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ON TOPOLOGICAL RIGIDITY OF PROJECTIVE FOLIATIONS

BY A. LINS NETO, P. SAD, B. SCÁRDUA (*)

ABSTRACT. — Let us denote by $\mathcal{X}(n)$ the space of degree $n \in \mathbb{N}$ foliations of the complex projective plane $\mathbb{C}P(2)$ which leave invariant the line at infinity. We prove that for each $n \geq 2$ there exists an open dense subset $\mathrm{Rig}(n) \subset \mathcal{X}(n)$ such that any topologically trivial analytic deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of an element $\mathcal{F}_0 \in \mathrm{Rig}(n)$, with $\mathcal{F}_t \in \mathcal{X}(n)$, for all $t \in \mathbb{D}$, is analytically trivial. This is an improvement of a remarkable result of Ilyashenko. Other generalizations of these results are given as well as a description of the class of nonrigid foliations.

RÉSUMÉ. — SUR LA RIGIDITÉ TOPOLOGIQUE DES FEUILLETAGES PROJECTIFS. Nous désignons par $\mathcal{X}(n)$ l'espace des feuilletages de degré $n \in \mathbb{N}$ du plan projectif complexe qui laissent invariante la droite de l'infini. Nous démontrons que, pour chaque $n \geq 2$, il existe un sous-ensemble ouvert et dense $\mathrm{Rig}(n) \subset \mathcal{X}(n)$ tel que toute déformation analytique et topologiquement triviale $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ d'un élément $\mathcal{F}_0 \in \mathrm{Rig}(n)$, avec $\mathcal{F}_t \in \mathcal{X}(n)$ pour tout $t \in \mathbb{D}$, est analytiquement triviale. Cela améliore un résultat remarquable de Ilyashenko. On donne aussi d'autres généralisations de ces résultats ainsi qu'une description de la classe des feuilletages non rigides.

Introduction

Let $\mathrm{Fol}(M)$ denote the set of singular (holomorphic) foliations on a complex manifold M. An *analytic deformation* of $\mathcal{F}$ is an analytic family $\{\mathcal{F}_t\}_{t\in Y}$ of foliations on M, with parameters on an analytic space Y, such that there exists a point $o \in Y$ with $\mathcal{F}_0 = \mathcal{F}$. For reasons of simplicity we will only consider deformations where the germ of analytic space (Y, o) is $(\mathbb{D}, 0)$ where $\mathbb{D} \subset \mathbb{C}$ is the unitary disk and $0 \in \mathbb{C}$ is the origin.

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A *topological equivalence* (resp. *analytical equivalence*) between two foliations $\mathcal{F}, \mathcal{F}_1$ is a homeomorphism (resp. biholomorphism) $\phi : M \to M$, which takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}_1$, and such that $\phi(\operatorname{sing}\mathcal{F}) = \operatorname{sing}\mathcal{F}_1$. The deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is *topologically trivial* (resp. *analytically trivial*) if there exists a continuous map (resp. holomorphic map) $\phi : M \times \mathbb{D} \to M$, such that each map $\phi_t : M \to M$ is a topological equivalence (respectively an analytical equivalence) between $\mathcal{F}_t$ and $\mathcal{F}_0$.

Let $\mathcal{C} \subset \operatorname{Fol}(M)$ be a class of foliations on M; $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is a deformation in the class $\mathcal{C}$, when $\mathcal{F}_t \in \mathcal{C}$, for all $t \in \mathbb{D}$. A holomorphic foliation $\mathcal{F} \in \mathcal{C}$ will be called *topologically rigid in the class $\mathcal{C}$* if any topologically trivial deformation in the class $\mathcal{C}$ is analytically trivial.

We denote by $\mathcal{F}(n)$ the class of foliations of degree $n \in \mathbb{N}$ of $\mathbb{C}P(2)$ (see [8]). Let us fix an affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$ and denote by $\mathcal{X}(n)$, the space of foliations of $\mathcal{F}(n)$ which leave invariant the line at the infinity $L_\infty = \mathbb{C}P(2)\backslash\mathbb{C}^2$. A well-known theorem of Y. Ilyashenko establishes topological rigidity for a residual class of foliations on the 2-dimensional complex projective space $\mathbb{C}P(2)$, leaving invariant a fixed projective line.

THEOREM (see [14], [5], [9]). — *For any $n \geq 2$ there exists a residual subset $I(n) \subset \mathcal{X}(n)$ whose elements are topologically rigid foliations in the class $\mathcal{X}(n)$.*

In this paper we find an *open* and dense subset of $\mathcal{X}(n)$ of topologically rigid foliations. In fact, we will work with a weaker notion of trivial deformation. Instead of following continuously *all* the leaves of a foliation along the deformation, our idea is to follow only the leaves in some subset as small as possible, invariant under topological equivalence, and see if this control implies analytical triviality. Our choice here is the set of separatrices of the foliation, and the useful λ-lemma from Complex Analysis permits in general to follow all leaves in its closure. On the other hand, it can be shown that foliations which belong to a certain open and dense subset of $\mathcal{X}(n)$ have dense separatrices; these foliations will ultimately have only analytically trivial deformations.

To be more precise, let us denote by $S_t \subset M$, the set of separatrices of each foliation $\mathcal{F}_t$. The deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ is *s-trivial*, when there exists a continuous family of maps $\phi_t : S_0 \to M$, $t \in \mathbb{D}$ such that ϕ_0 is the inclusion map, $\phi_t(S_0) = S_t$, and ϕ_t is a continuous injective map from S_0 into M. By continuity we mean that we have fixed a metric d in M (compatible with the topology of M), such that for any compact set $K \subset M$ it holds

$$\max_{p\in K\cap S_0}\{d(\phi_t(p), \phi_{t_0}(p))\} \longrightarrow 0 \quad \text{as} \quad t \to t_0.$$

We have in mind the cases $M = \mathbb{C}P(2)$ and $M = \mathbb{C}^2$. We introduce the following terminology :

DEFINITION 1. — *Let $\mathcal{C} \subset$ Fol(M) and $\mathcal{F} \in \mathcal{C}$. Then $\mathcal{F}$ is s-rigid in the class $\mathcal{C}$, when any s-trivial deformation in the class $\mathcal{C}$ is analytically trivial.*

We remark that a topologically trivial deformation is also a s-trivial one. Our main result for $\mathbb{C}P(2)$ can be stated as follows :

THEOREM 1.1. — *For each $n \geq 2$, $\mathcal{X}(n)$ contains an open dense subset* Rig(n) *such that any foliation in this set is s-rigid in the class $\mathcal{X}(n)$.*

In particular, the foliations in Rig(n) are topologically rigid. These foliations are essentially characterized by the properties :

(i) L_∞ is the only algebraic solution of the foliation;

(ii) the singularities at L_∞ are hyperbolic.

Now we state the corresponding result for $\mathbb{C}^2$:

THEOREM 1.2. — *For $n \geq 2$, any deformation in* Rig(n) *which is s-trivial in $\mathbb{C}^2$ is analytically trivial in $\mathbb{C}P(2)$.*

As a consequence we have that : any deformation in Rig(n) which is topologically trivial in $\mathbb{C}^2$ is analytically trivial in $\mathbb{C}P(2)$. This answers to a question motivated by the rigidity result of [14].

A similar situation occurs in [11], where deformations of germs of certain singular foliations are considered : along the divisor introduced after desingularization the deformation is not topologically rigid *a priori.*

The reader may think of topological triviality instead of s-triviality in Theorems 1.1 and 1.2; we prefer to work with s-triviality to single out basic features of the foliations involved.

Although we do not describe the non s-rigid foliations, we are able to give some information about the non topologically rigid foliations. Let us denote by Fol(M, S) the subspace of Fol(M) of foliations which leave the compact analytic curve $S \subset M$ invariant.

THEOREM 2.1. — *Let $S \subset \mathbb{C}P(2)$ be an irreducible algebraic curve and $\mathcal{F} \in$ Fol$(\mathbb{C}P(2), S)$ be a foliation which satisfies :*

(i) *the singularities of $\mathcal{F}$ along S are hyperbolic.*

(ii) *any singularity of $\mathcal{F}$ has exactly two local transverse separatrices.*

If $\mathcal{F}$ is non rigid in the class Fol$(\mathbb{C}P(2), S)$ *then $\mathcal{F}$ is defined by a logarithmic 1-form (Darboux foliation).*

As in [13] we may obtain examples of non rigid Darboux foliations.

THEOREM 2.2. — *Let M be a projective surface with a very ample irreducible algebraic curve $S \subset M$. Let $\mathcal{F} \in \mathrm{Fol}(M,S)$ be a foliation with hyperbolic singularities along S and given by a holomorphic map of linear bundles $\alpha : L \to TM$, where L is a linear bundle such that $H^1(M, \mathcal{O}_M(L)) = 0$. Choose a rational 1-form ω which defines $\mathcal{F}$ on M. Then, either $\mathcal{F}$ is topologically rigid in the class $\mathrm{Fol}(M,S)$ or $\mathcal{F}$ admits a Liouvillian first integral of the form $F = \int \omega/H$, where $H = \exp \int \eta$ for some closed meromorphic 1-form η with simple poles on M.*

When M is simply-connected, Theorem 2.2 can be completed by a description of the foliations which admit a Liouvillian first integral : they are defined by closed rational 1-forms or by rational pull-backs of Riccati equations

$$(\mathcal{R}) \qquad p(x)\,\mathrm{d}y - \big(y^2 a(x) + y b(x)\big)\,\mathrm{d}x = 0$$

on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, where S corresponds to $\overline{(y=0)}$ (see [2]).

As a matter of fact, Theorem 2.1 is a particular case of Theorem 2.2 : conditions (i) and (ii) in its statement prohibit the presence of dicritical singularities, which appear in pull-backs of Riccati equations (when not of Darboux type).

In order to make clearer the proofs of these two theorems, we explain in § 5 how to get a topological rigidity theorem for foliations in $\mathrm{Fol}(M,S)$, where M is a projective surface and $S \subset M$ is an irreducible algebraic curve. We point out that some of the steps used to prove such a rigidity theorem are well-known from several authors [6], [10], [11], [12], [14], [16], [19].

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1. Preliminaries

A foliation $\mathcal{F} \in \mathcal{X}(n)$ can be described in $\mathbb{C}^2$ by a polynomial vector field

$$X = P(x,y)\frac{\partial}{\partial y} + Q(x,y)\frac{\partial}{\partial x}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials of degree $\leq n$, with some of the components of degree n, and without common factors (see [8]).

Let $q \in U$ be an isolated singularity of a foliation $\mathcal{F}$ defined on an open subset $U \subset \mathbb{C}^2$. We say that q is *nondegenerate* if there exists a holomorphic vector field X tangent to $\mathcal{F}$ in a neighborhood of q, such that $DX(q)$ is nonsingular. In particular q is an isolated singularity of X.

Let q be a nondegenerate singularity of $\mathcal{F}$, the *characteristic numbers* of q are the quotients λ and λ^{-1} of the eigenvalues of $DX(q)$, which do not depend on the vector field X chosen as above. If $\lambda \notin \mathbb{Q}_+$ then $\mathcal{F}$ exhibits exactly two (smooth and transverse) *local separatrices* (see [5] for a definition) at q say, S_q^+ and S_q^-, which are tangent to the characteristic directions of a vector field X as above, and with eigenvalues λ_q^+ and λ_q^- respectively. The *characteristic numbers* of these local separatrices are given by

$$I(\mathcal{F}, S_q^-) = \frac{\lambda_q^+}{\lambda_q^-}, \quad I(\mathcal{F}, S_q^+) = \frac{\lambda_q^-}{\lambda_q^+}\cdot$$

The singularity is *hyperbolic* if the characteristic numbers are nonreal. We introduce the following spaces of foliations :

$$\mathcal{S}(n) = \{\mathcal{F} \in \mathcal{F}(n) \,/\, \text{the singularities of } \mathcal{F} \text{ are nondegenerate}\},$$

$$T(n) = \{\mathcal{F} \in \mathcal{S}(n) \,/\, \text{any characteristic number } \lambda \text{ of } \mathcal{F} \text{ satisfies } \lambda \in \mathbb{C}\backslash\mathbb{Q}_+\},$$

$$A(n) = T(n) \cap \mathcal{X}(n).$$

It is well-known that $T(n)$ contains an open and dense subset of $\mathcal{F}(n)$ (see [8]).

The first step of the proof of Theorem 1.1 is to use the following theorem :

THEOREM 3. — *The space $A(n)$ contains an open dense subset $M_1(n)$, such that if $\mathcal{F} \in M_1(n)$ then :*

(i) *L_∞ is the only algebraic solution of $\mathcal{F}$.*

(ii) *The holonomy group of the leaf $L_\infty \backslash \operatorname{sing} \mathcal{F}$ is non solvable.*

(iii) *$\operatorname{sing} \mathcal{F} \cap L_\infty$ consists of hyperbolic singularities.*

Then we adapt the ideas of [14] to our situation. In the proof of Theorems 2.1 and 2.2 we must consider the case where the holonomy group of the leaf $S \setminus \operatorname{sing} \mathcal{F}$ is solvable. The following result is a particular case of a more general situation studied in [2] and plays an important role :

THEOREM. — *Let $\mathcal{F}$ be a foliation on a projective surface M and $S \subset M$ be a very ample irreducible algebraic curve invariant by $\mathcal{F}$. Assume that the singularities of $\mathcal{F}$ in S are hyperbolic and that the holonomy group of S is solvable. Then, given a rational 1-form ω which defines $\mathcal{F}$ in M, there exists a closed rational 1-form η with simple poles in M, which satisfies*

$$\mathrm{d}\omega = \eta \wedge \omega.$$

2. Proof of Theorem 3

In this section we prove Theorem 3. We begin with a preliminar result :

PROPOSITION 1. — *Let $\mathcal{F}_0 \in \mathcal{S}(n)$. Then*

$$\#\operatorname{sing}\mathcal{F}_0 = n^2 + n + 1 = N(n) = N.$$

Moreover if $\operatorname{sing}(\mathcal{F}_0) = \{p_1^0, \dots, p_N^0\}$ *where* $p_i^0 \neq p_j^0$ *if* $i \neq j$, *then there are connected neighborhoods* $U_j \ni p_j$, *pairwise disjoint, and holomorphic maps* $\varphi_j : \mathcal{U} \subset \mathcal{S}(n) \to U_j$, *where* $\mathcal{U} \ni \mathcal{F}_0$ *is an open neighborhood, such that for* $\mathcal{F} \in \mathcal{U}$, $\operatorname{sing}(\mathcal{F}) \cap U_j = \varphi_j(\mathcal{F})$ *is a nondegenerate singularity. Moreover, if* $\mathcal{F}_0 \in T(n)$ *then the two local separatrices as well as their associated eigenvalues depend analytically on* $\mathcal{F}$. *In particular* $\mathcal{S}(n)$ *is open in* $\mathcal{F}(n)$.

This result is proved in [5] as a consequence of the Implicit Function Theorem for holomorphic mappings. We remark that

$$\mathcal{F} \in A(n) \implies \#(\operatorname{sing}\mathcal{F} \cap L_\infty) = n + 1 \text{ and } \#(\operatorname{sing}\mathcal{F} \cap \mathbb{C}^2) = n^2.$$

Let us make a definition :

Let $\mathcal{F} \in A(n)$. We enumerate $\operatorname{sing}\mathcal{F} = \{p_1, \dots, p_N\}$ in such a way that $\{p_1, \dots, p_{n^2}\} \subset \mathbb{C}^2$ and $p_j \in L_\infty$ for all $j \geq n^2 + 1$. We also enumerate the local separatrices of the singularity p_j as S_j^+ and S_j^- for all $j \in \{1, \dots, n^2\}$ and denote by S_i^0 the separatrix of p_i transverse to L_∞ for all $i \in \{n^2 + 1, \dots, N\}$. We denote by

$$I(\mathcal{F}, S_j^+), \quad I(\mathcal{F}, S_j^-)$$

the characteristic numbers associated to the local separatrices S_j^+, S_j^- respectively. Let us choose a neighborhood $\mathcal{U}$ of $\mathcal{F}$ in $\mathcal{F}(n)$ as in Proposition 1 above, in such a way that $\mathcal{U} \ni \mathcal{F}_1 \mapsto I(\mathcal{F}_1, S_j^+)$ and $U \ni \mathcal{F}_1 \mapsto I(\mathcal{F}_1, S_j^-)$ are holomorphic maps. We denote

$$\mathcal{S}(\mathcal{F}) = \{S_j^+, S_j^-, S_i^0 \mid j \in \{1, \dots, n^2\},\ i \in \{n^2 + 1, \dots, N\}\}$$

and also denote

$$\mathcal{S}(\mathcal{F})_{\text{fin}} = \{S_j^+, S_j^- \mid j \in \{1, \dots, n^2\}\}.$$

DEFINITION 2. — A *configuration* is a subset $C \subset \mathcal{S}(\mathcal{F})$. The configuration C is *finite* if we have $C \subset \mathcal{S}_{\text{fin}}$. Given a configuration C we define

$$I(\mathcal{F}, C) = \sum_{S_j^+ \in C} I(\mathcal{F}, S_j^+) + \sum_{S_j^- \in C} I(\mathcal{F}, S_j^-) + \sum_{S_i^0 \in C} I(\mathcal{F}, S_i^0).$$

If $C = \emptyset$ then we define $I(\mathcal{F}, C) = 0$. If $S \subset \mathbb{C}P(2)$ is an invariant irreducible algebraic curve then we define the *configuration of* S as the configuration $C(S)$ defined by the local separatrices of $\mathcal{F}$ contained in S, and put $I(\mathcal{F}, S) = I(\mathcal{F}, C(S))$.

Let C be a configuration. Then we can split C in three parts

$$C = A \cup B \cup K,$$

where

$$\begin{aligned} K &= \{S_i^0 \in C\}, \\ A &= \{S_j^+ \in C \mid S_j^- \notin C\} \cup \{S_j^- \in C \mid S_j^+ \notin C\}, \\ B &= \{S_j^+ \in C \mid S_j^- \in C\} \cup \{S_j^- \in C \mid S_j^+ \in C\}. \end{aligned}$$

We also write

$$\alpha = \alpha(C) = \#A, \quad \beta = \beta(C) = \#B, \quad k = k(C) = \#K.$$

In what follows we will consider configurations C satisfying the following properties :

(a) $k = \#K \geq 1$,

(b) $C \neq \mathcal{S}(\mathcal{F})$.

PROPOSITION 2. — *Let $\mathcal{F} \in A(n)$ be as above, and let $S \neq L_\infty$ an irreducible invariant algebraic curve. Write $C(S) = A \cup B \cup K$ as above. Then $C(S)$ satisfies properties (a), (b) and the following :*

(c) $I(\mathcal{F}, C(S)) = k^2 - \beta \geq 1$.

Proof. — Part (a) follows from Bezout's Theorem.

In order to prove (c) we recall [8] where it is shown that

$$0 < I(\mathcal{F}, S) = 3k - \mathcal{X}(\widetilde{S})$$

where $\mathcal{X}(\widetilde{S})$ is the Euler characteristic of the normalization $\widetilde{S}$ of the curve S. Since S has only nodal singularities, which correspond to local separatrices in B which meet transversely, it follows from the Hurwitz formula that

$$\mathcal{X}(\widetilde{S}) = 2 - 2\Big(\frac{(k-1)(k-2)}{2} - \frac{1}{2}\beta\Big)$$

so that $I(\mathcal{F}, C(S)) = k^2 - \beta$.

Now we prove (b) : If $C(S) = \mathcal{S}(\mathcal{F})$ then $k = n+1, \beta = 2n^2$, so that by (c) we have

$$I\big(\mathcal{F}, C(S)\big) = (n+1)^2 - 2n^2 = -n^2 + 2n + 1.$$

Therefore, $I(\mathcal{F}, C(S)) = 1$ if $n = 2$, and $I(\mathcal{F}, C(S)) < 0$ if $n \geq 3$. On the other hand, in [8] it is proved that if $I(\mathcal{F}, C(S)) = 1$ then S is a straight line, which is not possible if $C(S) = \mathcal{S}(\mathcal{F})$. $\square$

DEFINITION 3. — Let $n \in \mathbb{N}$, we define the subset

$$\begin{aligned} M(n) = \big\{\mathcal{F} \in A(n) \mid \ &\text{for all configuration } C \subset \mathcal{S}(\mathcal{F}),\\ &\text{such that } k(C) \geq 1 \text{ and } C \neq \mathcal{S}(\mathcal{F})\\ &\text{we have } I(\mathcal{F}, C) \neq k(C)^2 - \beta(C)\big\}. \end{aligned}$$

REMARKS.

(1) If $n \geq 2$ and $\mathcal{F} \in M(n)$ then $\mathcal{F}$ admits no irreducible algebraic invariant curve $S \neq L_\infty$.

(2) $M(n)$ is open in $A(n)$.

(3) $M(1) = \emptyset$.

(4) $A(n) \setminus M(n)$ is an analytic subset of $A(n)$, because it is defined (locally) by a finite number of equations of the form $I(\mathcal{F}, C) = k^2 - \beta$.

We prove the following result :

THEOREM 4. — *$M(n)$ is dense in $A(n)$ if $n \geq 2$.*

Proof. — Since $A(n) \setminus M(n)$ is an analytic subset of $A(n)$, it suffices to prove that $M(n) \neq \emptyset$ (see also [8]). Given $n \geq 2$ and $b \in \mathbb{C}$, we consider the foliation

$$\mathcal{F}(b) : (a_0 x - y^n + bx^n)\,\mathrm{d}y - (y - x^n + byx^{n-1})\,\mathrm{d}x = 0$$

where a_0 is a root of $\dfrac{(1+a)^2}{(n^2-1)a} = -2 + \sqrt{2}$. We take

$$\begin{aligned} a_0 &= -1 - \ell + \frac{\sqrt{2}}{2}\ell + \sqrt{\alpha - \beta\sqrt{2}},\\ \frac{1}{a_0} &= -1 - \ell + \frac{\sqrt{2}}{2}\ell - \sqrt{\alpha - \beta\sqrt{2}}, \end{aligned}$$

where $\alpha = \frac{3}{2}\ell^2 + 2\ell$, $\beta = \ell + \ell^2$ and $\ell = n^2 - 1$. Notice that $a_0 < 0$. It is enough to prove the following lemma :

LEMMA 1. — *There exists $\epsilon > 0$, such that for an open dense subset $A \subset \{b \in \mathbb{C} \mid 0 < |b| < \epsilon\}$ we have $b \in A \Rightarrow \mathcal{F}(b) \in M(n)$.*

Proof. — Let us fix the following notation : given any configuration $C \subset \mathcal{S}(\mathcal{F}(b))$ we put

$$I(\mathcal{F}(b), C) = I_C(b).$$

The configuration $C = C(\mathcal{F}(b)) = C(b)$ depends continuously on b if we choose b in such a way that $\mathcal{F}(b)$ is as in Proposition 1, and in this case $I_C(b)$ depends holomorphically on b. Thus it is enough to show that for any configuration C satisfying properties (a) and (b) we have $I_C(b) \not\equiv k^2 - \beta$.

CLAIM 1. — *Let $C \subset \mathcal{S}_{\mathrm{fin}}$ be a finite configuration. Then $I_C(0) \in \mathbb{Z}$ if and only if either $C = \emptyset$ and $I_C(0) = 0$, or $C = \mathcal{S}_{\mathrm{fin}}$ and $I_C(0) = -2n^2$.*

Proof. — Let $C \subset \mathcal{S}_{\mathrm{fin}}$ be such that $I_C(0) \in \mathbb{Z}$. For $b = 0$ we have the following differential equation :

$$\mathcal{F}(0) \qquad\qquad \dot{x} = a_0 x - y^n, \quad \dot{y} = y - x^n.$$

The singularities in $\mathbb{C}^2$ are given by :

(1) $(0,0)$ which has characteristic numbers a_0, a_0^{-1}.

(2) The other singularities are given by $y^n = a_0 x$ and $y = x^n$, that is, $y^{n^2-1} = a_0^n$, so that we obtain the roots $y_1, \dots, y_\ell$ where $\ell = n^2 - 1$, and $x_j = {a_0}^{-1} y_j^n$, for $j = 1, \dots, \ell$. The characteristic numbers are given by the matrix

$$DX(x_j, y_j) = \begin{pmatrix} a_0 & -n y_j^{n-1} \\ -n x_j^{n-1} & 1 \end{pmatrix}$$

that is, these characterisc numbers are the roots of

$$\lambda + \lambda^{-1} + 2 = \frac{T^2}{D} = \frac{(1 + a_0)^2}{-\ell a_0} = 2 - \sqrt{2}$$

where T is the trace and D is the determinant of the matrix $DX(x_j, y_j)$. Therefore we obtain

$$\lambda = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i.$$

Now, since C is a finite configuration we have

$$C \subset \{S_0^+, S_0^-, S_1^+, S_1^-, \dots, S_\ell^+, S_\ell^-\},$$

where $S_0^\pm$ are the local separatrices of the singularity $(0,0)$, and $S_j^\pm$ are the local separatrices of (x_j, y_j), for $j = 1, \dots, \ell$. Assume that $C \neq \emptyset$.

We claim that $C \not\subset \{S_j^{\pm} \mid j = 1, \dots, \ell\}$. Indeed, otherwise according to the characteristic numbers calculated above we have

$$I_C(0) = -\frac{\sqrt{2}}{2} r + \frac{\sqrt{2}}{2} is$$

for some $r, s \in \mathbb{Z}$, where $r > 0$, but this is an absurd. Therefore C must contain at least one of the local separatrices $S_0^{\pm}$. We consider two cases :

- Case 1 : $\{S_0^{\pm}\} \subset C$. In this case

$$\begin{aligned} I_C(0) &= a_0 + a_0^{-1} + \sum_{S_j^+ \in C} I(\mathcal{F}(0), S_j^+) + \sum_{S_j^- \in C} I(\mathcal{F}(0), S_j^-) \\ &= a_0 + a_0^{-1} + r\big(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\big) + s\big(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\big) \\ &= -2 - 2\ell + \sqrt{2}\ell - (r+s)\frac{\sqrt{2}}{2} + (r-s)\frac{\sqrt{2}}{2} i. \end{aligned}$$

But since $I_C(0) \in \mathbb{Z}$ it follows that $r = s$ and

$$I_C(0) = -2 - 2\ell + \sqrt{2}\ell - r\sqrt{2} = -2 - 2\ell + \sqrt{2}(\ell - r),$$

which by its turn implies $\ell = r$, and therefore $C = \mathcal{S}_{\text{fin}}$. Finally, $C = \mathcal{S}_{\text{fin}}$ implies $I_C(0) = -2 - 2\ell = -2n^2$.

- Case 2 : $S_0^+ \subset C$ and $S_0^- \not\subset C$, or *vice-versa.* In this case

$$I_C(0) = -1 - \ell + \frac{\sqrt{2}}{2}\ell \pm \sqrt{\alpha - \beta\sqrt{2}} - m\sqrt{2} = r + s\frac{\sqrt{2}}{2} \pm \sqrt{\alpha - \beta\sqrt{2}}$$

where $r = -1 - \ell$, $s = \ell - 2m$ and $m = \#\{S_j^+ \mid S_j^+ \subset C\}$. In particular we have $m \leq \ell$. Assume by contradiction that $I_C(0) \in \mathbb{Z}$, say

$$r + s\frac{\sqrt{2}}{2} \pm \sqrt{\alpha - \beta\sqrt{2}} = u \in \mathbb{Z}.$$

Then we can write

$$\pm\sqrt{\alpha - \beta\sqrt{2}} = u - r - s\frac{\sqrt{2}}{2} = v - s\frac{\sqrt{2}}{2}$$

for $v = u - r$. Therefore,

$$\alpha - \beta\sqrt{2} = v^2 - vs\sqrt{2} + \frac{1}{2}s^2,$$

which implies $\beta = vs$ and $\alpha = v^2 + \frac{1}{2}s^2$. Thus we obtain

$$v = \frac{\beta}{s}, \quad \alpha = \frac{\beta^2}{s^2} + \frac{s^2}{2}$$

and then $2\alpha s^2 = 2\beta^2 + s^4$. Replacing the values

$$\alpha = \frac{3}{2}\ell^2 + 2\ell, \quad \beta = \ell + \ell^2, \quad \ell = n^2 - 1, \quad r = -1 - \ell, \quad s = \ell - 2m$$

on this last equation we obtain

$$(*) \qquad (3\ell^2 + 4\ell)(\ell - 2m)^2 = 2\ell^2(1+\ell)^2 + (\ell - 2m)^4.$$

In particular, either $\ell = 2m$ or $\ell \mid (\ell - 2m)^4$. Notice that $\ell = 2m$ implies $\ell = 0$ using (*). Therefore $\ell = 2m$ implies $m = 0$ and $n = 1$.

We claim that equation (*) has no other solution in $\mathbb{Z}$. Indeed, writting $x = \ell - 2m$ equation (*) becomes

$$4x^4 - 4(3\ell^2 + 4\ell)x^2 + 8\ell^2(1+\ell)^2 = 0$$

or also,

$$\big(2x^2 - (3\ell^2 + 4\ell)\big)^2 = \ell^2(\ell^2 + 8\ell + 8).$$

Thus it follows that $\ell^2 + 8\ell + 8 = y^2$ for some integer $y \in \mathbb{Z}$. This can be written as $\ell^2 + 8\ell + 16 - y^2 = 8$ and therefore as

$$(\ell + 4 - y)(\ell + 4 + y) = 8.$$

Since for any integer $a \in \mathbb{Z}$ the numbers $a + y$ and $a - y$ have the same parity, it follows that these integers can only take the values $\pm 2, \pm 4$. Replacing these values in equation (*) we obtain $\ell \in \{-1, -7\}$. These values for ℓ give no solution $x \in \mathbb{Z}$. This ends the proof of Claim 1. $\square$

Now we regard the singularities over the line L_∞. We consider the change of coordinates given by $u = 1/x$, $v = y/x$. In these coordinates, $\mathcal{F}(b)$ is given by

$$\dot{u} = u(-b + v^n - a_0 u^{n-1}), \quad \dot{v} = v^{n+1} - 1 + vu^{n-1}(1 - a_0).$$

In particular $L_\infty : \overline{(u = 0)}$ is invariant, and the singularities over this line are given by $v^{n+1} - 1 = 0$, so that they can be writen as $(0, \delta^j)$ where δ is a primitive $(n+1)$-th root of 1, and $j \in \{0, 1, \dots, n\}$. Let us write

$$\operatorname{sing} \mathcal{F}(b) \cap L_\infty = \big\{(0, v_j) \mid j = 1, \dots, n+1\big\}.$$

The characteristic numbers are given by

$$(**) \qquad I(\mathcal{F}(b), S_j^0) = \frac{\phi'(v)}{v^n - b}\Big|_{v = v_j} = \frac{(n+1)}{1 - bv_j}$$

where $\phi(v) = v^{n+1} - 1$ (recall that $v_j^n \cdot v_j = 1$).

Let $C = C(b) \subset \{S_1^0(b), \dots, S_{n+1}^0(b)\}$ be a nonempty configuration.

CLAIM 2. — *$I_C(b)$ is not a constant function of b.*

Proof. — In fact, let $r = \#C$. We have

$$\begin{aligned} I_C(b) &= \sum_{j=1}^{r} I(\mathcal{F}(b), S^0_{i_j}) = (n+1)\sum_{j=1}^{r} \frac{1}{1 - bv_{i_j}} \\ &= (n+1)\sum_{j=1}^{r}\Big(1 + \sum_{m=1}^{\infty} v_{i_j}^m b^m\Big) \\ &= (n+1)r + (n+1)\sum_{m=1}^{\infty}\Big(\sum_{j=1}^{r} v_{i_j}^m\Big)b^m. \end{aligned}$$

Thus, if $I_C(b)$ was a constant we should have $\sum_{j=1}^{r} v_{i_j}^m = 0$, for all $m \geq 1$. But for $m = n+1$ we have $\sum_{j=1}^{r} v_{i_j}^{n+1} = r$, which gives a contradiction. This proves the claim. $\square$

Now we finish the proof of the lemma. Let C be a configuration satisfying properties (a), (b). Assume that, for b near 0,

$$I_C(b) = k^2 - \beta \ (> 0).$$

In particular $I_C(0) = k^2 - \beta$. Let us split $C = A \cup B \cup K$ as before, with $\alpha = \#A$, $\beta = \#B$ and $k = \#K$. We have

$$I_C(0) = I_A(0) + I_B(0) + I_K(0) = I_{A\cup B}(0) + I_K(0).$$

It follows from the formula (**) above that $I_K(0) = k(n+1)$, and therefore $I_{A\cup B}(0) \in \mathbb{Z}$. Hence by Claim 1 we have either

$$A \cup B = \emptyset \quad \text{or} \quad A \cup B = \mathcal{S}_{\text{fin}}.$$

We consider these two cases :

Case 1 : $A \cup B = \emptyset$. In this case

$$I_C(0) = k(n+1) = k^2 - \beta = k^2$$

(notice that $\beta = \#B = 0$). Therefore $k = n + 1$. On the other hand

$$I_C(b) = (n+1)\sum_{j=1}^{r} \frac{1}{1 - bv_{i_j}}$$

is not constant (Claim 2), so that $I_C(b) \not\equiv k^2 - \beta$.

Case 2 : $A \cup B = \mathcal{S}_{\text{fin}}$. In this case necessarily $A = \emptyset$ and $B = \mathcal{S}_{\text{fin}}$, so that $I_B(0) = -2n^2$ recalling that (in this case) $\beta = \#B = 2n^2$. Hence

$$I_C(0) = k(n+1) - 2n^2 = k^2 - \beta = k^2 - 2n^2$$

and then $k = n+1$ which implies $C = \mathcal{S}(\mathcal{F}(0))$ and therefore C does not satisfy property (b). The proof of Theorem 4 is now finished. ☐

Now we complete the proof of Theorem 3. Let

$$\mathcal{H}(n) = \{\mathcal{F} \in A(n) \mid \text{all the singularities of } \mathcal{F} \text{ in } L_\infty \text{ are hyperbolic}\}$$

Proof of Theorem 3. — We define

$$M_1(n) = M(n) \cap \mathcal{H}(n).$$

According to Theorem 4 and Proposition 1, $M_1(n)$ is open and dense in $A(n)$ (recall that $\mathcal{H}(n)$ contains an open and dense subset of $\mathcal{X}(n)$, see [5]). We obtain (i) from Theorem 4. We proceed to prove (ii). Let $\mathcal{F} \in M_1(n)$ and assume by contradiction that the holonomy group of $L_\infty \setminus \operatorname{sing} \mathcal{F}$ is solvable. Let us fix a polynomial 1-form ω which defines $\mathcal{F}_{|\mathbb{C}^2}$. According to [2] we can construct a rational closed 1-form with simple poles η which satisfies $\mathrm{d}\omega = \eta \wedge \omega$ (by the hypothesis $\operatorname{sing} \mathcal{F} \cap L_\infty$ consists of only hyperbolic singularities so that [2] applies). We give an idea of this fact : we assume that the holonomy group $G = \operatorname{Hol}(\mathcal{F}, L_\infty)$ is nonabelian (for the abelian case we refer to [1]). According to [4] there exists an analytic embedding

$$G \subset \mathbb{H}_k := \Big\{\varphi \in \operatorname{Diff}(\mathbb{C}, 0) \mid \varphi(z)^k = \frac{\mu_\varphi z^k}{1 + a_\varphi z^k},\ \mu_\varphi \in \mathbb{C}^*, a_\varphi \in \mathbb{C}\Big\}$$

for some $k \in \mathbb{N}$. Using the fact that $\operatorname{Hol}(\mathcal{F}, L_\infty)$ contains linearizable nonperiodic elements, we conclude that there exists an open covering of a neighborhood of L_∞ by a collection $(U_\alpha)_{\alpha \in A}$ of open connected subsets of $\mathbb{C}P(2)$ with holomorphic coordinates (x_α, y_α) in U_α such that [15] :

(1) $D \cap U_\alpha = \{y_\alpha = 0\}$;

(2) for any $\alpha \in A$ such that $U_\alpha \cap \operatorname{sing} \mathcal{F} = \phi$, $\mathcal{F}_{|U_\alpha}$ is given by $\mathrm{d}y_\alpha = 0$ and for any $\varphi \in \operatorname{Hol}(\mathcal{F}, L_\infty, \Sigma^\alpha)$ where

$$\Sigma^\alpha = (x_\alpha = p_\alpha), \quad \varphi(y_\alpha)^k = \frac{\lambda_\varphi y_\alpha^k}{1 + a_\varphi y_\alpha^k}\,;$$

it follows that, if $U_\alpha \cap U_\beta \neq \phi$ and $\operatorname{sing} \mathcal{F} \cap (U_\alpha \cup U_\beta) = \phi$, then $y_\alpha^k = H_{\alpha\beta}(y_\beta^k)$ for some $H_{\alpha\beta} \in \mathbb{H}_1$;

(3) If $q_\alpha = U_\alpha \cap \operatorname{sing}\mathcal{F} \neq \phi$ then $U_\alpha \cap \operatorname{sing}\mathcal{F} = \{q_\alpha\}$ is a single point, $x_\alpha(q_\alpha) = y_\alpha(q_\alpha) = 0$ and $\mathcal{F}_{|U_\alpha}$ is given by the normal form

$$x_\alpha \, \mathrm{d}y_\alpha - \lambda_\alpha y_\alpha \, \mathrm{d}x_\alpha = 0, \quad \lambda_\alpha \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, if $U_\beta \cap U_\alpha \neq \phi$, then $U_\beta \cap \operatorname{sing}\mathcal{F} = \phi$ and $U_\beta \cap U_\alpha$ is simply connected.

We take the covering $\{(x_\alpha, y_\alpha) \in (U_\alpha)\}_{\alpha \in A}$ above. If $U_\alpha \cap \operatorname{sing}\mathcal{F} = \phi$ we write

$$\omega_{|U_\alpha} = g_\alpha \, \mathrm{d}y_\alpha$$

and define

$$\eta_\alpha = (k+1)\frac{\mathrm{d}y_\alpha}{y_\alpha} + \frac{dg_\alpha}{g_\alpha}.$$

Whenever $U_\alpha \cap U_\beta \neq \phi$ and $(U_\alpha \cup U_\beta) \cap \operatorname{sing}\mathcal{F} = \phi$ we have $y_\alpha^k = H_{\alpha\beta}(y_\beta^k)$ for some $H_{\alpha\beta}(z) = \lambda_{\alpha\beta} z/(1 + a_{\alpha\beta} z) \in \mathbb{H}_1$, so we conclude that

$$\frac{\mathrm{d}y_\alpha}{y_\alpha^{k+1}} = \frac{\mathrm{d}y_\beta}{\lambda_{\alpha\beta} y_\beta^{k+1}} \quad \text{and} \quad \eta_\beta = \eta_\alpha \text{ in } U_\alpha \cap U_\beta.$$

Clearly $\eta_{|U_\alpha} := \eta_\alpha$ defines a closed meromorphic 1-form η in a neighborhood of $L_\infty \backslash \operatorname{sing}\mathcal{F} \cap L_\infty$. We remark that η extends meromorphically to $\operatorname{sing}\mathcal{F} \cap L_\infty$. Indeed, given a singularity $q_0 \in \operatorname{sing}\mathcal{F} \cap L_\infty \cap U_\alpha$ we have that q_0 is a linearizable singularity of the form

$$x_\alpha \, \mathrm{d}y_\alpha - \lambda_\alpha y_\alpha \, \mathrm{d}x_\alpha = 0, \quad \lambda_\alpha \in \mathbb{C} \setminus \mathbb{R}.$$

We define

$$\eta_{q_0} := (k+1)\frac{dy_\alpha}{y_\alpha} + (1 - \lambda_\alpha k)\frac{\mathrm{d}x_\alpha}{x_\alpha} + \frac{\mathrm{d}g_\alpha}{g_\alpha},$$

where g_α is defined by

$$\omega_{|U_\alpha} = g_\alpha \cdot (x_a \, \mathrm{d}y_\alpha - \lambda_\alpha y_\alpha \, \mathrm{d}x_\alpha).$$

The local coordinate (defined on a transverse section $\Sigma : (x = 1)$ at $q \in L_\infty \setminus \operatorname{sing}\mathcal{F}_0$) $\widehat{y}_\alpha = y_\alpha x_\alpha^{-\lambda_\alpha}$ linearizes the local holonomy of L_∞ around q_0. We may write

$$\omega_{|U_\alpha} = g_\alpha(x_\alpha \, \mathrm{d}y_\alpha - \lambda_\alpha y_\alpha \, \mathrm{d}x_\alpha) = \widehat{g}_\alpha \, \mathrm{d}\widehat{y}_\alpha,$$

where g_α is a meromorphic function, $\widehat{g}_\alpha = g_\alpha x_\alpha^{1+\lambda_\alpha}$. We define

$$\eta_{q_0} := (k+1)\frac{\mathrm{d}\widehat{y}_\alpha}{\widehat{y}_\alpha} + \frac{\mathrm{d}\widehat{g}_\alpha}{\widehat{g}_\alpha} = (k+1)\frac{\mathrm{d}y_\alpha}{y_\alpha} + (1 - \lambda_\alpha k)\frac{\mathrm{d}x_\alpha}{x_\alpha} + \frac{\mathrm{d}g_\alpha}{g_\alpha}.$$

The difference $\eta - \eta_{q0}$ satisfies

$$(\eta - \eta_{q0}) \wedge \omega = \mathrm{d}\omega - \mathrm{d}\omega = 0,$$

so that it can be written $\eta - \eta_{q0} = h_{q0} \cdot \omega$, for some meromorphic function h_{q0} defined over $(x_\alpha \neq 0)$ and which satisfies

$$\mathrm{d}(h_{q0} \cdot \omega) = 0.$$

Clearly the poles of h_{q0} are contained in L_∞ (notice that the forms η and $\eta_{q0\,|\,(x_\alpha \neq 0)}$ have simple poles contained in the divisor L_∞). On the other hand, since $\mathrm{Res}_{L_\infty} \eta = \mathrm{Res}_{L_\infty} \eta_{q0}$, it follows that $h_{q0} \cdot \omega$ is holomorphic along L_∞ and therefore $h_{q0} \cdot \omega$ is a closed *holomorphic* 1-form which defines $\mathcal{F}$ in $(x_\alpha \neq 0)$. Since $\lambda_\alpha \notin \mathbb{Q}$ it follows that $h_{q0} \cdot \omega = 0$. In fact, as it follows from [15], if we have

$$\mathrm{d}f \wedge (x\,\mathrm{d}y - \lambda y\,\mathrm{d}x) = 0, \quad \lambda \notin \mathbb{Q},$$

for some holomorphic function f defined on $(x \neq 0)$, then f must be constant (work with Laurent series for $f(x,y)$). Therefore we have constructed η in a neighborhood of L_∞ in $\mathbb{C}P(2)$. According to Levi's Extension Theorem [17] the 1-form η extends as a closed rational 1-form on $\mathbb{C}P(2)$, with the announced properties. According to the construction above, the local separatrices of the singularities at L_∞ (and *a fortiori* the separatrices of $\mathcal{F}$) are contained in the polar divisor of η, which is an algebraic curve. This is impossible since $\mathcal{F} \in M_1(n)$. □

3. Holonomy and Rigidity

Let $\mathrm{Diff}(\mathbb{C}, 0)$ be the set of germs of holomorphic diffeomorphisms at $0 \in \mathbb{C}$, fixing the origin. We will rely heavily on the work of A.A Scherbakov [16] and I. Nakai [12] concerning the dynamics of non solvable subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$. A basic fact is that a non solvable subgroup $\Gamma \subset \mathrm{Diff}(\mathbb{C}, 0)$, either has all orbits dense in small neighborhoods of $0 \in \mathbb{C}$, (*dense orbits property*, D.O.P. for short), or else there exists a germ of real analytic curve at $0 \in \mathbb{C}$ (holomorphically equivalent to $\mathrm{Im}\, z^\ell = 0$ for some $\ell \in \mathbb{N}$), which is invariant under the action of Γ.

Here we deal with the holonomy group $\mathrm{Hol}(\mathcal{F}, L_\infty)$, of a foliation $\mathcal{F} \in M_1(n)$, relative to L_∞ (which is a non solvable group). We define for $\mathcal{F} \in M_1(n)$

$$\mathrm{ord}(\mathcal{F}) = \min\big\{k \in \mathbb{N} \mid k \geq 2,\ \exists g(z) = z + az^k + \cdots \in \mathrm{Hol}(\mathcal{F}, L_\infty),\ a \neq 0\big\}.$$

LEMMA 2. — *The function* $M_1(n) \ni \mathcal{F} \mapsto \operatorname{ord}(\mathcal{F}) \in \mathbb{Z}$ *is upper semicontinuous.*

Proof. — We take $\mathcal{F}_0 \in M_1(n)$ and let $k_0 = \operatorname{ord}(\mathcal{F}_0)$; there exists $g_0(z) = z + a_0 z^{k_0} + \cdots \in \operatorname{Hol}(\mathcal{F}_0, L_\infty)$. We select some singularity $p_0 \in L_\infty$ of $\mathcal{F}_0$, and denote by

$$f_0(z) = \lambda_0 z + \cdots \in \operatorname{Hol}(\mathcal{F}, L_\infty),$$

the local holonomy map of $\mathcal{F}_0$, relative to L_0, around $p_0 \in L_\infty$; we have $|\lambda_0| \neq 1$. We consider

$$\begin{aligned} h_0(z) &:= g_0^{-1} \circ f_0^{-1} \circ g_0 \circ f_0(z) \\ &= z + a_0(\lambda_0^{k_0-1} - 1) z^{k_0} + \cdots \\ &= z + b_0 z^{k_0} + \cdots \in \operatorname{Hol}(\mathcal{F}_0, L_\infty). \end{aligned}$$

This element can be analytically followed in a small neighborhood of $\mathcal{F}_0 \in M_1(n)$:

$$h_{\mathcal{F}}(z) = z + \cdots + b_{\mathcal{F}} z^{k_0} + \cdots \in \operatorname{Hol}(\mathcal{F}, L_\infty)$$

for $\mathcal{F}$ close enough to $\mathcal{F}_0$; we have $b_{\mathcal{F}} \neq 0$, since $b_0 \neq 0$. Therefore $\operatorname{ord}(\mathcal{F}) \leq \operatorname{ord}(\mathcal{F}_0)$. $\square$

Let $M_2(n) \subset M_1(n)$ be the set of foliations such that the holonomy group at L_∞ has the D.O.P.

LEMMA 3. — *For all* $n \geq 2$, $M_2(n)$ *contains an open and dense subset* $\operatorname{Rig}(n)$ *of* $M_1(n)$.

Proof. — We start by considering

$$M_1'(n) = \{\mathcal{F}_0 \in M_1(n) \mid \operatorname{ord}(\mathcal{F}) = \operatorname{ord}(\mathcal{F}_0) \text{ for } \mathcal{F} \text{ in a neighborhood of } \mathcal{F}_0 \text{ in } M_1(n)\}.$$

The set $M_1'(n)$ is clearly open in $M_1(n)$. Let V be an open subset of $M_1(n)$; we choose $\mathcal{F}_1 \in V$, such that

$$\operatorname{ord}(\mathcal{F}_1) = \min\{\operatorname{ord}(\mathcal{F}) \mid \mathcal{F} \in V\}.$$

It follows from Lemma 2 that $\mathcal{F}_1 \in M_1'(n)$, so that $M_1'(n)$ is dense in $M_1(n)$.

Let us fix an open connected subset $V \subset M_1'(n)$ and $\mathcal{F}_0 \in V \setminus M_2(n)$ (if it exists). We know that there exists a germ of analytic curve S at $0 \in \mathbb{C}$, which is invariant under the action of $\mathrm{Hol}(\mathcal{F}_0, L_\infty)$. Let

$$\ell = \mathrm{ord}(\mathcal{F}_0) \quad \text{and} \quad h_0(z) = z + az^\ell + \cdots \in \mathrm{Hol}(\mathcal{F}_0, L_\infty)(a \neq 0).$$

The sectors around $0 \in \mathbb{C}$ determined by S, that is, the connected components of $U \setminus S$, where U is a small neighborhood of $0 \in \mathbb{C}$, are fixed under the action of the elements of $\mathrm{Hol}(\mathcal{F}_0, L_\infty)$ which are tangent to the identity map at $0 \in \mathbb{C}$; let U_1 be one of these sectors. The two half-lines L_1, L_2 which are tangent to ∂U_1 at $0 \in \mathbb{C}$ satisfy $\mathrm{Im}\, az^{\ell-1} = 0$.

Let $p_0 \in L_\infty$ be a singularity of $\mathcal{F}_0$ and $f_0(z) = \lambda_0 z + \cdots \in \mathrm{Hol}(\mathcal{F}_0, L_\infty)$ be the local holonomy map of $\mathcal{F}_0$ at $p_0 \in L_\infty$, relative to L_∞. Since

$$f_0^{-1} \circ h_0 \circ f_0(z) = z + a\lambda_0^{\ell-1} z^\ell + \cdots \in \mathrm{Hol}(\mathcal{F}_0, L_\infty),$$

we conclude that also

$$\mathrm{Im}(a\lambda_0^{\ell-1} z^{\ell-1}) = 0$$

when $z \in L_1 \cup L_2$. It follows that $\lambda_0^{\ell-1} \in \mathbb{R}$. The same is true in a neighborhood W of $\mathcal{F}_0$, namely, we follow f_0 as $f_{\mathcal{F}}(z) = \lambda_{\mathcal{F}} z + \cdots$ and, if $\mathcal{F} \notin M_2(n)$, one has $\lambda_{\mathcal{F}}^{\ell-1} \in \mathbb{R}$. Therefore $M_2(n) \cap W$ is open and dense in W. Lemma 3 follows then easily. $\square$

We proceed now to study deformations of elements of $\mathrm{Rig}(n)$, which are s-trivial. Let then $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$, $\mathcal{F}_0 \in \mathrm{Rig}(n)$ be s-trivial. We select a singularity $p(t)$ and consider some separatrix S_t^1 ($\neq L_\infty$); from D.O.P. and the fact that S_t^1 is not algebraic, we have that $\overline{S_t^1} = \mathbb{C}P(2)$.

LEMMA 4. — *$\bigcup_{|t|<\epsilon} S_t^1$ is a local lamination in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$, away from its singular points, for ϵ small.*

Proof. — Proposition 1 implies that $\bigcup_{|t|<\epsilon} (S_t^1)_{\mathrm{loc}}$ is, for $\epsilon > 0$ small, a holomorphic embedded surface, where $(S_t^1)_{\mathrm{loc}}$ is the local separatrix that passes through $p(t)$ which has S_t^1 as satured set. Let us consider a regular point $q_0 \in \mathbb{C}P(2)$ of $\mathcal{F}_0$, $q_0 \in \Sigma \subset \mathbb{C}P(2)$ a small transverse disk to $\mathcal{F}_0$ and $W \supset \Sigma$ an open set where $\mathcal{F}_0$ is trivial. Let A_0 be the component of S_0^1 which passes through $q \in \Sigma$, $\overline{A}_0 \subset W$; if $q \in \overline{\Sigma}^1 \subset \Sigma$ is close enough to q_0 and $\epsilon > 0$ is small, one has

$$\bigcup_{|t|<\epsilon} \phi_t(\overline{A}_0) \subset \bigcup_{|t|<\epsilon} W \times \{t\}$$

(ϕ_t is the continuous family of maps given by the definition of s-triviality). If $\overline{A}_0'$ is any other component, clearly

$$\bigcup_{|t|<\epsilon} \phi_t(\overline{A}_0) \cap \bigcup_{|t|<\epsilon} \phi_t(\overline{A}_0') = \emptyset$$

(these sets will be the laminae of the lamination). Let us prove that $\bigcup_{|t|<\epsilon} \phi_t(\overline{A}_0)$ is a holomorphic surface; it is enough to prove that

$$c(t) = \phi_t(\overline{A}_0) \cap (\Sigma \times \mathbb{D}_\epsilon)$$

is a holomorphic curve (for simplicity, we may assume that $c(t) = \phi_t(q)$, for all $|t| < \epsilon$). We join $q \in A_0$ to a point $\tilde{q} \in (S_0^1)_{\text{loc}}$ by a simple path ℓ_0 along S_0^1, and take a section $\widetilde{\Sigma} \ni \tilde{q}$ transverse to $\mathcal{F}_0$. Again, we may assume $\phi_t(\tilde{q}) \in \widetilde{\Sigma} \times \{t\}$. There exists a holonomy map associated to ℓ_0 from $\widetilde{\Sigma}$ to Σ; it sends $\tilde{q} \in \widetilde{\Sigma}$ to $q \in \Sigma$, and can be extended to the holonomy maps ψ_t associated to $\ell_t = \phi_t(\ell_0)$, $\psi_t(\phi_t(\tilde{q})) = \phi_t(q)$, for ϵ small enough. The map ψ defined as ϕ_t for each $|t| < \epsilon$, is holomorphic (since $\mathcal{F}_t$ is a holomorphic family), and $t \mapsto \phi_t(\tilde{q})$ is holomorphic, so is $c(t)$ as well.

We have then proved that $\bigcup_{|t|<\epsilon} S_t^1$ laminates a neighborhood of q_0 in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$. The lemma follows from the density of S_t^1 in $\mathbb{C}P(2)$. $\square$

We add to the lamination the set $L_\infty \times \mathbb{D}_\epsilon$. *A priori* this lamination might not be transversely continuous, but surprisingly enough it can be extended to a holomorphic foliation of $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$, with singularities :

LEMMA 5. — *There exists a codimension one holomorphic foliation $\widetilde{\mathcal{F}}$ on $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ such that :*

(i) $\operatorname{sing} \widetilde{\mathcal{F}} := \bigcup_{|t|<\epsilon} (\operatorname{sing} \mathcal{F}_t \times \{t\})$

(ii) *The leaves of $\mathcal{F}_t$ are the intersections of the leaves of $\widetilde{\mathcal{F}}$ with $\mathbb{C}P(2) \times \{t\}$, for each $|t| < \epsilon$.*

Proof. — Let us keep the same notation as in the proof of the last lemma. We have then a lamination in the neighborhood $(W \times \overline{\Sigma}') \times \mathbb{D}_\epsilon$ of $q_0 \in \Sigma$, and denote by $A_0(q)$ the component of S_0^1 which passes through $q \in \overline{\Sigma}' \subset \Sigma$. Let

$$q(t) = \phi_t\big(A_0(q)\big) \cap \big(\Sigma \times \{t\}\big);$$

the curve $\mathbb{D}_\epsilon \ni t \mapsto q(t) \in \Sigma \times \{t\}$ is a holomorphic curve, and the map $h_t(q) := q(t)$ is an injective map defined in a dense subset of $\overline{\Sigma}'$, and by the λ-lemma [18] it can be extended to all points of $\overline{\Sigma}'$. The foliation

$\widetilde{\mathcal{F}}$ is defined in $(W \times \overline{\Sigma}') \times \mathbb{D}_\epsilon$ as follows : given $q \in \overline{\Sigma}'$, we take the curve $q(t)$ as before and ℓ_t as the leaf of $\mathcal{F}_t$ restricted to $(W \times \overline{\Sigma}') \times \mathbb{D}_\epsilon$ passing through $q(t)$; then $\bigcup_{|t|<\epsilon} \ell_t$ will be a leaf of $\widetilde{\mathcal{F}}$ (which is of course holomorphic). This method can be carried to any regular point of $\mathcal{F}_0$, yielding the foliation $\widetilde{\mathcal{F}}$, which is completed by the curves of singularities, obtained following the singularities of $\mathcal{F}_0$ (see Proposition 1).

We have now to prove that $\widetilde{\mathcal{F}}$ is a holomorphic foliation (up to now, it is a continuous foliation with holomorphic leaves). Let us remark firstly that $L_\infty \times \mathbb{D}_\epsilon$ is a leaf of $\widetilde{\mathcal{F}}$; in order to prove that $\widetilde{\mathcal{F}}$ is transversely holomorphic along $L_\infty \times \mathbb{D}_\epsilon$, it is enough to guarantee that each map $h_t : \overline{\Sigma}' \to \Sigma \times \{t\}$ is holomorphic. Consider $q \in \overline{\Sigma}' \cap S_0^1$ and some $f_0 \in \mathrm{Hol}(\mathcal{F}_0, L_\infty)$ (we may assume that f_0 corresponds to the local holonomy around a singularity of $\mathcal{F}_0$ in L_∞). This map can be holomorphically followed as f_t along the deformation $\mathcal{F}_t$, and since we have s-triviality :

$$h_t\big(f_0(q)\big) = f_t\big(h_t(q)\big)$$

From the D.O.P. we get then

$$h_t \circ f_0 = f_t \circ h_t$$

in $\overline{\Sigma}'$. Therefore h_t conjugates $\mathrm{Hol}(\mathcal{F}_0, L_\infty)$ and $\mathrm{Hol}(\mathcal{F}_t, L_\infty)$; we invoke the Topological Rigidity Theorem of [12], [16] to conclude that h_t is a holomorphic map (here is where we use strongly the fact that the foliations $\mathcal{F}_t$ have non solvable holonomy groups associated to the leaf L_∞). Finally, since the leaves of each $\mathcal{F}_t$ accumulate on $L_\infty \times \{t\}$ (Maximum Principle), we conclude that $\widetilde{\mathcal{F}}$ is transversely holomorphic (outside its singular set), and therefore holomorphic ($\widetilde{\mathcal{F}}$ extends holomorphically to the singular set as a consequence of Hartogs' Theorem [17], because this set has codimension 2). $\square$

4. Proofs of Theorems 1.1 and 1.2

In this section Theorems 1.1 and 1.2 are proved. The first is proved as in [14]. Let $\mathcal{F} \in \mathrm{Rig}(n) \subset M_1(n)$ be given (where $\mathrm{Rig}(n)$ is defined as in Lemma 3), with $n \geq 2$, and consider an analytic deformation $\{\mathcal{F}_t\}_{t\in\mathbb{D}}$ of $\mathcal{F} = \mathcal{F}_0$, which is s-trivial. Let $\widetilde{\mathcal{F}}$ be the holomorphic foliation in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ given by Lemma 5. Let us choose a polynomial 1-form

$$\omega = p(x,y)\,\mathrm{d}y - q(x,y)\,\mathrm{d}x,$$

with isolated singularities and which defines $\mathcal{F}$ in $\mathbb{C}^2$. The foliation $\widetilde{\mathcal{F}}$ can therefore be given by a holomorphic 1-form

$$\Omega = P(x,y,t)\,\mathrm{d}y - Q(x,y,t)\,\mathrm{d}x + R(x,y,t)\,\mathrm{d}t$$

in the coordinate system $(x,y,t) \in \mathbb{C}^2 \times \mathbb{D}_\epsilon$, where $P(x,y,t), Q(x,y,t)$ and $R(x,y,t)$ are polynomials in the variables (x,y). We state a preliminar result based on Noether's Theorem :

LEMMA 6 (see [5], [14]). — *Under the hypothesis of Theorem* 1.1 *there exists a complete holomorphic vector field* $\widetilde{X}$ *on* $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ *which is tangent to* $\widetilde{\mathcal{F}}$ *(that is,* $\Omega \cdot \widetilde{X} \equiv 0$ *). Moreover* $\widetilde{X}$ *is of the form*

$$\widetilde{X}(x,y,t) = -b(x,y,t)\,\frac{\partial}{\partial x} + a(x,y,t)\,\frac{\partial}{\partial y} + \frac{\partial}{\partial t},$$

where $a(x,y,t)$ *and* $b(x,y,t)$ *are (degree one) polynomials on the affine variables* (x,y).

Proof of Theorem 1.1. — The local flow $\widetilde{X}_t$ of the vector field $\widetilde{X}$ given by Lemma 6 is such that $\widetilde{X}_t(x,y,0) \in \mathbb{C}^2 \times \{t\}$, is defined for all $t \in \mathbb{D}_\epsilon$, and the curve $t \mapsto \widetilde{X}_t(x,y,0)$ is contained in the leaf of $\widetilde{\mathcal{F}}$ through $(x,y,0)$, because $\Omega \cdot \widetilde{X} \equiv 0$. Moreover, the maps $\psi_t : \mathbb{C}^2 \to \mathbb{C}^2$, defined by $\psi_t(x,y) = \widetilde{X}_t(x,y,0)$ are affine maps, so that they extend to biholomorphisms of $\mathbb{C}P(2)$, which provide the analytical trivialization of the family $\{\mathcal{F}_t\}_{t\in\mathbb{D}_\epsilon}$. In order to finish the proof of Theorem 1.1 it suffices to use the fact that the set $\mathrm{Rig}(n)$ of foliations is open and dense in $\mathcal{X}(n)$ (*cf.* Theorem 3 and Lemma 3). $\square$

Proof of Theorem 1.2. — We have to construct the same holomorphic foliation in $\mathbb{C}P(2) \times \mathbb{D}_\epsilon$ as before. We keep the same notations introduced in Lemmas 4 and 5. We start by considering the continuous foliation $\widetilde{\mathcal{F}}$ with holomorphic leaves in $\mathbb{C}^2 \times \mathbb{D}_\epsilon$ (outside the singular locus) given by those lemmas applied to the separatrix of some singularity in $\mathbb{C}^2$. The problem here is that the triviality of $\widetilde{\mathcal{F}}$ in the box $\bigcup_{|t|<\epsilon} W \times \{t\}$, when $q_0 \in L_\infty$ is a regular point of $\mathcal{F}_0$, is not evident : we can not guarantee that $\phi_t(\overline{A}_0) \subset W \times \{t\}$ for all sets $\overline{A}_0$ and $|t| < \epsilon$. We have then to change the description of the leaves of $\widetilde{\mathcal{F}}$ in $\bigcup_{|t|<\epsilon}\big[(W \setminus L_\infty) \times \{t\}\big]$. Let

$$\overline{A}_0(q) = \bigcup_{|t|<\epsilon} \phi_t(\overline{A}_0)$$

for $q \in S_0 \cap \Sigma$. From the s-triviality, if we fix a compact subset $K \subset \Sigma \setminus \{0\}$, we may suppose that $\overline{A}_0(q) \subset \bigcup_{|t|<\epsilon} W \times \{t\}$ when $q \in K \cap S_0$. This compact set K is chosen so that $K \times \{t\}$ contains fundamental domains for the generators $f_{0,t}, \ldots, f_{n,t}$ of $\mathrm{Hol}(\mathcal{F}_t, L_\infty)$; for the moment we work with one of these generators, say $f_{0,t}$, and put $F_0(q,t) = (f_{0,t}(q), t)$. If $q, f_0(q) \in K$, again s-triviality implies that

$$F_0\big(\overline{A}_0(q)\big) = \overline{A}_0\big(f_0(q)\big).$$

Now, if $q' \in \Sigma \setminus \{0\}$, there exists $n(q') \in \mathbb{N}$ such that $q'' = f_0^{-n(q')}(q') \in K$; define

$$\overline{A}_0(q') := F_0^{n(q')}(\overline{A}_0(q'')),$$

and this definition independs on the choice of $n(q')$.

We have then a lamination $\mathcal{L} = \{\overline{A}_0(q)\}_{q \in \Sigma \cap S_0}$ and we may add $(L_\infty \cap W) \times \mathbb{D}_\epsilon$ as a lamina. This lamination extends to a continuous foliation $\overline{\mathcal{L}}$ in $\bigcup_{|t|<\epsilon} W \times \{t\}$ by the λ-lemma, and the relation

$$h_t \circ f_0 = f_t \circ h_t$$

holds. We claim that $\overline{\mathcal{L}}$ and $\widetilde{\mathcal{F}}_{|\cup_{|t|<\epsilon} W \times \{t\}}$ coincide. In fact, through a point $q' \in \Sigma \cap S_0$ passes a leaf of $\mathcal{L}$ which coincides with $\overline{A}_0(q')$ when $q' \in K$; by s-triviality, we propagate this property to any point of $\Sigma \cap S_0$, perhaps taking $\epsilon \in \mathbb{D}$ smaller depending on the point. Since the leaf and $\overline{A}_0(q')$ are holomorphic surfaces, they have to coincide throughout. We have then extended $\widetilde{\mathcal{F}}$ as a continuous foliation by including $(L_\infty \cap W) \times \mathbb{D}_\epsilon$ as a leaf, and verifying $h_t \circ f_0 = f_t \circ h_t$. Finally, the construction does not depend on the choice of the generator, because we end up with an extension of $\widetilde{\mathcal{F}}$, which is unique. Consequently, $h_t \circ f_j = f_{j,t} \circ h_t$ for $0 \le j \le n$, and this allows us (as in Lemma 5), to prove that $\widetilde{\mathcal{F}}$ is a holomorphic foliation. ∎

5. Rigidity in projective surfaces

Throughout this section M is a projective complex surface, $S \subset M$ an *irreducible* algebraic curve and the spaces $\mathrm{Fol}(M)$ and $\mathrm{Fol}(M,S)$ are the ones introduced in the beginning of the paper. Any foliation $\mathcal{F} \in \mathrm{Fol}(M)$ can be described by a holomorphic map of fibre bundles $\alpha : L \to TM$, where L is a line bundle over M. The topological type of L is given by its Chern class in $H^2(M,\mathbb{Z})$ which replaces the notion of *degree* of a foliation. We denote by $\mathrm{Fol}(M,n)$ the space of foliations in M with a given degree $n \in H^2(M,\mathbb{Z})$. It is known that $\mathrm{Fol}(M,n)$ is a complex variety of finite dimension (see [6]).

Again we are led to consider the following situation : $\mathcal{F} \in \mathrm{Fol}(M)$, $S \subset M$ is a compact analytic solution and $\mathcal{F}_t$ is an analytic deformation in the class $\mathrm{Fol}(M, S) \subset \mathrm{Fol}(M)$, that is, S is a solution of $\mathcal{F}_t$, for all $t \in \mathbb{D}$. We assume that the family $\mathcal{F}_t$ is topologically trivial. We introduce a continuous foliation $\widetilde{\mathcal{F}}$ on $M \times \mathbb{D}$ by saying that its leaves are the sets

$$\widetilde{L} := \bigcup_{t \in \mathbb{D}} \phi_t(L) \times \{t\},$$

where ϕ_t is the continuous family of equivalences given by the topological triviality, and L is a leaf of $\mathcal{F}$. The singular set is defined by

$$\operatorname{sing} \widetilde{\mathcal{F}} := \bigcup_{t \in \mathbb{D}} \operatorname{sing} \mathcal{F}_t \times \{t\}.$$

The next proposition is proved in the same spirit of [14]. The basic idea of using the results of [10], [19] can be found in [11].

PROPOSITION 4. — *Assume that :*

(i) *The holonomy group* $\mathrm{Hol}(\mathcal{F}, S)$ *is non solvable*;

(ii) $M \backslash S$ *is a Stein manifold*;

(iii) $\operatorname{sing} \mathcal{F} \cap S$ *consists of hyperbolic singularities.*

Then $\widetilde{\mathcal{F}}$ *is a holomorphic foliation on* $M \times \mathbb{D}$.

Proof. — The proof is divided in three parts. First we consider the nonsingular foliation $\widetilde{\mathcal{F}}' = \widetilde{\mathcal{F}}_{|M \backslash \operatorname{sing} \widetilde{\mathcal{F}}}$ and prove that its leaves are analytic submanifolds of $(M \times \mathbb{D}) \backslash \operatorname{sing} \widetilde{\mathcal{F}}$. Then we prove that $\widetilde{\mathcal{F}}$ is holomorphic in a neighborhood of $S \times \{0\} \subset M \times \mathbb{D}$. Finally we use the fact that $M \setminus S$ is a Stein manifold, in order to conclude that $\widetilde{\mathcal{F}}$ is holomorphic in all $M \times \mathbb{D}$. Let us denote by

$$\{p_j(t) \mid j = 1, \dots, N\}$$

the set of singular points $\operatorname{sing} \mathcal{F} \cap S$, where $N = N(n)$. Since $\mathcal{F}_0$ has hyperbolic singularities, there are holomorphic maps $\mathbb{D} \ni t \mapsto p_j(t) \in M$, for $j = 1, \dots, N$ such that

$$\operatorname{sing}(\mathcal{F}_t) \cap S = \{p_1(t), \dots, p_N(t)\}$$

(see [7]). We fix a local transverse section $\Sigma \cong \mathbb{D}$, $\Sigma \cap L_\infty = \{p\}$ so that $\Sigma \times \{t\}$ is also transverse to $\mathcal{F}_t$ if $t \in \mathbb{D}$ is small enough. We denote by G_t the subgroup of the holonomy group $\mathrm{Hol}(\mathcal{F}_t, S, \Sigma \times \{t\})$, generated by the holonomy maps $f_{j,t}$ associated to the singularities $p_j(t)$. We know that *these generators of G_t depend holomorphically on the parameter t.* We use the following fixed-point result :

THEOREM (see [9], [16]). — *Let G be a nonsolvable subgroup of* $\mathrm{Diff}(\mathbb{C},0)$. *There exists an open neighborhood Ω of $0 \in \mathbb{C}$, where G has a dense subset of hyperbolic fixed points.*

Take a point $q_0 \in \Sigma$, which is a hyperbolic fixed point for some word $f_0 \in G_0$. Then, following the same word in G_t and using the fact that the generators of G_t depend analytically on t, we obtain a holomorphic family of diffeomorphisms $f_t \in \mathrm{Diff}(\mathbb{C},0)$, whose maps f_t have hyperbolic fixed points $q_t \in \Sigma_t$ for t small enough, describing a germ of analytic curve. This implies that the leaf

$$\widetilde{L}_{q_0} = \bigcup_{t \in \mathbb{D}} \phi_t(L_{q_0}) \times \{t\}$$

of $\widetilde{\mathcal{F}}$ through $(q_0,0)$ is analytic along the transverse cut $\Sigma \times \mathbb{D} \subset M \times \mathbb{D}$ in a neighborhood of the point $(q_0,0) \in \Sigma$. This implies that $\widetilde{L}_{q_0}$ is holomorphic. Now, we use the fact that the set of such hyperbolic fixed points is dense in a neighborhood Ω of the point $p \in \Sigma_0$, in order to conclude that each leaf $\widetilde{L}$ is in fact a uniform limit of holomorphic leaves in a neighborhood of $S \times \{0\}$ in $M \times \mathbb{D}$. This implies that each leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ is in fact holomorphic, that is, the foliation $\widetilde{\mathcal{F}}$ has holomorphic leaves in a neighborhood of $S \times \{0\}$ in $M \times \mathbb{D}$.

It remains to conclude that $\widetilde{\mathcal{F}}$ is transversely holomorphic. This is done as follows : the deformation $\mathcal{F}_t$ is topologically trivial. We may therefore use the construction in the proof of Lemma 3 in order to obtain a continuous family $\{h_t\}_{t \in \mathbb{D},0}$ of homeomorphisms $h_t : (\mathbb{C},0) \to (\mathbb{C},0)$ which conjugate G_0 and G_t, for all $t \in (\mathbb{D},0)$. Since G_0 is non-solvable it follows from the Topological Rigidity Theorem of [16], [12] that $z \mapsto h_t(z)$ is holomorphic for each fixed $t \in (\mathbb{D},0)$. This implies that $\widetilde{\mathcal{F}}$ is transversely holomorphic outside the set of separatrices. For each $t \in \mathbb{D}$ we denote by $\operatorname{sep} \mathcal{F}_t$ the germ of analytic subset of the local separatrices of $\mathcal{F}_t$ through the singularities $\operatorname{sing} \mathcal{F}_t \cap S$, defined in a neighborhood of S in M. We consider the set

$$\operatorname{sep} \widetilde{\mathcal{F}} = \bigcup_{t \in \mathbb{D}} \operatorname{sep} \mathcal{F}_t.$$

Then, since $\mathcal{F}_t$ depends analytically on $t \in \mathbb{D}_\epsilon$ and since $\operatorname{sing} \mathcal{F} \cap S$ consists of nondegenerate singularities it follows that $\operatorname{sep} \widetilde{\mathcal{F}}$ is a codimension 1 germ of analytic subset of $S \times \mathbb{D}$ in a neighborhood of $S \times \mathbb{D}$.

Using the fact that the foliation $\widetilde{\mathcal{F}}'$ extends continuously to $\operatorname{sep} \widetilde{\mathcal{F}}$ which has codimension 1, it follows that $\widetilde{\mathcal{F}}$ is a holomorphic foliation in a neighborhood of $S \times \{0\} \subset M$. Since $(M \times \mathbb{D}) \setminus (S \times \mathbb{D})$ is a Stein manifold, it follows that $\widetilde{\mathcal{F}}$ is in fact a holomorphic foliation on $M \times \mathbb{D}$. $\square$

Using now the techniques of [6] we get then :

PROPOSITION 5. — *Let $\mathcal{F} \in \mathrm{Fol}(M,S)$ where M is a projective surface and $S \subset M$ be an algebraic very ample curve. Assume that*

(i) *$\mathcal{F}$ is given by a holomorphic bundle map $\alpha : L_0 \to TM$, where L_0 satisfies $H^1(M, \mathcal{O}_M(L_0)) = 0$;*

(ii) *$\mathrm{Hol}(\mathcal{F},S)$ is non solvable;*

(iii) *$\mathrm{sing}\,\mathcal{F} \cap S$ consists of hyperbolic singularities.*

Then any $\mathcal{F}$ is topologically rigid in the class $\mathrm{Fol}(M,S)$.

We may then proceed proving Theorems 2.1 and 2.2.

Proof of Theorem 2.1. — If $\mathcal{F} \in \mathrm{Fol}(\mathbb{C}P(2),S)$ is not rigid, then each local separatrix of $\mathcal{F}$ through a singularity $q \in \mathrm{sing}\,\mathcal{F} \cap S$, is contained in some algebraic leaf of $\mathcal{F}$ (otherwise $\mathrm{Hol}(\mathcal{F},S)$ would be non solvable and Proposition 4 applies). Let us denote by $C \supset S$ the algebraic curve obtained as the union of the algebraic leaves of $\mathcal{F}$ passing through some singularity $q \in \mathrm{sing}\,\mathcal{F} \cap S$. By the hypothesis of transversality on the local separatrices of $\mathcal{F}$ in $\mathbb{C}P(2)$, C is a nodal curve. We claim that

$$\deg(C) = \deg(\mathcal{F}) + 2.$$

To prove this fact, we consider a meromorphic vector field $X_{\mathcal{F}}$ with poles at L_∞ which defines $\mathcal{F}$ (by a projective change of coordinates, L_∞ can be supposed without singularities of $\mathcal{F}$ and non invariant). This vector field has now L_∞ as its line of poles with order $\deg(\mathcal{F}) - 1$. Let δ_S be the number of nodal points of S, and $\ell \in \mathbb{N}$ the number of the remaining singularities of $\mathcal{F}$ along S; it follows that

$$\ell = \deg(\overline{C \setminus S}) \cdot \deg(S).$$

According to the Poincaré-Hopf Theorem (see [3]) applied to $X_{\mathcal{F}|S}$ (or rather to a normalization of S) :

$$2\delta_S + \ell - \big[\deg(\mathcal{F}) - 1\big] \cdot \deg(S) = 2 - 2g(S),$$

where $g(S)$ is the genus of S. But

$$g(S) = \tfrac{1}{2}(\deg S - 1)(\deg S - 2) - \delta_S.$$

Combining these two expressions we prove the claim.

From [3] we may conclude that $\mathcal{F}$ is given by a logarithmic 1-form. This ends the proof. $\square$

Proof of Theorem 2.2. — Theorem 2.2 follows from Proposition 5 : if $\mathrm{Hol}(\mathcal{F}, S)$ is a solvable group, we may find a closed meromorphic 1-form η with simple poles such that $\mathrm{d}\omega = \eta \wedge \omega$, in the same way we did in the proof of Theorem 3. It follows that $F = \int \omega/H$ is a Liouvillian first integral for ω, where $H = \exp \int \eta$. ☐

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