Bull. Soc. math. France
129 (1), 2001, p. 115–139

REMARKS ON YU'S 'PROPERTY A' FOR DISCRETE METRIC SPACES AND GROUPS

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ABSTRACT. — Guoliang Yu has introduced a property on discrete metric spaces and groups, which is a weak form of amenability and which has important applications to the Novikov conjecture and the coarse Baum–Connes conjecture. The aim of the present paper is to prove that property in particular examples, like spaces with subexponential growth, amalgamated free products of discrete groups having property A and HNN extensions of discrete groups having property A.

RÉSUMÉ (*Remarques sur la propriété A de Yu pour les espaces métriques et les groupes discrets*)

Guoliang Yu a introduit une propriété sur les espaces métriques et les groupes discrets, qui est une forme faible de moyennabilité et qui a d'importantes applications à la conjecture de Novikov et la conjecture de Baum–Connes "coarse". Le but de cet article est de démontrer cette propriété dans des cas particuliers, tels que les espaces à croissance sous-exponentielle, les produits libres amalgamés de groupes discrets ayant la propriété A et les extensions HNN de groupes discrets ayant la propriété A.

1. Introduction

Let X be a discrete metric space. It is said to be of *bounded geometry* if there exists $N\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the number of elements in balls of given

Texte reçu le 10 novembre 1999, révisé le 7 février 2000 et le 11 avril 2000

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2000 Mathematics Subject Classification. — 54E35, 20F32, 19K56, 58G12, 05C05.

Key words and phrases. — Metric spaces, Novikov conjecture, coarse Baum–Connes conjecture.

radius is uniformly bounded:

$$\forall x \in X, \quad \#B(x,R) \le N(R). \tag{1.1}$$

In [19, Definition 2.1], Yu introduces a property on discrete metric spaces he calls property A, which is a weak form of amenability. It is shown in [10], [11], [19] that

- For every discrete group G with a left-invariant distance such that the resulting metric space has bounded geometry, G has property A if and only if it admits an amenable action on some compact space (or, equivalently, on its Stone-Čech compactification βG) [11, Theorem 3.3].
- With the same assumptions, if G has property A, then the Baum–Connes map for G is split injective [10, Theorem 3.2], hence G satisfies the Novikov Conjecture (see [3] for an introduction to the Baum–Connes conjecture and its relation to the Novikov conjecture). Moreover, the reduced group C^*-algebra $C^*_r(G)$ is exact, meaning that for every exact sequence of C^*-algebras

  $$0 \to J \longrightarrow A \longrightarrow A/J \to 0,$$

  the sequence obtained by taking spatial tensor products

  $$0 \to J \otimes_{\min} C^*_r(G) \longrightarrow A \otimes_{\min} C^*_r(G) \longrightarrow A/J \otimes_{\min} C^*_r(G) \to 0$$

  is exact (see [17] for a survey on exactness).
- Every discrete metric space with bounded geometry with property A satisfies the coarse Baum–Connes conjecture [19, Theorem 1.1] (see [13], [18] for an introduction to that conjecture).

That such impressive consequences result from that elementary property (see Definition 3.1) is quite remarkable. It was conjectured for a while that every discrete metric space has property A, but Gromov recently announced the construction of Cayley graphs that do not satisfy the property [7]. It remains important to determine classes of metric spaces or groups for which the property holds.

It is known that property A is true for amenable groups, semi-direct products of groups that have property A, asymptotically finite dimensional metric spaces with bounded geometry, hyperbolic groups in the sense of Gromov (see [8]). In this paper, it is proven that property A is true in each of the following cases, for a discrete metric space with bounded geometry X:

- $X \subset Y$, where Y is a metric space with property A;
- X has subexponential growth;
- $X = Y_1 \cup Y_2$, where (Y_1, Y_2) is an excisive pair;
- X is hyperbolic in the sense of Gromov;

- X is a group acting on a tree, such that the stabilizer of each vertex has property A. In particular, property A for groups is stable by taking amalgamated free products and HNN extensions.

We have tried in this paper to keep proofs as elementary and self-contained as possible, hoping to spark the interest of a broad range of readers.

2. Basic definitions

Let us recall a few elementary definitions from [13].

A metric space is said to be *proper* if every closed ball is compact.

Let X and Y be metric spaces. A (not necessarily continuous) map $f\colon X \to Y$ is said to be *proper* if the inverse image of any bounded set is bounded, and it is *coarse* if it is proper and if for every $R > 0$, there exists $S > 0$ such that for every x, $x' \in X$, $d(x, x') \leq R$ implies $d(f(x), f(x')) \leq S$.

Two coarse maps f, $g\colon X \to Y$ are *bornotopic* if there exists $R > 0$ such that $d(f(x), g(x)) \leq R$ for every $x \in X$. A coarse map $f\colon X \to Y$ is a *coarse equivalence* if there exists a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bornotopic to the identity; X and Y are then said to be coarsely equivalent.

Two distances d and d' on X are *coarsely equivalent* if the identity $(X, d) \to (X, d')$ is a coarse equivalence.

A map $f\colon X \to Y$ is a *uniform embedding* if it induces a coarse equivalence between X and $f(X)$. This means that f is coarse, and that for every $R > 0$, there exists $S > 0$ such that $d(x, x') \geq S$ implies $d(f(x), f(x')) \geq R$ for all $x, x' \in X$.

LEMMA 2.1. — *Let G be a countable discrete group. Then up to coarse equivalence, there exists one and only one left-invariant distance on G for which the resulting metric space has bounded geometry.*

Proof. — Let e be the unit element in G. Let d and d' be such distances, and $\ell(g) = d(g, e)$, $\ell'(g) = d'(g, e)$ the associated length functions. Let $R > 0$. Since $\#B_d(e, R) < \infty$, there exists $S > 0$ such that for all $g \in B_d(e, R)$, $\ell'(g) \leq S$. By the left invariance, $\mathrm{Id}_G\colon (G, d) \to (G, d')$ is coarse. Similarly, $\mathrm{Id}_G\colon (G, d') \to (G, d)$ is coarse.

To prove the existence, let $f\colon G \to \mathbb{N}^*$ be a function such that $f^{-1}([0, n])$ is finite for every n, $f(g) = f(g^{-1})$ for all $g \in G$, and $f(g) = 0$ iff $g = 1$. Let

$$\ell(g) = \inf\bigl\{f(g_1) + \cdots + f(g_n)\colon g = g_1 \cdots g_n\bigr\}.$$

The distance $d(g, h) = \ell(g^{-1}h)$ is left-invariant and the resulting metric space has bounded geometry. □

If the group is finitely generated, one can take the distance associated to any finite system of generators. If G acts freely and co-compactly by isometries on a proper metric space X, and $x_0 \in X$ is arbitrary, then one can take $d(g,h) = \ell(g^{-1}h)$ where $\ell(g) = d(gx_0, x_0)$.

3. Property A, equivalent definitions

This section presents a few equivalent definitions of the property A introduced by Yu [19]. For a given metric space and $R > 0$, Δ_R will denote

$$\big\{(x,y) \in X \times X : d(x,y) \leq R\big\}.$$

Definition 3.1. — (See [19, Definition 2.1].) A discrete metric space X is said to have property A if for any $R > 0$, $\varepsilon > 0$, there exist $S > 0$ and a family $(A_x)_{x\in X}$ of finite, nonempty subsets of $X \times \mathbb{N}$, such that

(i) $(y,n) \in A_x$ implies $(x,y) \in \Delta_S$;

(ii) for all $(x,y) \in \Delta_R$,

$$\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \leq \varepsilon.$$

Let us first recall the definition of a positive type kernel [12, Definition 5.1]. Let X be a set. A function $\varphi\colon X \times X \to \mathbb{R}$ is said to be a positive type kernel if $\varphi(x,y) = \varphi(y,x)$ for all x, $y \in X$, and if for every finitely supported, real-valued function $(\lambda_x)_{x\in X}$ on X, the following inequality holds:

$$\sum_{x,y\in X} \lambda_x \lambda_y \varphi(x,y) \geq 0. \tag{3.1}$$

A function $\varphi\colon X \times X \to \mathbb{R}$ is of positive type if and only if there exists a map $x \mapsto \eta_x$ from X to a real Hilbert space H such that $\varphi(x,y) = \langle \eta_x, \eta_y\rangle$ [12, Proposition 5.3].

Equivalent definitions listed in the proposition below clearly show that property A is a weak form of amenability. Indeed, (ii) and (iii) are Reiter's property (P1) and (P2) respectively, and (v) is Hulanicki's property [5].

Proposition 3.2. — *Let X be a discrete metric space with bounded geometry. The following are equivalent:*

(i) *X has property A;*

(ii) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\xi_x)_{x\in X}$, $\xi_x \in \ell^1(X)$, $\operatorname{supp}(\xi_x) \subset B(x,S)$, $\|\xi_x\|_{\ell^1(X)} = 1$, *and* $\|\xi_x - \xi_y\|_{\ell^1(X)} \leq \varepsilon$ *whenever* $d(x,y) \leq R$*;*

(ii′) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\chi_x)_{x\in X}$, $\chi_x \in \ell^1(X)$, $\operatorname{supp}(\chi_x) \subset B(x,S)$, $\|\chi_x - \chi_y\|_{\ell^1(X)}/\|\chi_x\|_{\ell^1(X)} \leq \varepsilon$ *whenever* $d(x,y) \leq R$*;*

(iii) $\forall R > 0$, $\forall \varepsilon > 0$, $\exists S > 0$, $\exists (\eta_x)_{x\in X}$, $\eta_x \in \ell^2(X)$, $\operatorname{supp}(\eta_x) \subset B(x,S)$, $\|\eta_x\|_{\ell^2(X)} = 1$, *and* $\|\eta_x - \eta_y\|_{\ell^2(X)} \leq \varepsilon$ *whenever* $d(x,y) \leq R$*;*

(iv) *$\forall R>0$, $\forall \varepsilon>0$, $\exists S>0$, $\exists(\zeta_x)_{x\in X}$, $\zeta_x\in \ell^2(X\times\mathbb{N})$, $\operatorname{supp}(\zeta_x)\subset B(x,S)\times\mathbb{N}$, $\|\zeta_x\|_{\ell^2(X\times\mathbb{N})}=1$, and $\|\zeta_x-\zeta_y\|_{\ell^2(X\times\mathbb{N})}\le\varepsilon$ whenever $d(x,y)\le R$;*

(v) *$\forall R>0$, $\forall\varepsilon>0$, $\exists S>0$, $\exists\varphi\colon X\times X\to\mathbb{R}$ of positive type such that $\operatorname{supp}\varphi\subset\Delta_S$ and $|1-\varphi(x,y)|\le\varepsilon$ whenever $d(x,y)\le R$.*

Proof. — (i) $\Leftrightarrow$ (ii): noting that in (ii), ξ_x may be supposed to be nonnegative (since $\||\xi_x|-|\xi_y|\|_{\ell^1(X)}\le\|\xi_x-\xi_y\|_{\ell^1(X)}$), this is exactly [11, Lemma 3.5].

(ii) $\Rightarrow$ (ii′): obvious.

(ii′) $\Rightarrow$ (ii): let χ_x as in (ii′). Let $\xi_x=\chi_x/\|\chi_x\|_{\ell^1(X)}$. Then

$$\begin{aligned}\|\xi_x-\xi_y\|_1 &\le \frac{\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}+\|\chi_y\|_1\Big|\frac{1}{\|\chi_x\|_1}-\frac{1}{\|\chi_y\|_1}\Big|\\ &= \frac{\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}+\frac{|\|\chi_y\|_1-\|\chi_x\|_1|}{\|\chi_x\|_1}\le\frac{2\|\chi_x-\chi_y\|_1}{\|\chi_x\|_1}\,.\end{aligned}$$

(ii) $\Rightarrow$ (iii): let ξ_x as in (ii). Define $\eta_x=|\xi_x|^{1/2}$. Then, denoting by $\int_X$ the summation on X, *i.e.* the integral with counting measure on X, one has

$$\begin{aligned}\|\eta_x-\eta_y\|^2_{\ell^2(X)} &= \int_X|\eta_x-\eta_y|^2\\ &\le \int_X|\eta_x^2-\eta_y^2| = \big\||\xi_x|-|\xi_y|\big\|_{\ell^1(X)}\le\|\xi_x-\xi_y\|_{\ell^1(X)}.\end{aligned}$$

(iii) $\Rightarrow$ (ii): Let η_x as in (iii). We can suppose that $\eta_x\ge0$. Let $\xi_x=\eta_x^2$. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|\xi_x-\xi_y\|_{\ell^1(X)} &= \int_X|\eta_x^2-\eta_y^2| = \int_X|\eta_x-\eta_y|(\eta_x+\eta_y)\\ &\le \|\eta_x-\eta_y\|_{\ell^2(X)}\|\eta_x+\eta_y\|_{\ell^2(X)}\le 2\|\eta_x-\eta_y\|_{\ell^2(X)}.\end{aligned}$$

(iii) $\Rightarrow$ (iv): obvious.

(iv) $\Rightarrow$ (iii): Let ζ_x as in (iv). Let $\eta_x(z)=\|\zeta_x(z,\cdot)\|_{\ell^2(\mathbb{N})}$. Then

$$\begin{aligned}\|\eta_x-\eta_y\|^2_{\ell^2(X)} &= \sum_{z\in X}\big|\|\zeta_x(z,\cdot)\|_{\ell^2(\mathbb{N})}-\|\zeta_y(z,\cdot)\|_{\ell^2(\mathbb{N})}\big|^2\\ &\le \sum_{z\in X}\big\|\zeta_x(z,\cdot)-\zeta_y(z,\cdot)\big\|^2_{\ell^2(X\times\mathbb{N})}=\|\zeta_x-\zeta_y\|^2_{\ell^2(X\times\mathbb{N})}.\end{aligned}$$

(iii) $\Rightarrow$ (v): Let η_x as in (iii). Let $\varphi(x,y)=\langle\eta_x,\eta_y\rangle$. Then $\operatorname{supp}\varphi\subset\Delta_{2S}$ and if $d(x,y)\le R$, then $1-\varphi(x,y)=\frac12\|\eta_x-\eta_y\|^2_{\ell^2(X)}\le\frac12\varepsilon^2$.

(v) $\Rightarrow$ (iii) is inspired from [4], proof of Theorem 13.8.6. The parallel would have been more apparent, had we introduced the concept of positive definite function on groupoids and used results from [16], but we opted for a more elementary proof. Let φ as in (v). Suppose $\varepsilon \le \frac{1}{2}$. Let

$$(T_\varphi \eta)(x) = \sum_{x\in X} \varphi(x,y)\eta(y).$$

For all ξ, $\eta \in \ell^2(X)$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\big|\langle \xi, T_\varphi\eta\rangle\big| &\le \sum_{x,y\in X} \big|\varphi(x,y)\big|\cdot\big|\xi(x)\big|\cdot\big|\eta(y)\big| \\
&\le \Big(\sum_{x,y\in X} \big|\varphi(x,y)\big|\cdot\big|\xi(x)\big|^2\Big)^{1/2}\Big(\sum_{x,y\in X} \big|\varphi(x,y)\big|\cdot\big|\eta(y)\big|^2\Big)^{1/2} \\
&\le \Big(\sup_{x\in X}\sum_{y\in X}\big|\varphi(x,y)\big|\Big)\|\xi\|_{\ell^2(X)}\cdot\|\eta\|_{\ell^2(X)}.
\end{aligned}$$

Since φ is of positive type,

$$\sup_{x,y}\big|\varphi(x,y)\big| \le \sup_{x\in X}\varphi(x,x) \le 1+\varepsilon \le 2,$$

so, using Notation (1.1), $|\langle \xi, T_\varphi\eta\rangle| \le 2N(S)\|\xi\|_{\ell^2(X)}\|\eta\|_{\ell^2(X)}$. We conclude that T_φ is a bounded operator on $\ell^2(X)$ and $\|T_\varphi\| \le 2N(S)$. Also, note that T_φ is a positive operator since (from Equation (3.1))

$$\langle \eta, T_\varphi\eta\rangle = \sum_{x,y\in X}\varphi(x,y)\eta(x)\eta(y) \ge 0.$$

Let p be a polynomial such that $0 \le p(t)$ and $|p(t)^2 - t| \le \varepsilon$ on $[0, 2N(S)]$. Let $\varphi_1 = p(\varphi)$, where $p(\varphi)$ is obtained using the convolution product

$$(\varphi * \psi)(x,y) = \sum_{z\in X}\varphi(x,z)\psi(z,y).$$

Let $(e_x)_{x\in X}$ be the canonical basis of $\ell^2(X)$. Let

$$\eta'_x = \varphi_1(x,\cdot), \quad \eta_x = \frac{\eta'_x}{\|\eta'_x\|_{\ell^2(X)}}.$$

We have

$$\langle \eta'_x, \eta'_y\rangle = \sum_{z\in X}\varphi_1(x,z)\varphi_1(z,y) = (\varphi_1 * \varphi_1)(x,y) = \big(p^2(\varphi)\big)(x,y),$$

$$\begin{aligned}
\big|\langle \eta'_x, \eta'_y\rangle - \varphi(x,y)\big| &= \big|(p^2(\varphi)-\varphi)(x,y)\big| = \big|\langle e_y, (p^2(T_\varphi)-T_\varphi)e_x\rangle\big| \\
&\le \big\|p^2(T_\varphi)-T_\varphi\big\| \le \varepsilon,
\end{aligned}$$

which implies $|\langle \eta'_x, \eta'_y\rangle - 1| \le 2\varepsilon$ for all $(x,y) \in \Delta_R$. Thus,

$$1 - \langle \eta_x, \eta_y\rangle = 1 - \frac{\langle \eta'_x, \eta'_y\rangle}{\langle \eta'_x, \eta'_x\rangle^{1/2}\langle \eta'_y, \eta'_y\rangle^{1/2}} \le 1 - \frac{1-2\varepsilon}{1+2\varepsilon} \le 4\varepsilon.$$

Therefore, $\|\eta_x - \eta_y\|_2 = \sqrt{2 - 2\langle \eta_x, \eta_y\rangle} \le \sqrt{8\varepsilon}$. Finally, if p is of degree n, then it is not hard to see that $\operatorname{supp}(\eta_x) \subset B(x, nS)$. □

We note that each of these definitions make sense for any metric space. A close examination of proofs shows that in the general case,

$$\text{(i)} \implies \text{(ii)} \iff \text{(iii)} \iff \text{(iv)} \implies \text{(v)}.$$

In the sequel, we shall say that a *not necessarily discrete* space has property A if and only if it satisfies the equivalent properties (ii)–(iv) (we found these definitions easier to manipulate that Yu's).

4. First properties

LEMMA 4.1. — *Let (X, d) be a discrete metric space and let d' be a coarsely equivalent distance. Then (X, d) has property A if and only of (X, d') has property A.*

Proof. — Obvious. □

In particular, from Lemma 2.1, one can talk about property A for discrete countable groups without reference to a particular distance. See also Proposition 4.3 (i) below.

Now, we prove that property A is inherited by subspaces.

PROPOSITION 4.2. — *Let X and Y be discrete metric spaces. Suppose there exists a uniform embedding of X into Y, and that Y has property A. Then X has property A.*

Proof. — From Lemma 4.1, we can assume that X is a subspace of Y. For every $y \in Y$, let $p(y) \in X$ be a point such that $d(y, p(y)) \le 2d(y, X)$. Let

$$V\colon \ell^2(Y) \longrightarrow \ell^2(X \times Y)$$

be the isometry defined by

$$(V\eta)(x, y) = \begin{cases} \eta(y) & \text{if } x = p(y), \\ 0 & \text{otherwise.} \end{cases}$$

Let $R > 0$, $\varepsilon > 0$. There exist $(\eta_y)_{y \in Y}$ and $S > 0$ such that

$$\eta_y \in \ell^2(Y), \quad \|\eta_y\|_2 = 1, \quad \operatorname{supp}(\eta_y) \subset B(y, S),$$

and $\|\eta_y - \eta_{y'}\|_2 \le \varepsilon$ whenever $d(y, y') \le R$. Define $\zeta_x = V(\eta_x) \in \ell^2(X \times Y)$. Then $\operatorname{supp}(\zeta_x) \subset B(x, 3S) \times Y$ since

$$\operatorname{supp}(\zeta_x) \subset p\big(\operatorname{supp}(\eta_x)\big) \times Y \subset p\big(B_Y(x, S)\big) \times Y \subset B_X(x, 3S) \times Y.$$

Moreover, $\|\zeta_x\|_2 = 1$ and $\|\zeta_x - \zeta_{x'}\|_2 = \|\eta_x - \eta_{x'}\|_2 \le \varepsilon$ whenever x, $x' \in X$ and $d(x, x') \le R$. We deduce that Proposition 3.2(iv) is satisfied. □

Proposition 4.3. — *Let G be a discrete group. Then*

(i) *G has property A if and only if for every $\varepsilon > 0$ and every $F \subset G$ finite, there exists $F' \subset G$ finite and $(\xi_x)_{x\in G}$, $\xi_x \in \ell^1(G)$, $\|\xi_x\|_1 = 1$, $\operatorname{supp}\xi_x \subset xF'$, and $\|\xi_x - \xi_{xg}\|_1 \le \varepsilon$ for all $x \in G$, for all $g \in F$;*

(ii) *G has property A if and only if every finitely generated subgroup $G' \subset G$ has property A.*

Proof. — For (i), choose an arbitrary left-invariant distance on G such that (G,d) has bounded geometry (*cf.* Lemma 2.1), and use the fact that

$$\begin{aligned} &\exists R,\ \big\{(x,xg)\colon x\in G, g\in F\big\} \subset \Delta_R \\ &\qquad \iff \big\{d(x,xg)\colon x\in G,\ g\in F\big\} \text{ is bounded} \\ &\qquad \iff \big\{d(e,g)\colon g\in F\big\} \text{ is bounded} \\ &\qquad \iff F \text{ is finite,} \end{aligned}$$

and that for every $R > 0$, $\Delta_R = \{(x,xg)\colon x\in G,\ g\in F\}$ where $F = B(e,0)$ is finite.

For (ii), the "only if" part follows from Proposition 4.2. For the "if" part, suppose that every finitely generated subgroup has property A, and let $\varepsilon > 0$, $F \subset G$ finite. Let G' be the subgroup generated by F. By assumption, there exist $F' \subset G'$ finite, $(\xi_x)_{x\in G'}$, $\xi_x \in \ell^1(G')$, $\|\xi_x\|_1 = 1$, $\operatorname{supp}\xi_x \subset xF'$, such that $\|\xi_x - \xi_{xg}\|_1 \le \varepsilon$ for every $x \in X$ and $g \in F$. Write $G = \amalg_{i\in I} x_i G'$. For every $i \in I$ and $g' \in G'$, let $\eta_{x_i g'}(y) = \xi_{g'}(x_i^{-1}y)$. Then $(\eta_x)_{x\in G}$ satisfies (i). □

5. Excision

Recall [13, Definition 9.1] that if X is a metric space, $Y \subset X$ and $Z \subset X$, then (Y,Z) is said to be an *excisive pair* if

$$\forall R>0,\ \exists S>0,\quad B(Y,R)\cap B(Z,R) \subset B(Y\cap Z, S).$$

Here, $B(Y,R)$ denotes $\{x\in X\colon d(x,Y)\le R\}$.

Let R, $\varepsilon > 0$. We shall say that $(\xi_x)_{x\in X}$ satisfies property $(\mathrm{P})_{R,\varepsilon,S}$ if

$$\xi_x \in \ell^1(X),\quad \xi_x \ge 0,\quad \|\xi_x\|_1 = 1,\quad \operatorname{supp}\xi_x \subset B(x,S) \text{ and } \|\xi_x - \xi_y\|_1 \le \varepsilon$$

whenever $d(x,y) \le R$.

Lemma 5.1. — *Let X be a discrete metric space, $Y \subset X$, $R > 0$, $\varepsilon \in (0,1]$, $S' > 0$, $S = S' + 16R/\varepsilon$, $R' = 33R/\varepsilon$ and $\varepsilon' = \varepsilon/2$. Suppose that $(\xi^0_x)_{x\in X}$ and $(\eta_y)_{y\in Y}$ satisfy $(\mathrm{P})_{R',\varepsilon',S'}$ for the spaces X and Y respectively. Then there exists $(\xi_x)_{x\in X}$ satisfying $(\mathrm{P})_{R,\varepsilon,S}$, such that $\xi_y = \eta_y$ for all $y \in Y$.*

Proof. — Let $c = 8R/\varepsilon$. Denote $\{t\} = \inf(1, \sup(t, 0))$. Define
$$\xi_x = \Big\{\frac{d(x,Y)}{c}\Big\}\xi_x^0 + \Big\{1 - \frac{d(x,Y)}{c}\Big\}\eta_{p(x)}$$
where $p\colon X \to Y$ is a projection such that $d(x, p(x)) \le 2d(x, Y)$.

Let x, $x' \in X$ such that $d(x, x') \le R$.

- If $\inf(d(x,Y), d(x',Y)) \ge c - R$, then
$$\|\xi_x - \xi_x^0\|_1 \le \frac{2R}{c} \quad \text{and} \quad \|\xi_{x'} - \xi_{x'}^0\|_1 \le \frac{2R}{c},$$
hence
$$\|\xi_x - \xi_{x'}\|_1 \le \|\xi_x - \xi_x^0\|_1 + \|\xi_{x'} - \xi_{x'}^0\|_1 + \|\xi_x^0 - \xi_{x'}^0\|_1 \le \frac{4R}{c} + \varepsilon' \le \varepsilon.$$
- If $d(x,Y) \le c$ and $d(x',Y) \le c$, then
$$d\big(p(x), p(x')\big) \le d(x,x') + d\big(x, p(x)\big) + d\big(x', p(x')\big) \le R + 4c \le R',$$
hence $\|\eta_{p(x)} - \eta_{p(x')}\|_1 \le \varepsilon'$. It follows that
$$\begin{aligned}\|\xi_x - \xi_{x'}\|_1 \le{} & \frac{d(x,Y)}{c}\|\xi_x^0 - \xi_{x'}^0\|_1 + \|\xi_{x'}^0\|_1 \frac{|d(x,Y) - d(x',Y)|}{c} \\ & + \Big\{1 - \frac{d(x,Y)}{c}\Big\}\|\eta_{p(x)} - \eta_{p(x')}\|_1 \\ & + \|\eta_{p(x')}\|_1 \frac{|d(x,Y) - d(x',Y)|}{c} \\ \le{} & \frac{d(x,Y)}{c}\varepsilon' + \frac{R}{c} + \Big\{1 - \frac{d(x,Y)}{c}\Big\}\varepsilon' + \frac{R}{c} = \varepsilon' + \frac{2R}{c} \le \varepsilon.\end{aligned}$$

The assertion about S is easy to check. □

Proposition 5.2. — *Let X be a metric space, and Y, Z be subspaces of X having property A, such that (Y,Z) is an excisive pair. Then $Y \cup Z$ has property A.*

Proof. — Let $R > 0$ and $\varepsilon > 0$. Let $S > 0$ such that
$$B(Y,R) \cap B(Z,R) \subset B(Y \cap Z, S).$$
Since $Y \cap Z \subset Y$, $Y \cap Z$ satisfies property A (Proposition 4.2), so $B(Y,S)$, $B(Z,S)$ and $B(Y \cap Z, S)$, being coarsely equivalent to Y, Z and $Y \cap Z$ respectively, satisfy property A (Lemma 4.1).

From Lemma 5.1 applied to the inclusions
$$B(Y \cap Z, S) \subset B(Y,S) \quad \text{and} \quad B(Y \cap Z, S) \subset B(Z,S),$$
there exist $(\eta_y)_{y \in Y}$ and $(\zeta_z)_{z \in Z}$ satisfying $(\mathrm{P})_{R,\varepsilon,S''}$ for the spaces $B(Y,S)$ and $B(Z,S)$ respectively, where $S'' > 0$ is some real number, such that $\eta_t = \zeta_t$

if $t \in B(Y \cap Z, S)$. Let

$$\xi_x = \begin{cases} \eta_x \text{ if } x \in Y, \\ \zeta_x \text{ if } x \in Z. \end{cases}$$

Let x, $x' \in X$ such that $d(x,x') \le R$. We check that $\|\xi_x - \xi_{x'}\|_1 \le \varepsilon$. This is clear if x, $x' \in Y$ or x, $x' \in Z$. If $x \in Y$ and $x' \in Z$, then x, $x' \in B(Y,R) \cap B(Z,R) \subset B(Y \cap Z, S)$, which implies $\xi_x = \eta_x$ and $\xi_{x'} = \eta_{x'}$, whence the conclusion. □

6. Spaces with subexponential growth

A (discrete) metric space X is said to have *subexponential growth* if

$$\lim_{R\to\infty} \sup_{x\in X} \frac{\log \#B(x,R)}{R} = 0.$$

In [9], it is proven that such a space admits a uniform embedding into Hilbert space. The goal of this section is to prove the stronger proposition below (see [19, Theorem 2.2] for a proof that property A implies uniform embeddability into Hilbert space):

THEOREM 6.1. — *Let X be a discrete metric space with subexponential growth. Then X has property A.*

The proof is much more complicated than in the case of groups, due to lack of homogeneity of the space. We need a few preliminary lemmas.

LEMMA 6.2. — *Let $(\alpha_n)_{n\ge1}$ be a sequence with $\alpha_n \ge 0$ and $\lim\limits_{n\to+\infty} \alpha_n/n = 0$. Then there exists $(\beta_n)_{n\ge1}$ such that*

(i) *$\alpha_n \le \beta_n$ for all $n \ge 1$;*
(ii) *(β_n) is increasing;*
(iii) *$\lim_{n\to+\infty} \beta_n/n = 0$;*
(iv) *$\lim_{n\to+\infty} \beta_{n+1} - \beta_n = 0$.*

Proof. — Let $\gamma_n = n \sup_{p\ge n}(\alpha_p/p)$. Clearly, (γ_n/n) decreases and converges to 0. Let $\beta_n = \sup_{q\le n} \gamma_q$. By construction, (i) and (ii) hold.

Let us show that $\beta_{n+1}/(n+1) \le \beta_n/n$. This is obvious if $\beta_n = \beta_{n+1}$. If $\beta_n < \beta_{n+1}$, then for some $q \le n$, one has $\gamma_q = \beta_n < \beta_{n+1} = \gamma_{n+1}$, so

$$\frac{\beta_{n+1}}{n+1} = \frac{\gamma_{n+1}}{n+1} \le \frac{\gamma_n}{n} \le \frac{\sup_{q\le n}\gamma_q}{n} = \frac{\beta_n}{n},$$

thus proving our claim. Assertion (iii) is obvious if (β_n) is bounded. If (β_n) is unbounded, then $\beta_n = \gamma_{q(n)}$ for some $q(n) \le n$, $\lim_{n\to+\infty} q(n) = \infty$, so $\beta_n/n = \gamma_{q(n)}/n \le \gamma_{q(n)}/q(n) \to 0$.

Let us prove (iv): using the fact that (β_n/n) is decreasing, one has

$$0 \le \beta_{n+1} - \beta_n = \frac{\beta_{n+1}}{n+1}(n+1) - \beta_n \le \frac{\beta_n}{n}(n+1) - \beta_n \le \frac{\beta_n}{n} \longrightarrow 0.$$

$\square$

LEMMA 6.3. — *Let $(a_n)_{n\ge 1}$ be a sequence with $a_n \ge 1$ and $\lim\limits_{n\to\infty} \log a_n/n = 0$. Then there exists $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

(i) *$a_n \le f(n)$ for all $n \ge 1$;*

(ii) *f is increasing;*

(iii) *f is convex;*

(iv) *$\lim_{n\to\infty} f(n+1)/f(n) = 1$.*

Proof. — Taking $\alpha_n = \log a_n$, we can assume from Lemma 6.2 that (a_n) is increasing and that a_{n+1}/a_n tends to 1. Assume also $\lim_{n\to\infty} a_n = \infty$ (otherwise, f can be chosen to be a constant function). Let $a_n = 0$ for $n \le 0$, and $b_n = \sup_{k\le n}(a_k - a_{k-1})$. Define

$$f(n+h) = \sum_{k\le n} b_k + hb_{n+1} \quad \big(n \in \mathbb{N},\ h \in [0,1)\big).$$

Then $f(n) \ge \sum_{k\le n} b_k \ge \sum_{k\le n} a_k - a_{k-1} = a_n$, whence (i).

(ii) and (iii) result from the fact that (b_n) is a nonnegative increasing sequence.

To prove (iv), first note that

$$\frac{f(n+1)}{f(n)} - 1 = \frac{b_{n+1}}{f(n)} \le \frac{b_{n+1}}{a_n}\cdot$$

If (b_n) is bounded, then clearly b_{n+1}/a_n tends to 0. If (b_n) is unbounded, then $b_n = a_{k(n)} - a_{k(n)-1}$ for some $k(n) \le n$, $\lim_{n\to\infty} k(n) = \infty$, so

$$0 \le \frac{f(n+1)}{f(n)} - 1 \le \frac{a_{k(n+1)} - a_{k(n+1)-1}}{a_n} \le \frac{a_{k(n+1)} - a_{k(n+1)-1}}{a_{k(n+1)-1}} \longrightarrow 0.$$

$\square$

LEMMA 6.4. — *Let X be a space with subexponential growth. Then there exists a space Y such that*

(i) *for all $R \ge 0$, $\psi(R) := \#B_Y(y,R)$ is independent of $y \in Y$;*

(ii) $\lim\limits_{R\to\infty} \dfrac{\sup_{x\in X} \#B_X(x,R)}{\psi(R) - \psi(R-1)} = 0$*;*

(iii) *$\psi(R+1) - \psi(R) \sim \psi(R) - \psi(R-1)$;*

(iv) *$R \mapsto \psi(R+1) - \psi(R)$ is an increasing function.*

Proof. — Let $a_n = \sup_{x\in X} \#B_X(x,n)$ $(n \in \mathbb{N})$, and let f as in Lemma 6.3. If $f(0) \le 1$, replace $f(x)$ with $f(x) + (1 - f(0)) + x$, and if $f(0) > 1$, replace $f(x)$ with $f(x) + (1 - f(0))(1 - x)$. We can thus suppose $f(0) = 1$, and that f is a continuous bijection of $[0, +\infty)$ onto $[1, +\infty)$. Let φ be the inverse of the bijection $f - 1\colon [0, +\infty) \to [0, +\infty)$. Then $\varphi(0) = 0$ and φ is concave. This implies $\varphi(s + t) \le \varphi(s) + \varphi(t)$ for all $s, t \ge 0$, hence

$$d(m,n) = \varphi\big(|m-n|\big), \quad m,n \in \mathbb{Z}$$

defines a distance on $\mathbb{Z}$. Let Y_1 be the metric space obtained. Then for all $y \in Y_1$,

$$\begin{aligned}\#B(y,R) &= 1 + 2\sup\big\{n \in \mathbb{N}\colon \varphi(n) \le R\big\}\\ &= 1 + 2\sup\big\{n \in \mathbb{N}\colon n \le f(R) - 1\big\}\\ &= 2\big[f(R)\big] - 1.\end{aligned}$$

Let $\psi_1(R) = 2[f(R)]-1$. Let $Y_2 = Y_1 \times Y_1$ with the sup distance. Let $Y = Y_2 \times \mathbb{Z}$ with the distance

$$d\big((y,n),(y',n')\big) = d(y,y') + |n - n'|.$$

Then $\psi(R) = \psi_1(R)^2 + 2\sum_{k=1}^{[R]} \psi_1(R-k)^2$, so

$$\psi(R) - \psi(R-1) = \psi_1(R)^2 + \psi_1(R-1)^2.$$

(i) is clear. Let us prove (ii). We have

$$\frac{\sup_{x\in X} \#B(x,R)}{\psi(R) - \psi(R-1)} \le \frac{f(R+1)}{\psi_1(R)^2} \sim \frac{f(R+1)}{(2f(R))^2} \sim \frac{1}{4f(R)} \longrightarrow 0.$$

Let us prove (iii). Since $f(R) \sim f(R+1)$, we have

$$\psi(R+1) - \psi(R) \sim 8f(R+1)^2 \sim 8f(R)^2 \sim \psi(R) - \psi(R-1).$$

(iv) results from the fact that f is an increasing function. □

It will be convenient to use the following terminology:

Definition 6.5. — Let Z be a set, and u_n, $v_n\colon Z \to \mathbb{R}_+^*$ $(n \in \mathbb{N})$. We say that $u_n \sim v_n$ uniformly in $z \in Z$ if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \ge n_0,\ \forall z \in Z, \quad \Big|\frac{u_n(z)}{v_n(z)} - 1\Big| \le \varepsilon.$$

Lemma 6.6. — *Let Z be a discrete proper metric space. Suppose that*

$$\#B(z, n+1) \sim \#B(z,n)$$

uniformly in $z \in Z$. Then Z has property A.

Proof. — Let χ_z^n be the characteristic function of $B(z,n)$. Let $R > 0$, and $\Delta_R = \{(z,z') \in Z^2 \colon d(z,z') \le R\}$. Then since $d(z,z') \le R$ implies

$$\begin{cases} B(z,n-R) \subset B(z',R) \subset B(z,n+R), \\ B(z,n-R) \subset B(z,R) \subset B(z,n+R), \end{cases}$$

we have

$$\frac{\|\chi_z^n - \chi_{z'}^n\|_1}{\|\chi_z^n\|_1} \le 2\frac{\#B(z,n+R) - \#B(z,n-R)}{\#B(z,n-R)}.$$

By assumption, $\#B(z,n-R) \sim \#B(z,n+R)$ uniformly in $z \in Z$, so

$$\lim_{n\to\infty} \frac{\|\chi_z^n - \chi_{z'}^n\|_1}{\|\chi_z^n\|_1} = 0$$

uniformly in $(z,z') \in \Delta_R$. By Proposition 3.2 ((ii′) ⇒ (i)), Z has property A. □

Proof of Theorem 6.1. — Let $Z = X \times Y$ with Y as in Lemma 6.4. Endow Z with the distance $d((x,y),(x',y')) = d(x,x') + d(y,y')$. By Proposition 4.2, since X is a subspace of Z, it suffices to prove that Z satisfies the assumptions of Lemma 6.6. For given $z = (x,y) \in X \times Y$ and $n \ge 1$, let

$$\begin{aligned} b_n(x) &= \#\big\{x' \in X \colon n-1 < d(x,x') \le n\big\}, \\ c_n &= \#\big\{y' \in Y \colon n-1 < d(y,y') \le n\big\} = \psi(n) - \psi(n-1). \end{aligned}$$

For brevity, we shall write b_n instead of $b_n(x)$. Let $d_n = b_0c_n + \cdots + b_nc_0$. Using the fact that (c_n) is an increasing sequence,

$$\begin{aligned} &\#B(z,n+1) - \#B(z,n) \\ &\quad = \#\big\{z' \in Z \colon n < d(z,z') \le n+1\big\} \\ &\quad \le \#\big\{y' \in Y \colon n < d(y,y') \le n+1\big\} \\ &\qquad\quad + \sum_{k=1}^{n+1} \#\big\{x' \in X \colon k-1 < d(x,x') \le k\big\} \\ &\qquad\qquad\quad \times \#\big\{y' \in Y \colon n-k < d(y,y') \le n-k+2\big\} \\ &\quad \le c_{n+1} + b_1(c_{n+1}+c_n) + \cdots + b_{n+1}(c_1+c_0) \\ &\quad \le 2(b_0c_{n+2} + b_1c_{n+1} + \cdots + b_{n+2}c_0) = 2d_{n+2}. \end{aligned}$$

Similarly,

$$\begin{aligned}\#B(z,n) - \#B(z,n-2) &\\ &= \#\{z' \in Z \colon n-2 < d(z,z') \le n\}\\ &\ge \sum_{k=0}^{n} \#\{x' \in X \colon k-1 < d(x,x') \le k\}\\ &\qquad\qquad \times \#\{y \in Y \colon n-k-1 < d(y,y') \le n-k\}\\ &= b_0c_n + \cdots + b_nc_0 = d_n.\end{aligned}$$

It follows that $\#B(z,n) \ge d_n + d_{n-2} + d_{n-4} + \cdots$, whence

$$\frac{\#B(z,n+1) - \#B(z,n)}{\#B(z,n)} \le \frac{2d_{n+2}}{d_n + d_{n-2} + d_{n-4} + \cdots}.$$

It thus suffices to show that $d_n(x) \sim d_{n+1}(x)$ uniformly in X. To do this, fix $q \in \mathbb{N}$ and let $r \le q$. Since $c_{n+1} \sim c_{n-r}$ and $b_{n-r}/c_{n-r} \to 0$ uniformly in $x \in X$ (Lemma 6.4), we have that $b_{n-r}/b_0c_{n+1} \sim b_{n-r}/c_{n-r}$ converges to 0 uniformly in $x \in X$, hence

$$d_{n+1} \sim b_0c_{n+1} + \cdots + b_{n-q}c_{q+1}$$

uniformly in $x \in X$. If $\varepsilon > 0$ and q is chosen so that $c_{k+1} \le (1+\varepsilon)c_k$ for all $k \ge q$, then for n large enough, we have for every $x \in X$,

$$\begin{aligned} d_{n+1} &\le (1+\varepsilon)(b_0c_{n+1} + \cdots + b_{n-q}c_{q+1})\\ &\le (1+\varepsilon)^2(b_0c_n + + \cdots + b_{n-q}c_q),\end{aligned}$$

hence $d_n \le d_{n+1} \le (1+\varepsilon)^2 d_n$. □

7. Reduction to graphs

Recall a definition by Gromov [6]:

DEFINITION 7.1. — Let X be a metric space. It is said to be *large-scale connected* if there exists a constant $c > 0$ such that every two points x and y in X can be joined by a finite chain of points

$$x = x_0,\ x_1, \ldots, x_n = y$$

such that $d(x_i, x_{i-1}) \le c$ $(1 \le i \le n)$.

LEMMA 7.2. — *Let X be a discrete metric space with bounded geometry. Then X is a subspace of a discrete, large-scale connected metric space with bounded geometry.*

Proof. — Let X_n $(n \ge 0)$ be the equivalence classes of the relation: $x \approx y$ if there exist $x_0, \ldots, x_m \in X$ such that $x_0 = x$, $x_m = y$, $d(x_i, x_{i-1}) \le 2$. Define $d_n = [d(X_n, X_{n+1})]$, and let $a_n \in X_n$, $b_{n+1} \in X_{n+1}$ such that

$d(a_n, b_{n+1}) \leq 2d_n$. Let $Y_n = \{0, 1, \ldots, d_n\} \times \{n\}$, and let Y be the space obtained by attaching "line segments" to X as follows:

$$Y = \frac{\coprod_{n\in\mathbb{N}}(X_n \amalg Y_n)}{(0,n) \sim a_n \text{ and } (d_n, n) \sim b_{n+1}}.$$

Endow Y with the maximal metric which agrees with the one on X, and such that $d((i,n),(j,n)) = |i-j| d(a_n, b_{n+1})/d_n$. Since $d_n \geq 1$, Y is large-scale connected (with $c = 2$ in Definition 7.1).

Let us prove that Y has bounded geometry. Let $N(R)$ satisfy Equation (1.1). Let $y \in Y$.

• Suppose $y \in X$. Since $d((i,n),(j,n)) \geq |i-j|$, $B_Y(y,R) \cap Y_n$ has at most $2R+2$ elements, and since the a_n are all distinct, and the b_n are all distinct, $B_Y(y,R)$ intersects at most $2N(R)$ of the spaces Y_n. Therefore,

$$\begin{aligned}\#B_Y(y,R) &\leq \#(B_Y(y,R) \cap X) + \sum_{n\in\mathbb{N}} B_Y(y,R) \cap Y_n \\ &\leq N(R) + 2N(R)(2R+2) = N(R)(4R+5).\end{aligned}$$

• Suppose $y \in Y_n$ ($n \in \mathbb{N}$). If $B(y,R)$ doesn't intersect X, then $B_Y(y,R) \subset Y_n - X$, hence $\#B(y,R) \leq 2R$. If $B(y,R) \cap X$ contains an element x, then $B(y,R) \subset B(x,2R)$, therefore

$$\#B(y,R) \leq N(2R)(8R+5).$$

We deduce that for all $y \in Y$, $\#B(y,R) \leq N(2R)(8R+5)$. □

Let X be a discrete metric space. For every $R > 0$, let $P_R(X)$ be the Rips' complex, defined as follows: $\{x_1, \ldots, x_n\}$ spans a simplex if and only if $d(x_i, x_j) \leq R$ for every i, j. Let X_R be the 1-skeleton of $P_R(X)$, and $X_R^{(0)}$ its set of vertices.

Recall a few definitions: let $\lambda \geq 1$ and $\mu \geq 0$.

A map $f \colon X \to Y$ between two metric spaces is called a (λ,μ)-*quasi-isometry* if for every x, x' in X, $(d(x,x') - \mu)/\lambda \leq d(f(x), f(x')) \leq \lambda d(x,x') + \mu$.

A (λ,μ)-*quasi-geodesic* between two points x, x' of a metric space X is a (λ,μ)-quasi-isometry $\varphi \colon [0, d(x,x')] \to X$ with $\varphi(0) = x$ and $\varphi(d(x,x')) = x'$.

A space X is said to be *quasi-geodesic* if there exist $\lambda \geq 1$ and $\mu \geq 0$ such that every two points x, $y \in X$ can be joined by a (λ,μ)-quasi-isometry.

LEMMA 7.3. — *Let X be a discrete, large-scale connected metric space, with bounded geometry.*

(i) *There exists $R_0 > 0$ such that for $R \geq R_0$, X_R is a connected graph with bounded geometry.*

(ii) *If for every $R \geq R_0$, X_R has property A, then X has property A.*

(iii) *If X is quasi-geodesic, then for R large enough, the canonical inclusion $X \to X_R$ is a quasi-isometry.*

Proof. — To prove (i), take $R_0 = c$ where c is as in Definition 7.1. Let $N(R)$ as in Equation (1.1). Since each vertex has at most $N(R)$ neighbors, X_R has bounded geometry (with $N_{X_R}(R') = N(R)^{R'}$).

Let us prove (ii). Let $R \geq R_0$ and $\varepsilon > 0$. Let d_R be the distance on X_R and note that $d \leq Rd_R$. Since $X_R^{(0)}$ has property A (*cf.* Proposition 4.2), there exists $S > 0$ and a family $(\eta_x)_{x\in X}$ of vectors of norm one $\eta_x \in \ell^2(X)$, such that $\|\eta_x - \eta_y\|_2 \leq \varepsilon$ whenever $d_R(x,y) \leq 1$, and η_x is supported in the ball centered in x of radius S in $X_R^{(0)}$. Therefore, η_x is supported in the ball centered in x of radius RS in X, and $\|\eta_x - \eta_y\|_2 \leq \varepsilon$ whenever $d(x,y) \leq R$.

To prove (iii), let $R \geq \sup(R_0, \lambda + \mu)$. Let x, $y \in X$. Clearly, $d_X(x,y) \leq Rd_{X_R}(x,y)$. Let φ be a (λ,μ)-quasi-geodesic from x to y. Let $x_k = \varphi(k)$ $(0 \leq k \leq n = [d_X(x,y)])$. Put $x_{n+1} = y$. Then $d_X(x_k, x_{k+1}) \leq \lambda + \mu \leq R$, hence $d_{X_R}(x,y) \leq n+1 \leq d_X(x,y) + 2$. □

8. Hyperbolic spaces

The following result was observed by Yu [19] in the case of discrete hyperbolic groups and negatively curved manifolds (see [8] for an introduction to hyperbolicity in the sense of Gromov).

Proposition 8.1. — *Property A holds for discrete metric spaces with bounded geometry, which are hyperbolic in the sense of Gromov.*

Proof. — Let X be a metric space as stated. Since a hyperbolic space is quasi-geodesic (and thus large-scale connected), it follows from Lemma 7.3 that X is quasi-isometric to a connected graph with bounded geometry. By Lemma 4.1 and the fact that hyperbolicity is preserved under quasi-isometry (see [8]), we are reduced to the case where X is the 0-skeleton of a connected graph. Then, the proof by E. Germain [2, Appendix B] applies almost word by word. We outline the proof for the reader's convenience. Choose $a \in \partial X$ (the Gromov boundary of X). For all $x \in X$, let $[[x, a[[$ be the set of infinite geodesics from x to a, *i.e.* isometries $g\colon \mathbb{N} \to X$ such that $g(0) = x$ and $\lim_{n\to\infty} g(n) = a$. For every $x \in X$ and k, $n \in \mathbb{N}^*$, define elements of $\ell^1(X)$ as follows:

$$F(x,k,n) = \text{characteristic function of } \bigcup_{\substack{d(x,y)<k \\ g\in[[y,a[[}} g\big([n,2n]\big),$$

$$H(x,n) = \frac{1}{n^{3/2}} \sum_{k<\sqrt{n}} F(x,k,n).$$

Let $\delta > 0$ such that X is δ-hyperbolic. Then

(i) For $n \geq n_0 = 36 + 300\delta$, for all $x \in X$, for all $y \in X$ with $d(x,y) < \sqrt{n}$, for all $g_0 \in [[x,a[[$, $g \in [[y,a[[$, $p \in g([n,2n])$, one has

$$d\big(p, g_0\big([n-\sqrt{n}, 2n+\sqrt{n}\,]\big)\big) \leq 4\delta.$$

See Lemma 2.3 in [2, Appendix B] for a proof.

(ii) $\exists C > 0$, $\forall x \in X$, $\forall n \in \mathbb{N}^*$, $\forall k < \sqrt{n}$,

$$n \leq \big\|F(x,k,n)\big\|_{\ell^1(X)} \leq Cn.$$

The first inequality is obvious, since $F(x,k,n)$ is always greater than the characteristic function of a geodesic of length n. For the second inequality, let $x \in X$ and $n \in \mathbb{N}^*$. Suppose $n \geq n_0$. Using (i) and (1.1), one has

$$\big\|F(x,k,n)\big\|_{\ell^1(X)} \leq \big[(2n+\sqrt{n}\,) - (n-\sqrt{n}\,) + 1\big] N(4\delta) \leq 2N(4\delta)n.$$

If $n \leq n_0$, then $\|F(x,k,n)\|_{\ell^1(X)} \leq N(2n_0 + \sqrt{n_0}) \leq N(108 + 900\delta)n$.

(iii) For all $R > 0$,

$$\lim_{n\to\infty} \big\|H(x,n) - H(y,n)\big\|_{\ell^1(X)} = 0$$

uniformly on $\Delta_R = \{(x,y) \in X \times X \colon d(x,y) \leq R\}$.

Indeed, suppose $d(x,y) \leq R$. Since $F(x,k,n) \leq F(y,k+R,n)$,

$$\begin{aligned}\sum_{0\leq k<\sqrt{n}} F(x,k,n) &\leq \sum_{\sqrt{n}-R\leq k<\sqrt{n}} F(x,k,n) + \sum_{0\leq k<\sqrt{n}-R} F(y,k+R,n) \\ &\leq \sum_{\sqrt{n}-R\leq k<\sqrt{n}} F(x,k,n) + \sum_{0\leq k<\sqrt{n}} F(y,k,n).\end{aligned}$$

By symmetry, and using (ii),

$$\begin{aligned}\big|H(x,n) - H(y,n)\big| &\leq \frac{1}{n^{3/2}} \sum_{\sqrt{n}-R\leq k<\sqrt{n}} \big(F(x,k,n) + F(y,k,n)\big), \\ \big\|H(x,n) - H(y,n)\big\|_{\ell^1(X)} &\leq \frac{1}{n^{3/2}}(2C(R+1)n) \leq \frac{2C(R+1)}{\sqrt{n}}\cdot\end{aligned}$$

Now from (ii), we have $\|H(x,n)\|_1 \geq 1$. Using (iii), $\chi_x = H(x,n)$ satisfies Proposition 3.2 (ii′) for n large enough. □

9. Groups acting on trees

Let us recall a few facts about groups acting on trees. See [14], [15] for further details. Let $\mathcal{G}$ be an oriented graph, and denote by $\mathcal{G}^{(0)}$ (resp. $\mathcal{G}^{(1)}$) its set of vertices (resp. edges). For each edge e, let e^+ and e^- its terminal and initial vertices. A graph of groups is by definition a collection of groups $(G_v)_{v\in\mathcal{G}^{(0)}}$, $(G_e)_{e\in\mathcal{G}^{(1)}}$, together with injective homomorphisms

$$\pi_e^+ \colon G_e \longrightarrow G_{e^+}, \quad \pi_e^- \colon G_e \longrightarrow G_{e^-}.$$

A graph of spaces is a collection of topological spaces with preferred basepoint $(X_v)_{v\in\mathcal{G}^{(0)}}$, $(X_e)_{e\in\mathcal{G}^{(1)}}$, together with pointed, injective maps $f_e^+\colon X_e\to X_{e^+}$, $f_e^-\colon X_e\to X_{e^-}$. (If the graph is a tree, one talks about a tree of spaces.) The total space is defined by

$$X=\frac{(\amalg_{v\in\mathcal{G}^{(0)}}X_v)\amalg(\amalg_{e\in\mathcal{G}^{(1)}}X_e\times[0,1])}{f_e^-(x)\sim(x,0)\text{ and }f_e^+(x)\sim(x,1)}.$$

One of the ways to define the fundamental group of a graph of groups is as follows: take connected pointed spaces X_v, X_e such that $\pi_1(X_v)=G_v$, $\pi_1(X_e)=G_e$, and such that the group morphisms $\pi_e^\pm\colon G_e\to G_{e^\pm}$ are induced by the inclusions $f_e^\pm\colon X_e\to X_{e^\pm}$. Then the fundamental group of the total space does not depend on the choice of the spaces X_v and X_e, and is called the fundamental group of the graph of groups [14, Section 3].

It is known that the fundamental group of a graph $\mathcal{G}$ of groups admits an action on a tree such that the quotient is isomorphic (as a graph) to $\mathcal{G}$, and such that the stabilizer of each vertex (resp. edge) is isomorphic to the corresponding vertex group G_v (resp. edge group G_e). Conversely, any group acting on a tree is the fundamental group of a graph of groups with the same properties.

Consider a tree of discrete metric spaces $(X_v)_{v\in T^{(0)}}$, $(X_e)_{e\in T^{(1)}}$ over T. Fix $\bar v\in\partial T$ (recall that the boundary ∂T of T is the set of infinite geodesics starting from a given basepoint). For every $v\in T^{(0)}$, let $\alpha(v)\in T^{(0)}$ such that $[v,\alpha(v)]$ is an edge pointing towards $\bar v$. Orient the tree by

$$\big[v,\alpha(v)\big]^+=\alpha(v).$$

Let $v\in T^{(0)}$. Let $Y_v=f_e^-(X_e)$ and let f_v be the holonomy map

$$f_v=f_e^+\circ(f_e^-)^{-1}\colon Y_v\longrightarrow X_{\alpha(v)},$$

where $e=[v,\alpha(v)]$. We suppose that there exists a function $\rho\colon\mathbb{R}_+\to\mathbb{R}_+$ such that for all $v\in T^{(0)}$, for all $y,y'\in Y_v$,

$$d\big(f_v(y),f_v(y')\big)\le\rho\big(d(y,y')\big),\quad d(y,y')\le\rho\big(d(f_v(y),f_v(y'))\big).$$

We metrize the total space X as follows: if $x\in X_v$, $x'\in X_{v'}$, there exist $k,\ell\in\mathbb{N}$ such that $\alpha^k(v)=\alpha^\ell(v')$, $d_T(v,v')=k+\ell$. If $k\ge1$ and $\ell\ge1$, we let

$$\begin{aligned}d(x,x')\;=\;k+\ell+\inf\Big[&d(x,x_0)+\sum_{j=0}^{k-2}d(f_{\alpha^j(v)}(x_j),x_{j+1})\\&+d\big(f_{\alpha^{k-1}(v)}(x_{k-1}),f_{\alpha^{\ell-1}(v')}(x'_{\ell-1})\big)\\&\quad+\sum_{j=0}^{\ell-2}d\big(f_{\alpha^j(v')}(x'_j),x'_{j+1}\big)+d(x',x'_0)\Big]\end{aligned}$$

with the constraints $x_j\in Y_{\alpha^j(v)}$, $x'_j\in Y_{\alpha^j(v')}$.

If $\ell = 0$ and $k > 0$, let

$$d(x,x') = k + \Big[\inf d(x,x_0) + \sum_{j=0}^{k-2} d\big(f_{\alpha^j(v)}(x_j), x_{j+1}\big) + d\big(f_{\alpha^{k-1}(v)}(x_{k-1}), x'\big)\Big].$$

If $k = 0$ and $\ell > 0$, we use a similar formula, and if $k = \ell = 0$, the distance coincides with the one on X_v.

PROPOSITION 9.1. — *With these assumptions, the tree of discrete spaces with the metric defined above has property A if and only if each of the vertex spaces has property A, and if one can find S (in Proposition 3.2 (ii)) independent of the vertex.*

Proof. — The "only if" part results from Proposition 4.2. Let us prove the reverse implication. Let $R \geq 1$ and $\varepsilon \in (0,1]$. We construct $(\eta_x)_{x\in X}$, $\eta_x \in \ell^1(X)$, $\|\eta_x\|_1 = 1$, $\operatorname{supp}(\eta_x) \subset B(x,S)$, such that $\|\eta_x - \eta_{x'}\|_1 \leq \varepsilon$ whenever $d(x,x') \leq R$.

For every $x \in X_v$, let $p(x) \in Y_v$ such that $d(x,p(x)) \leq d(x,Y_v) + 1$. Let

$$\beta(x) = f_v\big(p(x)\big), \quad x_k = \beta^k(x) \in X_{\alpha^k(v)} \quad (k \geq 0),$$
$$\delta_k = d\big(x_k, Y_{\alpha^k(v)}\big), \quad \theta_k = \sup_{j\leq k} \delta_j.$$

Note that $d(x_{k-1}, x_k) \leq \delta_{k-1} + 2$.

Let $R' > 0$, $S' > 0$, $\varepsilon' > 0$, $R_1 > 0$, $n \in \mathbb{N}$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function which will be specified later. For every $v \in T^{(0)}$ and $x \in X_v$, let $\xi_x \in \ell^1(X_v)$, $\|\xi_x\|_1 = 1$, $\operatorname{supp}(\xi_x) \subset B(x,S')$, such that $\|\xi_x - \xi_{x'}\|_1 \leq \varepsilon'$ whenever $d(x,x') \leq R'$. Set

$$\varphi(t) = \Big(1 - \frac{\psi(t)}{\psi(R_1)}\Big)_+, \quad a_k = \varphi(\theta_k).$$

Define finite sequences $(r_k)_{0\leq k<n}$ and $(c_k)_{0\leq k<n}$ by induction as follows: $r_0 = n$, and for $0 \leq k \leq n-1$,

$$c_k = \frac{r_k}{n-k}\big(1 + (n-k-1)(1-a_k)\big),$$
$$r_{k+1} = r_k - c_k = r_k\Big(\frac{n-k-1}{n-k}\Big)a_k.$$

It is easily seen by induction that $0 \leq c_k \leq r_k \leq n-k$ $(0 \leq k \leq n-1)$. Since $c_k = r_k - r_{k+1}$ $(0 \leq k < n-1)$ and $c_{n-1} = r_{n-1}$, we have $n = c_0 + \cdots + c_{n-1}$. Define

$$\eta_x = \frac{1}{n}\sum_{k=0}^{n-1} c_k \xi_{x_k}.$$

We show that $(\eta_x)_{x\in X}$ satisfies the required properties. First, since

$$\begin{aligned}(c_k \neq 0) \implies (r_k \neq 0) \implies (a_{k-1} \neq 0)\\ \implies (\delta_{k-1} \leq R_1) \implies \big(d(x_{k-1}, x_k) \leq R_1 + 2\big),\end{aligned}$$

we can take $S = n(2 + R_1) + S'$.

We shall need a preliminary lemma, which says that η_x depends continuously on the a_k's.

LEMMA 9.2. — *Let $a'_k \in [0,1]$ be a sequence. Let r'_k, c'_k be defined recursively by $r'_0 = n$, and for $0 \leq k \leq n-1$,*

$$c'_k = \frac{r_k}{n-k}\big(1 + (n-k-1)(1-a'_k)\big),$$

$$r'_{k+1} = r'_k - c'_k = r'_k \frac{n-k-1}{n-k} a'_k.$$

Let $\eta' = (1/n)\sum_{k=0}^{n-1} c'_k \xi_{x_k}$. Suppose that $|a_k - a'_k| \leq \varepsilon_1$ $(0 \leq k \leq n-1)$. Then $\|\eta_x - \eta'\|_1 \leq 2^n \varepsilon_1$.

Proof. — It is easily shown by induction that $|c'_k - c_k| \leq 2^k n\varepsilon_1$ and $|r'_k - r_k| \leq (2^k - 1)n\varepsilon_1$. It follows that $\|\eta_x - \eta'\|_{\ell^1(X)} \leq \frac{1}{n}\sum_{k=0}^{n-1} 2^k n \varepsilon_1 \leq 2^n \varepsilon_1$. □

Let x, $x' \in X$ such that $d(x,x') \leq R$. Our objective is now to show that $\|\eta_x - \eta_{x'}\|_1 \leq \varepsilon$. We shall distinguish several cases.

(a) Suppose x, $x' \in X_v$. Let δ'_k, θ'_k, *etc.* be the sequences associated to x', defined like δ_k, θ_k, *etc.* We first need a

LEMMA 9.3. — *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, such that $f(t) \geq t$. There exists $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:*

(i) $\psi(0) = 0$*;*

(ii) ψ *is increasing,* $\lim_{t\to\infty} \psi(t) = +\infty$*;*

(iii) ψ *is* 1*-Lipschitz;*

(iv) $\psi(f(t)) - \psi(t) \leq 1$ *for all* $t \geq 0$.

Proof. — Replacing $f(t)$ by $\int_t^{t+1} f(s)\,ds$, we can suppose f continuous. Then, replacing $f(t)$ by $t+1+\int_t^{t+1} f(s)\,ds$, we can suppose f differentiable, $f'(t) \geq 1$ for all $t \in \mathbb{R}_+$ and $f(0) \geq 1$. Let $t_n = f^n(0)$. We have $1 \leq t_1 < \cdots < t_n \to \infty$. Define $\psi(t) = t/t_1$ $(0 \leq t < t_1)$, and if $t_k \leq t < t_{k+1}$, $\psi(t) = \psi(f^{-k}(t)) + k$ for all $k \geq 1$. □

We have

$$d(x_{k+1}, x'_{k+1}) \leq \rho\big(d(p(x_k), p(x'_k))\big) \leq \rho\big(2 + \theta_k + \theta'_k + d(x_k, x'_k)\big).$$

Let $f_1(\theta) = \rho(2 + 2\theta + R)$, and define by induction

$$f_{k+1}(\theta) = \rho\bigl(2 + 2\theta + f_k(\theta)\bigr).$$

Let $f(\theta) = 2f_{n-1}(\theta) + 2\theta$. By induction on j, we have for $1 \leq j \leq k$,

$$d(x_j, x'_j) \leq f_j\bigl(\sup(\theta_{k-1}, \theta'_{k-1})\bigr).$$

Letting $j = k$, we find

$$2d(x_k, x'_k) \leq f\bigl(\sup(\theta_{k-1}, \theta'_{k-1})\bigr) \quad \forall k \geq 1. \tag{9.1}$$

With the function f thus defined, let ψ as in Lemma 9.3. Let us prove

$$|a_k - a'_k| \leq |a_{k-1} - a'_{k-1}| + \frac{1}{\psi(R_1)}. \tag{9.2}$$

Without loss of generality, we can suppose $\theta_{k-1} \geq \theta'_{k-1}$.

- If $\sup(\theta_k, \theta'_k) \leq f(\theta_{k-1})$, then

$$\theta'_{k-1} \leq \theta_{k-1} \leq \theta_k \leq f(\theta_{k-1}), \quad \theta'_{k-1} \leq \theta'_k \leq f(\theta_{k-1})$$

From Lemma 9.3(iv),

$$\psi(\theta'_{k-1}) \leq \psi(\theta_k) \leq \psi(\theta_{k-1}) + 1, \quad \psi(\theta'_{k-1}) \leq \psi(\theta'_k) \leq \psi(\theta_{k-1}) + 1$$

which implies $a_{k-1} - 1/\psi(R_1) \leq a_k \leq a'_{k-1}$ and $a_{k-1} - 1/\psi(R_1) \leq a'_k \leq a'_{k-1}$, whence (9.2).

- If $\theta_k > f(\theta_{k-1})$ and $\theta_k \geq \theta'_k$, then $\theta_k > f(\theta_{k-1}) > \theta_{k-1}$ implies $\delta_k = \theta_k > \theta_{k-1}$. From Equation (9.1), $\delta_k \geq 2d(x_k, x'_k)$, hence

$$\delta'_k \geq \delta_k - d(x_k, x'_k) \geq \frac{1}{2}\delta_k.$$

We deduce $\theta_k \geq \theta'_k \geq \delta'_k \geq \frac{1}{2}\delta_k = \frac{1}{2}\theta_k$, hence $\theta'_k \leq \theta_k \leq 2\theta'_k \leq f(\theta'_k)$. From Lemma 9.3 (iv),

$$\psi(\theta'_k) \leq \psi(\theta_k) \leq \psi(\theta'_k) + 1,$$

which implies $a'_k \geq a_k \geq a'_k - 1/\psi(R_1)$, whence (9.2).

- If $\theta'_k > f(\theta_{k-1})$ and $\theta'_k \geq \theta_k$, then we similarly show

$$\delta_k \geq \frac{1}{2}\delta'_k, \quad \theta'_k \geq \theta_k \geq \frac{1}{2}\theta'_k$$

and $a_k \geq a'_k \geq a_k - 1/\psi(R_1)$, whence (9.2).

It follows by induction that

$$|a_k - a'_k| \leq \frac{\psi(R) + k}{\psi(R_1)} \leq \frac{\psi(R) + n}{\psi(R_1)}.$$

Thus, from Lemma 9.2,

$$\begin{aligned}\|\eta_x - \eta_{x'}\|_1 &\le \|\xi_x - \xi_{x'}\|_1 + \Big\|\eta_x - \frac{1}{n}\sum_{k=0}^{n-1} c'_k \eta_{x_k}\Big\|_1 \\ &\le \|\xi_x - \xi_{x'}\|_1 + 2^n \frac{\psi(R)+n}{\psi(R_1)} \le \varepsilon' + 2^n n \frac{\psi(R)+1}{\psi(R_1)}\cdot\end{aligned}$$

(b) The same proof shows that if η_x is constructed using another projection $\tilde p\colon X_v \to Y_v$ instead of p, then the resulting vector $\tilde\eta_x$ satisfies

$$\|\eta_x - \tilde\eta_x\|_1 \le 2^n n/\psi(R_1).$$

(c) Suppose $x' = x_1$. By definition,

$$\eta_{x_1} = \frac{1}{n}\sum_{k=1}^{n} c'_k \xi_{x_k},$$

with $(c'_k)_{k\ge 1}$ defined as follows. Let $r'_1 = n$, and for $1 \le k \le n$,

$$\begin{aligned} c'_k &= \frac{r'_k}{n-k+1}\big(1 + (n-k)(1-a'_k)\big), \\ r'_{k+1} &= r'_k - c'_k = r'_k\Big(\frac{n-k}{n-k+1}\Big)a'_k, \end{aligned}$$

where $a'_k = \inf(\varphi(\delta_1), \dots, \varphi(\delta_k))$. Put $a'_0 = 1$, $r''_0 = n$ and for $1 \le k \le n-1$, define

$$\begin{aligned} c''_k &= \frac{r''_k}{n-k}\big(1 + (n-k-1)(1-a'_k)\big), \\ r''_{k+1} &= r''_k - c''_k = r''_k\Big(\frac{n-k-1}{n-k}\Big)a'_k. \end{aligned} \tag{9.3}$$

Let $\eta_2 = (1/n)\sum_{k=0}^{n-1} c''_k \xi_{x_k}$. Since

$$|a'_k - a_k| = \big|a'_k - \inf(a_0, a'_k)\big| = \sup(a'_k - a_0, 0) \le 1 - a_0 \le \psi(R)/\psi(R_1),$$

we have from Lemma 9.2

$$\|\eta_2 - \eta_x\|_1 \le 2^n \frac{\psi(R)}{\psi(R_1)}\cdot \tag{9.4}$$

Let $s''_k = r''_k/(n-k)$, $s'_k = r'_k/(n-k+1)$. Since $s''_{k+1}/s''_k = a'_k = s'_{k+1}/s'_k$, we have $s''_{k+1}/s'_{k+1} = s''_k/s'_k$, so s''_k/s'_k is constant equal to

$$\frac{s''_1}{s'_1} = \frac{r''_1}{n-1}\cdot\frac{n}{r'_1} = \frac{r''_0 a'_0}{n}\cdot 1 = 1,$$

hence, from Equation (9.3),

$$c''_k = \frac{r'_k}{n-k+1}\big(1 + (n-k-1)(1-a'_k)\big).$$

Thus, $c''_k \le c'_k \le \inf\big((n-k)/(n-k-1)c''_k, n-k+1\big)$.

Suppose $n \geq 8$, and let $a = n^{1/3}$. We have

$$\begin{aligned}
\|\eta_2 - \eta_{x_1}\|_1 &\leq \frac{1}{n}\Big(c''_0 + \sum_{k=1}^{n-1}(c'_k - c''_k) + c'_n\Big) \\
&\leq \frac{1}{n}\Big(1 + \sum_{k=1}^{n-1} \inf\Big(\frac{c''_k}{n-k-1}, n-k+1\Big) + 1\Big) \\
&\leq \frac{1}{n}\Big(2 + \sum_{1\leq k\leq n-a} \frac{c''_k}{a-1} + \sum_{n-a<k<n} n-k+1\Big) \\
&\leq \frac{1}{n}\Big(2 + \sum_{1\leq k\leq n-1} \frac{c''_k}{a-1} + \sum_{2\leq \ell\leq n-a} \ell\Big) \\
&\leq \frac{1}{n}\Big(\frac{n}{a-1} + \frac{a^2+3a+4}{2}\Big) \leq 2\Big(\frac{1}{a} + \frac{a^2}{n}\Big),
\end{aligned}$$

$$\|\eta_2 - \eta_{x_1}\|_1 \leq 4n^{-1/3}. \tag{9.5}$$

We conclude from (9.4) and (9.5) that

$$\|\eta - \eta_{x_1}\|_1 \leq 4n^{-1/3} + 2^n \frac{\psi(R)}{\psi(R_1)}\cdot \tag{9.6}$$

(d) Suppose $d(x, x_k) \leq R$. Then $k \leq R$, so it follows from (9.6) that

$$\|\eta_x - \eta_{x_k}\|_1 \leq \frac{4R}{n^{1/3}} + 2^n \frac{R\psi(R)}{\psi(R_1)}\cdot$$

(e) Finally, let x, x' such that $d(x, x') \leq R$. After possibly changing the projection $p\colon X_v \to Y_v$ into another projection $\tilde{p}\colon X_v \to Y_v$, there exist k, $\ell \in \mathbb{N}$ such that $d(x, \tilde{x}_k) \leq R$, $d(x', \tilde{x}_\ell) \leq R$, $\alpha^k(v) = \alpha^\ell(v')$ and $d(\tilde{x}_k, \tilde{x}_\ell) \leq R$ where $\tilde{x}_k$, $\tilde{\eta}_x$, *etc.* are constructed like x_k, η_x, *etc.*, but using $\tilde{p}$ instead of p. Putting together cases (a) and (d),

$$\begin{aligned}
\|\tilde{\eta}_x - \tilde{\eta}_{x'}\|_1 &\leq \|\tilde{\eta}_{x_k} - \tilde{\eta}_{x'_l}\|_1 + \big(\|\tilde{\eta}_x - \tilde{\eta}_{x_k}\|_1 + \|\tilde{\eta}_{x'_l} - \tilde{\eta}_{x'}\|_1\big) \\
&\leq \Big(2^n n\frac{\psi(R)+1}{\psi(R_1)} + \varepsilon'\Big) + 2\Big(\frac{4R}{n^{1/3}} + 2^n \frac{R\psi(R)}{\psi(R_1)}\Big).
\end{aligned}$$

Now, from (b),

$$\begin{aligned}
\|\eta_x - \eta_{x'}\|_1 &\leq \|\tilde{\eta}_x - \tilde{\eta}_{x'}\|_1 + 2\Big(\frac{2^n n}{\psi(R_1)}\Big) \\
&\leq \frac{8R}{n^{1/3}} + 2^{n+1}\frac{R\psi(R)}{\psi(R_1)} + 2^n n \frac{\psi(R)+3}{\psi(R_1)} + \varepsilon' \\
&\leq \frac{8R}{n^{1/3}} + 2^{n+2} n \frac{R\psi(R)+1}{\psi(R_1)} + \varepsilon'.
\end{aligned}$$

If we choose $\varepsilon' = \frac{1}{4}\varepsilon$, an integer n satisfying $n \geq (16R/\varepsilon)^3$ and R_1 such that $\psi(R_1) \geq 2^{n+4} n(R\psi(R)+1)/\varepsilon$, then $\|\eta_x - \eta_{x'}\|_1 \leq \varepsilon$. □

THEOREM 9.4. — *Let G be a discrete group acting on a tree, with finite quotient. Then G has property A if and only if the stabilizer of each vertex group has property A.*

Proof. — Analogous to [1, Theorem 3.1]. We reproduce here the argument for the reader's convenience. The group G is the fundamental group of a finite graph of groups $\mathcal{G}$, such that every vertex group has property A (see [15]). From Proposition 4.3, we can suppose that each vertex group G_v and each edge group G_e is finitely generated.

For each vertex group G_v and each edge group G_e, we fix a presentation such that the presentation of G_v contains the one of G_e if $v = e^+$ or $v = e^-$. Denote the standard 2-CW-complex associated to this presentation by X_v or X_e respectively (recall that it is obtained from a bouquet of loops, one for each generator, and attaching a 2-cell for each relation). Let $f_e^{\pm}\colon X_e \to X_{e^{\pm}}$ be the cellular maps associated to the homomorphisms $G_e \to G_{e^{\pm}}$. This defines a graph of spaces over $\mathcal{G}$, and by definition G is the fundamental group of the total space X.

Let $\tilde{X}$ be the universal cover of X, and $\tilde{X}^{(1)}$ its 1-skeleton. We shall call *vertex spaces* (resp. *edge spaces*) the connected components of the preimage of a space X_v (resp. $X_e \times [0,1]$) under the projection $\tilde{X}^{(1)} \to X$. Then $\tilde{X}^{(1)}$ is a tree of spaces, with each vertex space (resp. edge space) isomorphic to the Cayley graph of some G_v (resp. G_e). (Recall that the Cayley graph associated to a group G and a symmetric system of generators S is the graph whose vertex space is G, and such that g and h are endpoints of a common edge if and only if $g^{-1}h \in S$.)

$\tilde{X}^{(1)}$ is metrized as explained before Proposition 9.1. It is easy to see that G acts freely, cocompactly and by isometries on $\tilde{X}^{(1)}$. By the remark following Lemma 2.1, we are reduced to showing that $\tilde{X}^{(1)}$ has property A: this is true by Proposition 9.1. □

The corollaries below are consequences of [15].

COROLLARY 9.5. — *If G and H are countable discrete groups having property A, and K is a group that injects in each, then the amalgamated free product $G *_K H$ has property A.*

COROLLARY 9.6. — *If G is a countable discrete group with property A, K is a subgroup and $\theta\colon K \to \theta(K)$ is an isomorphism, then the HNN extension HNN(K, G, θ) has property A.*

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