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SCHÉMAS EN GROUPES ET IMMEUBLES DES GROUPES EXCEPTIONNELS SUR UN CORPS LOCAL. DEUXIÈME PARTIE : LES GROUPES F_4 ET E_6

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RÉSUMÉ. — Nous obtenons une version explicite de la théorie de Bruhat-Tits pour les groupes exceptionnels des type F_4 ou E_6 sur un corps local. Nous décrivons chaque construction concrètement en termes de réseaux : l'immeuble, les appartements, la structure simpliciale, les schémas en groupes associés.

ABSTRACT. — We give an explicit Bruhat-Tits theory for the exceptional group of type F_4 or E_6 over a local field. We describe every construct concretely in terms of lattices: the building, the apartments, the simplicial structure, and the associated group schemes.

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Introduction

In this sequel to our paper [9], we give an explicit description of the Bruhat-Tits theory [4–8] for a split exceptional group G of type F_4 or E_6 over a local field. More precisely, we give a natural and explicit model of the Bruhat-Tits building $\mathcal{B}(G)$ as a topological space, describe its simplicial structure, the structure of apartments and the associated parahoric group schemes in terms of this model, and discuss the relations among buildings of different groups. We refer the reader to the introduction of [9] (where the case $G = G_2$ was handled) for the goal and the history of this programme.

Many techniques used in this paper have already been developed in [9], or can at least be implicitly found there. However, since the rank of G_2 is very small, a proof in [9] can occasionally be achieved by staring at the Coxeter complex which is an apartment of $\mathcal{B}(G_2)$ (the figure in [9, §9]). Here it is necessary to develop a more systematic approach. We now outline our general strategy for studying the building of a simply connected simple group G over a local field k.

Step 1. Choosing a geometric description of G. — Namely, we realize G as $\mathrm{Aut}(V,T)$, where V is a vector space over k and $T = \{t_i\}$ is a set of tensors on V. Naturally, we prefer to make $\dim V$ small and the description of T economical.

Step 2. Embedding of buildings. — Let $\iota : G \to \mathrm{GL}(V)$ be the natural embedding and show that this extends to a strong descent datum $\iota_* : \mathcal{B}(G) \to \mathcal{B}(\mathrm{GL}(V))$ of the Bruhat-Tits buildings. In general, there may be many choices for ι_*, but in the cases treated in this paper, the choice of ι_* is essentially unique.

Step 3. Determination of the image of ι_*. — This can be achieved using the formalism in [9, §3]. Recall from the fundamental work [6] of Bruhat and Tits that $\mathcal{B}(\mathrm{GL}(V))$ can be identified with the set of norms on V. Hence, determining the image of ι_* amounts to describing $\mathcal{B}(G)$ as the set of norms on V satisfying suitable conditions (expressed in terms of the tensors $\{t_i\}$), and this gives the desired model of $\mathcal{B}(G)$ as a topological space. We remark that the key input for the formalism of [9, §3] is usually an arithmetic result. In [9], this key arithmetic result is the fact that any two maximal orders in the split octonion algebra are isomorphic. Here, the key input is a theorem of Racine [16] that any two distinguished orders in the split simple exceptional Jordan algebra are isomorphic.

Step 4. Making a list of graded lattice chains and their properties. — Recall from [6] that the norms on V are in natural bijection with graded lattice chains in V. For a "standard" closed chamber C on $\mathcal{B}(G)$ and each vertex $v \in C$, one can actually write down the norm $\alpha_v = \iota_*(v)$, and its associated graded lattice chain $(L_\bullet, c)$. The stabilizer in $G(k)$ of the graded lattice chain $(L_\bullet, c)$ is then equal to the stabilizer of the vertex v, and hence is a maximal parahoric subgroup of $G(k)$. In fact, since G is simply-connected in our case, the stabilizer of any member of the lattice chain $L_\bullet$ must already be the maximal parahoric subgroup. This suggests that the graded lattice chain $(L_\bullet, c)$ (and hence the vertex v) can be reconstructed from one particular member $L(v)$ of $L_\bullet$, as a consequence of certain properties that $L(v)$ possesses. Usually, we simply take

$$L(v) = \big\{x \in V : \alpha_v(x) \geq 0\big\}.$$

By examining the graded lattice chain for each vertex v on C, we make such a list P_v of properties that $L(v)$ satisfies. We distinguish two kinds of properties: (i) the basic numerical invariants of $L(v)$ and its associated graded lattice chain $(L_\bullet, c)$, such as the image of c and the volumes of the members of $L_\bullet$ (see the beginning of §5 for the notion of volume); (ii) other properties, whose description usually involve the tensors $\{t_i\}$.

Step 5. Lattice-theoretic description of the vertices. — By Step 4, we have an injective map

$$\{\text{vertices of } \mathcal{B}(G) \text{ conjugate to } v\} \longrightarrow \{\text{lattices in } V \text{ satisfying property } P_v\},$$
$$x \longmapsto L(x),$$

and we would like to show that it is surjective. This is achieved systematically as follows.

- ▷ Given a lattice $L \subset V$ satisfying P_v, we reconstruct a graded lattice chain $(L_\bullet, c)$ which corresponds to a norm α_L on V.
- ▷ Using the description of $\mathcal{B}(G)$ in Step 3, we check that α_L lies on $\mathcal{B}(G)$. Hence, we can conjugate it to a point in the closed "standard" chamber C using $G(k)$, and we need to show that $\alpha_L = v$.
- ▷ If $\mathcal{A}$ is a "standard" apartment of $\mathcal{B}(G)$ containing C, we identify the subset of $\mathcal{A}$ consisting of those norms α whose associated lattices $L(\alpha)$ satisfy part (i) of P_v. This is practicable and very useful since this subset lies in a lattice M_v in the affine space $\mathcal{A}$. The point α_L thus lie on $M_v \cap C$, which is a finite set.

▷ Using part (ii) of P_v, we show that v can be distinguished from other points in $M_v \cap C$.

This gives the desired description of the vertices of $\mathcal{B}(G)$ in terms of certain lattices in V.

Step 6. Determination of the simplicial structure. — We would like to show that if x and y are vertices, then x is incident to y if and only if there is an inclusion relation, say $L(y) \subset L(x)$. From the explicit list of graded lattice chains made in Step 4, such an inclusion relation is easily seen to be necessary, and it remains to show that it is also sufficient. After this is done, we would have a purely lattice-theoretical description of the simplicial complex $\mathcal{B}(G)$.

To prove the expected characterization of incidence relation, we may assume that x and y lie on the "standard" apartment $\mathcal{A}$. If N_{xy} is the number of vertices of type y incident to x, and N'_{xy} is the number of vertices z of type y such that $L(z) \subset L(x)$, then it suffices to show that $N_{xy} = N'_{xy}$. The number N_{xy} can be computed using the theory of Coxeter complexes, whereas the number N'_{xy} can be found with the aid of the computer. Indeed, we first identify the bounded set $B_x = \{z \in \mathcal{A} : L(z) \subset L(x)\}$. The points in B_x which satisfy part (i) of P_y lie in the finite set $B_x \cap M_y$. One can then use the computer to count the points in $B_x \cap M_y$ which satisfy P_y, and show that $N_{xy} = N'_{xy}$.

Step 7. Construction of the Bruhat-Tits schemes. — Let x be a vertex on $\mathcal{B}(G)$. We would like to describe its associated smooth model $\underline{G}_x$ of G over A (the ring of integers of k). In many cases, it can be shown that $\underline{G}_x$ is simply the schematic closure of G in $\mathrm{Aut}(L(x))$. The proof, following the paradigm laid out by Bruhat and Tits [5], relies on detailed analysis of the smoothness of schematic closures of root subgroups. More generally, one can construct the Bruhat-Tits scheme associated to a bounded convex set in an apartment by taking a suitable schematic closure.

It is instructive to compare the above programme to the analogous problem of determining the spherical building of G. In the latter case, we do not have the key formalism developed in [9, §3] and used in Step 3. Also, the apartments are simplicial spheres rather than affine spaces, and hence the geometric tricks in Steps 5 and 6 are not available. Indeed, the remarkable paper [1] of Aschbacher, which gives a description of the spherical building of F_4 or E_6 analogous to the conclusions of Steps 5 and 6, involves very different techniques. Since the spherical building of G (over the residue field of A) can be obtained as the link of a hyperspecial vertex in the Bruhat-Tits building, it is natural to ask whether our results can be used to recover Aschbacher's description of the spherical building of a split group of type F_4 or E_6, at least over a perfect field. We do not pursue this here, but in this connection, it is worth pointing out that this paper relies on [1] only in the proof of Proposition 5.3 where we have used [1, (3.16)].

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1. Cubic Forms and Jordan Algebras

We begin with some generalities on cubic forms and Jordan algebras. Let A be a (unital, commutative and associative) ring and J a projective A-module of finite rank. Let N be a cubic form on J, and t its associated symmetric trilinear form. The cubic form N determines a tensor Q on $J \times J$, characterized by the requirement that:

- ▹ for fixed y, $L_y : x \mapsto Q(x, y)$ is a linear form;
- ▹ for fixed x, $Q_x : y \mapsto Q(x, y)$ is a quadratic form;
- ▹ $N(x + y) - N(x) - N(y) = Q(x, y) + Q(y, x)$.

The 3-tuple (N, Q, t) satisfies

- ▹ the symmetric bilinear form associated to Q_x is $t(x, -, -)$, *i.e.*

$$t(x, y, y) = 2 \cdot Q(x, y),$$

- ▹ $Q(x, x) = 3 \cdot N(x)$,

and is called a regular 3-form in [1].

Let $e \in J$ be such that $N(e) = 1$. Then we obtain a symmetric bilinear T by setting

$$T(x, y) = Q(x, e)Q(y, e) - t(e, x, y).$$

If this symmetric bilinear form is non-degenerate, *i.e.* induces an isomorphism $J \to \mathrm{Hom}_A(J, A)$, then we can define a quadratic map $\#$ on J by the formula

$$T(x^{\#}, y) = Q(y, x).$$

In that case, we set

$$x \times y = (x + y)^{\#} - x^{\#} - y^{\#}.$$

Following Jacobson [12, §2.4], the pair (N, e) is said to be admissible if:

- ▹ T is non-degenerate,
- ▹ the quadratic map $\#$ satisfies $x^{\#\#} = N(x) \cdot x$.

Given an admissible pair (N, e), we have the following useful identities:

$$e^{\#} = e, \tag{1}$$

$$T(x \times y, z) = T(x, y \times z) = t(x, y, z), \tag{2}$$

$$e \times x = T(e, x)e - x. \tag{3}$$

Some more complicated identities (*cf.* [16, p. 99]), which will prove to be useful later on, are:

$$x \times x^{\#} = \big\{T(e,x^{\#})T(e,x) - N(x)\big\}e - T(e,x^{\#})x - T(e,x)x^{\#}, \tag{4}$$

$$(x \times y)^{\#} = T(x^{\#},y)y + T(y^{\#},x)x - x^{\#} \times y^{\#}. \tag{5}$$

The 5-tuple $(J, N, e, \#, T)$ is called a cubic norm structure; we refer the reader to [13, § 38] for its definition and further properties. We have seen that an admissible pair (N, e) on J gives rise to a cubic norm structure; the cubic norm structures thus obtained are those for which T is non-degenerate. Given any cubic norm structure, we set

$$yU_x = T(x,y)x - x^{\#} \times y. \tag{6}$$

Hence U_x is a linear operator on J, and $U : x \mapsto U_x$ is a quadratic map. Moreover, the triple (J, U, e) is a quadratic Jordan algebra of degree 3 (see [12, 1.3.4]). This allows us to define the positive-integer powers of $x \in J$: $x^2 = eU_x$, $x^3 = xU_x$ and so on. If 2 is invertible in A, then the product

$$x \circ y = \frac{1}{2}\big[(x+y)^2 - x^2 - y^2\big]$$

endows J with the structure of an honest Jordan algebra, i.e. a commutative algebra with unit e satisfying the identity

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \quad \text{for all } x, y \in J.$$

Conversely, given an honest Jordan algebra, one obtains a quadratic Jordan algebra by setting:

$$yU_x = 2x \circ (x \circ y) - (x \circ x) \circ y,$$

so that the two notions coincide when 2 is invertible in A.

The characteristic polynomial of $x \in J$ is the degree 3 polynomial given by:

$$p_x(\lambda) = N(\lambda \cdot e - x) = \lambda^3 - Q(x,e)\lambda^2 + Q(e,x)\lambda - N(x).$$

The analogue of Cayley-Hamilton theorem holds, *i.e.* $p_x(x) = 0$. Moreover, there is a notion of the minimal polynomial of x, at least when A is a field.

2. The Exceptional Jordan Algebra

In this section, we describe the principal examples of cubic form and Jordan algebra used in the paper. Taking $A = \mathbb{Z}$, we let $\underline{M}$ be the $\mathbb{Z}$-algebra of 3×3 matrices, and set $\underline{J} = \underline{M}^{\oplus 3}$. Let N be the cubic form on $\underline{J}$ defined by:

$$N : (a,b,c) \longmapsto \det(a) + \det(b) + \det(c) - \mathrm{Tr}(abc).$$

As in the previous section, N gives rise to the forms t and Q.

Now let $e = (1,0,0) \in \underline{J}$. Then one can check that the pair (N,e) is admissible. Hence, we have the non-degenerate symmetric bilinear form T and the quadratic map $\#$. These are given explicitly by:

$$\begin{cases} T\big((a_1,b_1,c_1),(a_2,b_2,c_2)\big) = \mathrm{Tr}(a_1a_2) + \mathrm{Tr}(b_1c_2) + \mathrm{Tr}(c_1b_2), \\ (a,b,c)^{\#} = (a^{\#} - bc, c^{\#} - ab, b^{\#} - ca), \end{cases}$$

where $a^{\#}$ is the adjoint matrix of a, *etc.* The Jordan algebra arising from the admissible pair (N,e) is called the split simple exceptional Jordan algebra, and the above construction is a special case of Tits' first construction of Jordan algebras. Hence, we shall call $(\underline{J},N,e)$ the Tits model.

Another model for $(\underline{J},N)$ can be described as follows. Let Λ be the split octonion algebra over $\mathbb{Z}$ [9, §5]. This has an anti-involution $x \mapsto \overline{x}$, and a quadratic norm form q with associated symmetric bilinear form f. Moreover, we have an explicit model for Λ, namely the Zorn's model, which comes equipped with a basis $\{e_{\pm 1}, e_{\pm 2}, e_{\pm 3}, e_{\pm 4}\}$, as discussed in [9, §5]. Let $\underline{J}_3$ be the set of 3×3 Hermitian matrices with entries in Λ. More precisely, an element in $\underline{J}_3$ has the form:

$$X = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}, \tag{7}$$

with $a,b,c \in \mathbb{Z}$ and $x,y,z \in \Lambda$. There is a natural cubic form N on $\underline{J}_3$ defined by:

$$N(X) = abc + \mathrm{Tr}(xyz) - aq(x) - bq(y) - cq(z), \tag{8}$$

and a linear form $\mathrm{Tr} : X \mapsto a+b+c$.

Taking $e \in \underline{J}_3$ to be the identity matrix I, we obtain an admissible pair (N,e). The corresponding non-degenerate symmetric bilinear form is

$$T(X_1,X_2) = a_1a_2 + b_1b_2 + c_1c_2 + f(x_1,x_2) + f(y_1,y_2) + f(z_1,z_2),$$

which is related to Tr by $T(X,I) = \mathrm{Tr}(X)$, and the quadratic map $\#$ is given by:

$$X^{\#} = \begin{pmatrix} bc - q(x) & \overline{x\cdot y} - cz & z\cdot x - b\overline{y} \\ x\cdot y - c\overline{z} & ca - q(y) & \overline{y\cdot z} - ax \\ \overline{z\cdot x} - by & y\cdot z - a\overline{x} & ab - q(z) \end{pmatrix}.$$

The triple $(\underline{J}_3,N,e)$ is isomorphic to that defined using Tits model. An explicit isomorphism j is given by:

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \longmapsto \begin{pmatrix} a & z_4 & y_{-4} \\ z_{-4} & b & x_4 \\ y_4 & x_{-4} & c \end{pmatrix} \oplus \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \oplus \begin{pmatrix} x_{-1} & y_{-1} & z_{-1} \\ x_{-2} & y_{-2} & z_{-2} \\ x_{-3} & y_{-3} & z_{-3} \end{pmatrix},$$

where $x = \sum_i x_ie_i$ and so on. The model $(\underline{J}_3,N,e)$ will be called the Freudenthal model.

We conclude the section by giving a few other examples of admissible pairs, which will appear later in the paper.

Example 2.1 (Jordan algebra associated to an associative algebra)
Given an associative A-algebra M, one can endow M with the structure of a quadratic Jordan algebra by setting:

$$yU_x = xyx.$$

The Jordan algebra thus obtained is denoted M^+, and any Jordan sub-algebra of M^+ is said to be special.

Taking M to be the algebra of 3×3 matrices with entries in A, one can check that M^+ also arises from the admissible pair $(N, 1)$, where $N = \det$ is the determinant. The pair $(N, 1)$ is admissible since T is simply the trace bilinear form.

Example 2.2 (Freudenthal's construction of Jordan algebra)
Let C be a composition algebra over A, such that the trace form f is non-degenerate. Then the A-module $H_3(C)$ of Hermitian matrices with entries in C consists of elements of the form (7), where the off-diagonal entries lie in C. The formula (8) furnishes a natural cubic form det on $H_3(C)$, and taking e to be the identity matrix, the pair $(\det, e)$ is admissible, and thus gives rise to a Jordan algebra. Unless C is an octonion algebra, the Jordan algebra so obtained is special. Of course, if one takes C to be the split octonion algebra over A, one obtains the Freudenthal model described above.

Example 2.3. — (Jordan algebra associated to a pointed quadratic space) Let (V, q, v) be a pointed quadratic space over A, *i.e.* a projective A-module V equipped with a quadratic form q and an element $v \in V$ with $q(v) = 1$. Assume that the symmetric bilinear form associated to q is non-degenerate, and let $J = A \times V$. Then J has a natural cubic form defined by

$$N : (x, y) \longmapsto x \cdot q(y).$$

If we let $e = (1, v)$, then the pair (N, e) is admissible, and the resulting Jordan algebra is again special.

3. Groups

We continue to work over $\mathbb{Z}$ in this section. Let $\underline{H}$ be the automorphism group of the cubic form N on $\underline{J}$; it is the Chevalley group over $\mathbb{Z}$ which is simply-connected of type E_6, and is a closed sub-scheme of $\mathrm{SL}(\underline{J})$ (*cf.* [1], [10] and [9, Prop. 6.1]). Further, it follows necessarily that $\underline{H}$ fixes the tensors Q and t. Let $\underline{H}' \subset \mathrm{GL}(\underline{J})$ be the group of similitudes:

$$\underline{H}'(B) = \big\{(h, \lambda(h)) \in \mathrm{GL}(\underline{M} \otimes B) \times \mathbb{G}_m(B) : N \circ h = \lambda(h) \cdot N\big\},$$

for any $\mathbb{Z}$-algebra B. Then we have an exact sequence:

$$1 \to \underline{H} \longrightarrow \underline{H}' \xrightarrow{\lambda} \mathbb{G}_m \to 1.$$

Moreover, it follows necessarily that for any $h \in \underline{H}'$,

$$Q \circ h = \lambda(h) \cdot Q, \quad \text{and} \quad t \circ h = \lambda(h) \cdot t.$$

To describe a maximal split torus $\underline{T}$ for $\underline{H}$, we first describe an embedding

$$(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_3 \hookrightarrow \underline{H}.$$

Let $(g_1, g_2, g_3) \in \mathrm{SL}_3^3$ act on $\underline{J}$ by

$$(x, y, z) \longmapsto (g_2 x g_3^{-1}, g_3 y g_1^{-1}, g_1 z g_2^{-1}).$$

It is clear that the action factors through $\mathrm{SL}_3^3/\Delta\mu_3$, and gives the desired embedding over $\mathbb{Z}$. We then take $\underline{T}$ to be the image of the product of diagonal tori

$$\widetilde{\underline{T}} = \widetilde{\underline{T}}_1 \times \widetilde{\underline{T}}_2 \times \widetilde{\underline{T}}_3$$

in SL_3^3. Let $\underline{T}'$ be a maximal split torus of $\underline{H}'$ containing $\underline{T}$. To be specific, let $\underline{T}' \subset \underline{H}'$ be the split torus generated by $\underline{T}$ and the 1-dimensional torus which acts on $\underline{J}_3$ by:

$$t : \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \longmapsto \begin{pmatrix} ta & tz & \overline{y} \\ t\overline{z} & tb & x \\ y & \overline{x} & t^{-1}c \end{pmatrix}, \quad \text{for } t \in \mathbb{G}_m.$$

Then we have an exact sequence:

$$1 \to \underline{T} \longrightarrow \underline{T}' \xrightarrow{\lambda} \mathbb{G}_m \to 1.$$

This is a split exact sequence, since the 1-dimensional torus defined above provides a splitting. In particular, we deduce:

Lemma 3.1. — *For any commutative $\mathbb{Z}$-algebra A, the map*

$$\lambda : \underline{H}'(A) \longrightarrow A^\times$$

is surjective.

It is clear that there exists a maximal split torus $\underline{C}$ of $\mathrm{SL}(\underline{J})$ such that $\underline{C} \cap \underline{H} = \underline{T}$. Indeed, it suffices to take $\underline{C}$ to be the maximal split torus determined by the natural basis of $\underline{J}$. We thus have a sequence of maps of $\mathbb{Z}$-modules:

$$X_*(\widetilde{\underline{T}}) \hookrightarrow X_*(\underline{T}) \hookrightarrow X_*(\underline{C}),$$

and dually,

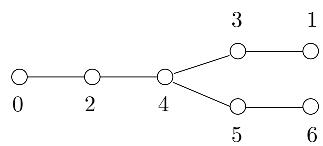
$$X^*(\underline{C}) \twoheadrightarrow X^*(\underline{T}) \hookrightarrow X^*(\widetilde{\underline{T}}). \tag{9}$$

For the purpose of explicit computation, we need to set up some explicit coordinates for the real vector spaces $X_*(\underline{T}) \otimes \mathbb{R}$ and $X_*(\underline{C}) \otimes \mathbb{R}$. For $i, j = 1, 2$ or 3, let $\epsilon_i[j]$ be the character of $\widetilde{\underline{T}}_i$ which sends an element of $\widetilde{\underline{T}}_i$ to its j-th diagonal entry. Then for each i, the elements $\epsilon_i[1]$, $\epsilon_i[2]$ and $\epsilon_i[3]$ generate the lattice $X^*(\underline{\widetilde{T}}_i)$, with the relation

$$\epsilon_i[1] + \epsilon_i[2] + \epsilon_i[3] = 0,$$

and the elements $\{\epsilon_i[j]\}$ span the lattice $X^*(\underline{\widetilde{T}})$.

Now the roots of $\underline{H}$ relative to $\underline{T}$ can be regarded as elements of $X^*(\underline{\widetilde{T}})$, in view of (9). We index the simple roots as shown in the following extended Dynkin diagram:



Then the simple roots can be chosen such that:

$$r_1 = \epsilon_2[1] - \epsilon_2[2],\ r_3 = \epsilon_2[2] - \epsilon_2[3], \quad r_6 = \epsilon_3[1] - \epsilon_3[2],$$
$$r_5 = \epsilon_3[2] - \epsilon_3[3],\ r_0 = -\epsilon_1[1] + \epsilon_1[2],\ r_2 = \epsilon_1[2] - \epsilon_1[3],$$

where r_0 is the highest root. Since

$$r_0 = r_1 + 2r_2 + 2r_3 + 3r_4 + 2r_5 + r_6,$$

we can work out what r_4 is. Now the simple roots $\{r_i\}$ serve as coordinate functions on the real vector space $X_*(\underline{T}) \otimes \mathbb{R}$, and we can obviously express the $\epsilon_j[i]$'s as rational linear combinations of the r_i's.

As for the coordinates on $X_*(\underline{C}) \otimes \mathbb{R}$, we shall use the natural ones furnished by the natural basis of $\underline{J}$. More precisely, for $i, j, k \in \{1, 2, 3\}$, let $e_{jk}[i]$ be the element of

$$\underline{J} = \underline{M} \oplus \underline{M} \oplus \underline{M},$$

all of whose entries are 0 except for the (j, k)-entry of its i-th component, which is 1. Let $a^i_{jk} : \underline{C} \to \mathbb{G}_m$ be the weight of $\underline{C}$ corresponding to the weight vector $e_{jk}[i]$. Then we have

$$\sum_{i,j,k} a^i_{jk} = 0.$$

Hence, we shall think of a point p of $X_*(\underline{C}) \otimes \mathbb{R}$ as a 3-tuple of 3×3 real matrices

$$\big[(a^1_{jk}(p)), (a^2_{jk}(p)), (a^3_{jk}(p))\big],$$

whose entries sum to zero.

From the definition of the map $\mathrm{SL}_3^3 \to \underline{H}$, it is easy to determine the pull-back of a^i_{jk} to $\underline{\widetilde{T}}$ in terms of $\{\epsilon_j[i]\}$. Indeed, we have:

$$a^i_{jk} \longmapsto \epsilon_{i+1}[j] - \epsilon_{i-1}[k],$$

where the indices are taken modulo 3. It is thus a simple matter to express the pull-back of the coordinates a^i_{jk} to $X_*(\underline{T})\otimes\mathbb{R}$ in terms of the coordinates $\{r_i\}$. We summarize the preceding discussion with the following lemma:

LEMMA 3.2. — *In terms of the coordinates $\{r_i\}$ and $\{a^i_{jk}\}$ introduced above, the embedding*

$$X_*(\underline{T})\otimes\mathbb{R}\hookrightarrow X_*(\underline{C})\otimes\mathbb{R}$$

is completely determined by:

$$(1,0,0,0,0,0)\longmapsto\begin{pmatrix}\frac23&\frac23&\frac23\\-\frac13&-\frac13&-\frac13\\-\frac13&-\frac13&-\frac13\end{pmatrix}\oplus\begin{pmatrix}\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\end{pmatrix}\oplus\begin{pmatrix}-\frac43&-\frac13&-\frac13\\-\frac13&\frac23&\frac23\\-\frac13&\frac23&\frac23\end{pmatrix},$$

$$(0,\tfrac12,0,0,0,0)\longmapsto\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\oplus\begin{pmatrix}\frac12&-\frac12&0\\\frac12&-\frac12&0\\\frac12&-\frac12&0\end{pmatrix}\oplus\begin{pmatrix}-\frac12&-\frac12&-\frac12\\\frac12&\frac12&\frac12\\0&0&0\end{pmatrix},$$

$$(0,0,\tfrac12,0,0,0)\longmapsto\begin{pmatrix}\frac16&\frac16&\frac16\\\frac16&\frac16&\frac16\\-\frac13&-\frac13&-\frac13\end{pmatrix}\oplus\begin{pmatrix}\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\end{pmatrix}\oplus\begin{pmatrix}-\frac56&-\frac56&-\frac13\\\frac16&\frac16&\frac23\\\frac16&\frac16&\frac23\end{pmatrix},$$

$$(0,0,0,\tfrac13,0,0)\longmapsto\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\oplus\begin{pmatrix}\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\\\frac23&-\frac13&-\frac13\end{pmatrix}\oplus\begin{pmatrix}-\frac23&-\frac23&-\frac23\\\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\end{pmatrix},$$

$$(0,0,0,0,\tfrac12,0)\longmapsto\begin{pmatrix}-\frac16&-\frac16&\frac13\\-\frac16&-\frac16&\frac13\\-\frac16&-\frac16&\frac13\end{pmatrix}\oplus\begin{pmatrix}\frac56&-\frac16&-\frac16\\\frac56&-\frac16&-\frac16\\\frac13&-\frac23&-\frac23\end{pmatrix}\oplus\begin{pmatrix}-\frac23&-\frac23&-\frac23\\\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\end{pmatrix},$$

$$(0,0,0,0,0,1)\longmapsto\begin{pmatrix}-\frac23&\frac13&\frac13\\-\frac23&\frac13&\frac13\\-\frac23&\frac13&\frac13\end{pmatrix}\oplus\begin{pmatrix}\frac43&\frac13&\frac13\\\frac13&-\frac23&-\frac23\\\frac13&-\frac23&-\frac23\end{pmatrix}\oplus\begin{pmatrix}-\frac23&-\frac23&-\frac23\\\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\end{pmatrix}.$$

Now let $\underline{G}$ be the subgroup scheme of $\underline{H}$ which fixes the element e. Then $\underline{G}$ is the Chevalley group over $\mathbb{Z}$ of type F_4. It is also the automorphism group of the cubic norm structure $(\underline{J},N,e,\#,T)$ or the quadratic Jordan algebra structure on $\underline{J}$. In any case, we have closed embeddings

$$\underline{G}\hookrightarrow\underline{H}\hookrightarrow\mathrm{SL}(\underline{J}).$$

The symmetric bilinear form T defines an involution $*$ on $\underline{H}$ by

$$T(gx,g^*y)=T(x,y).$$

This is an outer automorphism of $\underline{H}$ and the subgroup of $\underline{H}$ fixed by $*$ is precisely $\underline{G}$ [18, pp. 150–151]. Hence, yet another way to describe G is to say that it is the automorphism group of the pair (N,T). For ease of reference, we list the various descriptions of $\underline{G}$ below:

Proposition 3.3. — *The group $\underline{G}$ can be described in any one of the following ways:*

- ▷ *the automorphism group of the pair (N, e), i.e. the subgroup of $\underline{H}$ which fixes e;*
- ▷ *the automorphism group of the pair (N, T), i.e. the subgroup of $\underline{H}$ which fixes T, or equivalently, is fixed by the involution $*$ determined by T;*
- ▷ *the automorphism group of the Jordan algebra structure (U, e).*
- ▷ *the automorphism group of the pair $(\#, e)$.*

A maximal split torus of $\underline{G}$ is

$$\underline{S} = \underline{G} \cap \underline{T} = \underline{G} \cap \underline{C}.$$

More concretely, we have an embedding

$$\mathrm{SL}_3 \times \mathrm{SL}_3 / \Delta\mu_3 \hookrightarrow \mathrm{SL}_3^3 / \Delta\mu_3 \hookrightarrow \underline{H}$$

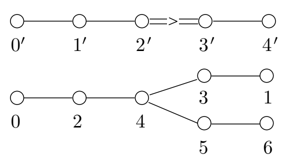
given by $(g_1, g_2) \mapsto (g_1, g_2, g_2)$. This embedding factors through $\underline{G}$, and $\underline{S}$ is the image of the product of diagonal tori in SL_3^2. Hence, we have an embedding of real vector spaces

$$X_*(\underline{S}) \otimes \mathbb{R} \hookrightarrow X_*(\underline{T}) \otimes \mathbb{R}.$$

On restricting the simple roots $\{r_i\}$ in $X^*(\underline{T})$ to $\underline{S}$, we see that a system of simple roots $\{r_i'\}$ for $\underline{G}$ relative to $\underline{S}$ can be taken to be:

$$r_1' = r_{2|\underline{S}}, \quad r_2' = r_{4|\underline{S}}, \quad r_3' = r_{3|\underline{S}} = r_{5|\underline{S}}, \quad r_4' = r_{1|\underline{S}} = r_{6|\underline{S}}. \tag{10}$$

Here the simple roots are indexed according to the following extended Dynkin diagram (for the sake of comparison, the E_6 diagram is shown below the diagram of F_4):



4. Orders and Radicals

Henceforth, we let k be a field which is complete with respect to a discrete valuation ord, with ring of integers A, uniformizer π and *perfect* residue field A/π of characteristic p (possibly zero). Assume that $\mathrm{ord}(\pi) = 1$. We have described various algebraic structures and algebraic groups over $\mathbb{Z}$ in the previous sections, and by base extension, we obtain the corresponding structures and groups over A and k. More precisely, we let

$$J = \underline{J} \otimes A, \quad V = \underline{J} \otimes k, \quad H = \underline{H} \otimes k, \quad \text{and so on.}$$

Suppose that $L \subset V$ is an A-lattice on which N is integral, so that (L, N) is a cubic space over A.

Definition 4.1. — The radical $\mathfrak{R}(L,N)$ of the cubic space (L,N) is:

$$\mathfrak{R}(L,N) = \big\{x \in L : N(x),\, Q(x,y) \text{ and } Q(y,x) \text{ lie in } \pi\widetilde{A} \text{ for all } y \in L \otimes_A \widetilde{A}\,\big\}$$

where $\widetilde{A}$ is the strict henselization of A. Further, we define recursively, for $i \geq 1$,

$$\mathfrak{R}^i(L,N) = \mathfrak{R}\big(\mathfrak{R}^{i-1}(L,N), \pi^{-i+1}N\big),$$

and call the sequence $\{\mathfrak{R}^i(L,N)\}$ the radical series of (L,N).

Definition 4.2. — An order in the quadratic Jordan algebra V is an A-lattice $L \subset V$ such that

- ▹ $e \in L$;
- ▹ $LU_x \subset L$, for all $x \in L$.

In particular, an order L is a Jordan algebra over A. Every element x in an order L is integral, in the sense that its characteristic polynomial p_x has coefficients in A [16, p. 9].

There is a notion of Jacobson radical for Jordan algebras. In the setting of this paper, it can be defined as follows. First, we recall the notion of ideals in an arbitrary Jordan algebra L. A submodule $B \subset L$ is

- ▹ an inner ideal if $yU_x \in B$ for all $x \in B$ and $y \in L$;
- ▹ an outer ideal if $yU_x \in B$ for all $x \in L$ and $y \in B$;
- ▹ an ideal if it is both an inner ideal and an outer ideal.

Say that L is simple if it has no non-trivial ideals, and is semisimple if it is a product of simple Jordan algebras.

Definition 4.3. — The radical of a Jordan algebra L is the smallest ideal $\mathfrak{R}(L) \subset L$ such that $L/\mathfrak{R}(L)$ is semisimple.

It is a remarkable theorem of Petersson and Racine [14, Thm. 9] that

$$\mathfrak{R}(L) = \mathfrak{R}(L,N).$$

Definition 4.4. — An order is said to be *distinguished* if it is a maximal lattice of integral elements.

For example, the order $J \subset V$ is distinguished. Note that a distinguished order is necessarily a maximal order. Now we have the following crucial result:

Theorem 4.5. — *Let L be a lattice in V such that $e \in L$. Then*

(i) *L is an order if and only if L is closed under the quadratic map $\#$.*

(ii) *Any two distinguished orders are isomorphic, and thus conjugate under $G(k)$.*

Proof. — (i) is [16, p. 102, Prop. 1], and (ii) is [16, p. 115, Prop. 5]. □

It should be noted that there exists maximal orders in V which are not distinguished. We describe an example below, following Racine [16]. In [9, §11], we have given an explicit description of the building of the split group Spin_8 in terms of norms on the (reducible) 24-dimensional representation. The building $\mathcal{B}(\mathrm{Spin}_8)$ can be canonically identified with $\mathcal{B}(\mathrm{SO}_8)$, and a particular chamber of the latter correspond to 4 lattices in the split octonion algebra $\Lambda \otimes k$. These are labelled by R, M, N and N' in [9, §9], with R a maximal order in $\Lambda \otimes k$, M is a self-dual lattice, whereas N and N' satisfy $\pi N^* = N$ and $\pi N'^* = N'$. Moreover, they satisfy

$$R * R \subset R, \quad M * N \subset N', \quad N * N' \subset M, \quad N' * M \subset N,$$

where $*$ is the natural symmetric composition on the octonion algebra (*cf.* [9, §5]). The distinguished order J_3 in V is then given by the lattice

$$J_3 = \begin{pmatrix} A & R & \\ & A & R \\ R & & A \end{pmatrix},$$

and an example of a maximal order which is not distinguished is given by the lattice:

$$J' = \begin{pmatrix} A & N & \\ & A & M \\ N' & & A \end{pmatrix}.$$

Whereas the reduction modulo π of J_3 is the split exceptional simple Jordan algebra over A/π, the Jordan algebra $J'/\pi J'$ is not semisimple. The maximal semisimple quotient $J'/\mathfrak{R}(J')$ is isomorphic to the Jordan algebra over A/π associated as in Example 2.3 to the pointed quadratic space of 2×2 Hermitian matrices with entries in the octonion algebra $\Lambda \otimes A/\pi$. In particular, $J'/\mathfrak{R}(J')$ has dimension 11 over A/π. Are these the only two isomorphism classes of maximal orders in V?

5. Buildings

In this section, using the formalism developed in [9, §3], we give explicit models for the buildings $\mathcal{B}(H)$ and $\mathcal{B}(G)$ of H and G in terms of norms on the 27-dimensional representation V.

The embedding $\iota : \underline{H} \hookrightarrow \mathrm{SL}(\underline{J})$ of Chevalley groups induces a strong descent datum [4, p. 203]

$$\iota_* : \mathcal{B}(H) \longrightarrow \mathcal{B}(\mathrm{SL}(V)).$$

By [6], we regard $\mathcal{B}(\mathrm{SL}(V))$ as the set of norms α on V with a fixed volume. More precisely, each norm α on V determines a norm $\mathrm{vol}(\alpha)$ on $\wedge^{\mathrm{top}} V$, and $\mathcal{B}(\mathrm{SL}(V))$ can be realized as the set of those α such that $\mathrm{vol}(\alpha)$ takes value 0 on a basis element of the lattice $\wedge^{\mathrm{top}} J \subset \wedge^{\mathrm{top}} V$. In general, if $\mathrm{vol}(\alpha)$ takes value r on a basis element of $\wedge^{\mathrm{top}} J$, we say that α has volume r. Similarly, if L is a lattice in V, say that L has volume r if $\wedge^{\mathrm{top}} L = \pi^r \wedge^{\mathrm{top}} J$. Note that with this notion of volume, if $L_1 \subset L_2$, then $\mathrm{vol}(L_1) \geq \mathrm{vol}(L_2)$. Moreover, every lattice L determines a graded period 1 lattice chain and the resulting norm α_L has volume $-\mathrm{vol}(L)$.

For each finite extension E of k, let $\mathcal{N}_E \subset \mathcal{B}(\mathrm{SL}(V_E))$ be the subset of those norms α which minorize N:

$$\begin{cases} \operatorname{ord} N(x) \geq 3\alpha(x), & \text{for all } x \in V_E, \\ \operatorname{ord} Q(x,y) \geq \alpha(x) + 2\alpha(y), & \text{for all } x, y \in V_E, \\ \operatorname{ord} t(x,y,z) \geq \alpha(x) + \alpha(y) + \alpha(z), & \text{for all } x, y, z \in V_E. \end{cases}$$

Then we have:

THEOREM 5.1. — *The image of ι_* is the set $\mathcal{N} = \mathcal{N}_k$. In other words, the building of H can be described as the set of norms on V which have volume* 0 *and minorize the cubic form N.*

We now begin the proof of the theorem. Firstly, by checking on a splitting basis for a norm α, it is not difficult to verify:

LEMMA 5.2. — *For each E, $\mathcal{N}_E$ is convex and $H(E)$-invariant, and the intersection of $\mathcal{N}_E$ with any apartment of $\mathcal{B}(H)$ is a rational polytope. Moreover, the collection $\{\mathcal{N}_E\}$ is compatible with base change. In other words, conditions* (BC) *and* (RAT) *of [9, Thm. 3.5] hold.*

To prove the theorem, it remains to verify the condition (TRANS) in [9], Theorem 3.5. Let $x_0 \in \mathcal{B}(\mathrm{SL}(V))$ be the rational point corresponding to the period 1 graded lattice chain determined by J. It is clear that $x_0 \in \mathcal{N}$, and by construction, x_0 is the image of a hyperspecial point on $\mathcal{B}(H)$ under ι_*. To verify (TRANS), it suffices to show:

PROPOSITION 5.3. — *For each E, $H(E)$ acts transitively on $\mathcal{N}_E \cap \mathrm{SL}(V_E)x_0$.*

Let $x \in \mathcal{N}_E \cap \mathrm{SL}(V_E)x_0$. Then x corresponds to a period 1 lattice chain determined by a certain lattice L in V_E with $\mathrm{vol}(L) = 0$. The assumption that $x \in \mathcal{N}_E$ simply says that N, Q and t are integer-valued on L. We need to show that L is conjugate to $J \otimes A_E$ under $H(E)$.

• *Special Case: suppose that $e \in L$.* — In this case, we claim that L is a distinguished order of the Jordan algebra $V \otimes E$. By Theorem 4.5 (i), we need to show that

▷ L is a maximal lattice of integral elements;
▷ L is closed under #.

Since $e \in L$ and the triple (N, Q, t) is integer-valued on L, we deduce that the symmetric bilinear form T is integer-valued on L. Since $J \otimes A_E$ is self-dual with respect to T, and has the same volume as L, we see that L must also be self-dual with respect to T. In particular, L is a maximal lattice on which T is integral. Since T must be integer-valued on any lattice consisting of integral elements, L is indeed a maximal lattice of integral elements. On the other hand, since

$$T(x, y^{\#}) = Q(x, y) \in A$$

for all $x, y \in L$, the self-duality of L relative to T implies that $L^{\#} \subset L$. This establishes the claim.

Hence by Theorem 4.5 (ii), we conclude that if $e \in L$, then L is conjugate to $J \otimes A_E$ under $G(E) \subset H(E)$.

- *General Case.* — Let $c = \min_{x\in L} \operatorname{ord} N(x) \geq 0$. Then the forms Q and t take value in $\pi^c A$ as well.

Now we claim in fact that $c = 0$. Indeed, let $x_0 \in L$ be such that $\operatorname{ord} N(x_0) = c$. By [1, 3.16 (1)] (we remind the reader that what we denote by (N, Q, t) is denoted by (T, Q, f) in [1], and a "point" in [1, 3.16 (1)] means a 1-dimensional vector subspace in V), there exists $h' \in H'(E)$ such that $h'(x_0) = e$. Note that such a h' satisfies $\lambda(h') = N(x_0)^{-1}$. Hence the lattice $h'(L)$ contains e, and the triple (N, Q, t) is still integer-valued on $h'(L)$. This implies that T is integer-valued on $h'(L)$, and so

$$\operatorname{vol}(L) \geq \operatorname{vol}\bigl(h'(L)\bigr) \geq \operatorname{vol}(J \otimes A_E),$$

since the latter is self-dual. Since L is assumed to have the same volume as $J \otimes A_E$, we conclude that equality holds throughout, so that $c = 0$, and $h'(L)$ has the same volume as $J \otimes A_E$. By the special case treated above, there exists $g \in G(E)$ such that $gh'(L) = J \otimes A_E$.

Now the map $\lambda : \underline{H}'(A_E) \to A_E^{\times}$ is surjective, by Lemma 3.1. Since $\lambda(gh')$ lies in $A_E^{\times}$, we can find $h'' \in \underline{H}'(A_E)$ such that $\lambda(h'') = \lambda(gh')^{-1}$. Now the element $h = h''gh'$ satisfies $\lambda(h) = 1$ and hence h lies in $H(F)$. Moreover,

$$h(L) = h''gh'(L) = h''(J \otimes A_E) = J \otimes A_E,$$

since by construction, $\underline{H}'(A_E)$ stabilizes $J \otimes A_E$. This proves the general case, and hence Proposition 5.3.

Finally, by combining Lemma 5.2, Proposition 5.3 and [9, Thm. 3.5], Theorem 5.1 is proved.

Corollary 5.4. — *The building $\mathcal{B}(H')$ of H' is the set of norms α on V such that $\alpha - \frac{1}{27}\operatorname{vol}(\alpha)$ minorizes N.*

Observe that, H is also the automorphism group of $\lambda \cdot N$ for any $\lambda \in k^{\times}$. We can thus also realize the building of H as a set of norms minorizing $\lambda \cdot N$. More precisely, we have:

Corollary 5.5. — *The building of H can be realized as the set $\mathcal{B}(H)_\lambda$ of norms α on V, which minorize $\lambda \cdot N$, and such that* $\mathrm{vol}(\alpha) = 9\,\mathrm{ord}(\lambda)$. *Moreover, the map $\mathcal{B}(H) \to \mathcal{B}(H)_\lambda$ given by $\alpha \mapsto \alpha + \frac{1}{3}\,\mathrm{ord}(\lambda)$ is an isomorphism of buildings.*

We come now to the building $\mathcal{B}(G)$ of G. There are a number of ways of describing $\mathcal{B}(G)$, corresponding to the different ways of realizing the group G in Proposition 3.3. As a piece of terminology, we say that a norm α on V is a *norm of Jordan algebra* if it satisfies:

- ▷ $\alpha(yU_x) \geq 2\alpha(x) + \alpha(y)$ for all $x, y \in V$;
- ▷ $\alpha(zU_{x,y}) \geq \alpha(x) + \alpha(y) + \alpha(z)$ for all $x, y, z \in V$, where $U_{x,y} = U_{x+y} - U_x - U_y$;
- ▷ $\alpha(e) \geq 0$.

We now have the following theorem:

Theorem 5.6. — *Let $\iota' : G \hookrightarrow \mathrm{GL}(V)$. Then the image of $\mathcal{B}(G)$ under ι'_* can be described in any one of the following ways:*

(i) *the set of $\alpha \in \mathcal{B}(H)$ such that $\alpha(e) \geq 0$.*

(ii) *the set of self-dual (relative to T) norms α on V which minorize N.*

(iii) *the set of self-dual norms of Jordan algebra;*

(iv) *the set of self-dual norms α on V satisfying:*
- ▷ $\alpha(x^\#) \geq 2\alpha(x)$ *for all* $x \in V$;
- ▷ $\alpha(x \times y) \geq \alpha(x) + \alpha(y)$ *for all* $x, y \in V$;
- ▷ $\alpha(e) \geq 0$.

The rest of the section is devoted to the proof of the theorem. In each case, we let $\mathcal{N}_E$ be the relevant set of norms on V_E, and it is then easy to check that the collection $\{\mathcal{N}_E\}$ satisfies conditions (BC) and (RAT) in [9, Thm. 3.5]. The verification of the condition (TRANS) is similar in the different cases and rests ultimately on Theorem 4.5 (ii).

(i) In this case, the verification of (TRANS) is precisely the special case in the proof of Theorem5.1 above.

(ii) This case is slightly more involved. Suppose that L is an A-lattice on which (N, Q, t) is integer-valued and L is self-dual with respect to T. Since

$$T(x^\#, y) = Q(y, x)$$

we see that L is closed under $\#$. To see that L is a distinguished order, it remains to verify that $e \in L$. Suppose that $\lambda \cdot e \in L$, but $\lambda' \cdot e \notin L$ for any λ' such that $\mathrm{ord}(\lambda') < \mathrm{ord}(\lambda)$. Then $\mathrm{ord}(\lambda) \geq 0$, and we need to show that equality holds. But by (3), we have:

$$T(x, e)(\lambda \cdot e) = (\lambda \cdot e) \times x + \lambda \cdot x,$$

so that $T(x,e)(\lambda\cdot e)$ lies in L for any $x\in L$. In particular, by the minimality of $\operatorname{ord}(\lambda)$, we deduce that $T(x,e)\in A$ for all $x\in L$. By the self-duality of L, we conclude that $e\in L$, and thus L is a distinguished order. (TRANS) then follows by Theorem 4.5 (ii).

The remaining cases (iii) and (iv) are even simpler, and we leave them to the reader. Theorem 5.6 is proved.

As we mentioned before, the symmetric bilinear form T defines an outer automorphism of H, with G as the group of fixed points. The induced action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{B}(H)$ is the map which sends a norm α to its dual α^* relative to T. Hence, Theorem 5.6 (ii) implies:

Corollary 5.7. — *The building $\mathcal{B}(G)$ is the subset of $\mathcal{B}(H)$ which is fixed pointwise by the action of $\mathbb{Z}/2\mathbb{Z}$.*

When $p\neq 2$, this corollary also follows from a general result in [15].

6. Apartments

We now describe the apartments of $\mathcal{B}(G)$ using its realization provided by Theorem 5.6. For this, we need to know how to specify a maximal split torus of G in terms of the Jordan algebra structure on V. From the construction in §3, we see that under the action of the maximal split torus S on V, the trivial character appears with multiplicity 3. Moreover, this 3-dimensional space of trivial weight has a canonically determined basis $\{e_1,e_2,e_3\}$ of orthogonal primitive idempotents. In the Freudenthal model, the e_i's are the elements $\operatorname{diag}(1,0,0)$ and so on. Recall that a primitive idempotent of V is a non-zero element x such that $x^{\#}=0$ and $x^2=x$, and a set $\{e_1,e_2,e_3\}$ of primitive idempotents of V is orthogonal if $e_jU_{e_i}=0$ and $(e_i+e_j)^2=e_i^2+e_j^2$ for all $i\neq j$ (see [12, 5.1]).

Now the algebraic subgroup of G which fixes the set $\{e_1,e_2,e_3\}$ pointwise is isomorphic to the split group Spin_8. When $p\neq 2$, this is a special case of a result of Soda [17]. However, the restriction on p is not necessary: the result follows quickly from the theory of triality (which works over any field) discussed in [9, Section 11]. The action of Spin_8 preserves the three subspaces VU_{e_i,e_j}, for $i\neq j$, which are all 8-dimensional. Moreover, VU_{e_i,e_j} is endowed with the quadratic form

$$x\longmapsto N(e_k+x)=Q(e_k,x),$$

where $\{i,j,k\}=\{1,2,3\}$. Indeed, the action of Spin_8 on the subspaces VU_{e_i,e_j} gives a realization of the three 8-dimensional representations of Spin_8. Further, the algebraic subgroup of G fixing $\{e_1,e_2,e_3\}$ setwise is the semi-direct product $\mathrm{Spin}_8\rtimes S_3$, and this group permutes the set $\{VU_{e_1,e_2},VU_{e_2,e_3},VU_{e_3,e_1}\}$, with its identity component Spin_8 acting trivially.

In any case, the split torus S is also a maximal split torus of Spin_8, and its action on VU_{e_i,e_j} has distinct weights. Hence, its weight vectors determine a decomposition of VU_{e_i,e_j} into lines. Further, by choosing a suitable vector from each of these lines, we obtain a Witt basis of the quadratic space VU_{e_i,e_j}. We call such a decomposition into lines of a quadratic space a *Witt decomposition*. Hence, we have seen that a maximal split torus S of G determines a complete system of orthogonal primitive idempotents, as well as Witt decompositions of the quadratic spaces VU_{e_i,e_j}.

Conversely, given a system $\eta = \{\eta_1, \eta_2, \eta_3\}$ of orthogonal primitive idempotents, one can find an element of $G(k)$ which conjugates η to the standard triple $\{e_1, e_2, e_3\}$. When $p \neq 2$, this is a special case of [11, Thm. 10, p. 389]; in general, this transitivity can be deduced from the Strong Coordinatization Theorem [12, 5.4.2]. Hence, the pointwise stabilizer of η is a subgroup G_η of G isomorphic to Spin_8. A Witt decomposition of VU_{e_1,e_2} determines a maximal split torus of G_η, and hence a maximal split torus of G. Observe that once we have chosen a Witt decomposition for VU_{e_1,e_2}, we obtain canonical Witt decompositions of VU_{e_2,e_3} and VU_{e_3,e_1}. In summary, we have shown:

Proposition 6.1. — *To give an apartment of $\mathcal{B}(G)$, it is necessary and sufficient to give the following data:*

- ▷ *a system of orthogonal primitive idempotents $\{e_1, e_2, e_3\}$ of V;*
- ▷ *a Witt decomposition $VU_{e_1,e_2} = \bigoplus_i ke^i_{12}$.*

Given such a data, we obtain canonical Witt decompositions

$$VU_{e_2,e_3} = \bigoplus_i ke^i_{23} \quad \textit{and} \quad VU_{e_3,e_1} = \bigoplus_i ke^i_{31},$$

and thus a decomposition of V into lines. The subgroup of G which preserves each of these lines is a maximal split torus of G, and its corresponding apartment is the subset of $\mathcal{N}$ consisting of norms which are split by the basis $\{e_1, e_2, e_3, e^i_{12}, e^i_{23}, e^i_{31}\}$.

7. Simplicial Structures of $\mathcal{B}(H)$

In this section, we describe the simplicial complex structure of $\mathcal{B}(H)$ in terms of graded lattice chains in V.

By the construction of the strong descent datum $\iota_* : \mathcal{B}(H) \hookrightarrow \mathcal{B}(\mathrm{SL}(V))$, we have an embedding of apartments

$$\iota_* : \mathcal{A}(T) \hookrightarrow \mathcal{A}(C).$$

Fixing the point $x_0 \in \iota_*(\mathcal{A}(T))$ corresponding to the lattice J as the origin, we identify $\mathcal{A}(T)$ and $\mathcal{A}(C)$ with $X_*(T) \otimes \mathbb{R}$ and $X_*(C) \otimes \mathbb{R}$ respectively. The embedding ι_* is then identified with the canonical embedding of real vector spaces induced by the inclusion $T \hookrightarrow C$, and has been described in Lemma 3.2.

Using the coordinate functions $\{r_i\}$ introduced in §3, we let C_H be the closed chamber of $\mathcal{A}(T)$ bounded by the hyperplanes

$$H_0 : r_0 = 1, \quad H_i : r_i = 0, \text{ for } i = 1, 2, \dots, 6.$$

Let v_i be the vertex of C_H lying on H_j for all $j \neq i$. Then the seven vertices of C_H are the origin 0, and the six points whose images under ι_* are given in Lemma 3.2.

Now a point p on $\mathcal{A}(C)$, with coordinates $(a^i_{jk}(p))$ such that $\sum_{i,j,k} a^i_{jk}(p) = 0$, determines a norm α_p on V, characterized by the requirement that α_p is split by the basis $\{e^i_{jk}\}$, and

$$\alpha_p(e^i_{jk}) = -a^i_{jk}(p).$$

By Theorem 5.1, the norms determined by points $p \in \mathcal{A}(T)$ have volume 0, and minorize N. As discussed in [9, §2], these norms give rise to graded lattices chains in V. We shall later need to detect those points $p \in \mathcal{A}(T)$ whose corresponding graded lattice chains have period 1, $\frac{1}{2}$ or $\frac{1}{3}$. For this, we have the following useful lemma.

Lemma 7.1. — *The set of those points in $\mathcal{A}(T)$ whose associated graded lattice chain has period 1 is precisely the lattice M of the real vector space $\mathcal{A}(T) \cong X_*(T) \otimes \mathbb{R}$ spanned by the vectors*

$$\begin{aligned} v_1 &= (1,0,0,0,0,0), & 2v_2 &= (0,1,0,0,0,0), & 2v_3 &= (0,0,1,0,0,0),\\ 3v_4 &= (0,0,0,1,0,0), & 2v_5 &= (0,0,0,0,1,0), & v_6 &= (0,0,0,0,0,1). \end{aligned}$$

The set of those norms whose graded lattice chain has period 1 or $\frac{1}{2}$ (resp. $\frac{1}{3}$) is precisely the lattice $\frac{1}{2}M$ (resp. $\frac{1}{3}M$).

The set of points in $\mathcal{A}(C)$ satisfying the same condition is obviously a lattice in $X_*(C) \otimes \mathbb{R} = \mathcal{A}(C)$. The above lemma is proved by computing the intersection of this lattice with $X_*(T) \otimes \mathbb{R}$. We omit the details.

The following proposition lists the graded lattice chains associated to each vertex v_i, and observes some properties of these lattice chains, some of which follow from Theorem 5.1.

Proposition 7.2. — (i) *The graded lattice chain corresponding to v_0 has period 1 and is determined by the lattice $L_0 = J$. Moreover,* $\mathrm{vol}(L_0) = 0$ *and N is integer-valued on L_0.*

(ii) *The graded lattice chain corresponding to v_1 has period 1 and is determined by the lattice*

$$L_{\frac{1}{3}} = \begin{pmatrix} \pi & \pi & \pi \\ A & A & A \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi^{-1} & A & A \\ A & \pi & \pi \\ A & \pi & \pi \end{pmatrix}.$$

Moreover, $\mathrm{vol}(L_{\frac{1}{3}}) = 9$, $\pi^{-1}N$ *is integer-valued on* $L_{\frac{1}{3}}$ *and*

$$(L_{\frac{1}{3}}/\pi L_{\frac{1}{3}}, \pi^{-1}N) \cong (J/\pi J, N).$$

(iii) *The graded lattice chain corresponding to* v_6 *has period* 1, *and is determined by:*

$$L_{\frac{2}{3}} = \begin{pmatrix} A & \pi & \pi \\ A & \pi & \pi \\ A & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix}.$$

Moreover, $\mathrm{vol}(L_{\frac{2}{3}}) = 18$, $\pi^{-2}N$ *is integer-valued on* $L_{\frac{2}{3}}$ *and*

$$(L_{\frac{2}{3}}/\pi L_{\frac{2}{3}}, \pi^{-2}N) \cong (J/\pi J, N).$$

(iv) *The graded lattice chain corresponding to* v_2 *has period* $\frac{1}{2}$ *and is given by:*

$$L_0 = \begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ A & A & A \end{pmatrix},$$

$$L_{\frac{1}{2}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi & A & \pi \\ \pi & A & \pi \\ \pi & A & \pi \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix}.$$

Moreover,

$$\mathrm{vol}(L_0) = 6 \text{ and } N \text{ is integer-valued on } L_0;$$

$$\mathrm{vol}(L_{\frac{1}{2}}) = 21, \text{ and } N \text{ takes value in } \pi^2 A \text{ on } L_{\frac{1}{2}}.$$

Further, $L_{\frac{1}{2}} = \mathfrak{R}(L_0, N)$, *and*

$$(L_0/L_{\frac{1}{2}}, N) \cong \big(H_3(C), \det\big),$$

with C *equal to the algebra of* 2×2 *matrices over* A/π (*cf. Example* 2.3).

(v) *The graded lattice chain corresponding to* v_3 *has period* $\frac{1}{2}$, *and is given by:*

$$L_{\frac{1}{3}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{5}{6}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} A & A & \pi \\ \pi & \pi & \pi^2 \\ \pi & \pi & \pi^2 \end{pmatrix}.$$

Moreover,

$$\mathrm{vol}(L_{\frac{1}{3}}) = 15 \text{ and } \pi^{-1}N \text{ is integer-valued on } L_{\frac{1}{3}};$$

$$\mathrm{vol}(L_{\frac{5}{6}}) = 30 \text{ and } \pi^{-1}N \text{ takes value in } \pi^2 A \text{ on } L_{\frac{5}{6}}.$$

Further, $\mathfrak{R}(L_{\frac{1}{3}}, \pi^{-1}N) = L_{\frac{5}{6}}$ *and*

$$(L_{\frac{1}{3}}/L_{\frac{5}{6}}, \pi^{-1}N) \cong \big(H_3(C), \det\big),$$

with C *equal to the algebra of* 2×2 *matrices over* A/π.

(vi) *The graded lattice chain corresponding to v_5 has period $\frac{1}{2}$, and is given by:*

$$L_{\frac{2}{3}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{7}{6}} = \begin{pmatrix} \pi & \pi & \pi^2 \\ \pi & \pi & \pi^2 \\ \pi & \pi & \pi^2 \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^2 & \pi^2 & \pi^2 \end{pmatrix}.$$

Moreover,

$$\begin{aligned} &\mathrm{vol}(L_{\frac{2}{3}}) = 24 \text{ and } \pi^{-2}N \text{ is integer-valued on } L_{\frac{2}{3}};\\ &\mathrm{vol}(L_{\frac{7}{6}}) = 39 \text{ and } \pi^{-2}N \text{ takes value in } \pi^2 A \text{ on } L_{\frac{7}{6}}. \end{aligned}$$

Further, $\mathfrak{R}(L_{\frac{2}{3}}, \pi^{-2}N) = L_{\frac{7}{6}}$ *and*

$$(L_{\frac{2}{3}}/L_{\frac{7}{6}}, \pi^{-2}N) \cong (H_3(C), \det),$$

with C equal to the algebra of 2×2 *matrices over A/π.*

(vii) *The graded lattice chain corresponding to v_4 has period $\frac{1}{3}$ and is given by:*

$$L_0 = \begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{1}{3}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{2}{3}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix}.$$

Moreover,

$$\begin{aligned} &\mathrm{vol}(L_0) = 9 \text{ and } N \text{ is integer-valued on } L_0;\\ &\mathrm{vol}(L_{\frac{1}{3}}) = 18 \text{ and } \pi^{-1}N \text{ is integer-valued on } L_{\frac{1}{3}};\\ &\mathrm{vol}(L_{\frac{2}{3}}) = 27 \text{ and } \pi^{-2}N \text{ is integer-valued on } L_{\frac{2}{3}}. \end{aligned}$$

Indeed, $(L_0, N) \cong (L_{\frac{1}{3}}, \pi^{-1}N) \cong (L_{\frac{2}{3}}, \pi^{-2}N)$, *and*

$$\mathfrak{R}(L_0, N) = L_{\frac{1}{3}}, \quad \mathfrak{R}(L_{\frac{1}{3}}, \pi^{-1}N) = L_{\frac{2}{3}}, \quad \mathfrak{R}(L_{\frac{2}{3}}, \pi^{-2}N) = L_1 = \pi L_0.$$

Further,

$$(L_0/L_{\frac{1}{3}}, N) \cong (M, \det),$$

where M is the algebra of 3×3 *matrices over A/π (cf. Example* 2.1).

There is an evident $\mathbb{Z}/3\mathbb{Z}$-symmetry in the description of the graded lattice chains given in Proposition 7.2, which is better appreciated in silent contemplation than made precise in words. It reflects the $\mathbb{Z}/3\mathbb{Z}$-symmetry in the extended Dynkin diagram of H, and will be exploited in our description of the vertices of $\mathcal{B}(H)$ and their incidence relations. Motivated by this symmetry and Proposition 7.2, we define the following sets of lattices in V:

(i) Let V_0 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 0$;
- ▷ N is integer-valued on L.

(ii) Let V_1 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 9$;
- ▷ $\pi^{-1}N$ is integer-valued on L.

(iii) Let V_6 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 18$;
- ▷ $\pi^{-2}N$ is integer-valued on L.

(iv) Let V_2 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 6$;
- ▷ N is integer-valued on L;
- ▷ $\dim_{A/\pi} L/\mathfrak{R}(L,N) = 15$;
- ▷ N takes values in $\pi^2 A$ on $\mathfrak{R}(L,N)$.

(v) Let V_3 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 15$;
- ▷ $\pi^{-1}N$ is integer-valued on L;
- ▷ $\dim_{A/\pi} L/\mathfrak{R}(L,\pi^{-1}N) = 15$;
- ▷ $\pi^{-1}N$ takes values in $\pi^2 A$ on $\mathfrak{R}(L,\pi^{-1}N)$.

(vi) Let V_5 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 24$;
- ▷ $\pi^{-2}N$ is integer-valued on L;
- ▷ $\dim_{A/\pi} L/\mathfrak{R}(L,\pi^{-2}N) = 15$;
- ▷ $\pi^{-2}N$ takes values in $\pi^2 A$ on $\mathfrak{R}(L,\pi^{-2}N)$.

(vii) Let V_4 be the set of lattices L in V such that
- ▷ $\mathrm{vol}(L) = 9$;
- ▷ N is integer-valued on L;
- ▷ $(L,N) \cong (\mathfrak{R}(L,N), \pi^{-1}N)$;
- ▷ $\mathfrak{R}^3(L,N) = \pi L$.
- ▷ the induced cubic form on $L/\mathfrak{R}(L,N)$ is irreducible.

Again, notice the $\mathbb{Z}/3\mathbb{Z}$-symmetry in the definitions of the sets V_i. Indeed, if we let $h \in H'(k)$ be an element such that $\lambda(h) = \pi$, then h maps the set V_0 bijectively to V_1, and V_1 bijectively to V_6. The sets V_2, V_3 and V_5 are related in the same way.

THEOREM 7.3. — *There is a natural bijection between V_i and the set of vertices of type i in $\mathcal{B}(H)$.*

Proof. — In each case, Proposition 7.2 furnishes an injection from the set of vertices of type i to the set V_i. Hence it remains to show that this injection is also a surjection.

Suppose that L lies in V_0, V_1 or V_6. Consider the graded lattice chain of period 1 determined by L, with grading

$$c(L) = \begin{cases} 0 \text{ if } L \in V_0, \\ \frac{1}{3} \text{ if } L \in V_1, \\ \frac{2}{3} \text{ if } L \in V_6. \end{cases}$$

By hypothesis, the corresponding norm α on V has volume 0 and minorizes N. Thus α lies on $\mathcal{B}(H)$, in view of Theorem 5.1. We need to show that α is a vertex of the appropriate type. To see this, we may assume that α lies in the chamber C_H defined earlier in the section. Now it follows by Lemma 7.1 that the only norms in C_H for which the corresponding graded lattice chain has period 1 are the 3 vertices v_0, v_1 and v_6. Moreover, these are distinguished by the fact that

$$\alpha \text{ takes value in } \begin{cases} \mathbb{Z} & \text{if } L \in V_0, \\ \frac{1}{3} + \mathbb{Z} & \text{if } L \in V_1, \\ \frac{2}{3} + \mathbb{Z} & \text{if } L \in V_6. \end{cases}$$

This proves the result for $i = 0$, 1 or 6.

Now suppose that L lies in V_2, V_3 or V_5. Consider the graded lattice chain of period $\frac{1}{2}$:

$$L \supset R \supset \pi L \supset \cdots$$

with grading

$$c(L) = \begin{cases} 0 \text{ if } L \in V_2, \\ \frac{1}{3} \text{ if } L \in V_3, \\ \frac{2}{3} \text{ if } L \in V_5, \end{cases}$$

where R is $\mathfrak{R}(L, N)$ if $L \in V_2$, $\mathfrak{R}(L, \pi^{-1}N)$ if $L \in V_3$, and $\mathfrak{R}(L, \pi^{-2}N)$ if $L \in V_5$. By hypothesis, the corresponding norm α has volume 0 and minorizes N, and hence lies on $\mathcal{B}(H)$. Again we may assume that α lies in the chamber C_H. Lemma 7.1 implies that, besides the vertices v_2, v_3 and v_5, the only norms in C_H which give rise to period $\frac{1}{2}$ lattice chains are

$$\frac{1}{2}(v_i + v_j) \quad \text{where } i, j \text{ are distinct elements of } \{0, 1, 6\}.$$

However, the graded lattice chains corresponding to these norms have successive quotients of dimensions 11 and 16, unlike those for the three vertices, which are 15 and 12. Moreover, since the three vertices can be distinguished by the value group of the norms, this proves the result for $i = 2$, 3 and 5.

Finally, we suppose that $L \in V_4$ and consider the graded lattice chain of period $\frac{1}{3}$:

$$L \supset \mathfrak{R}(L, N) \supset \mathfrak{R}^2(L, N) \supset \pi L \supset \cdots$$

with grading $c(L) = 0$. By hypothesis, the corresponding norm has volume 0 and minorizes N. Hence we may assume it lies in C_H. By Lemma 7.1, we see that besides v_4, the other norms in C_H which give rise to lattice chains of period $\frac{1}{3}$ are $\frac{1}{3}(v_0 + v_1 + v_6)$ and the points

$$\frac{1}{3}(v_i + 2v_j), \quad \text{where } i \in \{0, 1, 6\} \text{ and } j \neq 4, i.$$

However, by computing the dimensions of the successive quotients of the lattice chains corresponding to these latter points, we see that α is not one of them; we will omit the details here. The only possibility left, besides v_4, is the point $\frac{1}{3}(v_0 + v_1 + v_6)$. On examining its associated graded lattice chain, one sees that L_0 satisfies all the conditions in the definition of an element of V_4, except the last one, i.e. the induced cubic form on $L_0/L_{\frac{1}{3}}$ is reducible for the point $\frac{1}{3}(v_0 + v_1 + v_6)$. More precisely, the cubic space $L_0/L_{\frac{1}{3}}$ is isomorphic to the cubic space associated to the split octonion algebra, as in Example 2.3. This proves the theorem. □

Corollary 7.4. — *If $L \in V_i$, then the quotient of L by its radical is isomorphic to the cubic space*

$(J/\pi J, N)$, *if* $i = 0, 1, 6$;

$(H_3(C), \det)$, *with C the algebra of* 2×2 *matrices, if* $i = 2, 3, 5$;

$(M, \det)$, *with M the algebra of* 3×3 *matrices, if* $i = 4$.

Moreover, if V_4^* is the set of lattices L satisfying all the conditions in the definition of V_4, except possibly the last one, then $H(k)$ has two orbits on V_4^*, and these are represented by v_4 and $\frac{1}{3}(v_0 + v_1 + v_6)$. If $L \in V_4^* \smallsetminus V_4$, then the quotient of L by its radical is isomorphic to the cubic space $A/\pi \times C$, where C is the split octonion algebra.

The remainder of the section is devoted to describing the incidence relations of the vertices of $\mathcal{B}(H)$. These are all given by appropriate inclusions. Thanks to the $\mathbb{Z}/3\mathbb{Z}$-symmetry, it will not be necessary to write down all the 21 incidence relations. Instead, as we shall explain later, it suffices to prove the following theorem.

Theorem 7.5. — *Let L^i be a lattice in V_i. Then we have:*

(i) *L^0 is incident to L^2 if and only if $L^2 \subset L^0$.*

(ii) *L^0 is incident to L^3 if and only if $L^3 \subset L^0$.*

(iii) *L^0 is incident to L^4 if and only if $L^4 \subset L^0$.*
(iv) *L^0 is incident to L^5 if and only if $\mathfrak{R}(L^5, \pi^{-2}N) \subset \pi L^0$.*
(v) *L^0 is incident to L^6 if and only if $L^6 \subset L^0$.*
(vi) *L^2 is incident to L^3 if and only if $L^3 \subset L^2$.*
(vii) *L^2 is incident to L^4 if and only if $L^4 \subset L^2$.*

Proof. — In each case, the inclusion condition is easily seen to be necessary, by Proposition 7.2. The proof of its sufficiency can be reduced to a computer calculation (using Mathematica). Let us first explain the strategy of the proof.

Suppose we are interested in showing that the incidence of L^i and L^j is determined by $L^j \subset L^i$. Without loss of generality, let us assume that L^i is the lattice corresponding to the vertex v_i in C_H, and L^j corresponds to a vertex on the apartment $\mathcal{A}(T)$. To prove the theorem, it suffices to show that the number N_{ij} of vertices on $\mathcal{A}(T)$ which is of type j and incident to v_i is equal to the number N'_{ij} of vertices v' on $\mathcal{A}(T)$ which is of type j and for which the corresponding lattice L' is contained in L^i.

The number N_{ij} can be computed by well-known theory. Indeed, by [3, Thm. 5F, p. 24], we have

$$N_{ij} = \#W_i/W_{ij},$$

where W_i (resp. W_{ij}) is the Weyl group whose associated Dynkin diagram is obtained from the extended Dynkin diagram of H by removing the vertex i (resp. the vertices i and j). We tabulate the value of N_{ij} below:

(i,j)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(2,3)	(2,4)
N_{ij}	72	216	720	216	27	15	20

It remains to calculate N'_{ij}. Recall that the norms in $\mathcal{A}(T)$ are precisely those split by the natural basis $\{e^{\ell}_{mn}\}$ of J introduced in §3. Regarding a point $p \in \mathcal{A}(T)$ as a point in $\mathcal{A}(C)$ via the embedding ι_*, the (a^{ℓ}_{mn})-coordinates of p is given by

$$a^{\ell}_{mn}(p) = -\alpha_p(e^{\ell}_{mn}),$$

where α_p is the associated norm. Suppose that in the graded lattice chain associated to L^i (resp. L^j), the grading of L^i (resp. L^j) is r_i (resp. r_j). Then the set of norms α in $\mathcal{A}(T)$ satisfying $L_{\alpha,r_j} \subset L^i$ is precisely the region U_{ij} of $\mathcal{A}(T)$ consisting of those points p satisfying the 27 inequalities:

$$a^{\ell}_{mn}(p) > \lceil r_i + a^{\ell}_{mn}(v_i) \rceil - r_j - 1, \quad \text{for all } \ell, m, n \in \{1,2,3\}. \tag{11}$$

Expressing the restriction of a^{ℓ}_{mn} to $\mathcal{A}(T)$ as linear combinations of the coordinate functions $r_1, r_2, \ldots, r_6$ on $\mathcal{A}(T)$, we can write the inequalities in (11), and thus describe the region U_{ij}, using $\{r_k\}$.

It remains to count the number of vertices of type j contained in U_{ij}. This is the part where the computer is used, and is carried out as follows:

- ▷ Using linear programming, we find a box B, defined by $|r_k(p)| < c_k$ for $k = 1, 2, ..., 6$, which contains the region U_{ij}.
- ▷ Since a vertex of type j has graded lattice chains of period 1, $\frac{1}{2}$ or $\frac{1}{3}$, it follows by Lemma 7.1 that it lies in the lattice M, $\frac{1}{2}M$ or $\frac{1}{3}M$, where M is the lattice spanned by v_1, v_6, $2v_2$, $2v_3$, $2v_5$ and $3v_4$. We enumerate all the elements of the appropriate lattice which lie in B, calling this finite set S.
- ▷ We enumerate the elements in the finite set S which satisfy all the 27 inequalities in (11), calling this finite set S'.
- ▷ For each element in S', we compute the numerical invariants (the grading and the volumes of lattices) of its associated graded lattice chain. Using Theorem 7.3, pick out those elements of S' which are possibly vertices of type j, calling the resulting set S''.
- ▷ Now we have $N_{ij} \leq N'_{ij} \leq \#S''$. In all but one case, we find out that $N_{ij} = \#S''$ and this proves the incidence relation. The exceptional case is when $i = 0$, $j = 4$, and we obtain $\#S'' = 990$ in that case. However, it is clear that S'' contains points in $V_4^* \setminus V_4$, and the number of such points lying on a closed chamber of $\mathcal{A}(T)$ containing L^0 is $\#W(E_6)/\#W(D_4) = 270$. This shows that $N'_{ij} \leq 990 - 270 = 720 = N_{ij}$ and implies the desired conclusion.

On carrying out the above steps, the theorem is proved. □

As we mentioned above, Theorem 7.5 is sufficient to give all incidence relations between the vertices of $\mathcal{B}(H)$. To see this, recall that the group $H'(k)$ acts on the building $\mathcal{B}(H)$ as automorphisms of simplicial complex. The action is given by the formula

$$h\alpha(v) = \alpha\big(h^{-1}(v)\big) + \frac{1}{3}\operatorname{ord}\big(\lambda(h)\big) \text{ with } h \in H'(k),\ \alpha \in \mathcal{B}(H) \text{ and } v \in V. \tag{12}$$

On the level of graded lattice chains, if $(L_\bullet, c)$ corresponds to α, then the graded lattice chain corresponding to $h\alpha$ is $(h \cdot L_\bullet, c')$ with

$$c'(h \cdot L) = c(L) + \frac{1}{3}\operatorname{ord}\big(\lambda(h)\big). \tag{13}$$

The stabilizer in $H'(k)$ of the closed chamber C_H induces the action of $\mathbb{Z}/3\mathbb{Z}$ on the extended Dynkin diagram. The seven pairs of indices in the above theorem are precisely a set of representatives for the orbits of $\mathbb{Z}/3\mathbb{Z}$ on the set of pairs of distinct vertices of the extended Dynkin diagram. Hence, given an arbitrary pair of indices (i, j), let (i_0, j_0) be the unique pair of indices in the theorem which is conjugate to (i, j). Then, given vertices x and y of type i and j respectively, there exists an element $h \in H'(k)$ which sends x and y to vertices $h(x)$ and $h(y)$ of type i_0 and j_0 in the building $\mathcal{B}(H)$. Since h induces an automorphism of simplicial complex, x and y are incident in $\mathcal{B}(H)$

if and only if hx and hy are incident in $\mathcal{B}(H)$. Since Theorem 7.5 already gives necessary and sufficient conditions for the incidence of hx and hy in terms of their graded lattice chains, we obtain necessary and sufficient conditions for the incidence of x and y.

As an illustration, suppose we are interested in the incidence relation between a norm α of type 0 and a norm β of type 1. Observe that the pair of vertices $(0,6)$ in the extended Dynkin diagram can be brought to the pair $(1,0)$ by the action of $\mathbb{Z}/3\mathbb{Z}$. Hence, there exists $h \in H'(k)$, with $\lambda(h) = \pi^{-1}$, such that $h\alpha$ and $h\beta$ correspond to vertices of type 6 and 0 respectively. Now by Theorem 7.5 (v), we know that $h\alpha$ is incident to $h\beta$ if and only if

$$L_{h\alpha,\frac{2}{3}} \subset L_{h\beta,0}.$$

By (13), we deduce that α is incident to β if and only if

$$L_{\alpha,1} = \pi L_{\alpha,0} \subset L_{\beta,\frac{1}{3}}.$$

8. Simplicial Structures of $\mathcal{B}(G)$

In this section, we describe the simplicial structure of $\mathcal{B}(G)$. Because of the various natural descriptions of $\mathcal{B}(G)$ given in Theorem 5.6, it is possible to describe the simplicial structure in various ways. Since we like to think of G as the automorphism group of the split exceptional Jordan algebra V, we shall describe the simplicial structure using orders in V. More precisely, we shall think of $\mathcal{B}(G)$ as the set of norms satisfying the conditions in Theorem 5.6 (iv), and by an order of V, we shall mean a lattice L which contains e, and is stable under the quadratic map $\#$.

Recall that the strong descent datum $\iota'_* : \mathcal{B}(G) \hookrightarrow \mathcal{B}(\mathrm{GL}(V))$ factors through ι_*, so that we have $\iota'_* : \mathcal{B}(G) \hookrightarrow \mathcal{B}(H)$. This induces an embedding of apartments:

$$\iota'_* : \mathcal{A}(S) \longhookrightarrow \mathcal{A}(T).$$

Having chosen the vertex v_0 as the origin in $\mathcal{A}(T)$, we identify $\mathcal{A}(S)$ and $\mathcal{A}(T)$ with $X_*(S)\otimes\mathbb{R}$ and $X_*(T)\otimes\mathbb{R}$. Further, let C_G be the closed chamber bounded by the hyperplanes

$$H'_i : r'_i = 0, \text{ for } i = 1,2,3,4; \quad H'_0 : r'_0 = 2r'_1 + 3r'_2 + 4r'_3 + 2r'_4 = 1,$$

where we recall that $\{r'_1, r'_2, r'_3, r'_4\}$ are coordinate functions on $\mathcal{A}(S)$ furnished by the simple roots. Let v'_i be the vertex of C_G lying on H_j for all $j \neq i$. From (10), we see that the embedding ι'_* is given by:

$$v'_0 \mapsto v_0, \quad v'_1 \mapsto v_2, \quad v'_2 \mapsto v_4, \quad v'_3 \mapsto \frac{1}{2}(v_3 + v_5), \quad v'_4 \mapsto \frac{1}{2}(v_1 + v_6).$$

By Theorem 5.6, we know that $\mathcal{A}(S)$ is the subset of $\mathcal{A}(T)$ consisting of those norms which are self-dual with respect to the symmetric bilinear form T. Moreover, such norms are norms of Jordan algebra.

For a lattice $L \subset V$, we write $L^{\#}$ for the lattice of V spanned by $x^{\#}$ for all $x \in L$. Similarly, if M is another lattice, $L \times M$ will denote the lattice spanned by $x \times y$ for all $x \in L$ and $y \in M$. Now the following proposition enumerates the graded lattice chains associated to the vertices v_i'.

PROPOSITION 8.1. — (i) *The graded lattice chain associated to v_0' is the same as that of v_0. Moreover, $L_0 = J$ is a distinguished order, and is self-dual relative to the symmetric bilinear form T.*

(ii) *The graded lattice chain associated to v_1' is the same as that of v_2. Moreover, L_0 is a order with radical*

$$L_{\frac{1}{2}} = \pi L_0^*,$$

and $L_0/L_{\frac{1}{2}}$ is isomorphic to the Jordan algebra $H_3(C)$ where C is the algebra of 2×2 matrices over A/π (cf. example 2 of §2). We have $L_{1/2}^{\#} \subset \pi L_0$.

(iii) *The graded lattice chain associated to v_2' is the same as that of v_4. Moreover, L_0 is an order with radical*

$$L_{\frac{1}{3}} = \pi L_0^*,$$

and $L_{\frac{2}{3}} = L_{\frac{1}{3}}^{\#} + \pi L_0$. Further, $L_0/L_{\frac{1}{3}}$ is isomorphic to the Jordan algebra M^+ (cf. example 1 of §2), and $L_{-1/3}$ is self-dual.

(iv) *The graded lattice chain associated to v_3' has period $\frac{1}{4}$ and is given by*

$$L_0 = \begin{pmatrix} A & A & \pi \\ A & A & \pi \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{1}{4}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ A & A & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{2}{4}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix},$$

$$L_{\frac{3}{4}} = \begin{pmatrix} \pi & \pi & \pi \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} A & A & \pi \\ \pi & \pi & \pi^2 \\ \pi & \pi & \pi^2 \end{pmatrix}.$$

Moreover, L_0 is an order with radical

$$L_{\frac{1}{4}} = \pi L_0^*,$$

and $L_0/L_{\frac{1}{4}}$ is isomorphic to the Jordan algebra associated to the pointed quadratic space $(C, \det, 1)$, where C is the algebra of 2×2 matrices over A/π. Further, we have:

$$L_{\frac{2}{4}} = L_{\frac{1}{4}}^{\#} + \pi L_0, \quad L_{\frac{3}{4}} = L_{\frac{1}{4}} \times L_{\frac{2}{4}} + \pi L_0,$$

and the lattices $L_{-\frac{2}{4}}$ and $L_{-\frac{1}{4}}$ are dual to each other.

(v) *The graded lattice chain associated to v_4' has period $\frac{1}{2}$, and is given by*

$$L_0 = \begin{pmatrix} A & \pi & \pi \\ A & A & A \\ A & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi & A & A \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} \pi^{-1} & A & A \\ A & \pi & \pi \\ A & \pi & \pi \end{pmatrix},$$

$$L_{\frac{1}{2}} = \begin{pmatrix} \pi & \pi & \pi \\ A & \pi & \pi \\ A & \pi & \pi \end{pmatrix} \oplus \begin{pmatrix} \pi^2 & \pi & \pi \\ \pi & A & A \\ \pi & A & A \end{pmatrix} \oplus \begin{pmatrix} A & A & A \\ \pi & \pi & \pi \\ \pi & \pi & \pi \end{pmatrix}.$$

We have $L_{1/2}^{\#} \subset \pi L_0$. Moreover, L_0 is isomorphic to the non-distinguished maximal order described in §4, with radical

$$L_{\frac{1}{2}} = \pi L_0^*.$$

We define the following sets of orders in V:

(i) Let V_0' be the set of orders which are self-dual relative to T; equivalently, V_0' is the set of distinguished orders.

(ii) Let V_1' be the set of orders L such that
 ▷ $\dim L/\pi L^* = 15$;
 ▷ $L^{*\#} \subset \pi^{-1} L$.

(iii) Let V_2' be the set of orders L such that
 ▷ $L/\pi L^* \cong M_3(A/\pi)$, the algebra of 3×3 matrices over A/π;
 ▷ $M = \pi L^{*\#} + L$ is self-dual.

(iv) Let V_3' be the set of orders L such that
 ▷ $\dim L/\pi L^* = 5$,
 ▷ $M^* = \pi L^{*\#} + L$ is dual to $M = \pi M^* \times L^* + L$.

(v) Let V_4' be the set of orders L such that
 ▷ $\dim L/\pi L^* = 11$,
 ▷ $L^{*\#} \subset \pi^{-1} L$.

THEOREM 8.2. — *There is a natural bijection between V_i' and the set of vertices of $\mathcal{B}(G)$ of type i.*

Proof. — By Proposition 8.1, there is a natural inclusion of the set of vertices of type i into the set V_i. It remains to see that it is also surjective, and the result for $i = 0$ follows from Theorem 4.5 (ii).

Given L in V_1' or V_4', we form the period $\frac{1}{2}$ graded lattice chain

$$L \supset \pi L^* \supset \pi L \supset \cdots$$

with grading $c(L) = 0$. The definitions of these sets guarantee that the associated norm is a self-dual norm of Jordan algebra, and hence lies on $\mathcal{B}(G)$. Assume without loss of generality that α lies in the chamber C_G. But it is easy to see that the only points in C_G whose associated lattice chains have period $\frac{1}{2}$ are the two vertices v_1' and v_4'. Since we can distinguish these two vertices by the dimension of $L/\pi L^*$, the result for $i = 1$ or 4 follows.

Now assume that $L \in V_2'$. We form the period $\frac{1}{3}$ lattice chain:

$$L^* \supset M \supset L \supset \cdots$$

with grading $c(L) = 0$. The corresponding norm α is evidently self-dual. To show that α lies on $\mathcal{B}(G)$, we need to see that it satisfies the conditions in Theorem 5.6(iv). We do this systematically as follows:

- $L^{\#} \subset L$, since L is an order.
- $L^{*\#} \subset \pi^{-1}M$, by definition of M.
- $L \times L^* \subset L^*$. Indeed, given $x \in L$ and $y \in L^*$, we have

  $$T(x \times y, z) = T(y, x \times z) \in A$$

  for all $z \in L$. Hence, $x \times y \in L^*$.
- $L^* \times M \subset \pi^{-1}L$. Indeed, given $x \in L^*$ and $y \in M$, we have, for any $z \in L^*$,

  $$T(x \times y, \pi z) = T(y, \pi x \times z) \in A,$$

  since $\pi x \times z \in M$. Hence, $x \times y \in \pi^{-1}L$.
- $L \times M \subset M$. It suffices to show that if $x \in L$ and $y \in L^*$, then $x \times y^{\#}$ lies in $\pi^{-1}M$. This can be seen using the linearization of (4) and what we have proven so far.
- $M^{\#} \subset L^*$. We first observe that $M \times M \subset L^*$; for any $x, y \in M$ and $z \in L$, we have

  $$T(x \times y, z) = T(x, y \times z) \in A$$

  since $y \times z \in M$ by what we just showed above. It now remains to show that if $x \in L^*$, then $(\pi x^{\#})^{\#}$ lies in L^*. But

  $$(\pi x^{\#})^{\#} = \pi^2 N(x) \cdot x.$$

  Hence we need to show that $N(x) \in \pi^{-2}A$. This follows from (4); indeed, one checks that every term in (4) lies in $\pi^{-2}L$, so that the coefficient of e lies in $\pi^{-2}A$.

We conclude thus that α is a norm of Jordan algebra, and hence lies in $\mathcal{B}(G)$. Assume without loss of generality that α lies in C_G. Besides v_2', the only other points in C_G whose associated graded lattice chain has period $\frac{1}{3}$ are:

$$\frac{1}{3}(v_0' + 2v_1') \quad \text{and} \quad \frac{1}{3}(v_0' + 2v_4').$$

However their graded lattice chains can be distinguished from those arising from orders in V_2' as follows. For the first point, $\dim L_0/L_{\frac{1}{3}} = 15 \neq 9$, and for the second point, $L_0/L_{\frac{1}{3}}$ is isomorphic to the Jordan algebra $A/\pi \times C$ associated to the pointed quadratic space of the octonion algebra. Hence, we deduce that α is equal to v_2', as required.

It remains to deal with the case $i = 3$. Given $L \in V_3'$, we form the period $\frac{1}{4}$ lattice chain

$$L^* \supset M^* \supset M \supset L \supset \cdots$$

with grading $c(L) = 0$. Again, the corresponding norm is self-dual relative to T, and we need to show that α is a norm of Jordan algebra. This is even more tedious than the case $i = 2$, and so we shall be brief in the following:

- ▷ $L^{\#} \subset L$, since L is an order.
- ▷ $L^{*\#} \subset \pi^{-1} M^*$, by definition of M^*.
- ▷ $L \times L^* \subset L^*$; this follows easily using (2).
- ▷ $L \times M^* \subset M^*$. This follows from the linearization of (4).
- ▷ $L \times M \subset M$. This follows from the previous item, using (2).
- ▷ $M \times M^* \subset L^*$. This follows from the previous item, using (2).
- ▷ $M \times L^* \subset \pi^{-1} L$. This follows by (2) and the assumption that M is dual to M^*.
- ▷ $M^* \times L^* \subset \pi^{-1} M$, by definition of M.
- ▷ $M^{*\#} \subset \pi^{-1} L$. First, we observe that $M^* \times M^* \subset \pi^{-1} L$, using (2) and $M^* \times L^* \subset \pi^{-1} M$. It then remains to check that if $x \in L^*$, then $(\pi x^{\#})^{\#}$ lies in $\pi^{-1} L$. This reduces to showing that $N(x) \in \pi^{-2} A$, which follows by an application of (4).
- ▷ $M^{\#} \subset M^*$. It suffices to show that if $x \in M^*$ and $y \in L^*$, then $(x \times y)^{\#}$ lies in $\pi^{-2} M^*$. This follows by an application of (5). We leave the details to the reader.

It follows from the above that we can assume α to be an element of C_G. The other points in C_G whose associated lattice chains have period $\frac{1}{4}$ are

$$\frac{1}{2}(v_0' + v_1'), \quad \frac{1}{4}(v_0' + 3v_2'), \quad \frac{1}{2}(v_0' + v_4'), \quad \frac{1}{2}(v_1' + v_4').$$

However, the graded lattice chains of these points can be distinguished from those arising from orders in V_3'; one simply observes that $\dim L_0/L_{\frac{1}{4}} = 15, 9, 9, 7$ respectively. The theorem is proved completely. □

Corollary 8.3. — *If $L \in V_4'$, then L is a non-distinguished maximal order isomorphic to the one described in* §4.

Theorem 8.4. — *Given a vertex of type i, let $L^i \in V_i'$ be the corresponding order if $i \in \{0,1,2,3\}$, and let L^i be the radical of the corresponding order in V_4' if $i = 4$. Then L^i is incident to L^j if and only if one of them contains the other.*

If L^i, $i = 0,1,2,3,4$ are the lattices corresponding to vertices of the standard chamber C_G, we have $L^0 \supset L^1 \supset L^2 \supset L^3 \supset L^4$ from the description in Proposition 8.1. This proves the "only if" part. The other implication is verified by a computer-assisted calculation, similar to that in the proof of Theorem 7.5. We omit the details.

We conclude this section by giving natural constructions of the orders of the Jordan algebra V which arise in the simplicial description of $\mathcal{B}(G)$. These constructions will be given in terms of the Freudenthal model of V, analogous to the description of the two maximal orders of V given in Section 4. For this, we need to recall some results of [9] on orders in the split octonion algebra $\Lambda \otimes k$.

In [9], we described the building of the automorphism group of $\Lambda \otimes k$ (which is split of type G_2) in terms of orders in $\Lambda \otimes k$. More precisely, there are 3 conjugacy class of orders whose stabilizers are precisely the maximal compact subgroups of $G_2(k)$. The hyperspecial vertices of $\mathcal{B}(G_2)$ correspond to the maximal orders R. On the other hand, a vertex of $\mathcal{B}(G_2)$ whose parahoric group scheme has maximal reductive quotient SO_4 (called a vertex of type 2 in [9]) corresponds to an order R_2 with maximal semisimple quotient

$$R_2/\mathfrak{R}(R_2) \cong M_2(A/\pi),$$

whereas a vertex whose parahoric group scheme has maximal reductive quotient SL_3 (called a vertex of type 3 in [9]) is associated to an order R_3 with maximal semisimple quotient

$$R_3/\mathfrak{R}(R_3) \cong A/\pi \times A/\pi.$$

Let I (resp. J) be the inverse image of the ideal $A/\pi \times \{0\}$ (resp. $\{0\} \times A/\pi$) under the natural projection map $R_3 \to R_3/\mathfrak{R}(R_3)$. Then I and J are ideals of R_3.

Moreover, we have a natural embedding $\mathcal{B}(G_2) \hookrightarrow \mathcal{B}(\mathrm{SO}(q))$ where we recall that q is the quadratic norm form on $\Lambda \otimes k$, and the image of a vertex of type 3 is the barycenter of a 3-simplex in $\mathcal{B}(\mathrm{SO}_8)$. Such a 3-simplex is in turn associated to three lattices in $\Lambda \otimes k$, which we denote by M, N and N'; these lattices are precisely the ones arising in the description of the non-distinguished maximal order of V in Section 4. In particular, M is self-dual, whereas N and N' satisfy $\pi N^* = N$ and $\pi N'^* = N'^*$. More importantly,

$$M * N \subset N', \quad N * N' \subset M, \quad N' * M \subset N,$$

where $*$ is the natural symmetric composition on $\Lambda \otimes k$.

Now we're going to construct an order L in V whose underlying lattice has the form

$$L = \begin{pmatrix} A & \Lambda_3 & \\ & A & \Lambda_1 \\ \Lambda_2 & & A \end{pmatrix} \subset V,$$

where Λ_i is a lattice in the split octonion algebra $\Lambda \otimes k$. By [16, p. 106], such a lattice is an order in V if and only if the norm form q is integral on each Λ_i and $\Lambda_i * \Lambda_{i+2} \subset \Lambda_{i+1}$ for $i = 1, 2, 3$ and the indices are taken modulo 3. Now we have:

Proposition 8.5. — (i) *If $\Lambda_1 = \Lambda_2 = \Lambda_3 = R$, then $L \in V_0'$, so that L is a distinguished order of V.*

(ii) *If $\Lambda_1 = \Lambda_2 = \Lambda_3 = R_2$, then $L \in V_1'$.*

(iii) *If $\Lambda_1 = \Lambda_2 = \Lambda_3 = R_3$, then $L \in V_2'$.*

(iv) *If $\Lambda_1 = I$, $\Lambda_2 = R_3$ and $\Lambda_3 = J$, then $L \in V_3'$.*

(v) *If $\Lambda_1 = M$, $\Lambda_2 = N'$ and $\Lambda_3 = N$, then $L \in V_4'$, so that L is a non-distinguished maximal order.*

9. Group Schemes

The results of §7 and 8 furnish a set of lattices and orders whose stabilizers in $H(k)$ and $G(k)$ are the maximal parahoric subgroups. In this section, we shall see that the smooth group schemes associated to the parahoric subgroups can be described in terms of these orders. In the following, put $\mathcal{V} = \bigcup_i V_i$ and $\mathcal{V}' = \bigcup_i V_i'$.

Theorem 9.1. — *Given a lattice L in $\mathcal{V}$, the schematic closure of H in $\mathrm{Aut}(L)$ is the smooth integral model $\underline{H}_x$ associated to the vertex x corresponding to L. More generally, let X be a finite set of vertices contained in an apartment $\mathcal{A}$ with convex hull Ω, and let $\mathcal{L} \subset \mathcal{V}$ be the set of orders corresponding to the elements of X. Then the schematic closure of H in $\prod_{L\in\mathcal{L}} \mathrm{Aut}(L)$ is the smooth integral model $\underline{H}_\Omega$ of H associated to Ω.*

Let $\underline{H}$ be the schematic closure of H in $\prod_{L\in\mathcal{L}} \mathrm{Aut}(L)$. Then by Theorem 7.3, for any unramified extension $\widetilde{k}/k$ with ring of integers $\widetilde{A}$, $\underline{H}(\widetilde{A})$ is precisely the subgroup of $H(\widetilde{k})$ which fixes X pointwise. Hence it remains to show that $\underline{H}$ is smooth [2, 3.4.1].

Let T be the maximal split torus of H corresponding to $\mathcal{A}$, and $\{U_b\}_{b\in\Phi(H,T)}$ the root subgroups of H relative to T. By [5, 2.2.5], to prove the smoothness of $\underline{H}$, it suffices to show

(i) The schematic closure $\underline{T}$ of T in $\prod_{L\in\mathcal{L}} \mathrm{Aut}(L)$ is smooth;

(ii) for each $b \in \Phi(H,T)$, the schematic closure $\underline{U}_b$ of U_b in $\prod_{L\in\mathcal{L}} \mathrm{Aut}(L)$ is smooth.

The first assertion being immediate in our case, it remains to verify the second assertion, which will follow from the following lemma:

LEMMA 9.2. — *Fix $b \in \Phi(H,T)$ and let $H_b \cong \mathrm{SL}_2$ be its associated rank one subgroup. Then there exists a decomposition*

$$V = \oplus_i V_i$$

of V as a representation of H_b, such that:

- ▷ *each V_i isomorphic to the trivial or the standard representation of H_b;*
- ▷ $L = \bigoplus_i L \cap V_i$.

Proof. — We assume without loss of generality that T is the maximal split torus described in §3. In particular, we have:

$$T \hookrightarrow (\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_3 \hookrightarrow H.$$

The basis $\{e^i_{jk}\}$ of $V = M^{\oplus 3}$ introduced in §3 is a basis of weight vectors for T, corresponding to distinct weights. Hence, the normalizer $N(T)$ of T in H permutes the set of lines $B = \{V^i_{jk}\}$ spanned by these basis vectors. Further, any $L \in \mathcal{L}$ satisfies $L = \bigoplus_{i,j,k} L \cap V^i_{jk}$.

Assume first that U_b is a standard root subgroup (relative to the diagonal torus) of one of the copies of SL_3. Then it is clear that a desired decomposition $V = \bigoplus V_i$ exists; more precisely, there is a partition $B = \bigcup_i B_i$ such that V_i is the span of the lines in B_i. The general result now follows from the fact that the Weyl group $N(T)/T$ acts transitively on the elements of $\Phi(H,T)$. □

By virtue of the lemma, it suffices to show that the schematic closure of U_b in $\mathrm{Aut}(L \cap V_i)$ is smooth for each i and for each $L \in \mathcal{L}$. But this follows from [6, 3.9.2]. Theorem 9.1 is proved completely.

LEMMA 9.3. — *Let x be a vertex of $\mathcal{B}(H)$. The maximal reductive quotient $\overline{H}$ of $\underline{H}_x \otimes A/\pi$ is given by:*

$$\overline{H} \cong \begin{cases} \mathrm{Aut}(J/\pi J, N) \cong E_6, & \text{if } x \in V_0,\ V_1 \text{ or } V_6; \\ (\mathrm{SL}_2 \times \mathrm{SL}_6)/\Delta\mu_2, & \text{if } x \in V_2,\ V_3 \text{ or } V_5; \\ (\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_3, & \text{if } x \in V_4. \end{cases}$$

Proof. — The isogeny class of $\overline{H}$ is determined by its Dynkin diagram, and can be read off from the local Dynkin diagram of H by [2, 3.5.1]. To figure out the isomorphism class, we need to know the root datum $(\overline{X}, \overline{\Phi}, \overline{X}^\vee, \overline{\Phi}^\vee)$ of $\overline{H}$. Indeed, by Bruhat-Tits theory, if $(X, \Phi, X^\vee, \Phi^\vee)$ is the root datum of H, then we can identify $\overline{X}$ with X, $\overline{X}^\vee$ with $X^\vee$, and $\overline{\Phi}$ with the set of affine roots vanishing at x. A system of simple roots of $\overline{\Phi}$ is given by the rule described in [2, 1.9, 3.5.1].

In fact, we can derive our conclusion with very little computation. Let $\widetilde{\overline{H}}$ be the simply connected cover of $\overline{H}$ and C the kernel of $\widetilde{\overline{H}} \to \overline{H}$. The Cartier dual $C^\vee$ of C is the étale (constant) group isomorphic to $\overline{X}^\vee/\mathbb{Z}\langle r_j^\vee : j \neq i\rangle$ if $x \in V_i$. Since H is simply connected, $r_1^\vee, \ldots, r_6^\vee$ is a basis of $X^\vee = \overline{X}^\vee$. From the relation

$$r_0^\vee + r_1^\vee + 2r_2^\vee + 2r_3^\vee + 3r_4^\vee + 2r_5^\vee + r_6^\vee = 0,$$

we see that C is cyclic of the order equal to the coefficient of $r_i^\vee$ in the above relation. Moreover, the above relation also shows that if C is non-trivial and $\widetilde{\overline{H}}$ decomposes into simple factors $\prod_{j\in S} \widetilde{\overline{H}}_j$, then C does not lie in $\prod_{j\in T} \widetilde{\overline{H}}_j$ for any $T \subsetneq S$. This gives the desired result immediately. $\square$

Let x be a vertex of $\mathcal{B}(H)$, with corresponding lattice $L \in \mathcal{V}$, associated graded lattice chain $(L_{x,r})$ and smooth group scheme $\underline{H}_x$. Theorem 9.1 implies that $\underline{H}_x \otimes A/\pi$ is a closed subgroup of $\mathrm{Aut}(L/\pi L)$, and thus acts on the vector space $L/\pi L$ over A/π, preserving the flag of subspaces determined by $L_{x,r}/\pi L$ for $0 \leq r \leq 1$. The maximal reductive quotient $\overline{H}$ of $\underline{H}_x \otimes A/\pi$, given by the above lemma, acts on the successive quotients of this flag. By consideration of weights, we can determine the representations of $\overline{H}$ thus obtained.

Proposition 9.4. — (i) *If $x \in V_0$, V_1 or V_6, then $L/\pi L$ isomorphic to the standard representation $J/\pi J$ of $\overline{H} \cong \mathrm{Aut}(J/\pi J, N)$.*

(ii) *If $x \in V_2$, V_3 or V_5, then as representations of $\overline{H} \cong (\mathrm{SL}_2 \times \mathrm{SL}_6)/\Delta\mu_2$, we have:*

$$L/\mathfrak{R}(L,N) \cong \mathbf{1} \otimes \wedge^2(\mathrm{st}^\vee), \quad \mathfrak{R}(L,N)/\pi L \cong \mathrm{st} \otimes \mathrm{st},$$

where st *is the standard representation of* SL_2 *or* SL_6. *More precisely,* 'st' *for* SL_6 *is the standard representation associated to the fundamental weight corresponding to r_6 when $x \in V_2$, to r_0 when $x \in V_3$, to r_1 when $x \in V_5$.*

(iii) *If $x \in V_4$, then the successive quotients of the graded lattice chain of x are isomorphic to the representations*

$$\mathbf{1} \otimes \mathrm{st}(r_1) \otimes \mathrm{st}(r_5), \quad \mathrm{st}(r_2) \otimes \mathbf{1} \otimes \mathrm{st}(r_6) \quad \mathrm{st}(r_0) \otimes \mathrm{st}(r_3) \otimes \mathbf{1}$$

of $\overline{H} \cong (\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_3$, where $\mathrm{st}(r_i)$ denotes the standard representation of SL_3 associated to the fundamental weight corresponding to r_i.

We also have the analogues of Theorem 9.1 and Proposition 9.4 for the group G.

Theorem 9.5. — *Given any order L in $\mathcal{V}'$, the schematic closure of G in $\mathrm{Aut}(L)$ is the smooth integral model $\underline{G}_x$ associated to the vertex corresponding to L. More generally, let X be a finite set of vertices contained in an apartment $\mathcal{A}$ with convex hull Ω, and let $\mathcal{L} \subset \mathcal{V}'$ be the set of orders corresponding to the elements of X. Then the schematic closure of G in $\prod_{L\in\mathcal{L}} \mathrm{Aut}(L)$ is the smooth integral model $\underline{G}_\Omega$ of G associated to Ω.*

As before, we may assume without loss of generality that $\mathcal{A} = \mathcal{A}(S)$, where S is the maximal split torus of G introduced in §3, and it suffices to show that for each $b \in \Phi(G, S)$, with corresponding root subgroup U_b, the schematic closure of U_b in $\prod_{L \in \mathcal{L}} \mathrm{Aut}(L)$ is smooth. If b is a long root, then $U_b \subset G \subset H$ is also a root subgroup of H, for which we already know the result. Henceforth, we assume that b is a short root.

Now the basis $\{e^i_{jk}\}$ of V is certainly a basis of weight vectors for S, and each weight occurs with multiplicity one, except the trivial character which occurs with multiplicity 3. As we explain in §6, the trivial weight space has a canonical basis, consisting of orthogonal primitive idempotents, and these are the vectors e^1_{11}, e^1_{22} and e^1_{33}. As a result, the normalizer $N(S)$ of S in G permutes the set B of lines spanned by the vectors $\{e^i_{jk}\}$. It follows from this that for any given short root b, there is a partition $B = \bigcup_i B_i$ such that as a representation of the rank one subgroup $G_b \cong \mathrm{SL}_2$ attached to b, $V = \oplus_i V_i$, where V_i is spanned by the elements of B_i, and such that $L = \bigoplus_i L \cap V_i$. Moreover, V_i is either the trivial representation $\mathbf{1}$, the standard representation st or the representation $\mathrm{st} \otimes \mathrm{st}^\vee$, and the latter occurs with multiplicity 1.

As before, it remains to show that the schematic closure of U_b in $\mathrm{Aut}(L \cap V_i)$ is smooth for each i and for each $L \in \mathcal{L}$. If V_i is the trivial representation or the standard representation, then we are done by [6, 3.9.2]. Suppose that V_i is the unique subspace affording the representation $\mathrm{st} \otimes \mathrm{st}^\vee$. Using the observation that $N(S)$ permutes the basis elements $\{e^i_{jk}\}$ (up to signs), and the explicit information contained in Proposition 8.1, it is not difficult to see that there exists a vector space isomorphism

$$\phi : V_i \longrightarrow M_2(k), \text{ the space of } 2 \times 2 \text{ matrices over } k,$$

as well as an isomorphism

$$\varphi : U_b \cong U, \text{ the group of upper triangular } 2 \times 2 \text{ unipotent matrices},$$

such that

$$\varphi(g) \cdot \phi(v) = \phi(g \cdot v), \quad \text{for } g \in U_b \text{ and } v \in V_i,$$

and more importantly,

$$\phi(L \cap V_i) = \begin{pmatrix} A & A \\ \pi^n & A \end{pmatrix}, \quad \text{with } n = 0 \text{ or } 1.$$

The desired result now follows by [9, Lemma 10.3] and [6, 3.9.2]. Thus Theorem 9.5 is proved completely.

Finally, we come to the analogue of Proposition 9.4 for G. For a vertex x with corresponding order L, the maximal reductive quotient of the special fiber of $\underline{G}_x$ is given by:

$$\overline{G} = \begin{cases} \underline{G} \otimes A/\pi & \text{if } x \in V_0', \\ (\mathrm{SL}_2 \times \mathrm{Sp}_6)/\Delta\mu_2 & \text{if } x \in V_1', \\ (\mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_3 & \text{if } x \in V_2', \\ (\mathrm{SL}_2 \times \mathrm{SL}_4)/\Delta\mu_2 & \text{if } x \in V_3', \\ \mathrm{Spin}_9 & \text{if } x \in V_4', \end{cases}$$

and is proved by the same argument as that in Lemma 9.3. By consideration of weights, we can determine the representation of $\overline{G}$ on the successive quotients of the lattice chain associated to x.

Proposition 9.6. — (i) *If $x \in V_0'$, then $L/\pi L$ isomorphic to the standard representation $J/\pi J$ of $\overline{G} \cong \mathrm{Aut}(J/\pi J, N, e)$.*

(ii) *If $x \in V_1'$, then as representations of $\overline{G} \cong (\mathrm{SL}_2 \times \mathrm{Sp}_6)/\Delta\mu_2$, we have:*

$$L/\mathfrak{R}(L) \cong \mathbf{1} \otimes \wedge^2 \mathrm{st}, \quad \mathfrak{R}(L)/\pi L \cong \mathrm{st} \otimes \mathrm{st},$$

where st *is the standard representation of* SL_2 *or* Sp_6.

(iii) *If $x \in V_2'$, then as representations of $\overline{G} \cong (\mathrm{SL}_3 \times \mathrm{SL}_3)/\Delta\mu_2$, we have:*

$$L/\mathfrak{R}(L) \cong \mathbf{1} \otimes (\mathrm{st} \otimes \mathrm{st}^\vee), \quad \mathfrak{R}(L)/L_{\frac{2}{3}} \cong \mathrm{st}(r_1') \otimes \mathrm{st}(r_4'),$$
$$L_{\frac{2}{3}}/\pi L \cong \mathrm{st}(r_0') \otimes \mathrm{st}(r_3'),$$

where $\mathrm{st}(r_i')$ denotes the standard representation of SL_3 associated the the fundamental weight corresponding to r_i'.

(iv) *If $x \in V_3'$, then as representations of $\overline{G} \cong (\mathrm{SL}_2 \times \mathrm{SL}_4)/\Delta\mu_2$, we have:*

$$L/\mathfrak{R}(L) \cong (\mathbf{1} \oplus (\mathrm{st} \otimes \mathrm{st})) \otimes \mathbf{1}, \qquad \mathfrak{R}(L)/L_{\frac{2}{4}} \cong \mathrm{st} \otimes \mathrm{st}(r_2'),$$
$$L_{\frac{2}{4}}/L_{\frac{3}{4}} \cong \mathbf{1} \otimes \wedge^2 \mathrm{st}, \qquad L_{\frac{3}{4}}/\pi L \cong \mathrm{st} \otimes \mathrm{st}(r_0').$$

(v) *If $x \in V_4'$, then as representations of $\overline{G} \cong \mathrm{Spin}_9$, we have:*

$$L/\mathfrak{R}(L) \cong \mathbf{1} \oplus V_{10}, \quad \mathfrak{R}(L)/\pi L \cong \text{ the spin representation,}$$

where V_{10} is the representation obtained by regarding SO_9 as the stabilizer of a vector of norm 1 in the split quadratic space of dimension 10.

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