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ON p-ADIC NEARBY CYCLES OF LOG SMOOTH FAMILIES

BY TAKESHI TSUJI (*)

ABSTRACT. — We prove isomorphisms between p-adic nearby cycles and syntomic complexes for fs (= fine and saturated) log schemes log smooth over a p-adic henselian discrete valuation ring. This is a generalization of the results of M. Kurihara in the good reduction case and of K. Kato in the case of perfect residue field and semi-stable reduction. Combining with a result of C. Breuil, we obtain a comparison theorem between p-torsion étale cohomology and log crystalline cohomology for proper log smooth families.

RÉSUMÉ. — SUR LES CYCLES PROCHES p-ADIQUES DE FAMILLES LOG LISSES. — On prouve des isomorphismes entre des cycles proches p-adiques et des complexes syntomiques pour les fs log schémas log lisses sur un anneau de valuation discrète hensélien p-adique. Ceci généralise des résultats de M. Kurihara dans le cas de bonne réduction et de K. Kato dans le cas d'un corps résiduel parfait pour la réduction semi-stable. En combinant avec un résultat de C. Breuil, on obtient un théorème de comparaison entre la cohomologie étale p-torsion et la cohomologie log cristalline pour les familles propres et log lisses.

Introduction

Let K be a henselian discrete valuation field of characteristic 0 with residue field k (not necessarily perfect) of positive characteristic p. Let O_K denote the ring of integers of K, and let (S,N) denote the scheme $\mathrm{Spec}(O_K)$ endowed with the log structure defined by its closed point.

We consider an fs (= fine and saturated) smooth log scheme (X,M) over (S,N). Let X_{triv} denote the maximal open subset of X on which the log structure M is trivial and let (Y,M_Y) denote $(X,M)\otimes_{O_K}k$. Let i (resp. j) denote the immersion $Y\to X$ (resp. $X_{\mathrm{triv}}\to X$). We denote by $\mathcal{E}_n(r)_{(X,M)}$ the complex $i^*Rj_*\mathbb{Z}/p^n\mathbb{Z}(r)\in D^+(Y_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$ for integers $n\geq 1$ and r. Then we have the syntomic complex with log poles $\mathcal{S}_n(r)_{(X,M)}\in D^+(Y_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$ and the

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canonical morphism

$$\alpha\colon \mathcal{S}_n(r)_{(X,M)} \longrightarrow \mathcal{E}_n(r)_{(X,M)}$$

for integers $0 \le r \le p-2$ and $n \ge 1$ (see [Ka1], [Ku], [Ka3], [T1] and [T2]). The purpose of this paper is to prove that α induces an isomorphism (Theorem 5.1):

$$\mathcal{S}_n(r)_{(X,M)} \xrightarrow{\sim} \tau_{\le r}\mathcal{E}_n(r)_{(X,M)}$$

This was already proven by K. Kato and M. Kurihara for a semi-stable scheme X over S endowed with the log structure M defined by its special fiber and also for its base change by any finite extension of O_K (*loc. cit.*).

We follow the same strategy as the Hyodo's calculation [H] of the p-adic vanishing cycles in the semi-stable reduction case. The key observation for our proof is as follows (*cf.* Lemma 3.2 and Lemma 5.2): For a log blowing up (§3) $f\colon (X',M') \to (X,M)$ whose fibers are of dim≤ 1 and $y \in Y$, if the claim is true at y and at all non-closed points of $f^{-1}(y)$, then it is also true at all closed points of $f^{-1}(y)$. Using this fact, we are reduced to the good reduction case with horizontal divisor at infinity. In the good reduction case, we prove the claim by using spectral sequences and the calculation of p-adic vanishing cycles by S. Bloch and K. Kato [Bl-Ka] (in the case without horizontal divisor).

Assume that Y is reduced, which is equivalent to saying that (Y, M_Y) is of Cartier type over $(S,N) \otimes k$ and also to saying that (X,M) is universally saturated over (S,N) (see [T3]). Then, combining the above isomorphism with the explicit calculation of $\mathcal{H}^q(\mathcal{S}_1(q)_{(X,M)})$ $(0 \le q \le p-2)$ in Proposition A15 for $q = r$ (see also Proposition A5, A10, A11 and Proposition 2.11), we obtain a generalization of [Bl-Ka, Cor. 1.4.1] and [H, 1.7, Cor.] to $\mathcal{H}^q(\mathcal{E}_1(q)_{(X,M)})$ $(= i^* R^q j_* \mathbb{Z}/p\mathbb{Z}(q))$ under the assumption $0 \le q \le p-2$.

We still assume that Y is reduced. Let $\overline{K}$ be an algebraic closure of K and let $O_{\overline{K}}$ (resp. $\bar{k}$) be the ring of integers (resp. the residue field) of $\overline{K}$. Set

$$\overline{X} := X \otimes_{O_K} O_{\overline{K}}, \quad \overline{X}_{\text{triv}} := X_{\text{triv}} \otimes_K \overline{K}, \quad \overline{Y} := Y \otimes_k \bar{k}.$$

Let $\bar{\imath}$ and $\bar{\jmath}$ denote the immersions $\overline{Y} \to \overline{X}$ and $\overline{X}_{\text{triv}} \to \overline{X}$ respectively. Then, by taking the inductive limit to an algebraic closure $\overline{K}$ of K (see the end of §2), we obtain an isomorphism

$$\mathcal{S}_n(r)_{(\overline{X},\overline{M})} \xrightarrow{\sim} \tau_{\le r}\bar{\imath}^* R\bar{\jmath}_* \mathbb{Z}/p^n\mathbb{Z}(r)$$

for integers $0 \le r \le p-2$ and $n \ge 1$. Assume that X is proper over S. Then by taking $H^q_{\text{ét}}(\overline{Y}, -)$ and using the proper base change theorem for étale cohomology, we obtain an isomorphism

$$H^q_{\text{ét}}\big(\overline{Y}, \mathcal{S}_n(r)_{(\overline{X},\overline{M})}\big) \cong H^q_{\text{ét}}\big(\overline{X}_{\text{triv}}, \mathbb{Z}/p^n\mathbb{Z}(r)\big)$$

for integers $n \geq 1$ and $0 \leq q \leq r \leq p-2$. Assume further that k is perfect and K is absolutely unramified. In [Br], C. Breuil proved the isomorphism between $H^q_{\text{ét}}(\overline{Y}, \mathcal{S}_n(r)_{(\overline{X},\overline{M})})(-r)$ and the Galois representation associated to the q-th crystalline cohomology of (X_n, M_n) over the PD-polynomial ring $W_n\langle u\rangle$ with the log structure $\mathcal{L}(u)$ at "$u = 0$" for $n \geq 1$ and $0 \leq q \leq r \leq p-2$ (see [Br Thm. 3.2.4.1 and Prop. 3.2.1.7]). Composing these two isomorphisms, we obtain a comparison theorem between the étale cohomology $H^q_{\text{ét}}(\overline{X}_{\text{triv}}, \mathbb{Z}/p^n\mathbb{Z})$ and the log crystalline cohomology $H^q_{\text{crys}}((X_n, M_n)/(W_n\langle u\rangle, \mathcal{L}(u)))$ for $0 \leq q \leq p-2$ (see [Br, Thm. 3.2.4.7]).

This paper is organized as follows: In §1, we generalize the definition of p-bases and some related results in [Ka5, §1] to fine log schemes. In §2, we define the syntomic complex $\mathcal{S}_n(r)_{(X,M)}$ ($n \geq 1$, $r \leq p-1$) and construct the map to $\mathcal{E}_n(r)_{(X,M)}$. As in [Ka5, §2], we allow embeddings not only into smooth fine log schemes but also into fine log schemes with p-bases. This generalization is necessary to define the syntomic complex in the case where k is not perfect. Except this point, the definition and the construction are the same as [Ka3, §5] and [T2, §§2.1, 2.2, 3.1]. So we only give an outline. In §3, we study the behavior of the syntomic complex under log blowing ups. In §4, we prove the main theorem in the case where $X \to S$ is smooth and M is defined by a relative divisor with normal crossings by using [Bl-Ka, Cor. 1.4.1] mentioned above, and then in §5, we reduce the general case to this special case using the result of §3. In the Appendix, we give an explicit description of $\mathcal{H}^q(\mathcal{S}_1(q)_{(X,M)})$ ($0 \leq q \leq p-2$) and some following results necessary in §4 in the case where (Y, M_Y) is of Cartier type over $(S, N) \otimes k$. This is a log version of [Ku] and we follow and generalize the method of [Ku]. In fact, this was already done in [T1] in the case where k is perfect and the generalization to the imperfect residue field case is straightforward. However, in [T1, §7], we treat some non-constant coefficients as well and it makes the proof very hard to read. So, for the convenience of the readers, I will give an outline here again in the constant coefficients case, generalizing to the imperfect residue field case.

Notation. — Throughout this paper, we fix a prime number p and we denote by the subscript n the reduction mod p^n of schemes, log schemes, *etc.* Except §1, we use the following notation: Let K, k, O_K and (S, N) be as above. We choose a uniformizer π of O_K. We denote by $\widehat{K}$ the completion of K with respect to the discrete valuation and by $O_{\widehat{K}}$ its ring of integers. By [EGA IV, Chap. 0, Thm. 19.8.6], there exists a discrete valuation subring W of $O_{\widehat{K}}$ which is absolutely unramified and over which $O_{\widehat{K}}$ is a finite totally ramified extension. Furthermore, such a subring W admits a lifting of Frobenius. We choose and fix such a subring W and a lifting of Frobenius σ. We denote by (V, M_V) the scheme $\operatorname{Spec}(W[\mathbb{N}]) = \operatorname{Spec}(W[T])$ endowed with the log structure associated to the inclusion $\mathbb{N} \to W[\mathbb{N}]$. For each integer $n \geq 1$, let i_{V_n} be the exact closed immersion $(S_n, N_n) \hookrightarrow (V_n, M_{V_n})$ over W_n defined by the morphism of monoids $\mathbb{N} \to \Gamma(S_n, N_n)$; $1 \mapsto \pi$, and let F_{V_n} be the lifting of Frobenius of (V_n, M_{V_n}) defined by σ on W and the multiplication by p on $\mathbb{N}$.

1. Logarithmic p-bases, differential modules with log poles and PD-envelopes

To define the syntomic complex $\mathcal{S}_n(r)_{(X,M)}$ for (X, M) as in the Introduction without assuming that the residue field k is perfect, we first extend the definition of p-bases in [Ka5, §1] to fine log schemes.

DEFINITION 1.1 (see [Ka5, Def. 1.1, 1.2]). — 1) We say that a morphism of schemes $f: X \to Y$ over $\mathbb{F}_p$ is *relatively perfect* if the following diagram is cartesian, where F denotes the absolute Frobenius:

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{F} & Y. \end{array}$$

2) Let n be a positive integer. We say that a morphism of schemes $f: X \to Y$ over $\mathbb{Z}/p^n\mathbb{Z}$ is *relatively perfect* if f is formally étale and its reduction mod p is relatively perfect.

For a morphism of schemes over $\mathbb{F}_p$, we have the following implications:

$$\text{étale} \Longrightarrow \text{relatively perfect} \Longrightarrow \text{formally étale}$$

(see [EGA IV, Chap. 0, Thm. 21.2.7]). So the above definition is consistent. Relatively perfect morphisms are stable under compositions and base changes.

Let us recall the definition of p-bases and a criterion for p-bases in [Ka5].

DEFINITION 1.2 (see [Ka5, Def. 1.3]). — Let n be a positive integer and let $f: X \to Y$ be a morphism of schemes over $\mathbb{Z}/p^n\mathbb{Z}$. Then we say that a set $(b_\lambda)_{\lambda \in \Lambda}$ of elements of $\Gamma(X, \mathcal{O}_X)$ is a *p-basis* of X over Y (or of f) if the morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[T_\lambda]_{\lambda \in \lambda})$ defined by $T_\lambda \mapsto b_\lambda$ is relatively perfect.

PROPOSITION 1.3 (see [Ka5, Prop. 1.4]). — *Let n be a positive integer, let X be a scheme over $\mathbb{Z}/p^n\mathbb{Z}$, and let $(b_\lambda)_{\lambda \in \Lambda}$ be a set of elements of $\Gamma(X, \mathcal{O}_X)$. Then the following two conditions are equivalent.*

(i) *$(b_\lambda)_{\lambda \in \Lambda}$ forms a p-basis of X over $\mathbb{Z}/p^n\mathbb{Z}$.*

(ii) *$(b_\lambda \bmod p)_{\lambda \in \Lambda}$ forms a p-basis of X_1 over $\mathbb{F}_p$ and X is flat over $\mathbb{Z}/p^n\mathbb{Z}$.*

We extend the above definition of p-bases to fine log schemes as follows:

DEFINITION 1.4. — Let n be a positive integer and let $f:(X,M) \to (Y,N)$ be a morphism of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$.

1) We say that a pair of a set $(b_\lambda)_{\lambda\in\Lambda}$ of elements of $\Gamma(X,\mathcal{O}_X)$ and a chart $(P_X \to M, Q_Y \to N, Q \to P)$ of f is a *p-basis of* (X,M) *over* (Y,N) (or *of* f) if the kernel and the torsion part of the cokernel of $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ are finite groups of orders prime to p, and the morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P][T_\lambda]_{\lambda\in\Lambda})$ defined by the chart and $T_\lambda \mapsto b_\lambda$ is relatively perfect.

2) We say that (X,M) *has p-bases over* (Y,N) *locally* (or f *has p-bases locally*) if (X,M) has p-bases over (Y,N) étale locally on X and on Y.

By the criterion [Ka2, Thm. 3.5] of log smooth morphisms, we obtain the following:

LEMMA 1.5. — *Let n be a positive integer and let $f:(X,M) \to (Y,N)$ be a morphism of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$. Then f has p-bases locally if and only if, étale locally on X, there exists a factorization*

$$(X,M) \xrightarrow{g} (X',M') \xrightarrow{h} (Y,N)$$

of f such that h is smooth, $g^*M' \cong M$ and $X \to X'$ has p-bases (Def. 1.2).

Especially smooth morphisms of fine log schemes have p-bases locally.

PROPOSITION 1.6. — *Let n be a positive integer.*

1) *Let $f:(X,M) \to (Y,N)$ and $g:(Y',N') \to (Y,N)$ be morphisms of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$ and let $f':(X',M') \to (Y',N')$ be the base change of f by g in the category of fine log schemes. If f has p-bases locally, then f' also has p-bases locally.*

2) *Let $f:(X,M) \to (Y,N)$, $g:(Y,N) \to (Z,L)$ be morphisms of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$. If f and g have p-bases locally, then $g \circ f$ also has p-bases locally.*

Proof. — (1) follows from Lemma 1.5. Let us prove 2). We may assume that g has a p-basis $(b_\lambda)_{\lambda\in\Lambda}$, $\{P_{2,Y} \to N, P_{1,Z} \to L, P_1 \to P_2\}$. Then, by Lemma 1.5 and [Ka2, Thm. 3.5], étale locally on X, f has a p-basis $(c_\mu)_{\mu\in M}$, $\{P_{3,X} \to M, P_{2,Y} \to N, P_2 \to P_3\}$ with the same chart $P_{2,Y} \to N$ of N. Now $(b_\lambda)_{\lambda\in\Lambda} \cup (c_\mu)_{\mu\in M}$ and $\{P_{3,X} \to M, P_{1,Z} \to L, P_1 \to P_2 \to P_3\}$ form a p-basis of the composite $g \circ f$. □

EXAMPLE 1.7. — 1) Let W be a complete discrete valuation ring such that p is a uniformizer of W. Let $(b_\lambda)_{\lambda\in\Lambda}$ be a p-basis of the residue field k over $\mathbb{F}_p$, that is, $\prod_{\lambda\in\Lambda} b_\lambda^{n(\lambda)}$ for functions $\lambda:\Lambda \mapsto \{0,1,\dots,p-1\}$ with finite supports, form a basis of k over k^p. (Such a p-basis always exists [EGA IV, Chap. 0, Thm. 21.4.2].) Then a lifting $(\tilde{b}_\lambda)_{\lambda\in\Lambda}$ of $(b_\lambda)_{\lambda\in\Lambda}$ forms a p-basis of W/p^nW over $\mathbb{Z}/p^n\mathbb{Z}$. This follows from Proposition 1.3.

2) Let W be as above and let (X,M) be a smooth fine log scheme over $\mathrm{Spec}(W/p^nW)$ with the trivial log structure. Then (X,M) has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally. Indeed, étale locally on X, there exists a chart $P_X \to M$ such that the induced morphism $X \to \mathrm{Spec}(W/p^nW[P])$ is étale, and this chart and the p-basis $(\tilde{b}_\lambda)_{\lambda\in\Lambda}$ in (1) obviously give a p-basis of(X,M) over $\mathbb{Z}/p^n\mathbb{Z}$.

We can generalize [Ka5, Lemma 1.8] on PD-envelopes to fine log schemes as follows:

PROPOSITION 1.8. — *Let n be a positive integer, let (S,L) be a fine log scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and let (I,γ) be a quasi-coherent PD-ideal of $\mathcal{O}_S$. Let $i:(X,M) \to (Y,N)$ and $i':(X,M) \to (Y',N')$ be two closed immersions of fine log schemes over (S,L) with the same source, and let $f:(Y',N') \to (Y,N)$ be an (S,L)-morphism with a p-basis $(b_\lambda)_{\lambda\in\Lambda}$, $(P'_{Y'} \to N', P_Y \to N, h: P \to P')$ such that $f\circ i' = i$. Let $(c_\mu)_{\mu\in M}$ be a finite set of elements of P' whose image in $((P')^{\mathrm{gp}}/h^{\mathrm{gp}}(P^{\mathrm{gp}}))\otimes\mathbb{F}_p$ forms a basis, and suppose that there exist $x_\lambda \in \Gamma(Y,\mathcal{O}_Y)$ and $y_\mu \in \Gamma(Y,N)$ whose images in $\Gamma(X,\mathcal{O}_X)$ and $\Gamma(X,M)$ coincide with the images of b_λ and c_μ. (Such sections always exist étale locally on Y.) Choose such x_λ and y_μ. Assume that γ extends to X. Let (D,M_D) and $(D',M_{D'})$ be the PD-envelopes compatible with γ (resp. the m-th infinitesimal neighbourhood* [Ka2, Rem. 5.8]) *of (X,M) in (Y,N) and in (Y',N') respectively. Let u_μ be the unique section of* $\mathrm{Ker}\,(\mathcal{O}^*_{D'} \to \mathcal{O}^*_X)$ *such that $c_\mu = f^*(y_\mu)\cdot u_\mu$ in $M_{D'}$. Then, we have an $\mathcal{O}_D$-PD-isomorphism (resp. $\mathcal{O}_D$-isomorphism)*

$$\mathcal{O}_D\langle T_\lambda,S_\mu\rangle_{\lambda\in\Lambda,\mu\in M} \xrightarrow{\sim} \mathcal{O}_{D'},\quad T_\lambda,S_\mu \longmapsto b_\lambda - f^*(x_\lambda),u_\mu - 1.$$

$$(\textit{resp. } \mathcal{O}_D[T_\lambda,S_\mu]_{\lambda\in\Lambda,\mu\in M}/(J_D\cdot\mathcal{O}[T_\lambda,S_\mu]_{\lambda\in\Lambda,\mu\in M} + (T_\lambda,S_\mu))^m \xrightarrow{\sim} \mathcal{O}_{D'}).$$

Proof. — In the case $f^*N \cong N'$, $\Lambda = \emptyset$, $h = \mathrm{id}$, we can easily reduce to the case i and i' are exact closed immersions and then prove the claim in the same way as in [Ka5, §1]. Hence we may assume that the morphism

$$Y' \longrightarrow Y \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}\big(\mathbb{Z}[P'][T_\lambda]_{\lambda\in\Lambda}\big)$$

defined by the chart and $T_\lambda \mapsto b_\lambda$ is an isomorphism. Let x be any point of X and let Q (resp. Q') be the inverse image of $M_{\bar{x}}$ under the morphism $P^{\mathrm{gp}} \to N_{\bar{x}}^{\mathrm{gp}} \to M_{\bar{x}}^{\mathrm{gp}}$ (resp. $(P')^{\mathrm{gp}} \to (N'_{\bar{x}})^{\mathrm{gp}} \to M_{\bar{x}}^{\mathrm{gp}}$). Then, there exists an étale neighbourhood $U \to X$ of x such that $Q \to M_{\bar{x}}$ and $Q' \to M_{\bar{x}}$ are extended to charts (β: $Q_U \to M_{|U}, P_Y \to N, P \subset Q$) and ($\beta'$: $Q'_U \to M_{|U}, P'_{Y'} \to N'$, $P' \subset Q'$) respectively such that the composite

$$Q_U \xrightarrow{h^{\mathrm{gp}}|_Q} Q'_U \xrightarrow{\beta'} M_{|U}$$

coincides with β. By shrinking U if necessary, we can choose an étale lifting $V \to Y$ of $U \to X$. Replacing (Y,N) and (Y',N') by $(V,N_{|V})$ and $(V',N'_{|V'})$

where $V' = Y' \times_Y V$, we may assume $U = X$. Furthermore, by replacing (Y,N) (resp. (Y',N')) by $Y \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$ (resp. $Y' \times_{\mathrm{Spec}(\mathbb{Z}[P'])} \mathrm{Spec}(\mathbb{Z}[Q']))$ endowed with the log structure defined by Q (resp. Q'), we may assume that i and i' are exact closed immersions and $f^*N \cong N'$. In a Zariski open neighbourhood of X in Y', we can choose sections $v_\mu \in \mathcal{O}^*_{Y'}$ such that

$$c_\mu = f^*(y_\mu) \cdot v_\mu \quad \text{in } N' \quad \text{and} \quad dv_\mu = v_\mu \cdot d\log c_\mu$$

form a basis of $\Omega^1_{Y'/Y[T_\lambda]_{\lambda\in\Lambda}}$. Hence the morphism

$$Y' \longrightarrow Y \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\big(\mathbb{Z}[T_\lambda, S_\mu]\big)$$

defined by $T_\lambda \mapsto b_\lambda$, $S_\mu \mapsto v_\mu - 1$ is étale. Thus we are further reduced to the case $P = P'$. Now, in the second case, the claim is trivial. In the first case, we can prove the isomorphism by verifying that the left hand side of the homomorphism in question satisfies the universal property required for the right hand side. □

COROLLARY 1.9. — *Under the same notation as Proposition* 1.8,

$$\Omega^1_{Y'/Y}\big(\log(N'/N)\big)$$

is a free $\mathcal{O}_{Y'}$-module with a basis $\{db_\lambda\}_{\lambda\in\Lambda} \cup \{d\log c_\mu\}_{\mu\in M}$.

Proof. — Let (D, M_D) be the first infinitesimal neighbourhood of the diagonal immersion $\Delta\colon (Y',N') \to (Y',N') \times_{(Y,N)} (Y',N')$. Then, for the ideal J_D of $\mathcal{O}_D$ defining Y', we have an isomorphism:

$$J_D \cong \Omega^1_{Y'/Y}\big(\log(N'/N)\big),$$
$$p_2^*(x) - p_1^*(x), p_2^*(y)p_1^*(y)^{-1} - 1 \longmapsto dx, d\log(y) \quad (x \in \mathcal{O}_{Y'}, y \in N').$$

(See [Ka2, Rem. 5.8].) Hence the claim follows from Proposition 1.8 with $i = \mathrm{id}_{(Y',N')}$, $i' = \Delta$ and $f = p_1$. □

COROLLARY 1.10 (*cf.* [Ber-O, 6.4]). — *Let n be a positive integer, let (S,L) be a fine log scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and let (I,γ) be a quasi-coherent PD-ideal of $\mathcal{O}_S$. Let (X,M) be a fine log scheme over (S,L) and let i be an (S,L)-closed immersion of (X,M) into a fine log scheme (Y,N) having p-bases over (S,L) locally. Assume that γ extends to X and let (D,M_D) be the PD-envelope of (X,M) in (Y,N) compatible with γ. Then there exists a canonical derivation $d\colon \mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}(\log(N/L))$ over S compatible with $d\colon \mathcal{O}_Y \to \Omega^1_{Y/S}(\log(N/L))$ and a canonical homomorphism $d\log : M_D^{\mathrm{gp}} \to \mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}(\log(N/L))$ compatible with $d\log : N^{\mathrm{gp}} \to \Omega^1_{Y/S}(\log(N/L))$ characterized by the following properties:*

1) *d is an integrable connection with log poles on $\mathcal{O}_D$ as an $\mathcal{O}_Y$-module.*

2) $d(x^{[m]}) = x^{[m-1]} \cdot dx$ *for* $x \in J_D$ *and* $m \geq 1$.

3) $d\log(u) = u^{-1}du$ *for* $u \in \mathcal{O}_D^*$.

Proof. — We only give a construction of the homomorphisms. Let $(D(1), M_{D(1)})$ be the PD-envelope of $(i,i)\colon (X,M) \to (Y,N) \times_{(S,L)} (Y,N)$ compatible with γ and let (P^1, M_{P^1}) be the first infinitesimal neighbourhood of the diagonal immersion $(Y,N) \to (Y,N) \times_{(S,L)} (Y,N)$. Let $J_{D(1)}$ be the ideal of $\mathcal{O}_{D(1)}$ defining the exact closed immersion $(D, M_D) \hookrightarrow (D(1), M_{D(1)})$ induced by the diagonal immersion $(Y,N) \hookrightarrow (Y,N) \times_{(S,L)} (Y,N)$. Then, by Proposition 1.8 and the isomorphism in the proof of Corollary 1.9, the canonical morphism $\operatorname{Spec}(\mathcal{O}_{D(1)}/J_{D(1)}^{[2]}) \to P^1$ induces an isomorphism

$$\mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}(\log(N/L)) \cong J_{D(1)}/J_{D(1)}^{[2]},$$

$$1 \otimes dx, 1 \otimes d\log(y) \longmapsto p_2^*(x) - p_1^*(x), p_2^*(y)p_1^*(y)^{-1} - 1 \quad (x \in \mathcal{O}_Y, y \in N).$$

We define $d\colon \mathcal{O}_D \to J_{D(1)}/J_{D(1)}^{[2]}$ by

$$d(x) = p_2^*(x) - p_1^*(x)$$

and $d\log : M_D^{\mathrm{gp}} \to J_{D(1)}/J_{D(1)}^{[2]}$ by

$$d\log(y) = p_2^*(y)p_1^*(y)^{-1} - 1. \quad \square$$

COROLLARY 1.11. — *Let n, (S,L) and (I,γ) be the same as Corollary 1.10 and consider a commutative diagram*

$$\begin{array}{ccc} (X',M') & \xrightarrow{i'} & (Y',N') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (X,M) & \xrightarrow{i} & (Y,N) \end{array}$$

of fine log schemes over (S,L) such that $X' \to X$ is étale, $f^*M \cong M'$, γ extends to X, i and i' are closed immersions, and (Y,N) and (Y',N') have p-bases over (S,L) locally. Let (D,M_D) and $(D',M_{D'})$ denote the PD-envelopes of i and i' compatible with γ. Then the natural homomorphism of complexes:

$$\big(J_D^{[r-\bullet]} \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}(\log(N/L))\big)_{|X'} \longrightarrow J_{D'}^{[r-\bullet]} \otimes_{\mathcal{O}_{Y'}} \Omega^\bullet_{Y'/S}(\log(N'/L))$$

is a quasi-isomorphism for any $r \geq 0$.

Proof. — By considering the graph $(g, \mathrm{id})\colon (Y', N') \to (Y, N) \times_{(S,L)} (Y', N')$ of the morphism g, we are reduced to the case g has p-bases locally. By the proof of Proposition 1.6, 2) and Corollary 1.9, we have an exact sequence

$$0 \to g^*\Omega^1_{Y/S}(\log(N/L)) \longrightarrow \Omega^1_{Y'/S}(\log(N'/L)) \longrightarrow \Omega^1_{Y'/Y}(\log(N'/N)) \to 0.$$

We may assume that g has a p-basis $(b_\lambda)_{\lambda\in\Lambda}$, $(P'_{Y'} \to N', P_Y \to N, h\colon P \to P')$. Choose a set $\{c_\mu\}_{\mu\in M}$ of elements of P' whose image in $((P')^{\mathrm{gp}}/h^{\mathrm{gp}}(P^{\mathrm{gp}})) \otimes \mathbb{F}_p$ forms a basis. By taking an étale lifting $Y'' \to Y$ of $X' \to X$ Zariski locally on X' and considering the fiber product of $(Y'', N_{|Y''})$ and (Y', N') over (Y, N), we are reduced to the case $X = X'$. Then we may further assume that there exist liftings $x_\lambda \in \Gamma(Y, \mathcal{O}_Y)$ and $y_\mu \in \Gamma(Y, N)$ of the images of b_λ and c_μ in $\Gamma(X, \mathcal{O}_X)$ and $\Gamma(X, M)$. Choose such x_λ and y_μ, and let u_μ be the element of $\Gamma(D', 1+J_{D'})$ such that $c_\mu = g^*(y_\mu)\cdot u_\mu$ in $\Gamma(D', M_{D'})$. Then, by Proposition 1.8 and Corollary 1.9, $\mathcal{O}_{D'}$ is a PD-polynomial ring over $\mathcal{O}_D$ with indeterminates $b_\lambda - g^*(x_\lambda)$, $u_\mu - 1$, and

$$d\big(b_\lambda - g^*(x_\lambda)\big) = db_\lambda, \quad d(u_\mu - 1) = u_\mu \cdot d\log c_\mu$$

form a basis of $\mathcal{O}_{D'} \otimes_{\mathcal{O}_Y} \Omega^1_{Y'/Y}(\log(N'/N))$. Set

$$\begin{aligned}
\{t_i\}_{i\in I} &:= \big\{b_\lambda - g^*(x_\lambda)\big\} \cup \{u_\mu - 1\},\\
F^rC^\bullet &:= J_{D'}^{[r-\bullet]} \otimes_{\mathcal{O}_{Y'}} \Omega^\bullet_{Y'/S}\big(\log(N'/L)\big),\\
C^{q,0} &:= \mathcal{O}_{D'} \otimes_{\mathcal{O}_{Y'}} g^*\Omega^q_{Y/S}\big(\log(N/L)\big),\\
F^rC^{0,q} &:= J_{D'}^{[r-q]} \otimes_{\mathcal{O}_{D'}} \bigwedge^q\Big(\bigoplus_{i\in I} \mathcal{O}_{D'}\cdot dt_i\Big) \quad (\subset F^rC^q),\\
F^rC^{q_1,q_2} &:= C^{q_1,0} \otimes_{\mathcal{O}_{D'}} F^{r-q_1}C^{0,q_2}.
\end{aligned}$$

Then we have

$$F^rC^q = \bigoplus_{q_1+q_2=q} F^rC^{q_1,q_2}, \quad d^q_C(F^rC^{q_1,q_2}) \subset F^rC^{q_1+1,q_2} \bigoplus F^rC^{q_1,q_2+1}.$$

We define $d_I^{q_1,q_2}$ and $d_{II}^{q_1,q_2}$ by the formula

$$d^q_C(\omega) = d_I^{q_1,q_2}(\omega) + d_{II}^{q_1,q_2}(\omega),$$

where $\omega \in F^rC^{q_1,q_2}$, $d_I^{q_1,q_2}(\omega) \in F^rC^{q_1+1,q_2}$ and $d_{II}^{q_1,q_2}(\omega) \in F^rC^{q_1,q_2+1}$. Then $(F^rC^{\bullet,\bullet}, d_I, d_{II})$ forms a double complex whose associated simple complex is

$F^rC^\bullet$ and the morphism in question factors through the complex $(F^rC^{\bullet,0}, d_I^{\bullet,0})$. Hence it suffices to prove that $\varepsilon\colon J_D^{[r-q]} \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}(\log(N/L)) \to F^rC^{q,\bullet}$ is a resolution. Since

$$F^rC^{q,\bullet} \cong \big(\mathcal{O}_D \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}(\log(N/L))\big) \otimes_{\mathcal{O}_D} F^{r-q}C^{0,\bullet},$$

we may assume $q = 0$. Give a linear order on the index set I, and define the homomorphisms $k^0\colon F^rC^{0,0} \to J_D^{[r]}$ and $k^q\colon F^rC^{0,q} \to F^rC^{0,q-1}$ $(q \geq 1)$ by

$$k^0\Big(a\prod_i t_i^{[\underline{n}(i)]}\Big) = \begin{cases} a & \text{if } \underline{n} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

($\underline{n}\colon I \to \mathbb{N}$ finite support, $a \in J_D^{[r-|\underline{n}|]}$), and

$$k^q\Big(a\prod_i t_i^{[\underline{n}(i)]} \cdot dt_{i_1} \wedge \ldots \wedge dt_{i_q}\Big) = \begin{cases} a\prod_i t_i^{[\underline{n}(i)+\delta_{ii_1}]} dt_{i_2} \wedge \ldots \wedge dt_{i_q} \\ \qquad\qquad \text{if } n(i) = 0 \text{ for all } i < i_1, \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

($\underline{n}\colon I \to \mathbb{N}$, finite support, $a \in J_D^{[r-q-|\underline{n}|]}$, $i_\nu \in I$, $i_1 < i_2 < \cdots < i_q$). Then $k^0\varepsilon = 1$, $\varepsilon k^0 + k^1 d = 1$ and $dk^q + k^{q+1}d = 1$ $(q \geq 1)$. This completes the proof. ▯

2. Syntomic complexes

In this section, we say that an inductive system of fine log schemes $\{(X_n, M_n)\}_{n\geq 1}$ over $\mathbb{Z}/p^n\mathbb{Z}$ is *adic* if the homomorphisms

$$(X_n, M_n) \longrightarrow (X_{n+1}, M_{n+1}) \otimes_{\mathbb{Z}/p^{n+1}\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$$

are isomorphisms for all $n \geq 1$. A morphism

$$u = \{u_n\}\colon \{(X_n, M_n)\} \longrightarrow \{(Y_n, N_n)\}$$

of adic inductive systems of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$ is a set of morphisms $u_n\colon (X_n, M_n) \to (Y_n, M_n)$ such that the reduction mod p^n of u_{n+1} coincides with u_n for each $n \geq 1$. We say that u is a closed immersion, smooth,... if each u_n is a closed immersion, smooth, ...

We will define the syntomic complexes

$$\mathcal{S}_n(r)_{(X,M)} \in D^+(X_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \quad (r \leq p-1)$$

for a fine log scheme (X, M) over $\mathbb{Z}_{(p)}$ having the following property (*cf.* [Ka5, §2]): Étale locally on X, there exists a closed immersion of $\{(X_n, M_n)\}$ into an adic inductive system of fine log schemes $\{(Z_n, M_{Z_n})\}$ over $\mathbb{Z}/p^n\mathbb{Z}$ such that (Z_n, M_{Z_n}) has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally for every $n \geq 1$, which satisfies the following condition:

2.1. — If we denote by (D_n, M_{D_n}) the PD-envelope of (X_n, M_n) in (Z_n, M_{Z_n}) compatible with the canonical PD-structure on $p\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 1$, then $J_{D_n}^{[i]}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $J_{D_{n+1}}^{[i]} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J_{D_n}^{[i]}$ for $n \geq 1$, $i \geq 0$.

LEMMA 2.2 (*cf.* [Ka5, Lemma 2.1]). — *Let (X,M) be a fine log scheme over $\mathbb{Z}_{(p)}$ and let $\{(X_n,M_n)\} \to \{(Z_n,M_{Z_n})\}$ be an exact closed immersion of adic inductive systems of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$ such that (Z_n,M_{Z_n}) has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally for every $n \geq 1$. If*

- *Z_n is locally noetherian for $n \geq 1$;*
- *X is flat over $\mathbb{Z}_{(p)}$;*
- *and for any $x \in X_1$, $n \geq 1$, $\operatorname{Ker}(\mathcal{O}_{Z_n,x} \to \mathcal{O}_{X_n,x})$ is generated by an $\mathcal{O}_{Z_n,x}$-regular sequence;*

then the condition 2.1 *is satisfied.*

Proof. — The same as the proof of [Ka1, I, Lemma 1.3, 2]. □

LEMMA 2.3 (*cf.* [Ka5, Lemma 2.2]). — *Let (X,M) be a fine log scheme over $\mathbb{Z}_{(p)}$, and let*

$$\{i_n\}:\{(X_n,M_n)\} \longrightarrow \{(Z_n,M_{Z_n})\}, \quad \{i'_n\}:\{(X_n,M_n)\} \longrightarrow \{(Z'_n,M_{Z'_n})\}$$

be closed immersions of adic inductive systems of fine log schemes over $\mathbb{Z}/p^n\mathbb{Z}$ such that (Z_n,M_{Z_n}) and $(Z'_n,M_{Z'_n})$ have p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally for every $n \geq 1$. Then $\{i_n\}$ satisfies the condition 2.1 *if and only if $\{i'_n\}$ satisfies it.*

Proof. — The same as [Ka5, Lemma 2.2] using Proposition 1.8 instead of [Ka5, Lemma 1.8]. □

Now let us define the syntomic complex. We follow faithfully the construction in [Ka3, §5]. Let (X, M) be a fine log scheme over $\mathbb{Z}_{(p)}$ having the property in the beginning of this section. Choose a hypercovering $X^\bullet \to X$ in the étale topology and a closed immersion of $\{(X_n^\bullet, M_n^\bullet)\}$ into an adic inductive system of simplicial fine log schemes $\{(Z_n^\bullet, M_{Z_n^\bullet})\}$ over $\mathbb{Z}/p^n\mathbb{Z}$ with a lifting of Frobenius

$$\{F_{Z_n^\bullet}\}:\{(Z_n^\bullet, M_{Z_n^\bullet})\} \longrightarrow \{(Z_n^\bullet, M_{Z_n^\bullet})\}$$

such that $(Z_n^\nu, M_{Z_n^\nu})$ has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally for any $n \geq 1$ and $\nu \geq 0$. Here $M^\bullet$ denotes the inverse image of M on $X^\bullet$. Let $(D_n^\nu, M_{D_n^\nu})$ be the PD-envelope of (X_n^ν, M_n^ν) in $(Z_n^\nu, M_{Z_n^\nu})$ compatible with the canonical PD-structure on $p\mathbb{Z}/p^n\mathbb{Z}$ and let $J_{D_n^\nu}$ be the PD-ideal of $\mathcal{O}_{D_n^\nu}$ defining X_n^ν. Since $(D_n^\nu, M_{D_n^\nu})$ is also the PD-envelope of (X_1^ν, M_1^ν) in $(Z_n^\nu, M_{Z_n^\nu})$, $F_{Z_n^\nu}$ induces a lifting of Frobenius $F_{D_n^\nu}$ of $(D_n^\nu, M_{D_n^\nu})$. Since the closed immersion $X_1^\nu \to X_n^\nu$ (resp. $X_1^\nu \to D_n^\nu$) is a nilimmersion, we regard sheaves on $(X_n^\nu)_{\text{ét}}$ (resp. $(D_n^\nu)_{\text{ét}}$) as sheaves on $(X_1^\nu)_{\text{ét}}$. Let

$$\varphi\colon \mathcal{O}_{D_n^\nu} \cong F_{X_1^\nu}^{-1}(\mathcal{O}_{D_n^\nu}) \longrightarrow \mathcal{O}_{D_n^\nu}$$

denote the homomorphism induced by $F_{D_n^\nu}$. Then we have $\varphi(J_{D_n^\nu}^{[r]}) \subset p^r\mathcal{O}_{D_n^\nu}$ $(0 \leq r \leq p-1)$ (see [Ka1, I, Lemma 1.3, (1)]). On the other hand, by the assumption on (X, M) and Lemma 2.3, $J_{D_n^\nu}^{[r]}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $J_{D_{n+1}^\nu}^{[r]} \otimes \mathbb{Z}/p^n\mathbb{Z} \cong J_{D_n^\nu}^{[r]}$ for every $n \geq 1$ and $r \geq 0$. Hence, for $0 \leq r \leq p-1$, there exists a unique homomorphism $\varphi_r\colon J_{D_n^\nu}^{[r]} \to \mathcal{O}_{D_n^\nu}$ which makes the following diagram commute:

$$\begin{array}{ccc} J_{D_{n+r}^\nu}^{[r]} & \xrightarrow{\varphi} & \mathcal{O}_{D_{n+r}^\nu} \\ \downarrow & & \downarrow{\scriptstyle p^r} \\ J_{D_n^\nu}^{[r]} & \xrightarrow{\varphi_r} & \mathcal{O}_{D_n^\nu}. \end{array}$$

For an integer $r < 0$, we set

$$J_{D_n^\nu}^{[r]} := \mathcal{O}_{D_n^\nu}, \quad \varphi_r = p^{-r}\varphi.$$

Set

$$\omega_{Z_n^\nu}^q := \Omega_{Z_n^\nu}^q\big(\log(M_{Z_n^\nu})\big)$$

to simplify the notation. Let φ also denote the homomorphism

$$\mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q \longrightarrow \mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q$$

induced by $F_{Z_n^\nu}$ and $F_{D_n^\nu}$. Then $\varphi(\mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q) \subset p^q\mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q$. By Corollary 1.9, $\mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$, and hence we can define

$$\varphi_q\colon \mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q \longrightarrow \mathcal{O}_{D_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega_{Z_n^\nu}^q$$

in the same way as φ_r for $J_{D_n^\nu}^{[r]}$ above. Now, for an integer $r \leq p-1$, we have a morphism of complexes

$$\varphi_r\colon J_{D_n^\bullet}^{[r-\bullet]} \otimes_{\mathcal{O}_{Z_n^\bullet}} \omega_{Z_n^\bullet}^\bullet \longrightarrow \mathcal{O}_{D_n^\bullet} \otimes_{\mathcal{O}_{Z_n^\bullet}} \omega_{Z_n^\bullet}^\bullet$$

on the étale site $(X_1^\bullet)_{\text{ét}}$ of the simplicial scheme $X_1^\bullet$ whose degree q-part is $\varphi_{r-q} \otimes \varphi_q$. We define $\mathcal{S}_n(r)_{(X^\bullet, M^\bullet), (Z^\bullet, M_{Z^\bullet})}$ to be the mapping fiber of

$$1 - \varphi_r\colon J_{D_n^\bullet}^{[r-\bullet]} \otimes_{\mathcal{O}_{Z_n^\bullet}} \omega_{Z_n^\bullet}^\bullet \longrightarrow \mathcal{O}_{D_n^\bullet} \otimes_{\mathcal{O}_{Z_n^\bullet}} \omega_{Z_n^\bullet}^\bullet.$$

LEMMA 2.4 (*cf.* [Ka1, I, Thm. 3.6]). — $\mathcal{H}^q(\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}) = 0$ *for any integers* $q > r$.

Proof. — By the flatness of $J^{[s]}_{D^\nu_n} \otimes_{\mathcal{O}_{Z^\nu_n}} \omega^q_{Z^\nu_n}$, we are reduced to the case $n = 1$. If $n = 1$, φ_r is 0 in degree $> r$. Hence it remains to prove that

$$1 - \varphi_r : Z^r(\mathcal{O}_{D^\nu_1} \otimes \omega^\bullet_{Z^\nu_1}) \longrightarrow \mathcal{H}^r(\mathcal{O}_{D^\nu_1} \otimes \omega^\bullet_{Z^\nu_1})$$

is surjective when $r \geq 0$. Since $\varphi_r(\mathcal{O}_{D^\nu_1} \otimes \omega^r_{Z^\nu_1}) \subset Z^r(\mathcal{O}_{D^\nu_1} \otimes \omega^\bullet_{Z^\nu_1})$, it suffices to prove that

$$1 - \varphi_r : \mathcal{O}_{D^\nu_1} \otimes \omega^r_{Z^\nu_1} \longrightarrow (\mathcal{O}_{D^\nu_1} \otimes \omega^r_{Z^\nu_1})/B^r(\mathcal{O}_{D^\nu_1} \otimes \omega^\bullet_{Z^\nu_1})$$

is surjective. Since for $a_i \in M_{Z^\nu_1}$,

$$\varphi_r(d\log a_1 \wedge \ldots \wedge d\log a_r) \equiv d\log a_1 \wedge \ldots \wedge d\log a_r \bmod B^r(\mathcal{O}_{D^\nu_1} \otimes \omega^\bullet_{Z^\nu_1})$$

(*cf.* [T1, Lemma 7.1.4]), this follows from the surjectivity of $1 - \varphi : \mathcal{O}_{D^\nu_1} \to \mathcal{O}_{D^\nu_1}$. Note that we work on the étale site $(X^\nu_1)_{\text{ét}} \cong (D^\nu_1)_{\text{ét}}$. ☐

For any non-decreasing map $s : \{0,1,\ldots,\nu\} \to \{0,1,\ldots,\mu\}$ the canonical morphism $\underline{s}^{-1}\mathcal{S}_n(r)_{(X^\nu,M^\nu),(Z^\nu,M_{Z^\nu})} \to \mathcal{S}_n(r)_{(X^\mu,M^\mu),(Z^\mu,M_{Z^\mu})}$ is a quasi-isomorphism by Corollary 1.11, where $\underline{s}$ denotes the morphism $X^\mu_1 \to X^\nu_1$ corresponding to s. Hence $\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}$ is contained in the essential image of the fully faithful functor

$$L^+\theta^* : D^+(X_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow D^+(X^\bullet_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}),$$

where θ denotes the canonical morphism of topoi $(X^\bullet_1)^\sim_{\text{ét}} \to X_{1,\text{ét}}$ (see [SD, Thm. 2.4.12, Cor. 3.3.5, Prop. 4.3.3]). We define the syntomic complex $\mathcal{S}_n(r)_{(X,M)}$ to be the corresponding object

$$R\theta_*\big(\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}\big)$$

of $D^+(X^\bullet_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$.

If we choose another $X'^\bullet$, $\{(X'^\bullet_n, M'^\bullet_n)\} \to \{(Z'^\bullet_n, M_{Z'^\bullet_n})\}$ and $\{F_{Z'^\bullet_n}\}$, then by taking the fiber products

$$X''^\bullet := X^\bullet \times_X X'^\bullet, \quad (Z''^\bullet_n, M_{Z''^\bullet_n}) := (Z^\bullet_n, M_{Z^\bullet_n}) \times_{\mathbb{Z}/p^n\mathbb{Z}} (Z'^\bullet_n, M_{Z'^\bullet_n}),$$
$$F_{Z''^\bullet_n} := F_{Z^\bullet_n} \times F_{Z'^\bullet_n}$$

and using Corollary 1.11, we obtain canonical quasi-isomorphisms

$$\mathrm{pr}^{-1}\,\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \longrightarrow \mathcal{S}_n(r)_{(X''^\bullet,M''^\bullet),(Z''^\bullet,M_{Z''^\bullet})},$$
$$(\mathrm{pr}')^{-1}\mathcal{S}_n(r)_{(X'^\bullet,M'^\bullet),(Z'^\bullet,M_{Z'^\bullet})} \longrightarrow \mathcal{S}_n(r)_{(X''^\bullet,M''^\bullet),(Z''^\bullet,M_{Z''^\bullet})}$$

and hence a canonical quasi-isomorphism

$$R\theta_*\big(\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}\big) \xrightarrow{\sim} R\theta''_*\big(\mathcal{S}_n(r)_{(X''^\bullet,M''^\bullet),(Z''^\bullet,M_{Z''^\bullet})}\big)$$
$$\xleftarrow{\sim} R\theta'_*\big(\mathcal{S}_n(r)_{(X'^\bullet,M'^\bullet),(Z'^\bullet,M_{Z'^\bullet})}\big),$$

where pr, pr$'$, θ' and θ'' denote the canonical morphisms of topoi

$$(X_1''^\bullet)^\sim_{\text{ét}} \longrightarrow (X_1^\bullet)^\sim_{\text{ét}}, \quad (X_1''^\bullet)^\sim_{\text{ét}} \longrightarrow (X_1'^\bullet)^\sim_{\text{ét}},$$
$$(X_1'^\bullet)^\sim_{\text{ét}} \longrightarrow (X_1')^\sim_{\text{ét}}, \quad (X_1''^\bullet)^\sim_{\text{ét}} \longrightarrow (X_1'')^\sim_{\text{ét}}.$$

This quasi-isomorphism satisfies the transitivity, and hence $\mathcal{S}_n(r)_{(X,M)}$ is independent of the choice of $X^\bullet$, $\{(X_n^\bullet, M_n^\bullet)\} \to \{(Z_n^\bullet, M_{Z_n^\bullet})\}$ and $\{F_{Z_n^\bullet}\}$ up to canonical isomorphisms.

For another fine log scheme (X', M') over $\mathbb{Z}_{(p)}$ having the property in the beginning of this section and a morphism $f\colon (X', M') \to (X, M)$, we can construct $X'^\bullet \to X'$, $\{(X_n'^\bullet, M_n'^\bullet)\} \to \{(Z_n'^\bullet, M_{Z_n'^\bullet})\}$, $\{F_{Z_n'^\bullet}\}$, a morphism $g^\bullet\colon (X^\bullet, M^\bullet) \to (X'^\bullet, M'^\bullet)$ covering f, and a morphism $\{(Z_n^\bullet, M_{Z_n^\bullet})\} \to \{(Z_n'^\bullet, M_{Z_n'^\bullet})\}$ compatible with the closed immersions and the liftings of Frobenii. Then, we have a natural morphism

$$(g_1^\bullet)^{-1}\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \longrightarrow \mathcal{S}_n(r)_{(X'^\bullet,M'^\bullet),(Z'^\bullet,M_{Z'^\bullet})}$$

and hence

$$Lf_1^{-1}\mathcal{S}_n(r)_{(X,M)} \longrightarrow \mathcal{S}_n(r)_{(X',M')}.$$

We can verify that this morphism is independent of all choices.

We can define a product

$$\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \otimes \mathcal{S}_n(r')_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \longrightarrow \mathcal{S}_n(r+r')_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}$$

for any $0 \le r, r', r+r' \le p-1$ and a symbol map

$$(M_{n+1}^\bullet)^{\text{gp}} \xleftarrow{\sim} [1 + J_{D_{n+1}^\bullet} \to M_{D_{n+1}^\bullet}^{\text{gp}}] \longrightarrow \mathcal{S}_n(1)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}[1]$$

in the same way as [T2, §2.2] (*cf.* [Ka1, I, §§2–3]). We can also define a homomorphism

$$\mu_{p^n}(\mathcal{O}_{X_n^\bullet}) \longrightarrow \mathcal{H}^0\big(\mathcal{S}_n(1)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}\big)$$

by $\varepsilon \mapsto \log(\tilde{\varepsilon}^{p^n})$, where $\tilde{\varepsilon}$ denotes a lifting of ε in $\mathcal{O}_{D_n^\bullet}$. By taking $R\theta_*$, we obtain

$$\text{(2.5)}\quad \mathcal{S}_n(r)_{(X,M)} \otimes^{\mathbb{L}} \mathcal{S}_n(r')_{(X,M)} \longrightarrow \mathcal{S}_n(r+r')_{(X,M)} \qquad (0 \leq r, r', r+r' \leq p-1),$$

$$\text{(2.6)}\qquad M_{n+1}^{\mathrm{gp}} \longrightarrow \mathcal{S}_n(1)_{(X,M)}[1],$$

$$\text{(2.7)}\qquad \mu_{p^n}(\mathcal{O}_{X_n}) \longrightarrow \mathcal{S}_n(1)_{(X,M)}.$$

For (2.5), note $\mathcal{S}_n(r)_{(X,M)} \in D^b(X_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ by Lemma 2.4 and $L\theta^*$ is compatible with $\otimes^{\mathbb{L}}$.

These structures are independent of the choice of $X^\bullet$, $\{(X_n^\bullet, M_n^\bullet)\} \to \{(Z_n^\bullet, M_{Z_n^\bullet})\}$, $\{F_{Z_n^\bullet}\}$ and functorial on (X, M).

If there exist a closed immersion $\{(X_n, M_n)\} \to \{(Z_n, M_{Z_n})\}$ and $\{F_{Z_n}\}$ globally, then the syntomic complex $\mathcal{S}_n(r)_{(X,M),(Z,M_Z)}$ $(r \leq p-1)$ is defined in the same way as $\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}$ above and it is canonically isomorphic to $\mathcal{S}_n(r)_{(X,M)}$ in $D^+(X_{1,\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$. The isomorphisms are also compatible with the product structures, the symbol maps and the homomorphisms from $\mu_{p^n}(\mathcal{O}_{X_n})$.

Now let us return to the situation in the Introduction, and let K, k, O_K, (S,N), (X,M), X_{triv}, (Y, M_Y), i, j and $\mathcal{E}_n(r)_{(X,M)}$ be as in the Introduction. Then (X,M) have the property in the beginning of this section. Indeed, for W as in the Notation, (S,N) and hence (X,M) are syntomic over $\operatorname{Spec}(W)$ with the trivial log structure (see [Ka3, 2.5]). Therefore, by Lemma 2.2, the condition 2.1 is satisfied for any closed immersion of $\{(X_n, M_n)\}$ into an adic inductive system of smooth fine log schemes $\{(Z_n, M_{Z_n})\}$ over $\operatorname{Spec}(W/p^nW)$. Note that (Z_n, M_{Z_n}) has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally (Example 1.7). We will construct a canonical morphism in $D^+(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$:

$$\text{(2.8)}\qquad \mathcal{S}_n(r)_{(X,M)} \longrightarrow \mathcal{E}_n(r)_{(X,M)}$$

for $0 \leq r \leq p-2$ compatible with the product structures and functorial on X. Since the construction is essentially the same as [T2, §3.1], we only give an outline referring the readers to [T2] for details.

First, in [T2, §1.5], we did not use the fact that the residue field k is perfect. Hence, for an affine étale X-scheme $U = \operatorname{Spec}(A)$ with the log structure $M_U = M|_U$ satisfying [T2] Condition 1.5.2, if we choose an algebraic closure $\overline{\operatorname{Frac}(A^h)}$ of the field of fractions $\operatorname{Frac}(A^h)$ of the p-adic henselization A^h of A, we obtain an (U, M_U)-fine log scheme $(\overline{U}, M_{\overline{U}})$ with an action of $\operatorname{Gal}(\overline{A^h}/A^h)$, a Galois equivariant PD-thickening of $(\overline{U}, M_{\overline{U}})$ into a fine log scheme $(\overline{D}, M_{\overline{D}})$ over $\mathbb{Z}_p$ with an action of $\operatorname{Gal}(\overline{A^h}/A^h)$, where $\overline{A^h}$ is the integral closure of A^h in the maximal unramified extension of

$$U_{\text{triv}}^h = \operatorname{Spec}(A_{\text{triv}}^h) := X_{\text{triv}} \times_X \operatorname{Spec}(A^h).$$

$(\overline{D}, M_{\overline{D}})$ also has a canonical lifting of Frobenius $F_{\overline{D}}$. We have

$$A_{\mathrm{crys}}(\overline{A^h}) = \Gamma(\overline{D}, \mathcal{O}_{\overline{D}}), \quad \mathrm{Fil}^r A_{\mathrm{crys}}(\overline{A^h}) = \Gamma\big(\overline{D}, J_{\overline{D}}^{[r]}\big)^{\wedge}$$

for $r \geq 0$ and an exact sequence

$$(2.9) \qquad 0 \to \mathbb{Z}_p(r) \longrightarrow \mathrm{Fil}^r A_{\mathrm{crys}}(\overline{A^h}) \xrightarrow{1-p^{-r}\varphi} A_{\mathrm{crys}}(\overline{A^h}) \to 0$$

for $0 \leq r \leq p-2$. Here $J_{\overline{D}}$ denotes the PD-ideal of $\mathcal{O}_{\overline{D}}$ defining $\overline{U}$ and $\hat{}$ denotes the p-adic completion.

Choose $X^\bullet \to X$, $\{(X_n^\bullet, M_n^\bullet)\} \to \{(Z_n^\bullet, M_{Z_n^\bullet})\}$ and $\{F_{Z_n^\bullet}\}$ as in the definition of $\mathcal{S}_n(r)_{(X,M)}$. For $\nu \geq 0$ and an étale scheme $U \to X^\nu$ satisfying [T2, Cond. 1.5.2], let $(\overline{E}_n^\nu, M_{\overline{E}_n^\nu})$ be the PD-envelope of $(\overline{U}_n, M_{\overline{U}_n})$ in

$$(\overline{Z}_n^\nu, M_{\overline{Z}_n^\nu}) := (\overline{D}_n, M_{\overline{D}_n}) \times_{\mathbb{Z}/p^n\mathbb{Z}} (Z_n^\nu, M_{Z_n^\nu})$$

compatible with the PD-structure on $J_{\overline{D}_n} + p\mathcal{O}_{\overline{D}_n}$. (Note that we used the fiber product over W_n and not over $\mathbb{Z}/p^n\mathbb{Z}$ in [T2, §3.1].) Then the liftings of Frobenii $F_{\overline{D}_n}$ and $F_{Z_n^\nu}$ induce a lifting of Frobenius $F_{\overline{E}_n^\nu}$ on $(\overline{E}_n^\nu, M_{\overline{E}_n^\nu})$ and the action of $\mathrm{Gal}(\overline{A^h}/A^h)$ on $(\overline{D}_n, M_{\overline{D}_n})$ induces an action of $\mathrm{Gal}(\overline{A^h}/A^h)$ on $(\overline{E}_n^\nu, M_{\overline{E}_n^\nu})$. Let $J_{\overline{E}_n^\nu}$ denote the PD-ideal of $\mathcal{O}_{\overline{E}_n^\nu}$ defining $\overline{U}_n$. By Corollary 1.11, we have a resolution

$$J_{\overline{D}_n}^{[r]} \longrightarrow J_{\overline{E}_n^\nu}^{[r-\bullet]} \otimes_{\mathcal{O}_{\overline{Z}_n^\nu}} \Omega^\bullet_{\overline{Z}_n^\nu/\overline{D}_n}\big(\log(M_{\overline{Z}_n^\nu}/M_{\overline{D}_n})\big) \cong J_{\overline{E}_n^\nu}^{[r-\bullet]} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega^\bullet_{Z_n^\nu}.$$

For an integer $0 \leq r \leq p-1$, we define the complex $\overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})}$ with an action of $\mathrm{Gal}(\overline{A^h}/A^h)$ to be the mapping fiber of

$$1 - \varphi_r : \Gamma(\overline{E}_n^\nu, J_{\overline{E}_n^\nu}^{[r-\bullet]} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega^\bullet_{Z_n^\nu}) \longrightarrow \Gamma(\overline{E}_n^\nu, \mathcal{O}_{\overline{E}_n^\nu} \otimes_{\mathcal{O}_{Z_n^\nu}} \omega^\bullet_{Z_n^\nu}),$$

where φ_r is defined in the same way as the definition of $\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}$. We regard $\overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})}$ as a complex of sheaves on $(U^h_{\mathrm{triv}})_{\text{ét}}$. Then, by (2.9), for an integer $0 \leq r \leq p-2$, we have a canonical resolution

$$\mathbb{Z}/p^n\mathbb{Z}(r) \longrightarrow \overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})}$$

on $(U^h_{\mathrm{triv}})_{\text{ét}}$. On the other hand, there exists a natural homomorphism

$$\Gamma\big(U, i^\nu_* \mathcal{S}_n(r)_{(X^\nu,M^\nu),(Z^\nu,M_{Z^\nu})}\big) \longrightarrow \Gamma\big(U^h_{\mathrm{triv}}, \overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})}\big),$$

where $i^\bullet$ denotes the morphism $X^\bullet \otimes_{O_K} k \to X^\bullet$. We can define a product structure on the complex $\overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})}$ in the same way as that on $\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})}$ and the above two morphisms are compatible with the product structures.

Choose a sufficiently large algebraically closed field Ω of characteristic 0 and denote by C^* the Godement resolution with respect to all Ω-rational points. We define $F_n(r)_{(X^\nu,M^\nu),(Z^\nu,M_{Z^\nu})}$ (resp. $G_n(r)_{(X^\nu,M^\nu),(Z^\nu,M_{Z^\nu})}$) to be the complex of sheaves on $X^\nu_{\text{ét}}$ associated to the complexes of presheaves

$$U \longmapsto \begin{cases} \Gamma\big(U^h_{\text{triv}}, \operatorname{tot} C^*(\overline{\mathcal{S}}_n(r)_{(U,M_U),(Z^\nu,M_{Z^\nu})})\big) & \text{if } U_1 \neq \emptyset, \\ \qquad \big(\text{resp. } \Gamma(U^h_{\text{triv}}, C^*(\mathbb{Z}/p^n\mathbb{Z}(r)_{X_{\text{triv}}})_{|U^h_{\text{triv}}})\big) & \\ 0 & \text{if } U_1 = \emptyset, \end{cases}$$

where U ranges over all affine étale X^ν-schemes satisfying [T2, Cond. 1.5.2] and all étale X^ν_K-schemes. Then, as in [T2, §3.1], we have the following morphisms of complexes of sheaves on $(X^\bullet_{\text{ét}})^\sim$:

$$\begin{aligned} i^\bullet_*\mathcal{S}_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} &\longrightarrow F_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \\ &\xleftarrow{(1)} G_n(r)_{(X^\bullet,M^\bullet),(Z^\bullet,M_{Z^\bullet})} \\ &\longrightarrow \theta^* i_* i^* j_* C^*\big(\mathbb{Z}/p^n\mathbb{Z}(r)_{X_{\text{triv}}}\big) \end{aligned}$$

for $0 \leq r \leq p-2$, where θ denotes the canonical morphism of topoi $(X^\bullet_{\text{ét}})^\sim \to (X_{\text{ét}})^\sim$ and (1) is a quasi-isomorphism. Taking $R\theta_*$, we obtain the required morphism (2.8). This is independent of the choice of $X^\bullet$, $\{(X^\bullet_n, M^\bullet_n)\} \to \{(Z^\bullet_n, M_{Z^\bullet_n})\}$, $\{F_{Z^\bullet_n}\}$, and functorial on (X,M) and (S,N).

We can also prove the following compatibility with the symbol maps in the same way as in [T2, §3.2]. We have $M^{\text{gp}} = j_*\mathcal{O}^*_{X_{\text{triv}}}$ and, from the Kummer sequence, we obtain a morphism

$$i^*M^{\text{gp}} = i^*j_*\mathcal{O}^*_{X_{\text{triv}}} \longrightarrow \mathcal{E}_n(1)_{(X,M)}[1]. \tag{2.10}$$

PROPOSITION 2.11. — *If $p \geq 3$, then the following diagram is commutative*

$$\begin{array}{ccc} M^{\text{gp}}_{n+1} & \xrightarrow{(2.6)} & \mathcal{S}_n(1)_{(X,M)}[1] \\ \uparrow & & \downarrow (2.8) \\ i^*M^{\text{gp}} & \xrightarrow{(2.10)} & \mathcal{E}_n(1)_{(X,M)}[1] \end{array}$$

for any integer $n \geq 1$.

By the same argument as [T2, Lemma 4.9.1], we obtain the following lemma:

LEMMA 2.12. — *The following diagram is commutative*:

$$\begin{array}{ccc} \mu_{p^n}(\mathcal{O}_{X_n}) & \xrightarrow{(2.7)} & \mathcal{H}^0(\mathcal{S}_n(1)_{(X,M)}) \\ & \searrow & \downarrow{\scriptstyle (2.8)} \\ & & i^*j_*\mathbb{Z}/p^n\mathbb{Z}(1). \end{array}$$

Finally let us consider the base change to an algebraic closure of K. Assume that (X, M) is universally saturated over (S, N), which is equivalent to assuming that the special fiber Y is reduced (see [T3]). Let $\overline{K}$, $O_{\overline{K}}$, $\overline{X}$, $\bar{\imath}$ and $\bar{\jmath}$ be as in the Introduction. Then, by the same method as in the end of [T2, §2.1] (see also [Ka1, I, Rem. 1.7]), we can define the syntomic complex $\mathcal{S}_n(r)_{(\overline{X},\overline{M})}$ $(r \leq p-1, n \geq 1)$ of the base change of (X, M) to $O_{\overline{K}}$ as the "inductive limit" of the syntomic complexes for all finite base changes of (X, M). By the same method as in [T2, §3.1], we can define a morphism

$$\mathcal{S}_n(r)_{(\overline{X},\overline{M})} \longrightarrow \bar{\imath}^* R\bar{\jmath}_*\mathbb{Z}/p^n\mathbb{Z}(r) \tag{2.13}$$

for $0 \leq r \leq p-2$, $n \geq 1$ as the "inductive limit" of the morphisms (2.8) for all finite base changes of (X, M).

3. Log blowing ups

In this section, we study the behavior of the syntomic complex under log blowing ups.

For a ring R and a monoid P, we call the log structure on $\mathrm{Spec}(R[P])$ associated to the inclusion $P \to R[P]$ the canonical log structure and we often denote it by canlog.

Let P be a torsion free finitely generated saturated monoid and let (Z, M_Z) denote the scheme $\mathrm{Spec}(W[P])$ endowed with the canonical log structure. Then, we can associate to each proper subdivision $f: F \to \mathrm{Spec}(P)$ (*cf.* [Ka4, Def. 9.7]), a log étale morphism of fs log schemes $f: (Z', M_{Z'}) \to (Z, M_Z)$ (*cf.* [Ka4, Prop. 9.9]) which satisfies the following properties:

The underlying morphism of schemes of f is proper and surjective (*cf.* [Ka4, Prop. 9.11]). Let Z_{triv} denote the open affine subscheme $\mathrm{Spec}(W[P^{\mathrm{gp}}])$ of Z. Note that Z_{triv} is the maximal open subset of Z on which the log structure M_Z is trivial. Then, $Z'_{\mathrm{triv}} := f^{-1}(Z_{\mathrm{triv}})$ is the maximal open subset of Z' on which the log structure $M_{Z'}$ is trivial, and the morphism f induces an isomorphism $Z'_{\mathrm{triv}} \xrightarrow{\sim} Z_{\mathrm{triv}}$. Finally, the canonical morphism

$$\mathcal{O}_{Z_1} \longrightarrow Rf_{1*}\mathcal{O}_{Z'_1} \tag{3.1}$$

is an isomorphism (see [Ka4, Thm. 11.3], [KKMS, Chap. I, §3, Cor. 1]).

For each open affine subfan $\mathrm{Spec}(Q) \subset F$, we have the corresponding open fs log subscheme $(\mathrm{Spec}(W[Q]), \mathrm{canlog}) \subset (Z', M_{Z'})$. Indeed, $(Z', M_{Z'})$ is constructed by gluing these $(\mathrm{Spec}(W[Q]), \mathrm{canlog})$'s. For a monoid Q, we call the morphism

$$F\colon \big(\mathrm{Spec}(W[Q]), \mathrm{canlog}\big) \longrightarrow \big(\mathrm{Spec}(W[Q]), \mathrm{canlog}\big)$$

induced by $\sigma\colon W \to W$ and the multiplication by p on Q, the canonical lifting of Frobenius of $(\mathrm{Spec}(W[Q]), \mathrm{canlog})$. By gluing the canonical lifting of Frobenius of $(\mathrm{Spec}(W[Q]), \mathrm{canlog})$ for each open affine subfan $\mathrm{Spec}(Q) \subset F$, we obtain a lifting of Frobenius $F\colon (Z', M_{Z'}) \to (Z', M_{Z'})$, which we call the canonical lifting of Frobenius of $(Z', M_{Z'})$.

Suppose that we are given an injective morphism of monoids $h\colon \mathbb{N} \to P$ such that the torsion part of the cokernel of h^{gp} is of order prime to p, and we regard (Z, M_Z) (resp. $(\mathrm{Spec}(\mathbb{Z}_{(p)}[P]), \mathrm{canlog})$) as a smooth fs log scheme over (V, M_V) (resp. $(\mathrm{Spec}(\mathbb{Z}_{(p)}[\mathbb{N}]), \mathrm{canlog})$) by the morphism induced by h. Let

$$f_X\colon (X', M') \longrightarrow (X, M)$$

be the base change of the log étale morphism

$$(\overline{Z}', M_{\overline{Z}'}) \longrightarrow \big(\mathrm{Spec}(\mathbb{Z}_{(p)}[P]), \mathrm{canlog}\big)$$

associated to $f\colon F \to \mathrm{Spec}(P)$ (*cf.* [Ka4, Prop. 9.9]) by the morphism $(S, N) \to (\mathrm{Spec}(\mathbb{Z}_{(p)}[\mathbb{N}]), \mathrm{canlog})$ defined by $\mathbb{N} \to \Gamma(S, N); 1 \mapsto \pi$. We have an inductive system of cartesian diagrams

$$\begin{array}{ccccc}
(X'_n, M'_n) & \xrightarrow{f_{X,n}} & (X_n, M_n) & \longrightarrow & (S_n, N_n) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle i_{V_n}} \\
(Z'_n, M_{Z'_n}) & \xrightarrow{f_n} & (Z_n, M_{Z_n}) & \longrightarrow & (V_n, M_{V_n})
\end{array}$$

for $n \geq 1$. Let g be the reduction mod π of f_X.

LEMMA 3.2. — *Let the notation and the assumption be as above.*

1) *For any integers $n \geq 1$ and $r \leq p-2$, the canonical morphism*

$$\mathcal{S}_n(r)_{(X,M)} \longrightarrow Rg_*\big(\mathcal{S}_n(r)_{(X',M')}\big)$$

is an isomorphism.

2) *For any integers $n \geq 1$ and r, the canonical morphism*

$$\mathcal{E}_n(r)_{(X,M)} \longrightarrow Rg_*\big(\mathcal{E}_n(r)_{(X',M')}\big)$$

is an isomorphism.

Proof. — The claim (2) easily follows from the facts that the restriction of f to the maximal open subsets of X and X' on which the log. structures are trivial, is an isomorphism, and that the underlying morphism of scheme of f is proper. We prove 1). The claim is reduced to the case $n = 1$ by the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{S}_{n-1}(r)_X & \xrightarrow{''p''} & \mathcal{S}_n(r)_X & \longrightarrow & \mathcal{S}_1(r)_X & \longrightarrow & \mathcal{S}_{n-1}(r)_X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Rg_*\mathcal{S}_{n-1}(r)_{X'} & \xrightarrow{''p''} & Rg_*\mathcal{S}_n(r)_{X'} & \longrightarrow & Rg_*\mathcal{S}_1(r)_{X'} & \longrightarrow & Rg_*\mathcal{S}_{n-1}(r)_{X'}[1], \end{array}$$

where we abbreviate (X, M) (resp. (X', M')) to X (resp. X'). Using the canonical lifting of Frobenius of (Z, M_Z) (resp. $(Z', M_{Z'})$), we obtain a representation $\mathcal{S}_1(r)_{(X,M),(Z,M_Z)}$ (resp. $\mathcal{S}_1(r)_{(X',M'),(Z',M_{Z'})}$) of $\mathcal{S}_1(r)_{(X,M)}$ (resp. $\mathcal{S}_1(r)_{(X',M')}$). Since for $s \leq p-2$,

$$\varphi_{s|J_{D_1}^{[p-1]}} = p^{p-1-s}\varphi_{p-1} = 0,$$

$\mathcal{S}_1(r)_{(X,M),(Z,M_Z)}$ is quasi-isomorphic to the mapping fiber of

$$1 - \varphi_r : J_{D_1}^{[r-\bullet]}/J_{D_1}^{[p-1]} \otimes_{\mathcal{O}_{Z_1}} \omega_{Z_1}^{\bullet} \longrightarrow \mathcal{O}_{D_1}/J_{D_1}^{[p-1]} \otimes_{\mathcal{O}_{Z_1}} \omega_{Z_1}^{\bullet}.$$

The same assertion is true for $\mathcal{S}_1(r)_{(X',M'),(Z',M_{Z'})}$. Hence it suffices to prove that

$$J_{D_1}^{[s]}/J_{D_1}^{[p-1]} \otimes_{\mathcal{O}_{Z_1}} \omega_{Z_1}^{q} \longrightarrow Rg_*\big(J_{D_1'}^{[s]}/J_{D_1'}^{[p-1]} \otimes_{\mathcal{O}_{Z_1'}} \omega_{Z_1'}^{q}\big)$$

are isomorphisms for $q \geq 0$ and $s = 0,\ r-q$. We have isomorphisms

$$\begin{gathered} \mathcal{O}_{X_1} = \mathcal{O}_{Z_1}/T^e\mathcal{O}_{Z_1} \xrightarrow{\sim} J_{D_1}^{[s]}/J_{D_1}^{[s+1]}; \quad 1 \mapsto (T^e)^{[s]} \quad (s \geq 0), \\ \mathcal{O}_{X_1'} = \mathcal{O}_{Z_1'}/T^e\mathcal{O}_{Z_1'} \xrightarrow{\sim} J_{D_1'}^{[s]}/J_{D_1'}^{[s+1]}; \quad 1 \mapsto (T^e)^{[s]} \quad (s \geq 0), \\ \omega_{Z_1'}^{q} \cong f_1^*\omega_{Z_1}^{q}, \end{gathered}$$

where e is the absolute ramification index of K. On the other hand, since Z_1 and Z_1' are flat over V_1, T^e is a non-zero divisor in $\mathcal{O}_{Z_1}$ and $\mathcal{O}_{Z_1'}$. Hence, from (3.1), we obtain

$$\mathcal{O}_{Z_1}/T^e\mathcal{O}_{Z_1} \xrightarrow{\sim} Rg_*(\mathcal{O}_{Z_1'}/T^e\mathcal{O}_{Z_1'}).$$

We can derive easily the required isomorphisms from these facts. $\square$

We use the following proper subdivisions in the proof of the main theorem (§5).

THEOREM 3.3 (see [KKMS, Chap. I, Thm. 11]). — *Let P be a torsion free finitely generated saturated monoid. Then, there exists a proper subdivision $f: F \to \mathrm{Spec}(P)$ such that, locally on F, f is of the form $\mathrm{Spec}(Q) \to \mathrm{Spec}(P)$ for a submonoid Q of P^{gp} containing P and isomorphic to $\mathbb{N}^r \oplus \mathbb{Z}^s$ for some integers $r \geq 0$ and $s \geq 0$.*

LEMMA 3.4. — *Let d be an integer ≥ 2 and put $P = P_1 = P_2 = \mathbb{N}^d$. Let F be the fan obtained by gluing the two fans $\mathrm{Spec}(P_1)$ and $\mathrm{Spec}(P_2)$ via the isomorphism of affine open subfans*

$$\mathrm{Spec}(S_1^{-1}P_1) = \mathrm{Spec}(\mathbb{N} \oplus \mathbb{Z} \oplus \mathbb{N}^{d-2}) \cong \mathrm{Spec}(\mathbb{Z} \oplus \mathbb{N} \oplus \mathbb{N}^{d-2}) = \mathrm{Spec}(S_2^{-1}P_2),$$

where $S_1 = \{(0,n,0,\dots,0) \mid n \in \mathbb{N}\}$, $S_2 = \{(n,0,0,\dots,0) \mid n \in \mathbb{N}\}$, and the middle isomorphism is the one defined by

$$\mathbb{N} \oplus \mathbb{Z} \oplus \mathbb{N}^{d-2} \cong \mathbb{Z} \oplus \mathbb{N} \oplus \mathbb{N}^{d-2}$$
$$(n_1,n_2,n_3,\dots,n_d) \mapsto (-n_2 + n_1, n_1, n_3, \dots, n_d).$$

Let $h_1: P \to P_1$ (resp. $h_2: P \to P_2$) be the morphism which sends $(n_1,n_2,\dots,n_d)$ to $(n_1+n_2,n_2,n_3,\dots,n_d)$ (resp. $(n_1,n_2+n_1,n_3,\dots,n_d)$). Let $f: F \to \mathrm{Spec}(P)$ be the morphism obtained by gluing $\mathrm{Spec}(h_1)$ and $\mathrm{Spec}(h_2)$. Then f is a proper subdivision of $\mathrm{Spec}(P)$.

Proof. — This follows from the fact h_1^{gp} and h_2^{gp} are isomorphisms, and, for any $\varphi \in \mathrm{Hom}(P^{\mathrm{gp}},\mathbb{Z})$, $\varphi(P) \subset \mathbb{N}$ if and only if $\varphi \circ (h_1^{\mathrm{gp}})^{-1}(P_1) \subset \mathbb{N}$ or $\varphi \circ (h_2^{\mathrm{gp}})^{-1}(P_2) \subset \mathbb{N}$. □

The étale morphism $(Z', M_{Z'}) \to (\mathrm{Spec}(W[\mathbb{N}^d]), \mathrm{canlog})$ associated to $f: F \to \mathrm{Spec}(P)$ in Lemma 3.4 is the blowing up along the intersection of the two hyperplanes $T_1 = 0$ and $T_2 = 0$ endowed with the log structure defined by the inverse image of of the union of the hyperplanes $T_i = 0$ $(1 \leq i \leq d)$, which is a divisor with normal crossings on Z'. Here T_i denotes the image in $W[\mathbb{N}^d]$ of the element of $\mathbb{N}^d$ whose i-th component is 1 and other component is 0.

4. Preliminary Theorem

Let (X,M) be a smooth fs log scheme over (S,N). By Lemma 2.4, the morphism (2.8) induces a morphism in $D^+(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$

$$\mathcal{S}_n(r)_{(X,M)} \longrightarrow \tau_{\leq r}\mathcal{E}_n(r)_{(X,M)} \tag{4.1}$$

for $n \geq 1$ and $0 \leq r \leq p-2$.

In this section, we will prove the following special case of our main theorem: Theorem 5.1.

THEOREM 4.2 (*cf.* [T2, Thm. 3.4.3]. — *Assume that X is smooth over S and that there exists a reduced divisor D on X with normal crossings relative to S such that, if we denote by M_D the log structure on X defined by D, there exists an isomorphism $(X,M) \cong (X,M_D) \times_S (S,N)$ over (S,N). Then, for any integers $n \geq 1$ and $0 \leq r \leq p-2$, the morphism* (4.1) *above is an isomorphism.*

In the special case $D = \emptyset$, M. Kurihara proved in [Ku] a similar result for syntomic complexes without log poles along the special fiber.

To prove Theorem 4.2, since the question is étale local on X, we may assume

$$(X, M) = (S, N) \times_{\mathrm{Spec}(\mathbb{Z})} \big(\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^d]), \mathrm{canlog}\big)$$

for an integer $d \geq 0$. Let (Z, M_Z) be $(V, M_V) \times_{\mathrm{Spec}(\mathbb{Z})} (\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^d]), \mathrm{canlog})$ and let F_Z be the lifting of Frobenius of (Z, M_Z) induced by F_V and the multiplication by $\mathbb{N}^d$. We have an inductive system of cartesian diagrams for $n \geq 1$:

$$\begin{array}{ccc} (X_n, M_n) & \longrightarrow & (Z_n, M_{Z_n}) \\ \downarrow & & \downarrow \\ (S_n, N_n) & \xrightarrow{i_{V_n}} & (V_n, M_{V_n}). \end{array}$$

Hence we can apply the results of the Appendix to (X, M), (Z, M_Z) and F_Z. Note that the special fiber (Y, M_Y) is of Cartier type over $(S, N) \otimes_{O_K} k$.

Since we can use the result of Bloch-Kato [Bl-Ka] on $\mathcal{H}^q(\mathcal{E}_1(q)_{(X,M)})$ only in the case where the divisor D at infinity is empty, we will reduce to that case by describing $\mathcal{H}^q(\mathcal{S}_1(q)_{(X,M),(Z,M_Z)})$ and $\mathcal{H}^q(\mathcal{E}_1(q)_{(X,M)})$ in terms of the corresponding sheaves for X, the smooth components of D and their intersections without divisors at infinity. We just follow the argument in [T2, §3.4] faithfully. So we give only an outline.

In the following, we change the notation M and M_Z to M^0 and M_Z^0 and use M and M_Z to denote the inverse images of N and M_V on X and Z respectively. Let T_i $(1 \leq i \leq d)$ denote the image of 1 of the i-th component of $\mathbb{N}^d$ in $\Gamma(Z, M_Z^0)$ and also its images in $\Gamma(X, M^0)$, $\Gamma(Z, \mathcal{O}_Z)$ and $\Gamma(X, \mathcal{O}_X)$. For each integer $\nu \geq 1$, set

$$I_\nu := \big\{\alpha = (\alpha_1, \ldots, \alpha_\nu) \mid \alpha_\nu \in \mathbb{Z}, 1 \leq \alpha_1 < \cdots < \alpha_\nu \leq d\big\},$$

and for $\nu = 0$, set $I_0 := \{\emptyset\}$. For $\nu \geq 0$ and $\alpha \in I_\nu$, we define (X_α, M_α) (resp. (Z_α, M_{Z_α})) to be the reduction mod $(T_{\alpha_1}, \ldots, T_{\alpha_\nu})$ of (X, M) (resp. (Z, M_Z)). The lifting of Frobenius on Z induces the lifting of Frobenius F_{Z_α} on (Z_α, M_{Z_α}) compatible with the lifting of Frobenius F_V of (V, M_V). We can apply the results of the Appendix to (X_α, M_α), (Z_α, M_{Z_α}) and F_{Z_α}.

First let us describe $\mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)})$ in terms of $\mathcal{H}^{q-\nu}(\mathcal{E}_n(q-\nu)_{(X_\alpha,M_\alpha)})$ $(\nu \geq 0, \alpha \in I_\nu)$. As in the Introduction, let X_{triv} denote the maximal open subset of X on which the log structure M^0 is trivial, let (Y, M_Y^0) denote the special fiber $(X, M^0) \otimes_{O_K} k$ and let i (resp. j) denote the immersion $Y \to X$ (resp. $X_{\mathrm{triv}} \to X$). We will often regard étale sheaves on the special fiber Y_α of X_α as those on Y. Set $X_K := X \otimes_{O_K} K$ and let j_1 (resp. j_2) denote the open immersion $X_{\mathrm{triv}} \to X_K$ (resp. $X_K \to X$). We have $j = j_2 \circ j_1$. We define the weight filtration

$$W_\nu \mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)}) \quad (\nu \in \mathbb{Z})$$

to be the image of $i^* R^q j_{2*}(\tau_{\leq \nu} Rj_{1*}\mathbb{Z}/p^n\mathbb{Z}(q))$. Then, by the same argument as the proof of [T2] Lemma 3.4.7, we obtain the following lemma using the (relative) purity theorem for étale cohomology and the surjectivity of the symbol map [Bl-Ka, Thm. 1.4].

LEMMA 4.3. — *Assume that the primitive p^n-th roots of unity are contained in K. Then, for each $0 \leq \nu \leq q$, the symbol map $i^*(M^{0\ \mathrm{gp}})^{\otimes q} \to \mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)})$ (induced by* (2.10) *and cup products) induces a surjective homomorphism*

$$(i^* M^{\mathrm{gp}})^{\otimes q-\nu} \otimes (i^* M^{0\ \mathrm{gp}})^{\otimes \nu} \longrightarrow W_\nu \mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)}).$$

Furthermore, there exists a unique isomorphism

$$\bigoplus_{\alpha \in I_\nu} \mathcal{H}^{q-\nu}(\mathcal{E}_n(q-\nu)_{(X_\alpha, M_\alpha)}) \xrightarrow{\sim} \mathrm{gr}_\nu^W \mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)})$$

which makes the following diagram commute:

$$\begin{array}{ccc} \bigoplus_{\alpha \in I_\nu} i^*(M^{\mathrm{gp}})^{\otimes q-\nu} & \xrightarrow{\kappa} & i^*(M^{\mathrm{gp}})^{\otimes q-\nu} \otimes i^*(M^{0\ \mathrm{gp}})^{\otimes \nu} \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha \in I_\nu} \mathcal{H}^{q-\nu}(\mathcal{E}_n(q-\nu)_{(X_\alpha, M_\alpha)}) & \xrightarrow{\sim} & \mathrm{gr}_\nu^W \mathcal{H}^q(\mathcal{E}_n(q)_{(X,M^0)}). \end{array}$$

Here κ denotes the homomorphism which sends a section a of the α-component to $a \otimes (T_{\alpha_1} \otimes \cdots \otimes T_{\alpha_\nu})$, and the left vertical arrow is the composite of the symbol maps $(i^* M_\alpha^{\mathrm{gp}})^{\otimes q-\nu} \to \mathcal{H}^{q-\nu}(\mathcal{E}_n(q-\nu)_{(X_\alpha, M_\alpha)})$ with the natural surjective homomorphism $(i^* M^{\mathrm{gp}})^{\otimes q-\nu} \to (i^* M_\alpha^{\mathrm{gp}})^{\otimes q-\nu}$.

Next let us describe $\mathcal{H}^q(\mathcal{S}_n(q)_{(X,M^0),(Z,M_Z^0)})$ in terms of

$$\mathcal{H}^{q-\nu}(\mathcal{S}_n(q-\nu)_{(X_\alpha, M_\alpha),(Z_\alpha, M_{Z_\alpha})}) \quad (\nu \geq 0,\ \alpha \in I_\nu).$$

Let $(D_n, M_{D_n}^0)$ (resp. $(D_{\alpha,n}, M_{D_{\alpha,n}})$) be the PD-envelope of (X_n, M_n^0) (resp. $(X_{\alpha,n}, M_{\alpha,n})$) in $(Z_n, M_{Z_n}^0)$ (resp. $(Z_{\alpha,n}, M_{Z_{\alpha,n}})$) compatible with the canonical

PD-structure on $p\mathbb{Z}/p^n\mathbb{Z}$, and let J_{D_n} (resp. $J_{D_{\alpha,n}}$) be the PD-ideal of $\mathcal{O}_{D_n}$ (resp. $\mathcal{O}_{D_{\alpha,n}}$) defining X_n (resp. $X_{\alpha,n}$). If we denote by (E_n, M_{E_n}) the PD-envelope of (S_n, N_n) in (V_n, M_{V_n}) compatible with the PD-structure on $p\mathbb{Z}/p^n\mathbb{Z}$, then we have

$$(D_n, M^0_{D_n}) = (E_n, M_{E_n}) \times_{(V_n, M_{V_n})} (Z_n, M^0_{Z_n}),$$
$$(D_{\alpha,n}, M_{D_{\alpha,n}}) = (E_n, M_{E_n}) \times_{(V_n, M_{V_n})} (Z_{\alpha,n}, M_{Z_{\alpha,n}}).$$

This implies that, for $\nu \geq 0$, $\alpha \in I_\nu$ and $r \in \mathbb{Z}$, we have

$$J^{[r]}_{D_n} / \sum_{1 \leq i \leq \nu} T_{\alpha_i} J^{[r]}_{D_n} \cong J^{[r]}_{D_{\alpha,n}}$$

and, for $1 \leq \beta \leq d$, $\beta \neq \alpha_i$ $(1 \leq i \leq \nu)$, T_β is a non-zero divisor on $J^{[r]}_{D_{\alpha,n}}$.

We define the increasing filtration W_ν $(\nu \in \mathbb{Z}, \nu \geq 0)$ on $J^{[r]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n}(\log(M^0_{Z_n}))$ by $\mathcal{O}_{D_n}$-submodules generated by

$$\omega \wedge d\log T_{\alpha_1} \wedge \ldots \wedge d\log T_{\alpha_\mu}$$
$$\big(\omega \in J^{[r]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{q-\mu}_{Z_n}(\log(M_{Z_n})),\ 1 \leq \alpha_1 < \cdots < \alpha_\mu \leq s, \mu \leq \nu\big).$$

Then, the filtration W_ν is compatible with the differentials and φ_r $(r \leq p-1)$ and induces the filtration W_ν on $\mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)}$ $(r \leq p-1)$ (*cf.* [D, 3.1.5]). We have

$$W_0 \mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)} = \mathcal{S}_n(r)_{(X,M),(Z,M_Z)}.$$

For $\nu < 0$, we set $W_\nu = 0$. We see easily that the product structure of $\mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)}$ induces

$$W_\nu \mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)} \otimes W_{\nu'} \mathcal{S}_n(r')_{(X,M^0),(Z,M^0_Z)} \longrightarrow W_{\nu+\nu'} \mathcal{S}_n(r+r')_{(X,M^0),(Z,M^0_Z)}$$

for $r, r' \in \mathbb{Z}$ such that $0 \leq r, r', r+r' \leq p-1$. Hence, from the symbol maps

$$M^{\mathrm{gp}}_{n+1} \longrightarrow \mathcal{H}^1\big(\mathcal{S}_n(1)_{(X,M),(Z,M_Z)}\big) = \mathcal{H}^1\big(W_0 \mathcal{S}_n(1)_{(X,M^0),(Z,M^0_Z)}\big),$$
$$M^{0\ \mathrm{gp}}_{n+1} \longrightarrow \mathcal{H}^1\big(\mathcal{S}_n(1)_{(X,M^0),(Z,M^0_Z)}\big) = \mathcal{H}^1\big(W_1 \mathcal{S}_n(1)_{(X,M^0),(Z,M^0_Z)}\big),$$

we obtain

$$(M^{\mathrm{gp}}_{n+1})^{\otimes q-\nu} \otimes (M^{0\ \mathrm{gp}}_{n+1})^{\otimes \nu} \longrightarrow \mathcal{H}^q\big(W_\nu \mathcal{S}_n(q)_{(X,M^0),(Z,M^0_Z)}\big)$$

for $0 \leq \nu \leq q \leq p-1$.

On the other hand, for any integers $0 \le \nu \le q$ and r, we have an isomorphism

$$\mathrm{gr}_\nu^W(J_{D_n}^{[r]} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n}(\log(M^0_{Z_n}))) \cong \bigoplus_{\alpha\in I_\nu} J^{[r]}_{D_{\alpha,n}} \otimes_{\mathcal{O}_{Z_{\alpha,n}}} \Omega^{q-\nu}_{Z_{\alpha,n}}(\log(M_{Z_{\alpha,n}}))$$

which sends $\omega \wedge d\log T_{\alpha_1} \wedge \ldots \wedge d\log T_{\alpha_\nu}$, $\omega \in J^{[r]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{q-\nu}_{Z_n}(\log(M_{Z_n}))$ to the inverse image of ω in the α-component. By $\nabla(d\log T_i) = 0$ and $\varphi_1(d\log T_i) = d\log T_i$ $(1 \le i \le d)$, this isomorphism is compatible with the differentials and φ_r $(r \le p-1)$ and induces an isomorphism of complexes (*cf.* [D, 3.1.5.2])

$$\mathrm{gr}_\nu^W \mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)} \xrightarrow{\sim} \bigoplus_{\alpha\in I_\nu} \mathcal{S}_n(r-\nu)_{(X_\alpha,M_\alpha),(Z_\alpha,M_{Z_\alpha})}[-\nu].$$

for $r \le p-1$, $\nu \ge 0$.

We define the weight filtration W_ν $(\nu \in \mathbb{Z})$ on $\mathcal{H}^q(\mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)})$ $(q \ge 0,\ r \le p-1)$ by the images of $\mathcal{H}^q(W_\nu \mathcal{S}_n(r)_{(X,M^0),(Z,M^0_Z)})$. Then, using the surjectivity of the symbol map for $(X_\alpha, M_\alpha) \to (S, N)$ (Prop. A15 for $r = q$), we can prove the following lemma in the same way as [T2, Lemma 3.4.11].

LEMMA 4.4. — *For any integers $0 \le \nu \le q \le p-2$, the symbol map $(M^{0\,\mathrm{gp}}_{n+1})^{\otimes q} \to \mathcal{H}^q(\mathcal{S}_n(q)_{(X,M^0),(Z,M^0_Z)})$ (induced by (2.5) and (2.6)) induces a surjective homomorphism*

$$(M^{\mathrm{gp}}_{n+1})^{\otimes q-\nu} \otimes (M^{0\,\mathrm{gp}}_{n+1})^{\otimes \nu} \longrightarrow W_\nu \mathcal{H}^q(\mathcal{S}_n(q)_{(X,M^0),(Z,M^0_Z)}).$$

Furthermore, there exists a unique surjective homomorphism

$$\bigoplus_{\alpha\in I_\nu} \mathcal{H}^{q-\nu}(\mathcal{S}_n(q-\nu)_{(X_\alpha,M_\alpha),(Z_\alpha,M_{Z_\alpha})}) \longrightarrow \mathrm{gr}_\nu^W \mathcal{H}^q(\mathcal{S}_n(q)_{(X,M^0),(Z,M^0_Z)})$$

which makes the following diagram commute:

$$\begin{array}{ccc} \bigoplus_{\alpha\in I_\nu} (M^{\mathrm{gp}}_{n+1})^{\otimes q-\nu} & \xrightarrow{\kappa_{n+1}} & (M^{\mathrm{gp}}_{n+1})^{\otimes q-\nu} \otimes (M^{0\,\mathrm{gp}}_{n+1})^{\otimes \nu} \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha\in I_\nu} \mathcal{H}^{q-\nu}(\mathcal{S}_n(q-\nu)_{(X_\alpha,M_\alpha),(Z_\alpha,M_{Z_\alpha})}) & \longrightarrow & \mathrm{gr}_\nu^W \mathcal{H}^q(\mathcal{S}_n(q)_{(X,M^0),(Z,M^0_Z)}). \end{array}$$

Here κ_{n+1} is defined in the same way as κ in Lemma 4.3 and the left vertical homomorphism is the composite of $(M^{\mathrm{gp}}_{n+1})^{\otimes q-\nu} \to (M^{\mathrm{gp}}_{\alpha,n+1})^{\otimes q-\nu}$ with the symbol map for (X_α, M_α).

Now let us prove Theorem 4.2. The first half of the proof is the same as the proof of [T2, Thm. 3.4.3, 1].

Proof of Theorem 4.2. — We keep the notation and the assumption above. The claim is reduced to the case $n = 1$ using the commutative diagram

$$\begin{array}{ccccccc} \mathcal{S}_{n-1}(r)_X & \xrightarrow{''p''} & \mathcal{S}_n(r)_X & \longrightarrow & \mathcal{S}_1(r)_X & \longrightarrow & \mathcal{S}_{n-1}(r)_X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_{n-1}(r)_X & \xrightarrow{''p''} & \mathcal{E}_n(r)_X & \longrightarrow & \mathcal{E}_1(r)_X & \longrightarrow & \mathcal{E}_{n-1}(r)_X[1] \end{array}$$

where we abbreviate (X, M^0) to X, and $\mathcal{H}^q(\mathcal{S}_n(r)_{(X,M^0)}) = 0$ $(q > r)$.

First we assume that K contains the primitive p-th roots of unity, and let q be an integer such that $0 \leq q \leq p-2$. Since the morphism (4.1) is compatible with the symbol maps (Proposition 2.11) and the product structures, by Lemma 4.3 and Lemma 4.4, the homomorphism

$$\mathcal{H}^q\big(\mathcal{S}_1(q)_{(X,M^0),(Z,M_Z^0)}\big) \longrightarrow \mathcal{H}^q\big(\mathcal{E}_1(q)_{(X,M^0)}\big)$$

induced by (4.1) is compatible with the filtrations $W_\bullet$. To prove that it is an isomorphism, it suffices to show that the composite:

$$\begin{aligned} \bigoplus_{\alpha\in I_\nu} \mathcal{H}^{q-\nu}\big(\mathcal{S}_1(q-\nu)_{(X_\alpha,M_\alpha),(Z_\alpha,M_{Z_\alpha})}\big) &\xrightarrow{\text{Lemma 4.4}} \operatorname{gr}_\nu\big(\mathcal{H}^q(\mathcal{S}_1(q)_{(X,M^0),(Z,M_Z^0)})\big) \\ &\longrightarrow \operatorname{gr}_\nu^W \mathcal{H}^q(\mathcal{E}_1(q)_{(X,M^0)}) \\ &\xrightarrow[\sim]{\text{Lemma 4.3}} \bigoplus_{\alpha\in I_\nu} \mathcal{H}^{q-\nu}\big(\mathcal{E}_1(q-\nu)_{(X_\alpha,M_\alpha)}\big) \end{aligned}$$

is an isomorphism. By Lemma 4.3 and Lemma 4.4, this is compatible with the symbol maps. Hence, by comparing [Bl-Ka, Cor. 1.4.1] for $X_\alpha \to S$ with Proposition A15 for $(X_\alpha, M_\alpha) \to (S, N)$ and $r = q$, we see that the above composite is an isomorphism. Note that, with the notation in the Appendix, we have $\omega^\bullet_{Y_\alpha} = \Omega^\bullet_{Y_\alpha}$, $\omega^\bullet_{Y_\alpha,\log} = \Omega^\bullet_{Y_\alpha,\log}$, *etc.*

By Lemma 2.12 and Proposition A17, the homomorphism

$$\mathcal{H}^0\big(\mathcal{S}_1(r)_{(X,M^0)}\big) \longrightarrow \mathcal{H}^0\big(\mathcal{E}_1(r)_{(X,M^0)}\big)$$

induced by (4.1) is an isomorphism for an integer $0 \leq r \leq p-2$. Since the morphism (4.1) is compatible with the product structures, for integers r and q satisfying $0 \leq q \leq r \leq p-2$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^0\big(\mathcal{S}_1(r-q)_{(X,M^0)}\big) \otimes \mathcal{H}^q\big(\mathcal{S}_1(q)_{(X,M^0)}\big) & \longrightarrow & \mathcal{H}^q\big(\mathcal{S}_1(r)_{(X,M^0)}\big) \\ \downarrow & & \downarrow \\ \mathcal{H}^0\big(\mathcal{E}_1(r-q)_{(X,M^0)}\big) \otimes \mathcal{H}^q\big(\mathcal{E}_1(q)_{(X,M^0)}\big) & \longrightarrow & \mathcal{H}^q\big(\mathcal{E}_1(r)_{(X,M^0)}\big), \end{array}$$

where the two vertical homomorphisms is the ones induced by (4.1). The lower horizontal arrow is an isomorphism, and we already proved that the left vertical one is also an isomorphism. Furthermore, by Corollary A16, the upper horizontal one is also an isomorphism. Hence, for integers r and q such that $0 \le q \le r \le p-2$, the homomorphism induced by (4.1)

$$\mathcal{H}^q\big(\mathcal{S}_1(r)_{(X,M^0)}\big) \longrightarrow \mathcal{H}^q\big(\mathcal{E}_1(r)_{(X,M^0)}\big)$$

is an isomorphism.

Next we consider the general case. By replacing K with an unramified extension, we may assume that $K((\pi)^{1/(p-1)})$ contains a primitive p-th root of unity. Set $K' := K(\pi^{1/(p-1)})$ and choose a $(p-1)$-th root $\pi' \in K'$ of π. We follow the notation before the statement of Lemma A18. Since K'/K is a totally tamely ramified extension of degree $p-1$, we have

$$\mathcal{H}^q\big(\mathcal{E}_1(r)_{(X',M'^0)}\big)^{\mathrm{Gal}(K'/K)} = \mathcal{H}^q\big(\mathcal{E}_1(r)_{(X,M^0)}\big).$$

On the other hand, by Proposition A19, we have

$$\mathcal{H}^q\big(\mathcal{S}_1(r)_{(X',M'^0)}\big)^{\mathrm{Gal}(K'/K)} = \mathcal{H}^q\big(\mathcal{S}_1(r)_{(X,M^0)}\big)$$

for $0 \le q \le r \le p-2$. Thus the general case is reduced to the case where K contains the primitive p-th roots of unity. □

5. Main theorem

Our objective in this section is to prove the following theorem:

THEOREM 5.1. — *Let (X,M) be a smooth fs log scheme over (S,N). Then, for any integers $n \ge 1$ and $0 \le r \le p-2$, the morphism (4.1) is an isomorphism.*

Let n and r be integers such that $n \ge 1$ and $0 \le r \le p-2$. Let (X,M) be a smooth fs log scheme over (S,N) and let (Y,M_Y) denote the reduction mod π of (X,M). Let y be a point of Y and let q be an integer such that $q \le r$. We denote by $C_{n,r}((X,M),y,q)$ the following claim:

> *The homomorphism $\mathcal{H}^q(\mathcal{S}_n(r)_{(X,M)})_{\bar y} \to \mathcal{H}^q(\mathcal{E}_n(r)_{(X,M)})_{\bar y}$ induced by (2.8) is an isomorphism, and the homomorphism*
> $$\mathcal{H}^{q+1}(\mathcal{S}_n(r)_{(X,M)})_{\bar y} \to \mathcal{H}^{q+1}(\mathcal{E}_n(r)_{(X,M)})_{\bar y}$$
> *induced by (2.8) is injective.*

LEMMA 5.2. — *Let n and r be integers such that $n \ge 1$ and $0 \le r \le p-2$. Let $f\colon (X',M') \to (X,M)$ be a morphism of smooth fs log schemes over (S,N) and* *let g denotes the reduction mod π of f. Let y be a point of $Y := X \otimes_{O_K} k$ and let q be an integer such that $q \leq r$. Assume that the following conditions hold:*

(0) *The underlying morphism of schemes of f is proper.*

(i) *The dimension of $g^{-1}(y)$ is ≤ 1.*

(ii) *The canonical morphisms*

$$\mathcal{S}_n(r)_{(X,M)} \longrightarrow Rg_*\big(\mathcal{S}_n(r)_{(X',M')}\big), \quad \mathcal{E}_n(r)_{(X,M)} \longrightarrow Rg_*\big(\mathcal{E}_n(r)_{(X',M')}\big)$$

are isomorphisms.

(iii) *The claim $C_{n,r}((X,M),y,q)$ is true.*

(iv) *For every non-closed point z of $g^{-1}(y)$, $C_{n,r}((X',M'),z,q)$ is true.*

(v) *For every point w of $g^{-1}(y)$, the claim $C_{n,r}((X',M'),w,q-1)$ is true.*

Then the claim $C_{n,r}((X',M'),y',q)$ is also true for every closed point y' of $g^{-1}(y)$.

Proof. — From the morphism (2.8):

$$\mathcal{S}_n(r)_{(X',M')} \longrightarrow \mathcal{E}_n(r)_{(X',M')},$$

we obtain a morphism of spectral sequences

$$\begin{array}{ccc} E_2^{a,b} = R^a g_*\big(\mathcal{H}^b(\mathcal{S}_n(r)_{(X',M')})\big) & \longrightarrow & R^{a+b} g_*\big(\mathcal{S}_n(r)_{(X',M')}\big) \\ \downarrow & & \downarrow \\ E_2^{a,b} = R^a g_*\big(\mathcal{H}^b(\mathcal{E}_n(r)_{(X',M')})\big) & \longrightarrow & R^{a+b} g_*\big(\mathcal{E}_n(r)_{(X',M')}\big). \end{array}$$

By the proper base change theorem for étale cohomology and the assumption (0), we have isomorphisms

$$R^a g_*\big(\mathcal{H}^b(\mathcal{S}_n(r)_{(X',M')})\big)_{\bar{y}} \cong R^a g_{\bar{y}*}\big(\mathcal{H}^b(\mathcal{S}_n(r)_{(X',M')})_{|Y'_{\bar{y}}}\big),$$
$$R^a g_*\big(\mathcal{H}^b(\mathcal{E}_n(r)_{(X',M')})\big)_{\bar{y}} \cong R^a g_{\bar{y}*}\big(\mathcal{H}^b(\mathcal{E}_n(r)_{(X',M')})_{|Y'_{\bar{y}}}\big),$$

where $g_{\bar{y}}: Y'_{\bar{y}} \to \bar{y}$ denotes the base change of g under $\bar{y} \to Y$. By the assumption (i), these groups vanish if $a \geq 2$. Hence, using the assumption (ii), we obtain the following morphisms of short exact sequences:

$$\begin{array}{ccccccccc} 0 \to & R^1 g_{\bar{y}*}\big(\mathcal{H}^{q-1}(\mathcal{S}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to & \mathcal{H}^q\big(\mathcal{S}_n(r)_X\big)_{\bar{y}} & \to & g_{\bar{y}*}\big(\mathcal{H}^q(\mathcal{S}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & R^1 g_{\bar{y}*}\big(\mathcal{H}^{q-1}(\mathcal{E}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to & \mathcal{H}^q\big(\mathcal{E}_n(r)_X\big)_{\bar{y}} & \to & g_{\bar{y}*}\big(\mathcal{H}^q(\mathcal{E}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to 0, \end{array}$$

$$\begin{array}{ccccccc} 0 \to R^1 g_{\bar{y}*}\big(\mathcal{H}^q(\mathcal{S}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to & \mathcal{H}^{q+1}\big(\mathcal{S}_n(r)_X\big)_{\bar{y}} & \to & g_{\bar{y}*}\big(\mathcal{H}^{q+1}(\mathcal{S}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to R^1 g_{\bar{y}*}\big(\mathcal{H}^q(\mathcal{E}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) & \to & \mathcal{H}^{q+1}\big(\mathcal{E}_n(r)_X\big)_{\bar{y}} & \to & g_{\bar{y}*}\big(\mathcal{H}^{q+1}(\mathcal{E}_n(r)_{X'})_{|Y'_{\bar{y}}}\big) \to 0. \end{array}$$

Here we abbreviate (X,M) (resp. (X',M')) to X (resp. X') to simplify the notation. By the assumptions (iii) and (v), the left and middle vertical homomorphisms in the first diagram are isomorphisms, and the middle vertical homomorphism in the second diagram is injective. Hence the right vertical homomorphism in the first diagram is an isomorphism, and the left vertical homomorphism in the second diagram is injective. Using the assumption (iv), we see that the homomorphism

$$\mathcal{H}^q\big(\mathcal{S}_n(r)_{(X',M')}\big)_{|Y'_{\bar{y}}} \longrightarrow \mathcal{H}^q\big(\mathcal{E}_n(r)_{(X',M')}\big)_{|Y'_{\bar{y}}}$$

induced by (2.8) is an isomorphism. Using the second diagram again, we find that the right vertical homomorphism of the diagram is injective. Hence, using the assumption (iv) again, we see that the homomorphism

$$\mathcal{H}^{q+1}\big(\mathcal{S}_n(r)_{(X',M')}\big)_{|Y'_{\bar{y}}} \longrightarrow \mathcal{H}^{q+1}\big(\mathcal{E}_n(r)_{(X',M')}\big)_{|Y'_{\bar{y}}}$$

induced by (2.8) is injective. □

LEMMA 5.3. — *Let d be a positive integer and let x be a point of codimension e $(0 \le e \le d)$ of* $\mathrm{Spec}(k[T_1,\dots,T_d])$. *Then there exist integers $1 \le i_1 < \cdots < i_{d-e} \le d$ such that the image of x in* $\mathrm{Spec}(k[T_{i_1},\dots,T_{i_{d-e}}])$ *is a generic point.*

Proof. — The transcendental degree of the residue field $k(x)$ over k is $d-e$ and the residue field is generated by the images of $T_1,\dots,T_d$. Hence there exist $1 \le i_1 < \cdots < i_{d-e} \le d$ such that the homomorphism $k[T_{i_1},\dots,T_{i_{d-e}}] \to k(x)$ is injective. □

DEFINITION 5.4. — We say that a morphism of finitely generated saturated monoids $h\colon \mathbb{N} \to P$ is *smooth (with respect to p)* if it is injective and the order of the torsion part of the cokernel is prime to p. For such a morphism h, we denote by $({}^hZ, M_{{}^hZ})$ the fs log scheme $(\mathrm{Spec}(W[P]), \mathrm{canlog})$ regarded as a smooth fs log scheme over (V, M_V) by the morphism induced by h, and denote by

$$({}^hX, {}^hM) \longrightarrow (S, N)$$

the base change of $(\mathrm{Spec}(\mathbb{Z}_{(p)}[P]), \mathrm{canlog}) \to (\mathrm{Spec}(\mathbb{Z}_{(p)}[\mathbb{N}]), \mathrm{canlog})$ induced by h by the morphism $(S,N) \to (\mathrm{Spec}(\mathbb{Z}_{(p)}[\mathbb{N}], \mathrm{canlog})$ defined by $\mathbb{N} \to \Gamma(S,N)$, $1 \mapsto \pi$. We denote by $({}^hY, M_{{}^hY})$ the reduction mod π of $({}^hX, {}^hM)$.

Recall that (Z^h, M_{Z^h}) has the canonical lifting of Frobenius compatible with F_V (§3), which we denote by F_{Z^h}.

We will use the following two special cases of Theorem 4.2:

PROPOSITION 5.5. — *Let (X,M) be a smooth fs log scheme over (S,N) and let Y be the reduction mod π of X. Then, for every point $y \in Y$ of codimension* 0, *the morphism*

$$\big(\mathcal{S}_n(r)_{(X,M)}\big)_{\bar{y}} \longrightarrow \big(\tau_{\leq r}\mathcal{E}_n(r)_{(X,M)}\big)_{\bar{y}}$$

induced by (4.1) *is an isomorphism for any integers $n \geq 1$ and $0 \leq r \leq p-2$.*

Proof. — By [Ka4, Thms. 4.1 and 8.2], the ring $\mathcal{O}_{X,y}$ is a discrete valuation ring. (See also [T2, Lemma 1.5.1].) Let $O_{K'}$ be the henselization of $\mathcal{O}_{X,x}$ and put $S' := \mathrm{Spec}(O_{K'})$. Then the inverse image of M under the natural morphism $S' \to X$ coincides with the log structure defined by its closed point (*cf.* [Ka4, Thm. 11.6]), which we denote by N'. If we denote by s' the closed point of S', the natural morphisms

$$\mathcal{S}_n(r)_{(X,M)}|_{s'} \longrightarrow \mathcal{S}_n(r)_{(S',N')} \quad \text{and} \quad \mathcal{E}_n(r)_{(X,M)}|_{s'} \longrightarrow \mathcal{E}_n(r)_{(S',N')}$$

are isomorphisms. Hence the claim follows from Theorem 4.2 with $(X,M) = (S,N)$. □

PROPOSITION 5.6. — *Let d be a positive integer and let $h\colon \mathbb{N} \to \mathbb{N}^d$ be a smooth morphism of monoids. Let e be an integer such that $0 \leq e \leq d-1$ and assume $m_1 > 0$ and $m_i = 0$ $(2 \leq i \leq e+1)$, where $h(1) = (m_1, m_2, \ldots, m_d)$. Let y be a point of ${}^hY$ whose image under the composite*

$${}^hY \longrightarrow ({}^hZ)_1 = \mathrm{Spec}\big(k[\mathbb{N}^d]\big) \longrightarrow \mathrm{Spec}\big(k[\mathbb{N}^{d-e-1}]\big)$$

is the generic point, where the second morphism is the one defined by the inclusion into the last $d-e-1$ components $\mathbb{N}^{d-e-1} \to \mathbb{N}^d$. Then the morphism

$$\big(\mathcal{S}_n(r)_{({}^hX,{}^hM)}\big)_{\bar{y}} \longrightarrow \big(\tau_{\leq r}\mathcal{E}_n(r)_{({}^hX,{}^hM)}\big)_{\bar{y}}$$

induced by (4.1) *is an isomorphism for any integers $n \geq 1$ and $0 \leq r \leq p-2$.*

Proof. — Let $\kappa\colon \mathbb{N}^{d-e} \to \mathbb{N}^d$ be the morphism of monoids which sends the first component to the first one and the other $d-e-1$ components to the last $d-e-1$ components of $\mathbb{N}^d$. Then h factors as $\mathbb{N} \xrightarrow{h'} \mathbb{N}^{d-e} \xrightarrow{\kappa} \mathbb{N}^d$ and h' is smooth. We regard $({}^hX, {}^hM)$ as an fs log scheme over $({}^{h'}X, {}^{h'}M)$ by the morphism induced by κ. Then

$$({}^hX, {}^hM) \cong ({}^{h'}X, {}^{h'}M) \times_S \big(\mathrm{Spec}(O_K[\mathbb{N}^e]), \mathrm{canlog}\big).$$

Moreover the assumption on y implies that the image y' of y in ${}^{h'}Y$ is of codimension 0. Let $O_{K'}$ be the henselization of $\mathcal{O}_{{}^{h'}X,y'}$ and let N' be the inverse image of ${}^{h'}M$ under the natural morphism $S' := \mathrm{Spec}(O_{K'}) \xrightarrow{h'} X$. Then, as in the proof of Proposition 5.5, $O_{K'}$ is a discrete valuation ring and N' is the log structure defined by the closed point. Set

$$\begin{aligned}(X', M') &:= ({}^hX, {}^hM) \times_{({}^{h'}X, {}^{h'}M)} (S', N') \\ &\cong (S', N') \times_{S'} \big(\mathrm{Spec}(O_{K'}[\mathbb{N}^e]), \mathrm{canlog}\big).\end{aligned}$$

If we denote by Y' the special fiber of X', the natural morphisms

$$\mathcal{S}_n(r)_{({}^hX,{}^hM)}|_{Y'} \longrightarrow \mathcal{S}_n(r)_{(X',M')} \quad \text{and} \quad \mathcal{E}_n(r)_{({}^hX,{}^hM)}|_{Y'} \to \mathcal{E}_n(r)_{(X',M')}$$

are isomorphisms. Applying Theorem 4.2 to $(X', M') \to (S', N')$, we obtain the proposition. ∎

Proof of Theorem 5.1. — Since the question is étale local on X, we may assume $(X, M) = ({}^hX, {}^hM)$ for some finitely generated saturated monoid P and some smooth morphism $h: \mathbb{N} \to P$ (Definition 5.4). Let G denote the torsion part of P^{gp}, whose order is prime to p. Since P is saturated, we have $G \subset P$ and $P \cong (P/G) \oplus G$ (non-canonical). By adding the primitive $\#G$-th roots of unity to W and O_K, we may assume $W[G]$ is isomorphic to a finite product of W. By considering each irreducible component of $\mathrm{Spec}(W[(P/G) \oplus G])$ separately and replacing the fixed π by $\pi \cdot \zeta$ for some $\#G$-th root of unity ζ, we may assume $G = \{1\}$. Furthermore, by using Theorem 3.3 and Lemma 3.2, we can reduce to the case where $P = \mathbb{N}^d$ for some integer $d \geq 1$.

If $d = 1$, the theorem follows from Proposition 5.5.

We fix integers $d \geq 2$, $n \geq 1$, and $0 \leq r \leq p-2$. We will prove the following claim by induction on $e \geq 0$:

For any smooth morphism $h: \mathbb{N} \to \mathbb{N}^d$ and any $y \in {}^hY$ such that the codimension of $\overline{\{y\}}$ in ${}^hY$ is $\leq e$, the stalk at y of the morphism (4.1) *for* $({}^hX, {}^hM)$ *is an isomorphism.*

If $e = 0$, the above claim follows from Proposition 5.5. Let e be a positive integer and assume that the claim is true for $e - 1$. We will prove the claim for e. Let $h: \mathbb{N} \to \mathbb{N}^d$ be a smooth morphism and let y be a point of ${}^hY$ such that $\mathrm{codim}(\overline{\{y\}}, {}^hY) = e$. We will reduce the claim to Proposition 5.6 by using Lemma 5.2.

The codimension of the image of y under ${}^hY \to ({}^hZ)_1 = \mathrm{Spec}(k[\mathbb{N}^d])$ is $e+1$. Hence, by Lemma 5.3, we may assume that the first component of $h(1)$ is non-zero and the image of y under the composite

$${}^hY \longrightarrow ({}^hZ)_1 = \mathrm{Spec}\big(k[\mathbb{N}^d]\big) \longrightarrow \mathrm{Spec}\big(k[\mathbb{N}^{d-e-1}]\big)$$

is the generic point, where the second morphism is the one induced by the inclusion into the last $d-e-1$ components $\mathbb{N}^{d-e-1} \to \mathbb{N}^d$.

It is easy to see that there exist an integer $s \geq 0$, a sequence of morphisms of monoids

$$\mathbb{N}^d \xrightarrow{\kappa^1} \mathbb{N}^d \xrightarrow{\kappa^2} \cdots \xrightarrow{\kappa^{s-1}} \mathbb{N}^d \xrightarrow{\kappa^s} \mathbb{N}^d,$$

and a smooth morphism of monoids $h^0 : \mathbb{N} \to \mathbb{N}^d$ satisfying the following properties:

$$h = \kappa^s \cdots \kappa^2 \kappa^1 h^0.$$

Each κ^ν is given by $\epsilon_i \mapsto \epsilon_i$ $(i \neq a)$, $\epsilon_a \mapsto \epsilon_a + \epsilon_b$ for some integers a and b satisfying $1 \leq a \leq e+1$, $1 \leq b \leq e+1$, and $a \neq b$. The i-th components of $h^0(1)$ vanish for all $2 \leq i \leq e+1$ and the first component of $h^0(1)$ does not vanish. Here ϵ_i denotes the element of $\mathbb{N}^d$ whose i-th component is 1 and other components are 0.

Choose such an integer and morphisms. Let h^ν denote the smooth morphism of monoids $\kappa^\nu \kappa^{\nu-1} \cdots \kappa^1 h^0$ for $1 \leq \nu \leq s$. We have $h^s = h$. For $0 \leq \nu \leq s$, we write (X^ν, M^ν) (resp. (Y^ν, M_{Y^ν})) for $({}^{h^\nu}X, {}^{h^\nu}M)$ (resp. $({}^{h^\nu}Y, M_{{}^{h^\nu}Y})$) to simplify the notation. We have $({}^hX, {}^hM) = (X^s, M^s)$ and $({}^hY, M_{{}^hY}) = (Y^s, M_{Y^s})$. For $1 \leq \nu \leq s$, let f_ν denote the morphism $(X^\nu, M^\nu) \to (X^{\nu-1}, M^{\nu-1})$ of smooth fs log schemes over (S, N) defined by κ^ν and let g_ν denote the reduction mod π of f_ν.

Let y^ν be the image of y in Y^ν for $0 \leq \nu \leq s$. For $0 \leq \nu \leq s$, let σ^ν denote the composite

$$Y^\nu \longrightarrow ({}^{h^\nu}Z)_1 = \operatorname{Spec}(k[\mathbb{N}^d]) \longrightarrow \operatorname{Spec}(k[\mathbb{N}^{d-e-1}]),$$

where the second morphism is the one defined by the inclusion into the last $d-e-1$ components $\mathbb{N}^{d-e-1} \to \mathbb{N}^d$. By the choice of κ^ν, we have $\sigma^{\nu-1} g^\nu = \sigma^\nu$ for $1 \leq \nu \leq s$. Hence the image of y^ν under σ^ν is the generic point for $0 \leq \nu \leq s$. Hence the codimension of y^ν in Y^ν is e for $0 \leq \nu \leq s$ and y^ν is a closed point of $(g^\nu)^{-1}(y^{\nu-1})$ for $1 \leq \nu \leq s$.

Now we will prove that the stalk at y^ν of the morphism (4.1) for (X^ν, M^ν) is an isomorphism by induction on ν. For $\nu = 0$, the claim follows from Proposition 5.6. Let ν be an integer such that $1 \leq \nu \leq s$ and assume that the claim is true for $\nu - 1$. By the choice of κ^ν, Lemma 3.4 and Lemma 3.2, there exists a factorization

$$(X^\nu, M^\nu) \xrightarrow{j^\nu} (\overline{X^\nu}, \overline{M^\nu}) \xrightarrow{\overline{f^\nu}} (X^{\nu-1}, M^{\nu-1})$$

of f^ν such that j^ν is an open immersion and $\overline{f^\nu}$ satisfies the conditions (0), (i), and (ii) of Lemma 5.2. By the induction hypothesis with respect to e (resp. ν), the condition (iv) (resp. (iii)) also holds for $y^{\nu-1} \in Y^{\nu-1}$ and any integer $q \leq r$. Hence, by Lemma 5.2 and the induction on q, we see that $C_{n,r}((X^\nu, M^\nu), w, q)$ is true for any $w \in \overline{f^\nu}^{-1}(y^{\nu-1})$ and any integer $q \leq r$. Especially, the stalk at y^ν of the morphism (4.1) for (X^ν, M^ν) is an isomorphism. ∎

Appendix. Calculation of syntomic complexes in the imperfect residue field case

In this appendix, we will show that the calculation of

$$\mathcal{H}^q(\mathcal{S}_1(r)) \quad (0 \leq q \leq r \leq p-2)$$

in [T1, §7] (the case $\mathfrak{M}$ is constant) still works in the case where the residue field of the base field is not perfect. First we note that M. Kurihara allowed imperfect residue field in his calculation [Ku] in the good reduction case, and that we followed his method in [T1, §7]. Once we formulate the statement, the generalization of the proof in [T1, §7] is completely straightforward. However, in [T1, §7], we treated non-constant coefficients as well and it made the proof much complicated. So we will give an outline of the proof here for the convenience of the readers.

We consider a smooth fine log scheme (X, M) over (S, N). Let (Y, M_Y) (resp. (s, N_s)) denote the reduction mod π of (X, M) (resp. (S, N)). We assume that (Y, M_Y) is of Cartier type over (s, N_s). We also assume that there exist smooth fine log schemes (Z_n, M_{Z_n}) over (V_n, M_{V_n}) with isomorphisms

$$\text{(A1)} \qquad (Z_{n+1}, M_{Z_{n+1}}) \times_{(V_{n+1}, M_{V_{n+1}})} (V_n, M_{V_n}) \cong (Z_n, M_{Z_n})$$

over (V_n, M_{V_n}), exact closed immersions $\iota_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$ compatible with the above isomorphisms (A1) which make the following diagrams cartesian

$$\begin{array}{ccc} (X_n, M_n) & \xrightarrow{\iota_n} & (Z_n, M_{Z_n}) \\ \downarrow & & \downarrow \\ (S_n, N_n) & \xrightarrow{i_{V_n}} & (V_n, M_{V_n}), \end{array}$$

and a system of liftings of Frobenius $\{F_{Z_n} : (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\}$ compatible with the above isomorphisms (A1) and F_{V_n}. Choose such (Z_n, M_{Z_n}), ι_n and F_{Z_n}. For integers $n \geq 1$ and $r \leq p-1$, we defined the syntomic complex $\mathcal{S}_n(r)_{(X,M),(Z,M_Z)}$ on $Y_{\text{ét}}$ in §2.

Let (D_n, M_{D_n}) denote the PD-envelope of the exact closed immersion ι_n compatible with the canonical PD-structure on $p(\mathbb{Z}/p^n\mathbb{Z})$. Since the closed immersion $Y \hookrightarrow X_n$ (resp. $Y \hookrightarrow D_n$) is a nilimmersion, we regard a sheaf on $(X_n)_{\text{ét}}$ (resp. $(D_n)_{\text{ét}}$) as that on $Y_{\text{ét}}$ in the following. Set

$$\omega^q_{Z_n} := \Omega^q_{Z_n/\mathbb{Z}}(\log(M_{Z_n})) \quad (q \geq 0),$$

which is locally free over $\mathcal{O}_{Z_n}$. In §2 we defined the Frobenius "divided by p^r (resp. p^q)"

$$\varphi_r : J^{[r]}_{D_n} \to \mathcal{O}_{D_n} \quad (r \leq p-1) \quad \big(\text{resp. } \varphi_q : \omega^q_{Z_n} \to \omega^q_{Z_n} \quad (q \geq 0)\big).$$

For integers $n \geq 1$ and $r \leq p-1$, the complex $\mathcal{S}_n(r)_{(X,M),\,(Z,M_Z)}$ on $Y_{\text{ét}}$ is the mapping fiber of

$$1 - \varphi_r : J^{[r-\bullet]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^\bullet_{Z_n} \longrightarrow \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^\bullet_{Z_n},$$

where $\varphi_r = \varphi_{r-q} \otimes \varphi_q$ in degree q. We will write $\mathcal{S}_n(r)$ for $\mathcal{S}_n(r)_{(X,M),(Z,M_Z)}$ to simplify the notation in the following.

LEMMA A2 (*cf.* [T2, Lemma 2.4.6]). — *For $a_1, \dots, a_q \in M^{\mathrm{gp}}_{D_{n+1}}$, the image of $\overline{a_1} \otimes \cdots \otimes \overline{a_q}$ in $\mathcal{H}^q(\mathcal{S}_n(q))$ under the symbol map $(M^{\mathrm{gp}}_{n+1})^{\otimes q} \to \mathcal{H}^q(\mathcal{S}_n(q))$ (induced by* (2.5) *and* (2.6)*), is the class of the cocycle*

$$\Big(d\log a_1 \wedge \ldots \wedge d\log a_q,\ \sum_{i=1}^{q} (-1)^{i-1} p^{-1} \log\big(a_i^p \varphi_{D_{n+1}}(a_i)^{-1}\big) d\log a_1 \wedge \ldots \wedge d\log a_{i-1} \wedge \varphi_1(d\log a_{i+1}) \wedge \ldots \wedge \varphi_1(d\log a_q)\Big) \in (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^q_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^{q-1}_{Z_n}),$$

where $\overline{a_i}$ denote the images of a_i in M^{gp}_{n+1}.

Proof. — Straightforward. ▯

Let (U, M_U) denote the scheme $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}]) = \mathrm{Spec}(\mathbb{Z}[T])$ endowed with the log structure associated to the inclusion $\mathbb{N} \to \mathbb{Z}[\mathbb{N}]$ and let (t, M_t) denote the reduction mod (p, T) of (U, M_U).

We have a canonical morphism $(V, M_V) \to (U, M_U)$ defined by $\mathrm{id} : \mathbb{N} \to \mathbb{N}$, and the following two squares are cartesian:

$$\begin{array}{ccc} (Y, M_Y) & \longrightarrow & (Z_n, M_n) \\ \downarrow & & \downarrow \\ (s, N_s) & \longrightarrow & (V, M_V) \\ \downarrow & & \downarrow \\ (t, M_t) & \longrightarrow & (U, M_U). \end{array}$$

We define the sheaf ω_Y^q on $Y_{\text{ét}}$ to be $\Omega^q_{Y/t}(\log(M_Y/M_t))$, and define the subsheaves Z_Y^q and B_Y^q of ω_Y^q by

$$Z_Y^q := \operatorname{Ker}\big(d^q\colon \omega_Y^q \to \omega_Y^{q+1}\big),\quad B_Y^q := \operatorname{Image}\big(d^{q-1}\colon \omega_Y^{q-1} \to \omega_Y^q\big).$$

Since we assume that $(Y, M_Y) \to (s, N_s)$ is of Cartier type, we have Cartier isomorphisms as follows:

THEOREM A3 (*cf.* [Ka2, Thm. 4.12]). — *For each integer $q \geq 0$, there exists an isomorphism*

$$C^{-1}\colon \omega_Y^q \xrightarrow{\sim} \mathcal{H}^q(\omega_Y^\bullet)$$

characterized by

$$\begin{aligned} C^{-1}\big(a d\log(b_1) \wedge d\log(b_2) \wedge \ldots \wedge d\log(b_q)\big) \\ = \text{the class of } a^p d\log(b_1) \wedge d\log(b_2) \wedge \ldots \wedge d\log(b_q) \\ (a \in \mathcal{O}_Y \text{ and } b_1, b_2, \ldots, b_q \in M_Y). \end{aligned}$$

Proof. — By the characterization, the question is étale local on Y. Hence we may assume $(Y, M_Y) = (\widetilde{Y}, M_{\widetilde{Y}}) \times_{(t,M_t)} (s, N_s)$ for a fine log affine scheme $(\widetilde{Y}, M_{\widetilde{Y}})$ smooth of Cartier type over (t, M_t). (Use [Ka2, Thm. 3.5].) For a finitely generated subfield k' of k, let $(s', N_{s'})$ denote Spec(k') with the inverse image of M_t and set

$$(Y', M_{Y'}) := (\widetilde{Y}, M_{\widetilde{Y}}) \times_{(t,M_t)} (s', N_{s'}).$$

Then the theorem is true for $(Y', M_{Y'})$ by [Ka2, Thm. 4.12]. Since $\Gamma(Y, \omega_Y^\bullet) = \varinjlim_{k'} \Gamma(Y', \omega_{Y'}^\bullet)$, this implies the claim for (Y, M_Y). Note that, if we regard $\omega_{Y'}^q$ as quasi-coherent $\mathcal{O}_{Y'}$-modules by the action $x \mapsto a^p x$ ($a \in \mathcal{O}_{Y'}$), $\omega_{Y'}^\bullet$ is a complex of $\mathcal{O}_{Y'}$-modules and C^{-1} becomes $\mathcal{O}_{Y'}$-linear. □

We define the sheaf $\omega^q_{Y,\log}$ to be the subsheaf of abelian groups of ω_Y^q which is generated by local sections of the form

$$d\log(b_1) \wedge d\log(b_2) \wedge \ldots \wedge d\log(b_q) \quad (b_1, b_2, \ldots, b_q \in M_Y).$$

THEOREM A4 (*cf.* [T1, Thm. 6.1.1]). — *Let the notation and the assumption be as above. Then, for each integer $q \geq 0$, the following sequence is exact:*

$$0 \to \omega^q_{Y,\log} \longrightarrow Z_Y^q \xrightarrow{1-C^{-1}} \mathcal{H}^q(\omega_Y^\bullet) \to 0.$$

Proof. — The claim is trivial except the exactness in the middle. (The surjectivity in the right is reduced to that of $1 - C^{-1}\colon \omega_Y^q \to \omega_Y^q/Z_Y^q$. Note that we work on $Y_{\text{ét}}$.) We are reduced to the special situation as in the proof of Theorem A3. Suppose $(1-C^{-1})(x)=0$ for an affine étale scheme U over Y and $x \in \Gamma(U, Z^q_Y)$. Then, with the notation in the proof of Theorem A3, there exist k', an affine étale scheme U' over Y' with $U \cong U' \times_{Y'} Y$ and $x' \in \Gamma(U', Z^q_{Y'})$ such that the image of x' in $\Gamma(U, Z^q_Y)$ is x and $(1-C^{-1})(x')=0$. Since the claim is true for $(Y', M_{Y'})$ by [T1, Thm. 6.1.1], $x' \in \Gamma(U', \omega^q_{Y',\log})$ and hence $x \in \Gamma(U, \omega^q_{Y,\log})$. □

Now let us give the statement of the main results.

Following [Ku], we define the descending filtration $\widetilde{U}^m$ $(m \in \mathbb{N})$ on $\mathcal{S}_1(r)$ for an integer $0 \le r \le p-2$ as follows: First we define the filtration $\widetilde{U}^m$ $(m \in \mathbb{N})$ on $\mathcal{O}_{D_1}$ (resp. $J^{[r]}_{D_1}$ $(r \le p-2)$) by

$$T^m \mathcal{O}_{D_1} + J^{[p]}_{D_1} \quad \big(\text{resp. } T^{\max\{m, er+\lceil m/p \rceil\}} \mathcal{O}_{D_1} + J^{[p]}_{D_1}\big).$$

Note $J^{[r]}_{D_1} = T^{er}\mathcal{O}_{D_1} + J^{[p]}_{D_1}$ $(0 \le r \le p-2)$. Here e denotes the absolute ramification index of $\mathcal{O}_K$ and, for $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer $\ge x$. We see easily that the sheaves $\widetilde{U}^m(\mathcal{O}_{D_1}) \otimes \omega^q_{Z_1}$ (resp. $\widetilde{U}^m(J^{[r-q]}_{D_1}) \otimes \omega^q_{Z_1}$) are compatible with the differentials and give a filtration $\widetilde{U}^m$ $(m \in \mathbb{N})$ on the complex $\mathcal{O}_{D_1} \otimes \omega^\bullet_{Z_1}$ (resp. $J^{[r-\bullet]}_{D_1} \otimes \omega^\bullet_{Z_1}$). Finally, for an integer $0 \le r \le p-2$, it is easy to see that the morphisms $1, \varphi_r : J^{[r-\bullet]}_{D_1} \otimes \omega^\bullet_{Z_1} \longrightarrow \mathcal{O}_{D_1} \otimes \omega^\bullet_{Z_1}$ are compatible with $\widetilde{U}^\bullet$. Thus we obtain the filtration $\widetilde{U}^\bullet$ on the complex $\mathcal{S}_1(r)$.

PROPOSITION A5 (*cf.* [Ku, Prop. 4.3], [T1, Prop. 7.3.5]). — *Let the notation and the assumption be as above. Let r and q be any integers such that $0 \le q \le r \le p-2$. Then, for each integer $m \ge 0$, the sheaf $\mathcal{H}^q(\operatorname{gr}^m_{\widetilde{U}} \mathcal{S}_1(r))$ has the following structure*:

1) *If $m < ep(r-q)/(p-1)$ or $m \ge ep(r-q+1)/(p-1)$, then*

$$\mathcal{H}^q\big(\operatorname{gr}^m_{\widetilde{U}} \mathcal{S}_1(r)\big) = 0.$$

2) *If $m = ep(r-q)/(p-1)$, then there exists an exact sequence*

$$\begin{aligned} 0 \to \operatorname{Ker}\big(1 - a_0^{p(r-q)} C^{-1} &: Z^{q-1}_Y \to \mathcal{H}^{q-1}(\omega^\bullet_Y)\big) \\ &\longrightarrow \mathcal{H}^q\big(\operatorname{gr}^m_{\widetilde{U}} \mathcal{S}_1(r)\big) \\ &\qquad \longrightarrow \operatorname{Ker}\big(1 - a_0^{p(r-q)} C^{-1} : Z^q_Y \to \mathcal{H}^q(\omega^\bullet_Y)\big) \to 0, \end{aligned}$$

where $a_0 = \big(N_{\mathcal{O}_{\widehat{K}}/W}(-\pi) \cdot p^{-1} \bmod p\big) \in k^*$.

3) *Suppose* $ep(r-q)/(p-1) < m < ep(r-q+1)/(p-1)$. *Then*:

3-1) *If* $p \nmid m$, *there exists an exact sequence*

$$0 \to \omega_Y^{q-2}/Z_Y^{q-2} \longrightarrow \mathcal{H}^q\big(\mathrm{gr}_{\widetilde{U}}^m \mathcal{S}_1(r)\big) \longrightarrow \omega_Y^{q-1}/B_Y^{q-1} \to 0.$$

3-2) *If* $p \mid m$, *there exists an exact sequence*

$$0 \to \omega_Y^{q-2}/Z_Y^{q-2} \longrightarrow \mathcal{H}^q\big(\mathrm{gr}_{\widetilde{U}}^m \mathcal{S}_1(r)\big) \longrightarrow \omega_Y^{q-1}/Z_Y^{q-1} \to 0.$$

PROPOSITION A6 (*cf.* [Ku, Lemma 4.2], [T1, Prop. 7.3.4]). — *Let r and q be integers such that $0 \le q \le r \le p-2$. Then, for $m \in \mathbb{N}$, the following sequence is exact*:

$$0 \to \mathcal{H}^q\big(\widetilde{U}^{m+1}\mathcal{S}_1(r)\big) \longrightarrow \mathcal{H}^q\big(\widetilde{U}^{m}\mathcal{S}_1(r)\big) \longrightarrow \mathcal{H}^q\big(\mathrm{gr}_{\widetilde{U}}^m \mathcal{S}_1(r)\big) \to 0.$$

Furthermore, $\mathcal{H}^q(\widetilde{U}^{ep}\mathcal{S}_1(r)) = 0$.

We first give a proof of Proposition A6 assuming Proposition A5.

Proof. — For the first claim, it suffices to prove the surjectivity in the right. By Proposition A5, we may assume $ep(r-q)/(p-1) \le m < ep(r-q+1)/(p-1)$ and it is enough to prove $\mathcal{H}^{q+1}(\widetilde{U}^{m+1}\mathcal{S}_1(r)) = 0$ in this case. We will prove

$$\mathcal{H}^{q+1}\big(\widetilde{U}^{m}\mathcal{S}_1(r)\big) = 0 \quad \text{for} \quad m > ep(r-q)/(p-1).$$

For $m > ep(r-q)/(p-1)$ ($\Leftrightarrow m > e(r-q) + m/p$), the complex $\widetilde{U}^m J_{D_1}^{[r-\bullet]} \otimes \omega_{Z_1}^\bullet$ coincides with $\widetilde{U}^m \mathcal{O}_{D_1} \otimes \omega_{Z_1}^\bullet$ in degree $\ge q$ and

$$\varphi_{r-q'}(\widetilde{U}^{m'}\mathcal{O}_{D_1}) \subset \widetilde{U}^{m'+1}\mathcal{O}_{D_1}, \quad \varphi_{r-q'}(\widetilde{U}^{ep}\mathcal{O}_{D_1}) = 0$$

for $m' > ep(r-q)/(p-1)$ and $q' \ge q$. Hence

$$1 - \varphi_r : \widetilde{U}^m J_{D_1}^{[r-\bullet]} \otimes \omega_{Z_1}^\bullet \longrightarrow \widetilde{U}^m \mathcal{O}_{D_1} \otimes \omega_{Z_1}^\bullet$$

is an isomorphism in degree $\ge q$, which implies $\mathcal{H}^{q+1}(\widetilde{U}^m\mathcal{S}_1(r)) = 0$. We see the second claim similarly (and more easily). ∎

To prove Proposition A5, we will prepare for several lemmas. For a complex $K^\bullet$, we denote by $B^q(K^\bullet)$ (resp. $Z^q(K^\bullet)$) the image (resp. kernel) of $d^{q-1}: K^{q-1} \to K^q$ (resp. $d^q: K^q \to K^{q+1}$). Note that the subsheaves $T^m\omega_{Z_1}^q$ (resp. $T^m\omega_{Z_1/U_1}^q$) are compatible with the differentials and give a subcomplex $T^m\omega_{Z_1}^\bullet$ (resp. $T^m\omega_{Z_1/U_1}^\bullet$) of $\omega_{Z_1}^\bullet$ (resp. $\omega_{Z_1/U_1}^\bullet$).

LEMMA A7 (*cf.* [T1, Lemma 7.4.3]). — *Let m be a non-negative integer.*

1) *From the reduction mod T of the short exact sequence*

$$0 \to \omega^{q-1}_{Z_1/U_1} \xrightarrow{\wedge d\log T} \omega^q_{Z_1} \longrightarrow \omega^q_{Z_1/U_1} \to 0$$

and the $\mathcal{O}_{Z_1}$-linear isomorphism

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{Z_1}} \omega^q_{Z_1} \xrightarrow{\sim} T^m\omega^q_{Z_1}/T^{m+1}\omega^q_{Z_1}$$

induced by the multiplication by T^m on $\omega^q_{Z_1}$ for each integer $q \geq 0$, we obtain a short exact sequence of complexes:

$$\text{(A7.1)} \qquad 0 \to \omega^\bullet_Y[-1] \longrightarrow T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1} \longrightarrow \omega^\bullet_Y \to 0.$$

Furthermore, for each integer $q \geq 0$, the connecting homomorphism $\mathcal{H}^q(\omega^\bullet_Y) \to \mathcal{H}^q(\omega^\bullet_Y)$ of the long exact sequence associated to (A7.1) *is the multiplication by $(-1)^q m$.*

2) *The homomorphism $\varphi_q\colon \omega^q_{Z_1} \to \omega^q_{Z_1}$ induces an isomorphism:*

$$C^{-1}\colon \omega^q_{Z_1}/T\omega^q_{Z_1} \xrightarrow{\sim} \mathcal{H}^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}).$$

3) *If $p \nmid m$, $\mathcal{H}^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1}) = 0$.*

4) *If $p \mid m$, the multiplication by T^m on $\omega^q_{Z_1}$ for $q \geq 0$ induces an isomorphism of complexes*

$$\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1} \xrightarrow{\sim} T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1}.$$

Proof. — We can prove this in the same way as [T1, Lemma 7.4.3] using Theorem A3 and an analogue of [T1, Lemma 7.1.4] for (Z_n, M_{Z_n}) $(n \geq 1)$ and φ_q. □

LEMMA A8 (*cf.* [T1, Lemma 7.4.6], [T2, Lemma 2.4.5]). — *Let m be a non-negative integer.*

1) *If $p \nmid m$, there is a short exact sequence*

$$0 \to \omega^{q-2}_Y/Z^{q-2}_Y \longrightarrow B^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1}) \longrightarrow \omega^{q-1}_Y/B^{q-1}_Y \longrightarrow 0$$

which is characterized by the following properties: For $x \in \mathcal{O}_{Z_1}$ and $a_1, \ldots, a_{q-1}$ in $M^{\mathrm{gp}}_{Z_1}$, the image of

$$d(T^m x\cdot d\log(a_1)\wedge\ldots\wedge d\log(a_{q-1}) \bmod T^{m+1})\in B^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1})$$

in ω_Y^{q-1}/B_Y^{q-1} *is* $\bar{x}d\log(\overline{a_1})\wedge\ldots\wedge d\log(\overline{a_{q-1}})$, *and*

$$d(T^m x\cdot d\log(a_1)\wedge\ldots\wedge d\log(a_{q-2})\wedge d\log(T) \bmod T^{m+1})\in B^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1})$$

is the image of $\bar{x}d\log(\overline{a_1})\wedge\ldots\wedge d\log(\overline{a_{q-2}})\in\omega_Y^{q-2}/Z_Y^{q-2}$, *where* $\overline{a_i}$ *denote the images of* a_i *in* M_Y^{gp} *and* $\bar{x}$ *denotes the image of* x *in* $\mathcal{O}_Y$.

2) *If* $p\mid m$, *there is a short exact sequence*

$$0\to\omega_Y^{q-2}/Z_Y^{q-2}\longrightarrow B^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1})\longrightarrow\omega_Y^{q-1}/Z_Y^{q-1}\to 0$$

which is characterized in the same way as 1).

3) *For* $a\in k^*$, *the homomorphism*

$$1-a^p\cdot C^{-1}\colon Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1})\longrightarrow\mathcal{H}^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1})$$

is surjective, where C^{-1} *is as in Lemma* A7, 2). *Its kernel* $\mathcal{K}$ *is the subsheaf of abelian groups of* $Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1})$ *generated by local sections of the form*

$$x\cdot(d\log(a_1)\wedge\ldots\wedge d\log(a_q) \bmod T),\qquad (x\in\mathrm{Ker}\,(1-a^p\varphi\colon\mathcal{O}_Y\to\mathcal{O}_Y),\ a_1,\ldots,a_q\in M^{\mathrm{gp}}_{Z_1})$$

and there exists a short exact sequence

$$0\to\mathrm{Ker}\left(1-a^pC^{-1}\colon Z_Y^{q-1}\to\mathcal{H}^{q-1}(\omega_Y^\bullet)\right)\longrightarrow\mathcal{K}\longrightarrow\mathrm{Ker}\left(1-a^pC^{-1}\colon Z_Y^q\to\mathcal{H}^q(\omega_Y^\bullet)\right)\to 0$$

which is characterized by the following properties: *For* $a_1,\ldots,a_q\in M^{\mathrm{gp}}_{Z_1}$ *and* $x\in\mathrm{Ker}\,(1-a^p\varphi\colon\mathcal{O}_Y\to\mathcal{O}_Y)$, *the image of*

$$x\cdot\big(d\log(a_1)\wedge\ldots\wedge d\log(a_q) \bmod T\big)\in\mathcal{K}$$

in the right term is $x\cdot d\log(\overline{a_1})\wedge\ldots\wedge d\log(\overline{a_q})$, *and*

$$x\cdot\big(d\log(a_1)\wedge\ldots\wedge d\log(a_{q-1})\wedge d\log(T) \bmod T\big)\in\mathcal{K}$$

is the image of the section $x\cdot d\log(\overline{a_1})\wedge\ldots\wedge d\log(\overline{a_{q-1}})$ *of the left term, where* $\overline{a_i}$ *denote the images of* a_i *in* M_Y^{gp}.

Proof. — We can prove this in the same way as [T2, Lemma 2.4.5] using Lemma A7 and Theorem A4. Note $d(x^p)=px^{p-1}dx=0$ in Ω^1_k for $x\in k$ and that 3) is reduced to the case $a=1$ by the commutative diagrams

$$\begin{array}{ccc} Z_Y^q & \xrightarrow{1-a^pC^{-1}} & \mathcal{H}^q(\omega_Y^\bullet) \\ {\scriptstyle b^{-p}}\uparrow\wr & & \wr\uparrow{\scriptstyle b^{-p}} \\ Z_Y^q & \xrightarrow{1-C^{-1}} & \mathcal{H}^q(\omega_Y^\bullet), \end{array}\qquad \begin{array}{ccc} Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}) & \xrightarrow{1-a^pC^{-1}} & \mathcal{H}^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}) \\ {\scriptstyle b^{-p}}\uparrow\wr & & \wr\uparrow{\scriptstyle b^{-p}} \\ Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}) & \xrightarrow{1-C^{-1}} & \mathcal{H}^q(\omega^\bullet/T\omega^\bullet_{Z_1}), \end{array}$$

where $b=a^{1/(p-1)}$ which exists étale locally on Spec(k). □

LEMMA A9 (*cf.* [T1, Lemma 7.4.4]). — *Let q and r be integers such that $0 \leq q \leq r \leq p-2$. Then the homomorphism*

$$\mathcal{H}^q(\mathrm{gr}^m_{\widetilde{U}}(1) - \mathrm{gr}^m_{\widetilde{U}}(\varphi_r)) \colon \mathcal{H}^q(\mathrm{gr}^m_{\widetilde{U}}(J^{[r-\bullet]}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^\bullet_{Z_1})) \longrightarrow \mathcal{H}^q(\mathrm{gr}^m_{\widetilde{U}}(\mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^\bullet_{Z_1}))$$

is surjective and its kernel is as follows:

1) *If $m < ep(r-q)/(p-1)$ or $m \geq ep(r-q+1)/(p-1)$, then the kernel vanishes.*

2) *If $m = ep(r-q)/(p-1)$, then the kernel is isomorphic to the kernel of*

$$1 - a_0^{p(r-q)} \cdot C^{-1} \colon Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}) \longrightarrow \mathcal{H}^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}),$$

where C^{-1} is as in Lemma A7, 2) *and $a_0 = (N_{O_{\widehat{K}}/W}(-\pi) \cdot p^{-1} \bmod p) \in k^*$.*

3) *If $ep(r-q)/(p-1) < m < ep(r-q+1)/(p-1)$, then the kernel is isomorphic to $B^q(T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1})$.*

Proof. — First note

$$\mathrm{gr}^m_{\widetilde{U}}(\mathcal{O}_{D_1} \otimes_{\mathcal{O}_{Z_1}} \omega^\bullet_{Z_1}) \xleftarrow{\sim} T^m\omega^\bullet_{Z_1}/T^{m+1}\omega^\bullet_{Z_1}$$

and $m \gtreqless e(r-q) + m/p \Leftrightarrow m \gtreqless ep(r-q)/(p-1)$.

If $m \geq ep(r-(q-1))/(p-1)$, $\widetilde{U}^m J^{[r-\bullet]}_{D_1} \otimes \omega^\bullet_{Z_1}$ coincides with $\widetilde{U}^m\mathcal{O}_{D_1} \otimes \omega^\bullet_{Z_1}$ in degree $\geq q-1$ and

$$\varphi_{r-q}(\widetilde{U}^m J^{[r-q]}_{D_1}) \subset \widetilde{U}^{m+1}\mathcal{O}_{D_1}.$$

Hence the morphism in question is the identity.

If $ep(r-q)/(p-1) \leq m < ep(r-(q-1))/(p-1)$, $\widetilde{U}^m J^{[r-\bullet]}_{D_1} \otimes \omega^\bullet_{Z_1}$ coincides with $\widetilde{U}^m\mathcal{O}_{D_1} \otimes \omega^\bullet_{Z_1}$ in degree $\geq q$ and

$$d^{q-1}(\widetilde{U}^m J^{[r-q+1]}_{D_1} \otimes \omega^{q-1}_{Z_1}) \subset \widetilde{U}^{m+1} J^{[r-q]}_{D_1} \otimes \omega^q_{Z_1}.$$

In the case $m > ep(r-q)/(p-1)$, we have

$$\varphi_{r-q}(\widetilde{U}^m J^{[r-q]}_{D_1}) \subset \widetilde{U}^{m+1}\mathcal{O}_{D_1}$$

and obtain 3).

If $m = ep(r-q)/(p-1)$, we obtain 2) from Lemma A7, 4), Lemma A8, 3) and

$$\varphi_{r-q}(T^{e(r-q)}) = (a_0^p + a_1^p T^p + \cdots + a_{e-1}^p T^{(e-1)p} + (T^e)^{[p]})^{r-q},$$

where $T^e + p(a_{e-1}T^{e-1} + \cdots + a_1 T + a_0)$ $(a_i \in W)$ denotes the Eisenstein polynomial of π over W.

If $m < ep(r-q)/(p-1)$ and $p \nmid m$, both sides of the homomorphism in question vanish by Lemma A7, 3).

If $m < ep(r-q)/(p-1)$ and $p \mid m$, the claim 1) is reduced to Lemma A7, 2) using Lemma A7, 4) and the above description of $\varphi_{r-q}(T^{e(r-q)})$. □

Now we immediately obtain Proposition A5 combining Lemma A9 with Lemma A8.

For integers $0 \le q \le r \le p-2$, we define the filtration $\widetilde{U}^m$ $(m \in \mathbb{N})$ on $\mathcal{H}^q(\mathcal{S}_1(r))$ to be the image of $\mathcal{H}^q(\widetilde{U}^m\mathcal{S}_1(r))$. By Proposition A6, we have

$$\text{(A10)} \qquad \operatorname{gr}^m_{\widetilde{U}} \mathcal{H}^q\big(\mathcal{S}_1(r)\big) \cong \mathcal{H}^q\big(\operatorname{gr}^m_{\widetilde{U}} \mathcal{S}_1(r)\big) \quad (m \ge 0),$$

$$\text{(A11)} \qquad \widetilde{U}^{ep}\mathcal{H}^q\big(\mathcal{S}_1(r)\big) = 0.$$

For $x \in \widetilde{U}^m\mathcal{H}^q(\mathcal{S}_1(r))$ and $x' \in \widetilde{U}^{m'}\mathcal{H}^{q'}(\mathcal{S}_1(r'))$ (where $m, m', q, q' \ge 0$ and $0 \le r, r', r+r' \le p-2$) the product $x{\cdot}x'$ is contained in $\widetilde{U}^{m+m'}\mathcal{H}^{q+q'}(\mathcal{S}_1(r{+}r'))$.

Assume that K contains a primitive p-th root of unity. Then it is easy to see $a_0 \in (k^*)^{p-1}$. (See the proof of Proposition A17 below.) Choose a $(p-1)$-th root $b_0 \in k$ of a_0. Then, by Theorem A4, for integers $q \ge 0$ and $u \ge 0$, we have an isomorphism

$$\text{(A12)} \qquad \omega^q_{Y,\log} \xrightarrow{\sim} \operatorname{Ker}\big(1 - a_0^{pu}C^{-1}\colon Z^q_Y \to \mathcal{H}^q(\omega^\bullet_Y)\big), \quad \omega \longmapsto {b_0}^{-pu} \cdot \omega.$$

By Proposition A5 and (A10), for each integer $0 \le r \le p-2$, we have an isomorphism

$$(A13) \quad \mathcal{H}^0\big(\mathcal{S}_1(r)\big) \xleftarrow{\sim} \widetilde{U}^{epr/(p-1)}(\mathcal{H}^0(\mathcal{S}_1(r))) \xrightarrow{\sim} \operatorname{gr}^{epr/(p-1)}_{\widetilde{U}}(\mathcal{H}^0(\mathcal{S}_1(r))) \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}.$$

In the last isomorphism, we use (A12) with $q = 0$, $u = r$.

Define the descending filtrations U^m and V^m $(m \in \mathbb{N})$ on $(M_2^{\mathrm{gp}})^{\otimes q}$ as follows:

For $q = 0$, $U^0 = \mathbb{Z}$ and $V^m = U^{m+1} = 0$ for $m \ge 0$.

For $q = 1$, $U^0(M_2^{\mathrm{gp}}) = M_2^{\mathrm{gp}}$, $V^0(M_2^{\mathrm{gp}}) = (1 + \pi\mathcal{O}_{X_2}) \cdot \pi^{\mathbb{N}}$, $U^m(M_2^{\mathrm{gp}}) = 1 + \pi^m\mathcal{O}_{X_2}$ for $m \ge 1$, and $V^m(M_2^{\mathrm{gp}}) = U^{m+1}M_2^{\mathrm{gp}}$ for $m \ge 1$.

For $q \ge 2$,

$$U^m(M_2^{\mathrm{gp}})^{\otimes q} = \big(\text{image of } U^m(M_2^{\mathrm{gp}})\big) \otimes (M_2^{\mathrm{gp}})^{\otimes(q-1)},$$
$$V^m(M_2^{\mathrm{gp}})^{\otimes q} = U^{m+1}(M_2^{\mathrm{gp}})^{\otimes q} + \big(\text{image of } U^m(M_2^{\mathrm{gp}})\big) \otimes (M_2^{\mathrm{gp}})^{\otimes(q-2)} \otimes (\pi^{\mathbb{N}}).$$

(See [H, 1.4]). Here and hereafter, we denote by the same letter π the images of $\pi \in \Gamma(S, N) = O_K\backslash\{0\}$ in $\Gamma(X, M)$ and $\Gamma(X_n, M_n)$ $(n \ge 1)$.

As in [T2, Lemma 2.5.2], we see that the image of $U^m(M_2^{\mathrm{gp}})^{\otimes q}$ under the symbol map is contained in $\widetilde{U}^m\mathcal{H}^q(\mathcal{S}_1(q))$. Hence, for integers r and q such that

$0 \le q \le r \le p-2$, and for integers $m \ge 0$, using the symbol map and (A13), we obtain a map

$$\text{(A14)}\quad U^m(M_2^{\text{gp}})^{\otimes q} \longrightarrow \mathbb{Z}/p\mathbb{Z} \otimes U^m(M_2^{\text{gp}})^{\otimes q} \longrightarrow \widetilde{U}^{ep(r-q)/(p-1)}\big(\mathcal{H}^0(\mathcal{S}_1(r-q))\big) \otimes \widetilde{U}^m\big(\mathcal{H}^q(\mathcal{S}_1(q))\big) \longrightarrow \widetilde{U}^{ep(r-q)/(p-1)+m}\big(\mathcal{H}^q(\mathcal{S}_1(r))\big).$$

PROPOSITION A15 (*cf.* [T1, Lemma 8.4.4], [T2, Prop. 2.4.1]). — *Let the notation and the assumption be as above. Let q and r be integers such that $0 \le q \le r \le p-2$. Then, for every integer $m \ge 0$, the homomorphism* (A14) *is surjective and it is related to the description in Proposition* A5 *as follows.*

1) *If $m = 0$, by* (A12) *and* (A10), *we get an exact sequence*:

$$0 \to \omega_{Y,\log}^{q-1} \longrightarrow \operatorname{gr}_{\widetilde{U}}^{ep(r-q)/(p-1)}\big(\mathcal{H}^q(\mathcal{S}_1(r))\big) \longrightarrow \omega_{Y,\log}^{q} \to 0.$$

The map (A14) *induces the following surjective maps*:

$$U^0(M_2^{\text{gp}})^{\otimes q} \longrightarrow \omega_{Y,\log}^{q}, \quad a_1 \otimes \cdots \otimes a_q \longmapsto d\log(\overline{a_1}) \wedge \ldots \wedge d\log(\overline{a_q}),$$
$$V^1(M_2^{\text{gp}})^{\otimes q} \longrightarrow \omega_{Y,\log}^{q-1}, \quad a_1 \otimes \cdots \otimes a_{q-1} \otimes \pi \longmapsto d\log(\overline{a_1}) \wedge \ldots \wedge d\log(\overline{a_{q-1}}),$$

where $\overline{a_i}$ denote the images of a_i in M_Y^{gp}.

2) *Suppose $1 \le m < ep/(p-1)$. If $p \nmid m$ (resp. $p \mid m$), by* (A10), *we get an exact sequence*:

$$0 \to \omega_Y^{q-2}/Z_Y^{q-2} \longrightarrow \operatorname{gr}_{\widetilde{U}}^{ep(r-q)/(p-1)+m}\big(\mathcal{H}^q(\mathcal{S}_1(r))\big) \longrightarrow \omega_Y^{q-1}/B_Y^{q-1} \longrightarrow 0$$

$$\big(\textit{resp.}\ 0 \longrightarrow \omega_Y^{q-2}/Z_Y^{q-2} \longrightarrow \operatorname{gr}_{\widetilde{U}}^{ep(r-q)/(p-1)+m}(\mathcal{H}^q(\mathcal{S}_1(r))) \longrightarrow \omega_Y^{q-1}/Z_Y^{q-1} \to 0\big).$$

The map (A14) *induces the following surjective maps*:

$$U^m(M_2^{\text{gp}})^{\otimes q} \longrightarrow \omega_Y^{q-1}/B_Y^{q-1} \quad (\textit{resp.}\ \omega_Y^{q-1}/Z_Y^{q-1}),$$
$$(1+\pi^m x) \otimes a_1 \otimes \cdots \otimes a_{q-1} \longmapsto \big(b_0^{-p(r-q)}\bar{x}\, d\log(\overline{a_1}) \wedge \ldots \wedge d\log(\overline{a_{q-1}})\big);$$
$$V^m(M_2^{\text{gp}})^{\otimes q} \longrightarrow \omega_Y^{q-2}/B_Y^{q-2},$$
$$(1+\pi^m x) \otimes a_1 \otimes \cdots \otimes a_{q-2} \otimes \pi \longmapsto \big(b_0^{-p(r-q)}\bar{x}\, d\log(\overline{a_1}) \wedge \ldots \wedge d\log(\overline{a_{q-2}})\big),$$

where $\overline{a_i}$ denote the images of a_i in M_Y^{gp} and $\bar{x}$ denotes the image of $x \in \mathcal{O}_{X_2}$ in $\mathcal{O}_Y$.

Proof. — Set $m_0 := ep(r-q)/(p-1)$ and denote by c the image of $1 \in \mathbb{Z}/p\mathbb{Z}$ in

$$\widetilde{U}^{m_0}\mathcal{H}^0(\mathcal{S}_1(r-q)) = \big(\widetilde{U}^{m_0}J_{D_1}^{[r-q]}\big)^{\varphi_{r-q}=1,\nabla=0}$$

under (A13). Then $c \equiv T^{m_0}b_0^{-p(r-q)} \pmod{\widetilde{U}^{m_0+1}J_{D_1}^{[r-q]}}$. By Lemma A2, the image of $a_1\otimes\cdots\otimes a_q$ (resp. $a_1\otimes\cdots\otimes a_{q-1}\otimes\pi$) by (A14) is the class of a cocycle of the form

$$(c\cdot d\log\tilde{a}_1\wedge\ldots\wedge d\log\tilde{a}_q, *)\quad\big(\text{resp. }(c\cdot d\log\tilde{a}_1\wedge\ldots\wedge d\log\tilde{a}_{q-1}\wedge d\log T, *)\big),$$

where $\tilde{a}_i$ denote liftings of a_i in M_{Z_2}. Its image in

$$\mathcal{H}^q(\operatorname{gr}^{m_0}_{\widetilde{U}}\mathcal{S}_1(r)) \cong \operatorname{Ker}\big(1-a_0^{p(r-q)}C^{-1}: Z^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1}) \to \mathcal{H}^q(\omega^\bullet_{Z_1}/T\omega^\bullet_{Z_1})\big)$$

(Lemma A9, 2) is

$$b_0^{-p(r-q)}d\log\tilde{a}_1\wedge\ldots\wedge d\log\tilde{a}_q \bmod T$$
$$(\text{resp. } b_0^{-p(r-q)}d\log\tilde{a}_1\wedge\ldots\wedge d\log\tilde{a}_{q-1}\wedge d\log T \bmod T).$$

Now the claim 1) follows from Lemma A8, 3). If $a_1 = 1+\pi^m x$ for $x\in\mathcal{O}_{X_2}$ and $\tilde{a}_1 = 1+T^m\tilde{x}$ for a lifting $\tilde{x}\in\mathcal{O}_{Z_2}$ of x, then the image in

$$\mathcal{H}^q(\operatorname{gr}^{m_0+m}_{\widetilde{U}}\mathcal{S}_1(r)) \cong B^q(T^{m_0+m}\omega^\bullet_{Z_1}/T^{m_0+m+1}\omega^\bullet_{Z_1})$$

(Lemma A9, 3) is

$$d(T^{m_0+m}b_0^{-p(r-q)}\tilde{x}d\log\tilde{a}_2\wedge\ldots\wedge d\log\tilde{a}_q \bmod T^{m_0+m+1})$$
$$(\text{resp. } d(T^{m_0+m}b_0^{-p(r-q)}\tilde{x}d\log\tilde{a}_2\wedge\ldots\wedge d\log\tilde{a}_{q-1}\wedge d\log T \bmod T^{m_0+m+1})).$$

Hence the claim 2) follows from Lemma A8, 1), 2). □

Corollary A16. — *If K contains a primitive p-th root of unity, for any integers q and r such that $0\le q\le r\le p-2$, the homomorphism*

$$\mathcal{H}^0(\mathcal{S}_1(r-q))\otimes\mathcal{H}^q(\mathcal{S}_1(q)) \longrightarrow \mathcal{H}^q(\mathcal{S}_1(r))$$

induced by the product structure is an isomorphism.

We still assume that K contains a primitive p-th root of unity and denote by $\mathbb{Z}/p\mathbb{Z}(r)_Y$ the constant sheaf on $Y_{\text{ét}}$ associated to $\mu_p(K)^{\otimes r}$.

PROPOSITION A17. — *If K contains a primitive p-th root of unity, for any integer $0 \le r \le p-2$, the homomorphism $\mathbb{Z}/p\mathbb{Z}(r)_Y \to \mathcal{H}^0(\mathcal{S}_1(r))$ induced by* (2.7) *and* (2.5) *is an isomorphism.*

Proof. — Choose a primitive p-th root $\zeta_p \in K$ of unity and put

$$u := N_{\widehat{K}/K_0(\zeta_p)}(-\pi)\cdot(1-\zeta_p)^{-1} \in W[\zeta_p]^*,$$

where K_0 denotes the field of fractions of W. Then, since $N_{K_0(\zeta_p)/K_0}(1-\zeta_p) = p$, we have $a_0 = N_{K_0(\zeta_p)/K_0}(u) \bmod p$. We choose $u \bmod (1-\zeta_p)$ as b_0. We assert that the image of $\zeta_p^{\otimes r}$ in $\mathcal{H}^0(\mathcal{S}_1(r))$ coincides with the image of $1 \in \mathbb{Z}/p\mathbb{Z}$ under (A13). Set

$$v := N_{\widehat{K}/K_0(\zeta_p)}(-\pi)\cdot \pi^{-e/(p-1)} \in O_K^*,$$

where $e = [K:K_0]$. We have

$$\zeta_p = 1 - u^{-1}v\cdot\pi^{e/(p-1)}.$$

If we choose a lifting w of $-u^{-1}v$ in $k[T]$, we obtain a lifting $1+w\cdot T^{e/(p-1)} \in \mathcal{O}_{Z_1}$ of ζ_p, and the image of ζ_p in $\mathcal{H}^0(\mathcal{S}_1(1))$ is

$$\log\big((1+w\cdot T^{e/(p-1)})^p\big) = \sum_{i\ge 1}(-1)^{i-1}(i-1)!\,\big(w^p\cdot T^{ep}/(p-1)\big)^{[i]} \in \widetilde{U}^{pe/(p-1)}J_{D_1} = \widetilde{U}^{pe/(p-1)}\mathcal{O}_{D_1}.$$

By looking at the Eisenstein polynomial of π over $K_0(\zeta_p)$, we see $v \equiv -1 \bmod \pi$ and hence $w \bmod T \equiv b_0^{-1}$. Hence the above element is congruent to $b_0^{-p}T^{ep/(p-1)}$ modulo $\widetilde{U}^{pe/(p-1)+1}\mathcal{O}_{D_1}$. Now the assertion follows from the remark in the beginning of the proof of Proposition A15. □

Again, we consider a general K, that is, we don't assume that K contains a primitive p-th root of unity. Let $O_{K'}$ be a totally ramified extension of O_K of degree d. Let (S',N') denote the scheme $\mathrm{Spec}(O_{K'})$ with the log structure defined by the closed point. Assume that there exists a prime π' of $O_{K'}$ such that $(\pi')^d = \pi$. Choose such a prime π'. Let $(V',M_{V'})$ denote the scheme $\mathrm{Spec}(W[\mathbb{N}]) = \mathrm{Spec}(W[T'])$ endowed with the log structure associated to the inclusion $\mathbb{N} \to W[\mathbb{N}]$ and define the exact closed immersion $i_{V'_n}:(S'_n,N'_n) \to (V'_n,M_{V'_n})$ in the same way as i_{V_n}, using π'. We have a cartesian diagram

$$\begin{array}{ccc} (S'_n,N'_n) & \longrightarrow & (V'_n,M_{V'_n}) \\ \downarrow & & \downarrow{\scriptstyle(*)} \\ (S_n,N_n) & \longrightarrow & (V_n,M_{V_n}), \end{array}$$

where the morphism (*) is defined by the multiplication by d on $\mathbb{N}$. Let (X', M') denote the base change of (X, M) by $(S', N') \to (S, N)$, and let $(Z'_n, M_{Z'_n})$ and $\{F_{Z'_n}\}$ denote the base changes of (Z_n, M_{Z_n}) and $\{F_{Z_n}\}$ under the morphism $(*)$ above. Then one can apply the above arguments to $O_{K'}$, π', (X', M'), $(Z'_n, M_{Z'_n})$ and $\{F_{Z'_n}\}$. We denote by $'$ the corresponding things. Since $(Y', M_{Y'}) = (t', M_{t'}) \times_{(t, M_t)} (Y, M_Y)$ and $t' = t$, $Y' = Y$, we have $\omega^\bullet_Y \xrightarrow{\sim} \omega^\bullet_{Y'}$. So we identify $\omega^\bullet_{Y'}$ with $\omega^\bullet_Y$. Then the filtrations $\widetilde{U}^m$ on $\mathcal{H}^q(\mathcal{S}_1(r))$ and $\mathcal{H}^q(\mathcal{S}_1(r)')$ and the description of $\mathrm{gr}^m_{\widetilde{U}}$ given by Proposition A5 and (A10) have the following relations:

LEMMA A18 (*cf.* [T1, Lemma 7.5.3, 3]). — *Let r and q be integers such that $0 \le q \le r \le p-2$. Then the canonical homomorphism $\mathcal{H}^q(\mathcal{S}_1(r)) \to \mathcal{H}^q(\mathcal{S}_1(r)')$ sends $\widetilde{U}^m$ into $\widetilde{U}^{dm}$ for $m \in \mathbb{N}$. Furthermore, if $ep(r-q)/(p-1) \le m < ep(r-q+1)/(p-1)$, the following diagram is commutative:*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K}_1 & \longrightarrow & \mathrm{gr}^m_{\widetilde{U}} \mathcal{H}^q(\mathcal{S}_1(r)) & \longrightarrow & \mathcal{K}_2 & \to & 0 \\ & & \downarrow{\scriptstyle d\cdot \mathrm{id}_{\mathcal{K}_1}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \\ 0 & \to & \mathcal{K}_1 & \longrightarrow & \mathrm{gr}^{md}_{\widetilde{U}} \mathcal{H}^q(\mathcal{S}_1(r)') & \longrightarrow & \mathcal{K}'_2 & \to & 0. \end{array}$$

Here

$$\begin{aligned} \mathcal{K}_1 &= \mathrm{Ker}\,\big(1 - a_0^{p(r-q)} C^{-1} : Z_Y^{q-1} \to \mathcal{H}^{q-1}(\omega^\bullet_Y)\big) \\ &\qquad (\textit{resp.}\ \omega_Y^{q-2}/Z_Y^{q-2},\ \textit{resp.}\ \omega_Y^{q-2}/Z_Y^{q-2}), \\ \mathcal{K}_2 &= \mathrm{Ker}\,\big(1 - a_0^{p(r-q)} C^{-1} : Z_Y^{q} \to \mathcal{H}^{q}(\omega^\bullet_Y)\big) \\ &\qquad (\textit{resp.}\ \omega_Y^{q-1}/B_Y^{q-1},\ \textit{resp.}\ \omega_Y^{q-1}/Z_Y^{q-1}), \\ \textit{if} \quad m &= ep(r-q)/(p-1) \\ &\qquad (\textit{resp.}\ m > ep(r-q)/(p-1),\ p \nmid m, \\ &\qquad\quad \textit{resp.}\ m > ep(r-q)/(p-1),\ p \mid m). \\ \mathcal{K}'_2 &= \omega_Y^{q-1}/Z_Y^{q-1} \quad \textit{if}\ m > ep(r-q)/(p-1),\ p \mid md \end{aligned}$$

and $\mathcal{K}'_2 = \mathcal{K}_2$ otherwise, and pr *denotes the canonical projection or the identity.*

Especially, if K' is tamely ramified over K, we have isomorphisms:

$$\mathrm{gr}^m_{\widetilde{U}}\big(\mathcal{H}^q(\mathcal{S}_1(r))\big) \xrightarrow{\sim} \mathrm{gr}^{dm}_{\widetilde{U}}\big(\mathcal{H}^q(\mathcal{S}_1(r)')\big) \quad (m \in \mathbb{N}).$$

Proof. — The first claim is trivial by $T = (T')^d$ and the second claim follows from $d\log T = d \cdot d\log T'$. □

PROPOSITION A19. — *Let the notation and the assumption be as above, and assume $d = p-1$ and that K' contains a primitive p-th root of unity. Then, for any integers r and q such that $0 \le q \le r \le p-2$, the canonical homomorphism*

$$\mathcal{H}^q(\mathcal{S}_1(r)) \longrightarrow \mathcal{H}^q(\mathcal{S}_1(r)')^{\mathrm{Gal}(K'/K)}$$

is an isomorphism.

Proof. — By Proposition A15, the filtration $\widetilde{U}^\bullet$ on $\mathcal{H}^q(\mathcal{S}_1(r)')$ is independent of the choice of a uniformizer of K', and hence it is $\mathrm{Gal}(K'/K)$-stable. Since $[K'\colon K] = p-1$ is prime to p, by Lemma A18, it remains to prove

$$\mathrm{gr}_{\widetilde{U}}^{m+m_0} \mathcal{H}^q(\mathcal{S}_1(r)')^{\mathrm{Gal}(K'/K)} = 0$$

for $m_0 = ep(r-q)$ and an integer $0 \le m < ep$ such that $(p-1) \nmid (m_0+m)$. Let χ_{cyclo} (resp. $\chi_{\pi'}$): $\mathrm{Gal}(K'/K) \to \mathbb{F}_p^*$ be the cyclotomic character (resp. the character defined by $\chi_{\pi'}(g) = g(\pi')(\pi')^{-1} \bmod \pi'$). Then, since $(\zeta_p - 1)\cdot(\pi')^{-e} \in O_{K'}^*$, we have

$$\chi_{\mathrm{cyclo}}(g) = g(\zeta_p - 1)(\zeta_p - 1)^{-1} \bmod \pi' = \chi_{\pi'}(g)^e.$$

The action of $\mathrm{Gal}(K'/K)$ on $\mathrm{gr}_{\widetilde{U}}^{m_0+m} \mathcal{H}^q(\mathcal{S}_1(r)')$, by Proposition A15 and Proposition A17, is given by the character

$$\chi_{\mathrm{cyclo}}^{r-q} \cdot \chi_{\pi'}^{m} = \chi_{\mathrm{cyclo}}^{(r-q)p} \chi_{\pi'}^{m} = \chi_{\pi'}^{m_0+m},$$

which is trivial if and only if $(p-1) \mid (m_0+m)$. (Precisely speaking, we need the fact that we may replace $\cdots \otimes \pi$ by $\cdots \otimes \pi \cdot u$ for any $u \in 1 + \pi O_K$ in Proposition A15). ∎

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