(\mathbb{R}\Gamma(F^{-1}(c) \cap G^{-1}(\infty), \psi_{1/G}(DR(\mathcal{O}_{\mathbb{X}}[*D \cup F^{-1}(c)]))))).$$

LEMMA 4.5. — *For η small enough, $|\rho|$ big enough, $\chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho))$ is equal to*

$$\chi(\mathbb{R}\Gamma(F^{-1}(c) \cap G^{-1}(\infty), \psi_{1/G}(DR(\mathcal{O}_{\mathbb{X}}[*D \cup F^{-1}(c)]))))).$$

Proof of Lemma 4.4. — This proof consists in applying Lemma 3.4 on irregularity of regular holonomic \mathcal{D} -modules twisted by an exponential and in globalizing the situation.

- First, we want to prove that

$$\text{IR}_c = -\chi(\mathbb{R}\Gamma(F^{-1}(c) \cap G^{-1}(\infty), \text{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)^{\text{an}})).$$

According to Definition 2.5 and Theorem 2.4, we have

$$\begin{aligned} \text{IR}_c(j_+ f_+(\mathcal{O}_{\mathbb{C}^2} e^g)) &= \text{IR}_c(F_+(\mathcal{O}_{\mathbb{X}}[*D]e^G)) \\ &= RF_*(\text{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G))[+1]. \end{aligned}$$

Then, $\mathrm{IR}_c = -\chi(\mathbb{R}\Gamma(F^{-1}(c), \mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)^{\mathrm{an}}))$. So we have to prove that the support of $\mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)^{\mathrm{an}}$ is included in $F^{-1}(c) \cap G^{-1}(\infty)$.

Let $x \notin G^{-1}(\infty)$. Then, G is holomorphic in a neighbourhood of x and $(\mathcal{O}_{\mathbb{X}}[*D]e^G)_x^{\mathrm{an}}$ is isomorphic to $(\mathcal{O}_{\mathbb{X}}[*D])_x^{\mathrm{an}}$. According to Theorem 2.7, $\mathcal{O}_{\mathbb{X}}$ is regular. Then $\mathcal{O}_{\mathbb{X}}[*D]$ is also regular and $(\mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)^{\mathrm{an}})_x = 0$.

- According to Theorem 3.4, the complexes $\mathrm{IR}_{F^{-1}(c)}(\mathcal{O}_{\mathbb{X}}[*D]e^G)^{\mathrm{an}}$ and $\psi_{1/G}(DR(\mathcal{O}_{\mathbb{X}}[*D] \cup F^{-1}(c)))$ have the same characteristic function on $F^{-1}(c) \cap G^{-1}(\infty)$.

We conclude using the following lemma.

LEMMA. — *Let X be an algebraic curve over \mathbb{C} . Let \mathcal{F}_1^\bullet and \mathcal{F}_2^\bullet be two constructible complexes on X which have the same characteristic function on X . Then*

$$\chi(\mathbb{R}\Gamma(X, \mathcal{F}_1^\bullet)) = \chi(\mathbb{R}\Gamma(X, \mathcal{F}_2^\bullet)).$$

Proof. — Let $Z \subset X$ be a finite subset of points such that $\mathcal{F}_{1|X \setminus Z}$ and $\mathcal{F}_{2|X \setminus Z}$ are some local systems \mathcal{L}_1 and \mathcal{L}_2 on $X \setminus Z$. For $i = 1, 2$, we have the following distinguished triangle

$$\mathbb{R}\Gamma_Z(\mathcal{F}_i^\bullet) \longrightarrow \mathcal{F}_i^\bullet \longrightarrow \mathbb{R}j_*j^{-1}(\mathcal{F}_i^\bullet) \xrightarrow{[+1]},$$

where j is the inclusion of $X \setminus Z$ in X . Then

$$\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_i^\bullet)) \longrightarrow \mathbb{R}\Gamma(X, \mathcal{F}_i^\bullet) \longrightarrow \mathbb{R}\Gamma(X, \mathbb{R}j_*j^{-1}(\mathcal{F}_i^\bullet)) \xrightarrow{[+1]}.$$

- First, we want to prove that

$$\chi(\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_1^\bullet))) = \chi(\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_2^\bullet))).$$

We have $\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_i^\bullet)) = \mathbb{R}\Gamma(Z, \mathbb{R}\Gamma_Z(\mathcal{F}_i^\bullet))$. In the same way as at the end of the proof of Lemma 3.4, using the Verdier duality (see [1]), we can prove that $\mathbb{R}\Gamma(Z, \mathbb{R}\Gamma_Z(\mathcal{F}_i^\bullet))$ and $\mathbb{R}\Gamma(X, \mathcal{F}_i^\bullet)$ have the same characteristic function on Z . Then, $\mathbb{R}\Gamma(Z, \mathbb{R}\Gamma_Z(\mathcal{F}_1^\bullet))$ and $\mathbb{R}\Gamma(Z, \mathbb{R}\Gamma_Z(\mathcal{F}_2^\bullet))$ have the same characteristic function on Z . As Z is a finite number of points, it is obvious that $\chi(\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_1^\bullet))) = \chi(\mathbb{R}\Gamma(X, \mathbb{R}\Gamma_Z(\mathcal{F}_2^\bullet)))$.

- Now, let us prove that

$$\chi(\mathbb{R}\Gamma(X, \mathbb{R}j_*j^{-1}(\mathcal{F}_1^\bullet))) = \chi(\mathbb{R}\Gamma(X, \mathbb{R}j_*j^{-1}(\mathcal{F}_2^\bullet))).$$

As $\chi(\mathbb{R}\Gamma(X, \mathbb{R}j_*j^{-1}(\mathcal{F}_i^\bullet))) = \chi(\mathbb{R}\Gamma(X \setminus Z, \mathcal{L}_i))$, we have to prove that

$$\chi(\mathbb{R}\Gamma(X \setminus Z, \mathcal{L}_1)) = \chi(\mathbb{R}\Gamma(X \setminus Z, \mathcal{L}_2)),$$

where \mathcal{L}_1 and \mathcal{L}_2 are two local systems with locally the same rank on $X \setminus Z$. Since X is an algebraic curve, a finite covering of $X \setminus Z$ by contractible open subsets and with contractible intersections can be built directly. We conclude using the theorem of Mayer-Vietoris.

Then, $\chi(\mathbb{R}\Gamma(X, \mathcal{F}_1^\bullet)) = \chi(\mathbb{R}\Gamma(X, \mathcal{F}_2^\bullet))$. □ □

Proof of Lemma 4.5. — Let us prove Lemma 4.5. A similar argument can be found in [3], p. 125. Denote by \mathcal{F}^\bullet the complex $\psi_{1/G}(DR(\mathcal{O}_{\mathbb{X}}[*D \cup F^{-1}(c)]))$.

• Let η_2 small enough such that $G : G^{-1}(\widetilde{D^*(\infty, \eta_2)}) \rightarrow D^*(\infty, \eta_2)$ is a locally trivial fibration. We denote by $\widetilde{D^*(\infty, \eta_2)}$ the universal covering of $D^*(\infty, \eta_2)$. Let (E, π, \widetilde{G}) be the fiber product over $D^*(\infty, \eta_2)$ of $G^{-1}(D^*(\infty, \eta_2))$ and $\widetilde{D^*(\infty, \eta_2)}$. Then, we have the following diagram

$$\begin{array}{ccccc} G^{-1}(\infty) & \xleftarrow{j} & \mathbb{X} & \xleftarrow{i} & G^{-1}(D^*(\infty, \eta_2)) & \xleftarrow{\pi} & E \\ & & & & \downarrow G & & \downarrow \\ & & & & D^*(\infty, \eta_2) & \xleftarrow{\quad} & \widetilde{D^*(\infty, \eta_2)}. \end{array}$$

By definition,

$$\mathcal{F}^\bullet = j^{-1}R(i \circ \pi)_*(i \circ \pi)^{-1}(DR(\mathcal{O}_{\mathbb{X}}[*D \cup F^{-1}(c)])).$$

• Let $\alpha : \mathbb{X} \setminus (F^{-1}(c) \cap D) \rightarrow \mathbb{X}$ open inclusion. As $\mathcal{O}_{\mathbb{X}}$ is regular in the sense of Definition 2.5 (Theorem 2.7),

$$DR(\mathcal{O}_{\mathbb{X}}[*D \cup F^{-1}(c)]) = R\alpha_*\alpha^{-1}(\underline{\mathbb{C}}_{\mathbb{X}}).$$

• Let $Z = F^{-1}(c) \cap G^{-1}(\infty)$, $Z_{\eta_1, \eta_2} = F^{-1}(D(c, \eta_1)) \cap G^{-1}(D(\infty, \eta_2))$ and $Z_{\eta_1, \rho} = F^{-1}(D(c, \eta_1)) \cap G^{-1}(\rho)$. As Z is closed, for all $k \in \mathbb{N}$,

$$\mathbb{R}^k\Gamma(Z, \mathcal{F}^\bullet) = \varinjlim_{\substack{Z \subset U \\ \text{open}}} \mathbb{R}^k\Gamma(U, R(i \circ \pi)_*(i \circ \pi)^{-1}(R\alpha_*\alpha^{-1}(\underline{\mathbb{C}}_{\mathbb{X}}))).$$

As Z is compact, the family $\{Z_{\eta_1, \eta_2}\}_{\eta_1, \eta_2 > 0}$ is a fundamental system of neighbourhoods of Z . Then,

$$\mathbb{R}^k\Gamma(Z, \mathcal{F}^\bullet) = \varinjlim_{Z_{\eta_1, \eta_2}} \mathbb{R}^k\Gamma(Z_{\eta_1, \eta_2}, R(i \circ \pi)_*(i \circ \pi)^{-1}(R\alpha_*\alpha^{-1}(\underline{\mathbb{C}}_{\mathbb{X}}))).$$

Now, we want to prove that, for η_1 and η_2 small enough,

$$\begin{aligned} \mathbb{R}^k\Gamma(Z_{\eta_1, \eta_2}, R(i \circ \pi)_*(i \circ \pi)^{-1}(R\alpha_*\alpha^{-1}(\underline{\mathbb{C}}_{\mathbb{X}}))) \\ = \mathbb{R}^k\Gamma(f^{-1}(D^*(c, \eta_1)) \cap g^{-1}(\rho), \alpha^{-1}\underline{\mathbb{C}}_{\mathbb{X}}), \end{aligned}$$

where $|\rho|$ is big enough. Let Σ be a Whitney stratification associated with the constructible sheaf $R\alpha_*\alpha^{-1}(\underline{\mathbb{C}}_{\mathbb{X}})$. Then, $F^{-1}(c)$ and D are union of strata. According to the proof of Lemma 4.2, for η_1 and η_2 small enough,

$$G : F^{-1}(D(c, \eta_1)) \cap G^{-1}(D^*(\infty, \eta_2)) \longrightarrow D^*(\infty, \eta_2)$$

is a locally trivial fibration with respect to Σ . Then, there exists a homotopy equivalence $p : (i \circ \pi)^{-1}(Z_{\eta_1, \eta_2}) \rightarrow Z_{\eta_1, \rho}$ compatible with Σ . Thus, while adapting Proposition I.3-4 of [11] to constructible sheaves,

$$\begin{aligned} \mathbb{R}\Gamma(Z_{\eta_1, \eta_2}, R(i \circ \pi)_*(i \circ \pi)^{-1}(R\alpha_*\alpha^{-1}(\mathbb{C}_{\mathbb{X}}))) &= \mathbb{R}\Gamma(Z_{\eta_1, \rho}, R\alpha_*\alpha^{-1}(\mathbb{C}_{\mathbb{X}})) \\ &= \mathbb{R}\Gamma(\alpha^{-1}(Z_{\eta_1, \rho}), \alpha^{-1}(\mathbb{C}_{\mathbb{X}})) = \mathbb{R}\Gamma(f^{-1}(D^*(c, \eta_1)) \cap g^{-1}(\rho), \alpha^{-1}(\mathbb{C}_{\mathbb{X}})). \end{aligned}$$

Then, $\chi(\mathbb{R}\Gamma(Z, \mathcal{F}^\bullet)) = \chi(f^{-1}(D^*(c, \eta_1)) \cap g^{-1}(\rho))$. □

5. When f and g are algebraically independant

Let $f, g \in \mathbb{C}[x, y]$ be two polynomials which are algebraically independant. In this section, we will prove that the complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is essentially concentrated in degree 0. Finally we obtain a formula for the irregularity number at $c \in \mathbb{P}^1$ in terms of some geometric data associated with f and g .

5.1. The complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is essentially concentrated in degree zero

PROPOSITION 5.1. — *The complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is concentrated in degree zero except at a finite number of points.*

Proof

- First of all, we recall the result of F. Maaref [7] about the generic fibre of the sheaf of horizontal analytic sections of $\mathcal{H}^{k-1}(f_+(\mathcal{O}_{\mathbb{C}^2} e^g))$.

THEOREM 5.2. — *There exists a finite subset Σ of \mathbb{C} such that for all $c \in \mathbb{C} \setminus \Sigma$ and all $\rho \in \mathbb{C}$, such that $Re(-\rho)$ is big enough,*

$$i_c^+ \mathcal{H}^{k-1}(f_+(\mathcal{O}_{\mathbb{C}^2} e^g)) \simeq H^k(f^{-1}(c), (f, g)^{-1}(c, \rho), \mathbb{C}),$$

where i_c is the inclusion of $\{c\}$ in \mathbb{C} .

- For all $c, \rho \in \mathbb{C}$, we have the long exact sequence of relative cohomology:

$$\begin{aligned} 0 \rightarrow H^0(f^{-1}(c), (f, g)^{-1}(c, \rho), \mathbb{C}) &\rightarrow H^0(f^{-1}(c), \mathbb{C}) \xrightarrow{\alpha} H^0((f, g)^{-1}(c, \rho), \mathbb{C}) \\ &\rightarrow H^1(f^{-1}(c), (f, g)^{-1}(c, \rho), \mathbb{C}) \rightarrow H^1(f^{-1}(c), \mathbb{C}) \xrightarrow{\beta} H^1((f, g)^{-1}(c, \rho), \mathbb{C}) \\ &\rightarrow H^2(f^{-1}(c), (f, g)^{-1}(c, \rho), \mathbb{C}) \rightarrow 0. \end{aligned}$$

We want to prove that

$$H^k(f^{-1}(c), (f, g)^{-1}(c, \rho)) = 0$$

for all $k \neq 1$. As $H^1((f, g)^{-1}(c, \rho), \mathbb{C}) = 0$, it is enough to prove that α is injective. Then, it is sufficient to prove that the fibre $g^{-1}(\rho)$ intersects all the connected components of $f^{-1}(c)$.

Let $(F, G) : \mathbb{X} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a compactification of $(f, g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. As $(F, G) : \mathbb{X} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is proper, we know that its image is closed in $\mathbb{P}^1 \times \mathbb{P}^1$.

Moreover, as f and g are algebraically independent, (F, G) is necessarily surjective.

According to the Stein Factorization Theorem (see [5], Corollary 11.5, p. 280), there exists $F' : X \rightarrow Y$, surjective morphism of projective varieties with connected fibres and a finite morphism $\Psi : Y \rightarrow \mathbb{P}^1$, such that $F = \psi \circ F'$. As (F, G) is surjective, (F', G) is also surjective. Then, for all $(c, \rho) \in \mathbb{P}^1 \times \mathbb{P}^1$, $G^{-1}(\rho)$ intersects all the connected components of $F^{-1}(c)$.

Furthermore, there exists $\Sigma \subset \mathbb{C}$ finite subset such that for all $c \in \mathbb{C} \setminus \Sigma$, the fibre $F^{-1}(c)$ is the union of $f^{-1}(c)$ with a finite number of points. Then, for a such c , there exists $\Sigma_c \subset \mathbb{C}$ finite subset such that for all $\rho \in \mathbb{C} \setminus \Sigma_c$, $g^{-1}(\rho)$ intersects all the connected components of $f^{-1}(c)$. Then, for all $(c, \rho) \in \mathbb{C}^2$ except a finite number, the fibre $g^{-1}(\rho)$ intersects all the connected components of $f^{-1}(c)$.

- Then, according to Theorem 5.2, for all $c \in \mathbb{C} \setminus \Sigma$, for all $k \neq 0$, $i_c^+(\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^2} e^g))) = 0$. As $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^2} e^g))$ is an integrable connection except at a finite number of points, we have that, for all $c \in \mathbb{C}$ except a finite number, $\mathcal{H}^k(f_+(\mathcal{O}_{\mathbb{C}^2} e^g))_c = 0$, if $k \neq 0$. Thus, $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is essentially concentrated in degree 0. □

COROLLARY 5.3. — *For all $c \in \mathbb{P}^1$, the complex $\mathrm{IR}_c(j_+ f_+(\mathcal{O}_{\mathbb{C}^2} e^g))$ is concentrated in degree 0.*

Proof. — Let $c \in \mathbb{P}^1$. Denote by \mathcal{M}^\bullet the complex $j_+ f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$. For $k \neq 0$, $\mathcal{H}^k(\mathcal{M}^\bullet)$ has punctual support. Let η be small enough such that for all $k \neq 0$, $\mathcal{H}^k(\mathcal{M}^\bullet)|_{D(c, \eta)^*} = 0$. Then

$$(\mathcal{M}^\bullet[*\{c\}])|_{D(c, \eta)} = (\mathcal{H}^0(\mathcal{M}^\bullet)[*\{c\}])|_{D(c, \eta)}.$$

Then, $\mathrm{IR}_c(\mathcal{M}^\bullet) = \mathrm{IR}_c(\mathcal{H}^0(\mathcal{M}^\bullet)[*\{c\}])$. Then, according to Positivity Theorem 2.3, this complex is just a vector space over \mathbb{C} . So, for all $k \neq 0$, $\mathcal{H}^k(\mathrm{IR}_c(j_+ f_+(\mathcal{O}_{\mathbb{C}^2} e^g))) = 0$. □

REMARK 5.4. — According to this corollary, the complex $\mathrm{IR}_c(j_+ f_+(\mathcal{O}_{\mathbb{C}^2} e^g))$ is entirely determined by its Euler characteristic IR_c .

5.2. Geometrical interpretation of the irregularity

NOTATION 5.5. — • Let Γ be the critical locus of (F, G) . In the case where f and g are algebraically independent, this variety has dimension 1.

- Let Δ be the discriminant variety of F and G . Δ is the image by (F, G) of the curve Γ (counted with multiplicity). Δ is an algebraic closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

- Denote by Δ_1 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $\Delta \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where Δ is counted with multiplicity.

- Denote by Δ_2 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(D) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where the image is counted with multiplicity (the

supports of Δ_1 and Δ_2 are two algebraic closed subsets in $\mathbb{P}^1 \times \mathbb{P}^1$. They are some union of curves and points).

- For all $c \in \mathbb{P}^1$, the germs at (c, ∞) of the supports of Δ_1 and Δ_2 are some germs of curves or are empty. We denote by $I_{(c, \infty)}(\Delta_i, \mathbb{P}^1 \times \{\infty\})$ the intersection number of the cycles Δ_i and $\mathbb{P}^1 \times \{\infty\}$. If the germ at (c, ∞) of Δ_i is empty, this number is equal to 0.

THEOREM 5.6. — *Let $f, g \in \mathbb{C}[x, y]$ be two polynomials algebraically independants. Let $c \in \mathbb{P}^1$. Then*

$$\text{IR}_c = I_{(c, \infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}) + I_{(c, \infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}).$$

Proof. — According to Theorem 4.3,

$$\text{IR}_c = -\chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)).$$

We want to study the topology of the fibre $f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)$. For $|\rho|$ big enough, $G^{-1}(\rho)$ is smooth and cut transversally D . Then,

$$\begin{aligned} \chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)) &= \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) \\ &\quad - \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D). \end{aligned}$$

- First, we want to prove that

$$\chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) = -I_{(c, \infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}).$$

As Δ_1 is a union of curves and points, for η small enough and $|\rho|$ big enough, $\Delta_1 \cap (D^*(c, \eta) \times \{\rho\})$ is a finite set $\{(c_1, \rho), \dots, (c_r, \rho)\}$. Moreover, $F : F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \rightarrow D^*(c, \eta)$ is a ramified covering of degree k . The ramified points are the points $P_1, \dots, P_s \in F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap \Gamma$ and the ramification index at $P_i, i = 1, \dots, s$, is $I_{P_i}(F^{-1}(F(P_i)), G^{-1}(\rho))$.

Let D_1, \dots, D_s , be some disjoint discs in $D^*(c, \eta)$ centered at P_1, \dots, P_s . According to Mayer-Vietoris Theorem,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) &= \chi(F^{-1}(D(c, \eta) \setminus \{c, P_1, \dots, P_s\}) \cap G^{-1}(\rho)) \\ &\quad + \sum_{i=1}^s \chi(F^{-1}(D_i) \cap G^{-1}(\rho)) \\ &\quad - \sum_{i=1}^s \chi(F^{-1}(D_i \setminus \{P_i\}) \cap G^{-1}(\rho)). \end{aligned}$$

As $F : F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \rightarrow D^*(c, \eta)$ is a ramified covering of degree k with ramification points P_1, \dots, P_s ,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) &= -ks + \sum_{i=1}^s (k - I_{P_i}(F^{-1}(F(P_i)), G^{-1}(\rho)) + 1) \\ &= \sum_{i=1}^s -I_{P_i}(F^{-1}(F(P_i)), G^{-1}(\rho)) + 1. \end{aligned}$$

Then, as $I_{P_i}(F^{-1}(F(P_i)), G^{-1}(\rho)) - 1 = I_{P_i}(\Gamma, G^{-1}(\rho))$,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) &= \sum_{i=1}^s -I_{P_i}(\Gamma, G^{-1}(\rho)) \\ &= - \sum_{P \in F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap \Gamma} I_P(\Gamma, G^{-1}(\rho)). \end{aligned}$$

According to the projection formula for a proper map,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)) &= - \sum_{i=1}^r I_{(c_i, \rho)}(\Delta, \mathbb{P}^1 \times \{\rho\}) \\ &= -I_{(c, \infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}). \end{aligned}$$

• Now, we want to prove that

$$\chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D) = I_{(c, \infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}).$$

As $G^{-1}(\rho)$ is smooth and cut transversally D ,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D) &= \text{Card}(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D) \\ &= \sum_{Q \in F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D} I_Q(D, G^{-1}(\rho)). \end{aligned}$$

According to the projection formula for a proper map,

$$\begin{aligned} \chi(F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap D) &= \sum_{(c', \rho) \in \Delta_2 \cap (D^*(c, \eta) \times \{\rho\})} I_{(c', \rho)}((F, G)(D), \mathbb{P}^1 \times \{\rho\}) \\ &= I_{(c, \infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}). \quad \square \end{aligned}$$

REMARK 5.7. — When $c \in \mathbb{C}$, we have another formulation of Theorem 5.6. Denote by $\tilde{\Gamma}$ the critical locus of (f, g) . Let $\tilde{\Delta}$ be the closure of $(f, g)(\tilde{\Gamma}) \setminus (\{c\} \times \mathbb{C})$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then, we can prove that the germ at (c, ∞) of Δ_1 is the germ at (c, ∞) of $\tilde{\Delta}$ and the one at (c, ∞) of Δ_2 is empty. Then, if $c \in \mathbb{C}$,

$$\text{IR}_c = I_{(c, \infty)}(\tilde{\Delta}, \mathbb{P}^1 \times \{\infty\}).$$

We deduce this remark from the following lemma.

LEMMA 5.8. — *Let $c \in \mathbb{C}$. For $\eta > 0$ small enough and $R > 0$ big enough, $F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho)$ does not intersect D , for $|\rho| > R$.*

Proof. — We recall that $F, G : \mathbb{X} \rightarrow \mathbb{P}^1$ are constructed in the following way. We consider an immersion of \mathbb{C}^2 in \mathbb{P}^2 . We construct a rational map $(\tilde{f}, \tilde{g}) : \mathbb{P}^2 \dots \rightarrow \mathbb{P}^1$ which extend (f, g) . On $\mathbb{P}^2 \setminus \mathbb{C}^2$, (\tilde{f}, \tilde{g}) takes the value (∞, ∞) or is not well defined. Then, we lift the indeterminacies of (\tilde{f}, \tilde{g}) after a finite number of blowing ups. Then, according to [13], we know that $F^{-1}(\infty)$ and $G^{-1}(\infty)$ are connected. As $F^{-1}(\infty)$ and $G^{-1}(\infty)$ have a non-empty intersection (they contain the strict transform of $\mathbb{P}^2 \setminus \mathbb{C}^2$), $F^{-1}(\infty) \cup G^{-1}(\infty)$ is connected.

Now let Z be an irreducible component of D . We want to prove that for η small enough and R big enough, $F^{-1}(D^*(c, \eta)) \cap G^{-1}(\rho) \cap Z = \emptyset$, $|\rho| > R$.

If it is not true, we can construct a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in Z$, $0 < |F(x_n) - c| < 1/n$ and $|G(x_n)| = \rho_n$, with $\lim_{n \rightarrow +\infty} \rho_n = \infty$. By compactness, there exists a point $x \in F^{-1}(c) \cap G^{-1}(\infty) \cap Z$.

As $Z \simeq \mathbb{P}^1$, $F|_Z$ and $G|_Z$ are necessarily surjective or constant. As for all $n \in \mathbb{N}^*$, $x_n \in Z$, $0 < |F(x_n) - c| < 1/n$, $F|_Z$ is necessarily surjective. Moreover, as for all $n \in \mathbb{N}^*$, $x_n \in Z$, $G(x_n) = \rho_n \neq \infty$, with $\lim_{n \rightarrow +\infty} \rho_n = \infty$, $G|_Z$ is also surjective. Then there exists another point $y \in Z \cap F^{-1}(\infty)$, $y \neq x$ and $G(y) \neq \infty$.

This contradicts the facts that $F^{-1}(\infty) \cup G^{-1}(\infty)$ is connected and $F|_Z$ is surjective. □

6. When f and g are algebraically dependant

Let $f, g \in \mathbb{C}[x, y]$ be two polynomials which are algebraically dependant. Then, the complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is not necessarily concentrated in degree 0. However, we give a formula for the irregularity number IR_c at $c \in \mathbb{P}^1$.

Let $(F, G) : \mathbb{X} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a compactification of the map $(f, g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. As f and g are algebraically dependant, $\tilde{\Delta} = \text{im}(F, G)$ is a closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$. We consider $\tilde{\Delta}$ as a reduced variety.

- Let Δ be the cycle which is the closure of $\tilde{\Delta} \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$ in $\mathbb{P}^1 \times \mathbb{P}^1$. In a neighbourhood of (c, ∞) , Δ is a curve or is empty. We denote by

$$I_{(c, \infty)}(\Delta, \mathbb{P}^1 \times \{\infty\})$$

the intersection number of the cycles Δ and $\mathbb{P}^1 \times \{\infty\}$. If the germ at (c, ∞) of Δ is empty, this number is equal to 0.

- Let F be the generic fiber of $(f, g) : \mathbb{C}^2 \rightarrow \text{im}(f, g)$.

THEOREM 6.1. — *Let $f, g \in \mathbb{C}[x, y]$ be two polynomials algebraically dependants. Let $c \in \mathbb{P}^1$.*

$$\text{IR}_c = -\chi(F) * I_{(c, \infty)}(\Delta, \mathbb{P}^1 \times \{\infty\}).$$

Proof. — According to Theorem 4.3, $\text{IR}_c = -\chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho))$, where η is small enough and $|\rho|$ is big enough.

As the support of Δ is a curve or is empty in a neighbourhood of (c, ∞) , $\Delta \cap (D^*(c, \eta) \times \{\rho\})$ is a finite union of points $(c_1, \rho), \dots, (c_r, \rho)$. Then

$$f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho) = \bigcup_{i=1}^r (f, g)^{-1}(c_i, \rho).$$

Then, we remark that $r = I_{(c, \infty)}(\Delta, \mathbb{P}^1 \times \{\infty\})$. Moreover, if $|\rho|$ is big enough,

$$\chi((f, g)^{-1}(c_i, \rho)) = \chi(F).$$

We conclude that $\chi(f^{-1}(D^*(c, \eta)) \cap g^{-1}(\rho)) = \chi(F) * I_{(c, \infty)}(\Delta, \mathbb{P}^1 \times \{\infty\})$. \square

Acknowledgements. — I am grateful to my thesis supervisor Michel Granger for having introduced me to this subject and for many comments and remarks. I also thank Claude Sabbah for his remarks while refereeing my thesis.

BIBLIOGRAPHY

- [1] DUALITÉ DE POINCARÉ – Séminaire Heidelberg-Strasbourg 1966–1967, t. **3** (1969).
- [2] DELIGNE (P.) – *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math., vol. 163, Springer-Verlag, 1970.
- [3] ———, *Comparaison avec la théorie transcendante*, in *Lecture Notes in Math.*, vol. 340, Springer-Verlag, 1973, pp. 116–164.
- [4] GROTHENDIECK (A.) – *On the De Rham cohomology of algebraic varieties*, Publ. Math. Inst. Hautes Études Sci., t. **29** (1966), pp. 93–103.
- [5] HARTSHORNE (R.) – *Algebraic Geometry*, in *Graduate Texts in Math.*, vol. 52, Springer-Verlag, New York, 1977.
- [6] KASHIWARA (M.) – *On the maximally overdetermined systems of differential equations*, Publ. RIMS, Kyoto Univ., t. **10** (1975), pp. 563–579.
- [7] MAAREF (F.) – *Sur un analogue irrégulier de la connexion de Gauss-Manin*, Ann. Fac. Sci. Toul., t. **VIII** (1999), pp. 117–124.
- [8] MEBKHOUT (Z.) – *Le théorème de comparaison entre cohomologies de De Rham d'une variété algébrique complexe et le théorème d'existence de Riemann*, Publ. Math. Inst. Hautes Études Sci., t. **69** (1989), pp. 47–89.
- [9] ———, *Le théorème de positivité de l'irrégularité pour les \mathcal{D}_X -modules*, *Grothendieck Festschrift III*, in *Progress in Math.*, vol. 88, 1990, pp. 84–131.
- [10] ———, *Le théorème de positivité, le théorème de comparaison, le théorème d'existence de Riemann*, Séminaires et Congrès, t. **8** (2004), pp. 165–307.
- [11] MEBKHOUT (Z.) & NARVÁEZ-MACARRO (L.) – *Le théorème de constructibilité de Kashiwara, Images directes et constructibilité*, Travaux en Cours, vol. 46, Hermann, Paris, 1993, pp. 47–98.
- [12] SABBAB (C.) – *On the comparison theorem for elementary irregular \mathcal{D} -modules*, Nagoya J. Math., t. **141** (1996), pp. 107–124.
- [13] TRÃNG (LÊ DŨNG) & WEBER (C.) – *Polynômes à fibres rationnelles et conjecture jacobienne à deux variables*, C. R. Acad. Sci. Paris Sér. I Math., t. **320** (1995), pp. 581–584.