BULLETIN DE LA S. M. F.

GREGORY MARGULIS

Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients

Bulletin de la S. M. F., tome 125, nº 3 (1997), p. 447-456
<http://www.numdam.org/item?id=BSMF_1997__125_3_447_0>

Bull. Soc. math. France,
125, 1997, p. 447–456.

EXISTENCE OF COMPACT QUOTIENTS OF HOMOGENEOUS SPACES, MEASURABLY PROPER ACTIONS, AND DECAY OF MATRIX COEFFICIENTS

BY GREGORY MARGULIS (*)

ABSTRACT. — The main purpose of the present paper is to give a new approach for constructing examples of homogeneous spaces G/H with no compact quotients where G is a Lie group and H is a closed noncompact subgroup. This approach is based on the study of the restriction to H of matrix coefficients of unitary representations of G. A similar method also gives a criterion when the restriction to H of an action of G on a locally compact space X with a G-invariant infinite measure is measurably proper in the sense that, for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper.

RÉSUMÉ. — Le but principal de cet article est de donner une nouvelle méthode pour construire des exemples d'espaces homogènes G/H qui n'admettent pas de quotients compacts où G est un groupe de Lie et H est un sous-groupe fermé non compact. Cette méthode est basée sur l'étude de la restriction à H des coefficients matriciels de représentations unitaires de G. Une méthode similaire donne un critère pour que la restriction à H d'une action de G sur un espace localement compact X qui admet une mesure G-invariante infinie soit mesurablement propre ce qui veut dire que l'application naturelle $H \to Hx$, $h \mapsto hx$, est propre pour presque tout $x \in X$.

Let G be a Lie group, and H a closed subgroup of G. There is a natural question: when does G/H have a compact quotient? More precisely when can one find a discrete subgroup Γ of G such that Γ acts properly on G/H and the quotient space $\Gamma\backslash G/H$ is compact? If G is semisimple and H is compact then according to a theorem of Borel G/H always has a compact form. But if H is not compact the answer to the question is unknown even for semisimple G. For a connected semisimple group G and a connected reductive subgroup H all known examples of homogeneous

(*) Texte reçu le 18 août 1997, accepté le 18 septembre 1997.
G. MARGULIS, Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06520-8283 (USA).
This work was supported in part by NSF Grant DMS-9424613.
AMS classification: 22E40,22E46,53C30,57S30.

spaces G/H which have compact quotients by discrete subgroups are based on the following construction. Suppose that there exists a connected closed subgroup $F \subset G$ such that $G = F \cdot H$ and $F \cap H$ is compact. Then G/H is naturally isomorphic to $F/F \cap H$. In particular if F is semisimple or, more generally, reductive we can use Borel's theorem to construct a compact quotient of G/H by a discrete subgroup.

On the other hand, there are many examples of homogeneous spaces G/H without compact quotients (see surveys [1], [2] and references therein). To prove that G/H has no compact quotients several criteria are used. These criteria are mostly based on considerations from topology, ergodic theory and the theory of linear groups. In this paper we give a new criterion which is based on the study of the restriction to H of matrix coefficients of unitary representations of G. This criterion gives many new examples of homogeneous spaces G/H without compact quotients.

The study of matrix coefficients also gives a criterion when the restriction to H of an action of G on a locally compact space X with a G-invariant (infinite) measure μ is measurably proper (in the sense that for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper).

ACKNOWLEDGEMENTS. — This work was completed during the author's stay at the University of Bielefeld in June–July, 1997. This stay was supported by the Humboldt Foundation. The author is grateful to Hee OH who made many useful comments on the preliminary version of the paper.

1. (G, K, H)-tempered actions

In this section G is a locally compact group, K is a compact subgroup of G, and H is a closed subgroup of G. Let θ denote a (left invariant) Haar measure on H.

Let G act continuously by measure preserving transformations on a (noncompact) locally compact space X with an infinite regular Borel measure μ. Consider the regular unitary representation ρ of G on $L^2(X,\mu)$:

$$\big(\rho(g)f\big)(x) = f(g^{-1}x); \quad g \in G,\ x \in X,\ f \in L^2(X,\mu).$$

DEFINITION 1. — We say that the action of G on X is (G,K,H)-*tempered* if there exists a (positive) function $q \in L^1(H,\theta)$ such that

$$|\langle \rho(h)f_1, f_2\rangle| \leq q(h)\|f_1\| \cdot \|f_2\| \tag{1}$$

for any $h \in H$ and any $\rho(K)$-invariant functions $f_1, f_2 \in L^2(X,\mu)$.

PROPOSITION 1. — *If the action of G on X is (G,K,H)-tempered then $\mu(X - HM) > 0$ and, consequently, $HM \neq X$ for any compact subset M of X.*

Proof. — Let M be a compact subset of X. Then there exists a non-negative K-invariant continuous function f on X with compact support such that $f(x) > 1$ for any $x \in M$. Consider a function

$$\varphi = \int_H \rho(h) f \,\mathrm{d}\theta(h), \quad \varphi(x) = \int_H f(h^{-1}x)\,\mathrm{d}\theta(h).$$

(The function φ can be infinite, and if HM is not compact then usually φ is not in $L^2(X,\mu)$.) Since f is continuous, M is compact and $f(x) > 1$ for any $x \in M$, there exists a neighborhood W of e in H such that $f(w^{-1}x) > \frac{1}{2}$ for all $x \in M$ and $w \in W$. Now if $x \in HM$ then $\varphi(x) > \frac{1}{2}\theta(W)$ (because if $h^{-1}x \in M$ then $f((hw)^{-1}x) > \frac{1}{2}$ for any $w \in W$). Thus

$$\varphi(x) > \tfrac{1}{2}\theta(W) \quad \text{for any} \quad x \in HM. \tag{2}$$

Take a compact subset L of H such that

$$\int_{H-L} q(h)\,\mathrm{d}\mu(h) < \frac{1}{2\|f\|}\theta(W) \tag{3}$$

where $\|f\| = \sup\{f(x) \mid x \in X\}$. Since the measure μ is Borel and infinite and the support $\mathrm{supp} f$ of f is not compact, there exists a K-invariant set $A \subset X$ such that $\mu(A) = 1$ and $(L \cdot \mathrm{supp} f) \cap A = \emptyset$. Let χ_A denote the characteristic function of A. Then using (1) and (3) we get

$$\begin{aligned}
(4) \quad \int_A \varphi(x)\,\mathrm{d}\mu(x) &= \int_X (\varphi \cdot \chi_A)(x)\,\mathrm{d}\mu(x) \\
&= \int_H \Big(\int_X ((\rho(h)f)\chi_A)(x)\,\mathrm{d}\mu(x) \Big)\,\mathrm{d}\theta(h) \\
&= \int_H \langle \rho(h)f, \chi_A \rangle\,\mathrm{d}\theta(h) = \int_{H-L} \langle \rho(h)f, \chi_A\rangle\,\mathrm{d}\theta(h) \\
&\le \int_{H-L} q(h)\|f\| \cdot \|\chi_A\|\,\mathrm{d}\theta(h) = \|f\| \int_{H-L} q(h)\,\mathrm{d}\theta(h) \\
&< \tfrac{1}{2}\theta(W).
\end{aligned}$$

The equality $\mu(A) = 1$ and the inequalities (2) and (4) imply that $\mu(A - HM) > 0$ and, consequently $\mu(X - HM) > 0$. □

PROPOSITION 2. — *Let A be a bounded Borel subset of X. For any $x \in X$, let $\psi_A(x)$ denote the θ-measure of the set $\{h \in H \mid hx \in A\}$. Suppose that the action of G on X is (G,K,H)-tempered.*

(a) *The function* ψ_A *is locally integrable, that is*

$$\int_B \psi_A(x)\,\mathrm{d}\mu(x) < \infty$$

for any bounded Borel subset B *of* X.

(b) *If* X *is* σ*-compact then* $\psi_A(x) < \infty$ *for almost all* $x \in X$.

Proof. — Clearly (a) implies (b). Let us prove (a). Replacing A by KA and B by KB, we can assume that A and B are K-invariant. Let χ_A and χ_B denote the characteristic functions of A and B. It is easy to see that

$$\psi_A = \int_H \rho(h)\chi_A\,\mathrm{d}\theta(h).$$

Then using (1) we get

$$\int_B \psi_A(x)\,\mathrm{d}\mu(x) = \langle\psi_A, \chi_B\rangle = \int_H \langle\rho(h)\chi_A, \chi_B\rangle\,\mathrm{d}\theta(h)$$
$$\leq \int_H q(h)\langle\chi_A, \chi_B\rangle\,\mathrm{d}\theta(h) < \infty. \quad \square$$

2. (G, K)-tempered subgroups

In this section G, K, H and θ denote the same as in §1.

DEFINITION 2. — We say that H is (G,K)-*tempered* if there exists a function $q \in L^1(H,\theta)$ such that

$$|\langle\pi(h)w_1, w_2\rangle| \leq q(h)\|w_1\| \cdot \|w_2\| \tag{5}$$

for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any unitary representation π of G without non-trivial $\pi(G)$-invariant vectors.

REMARK 1. — As in §1 let us consider a continuous action of G by measure preserving transformations on a locally compact space X with an infinite regular Borel measure μ, and let us denote by ρ the regular representation of G on $L^2(X,\mu)$. If $a > 0, f \in L^2(X,\mu)$ and $\rho(G)f = f$, then the sets

$$\{x \in X \mid f(x) > a\} \quad \text{and} \quad \{x \in X \mid f(x) < -a\}$$

have finite measure and they are G-invariant (modulo sets of measure 0). Hence if X has no G-invariant subsets of finite nonzero measure and the subgroup H is (G,K)-tempered then the action of G on X is (G,K,H)-tempered.

REMARK 2. — Let

$$\pi = \int_y \pi_y \,\mathrm{d}\sigma(y)$$

be a decomposition of π into a continuous sum of irreducible unitary representations, and let

$$W = \int_Y W_y \,\mathrm{d}\sigma(y), \quad w_1 = \int_Y w_{1y} \,\mathrm{d}\sigma(y), \quad w_2 = \int_Y w_{2y} \,\mathrm{d}\sigma(y),$$

w_{1y}, $w_{2y} \in W_y$, be corresponding decompositions of the space W of the representation π and of vectors $w_1, w_2 \in W$. Suppose that for all $y \in Y$

$$|\langle \pi_y(h) w_{1y}, w_{2y}\rangle| \leq q(h) \|w_{1y}\| \cdot \|w_{2y}\|.$$

Then using Cauchy-Schwartz inequality we get

$$\begin{aligned}
\big|\langle \pi(h) w_1, w_2\rangle\big| &= \Big| \int_Y \langle \pi_y(h) w_{1y}, w_{2y}\rangle \,\mathrm{d}\sigma(y)\Big| \\
&\leq q(h) \int_Y \|w_{1y}\| \cdot \|w_{2y}\| \,\mathrm{d}\sigma(y) \\
&\leq q(h) \sqrt{\int_Y \|w_{1y}\|^2 \,\mathrm{d}\sigma(y) \int_Y \|w_{2y}\|^2 \,\mathrm{d}\sigma(y)} \\
&= q(h) \|w_1\| \cdot \|w_2\|.
\end{aligned}$$

Thus H is (G,K)-tempered if and only if the inequality (5) is true for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any non-trivial irreducible unitary representation π of G.

Let us now give some examples of (G,K)-tempered subgroups. We give only indications of the proofs because more precise and general results are obtained by Hee Oh (see [3]).

EXAMPLES.

(a) Let G be a connected semisimple Lie group having Kazhdan's property (T) and K a maximal compact subgroup of G. Then any commutative diagonalizable subgroup H of G is (G,K)-tempered. To show this it is enough to use Howe-Moore estimates which provide uniform exponential decay for matrix coefficients corresponding to K-invariant vectors and irreducible nontrivial unitary representations of semisimple groups with property (T).

(b) Let $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n)$, and α_n the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Suppose that $n \geq 4$. Then the subgroup $H = \alpha_n(\mathrm{SL}(2,\mathbb{R}))$ is (G,K)-tempered. Let us show this in the case where $n = 4$ and

$$\alpha_4(d_t) = r_t, \quad d_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad r_t = \begin{pmatrix} e^{3t} & & & 0 \\ & e^t & & \\ & & e^{-t} & \\ 0 & & & e^{-3t} \end{pmatrix}.$$

It is well known that the restriction of any nontrivial irreducible unitary representation π of $\mathrm{SL}_4(\mathbb{R})$ to the subgroup

$$F = \left\{ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

does not contain complementary series. But r_t belongs to the subgroup

$$\left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ 0 & & & 0 \\ c & 0 & 0 & d \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

which is the direct product of two conjugates of F.

Using these facts and formulas for matrix coefficients of the principal series of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ we easily get that for some $c > 0$

$$\left|\langle \pi(r_t)w, w\rangle\right| \leq c e^{-4t} t^2 \cdot \left|\langle w, w\rangle\right|, \quad t \geq 0,$$

for any $\pi(K)$-invariant vector w. Now it remains to notice that the function

$$f(k_1 d_t k_2) = e^{-4t} t^2, \quad k_1, k_2 \in \mathrm{SO}(2),\ t \geq 0,$$

is integrable on $\mathrm{SL}_2(\mathbb{R})$ because the Haar measure of the set

$$\left\{ k_1 d_t k_2 \mid k_1, k_2 \in \mathrm{SO}(2),\ 0 \leq t \leq T \right\}$$

is asymptotically $c e^{2T}$ when $T \to +\infty$.

(c) Let L be a connected simple Lie group, $n \geq 3$, $\varphi : L \to \mathrm{SL}_n(\mathbb{R})$ an n-dimensional representation of L, and $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_i$ a decomposition of φ into the sum of irreducible representations of L. Let us denote by β the sum of the positive roots of L with respect to a maximal $\mathbb{R}$-split torus $S \subset L$ and an ordering on the character group $X(S)$ of S, and by χ_j the highest weight of the representation φ_j, $1 \leq j \leq i$. Then using arguments similar to those from the example (b) one can prove that the subgroup $\varphi(L)$ is $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$-tempered whenever

$$(*) \qquad \sum_{j \in \mathcal{J}} \chi_j > \beta(1+\varepsilon) \text{ for some } \varepsilon > 0, \text{ where } \mathcal{J} = \{j \mid \dim \varphi_j \geq 2\}.$$

From this we easily deduce the existence of $N > 0$ such that if

$$\sum_{j \in \mathcal{J}} \dim \varphi_j > N$$

then $\varphi(L)$ is $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n))$-tempered. (Let us note that $\sum_{j \in \mathcal{J}} \dim \varphi_j$ is the codimension in $\mathbb{R}^n$ of the subspace of $\varphi(L)$-invariant vectors.)

3. Compact quotients of homogeneous spaces

As usual we say that a continuous action of a locally compact group G on a locally compact space X is *proper* if, for every compact subset $L \subset X$, the set $\{g \in G \mid gL \cap L \neq \emptyset\}$ is compact. If G acts properly on X then the quotient space $G \backslash X$ is Hausdorff. We say that the action of G on X is *cocompact* if there exists a compact subset L of X such that $X = GL$. For proper actions this property is equivalent to the compactness of $G \backslash X$.

It is well known and easy to check that, for any locally compact group G and any closed subgroups P and Q of G, the following conditions are equivalent:

(I) the action of P on G/Q by left translations is proper (resp. cocompact);

(II) the action of Q on $P \backslash G$ by right translations is proper (resp. cocompact);

(III) the action $(p,q)g = pgq^{-1}$, $p \in P$, $q \in Q$, $g \in G$, of $P \times Q$ on G is proper (resp. cocompact).

It is natural to call the equivalence (I) $\Leftrightarrow$ (II) the *duality principle.*

THEOREM 1. — *Let G be a unimodular locally compact group, H a closed subgroup of G, and F a closed subgroup of H. Suppose that H is (G,K)-tempered for some compact subgroup K of G.*

(a) *If* Γ *is a discrete subgroup of* G *such that the volume of* $\Gamma\backslash G$ *with respect to Haar measure is infinite then the action of* Γ *on* G/F *by left translations is not cocompact.*

(b) *If* F *is not compact then there are no discrete subgroups* Γ *of* G *such that* Γ *acts properly on* G/F *by left translations and the quotient* $\Gamma\backslash(G/F)$ *is compact.*

Proof.

(a) The group G is unimodular. Therefore the action of G on $\Gamma\backslash G$ by right translations preserves Haar measure μ. Since $\mu(\Gamma\backslash G) = \infty$ there are no G-invariant subsets in $\Gamma\backslash G$ of finite measure. Hence (see Remark 1 after Definition 2) the action of G on $\Gamma\backslash G$ is (G, K, H)-tempered. Now applying Proposition 1 we get that the action of H on $\Gamma\backslash G$ and, consequently, the action of F on $\Gamma\backslash G$ are not cocompact. From this, using the above mentioned duality principle, we deduce that the action of Γ on G/F is not cocompact.

(b) In view of (a) it is enough to consider the case where $\mu(\Gamma\backslash G) < \infty$, but in this case F can not act properly on $\Gamma\backslash G$ because any continuous action of a noncompact group by transformations preserving a finite nonzero regular Borel measure is not proper. $\square$

Combining Theorem 1 with examples (b) and (c) from §2 we get the following two corollaries.

COROLLARY 1. — *Let* α_n *denote the* n*-dimensional irreducible representation of* $\mathrm{SL}_2(\mathbb{R})$. *Let* $G = \mathrm{SL}_n(\mathbb{R})$, $H = \alpha_n(\mathrm{SL}_2(\mathbb{R})) \subset G$, *and* F *a closed subgroup of* H. *Suppose that* $n \geq 4$. *Then for* G,H *and* F *the statements* (a) *and* (b) *in Theorem* 1 *are true. In particular* G/H *has no compact quotients by discrete subgroups.*

COROLLARY 2. — *Let* L *be a connected simple Lie group,* $n \geq 3$, *and let* $\varphi\colon L \to \mathrm{SL}_n(\mathbb{R})$ *be an* n*-dimensional representation of* L *such that the condition from example* (c) *of* §2 *is satisfied. Then the statements* (a) *and* (b) *of Theorem* 1 *are true for* $G = \mathrm{SL}_n(\mathbb{R})$,$H = \varphi(L)$ *and a closed subgroup* F *of* H.

4. Measurably proper actions

Let H be a locally compact second countable group acting continuously on a locally compact second countable space X with an H-quasi-invariant Borel measure μ. Let θ be a left invariant Haar measure on H. Then the following conditions are equivalent:

(a) for almost all (with respect to μ) points $x \in X$, the orbit Hx is closed in X and the stabilizer $H_x = \{h \in H \mid hx = x\}$ is compact;

(b) for almost all $x \in X$, the stabilizer H_x is compact and the natural map $hH_x \mapsto hx$ of H/H_x onto Hx is a homeomorphism;

(c) for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper or, in other words, the set $\{h \in H \mid Hx \in A\}$ is bounded in H for any bounded subset A of X;

(d) for almost all $x \in X$ and any bounded subset A of X, the θ-measure of the set $\{h \in H \mid hx \in A\}$ is finite.

The equivalences (a)$\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) are standard facts about group actions. The implication (c) $\Rightarrow$ (d) is trivial. To prove (d) $\Rightarrow$ (c) let us consider a bounded neighborhood U of e in H. Then

$$\{h \in H \mid hx \in UA\} = U\{h \in H \mid hx \in A\}.$$

Therefore if $\{h \in H \mid hx \in A\}$ is unbounded then $\{h \in H \mid hx \in UA\}$ has infinite measure. It remains to notice that if A is bounded then UA is also bounded.

If the conditions (a)–(d) are satisfied then we say the action of H on X is *measurable proper.* It is easy to see that if the action of H on X is measurably proper then almost all components in the decomposition of μ into H-ergodic measures are supported on closed H-orbits Hx with compact stabilizers H_x. In particular if the measure μ is H-ergodic then there exists $x \in X$ such that $\mu(X - Hx) = 0$, Hx is closed in X and H_x is compact. Let us also note that if H acts measurably proper on X and F is a closed subgroup of H then the action of F is also measurably proper.

THEOREM 2.—*Let G be a locally compact second countable group acting continuously on a locally compact second countable space X with a G-invariant (regular infinite) Borel measure μ, let H be a closed subgroup of G, and K a compact subgroup of G.*

(a) *If the action of G on X is (G,K,H)-tempered then the restriction of this action to H is measurably proper.*

(b) *If the subgroup H is (G,K)-tempered and X has no G-invariant subsets of finite nonzero measure then the action of H on X is measurably proper.*

Proof.

(a) follows from Proposition 2 (b). In view of Remark 1 from §2, (a) implies (b). □

REMARKS.

(I) In view of examples (a)–(c) from §2, Theorem 2 (b) can be applied in the following cases:

(a) G is a connected semisimple Lie group having Kazhdan's property (T) and H is a commutative diagonalizable subgroup of G;
(b) $G = \mathrm{SL}_n(\mathbb{R})$ and $H = \pi_n(\mathrm{SL}_2(\mathbb{R}))$ where $n \geq 4$ and π_n is the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$.
(c) $G = \mathrm{SL}_n(\mathbb{R})$ and $H = \varphi(L)$ where L is a connected simple Lie group and $\varphi : L \to \mathrm{SL}_n(\mathbb{R})$ is an n-dimensional representation of L such that the condition (*) from example (c) of §2 is satisfied.

(II) Let G be a unimodular locally compact second countable group, H a closed subgroup of G, and Γ a discrete subgroup of G. Suppose that the Haar measure of G/Γ is infinite and that H is (G, K)-tempered for some compact subgroup K of G. Then as a corollary of Theorem 2 we have that the action of H on G/Γ by left translations is measurably proper. In particular if in addition H is not open then the action of H on G/Γ is not ergodic.

BIBLIOGRAPHIE

[1] KOBAYASHI (T.). — *Discontinuous groups and Clifford-Klein forms on pseudoriemannian homogeneous manifolds*, in 'Algebraic and analytic methods in representation theory'. — B. Orsted and H. Schlichtkrull eds, Perspectives in Mathematics, vol. **17**, Academic Press, p. 99–165.
[2] LABOURIE (F.). — *Quelques résultats récents sur les espaces localement homogènes compacts*, in 'Manifolds and Geometry', actes d'un colloque (Pise 1993) en l'honneur d'Eugenio Calabi. — P. de Bartolomeis, F. Tricerri and E. Vesentini eds, Symposia Mathematica, vol. **XXXVI**, Cambridge University Press 1996.
[3] OH (H.). — *Representations with minimal decay of matrix coefficients and tempered subgroups*, Preprint.