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CHARACTERIZATION OF CYCLE DOMAINS VIA KOBAYASHI HYPERBOLICITY

by Gregor Fels & Alan Huckleberry

Abstract. — A real form G of a complex semi-simple Lie group $G^{\mathbb{C}}$ has only finitely many orbits in any given $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$. The complex geometry of these orbits is of interest, e.g., for the associated representation theory. The open orbits D generally possess only the constant holomorphic functions, and the relevant associated geometric objects are certain positive-dimensional compact complex submanifolds of D which, with very few well-understood exceptions, are parameterized by the Wolf cycle domains $\Omega_W(D)$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, where K is a maximal compact subgroup of G. Thus, for the various domains D in the various ambient spaces Z, it is possible to compare the cycle spaces $\Omega_W(D)$.

The main result here is that, with the few exceptions mentioned above, for a fixed real form G all of the cycle spaces $\Omega_W(D)$ are the same. They are equal to a universal domain Ω_{AG} which is natural from the the point of view of group actions and which, in essence, can be explicitly computed.

The essential technical result is that if $\widehat{\Omega}$ is a G-invariant Stein domain which contains Ω_{AG} and which is Kobayashi hyperbolic, then $\widehat{\Omega} = \Omega_{AG}$. The equality of the cycle domains follows from the fact that every $\Omega_W(D)$ is itself Stein, is hyperbolic, and contains Ω_{AG}.

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Résumé (*Caractérisation de domaines de cycles par l'hyperbolicité au sens de Kobayashi*)

Une forme réelle G d'un groupe de Lie semi-simple $G^{\mathbb{C}}$ n'admet qu'un nombre fini d'orbites dans toute $G^{\mathbb{C}}$-variété de drapeaux $Z = G^{\mathbb{C}}/Q$. La géométrie complexe de ces orbites est intéressante, par exemple pour la théorie de la représentation associée. Les fonctions holomorphes sur les orbites ouvertes D de G sont constantes en général ; les objets géométriques importants liés à ces orbites sont des sous-variétés complexes de D de dimension positives qui, à quelques rares exceptions bien comprises, sont paramétrées par les domaines de cycles de Wolf $\Omega_W(D) \in G^{\mathbb{C}}/K^{\mathbb{C}}$, où K est un sous-groupe maximal compact de G. Alors, pour les domaines D dans les variétés ambiantes Z, il est possible de comparer les domaines de cycles $\Omega_W(D)$.

Le résultat principal de cet article, aux exceptions près mentionnées ci-dessus, est que pour une forme réelle G fixée, les domaines $\Omega_W(D)$ sont les mêmes. Ils sont égaux à un domaine universel Ω_{AG}, qui est canonique du point de vue d'actions de groupe et qui peut être essentiellement calculé.

Le résultat technique important est que tout domaine de Stein hyperbolique au sens de Kobayashi $\widehat{\Omega}$ qui contient Ω_{AG} est égal à Ω_{AG}. L'égalité des domaines de cycles s'ensuit du fait que chaque $\Omega_W(D)$ est lui-même de Stein, hyperbolique et contient Ω_{AG}.

1. Introduction

Let G be a non-compact real semi-simple Lie group which is embedded in its complexification $G^{\mathbb{C}}$ and consider the associated G-action on a $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$. It is known that G has only finitely many orbits in Z; in particular, there exit open G-orbits D. In each such open orbit every maximal compact subgroup K of G has exactly one orbit C_0 which is a complex submanifold (see [42]).

Let $q := \dim_{\mathbb{C}} C_0$, regard C_0 as a point in the space $\mathcal{C}^q(Z)$ of q-dimensional compact cycles in Z and let

$$\Omega := G^{\mathbb{C}} \cdot C_0$$

be the orbit in $\mathcal{C}^q(Z)$. Define the Wolf cycle space $\Omega_W(D)$ to be the connected component of $\Omega \cap \mathcal{C}^q(D)$ which contains the base cycle C_0.

Since the above mentioned basic paper [42], there has been a great deal of work aimed at describing these cycle spaces. Even in situations where good matrix models are available this is not a simple matter. Using a variety of techniques, exact descriptions of $\Omega_W(D)$ have been given in a number of special situations (see *e.g.* [1] [4], [3], [13], [17], [23], [24], [34], [38], [41], [45]).

In [16] it was conjectured that, except in the holomorphic Hermitian case where $\Omega_W(D)$ is just the associated bounded symmetric space, the cycle spaces can be naturally identified with a certain universal domain Ω_{AG} which only depends on G. This domain, which is precisely defined below, is a certain G-invariant neighborhood of the Riemannian symmetric space $M = G/K$ in its complexification $\Omega = G^{\mathbb{C}}/K^{\mathbb{C}}$.

The inclusion $\Omega_{AG} \subset \Omega_W(D)$ was proved in most cases in [17] by analyzing concrete models and by using a nice general result which reduces this inclusion to special cases.

In [25], using incidence geometry given by Schubert varieties (see also [23] and [22]), it was shown that $\Omega_W(D)$ agrees with the Schubert domain $\Omega_S(D)$ which is defined by removing certain algebraic incidence divisors from Ω.

The Schubert domains in turn contain a universal domain Ω_I which is known to agree with Ω_{AG}. The inclusion $\Omega_{AG} \subset \Omega_I$ was proved by complex analytic methods (see [22]), but now there is an algebraic proof (see [33]) which may be more appropriate, because the situation would apriori seem to be algebraic in nature. The inclusion $\Omega_I \subset \Omega_{AG}$ was shown in [2]. Thus,

$$\Omega_{AG} \subset \Omega_I \subset \Omega_S(D) = \Omega_W(D).$$

In particular $\Omega_{AG} \subset \Omega_W(D)$, has now been proved in complete generality. Therefore, to prove the above mentioned conjecture it is necessary to prove the opposite inclusion $\Omega_W(D) \subset \Omega_{AG}$.

This is a consequence of the following complex geometric characterization of Ω_{AG} which is the main result of the present paper (see Theorem 3.4.5).

THEOREM 1.0.1. — *If $\widehat{\Omega}$ is a G-invariant domain which contains Ω_{AG} in Ω and which is in addition Stein and Kobayashi hyperbolic, then $\widehat{\Omega} = \Omega$.*

Obviously $\widehat{\Omega} = \Omega_W(D)$ is G-invariant. It follows directly from the definitions that Schubert domains are Stein (see [25]). Thus $\Omega_W(D) = \Omega_S(D)$ implies that the cycle spaces are Stein, a fact that has been known in the measurable case for some time (see [43]).

Using a slight refinement of the results in [22], we show here that, with the exception of the holomorphic Hermitian case where $\Omega_W(D)$ is just the associated bounded symmetric space, $\Omega_W(D)$ is naturally embedded in Ω as a Kobayashi hyperbolic domain.

Consequently, with this well-understood exception, the above theorem together with the inclusion $\Omega_{AG} \subset \Omega_W(D)$ shows that $\Omega_W(D) = \Omega_{AG}$.

Before going to the main body of our work, let us set the notation.

Let $M = G/K$ be the associated Riemannian symmetric space of non-positive curvature embedded in $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ as an orbit of the same base point x_0 as was chosen above in the discussion of cycle spaces.

Denote by θ a Cartan involution on $\mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution on $\mathfrak{g}$ such that $\mathrm{Fix}(\theta_{|\mathfrak{g}}) = \mathfrak{k}$ is the Lie algebra of the given maximal compact subgroup K. The anti-holomorphic involution $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which defines $\mathfrak{g}$ commutes with θ as well as with the holomorphic extension τ of $\theta_{|\mathfrak{g}}$ to $\mathfrak{g}^{\mathbb{C}}$.

Let $\mathfrak{u}$ be the fixed point set of θ in $\mathfrak{g}^{\mathbb{C}}$, U be the associated maximal compact subgroup of $G^{\mathbb{C}}$ and define Σ to be the connected component containing x_0 of $\{x \in U \cdot x_0 : G_x \text{ is compact}\}$. Set $\Omega_{AG} := G \cdot \Sigma$.

To cut down on the size of Σ, one considers a maximal Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$) and notes that $G \cdot (\exp(i\mathfrak{a}) \cap \Sigma) \cdot x_0 = \Omega_{AG}$.

In fact there is an explicitly defined neighborhood ω_{AG} of $0 \in \mathfrak{a}$ such that $i\omega_{AG}$ is mapped diffeomorphically onto its images $\exp(i\omega_{AG})$ and $\exp(i\omega_{AG}) \cdot x_0$ and $\Omega_{AG} = G \cdot \exp(i\omega_{AG}) \cdot x_0$.

The set ω_{AG} is defined by the set of roots $\Phi(\mathfrak{a})$ of the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$: It is the connected component containing $0 \in \mathfrak{a}$ of the set which is obtained from $\mathfrak{a}$ by removing the root hyperplanes $\{\mu = \frac{1}{2}\pi\}$ for all $\mu \in \Phi$. It is convex and is invariant under the action of the Weyl group $\mathcal{W}(\mathfrak{a})$ of the symmetric space G/K.

Modulo $\mathcal{W}(\mathfrak{a})$, the set $\exp(i\omega_{AG}) \cdot x_0$ is a geometric slice for the G-action on Ω_{AG}. From this root point of view, Σ can be seen to be the set of points which are at most half way from x_0 to the cut-point locus in the compact Riemannian symmetric space U/K (see [11]).

2. Spectral properties of $\mathbf{\Omega_{AG}}$

2.1. Linearization. — The map

$$\eta : G^{\mathbb{C}} \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}), \quad x \longmapsto \sigma \circ \mathrm{Ad}(x) \circ \tau \circ \mathrm{Ad}(x^{-1}),$$

provides a suitable linearization of the setting at hand. The idea of using this linearization in the context of double coset spaces is due to T. Masuki. Some of the results in this and the following section on the Jordan decomposition can be found in §4 of [32]. In particular, in §3.2 for the sake of completeness we give proofs of his Proposition 3 and Proposition 4. In this section elementary properties of η are summarized.

Let $G^{\mathbb{C}}$ act on $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by $h \cdot \varphi := \mathrm{Ad}(h) \circ \varphi \circ \mathrm{Ad}(h^{-1})$.

Lemma 2.1.1 (G-equivariance). — *For $h \in G$ it follows that $\eta(h \cdot x) = h \cdot \eta(x)$ for all $x \in G^{\mathbb{C}}$.*

Proof. — By definition $\eta(h \cdot x) = \sigma\,\mathrm{Ad}(h)\,\mathrm{Ad}(x)\tau\,\mathrm{Ad}(x^{-1})\,\mathrm{Ad}(h^{-1})$. Since h belongs to G, it follows that σ and $\mathrm{Ad}(h)$ commute and the desired result is immediate. □

The normalizer of $K^{\mathbb{C}}$ in $G^{\mathbb{C}}$ is denoted by $N^{\mathbb{C}} := N_{G^{\mathbb{C}}}(K^{\mathbb{C}})$. It is indeed the complexification of $N := N_U(K)$.

Lemma 2.1.2 ($N^{\mathbb{C}}$-invariance). — *The map η factors through a G-equivariant embedding of $G^{\mathbb{C}}/N^{\mathbb{C}}$:*

$$\eta(x) = \eta(y) \iff y = xg^{-1} \text{ for some } g \in N^{\mathbb{C}}.$$

Proof. — We may write $y = xg^{-1}$ for some $g \in G^{\mathbb{C}}$. Thus it must be shown that $\eta(x) = \eta(xg^{-1})$ if and only if $g \in N^{\mathbb{C}}$. But $\eta(x) = \eta(xg^{-1})$ is equivalent to $\mathrm{Ad}(g)\tau = \tau\,\mathrm{Ad}(g)$, which, in turn, is equivalent to the fact that $\mathrm{Ad}(g)$ stabilizes the complexified Cartan decomposition $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^{\tau} \oplus (\mathfrak{g}^{\mathbb{C}})^{-\tau} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$.

Now, if $\mathrm{Ad}(g)$ stabilizes $\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$, then $\mathrm{Ad}(g)(\mathfrak{k}^{\mathbb{C}}) = \mathfrak{k}^{\mathbb{C}}$, *i.e.*, $g \in N^{\mathbb{C}}$. On the other hand, given any $g \in N^{\mathbb{C}}$, it follows $\mathrm{Ad}(g)(\mathfrak{p}^{\mathbb{C}}) = \mathfrak{p}^{\mathbb{C}}$, because $\mathfrak{p}^{\mathbb{C}}$ is the orthogonal complement of $\mathfrak{k}^{\mathbb{C}}$ with respect to the Killing form of $\mathfrak{g}^{\mathbb{C}}$. □

Note that $N^{\mathbb{C}}/K^{\mathbb{C}}$ is a finite Abelian group (see [15] for a classification). Consequently, up to finite covers, η is an embedding of the basic space $G^{\mathbb{C}}/K^{\mathbb{C}}$.

The involutions σ and τ are regarded as acting on $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by conjugation. On $\mathrm{Im}(\eta)$ their behavior is particularly simple.

LEMMA 2.1.3 (Action of the basic involutions). — *For all $x \in G^{\mathbb{C}}$ it follows that*

1) $\eta(\tau(x)) = \tau(\eta(x))$,
2) $\sigma(\eta(x)) = \eta(x)^{-1}$,

In particular $\mathrm{Im}(\eta)$ *is both σ- and τ-invariant.*

Proof. — Let $\varphi_* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ denote the differential of $\varphi : G^{\mathbb{C}} \to G^{\mathbb{C}}$ and $\mathrm{Int}(x) : G^{\mathbb{C}} \to G^{\mathbb{C}}$ be defined by $\mathrm{Int}(x)(z) := xzx^{-1}$. The first statement follows directly from the facts that σ and τ commute and

$$\tau\,\mathrm{Ad}(x)\tau = \big(\tau\,\mathrm{Int}(x)\tau\big)_* = \mathrm{Int}\big(\tau(x)\big)_* = \mathrm{Ad}\big(\tau(x)\big).$$

For the second statement note that $\sigma\eta(x) = \mathrm{Ad}(x)\tau\,\mathrm{Ad}(x^{-1})$, and thus $\eta(x)\sigma\eta(x) = \sigma$. □

We have seen that η is a G-equivariant map which induces a finite equivariant map $\eta : G^{\mathbb{C}}/K^{\mathbb{C}} \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. We will shortly see that the image $\eta(G^{\mathbb{C}}/K^{\mathbb{C}})$ is also closed in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Hence, for a characterization of G-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and their topological properties we may identify $G^{\mathbb{C}}/K^{\mathbb{C}}$ with its image in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ on which G acts by conjugation.

The following special case of a general result on conjugacy classes (see [26, p. 117] and [7]) is of basic use.

LEMMA 2.1.4. — *Let V be a finite-dimensional $\mathbb{R}$-vector space, H a closed reductive algebraic subgroup of $\mathrm{GL}_{\mathbb{R}}(V)$ and $s \in \mathrm{GL}_{\mathbb{R}}(V)$ an element which normalizes H. Regard H as acting on $\mathrm{GL}_{\mathbb{R}}(V)$ by conjugation. Then, for a semi-simple s the orbit $H \cdot s$ is closed.*

COROLLARY 2.1.5. — *The image* $\mathrm{Im}(\eta)$ *is closed in* $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.

Proof. — It is enough to show that $G^{\mathbb{C}}.\tau = \{\mathrm{Ad}(g)\tau\,\mathrm{Ad}(g^{-1}) : g \in G^{\mathbb{C}}\}$ is closed in $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$. Since τ is semi-simple and normalizes $G^{\mathbb{C}}$ in this representation, this follows from Lemma 2.1.4. □

Using a bit of invariant theory over $\mathbb{R}$, we are able to carry over the standard result on orbits in the complex case (see [32, Prop. 4]).

PROPOSITION 2.1.6. — *If $\eta(x) = s$ is semi-simple, then $G\cdot x$ is closed.*

Proof. — It is enough to show that $G\cdot s$ is closed. By Lemma 2.1.4 the complex orbit $G^{\mathbb{C}}\cdot s$ is closed. Define $\hat\sigma : \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}) \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by $\hat\sigma(\varphi) = (\sigma(\varphi))^{-1}$. Here σ acts by conjugation as usual.

Now $\mathrm{Im}(\eta)$ belongs to the fixed point set $\mathrm{Fix}(\hat\sigma)$ and since $G^{\mathbb{C}}\cdot s \cap \mathrm{Fix}(\hat\sigma)$ consists of only finitely many G-orbits (see [9]), it follows that $G\cdot s$ is closed. □

2.2. Jordan Decomposition. — Here x denotes an arbitrary element of $G^{\mathbb{C}}$ and $su = us = \eta(x)$ is its Jordan decomposition in $\mathrm{GL}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Since $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is algebraic, $s,u \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ as well. If $\eta(x) = us$ is not semi-simple, *i.e.*, $u \neq 1$, consider $\xi = \log(u) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Since ξ is nilpotent, $t \mapsto \exp(t\xi)$ is an algebraic map and $\exp(\mathbb{Z}\xi) \subset \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. It follows that $\exp(t\xi) \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ for all $t \in \mathbb{R}$. In particular, u is in the connected component $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})^0$, and ξ is a derivation: $\xi = \mathrm{ad}(N)$ for some nilpotent $N \in \mathfrak{g}^{\mathbb{C}}$. Finally, $u = \mathrm{Ad}(\exp(N)) = \exp(\mathrm{ad}\, N)$.

Given an element $z \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$, let $(\mathfrak{g}^{\mathbb{C}})^z = \{X \in \mathfrak{g}^{\mathbb{C}} : z(X) = X\}$ denote the subalgebra of fixed points. Observe also that if $\nu : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is any involution such that $\nu(z) = z$ or $\nu(z) = z^{-1}$, then the subalgebra $(\mathfrak{g}^{\mathbb{C}})^z$ is ν-stable. For z semi-simple the subalgebra $(\mathfrak{g}^{\mathbb{C}})^z$ is reductive.

PROPOSITION 2.2.1 (Lifting of the Jordan decomposition)
For $x \in G^{\mathbb{C}}$ with Jordan decomposition $\eta(x) = u\cdot s$ there exists a nilpotent element $N \in (\mathfrak{g}^{\mathbb{C}})^s \cap i\mathfrak{g}$ such that

1) $u = \mathrm{Ad}(\exp(N))$,
2) $\eta(\exp(\frac{1}{2}N)\cdot x) = s$.

Proof. — Let $N \in \mathfrak{g}^{\mathbb{C}}$ be the element with $u = \mathrm{Ad}(\exp(N))$ as explained above. First we show that $N \in (\mathfrak{g}^{\mathbb{C}})^s \cap i\mathfrak{g}$. From Lemma 2.1.3(2) it follows that $\sigma(\eta(x)) = \sigma(us) = s^{-1}u^{-1}$. This implies $\sigma(u) = u^{-1}$ or, equivalently, $\sigma(N) = -N$, *i.e.*, $N \in i\mathfrak{g}$. Secondly, the statement that $\mathrm{Ad}(\exp(N))$ commutes with s is equivalent to $s\mathrm{e}^{\mathrm{ad}(N)}s^{-1} = \mathrm{e}^{\mathrm{ad}(N)}$ which is the same as $s(N) = N$ in the semi-simple case. Thus $N \in (\mathfrak{g}^{\mathbb{C}})^s \cap i\mathfrak{g}$.

Finally, since $N \in (\mathfrak{g}^{\mathbb{C}})^s$, it follows that $\mathrm{Ad}(\exp(tN))$ commutes with s for all $t \in \mathbb{R}$. Having also in mind that $\sigma(N) = -N$, it follows that

$$\begin{aligned}
\eta(\exp\tfrac{1}{2}N\cdot x) &= \sigma\,\mathrm{Ad}(\exp\tfrac{1}{2}N)\,\mathrm{Ad}(x)\tau\,\mathrm{Ad}(x^{-1})\,\mathrm{Ad}(\exp-\tfrac{1}{2}N)\\
&= \mathrm{Ad}(\exp-\tfrac{1}{2}N)\cdot\sigma\,\mathrm{Ad}(x)\tau\,\mathrm{Ad}(x^{-1})\cdot\mathrm{Ad}(\exp-\tfrac{1}{2}N)\\
&= \mathrm{Ad}(\exp-\tfrac{1}{2}N)\cdot su\cdot\mathrm{Ad}(\exp-\tfrac{1}{2}N)\\
&= s\cdot\mathrm{Ad}(\exp-\tfrac{1}{2}N)\cdot u\cdot\mathrm{Ad}(\exp-\tfrac{1}{2}N) = s.
\end{aligned}$$
□

Observe now that since $\sigma(s) = s^{-1}$, $(\mathfrak{g}^{\mathbb{C}})^s$ is a σ-stable reductive subalgebra. Let $(\mathfrak{g}^{\mathbb{C}})^s = \mathfrak{h} \oplus \mathfrak{q}$ be its σ-eigenspace decomposition. We now build an appropriate $\mathfrak{sl}_2$-triple (E, H, F) around $N = E$ in $(\mathfrak{g}^{\mathbb{C}})^s$.

LEMMA 2.2.2. — *Let $E \in (\mathfrak{g}^{\mathbb{C}})^s \cap i\mathfrak{g}$ be an arbitrary non-trivial nilpotent element. There exists an $\mathfrak{sl}_2$-triple (E, H, F) in $(\mathfrak{g}^{\mathbb{C}})^s$, i.e.,*

$$[E, F] = H, \quad [H, E] = 2E \quad \textit{and} \quad [H, F] = -2F$$

such that $E, F \in \mathfrak{q}$ and $H \in \mathfrak{h}$.

Proof. — Since $(\mathfrak{g}^{\mathbb{C}})^s$ is reductive, there exists a $\mathfrak{sl}_2$-triple (E, H, F) in $(\mathfrak{g}^{\mathbb{C}})^s$ by the theorem of Jacobson-Morozov. It can be chosen to be σ-compatible.

To see this, split $H = H^{\sigma} + H^{-\sigma}$ with respect to the σ-eigenspace decomposition of $(\mathfrak{g}^{\mathbb{C}})^s$. Since $[H, E] = 2E$ and $\sigma(E) = -E$, it follows that

$$[H^{-\sigma}, E] = 0.$$

Hence, we may assume that $H = H^{\sigma}$ (see [8, Chap. VIII, §11, Lemme 6]). Observe further that in this case one has $[E, F] = [E, (F)^{-\sigma}] = H$ and $[H, (F)^{-\sigma}] = (F)^{-\sigma}$. The desired result follows then from the uniqueness of the third element F in a $\mathfrak{sl}_2$-triple. □

Now we have all the ingredients which are needed to give a complete characterization of the closed orbits in $\mathrm{Im}(\eta)$ (see [32, Prop. 3]):

PROPOSITION 2.2.3 (Closed orbits). — *If $\eta(x) = us$ is the Jordan decomposition, then the orbit $G \cdot \eta(x) = G \cdot (su)$ contains the closed orbit $G \cdot s$ in its closure $\overline{G \cdot \eta(x)}$. In particular, $G \cdot \eta(x)$ is closed if and only if $\eta(x)$ is semi-simple and $s \in Im(\eta)$.*

Proof. — Let $u = \mathrm{Ad}(\exp N)$ with N as in Proposition 2.2.1. Hence, by Lemma 2.2.2 there is a $\mathfrak{sl}_2$-triple (N, H, F) $(E = N)$ such that $[tH, N] = 2tN$, *i.e.*, $\mathrm{Ad}(\exp tH)(N) = \mathrm{e}^{2t}N$ for every $t \in \mathbb{R}$. Note also that $\exp(\mathbb{R}H) \subset G \cap \exp(\mathfrak{g}^{\mathbb{C}})^s$ by construction of the $\mathfrak{sl}_2$-triple. It follows that

$$\begin{aligned}
\eta(\exp tH \cdot x) &= \exp tH \cdot (us) = \mathrm{Ad}(\exp tH) \cdot us \cdot \mathrm{Ad}(\exp -tH) \\
&= \mathrm{Ad}(\exp tH) \cdot u \cdot \mathrm{Ad}(\exp -tH) \cdot s \\
&= \mathrm{Ad}(\exp tH)\, \mathrm{Ad}(\exp N)\, \mathrm{Ad}(\exp -tH) \cdot s \\
&= \mathrm{Ad}(\exp \mathrm{e}^{2t} N) \cdot s.
\end{aligned}$$

For $t \to -\infty$ it follows $\lim \exp tH \cdot (us) = \mathrm{Ad}(\exp \mathrm{e}^{2t}N) \cdot s = s$. Hence, the closed orbit $G \cdot s$ lies in the closure of $G \cdot (us)$. In particular $G \cdot (us)$ is non-closed if $u \neq 1$, *i.e.*, if $\eta(x)$ is not semi-simple. This, together with Proposition 2.1.6 implies that $G \cdot \eta(x)$ is closed if and only if $\eta(x)$ is semi-simple. Recall that the image $\mathrm{Im}(\eta)$ is closed. This forces $s \in \mathrm{Im}(\eta)$ and the proof is now complete. □

2.3. Elliptic elements and closed orbits. — Every non-zero complex number z has the unique decomposition $r \cdot \mathrm{e}^{i\phi}$ into the hyperbolic part $r > 0$ and elliptic part $\mathrm{e}^{i\phi}$. This generalizes for an arbitrary semi-simple element $s \in \mathrm{GL}(\mathfrak{g}^{\mathbb{C}})$: By decomposing its eigenvalues one obtains the unique decomposition $s = s_{\mathrm{ell}} \cdot s_{\mathrm{hyp}} = s_{\mathrm{hyp}} \cdot s_{\mathrm{ell}}$. An element $x \in G^{\mathbb{C}}$ is said to be *elliptic* if $\eta(x) = s$ is semi-simple with eigenvalues lying in the unit circle. It should be remarked that x itself may in such a case not be a semi-simple element of the group $G^{\mathbb{C}}$, *e.g.*, $K^{\mathbb{C}}$ contains unipotent elements.

Let $\Omega_{\mathrm{ell}} \subset G^{\mathbb{C}}$ be the set of elliptic elements. This set is invariant by the right-action of $K^{\mathbb{C}}$, and therefore by choosing the same base point x_0 as in the case of Ω_{AG}, by abuse of notation we also regard Ω_{ell} as a subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$. We reiterate that, since the map η is not a group morphism, the classical notion of an elliptic element in $G^{\mathbb{C}}$ differs from the above definition.

LEMMA 2.3.1. — *For U the maximal compact subgroup of $G^{\mathbb{C}}$ defined by θ it follows that $U \subset \Omega_{\mathrm{ell}}$.*

Proof. — For θ the Cartan involution defining $\mathfrak{u}$, observe that

$$\widehat{U} := \{\varphi \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}) : \varphi\theta = \theta\varphi\}$$

is a maximal compact subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ (with identity component $\mathrm{Ad}(U)$).

Now θ commutes with every term in the definition of $\eta(u)$for every $u \in U$. It follows that $\theta\eta(u) = \eta(u)\theta$. Therefore $\eta(U)$ is contained in the compact group $\widehat{U}$ and consequently $U \subset \Omega_{\mathrm{ell}}$. □

PROPOSITION 2.3.2 (Elliptic elements). — *In the homogeneous space $G^{\mathbb{C}}/K^{\mathbb{C}}$ the set of elliptic elements is described as $\Omega_{\mathrm{ell}} = G \cdot \exp(i\mathfrak{a}) \cdot x_0$.*

Proof. — Observe that Ω_{ell} is G-invariant. Hence, the above lemma implies that $G \cdot \exp(i\mathfrak{a}) \cdot x_0 \subset \Omega_{\mathrm{ell}}$.

Conversely, suppose x is elliptic, *i.e.*, $\eta(x)$ is contained in some maximal compact subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Hence, there is a Cartan involution $\theta'' : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which commutes with $\eta(x)$. We now make the usual adjustments so that, after replacing x by an appropriate G-translate, $\eta(x)$ will commute with the given Cartan involution θ.

For this, if θ'' does not commute with σ, define the semi-simple element $\rho := \sigma\theta''\sigma\theta''$ which is diagonalizable with all positive eigenvalues over $\mathbb{R}$. It follows that ρ^t is defined for all $t \in \mathbb{R}$, and $\theta' := \rho^{\frac{1}{4}}\theta''\rho^{-\frac{1}{4}}$ commutes with σ (see [21, Chap. III, §7]). By direct calculation one verifies that ρ, hence ρ^t commutes with $\eta(x)$. Thus it follows that θ' and $\eta(x)$ commute.

Finally, since θ' and our original θ both commute with σ, there exists $h \in G$ such that $\mathrm{Ad}(h)\theta' \mathrm{Ad}(h^{-1}) = \theta$. Consequently, if x is replaced by $h^{-1} \cdot x$, then we may assume that $\eta(x)$ and θ commute.

Now we will adjust x so that it lies in U. With respect to the global Cartan decomposition of $G^{\mathbb{C}}$ defined by θ write $x = u\exp(Z)$, *i.e.*, $u \in U$ and $\theta(Z) = -Z$. We now show that in fact $\exp(Z) \in K^{\mathbb{C}}$.

Since θ commutes with σ, τ and u and anti-commutes with Z, we have

$$\begin{aligned}\theta\eta(x) &= \theta \cdot \big(\sigma \operatorname{Ad}(u) \operatorname{Ad}(\exp(Z))\tau \operatorname{Ad}\big(\exp(-Z) \operatorname{Ad}(u^{-1})\big)\\ &= \sigma \operatorname{Ad}(u) \operatorname{Ad}\big(\exp(-Z)\big)\tau \operatorname{Ad}\big(\exp(Z)\big) \operatorname{Ad}(u^{-1}) \cdot \theta.\end{aligned}$$

On the other hand

$$\theta\eta(x) = \eta(x)\theta = \sigma \operatorname{Ad}(u) \operatorname{Ad}\big(\exp(Z)\big)\tau \operatorname{Ad}\big(\exp(-Z)\big) \operatorname{Ad}(u^{-1}) \cdot \theta.$$

Combining these two equations, we obtain

$$\operatorname{Ad}\big(\exp(Z)\big)\tau \operatorname{Ad}\big(\exp(-Z)\big) = \operatorname{Ad}\big(\exp(-Z)\big)\tau \operatorname{Ad}\big(\exp(Z)\big)$$

and consequently $\operatorname{Ad}(\exp(2Z))$ commutes with τ. Since the restriction $\operatorname{Ad} : \exp(i\mathfrak{u}) \to \operatorname{Aut}(\mathfrak{g}^{\mathbb{C}})$ is injective, it follows that $\tau(\exp(Z)) = \exp(Z)$, *i.e.*, $\exp(Z) \in K^{\mathbb{C}}$. Replacing x by $x\exp(-Z)$, it follows that $x \cdot x_0 = x\exp(-Z) \cdot x_0$; hence, we may assume that $x \in U$.

Since $U = K \cdot \exp(i\mathfrak{a}) \cdot K$, we may assume that $x \in K\exp(i\mathfrak{a})$ and then translate it by left multiplication by an element of K to reach the following conclusion: If $x \in G^{\mathbb{C}}$ is elliptic, then there exists $h \in G$ and $\ell \in K^{\mathbb{C}}$ with $hx\ell$ in $\exp i\mathfrak{a}$ or, equivalently, there is $h \in G$ with $hx \cdot x_0 \in \exp(i\mathfrak{a}) \cdot x_0$. This proves the inclusion $\Omega_{\text{ell}} \subset G \cdot \exp(i\mathfrak{a}) \cdot x_0$. □

The following is a key ingredient for understanding the G-orbit structure in $\operatorname{bd}(\Omega_{AG})$.

Proposition 2.3.3. — *One has* $\exp(i\mathfrak{a}) \cdot x_0 \cap c\ell(\Omega_{AG}) = c\ell(\exp(i\omega_{AG}) \cdot x_0)$

Proof. — If $x \in c\ell(exp(i\omega_{AG}) \cdot x_0)$, then it is elliptic and therefore its orbit $G \cdot x$ is closed. In other words $\exp(i\mathfrak{a}) \cdot x_0 \cap c\ell(\Omega_{AG}) \supset c\ell(\exp(i\omega_{AG}) \cdot x_0)$.

For the opposite inclusion, observe that if $s, s' \in \exp(i\mathfrak{a}) \cdot x_0$ and $s' \in G \cdot s$, then $s' = k(s)$ for some element k of the Weyl group. Thus, if s belongs to $c\ell(\exp(i\omega_{AG}) \cdot x_0)$, then s' belongs to $c\ell(\exp(i\omega_{AG}) \cdot x_0)$ as well. Therefore, in order to prove the opposite inclusion it is enough to show that, given $s' \in \exp(i\mathfrak{a} \cdot x_0) \cap c\ell(\Omega_{AG})$, there exists $s \in c\ell(\exp(i\omega_{AG}) \cdot x_0)$ with $s' \in G \cdot s$.

Given s' as above, there exist sequences $\{s_n\} \subset \exp(i\omega_{AG}) \cdot x_0$ and $\{s'_n\} \subset \Omega_{AG}$ such that $s'_n \in G \cdot s_n$, $s'_n \to s'$ and $s_n \to s \in c\ell(\exp(i\omega_{AG}) \cdot x_0)$. Consider the (real) categorical quotient map $\pi : \operatorname{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}) \to \operatorname{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})/\!/G$. It is continuous, the base is Hausdorff and in every fiber there is exactly one closed G-orbit. Since $\pi(s_n) = \pi(s'_n)$, it follows that $G \cdot s = G \cdot s'$. □

Corollary 2.3.4. — *Let* $\Omega_{c\ell}$ *denote* $\{x \in \Omega : G \cdot x \textit{ is closed }\}$. *Then*

$$\Omega_{c\ell} \cap c\ell(\Omega_{AG}) = G \cdot c\ell(\exp(i\omega_{AG}) \cdot x_0) = \Omega_{\text{ell}} \cap c\ell(\Omega_{AG}).$$

Proof. — From Proposition 2.3.2, $\Omega_{AG} \subset \Omega_{\text{ell}}$. By continuity, the semi-simple part of $\eta(x)$ is elliptic for every $x \in c\ell(\Omega_{AG})$. Thus $\Omega_{c\ell} \cap c\ell(\Omega_{AG}) \subset \Omega_{\text{ell}} \cap c\ell(\Omega_{AG})$, because elements of closed orbits are semi-simple. Proposition 2.3.2 gives $\Omega_{c\ell} \cap c\ell(\Omega_{AG}) \subset G \cdot c\ell(\exp(i\omega_{AG}) \cdot x_0)$, and from Proposition 2.3.3 it follows that $G \cdot c\ell(exp(i\omega_{AG}) \cdot x_0) \subset \Omega_{c\ell} \cap c\ell(\Omega_{AG})$. So we have $\Omega_{c\ell} \cap c\ell(\Omega_{AG}) \subset G \cdot c\ell(\exp(i\omega_{AG}) \cdot x_0) = \Omega_{\text{ell}} \cap c\ell(\Omega_{AG})$. Finally, if $x \in \Omega_{\text{ell}}$, then in particular it is semi-simple and $G \cdot x$ is closed. This proves the remaining inclusion. □

3. Q_2-slices

At a generic point $y \in \text{bd}(\Omega_{\text{AG}})$ we determine a 3-dimensional, σ-invariant, semi-simple subgroup $S^{\mathbb{C}}$ such that $S = (S^{\mathbb{C}})^{\sigma} = \text{Fix}(\sigma : S^{\mathbb{C}} \to S^{\mathbb{C}})$ is a non-compact real form and such that the isotropy group $S^{\mathbb{C}}_y$ is either a maximal complex torus or its normalizer. Geometrically speaking, $Q_2 = S^{\mathbb{C}} \cdot y$ is either the 2-dimensional affine quadric, which can be realized by the diagonal action as the complement of the diagonal in $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$, or its (2-1)-quotient, which is defined by exchanging the factors and which can be realized as the complement of the (closed) 1-dimensional orbit of $\text{SO}_3(\mathbb{C})$ in $\mathbb{P}_2(\mathbb{C})$. By abuse of notation, we refer in both cases to $S^{\mathbb{C}} \cdot y$ as a 2-dimensional affine quadric.

The key property is that, up to the above mentioned possibility of a (2-1)-cover, the intersection $Q_2 \cap \Omega_{\text{AG}}$ is the Akhiezer-Gindikin domain in $S^{\mathbb{C}}/K^{\mathbb{C}}_S$ for the unit disk S/K_S.

For the sake of brevity we say that the orbit $S^{\mathbb{C}} \cdot y$ is a Q_2-*slice at* y whenever it has all of the above properties.

3.1. Existence. — Given a non-closed G-orbit $G \cdot y$ in $\text{bd}(\Omega_{\text{AG}})$, we may apply Proposition 2.2.1 to obtain a lifting of the semi-simple (elliptic) part of the Jordan decomposition of $\eta(x)$. For an appropriate base point z this lifting can be chosen in $\text{bd}(\exp(i\omega_{\text{AG}}))$. Recall that the action $G^{\mathbb{C}} \times \text{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}) \to \text{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is given by conjugation (see §3.1). Note that the isotropy Lie algebra at $\varphi \in \text{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is the totally real subalgebra of fixed points $(\mathfrak{g}^{\mathbb{C}})^{\varphi} = \{Z \in \mathfrak{g}^{\mathbb{C}} : \varphi(Z) = Z\}$.

LEMMA 3.1.1 (Optimal base point). — *Every non-closed G-orbit $G \cdot y$ in $\text{bd}(\Omega_{\text{AG}})$ contains a point $z = \exp E \cdot \exp iA \cdot x_0$ such that $E \in (\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} \cap i\mathfrak{g}$ is a non-trivial nilpotent element.*

Proof. — Let $\eta(y) = su$ be the Jordan decomposition and let $N \in (\mathfrak{g}^{\mathbb{C}})^s \cap i\mathfrak{g}$ be as in Proposition 2.2.1. We then have

$$\eta(\exp y) = \eta\Big(\exp\Big(-\frac{1}{2}N\Big)\exp\Big(\frac{1}{2}N\Big)\cdot y\Big) = \text{Ad}(\exp N)\circ\eta\Big(\exp\Big(\frac{1}{2}N\Big)\cdot y\Big) = u\cdot s.$$

By Proposition 2.2.3 and Corollary 2.3.4 the semi-simple element $\eta(\exp(\frac{1}{2}N)\cdot y)$ is elliptic. Hence, Proposition 2.3.2 implies the existence of $g \in G$ and $A \in \mathrm{bd}(\omega_{\mathrm{AG}})$ such that $\exp\frac{1}{2}N\cdot y = g^{-1}\exp iA\cdot x_0$.

Define now $E := \mathrm{Ad}(g)(-\frac{1}{2}N)$ and observe that $g\cdot y = \exp E \exp iA\cdot x_0$. Finally, $E\in(\mathfrak{g}^{\mathbb{C}})^{g.s} = (\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$, and the lemma is proved. □

Recall that $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$ is a σ-stable real reductive algebra. Let

$$(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} = \mathfrak{h}\oplus\mathfrak{q}$$

be the decomposition into σ-eigenspaces. In this notation, the nilpotent element E as in the above lemma belongs to $\mathfrak{q}$.

Let now an arbitrary non-closed orbit $G\exp E\exp iA\cdot x_0$ be given. Fix a $\mathfrak{sl}_2$-triple (E,H,F) as in Lemma 2.2.2. Let $S^{\mathbb{C}}$ be the complex subgroup of $G^{\mathbb{C}}$ defined by this triple. Set $e := iE$, $f := -iF$ and let S be the σ-invariant real form in $S^{\mathbb{C}}$. The Lie algebra of S is then the subalgebra generated by the $\mathfrak{sl}_2$ triple (e,H,f). Finally, let $x_1 = \exp(iA)\cdot x_0$ be the base point chosen as above in the closure of a given G-orbit.

Lemma 3.1.2. — *The connected component $(S^{\mathbb{C}}_{x_1})^0$ of the $S^{\mathbb{C}}$-isotropy at x_1 is the 1-parameter subgroup $\{\exp(zH) : z\in\mathbb{C}\}\cong\mathbb{C}^*$.*

Proof. — Since the action $S^{\mathbb{C}}\times G^{\mathbb{C}}/K^{\mathbb{C}}\to G^{\mathbb{C}}/K^{\mathbb{C}}$ is affine-algebraic, the orbit $S^{\mathbb{C}}\cdot\exp iA\cdot x_0 = S^{\mathbb{C}}\cdot x_1$ is an affine variety. Then the isotropy at $\exp iA\cdot x_0$ is 1-dimensional or $S^{\mathbb{C}}$. Note that $S^{\mathbb{C}}\cdot x_1$ cannot be a point, because by construction $\exp E\cdot x_1\neq x_1$; therefore $S^{\mathbb{C}}_{x_1}$ is 1-dimensional.

We now show that $\exp\mathbb{R}H\cdot x_1 = x_1$, or equivalently, $\exp tH\cdot\eta(x_1)=\eta(x_1)$. Define $\varphi := \mathrm{Ad}(\exp(iA))\tau\,\mathrm{Ad}(\exp(-iA))$ and note that $H\in(\mathfrak{g}^{\mathbb{C}})^{\sigma}\cap(\mathfrak{g}^{\mathbb{C}})^{\varphi}=\mathfrak{h}$ yields

$$\begin{aligned}\exp tH\cdot\eta(x_1) &= \mathrm{Ad}(\exp tH)\circ\sigma\varphi\circ\mathrm{Ad}(\exp -tH)\\ &= \mathrm{Ad}(\exp tH)\,\mathrm{Ad}(\exp -tH)\circ\sigma\varphi = \eta(x_1).\end{aligned}$$

It follows that $\exp\mathbb{C}H\cdot x_1 = x_1$. Since $S^{\mathbb{C}}_{x_1}$ is 1-dimensional and H semi-simple, we deduce $(S^{\mathbb{C}}_{x_1})^0 = \exp\mathbb{C}H\cong\mathbb{C}^*$. □

3.2. Genericity. — Without going into a technical analysis of $\mathrm{bd}(\Omega_{\mathrm{AG}})$, we will construct Q_2-slices only at its generic points. The purpose of this section is to introduce the appropriate notion of "generic" and prove that the set of such points is open and dense. The set of generic points is defined to be the complement of the union of small semi-algebraic sets $\mathcal{C}$ and $\mathcal{E}$ in $\mathrm{bd}(\Omega_{\mathrm{AG}})$. We begin with the definition of $\mathcal{C}$.

Let $R := \mathrm{bd}(\exp(i\omega_{\mathrm{AG}})\cdot x_0)$ and recall that for $y\in\mathrm{bd}(\Omega_{\mathrm{AG}})$ the orbit $G\cdot y$ is closed if and only if $G\cdot y\cap R\neq\varnothing$. In fact R parameterizes the closed orbits in $\mathrm{bd}(\Omega_{\mathrm{AG}})$ up to the orbits of a finite group. Recall also that R is naturally identified with $\mathrm{bd}(\omega_{\mathrm{AG}})$, which is the boundary of a convex polytope, and is defined by linear inequalities. Let E be the image in R of the lower-dimensional edges in $\mathrm{bd}(\omega_{\mathrm{AG}})$, *i.e.*, the set of points which are contained in at least two root hyperplanes $\{\alpha = c_\alpha\}$. Finally, let $R_{\mathrm{gen}} := R \setminus E$.

As we have seen in Corollary 2.3.4, the set of closed orbits in the boundary of Ω_{AG} can be described as $G \cdot \mathrm{bd}(\exp(i\omega_{\mathrm{AG}})x_0)$. This is by definition the set $\mathcal{C}$.

LEMMA 3.2.1. — *For $x \in R$ it follows that* $\dim G \cdot x \leq \mathrm{codim}_\Omega \mathrm{bd}(\Omega) - 2$.

Proof. — Note that $\mathrm{bd}(\Omega_{\mathrm{AG}})$ is connected and of codimension 1 in Ω. The G-isotropy group $C_K(\mathfrak{a})$ at generic points of $\exp(i\omega_{AG}) \cdot x_0$ fixes this slice pointwise and therefore is contained in a maximal compact subgroup of the isotropy subgroup G_x of each of its boundary points. Since by definition G_x is non-compact, it follows that $\dim G_x$ is larger than the dimension of the generic G-isotropy subgroup at points of $\exp(i\omega_{AG}) \cdot x_0$. □

REMARK. — For $x \in \mathrm{bd}(\Omega_{\mathrm{AG}})_{\mathrm{gen}}$ the isotropy subgroup G_x is precisely calculated in §3.3. This shows that $\dim G \cdot x = \mathrm{codim}_\Omega \mathrm{bd}(\Omega_{\mathrm{AG}}) - m$, where m is at least 2. Thus, by semi-continuity we have for all $x \in \mathrm{bd}(\Omega_{\mathrm{AG}})$ the estimate

$$\dim G \cdot x \leq \mathrm{codim}_\Omega \mathrm{bd}(\Omega_{\mathrm{AG}}) - m.$$

Now let $X := \mathrm{Im}(\eta) \subset \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. It is a connected component of a real algebraic submanifold in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. The complexification $X^{\mathbb{C}}$ of X which is contained in the complexification $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})$ of $\mathrm{Aut}_R(\mathfrak{g}^{\mathbb{C}})$ is biholomorphic to $G^{\mathbb{C}}/N^{\mathbb{C}} \times G^{\mathbb{C}}/N^{\mathbb{C}}$, where $N^{\mathbb{C}}$ denotes the normalizer of $K^{\mathbb{C}}$ in $G^{\mathbb{C}}$. The complexification of the piecewise real analytic variety R is a piecewise complex analytic subvariety $R^{\mathbb{C}}$ of $X^{\mathbb{C}}$ defined in a neighborhood of R in $X^{\mathbb{C}}$. Finally, let $\pi : X^{\mathbb{C}} \to X^{\mathbb{C}}/\!/G^{\mathbb{C}}$ be the complex categorical quotient.

Recall that in every π-fiber there is a unique closed $G^{\mathbb{C}}$-orbit. The closed G-orbits in X are components of the the real points of the closed $G^{\mathbb{C}}$-orbits which are defined over $\mathbb{R}$. For a more extensive discussion of the interplay between the real and complex points in complex varieties defined over $\mathbb{R}$ see [39], [40], [9].

Let $k := \dim_{\mathbb{R}} \Omega - \dim R - m$ be the dimension of the generic G-orbits of points of R and let S_k be the closure in $X^{\mathbb{C}}$ of

$$\big\{z \in X^{\mathbb{C}} : G^{\mathbb{C}} \cdot z \text{ is closed and } k\text{-dimensional}\big\}.$$

Define $\mathcal{C}_k := S_k \cap R^{\mathbb{C}}$. It follows that $\mathcal{C}_k$ is a piecewise complex analytic set of dimension $k + \dim_{\mathbb{C}} R^{\mathbb{C}}$.

PROPOSITION 3.2.2. — *The set* $G \cdot R = \{x \in \mathrm{bd}(\Omega_{\mathrm{AG}}) : G \cdot x \text{ is closed}\}$ *is contained in a closed semi-algebraic subset $\mathcal{C}$ of codimension at least* 1 *in* $\mathrm{bd}(\Omega_{\mathrm{AG}})$.

Proof. — The set $\mathcal{C}$ is defined to be the intersection of the real points of $\mathcal{C}_k$ with $\mathrm{bd}(\Omega_{\mathrm{AG}})$. The desired result follows from $\dim_{\mathbb{C}} \mathcal{C}_k = k + \dim_{\mathbb{C}} R^{\mathbb{C}}$. □

Recall that π denotes the categorical quotient map $\pi : X^{\mathbb{C}} \to X^{\mathbb{C}}//G^{\mathbb{C}}$. Define $\mathcal{E} := \eta^{-1}(\pi^{-1}(\pi(E))) \cap \mathrm{bd}(\Omega_{\mathrm{AG}})$. In particular it is a closed semi-algebraic subset of $\mathrm{bd}(\Omega_{\mathrm{AG}})$ which contains the set

$$\big\{x \in \mathrm{bd}(\Omega_{\mathrm{AG}}) : c\ell(G \cdot x) \cap E \neq \varnothing\big\}.$$

Definition. — A point $z \in \mathrm{bd}(\Omega_{\mathrm{AG}})$ is said to be *generic* if it is contained in the complement of $\mathcal{C} \cup \mathcal{E}$.

Let $\mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ denote the set of generic boundary points.

Proposition 3.2.3. — *The set of generic points* $\mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ *is open and dense in* $\mathrm{bd}(\Omega_{\mathrm{AG}})$.

It has already been noted that $\mathcal{C}$ and $\mathcal{E}$ are closed. Since $\mathcal{C}$ is of codimension two, the complement of $\mathcal{C}$ is dense. Thus this proposition is an immediate consequence of the following fact.

Proposition 3.2.4. — *The saturation* $\mathcal{E}$ *is at least* 1*-codimensional in* $\mathrm{bd}(\Omega_{\mathrm{AG}})$.

This in turn follows from a computation of the dimension of the fibers at points of E of the above mentioned categorical quotient. For this it is convenient to use the Jordan decomposition $\eta(z) = u \cdot s$ for $z \in \Omega$ such that $x = \exp(i\mathfrak{a}) \cdot x_0$ is in $c\ell(G \cdot z)$.

As in Lemma 3.1.1 we choose an optimal base point such that $\eta(x) = s$ and $u = \mathrm{Ad}(\exp(N))$ with $N \in \mathfrak{q}$, where $\mathfrak{h} \oplus \mathfrak{q}$ is the σ-decomposition of $\mathfrak{l} = (\mathfrak{g}^{\mathbb{C}})^s$. Let N_x be the the cone of nilpotent elements in $\mathfrak{q}$ and observe that the saturation $\mathcal{E}_x = \{z \in \mathrm{bd}(\Omega_{\mathrm{AG}}) : x \in c\ell(G \cdot z)\}$ is an N_x-bundle over the closed orbit $G \cdot x$. Thus it is necessary to estimate $\dim_{\mathbb{R}} N_x$.

Recall that any two maximal toral Abelian subalgebras of $\mathfrak{q}^{\mathbb{C}}$ are conjugate and therefore the dimension m of one such is an invariant. Since $\mathfrak{a}^{\mathbb{C}}$ is such an algebra, the following is quite useful (see [28]).

Lemma 3.2.5. — *The complex codimension in* $\mathfrak{q}^{\mathbb{C}}$ *of every component of the nilpotent cone in* $\mathfrak{q}^{\mathbb{C}}$ *is* m.

Proof of Proposition 3.2.4. — We prove the estimate $\mathrm{codim}_{\Omega} \mathcal{E}_x \geq \dim \mathfrak{a}$. For this observe that, since $G \cdot s$ is closed in $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$, an application of the Luna slice theorem for the (closed) complex orbit $G^{\mathbb{C}}$.s in the complexification of $\mathrm{Im}(\eta)$ yields the bundle structure $\mathrm{Im}(\eta) = G \times_{G_s} \mathfrak{q}$ locally near s; in particular $\mathrm{codim}_{\mathfrak{q}} N_x = \mathrm{codim}_{\Omega}(\mathcal{E}_x)$. The result follows from the above Lemma by noting that $\mathrm{codim}_{\mathfrak{q}} N_x$ is at most the complex codimension of the nilpotent cone in $\mathfrak{q}^{\mathbb{C}}$ and, as mentioned above, that $\dim \mathfrak{a} = m$. □

The group $S^{\mathbb{C}}$ constructed above for a generic boundary point has the property that the intersection of the $S^{\mathbb{C}}$-orbit, *i.e.*, a 2-dimensional affine quadric $Q_2 \cong \mathrm{SL}_2(\mathbb{C})/\mathbb{C}^*$ (or $\cong \mathrm{SL}_2(\mathbb{C})/N(\mathbb{C}^*)$) with Ω_{AG} contains an Akhiezer-Gindikin domain $\Omega_{\mathrm{AG}}^{SL} \cong D \times \overline{D}$ of Q_2. To see this, we will conjugate $S^{\mathbb{C}}$ by an element of G in order to relate $S^{\mathbb{C}}$ to the fixed Abelian Lie algebra $\mathfrak{a}$. This is carried out in the next section.

3.3. The intersection property. — To complete our task we conjugate the group $S^{\mathbb{C}}$ obtained in §3.1 above by an element h in the isotropy group G_{x_1} so that it can be easily seen that the resulting orbit $Q_2 = S^{\mathbb{C}} \cdot x_1$ intersects Ω_{AG} in the Akhiezer-Gindikin domain of Q_2.

The following is a first step in this direction.

Proposition 3.3.1. — *Let $G \cdot \exp E \cdot \exp iA \cdot x_0 = G \cdot x_1$ be any non-closed orbit in $\mathrm{bd}(\Omega_{\mathrm{AG}})$ and (E, H, F) a $\mathfrak{sl}_2$-triple in $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$ as in Lemma 2.2.2. Given $Z := E - F$, then there exits $h \in G_{x_1}$ so that $\mathrm{Ad}(h)(Z) \in i\mathfrak{a}$.*

This result is an immediate consequence of the following basic fact.

Lemma 3.3.2. — *Let $\mathfrak{l}$ be a real reductive Lie algebra, θ a Cartan involution and σ a further involution which commutes with θ. Let $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ be the eigenspace decomposition with respect to θ and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$ with respect to σ. Then, if $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ is a maximal Abelian subalgebra of $\mathfrak{q}$ and ξ is a hyperbolic semi-simple element of $\mathfrak{q}$, there exists $h \in \mathrm{Int}(\mathfrak{h})$ such that $\mathrm{Ad}(h)(\xi) \in \mathfrak{q}$.*

Proof. — Since ξ is hyperbolic, we may assume that there is a Cartan involution $\theta' : \mathfrak{l} \to \mathfrak{l}$ such that $\theta'(\xi) = -\xi$ and $\theta'\sigma = \sigma\theta'$. Then there exists $h \in \mathrm{Int}(\mathfrak{h})$ with $\mathrm{Ad}(h)\theta' \mathrm{Ad}(h^{-1}) = \theta$ (see [31]) and $\mathrm{Ad}(h)(\xi) \in \mathfrak{p} \cap \mathfrak{q}$.

To complete the proof, just note that $(\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{p} \oplus \mathfrak{q})$ is a Riemannian symmetric Lie algebra where any two maximal Abelian algebras in $\mathfrak{p} \cap \mathfrak{q}$ are conjugate by an element of $\mathrm{Int}(\mathfrak{h} \cap \mathfrak{k})$. □

Proof of Proposition 3.3.1. — Observe that $\mathrm{ad}(Z)$ has only imaginary eigenvalues. Replacing $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} = \mathfrak{h} \oplus \mathfrak{q}$ by the dual $\tilde{\mathfrak{l}} := \mathfrak{h} \oplus i\mathfrak{q} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{q}}$ and defining $\tilde{\sigma}$ and $\tilde{\theta}$ accordingly, we apply the above Lemma to $\xi := iZ$ and the Abelian Lie algebra $\mathfrak{a} \subset \tilde{\mathfrak{q}}$ to obtain $h \in \mathrm{Int}(\mathfrak{h})$ with $\mathrm{Ad}(h)(\xi) \in \mathfrak{a}$. Thus $\mathrm{Ad}(h)(Z)$ has the required property $\mathrm{Ad}(h)(Z) \in i\mathfrak{a}$. □

We now show that for $z \in \mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ the group $S^{\mathbb{C}}$ which is associated to the $\mathfrak{sl}_2$-triple constructed in the above proposition produces a Q_2-slice. For a precise formulation it is convenient to let $\mathrm{bd}_{\mathrm{gen}}(\omega_{\mathrm{AG}}) := \mathrm{bd}(\omega_{\mathrm{AG}}) \smallsetminus E$, where E is the union of the lower-dimensional strata as in §3.2.

Proposition 3.3.3. — *For $z \in \mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ and $x_1 = \exp iA \cdot x_0$ the associated point with $iA \in \mathrm{bd}_{\mathrm{gen}}(i\omega_{\mathrm{AG}})$ it follows that the line $\mathbb{R}(E - F)$ is transversal to $\mathrm{bd}_{\mathrm{gen}}(i\omega_{\mathrm{AG}})$ at iA in $i\mathfrak{a}$.*

The proof requires a more explicit description of $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} =: \mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$. For this, recall the root decompositions of $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}$ or $\mathfrak{a}^{\mathbb{C}}$, respectively: $\mathfrak{g}^{\mathbb{C}} = C_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) \oplus \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\Phi(\mathfrak{a})} \mathfrak{g}^{\mathbb{C}}_{\lambda}$. The behavior of this decomposition with respect to our involutions is the following: $\theta(\mathfrak{g}^{\mathbb{C}}_{\lambda}) = \mathfrak{g}^{\mathbb{C}}_{-\lambda}$ and $\tau(\mathfrak{g}^{\mathbb{C}}_{\lambda}) = \mathfrak{g}^{\mathbb{C}}_{-\lambda}$; furthermore, the root decomposition is σ-stable, *i.e.*, $\sigma(\mathfrak{g}^{\mathbb{C}}_{\lambda}) = \mathfrak{g}^{\mathbb{C}}_{\lambda}$. Fix a τ-stable basis of root covectors, *i.e.*, select any basis $L^1_{\lambda}, \dots, L^k_{\lambda}$ of $\mathfrak{g}_{\lambda} = (\mathfrak{g}^{\mathbb{C}}_{\lambda})^{\sigma}$ and define $L^j_{-\lambda} := \tau(L^j_{\lambda})$. Define $\mathfrak{g}[\lambda] := \mathfrak{g}_{\lambda} \oplus \mathfrak{g}_{-\lambda}$, $\mathfrak{g}^{\mathbb{C}}[\lambda] := \mathfrak{g}[\lambda] \oplus i\mathfrak{g}[\lambda]$ and notice that $\mathfrak{g}[\lambda] = (\mathfrak{g}[\lambda])^{\tau} \oplus (\mathfrak{g}[\lambda])^{-\tau}$. Finally, set

$$X^j_{[\lambda]} := L^j_{\lambda} + L^j_{-\lambda}, \quad Y^j_{[\lambda]} := L^j_{\lambda} - L^j_{-\lambda}$$

and observe that $X^j_{[\lambda]} \in (\mathfrak{g}[\lambda])^{\tau}$, $Y^j_{[\lambda]} \in (\mathfrak{g}[\lambda])^{-\tau}$. The reason for introducing such a basis is that the complex subspaces $\langle\langle X^j_{[\lambda]}, Y^j_{[\lambda]}\rangle\rangle_{\mathbb{C}}$ are $\mathrm{Ad}(t)$-stable for any $t := \exp iA$, $A \in \mathfrak{a}$.

Express $\mathrm{Ad}(t)$ as a matrix with respect to the basis $X^j_{[\lambda]}, Y^j_{[\lambda]}$:

$$\mathrm{Ad}(t)_{|\langle\langle X_{[\lambda]}, Y_{[\lambda]}\rangle\rangle} = \begin{pmatrix} \cosh\lambda(iA) & \sinh\lambda(iA) \\ \sinh\lambda(iA) & \cosh\lambda(iA) \end{pmatrix}.$$

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$ be the complexification of the Cartan decomposition of $\mathfrak{g}$. A simple calculation yields for $t = \exp iA$:

$$\begin{aligned}
\mathfrak{h} = \mathfrak{g} \cap \mathrm{Ad}(t)(\mathfrak{k}^{\mathbb{C}}) &= C_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\lambda(A) = \mathbb{Z}\pi} \mathfrak{g}[\lambda]^{\tau} \oplus \bigoplus_{\lambda(A) = \frac{1}{2}\pi + \mathbb{Z}\pi} \mathfrak{g}[\lambda]^{-\tau}, \\
\mathfrak{q} = i\mathfrak{g} \cap \mathrm{Ad}(t)(\mathfrak{p}^{\mathbb{C}}) &= i\mathfrak{a} \oplus \bigoplus_{\lambda(A) = \mathbb{Z}\pi} i\mathfrak{g}[\lambda]^{-\tau} \oplus \bigoplus_{\lambda(A) = \frac{1}{2}\pi + \mathbb{Z}\pi} i\mathfrak{g}[\lambda]^{\tau}.
\end{aligned}$$

Let $A \in \mathrm{bd}_{\mathrm{gen}}(\omega_{\mathrm{AG}})$ be boundary-generic, *i.e.*, there is a single $\lambda \in \Phi(\mathfrak{a})$ with

$$\lambda(A) = \pm\frac{\pi}{2}, \quad \mu(A) \notin \frac{1}{2}\pi\mathbb{Z} \text{ for all } \mu \in \Phi(\mathfrak{a}) \smallsetminus \{\pm\lambda\}.$$

The above general formulas imply that the centralizer subalgebra $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$ for such a boundary-generic point as above is given by

$$(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} = \mathfrak{m} \oplus \mathfrak{g}[\lambda]^{-\tau} \oplus i\mathfrak{a} \oplus i\mathfrak{g}[\lambda]^{\tau}.$$

To complete the proof of the proposition it is then enough to show that for the selected $\mathfrak{sl}_2$-triple $(E, H, F) \in (\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$ it follows that $E - F \in \mathbb{R}ih_{\lambda}$, where $h_{\lambda} \in \mathfrak{a}$ is the coroot determined by the root $\lambda \in \Phi(\mathfrak{a})$. This is the content of the following

LEMMA 3.3.4. — *Let $A \in \{\lambda = \frac{1}{2}\pi\} \cap \mathrm{bd}_{\mathrm{gen}}(\omega_{\mathrm{AG}})$ be boundary-generic as above. Then $E - F \in \mathbb{R}ih_{\lambda}$.*

Proof. — Let $\mathfrak{l} := (\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)} = \mathfrak{h} \oplus \mathfrak{q}$. Since $((E,H,F))_{\mathbb{R}}$ is semi-simple, it follows that $((E,H,F))_{\mathbb{R}} \subset [\mathfrak{l} : \mathfrak{l}]$. Hence, since $B([\mathfrak{g}_\lambda : \mathfrak{g}_{-\lambda}], \{\lambda = 0\}) = 0$ (B denotes the Killing form) we have

$$\begin{aligned}[\mathfrak{l} : \mathfrak{l}] &= [\mathfrak{m} \oplus \mathfrak{g}[\lambda]^{-\tau} \oplus i\mathfrak{a} \oplus i\mathfrak{g}[\lambda]^{\tau} : \mathfrak{m} \oplus \mathfrak{g}[\lambda]^{-\tau} \oplus i\mathfrak{a} \oplus i\mathfrak{g}[\lambda]^{\tau}] \\ &= \mathfrak{m} \oplus \mathbb{R}ih_\lambda \oplus \mathfrak{g}[\lambda]^{-\tau} \oplus i\mathfrak{g}[\lambda]^{\tau}.\end{aligned}$$

By Proposition 3.3.1 we have $E - F \in i\mathfrak{a}$. Finally, $E - F \in i\mathfrak{a} \cap [\mathfrak{l} : \mathfrak{l}] = \mathbb{R}ih_\lambda$. □

Recall that the set $\mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}}) = \mathrm{bd}(\Omega_{\mathrm{AG}}) \smallsetminus (\mathcal{C} \cup \mathcal{E})$ consists of certain non-closed orbits in the boundary of Ω_{AG}.

THEOREM 3.3.5. — *On every G-orbit in $\mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ there exists a point of the form $z := \exp E \cdot \exp iA \cdot x_0$, $A \in \mathrm{bd}_{\mathrm{gen}}(\omega_{\mathrm{AG}})$, E nilpotent, and a corresponding 3-dimensional simple subgroup $S^{\mathbb{C}} \subset G^{\mathbb{C}}$ such that*

1) *the 2-dimensional affine quadric $S^{\mathbb{C}} \cdot \exp iA \cdot x_0 =: S^{\mathbb{C}} \cdot x_1$ contains z;*
2) *the intersection $\Omega_{\mathrm{AG}} \cap S^{\mathbb{C}} \cdot x_1$ contains an Akhiezer-Gindikin domain $\Omega_{\mathrm{AG}}(S)$ of $S^{\mathbb{C}} \cdot x_1$, i.e., the orbit $S^{\mathbb{C}} \cdot x_1$ is a Q_2-slice.*

Proof. — Given a non-closed G-orbit in $\mathrm{bd}_{\mathrm{gen}}(\Omega_{\mathrm{AG}})$ let $z = \exp E \cdot \exp iA \cdot x_0$ be an optimal base point as in Lemma 3.1.1. By Proposition 3.3.1 we may choose an $\mathfrak{sl}_2$-triple (E,H,F) in $(\mathfrak{g}^{\mathbb{C}})^{\eta(\exp iA)}$ such that $E - F \in i\mathfrak{a}$. Let $S^{\mathbb{C}} \subset G^{\mathbb{C}}$ be the complex subgroup with Lie algebra $\mathfrak{s}^{\mathbb{C}} := ((E,H,F))_{\mathbb{C}}$. By construction $S^{\mathbb{C}} \cdot x_1$ contains z.

For a boundary-generic point x_1 with $\lambda(A) = \frac{1}{2}\pi$ and $\mu(A) \neq \frac{1}{2}\pi\mathbb{Z}$ for all $\mu \neq \pm\lambda$ we already know by 3.3.4 that $E - F \in \mathbb{R}ih_\lambda$. Assume that $h_\lambda \in \mathfrak{a}$ is the normalized coroot of λ, *i.e.*, $\lambda(h_\lambda) = 2$. Since ω_{AG} is invariant under the Weyl group, the image A' of A under the reflection on $\{\lambda = 0\}$ is also boundary-generic, and the intersection of $A - \mathbb{R}h_\lambda$ with ω_{AG} is the segment $\{A - th_\lambda : t \in (0, \frac{1}{2}\pi)\}$ with boundary points A and A'.

Recall that (e,H,f) with $E = ie$ and $F = -if$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{s}^{\mathbb{C}}$ such that $\mathfrak{s} := \mathfrak{g} \cap \mathfrak{s}^{\mathbb{C}} = ((e,H,f))_{\mathbb{R}}$. Let S denote the corresponding subgroup in $S^{\mathbb{C}}$ (isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$). The S-isotropy at all points $\exp((-\frac{1}{2}\pi, 0)ih_\lambda + iA) \cdot x_0$ is compact and it is non-compact at $\exp iA \cdot x_0$ and $\exp iA' \cdot x_0$. Hence, $S \cdot \exp((-\frac{1}{2}\pi, 0)ih_\lambda + iA) \cdot x_0$ is an Akhiezer-Gindikin domain in $S^{\mathbb{C}} \cdot x_1$ which is contained in Ω_{AG}. □

3.4. Domains of holomorphy. — Let $S^{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$, $S = \mathrm{SL}_2(\mathbb{R})$ be embedded in $S^{\mathbb{C}}$ as the subgroup of matrices which have real entries and let $K_S = \mathrm{SO}_2(\mathbb{R})$. To fix the notation, let D_0 and D_∞ be the open S-orbits in $\mathbb{P}_1(\mathbb{C})$. Further, choose $\mathbb{C} \subset \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ in such a way that $0 \in D_0$ and $\infty \in D_\infty$ are the K_S-fixed points.

Now let $S^{\mathbb{C}}$ act diagonally on $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ and note that the open orbit Ω, which is the complement of the diagonal $\mathrm{diag}(\mathbb{CP}^1)$ in Z, is the complex symmetric space $S^{\mathbb{C}}/K_S^{\mathbb{C}}$. Note that in $\mathbb{CP}^1 \times \mathbb{CP}^1$ there are four open

$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$-orbits: the bi-disks $D_\alpha \times D_\beta$ for any pair (α, β) from $\{0, \infty\}$. As S-spaces, the domains $D_0 \times D_\infty$ and $D_\infty \times D_0$ are equivariantly biholomorphic; further, they are actually subsets of Ω,and the Riemannian symmetric space S/K_S sits in each of them as the totally real S-orbit $S{\cdot}(0,\infty)$ (or $S{\cdot}(\infty, 0)$, respectively). Depending on which of these points is chosen as a reference point in Ω, both domains can be considered as the Akhiezer-Gindikin domain

$$\Omega_{\mathrm{AG}} = D_0 \times D_\infty = S \cdot \exp i\omega_{\mathrm{AG}} \cdot (0, \infty), \quad D_\infty \times D_0 = S \cdot \exp i\omega_{\mathrm{AG}} \cdot (\infty, 0)$$

with $\omega_{\mathrm{AG}} = (-\frac{1}{4}\pi, \frac{1}{4}\pi)h_\alpha$ and $h_\alpha \in \mathfrak{a}$ is the normalized coroot (*i.e.*, $\alpha(h_\alpha) = 2$).

Our main point here is to understand S-invariant Stein domains in Ω which properly contain Ω_{AG}. By symmetry we may assume that such has non-empty intersection with $D_0 \times D_0$. Observe that $(D_0 \times D_0) \cap \Omega = D_0 \times D_0 \smallsetminus \mathrm{diag}(D_0)$. Furthermore, other than $\mathrm{diag}(D_0)$, all S-orbits in $D_0 \times D_0$ are closed real hypersurfaces. For $D_0 \times D_0 \smallsetminus \mathrm{diag}(D_0)$ let $\Omega(p)$ be the domain bounded by $S \cdot p$ and $\mathrm{diag}(D_0)$. We shall show that a function which is holomorphic in a neighborhood of $S \cdot p$ extends holomorphically to $\Omega(p)$.

For this, define $\Sigma := \{(-s, s) : 0 \leq s < 1\} \subset D_0 \times D_0$. It is a geometric slice for the S-action. We say that a (1-dimensional) complex curve $C \subset \mathbb{C}^2 \subset Z$ is a supporting curve for $\mathrm{bd}(\Omega(p))$ at p if $C \cap \mathrm{c}\ell(\Omega(p)) = \{p\}$. Here, $\mathrm{c}\ell(\Omega(p))$ denotes the topological closure in $D_0 \times D_0$.

Proposition 3.4.1. — *For every $p \in D_0 \times D_0 \smallsetminus \mathrm{diag}(D_0)$ there exists a supporting curve for $\mathrm{bd}(\Omega(p))$ at p.*

Proof. — Recall that we consider D_0 embedded in $\mathbb{C}$ as the unit disc. It is enough to construct such a curve $C \subset \mathbb{C}^2$ at each point $p_s = (-s, s) \in \Sigma$, $s \neq 0$. For this we define $C_s := \{(-s{+}z, s{+}z) : z \in \mathbb{C}\}$. To prove $C_s \cap \mathrm{c}\ell(\Omega(p_s)) = \{p_s\}$ let d be the Poincare metric of the unit disc D_0, considered as the function $d : D_0 \times D_0 \to \mathbb{R}_{\geq 0}$. Note that it is an S-invariant function on $D_0 \times D_0$. In fact the values of d parameterize the S-orbits.

We now claim that $d(-s + z, s + z) \geq d(-s, s) = d(p_s)$ for $z \in \mathbb{C}$ and $(-s{+}z, s{+}z) \in D_0 \times D_0$, with equality only for $z = 0$, *i.e.*, C_s touches $\mathrm{c}\ell(\Omega(p_s))$ only at p_s. To prove the above inequality, it is convenient to compare the Poincaré length of the Euclidean segment $\mathrm{seg}(z - s, z + s)$ in D_0 with the length of $\mathrm{seg}(-s, s)$. Writing the corresponding integral for the length, it is clear, without explicit calculation, that $d(-s + x, s + x) > d(-s, s)$ for $z = x \in \mathbb{R} \smallsetminus 0$. The same argument shows also that $d(-s + x + iy, s + x + iy) > d(-s + x, s + x)$ for all non-zero $y \in \mathbb{R}$ and the proposition is proved. □

From the above construction it follows that the boundary hypersurfaces $S(p)$ are strongly pseudoconvex. Since then the smallest Stein domain containing a S-invariant neighborhood of $S(p)$ is $\Omega(p) \smallsetminus \mathrm{diag}(D_0)$, the following is immediate.

COROLLARY 3.4.2. — *For $p \in D_0 \times D_0 \setminus \operatorname{diag}(D_0)$ every function f which is holomorphic on some neighborhood of the orbit $S \cdot p$ extends holomorphically to $\Omega(p) \setminus \operatorname{diag}(D_0)$. An analogous statement is valid for $p \in D_\infty \times D_\infty \setminus \operatorname{diag}(D_\infty)$.*

Observe that the set $\mathrm{bd}_{\mathrm{gen}}(\mathrm{D}_0 \times \mathrm{D}_\infty)$ of generic boundary points, which was introduced in section 4.2, consists of the two S-orbits $\mathrm{bd}(D_0) \times D_\infty \cup D_0 \times \mathrm{bd}(D_\infty)$. Let $z \in \mathrm{bd}(D_0) \times D_\infty$ (or $z \in D_0 \times \mathrm{bd}(D_\infty)$ be such a boundary point.

COROLLARY 3.4.3. — *Let $\widehat{\Omega} \subset Q_2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be an S-invariant Stein domain which contains $D_0 \times D_\infty$ and the boundary point z. Then $\widehat{\Omega}$ also contains $D_0 \times \mathbb{CP}^1 \setminus \operatorname{diag}(\mathbb{CP}^1)$ (or $\mathbb{CP}^1 \times D_\infty \setminus \operatorname{diag}(\mathbb{CP}^1)$, respectively).*

Proof. — Let B be a ball around z which is contained in $\widehat{\Omega}$. For p in $B(z) \cap D_\infty \times D_\infty$ sufficiently close to z it follows that $S \cdot q \subset \widehat{\Omega}$ for all q in $B(z) \cap (D_\infty \times D_\infty)$. The result then follows from the previous corollary. □

If $\widehat{\Omega}$ is as in the above corollary, the fibers of the projection of $\widehat{\Omega} \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{P}^1$ can be regarded as non-constant holomorphic curves $f : \mathbb{C} \to \widehat{\Omega}$. One says that a complex manifold X is Brody hyperbolic if there are no such curves.

COROLLARY 3.4.4. — *If $\widehat{\Omega}$ is as above, then $\widehat{\Omega}$ is not Brody hyperbolic.*

A complex manifold X is said to be Kobayashi hyperbolic whenever the Kobayashi pseudo-metric is in fact a metric (see [27]). The pseudo-metric is defined in such a way that, if there exists a non-constant holomorphic curve $f : \mathbb{C} \to X$, then X is not hyperbolic, *i.e.*, Kobayashi hyperbolicity is a stronger condition than Brody hyperbolic. For an arbitrary semi-simple group G the domain Ω_{AG} is indeed Kobayashi hyperbolic (*cf.* [22], see §5 for stronger results).

The following is our main application of the existence of Q_2-slices at generic points of $\mathrm{bd}(\Omega_{\mathrm{AG}})$.

THEOREM 3.4.5. — *A G-invariant, Stein and Brody hyperbolic domain $\widehat{\Omega}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ which contains Ω_{AG} is equal to Ω_{AG}.*

Proof. — Arguing by contraposition, if Ω_{AG} is strictly contained in a G-invariant Stein domain $\widehat{\Omega}$, then by Theorem 3.3.5 there exists a Q_2-slice at a generic boundary point $z \in \mathrm{bd}(\Omega_{\mathrm{AG}}) \cap \widehat{\Omega}$ with $Q_2 \cap \widehat{\Omega}$ an S-invariant Stein domain properly containing the Akhiezer-Gindikin domain of Q_2. However, by Corollary 3.4.4, such a domain in Q_2 is not Brody hyperbolic. □

4. Hyperbolicity and the characterization of cycle domains

In this section it is shown the Wolf cycle domains $\Omega_W(D)$ are Kobayashi hyperbolic. The above theorem then yields their characterization (see 4.2.5).

4.1. Families of hyperplanes. — We start by proving a general result concerning families of hyperplanes in projective space and their intersections with locally closed subvarieties. Since such a subvariety is usually regarded as being embedded by sections of some line bundle, it is natural to regard the projective space as the projectivization $\mathbb{P}(V^*)$ of the dual space and a hyperplane in $\mathbb{P}(V^*)$ as a point in $\mathbb{P}(V)$.

We will think of a subset $S \in \mathbb{P}(V)$ as parameterizing a family of hyperplanes in $\mathbb{P}(V^*)$. A non-empty subset $S \subset \mathbb{P}(V)$ is said to have the normal crossing property if for every $k \in \mathbb{N}$ there exist $H_1, \dots H_k \in S$ so that for every subset $I \subset \{1, \dots, k\}$ the intersection $\bigcap_{i \in I} H_i$ is $|I|$-codimensional. If $|I| \geq \dim_{\mathbb{C}} V$, this means that the intersection is empty.

In the sequel $\langle S \rangle$ denotes the complex linear span of S in $\mathbb{P}(V)$, *i.e.*, the smallest plane in $\mathbb{P}(V)$ containing S.

Proposition 4.1.1. — *A locally closed, irreducible real analytic subset S with $\langle S \rangle = \mathbb{P}(V)$ has the normal crossing property.*

Proof. — We proceed by induction over k. For $k = 1$ there is nothing to prove. Given a set $\{H_{s_1}, \dots, H_{s_k}\}$ of hyperplanes with the normal crossing property and a subset $I \subset \{s_1, \dots, s_k\}$, define

$$\Delta_I := \bigcap_{s \in I} H_s, \quad \mathcal{H}(I) := \big\{ s \in S : H_s \supset \Delta_I \big\}, \quad \mathcal{C}\ell_k := \bigcup_{\substack{J \subset \{s_1, \dots, s_k\} \\ \Delta_J \neq \varnothing}} \mathcal{H}(J).$$

We wish to prove that $S \setminus \mathcal{C}\ell_k \neq \varnothing$. For this, note that each $\mathcal{H}(I)$ is a real analytic subvariety of S. Hence, if $S = \mathcal{C}\ell_k$, then $S = \mathcal{H}(J)$ for some J with $\Delta_J \neq \varnothing$. However, $\{H \in \mathbb{P}(V^*) : H \supset \Delta_J\}$ is a proper, linear plane $\mathcal{L}(J)$ of $\mathbb{P}(V)$. Consequently, $S \subset \mathcal{L}(J)$, and this would contradict $\langle S \rangle = \mathbb{P}(V)$. Therefore, there exists $s \in S \setminus \mathcal{C}\ell_k$, or equivalently, $\{H_{s_1}, \dots, H_{s_k}, H_s\}$ has the normal crossing property. □

It is known that if $H_1, \dots, H_{2m+1}$ are hyperplanes having the normal crossing property, where $m = \dim_{\mathbb{C}} \mathbb{P}(V)$, then $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic (*cf.* [12], see also [27, p. 137]).

Corollary 4.1.2. — *If S is a locally closed, irreducible and generating real analytic subset of $\mathbb{P}(V)$, then there exist hyperplanes $H_1, \dots H_{2m+1} \in S$ so that the complement $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic.*

Our main application of this result arises in the case where S is an orbit of the real form at hand.

Corollary 4.1.3. — *Let $G^{\mathbb{C}}$ be a reductive complex Lie group, G a real form, V^* an irreducible $G^{\mathbb{C}}$-representation space and S a G-orbit in $\mathbb{P}(V)$. Then there exist hyperplanes $H_1, \dots, H_{2m+1} \in S$ so that $\mathbb{P}(V^*) \setminus \bigcup H_j$ is Kobayashi hyperbolic.*

Proof. — From the irreducibility of the representation V^*, it follows that V is likewise irreducible and this, along with the identity principle, implies that for $\langle S\rangle = \mathbb{P}(V)$. □

4.2. Hyperbolic domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$. — Hypersurfaces H in $\Omega = G^{\mathbb{C}}/K^{\mathbb{C}}$ which are invariant under the action of an Iwasawa-Borel group B, *i.e.*, Borel groups which contain the AN part of some Iwasawa-decomposition $G = KAN$, play a key role in the study of G-invariant domains (see also [22], [23], [24] and [25]). In the sequel we shall simply refer to such H simply as a B-hypersurface.

Recall that if $H_1, \dots, H_m$ are all of the irreducible B-hypersurfaces in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and if $\bigcup_j(\bigcup_{g\in G} g(H_j))$ is removed from Ω, then the connected component Ω_I of the resulting domain is the Akhiezer-Gindikin domain Ω_{AG} (see [22], [2] and [33]). In particular, the Ω_I is non-empty.

Now if H is just one (possibly not irreducible) B-hypersurface, then the set $\bigcup_{g\in G} g(H) = \bigcup_{g\in K} g(H)$ is closed and its complement in $\Omega = G^{\mathbb{C}}/K^{\mathbb{C}}$ is open. Let Ω_H be the connected component of that open complement, containing the chosen base point x_0. It is likewise a non-empty G-invariant Stein domain in $\Omega = G^{\mathbb{C}}/K^{\mathbb{C}}$.

Here we shall prove that, if G is not Hermitian, any such Ω_H is Kobayashi hyperbolic. In the Hermitian case one easily describes the situation where Ω_H is not hyperbolic.

Let H be given as above and let L be the line bundle which it defines. Let σ_H be the corresponding section, *i.e.*, $\{\sigma_H = 0\} = H$.

Note that σ_H is a B-eigenvector in $\Gamma(\Omega, L)$. Let $V_H \subset \Gamma(\Omega, L)$ be the irreducible $G^{\mathbb{C}}$-representation space which contains σ_H. Define $\varphi_H : \Omega \to \mathbb{P}(V_H^*)$ to be the canonically associated $G^{\mathbb{C}}$-equivariant meromorphic map.

Lemma 4.2.1. — *The map $\varphi_H : \Omega \to \mathbb{P}(V_H^*)$ is a regular morphism onto a quasi-projective $G^{\mathbb{C}}$-orbit $G^{\mathbb{C}} \cdot v_0^* =: \widetilde{\Omega}$.*

Proof. — By definition φ_H is $G^{\mathbb{C}}$-equivariant; in particular its set E of base points is $G^{\mathbb{C}}$-invariant. Since Ω is $G^{\mathbb{C}}$-homogeneous, $E = \varnothing$. □

By definition every section $s \in V_H$ is the pull-back $\varphi_H^*(\widetilde{s})$ of a hyperplane section. Thus, there is a uniquely defined B-hypersurface $\widetilde{H}$ in $\mathbb{P}(V_H^*)$ with $\varphi_H^{-1}(\widetilde{H}) = H$. Let $\widetilde{\Omega}_{\widetilde{H}} \subset \mathbb{P}(V_H^*)$ be defined analogously to Ω_H, *i.e.*, $\widetilde{\Omega}_{\widetilde{H}} = \mathbb{P}(V_H^*) \smallsetminus \bigcup_{g\in G} g(\widetilde{H})$. Applying Corollary 4.1.3 to $\mathbb{P}(V_H^*)$ and $S := G\cdot\widetilde{H} \subset \mathbb{P}(V_H)$, it follows that the domain $\widetilde{\Omega}_{\widetilde{H}}$ is Kobayashi hyperbolic. Further, the connected component of $\varphi_H^{-1}(\widetilde{\Omega}_{\widetilde{H}})$ which contains the base point x_0 is just the original domain Ω_H.

If φ has positive dimensional fibers, which indeed can happen in the Hermitian case, then, since the connected components of its fibers contain many holomorphic curves $f : \mathbb{C} \to \Omega$, it follows that Ω_H is not Kobayashi hyperbolic.

In the case of finite fibers, since preimages under locally biholomorphic maps of hyperbolic manifolds are hyperbolic, the opposite is true.

THEOREM 4.2.2. — *If the φ_H-fibers are finite, then Ω_H is Kobayashi hyperbolic.*

COROLLARY 4.2.3. — *If G is not of Hermitian type, then Ω_H is Kobayashi hyperbolic.*

Proof. — If G is not of Hermitian type, then $K^{\mathbb{C}}$ is dimension theoretically maximal in $G^{\mathbb{C}}$ and, since φ_H is non-constant, it follows that it has finite fibers. □

THEOREM 4.2.4. — *The Wolf cycle domain $\Omega_W(D)$ of an open orbit D of an arbitrary real form G of an arbitrary complex semi-simple group $G^{\mathbb{C}}$ in an arbitrary flag manifold $Z = G^{\mathbb{C}}/Q$ ist Stein and Kobayashi hyperbolic.*

Proof. — It was shown in [25] that every Wolf cycle space $\Omega_W(D)$ is the intersection of certain of the Ω_H. In the notation of [25] such an intersection is referred to as the associated Schubert domain $\Omega_S(D)$. Thus the cycle domains $\Omega_W(D)$ are Stein.

If G is not of Hermitian type, then, since it is contained in Ω_H for certain B-hypersurfaces H, Corollary 4.2.3 implies that it is hyperbolic.

If G is of Hermitian type, then $\Omega_W(D)$ is either the associated bounded symmetric domain $\mathcal{B}$, its complex conjugate or, if Ω is non-compact, $\mathcal{B} \times \overline{\mathcal{B}}$ (see [43], [45], [46], [25]). Since bounded domains are hyperbolic, this completes the proof. □

We now give a characterization of all Wolf cycle domains, including the few exceptions mentioned above. For this recall that D is an open G-orbit in $Z = G^{\mathbb{C}}/Q$, C_0 the base cycle in D, and $G^{\mathbb{C}} \cdot C_0 = \Omega$ is the corresponding orbit in the cycle space $\mathcal{C}^q(Z)$.

THEOREM 4.2.5. — *If Ω is compact, then either $\Omega_W(D)$ consists of a single point or G is Hermitian and $\Omega_W(D)$ is either the associated bounded symmetric domain $\mathcal{B}$ or its complex conjugate $\bar{\mathcal{B}}$. If Ω is non-compact, then, regarding $\Omega_W(D)$ as a domain $G^{\mathbb{C}}/K^{\mathbb{C}}$, it follows that*

$$\Omega_W(D) = \Omega_{\mathrm{AG}}$$

for every open G-orbit in every $G^{\mathbb{C}}$-flag manifold $Z = G^{\mathbb{C}}/Q$.

Proof. — The exceptional case where Ω is compact is discussed in detail in the proof of Theorem 4.2.4 and therefore we restrict here to the non-compact case.

The statement $\Omega_W(D) = \Omega_S(D)$ is proved in [25]. In [22] and [33] it is proved that $\Omega_{AG} \subset \Omega_I$. By definition $\Omega_S(D) \supset \Omega_I = \Omega_{\mathrm{AG}}$. Since $\Omega_W(D)$ is Stein and hyperbolic (Theorem 4.2.4), by Theorem 3.4.5 it follows that $\Omega_W(D) = \Omega_{\mathrm{AG}}$, and all equalities are forced. □

REMARK. — The second author's proof [22] of the inclusion $\Omega_{AG} \subset \Omega_I$ only used the existence of a G-invariant strictly plurisubharmonic function on Ω_{AG}. It is in fact necessary to use the existence, shown in [10], of such a function which in addition restricts to and exhaustion of $\exp(i\omega_{AG})\cdot x_0$. In the meantime T. Matsuki [33] has given an algebraic proof which holds in greater generality.

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