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CONTROL THEOREMS OF p-NEARLY ORDINARY COHOMOLOGY GROUPS FOR SL(n)

BY

HARUZO HIDA (*)

ABSTRACT. — In this paper, we prove control theorems for the p-adic nearly ordinary cohomology groups for SL(n) over an aribitrary number field, generalizing the result already obtained for SL(2). The result should have various implications in the study of p-adic cohomological modulat forms on GL(n). In particular, in a subsequent paper, we will study p-adic analytic families of cuch Hecke eigenforms.

RÉSUMÉ. — Dans cet article, on démontre le théorème de contrôle pour les groupes de cohomologie quasi-ordinaire p-adique de SL(n) sur un corps de nombre arbitraire en généralisant le résultat déjà connu pour SL(2). Le résultat doit avoir des implications variées dans la théorie des formes modulaires p-adiques cohomologiques sur GL(n). En particulier, on étudiera des familles p-adiques analytiques des formes propres de Hecke dans un prochain article.

Introduction

Let p be a prime. In [H2], we have studied the control theorem for p-ordinary cohomology groups for the algebraic group SL(2) defined over an arbitrary number field F. Here we generalize the result to reductive algebraic groups G over $\mathbb{Q}$ whose group of $\mathbb{Q}_p$-points is isomorphic to $\mathrm{GL}_n(F_p)$ for $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We fix such an isomorphism

$$\text{(GL)} \qquad i : G(\mathbb{Q}_p) \cong \mathrm{GL}_n(F_p).$$

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Then for the derived group $G_{1/\mathbb{Q}}$ of G, the isomorphism i induces $G_1(\mathbb{Q}_p) \cong \mathrm{SL}_n(F_p)$. We further assume throughout the paper that

(SA) $G_1(\mathbb{Q})$ *is dense in* $G_1(A^{(\infty)})$ *for the ring of finite adèles* $\mathbb{A}^{(\infty)}$.

We consider a sufficiently large finite extension K of $\mathbb{Q}_p$ so that all rational absolutely irreducible representations of G are defined over K. We write $\mathcal{O}$ for the p-adic integer ring of K. For each partition $n = s + t$ of n into two parts by positive integers, we have the standard maximal proper parabolic subgroup P_s of $\mathrm{SL}(n)_{/\mathbb{Z}}$ given by

$$P_s = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}(n) \,;\; a \in \mathrm{GL}(t),\; d \in \mathrm{GL}(s) \right\}.$$

The set $\{P_1, \dots, P_{n-1}\}$ gives a complete representative set for conjugacy classes of maximal parabolic subgroups of $\mathrm{SL}(n)$. To each P_s, we can attach a Hecke operator T_s at p. It is given by the action of the double coset of $\xi_s \in G(\mathbb{Q})$ which is sufficiently close to $\begin{pmatrix} 1_t & 0 \\ 0 & p^m 1_s \end{pmatrix}$ in $G(\mathbb{A}^{(\infty)})$ for a suitable $m > 0$, where 1_t is the $t \times t$ identity matrix. More generally, the set of intersections

$$\mathcal{P} = \left\{ P = \bigcap_{s \in \Xi} P_s \,;\; \emptyset \neq \Xi \subset I \right\}$$

for $I = \{1, 2, \dots, n\}$ gives a complete representative set of conjugacy classes of all proper parabolic subgroups. Thus associated to P as above, we have a set of Hecke operators

$$\{T_s = T_s(p^m) \,;\; s \in \Xi\}$$

and a projector e_P (as an endomorphism of p-adic cohomology groups) attached to the product $\prod_{s \in \Xi} T_s$. We will prove the control theorems with finite error and the independence of weight for the p-nearly ordinary part associated to P, where the p-nearly ordinary part is the p-adic unit eigen-space of e_P (THEOREMS 4.1, 5.1, 5.2 and 6.1). When P is the Borel subgroup, we can also prove the exact control theorem for almost all primes p with some additional assumption (THEOREM 7.1). Here for simplicity, we state the result for the standard Borel subgroup B in $\mathcal{P}$ (made of upper triangular matrices). Let T be the standard split torus in B, that is, the torus made of diagonal matrices. For each dominant weight $\chi \in X(\mathrm{Res}_{F/\mathbb{Q}} T)$ with respect to $B^0 = \mathrm{Res}_{F/\mathbb{Q}} B$, we write $L(\chi; K)$

for the induced module $\mathrm{Ind}_{B^0}^S(\chi)$, where we regard B^0 as a subgroup of G_1 via (GL), and «Ind» indicates induction in the category of polynomial representations of algebraic groups. Let r be the integer ring of F, and put $r_p = r \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Since $\mathrm{SL}(n)$ has a natural integral structure, it induces an $\mathcal{O}$-integral structure on $L(\chi;K)$ that is, an $\mathcal{O}$-lattice $L(\chi;\mathcal{O})$ in $L(\chi;K)$ stable under $\mathrm{SL}_n(r_p)$ (we will explicitly specify the lattice in the text). We put

$$L(\chi) = L(\chi;K)/L(\chi;\mathcal{O}).$$

Let

$$I_B = \{x \in \mathrm{SL}_n(r_p) \mid x \mod p \text{ is upper triangular}\}.$$

We put $Y = I_B/U_B$, where U_B is the subgroup of (upper) unipotent matrices in I_B. We can naturally extend the left action of I_B on Y to the semi-group $\Delta = I_B D I_B$ (see Section 2) for the semi-group D of diagonal matrices with diagonal entries a_j with p-integral a_1 and satisfying $a_j \mid a_{j+1}$ for all j. The extension of the action depends on the choice of a prime element at each place of F over p. However the resulting idempotent e_B does not depend on such choices. There is also a natural right action on Y (induced by the right multiplication) of

$$T(r_p) \cong B(r_p)/U_B(r_p).$$

Then we consider the space $\mathcal{C}(Y;R)$ of continuous functions on Y having values in a topological module R. We regard $\mathcal{C}(Y;R)$ as a $(\Delta^{-1}, T(r_p))$-module by the action:

$$\gamma\phi(y) = \phi(\gamma^{-1}y) \quad \text{and} \quad z\phi(y) = \phi(yz)$$

for $\gamma \in \Delta^{-1} = \{\delta^{-1};\ \delta \in \Delta\}$ and $z \in T(r_p)$. If we take the standard lattice $L(\chi;\mathcal{O})$, we can show that $L(\chi)$ can be naturally embedded into $\mathcal{C} = \mathcal{C}(Y;K/\mathcal{O})$ as a Δ^{-1}-module. In other words, we have a natural embedding of I_B-modules from $L(\chi)$ into $\mathcal{C}$, whose image is stable under the action of Δ^{-1} on $\mathcal{C}$. Then we define the action of Δ^{-1} on $L(\chi)$ by the one induced from the action of Δ^{-1} on $\mathcal{C}$. Let Φ be a congruence subgroup of $G_1(\mathbb{Q})$. Write S for the closure of Φ in $G_1(\mathbb{A}^{(\infty)})$ and suppose

(S) $\quad S = \prod_\ell S_\ell \quad$ for a subgroup S_ℓ of $G_1(\mathbb{Q}_\ell)$ for each prime ℓ, and $S_p = I_B$.

Since $\Phi \subset \Delta^{-1}$, $\mathcal{C}$ is naturally a Φ-module. We put

$$\Phi_0(p^\alpha) = \{\gamma \in \Phi \mid \gamma \mod p^\alpha \in B(r/p^\alpha r)\},$$

$$\Phi_1(p^\alpha) = \{\gamma \in \Phi \mid \gamma \mod p^\alpha \in U_B(r/p^\alpha r)\}.$$

We write $H^q_{n\text{-ord}}$ for $e_B H^q$. Then we have:

THEOREM 1.1 (Independence of weight). — *We assume* (GL) *and* (S). *Then we have a canonical isomorphism of Hecke modules for each dominant weight* χ:

$$\iota_\chi : H^q_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\chi)\big) = \varinjlim_{\alpha} H^q_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\alpha), L(\chi)\big) \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}\big),$$

where the injective limit is taken with respect to the restriction maps, and ι_χ *commutes with Hecke operators supported outside* p.

Here ι_χ preserves the action of $T(r_p)$ in the following twisted way

$$\chi(z)\iota_\chi(zc) = z\iota_\chi(c) \quad \text{for } z \in T(r_p) \text{ and } c \in H^q_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\chi)\big),$$

where z acts on $H^q_{n\cdot\mathrm{ord}}(\Phi_1(p^\infty), L(\chi))$ through the projection

$$T(r_p) \longrightarrow \Phi_0(p^\alpha)/\Phi_1(p^\alpha)$$

and on $H^q_{n\cdot\mathrm{ord}}(\Phi_0(p), \mathcal{C})$ via the $T(r_p)$-module structure of $\mathcal{C}$.

THEOREM 1.2 (Control with finite error). — *We assume* (GL) *and* (S). *Let* χ *be a dominant weight of* $\mathrm{Res}_{F/\mathbb{Q}} T$. *Suppose that:*

$$H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi; K)\big) = 0 \quad \textit{for all} \quad 0 \le q < r.$$

Then the morphism ι_χ *and the inclusion* $L(\chi) \subset \mathcal{C}$ *induces a morphism of Hecke modules*

$$H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big) \longrightarrow H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}\big)[\chi]$$

which has finite kernel and cokernel.

Here for each $T(r_p)$-module M, we put:

$$M[\chi] = \big\{x \in M\,;\ wx = \chi(w)x \text{ for all } w \in T(r_p)\big\}.$$

Decompose $T(r_p) = W \times \mu$ for the maximal finite subgroup μ and the torsion-free part W. Then for each χ, we consider:

$$\mathcal{C}_\chi = \big\{\phi \in \mathcal{C}\,;\ \zeta\phi = \chi(\zeta)\phi \text{ for } \zeta \in \mu\big\}.$$

Then we have:

THEOREM 1.3 (Exact control). — *We assume* (GL) *and* (S). *Fix a regular dominant weight* χ *of* $\mathrm{Res}_{F/\mathbb{Q}} T$. *Suppose that:*

$$H^q\big(\Phi, L(\chi; \mathbb{C})\big) = 0 \quad \textit{for all} \quad 0 \le q < r.$$

Then for each dominant weight ψ *such that* $\psi_{|\mu} = \chi_{|\mu}$ *the morphism* ι_ψ *and the inclusion* $L(\psi) \subset \mathcal{C}_\chi$ *induces an isomorphism*

$$H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\psi)\big) \cong H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}_\chi\big)[\psi]$$

for almost all primes p.

Here almost all p means «except finitely many». We can give a slightly stronger result in the text dealing with characters χ which are only algebraic on a neighborhood of the identity element in the p-adic sense. We write S and $S_?(p^\alpha)$ for the closure in $G_1(\mathbb{A}^{(\infty)})$ of Φ and $\Phi_?(p^\alpha)$, respectively. Then, as is well known, we can consider a modular variety

$$Y_?(p^\alpha) = G_1(\mathbb{Q})\backslash G_1(\mathbb{A})/S_?(p^\alpha)C_{\infty+}$$

and a locally constant sheaf $\underline{L}$ on $Y_?(p^\alpha)$ associated to any S_p-module L, where $C_{\infty+}$ is a maximal compact subgroup of $G_1(\mathbb{R})$. The above theorems hold for sheaf cohomology groups of $\underline{L}(\chi)$ and $\underline{\mathcal{C}}$ (in place of group cohomology). We call the bottom degree for χ (with respect to B) the maximum degree $r = r(\chi)$ satisfying the condition in THEOREM 1.2. It is easy to show that $r > 0$ if Φ is an infinite group. When G is associated with a division algebra of degree n^2 over F split at p and ∞, assuming the existence of the generalized Jacquet-Langlands correspondence between automorphic representations of G and $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}(n)$ (which is known to hold under a certain local conditions on G by Arthur-Clozel and Vigneras), we will derive in Section 8 an explicit formula for $r(\chi)$ from a result of Clozel [C]. For example, if F is totally imaginary, then $r(\chi)$ is either $r(\chi) = \infty$ (that is, the nearly ordinary cohomology just vanishes for all degree) or $r(\chi)$ is given by $\frac{1}{4}n(n-1)[F:\mathbb{Q}]$ depending on χ. It is an interesting problem to determine the locus of χ with a given $r(\chi)$ in $\mathrm{Spec}(\mathcal{O}[[T(r_p)]])$ for the continuous group algebra $\mathcal{O}[[T(r_p)]]$ of $T(r_p)$. To stress the importance of the problem, we note here that the dimension of the locus is well related to the Leopoldt conjecture for p of F and extensions of F with degree n (see Conjecture 4.3 in [H4] in the case of $n = 2$). As an application of THEOREMS 1.1 and 1.2, we expect to have some finiteness theorems and control theorems for p-adic nearly ordinary Hecke algebras for G (as was given for GL(2) in [H3], [H4]). We hope to discuss this matter in future occasion.

The author already had the result for maximal parabolic subgroups before he wrote the previous paper [H2], since the argument given in [H2] works well without any modification in this maximally parabolic case. The existence of this paper owes much to L. CLOZEL who suggested the author to write down the result, and in the process of formulating the result for the maximally parabolic case, the author found the generalization to arbitrary parabolic subgroups, which requires a bit more work. The author is grateful to L. CLOZEL in this regard.

Here are some notations which we shall use throughout the paper. First of all, we keep the notation introduced in Section 1 throughout the paper. For two algebraic groups $G \supset H$ and a polynomial representation ρ

of H (that is, a morphism of algebraic groups from H to $\mathrm{GL}(d)$), $\mathrm{Ind}_H^G \rho$ indicates the induced representation in the category of polynomial representations of algebraic groups. For topological groups $G \supset H$ and a continuous representation ρ of H on a topological module V, $\mathrm{ind}_H^G \rho$ is the representation of G on

$$\mathrm{ind}_H^G V = \big\{\phi : G \to V \text{ continuous} \mid \phi(gh^\iota) = \rho(h)\phi(g) \text{ for } h \in H\big\},$$

where ι is a suitable involution specified in the text. For an $n_i \times n_i$ matrix A_i for $i = 1, \ldots, m$, we write $\mathrm{diag}(A_1, \ldots, A_m)$ for the $n \times n$ matrix $(n = n_1 + \ldots + n_m)$ whose i-th diagonal block is A_i $(i = 1, \ldots, m)$ and all the other off-diagonal blocks are zero. For a given parabolic subgroup $P \in \mathcal{P}$, we write $H^q_{P\text{-}n\cdot\mathrm{ord}}$ for the ordinary part with respect to the parabolic subgroup P. There is an exception: when $P = B$, the Borel subgroup, we just write $H^q_{n\cdot\mathrm{ord}}$ for $H^q_{B\text{-}n\cdot\mathrm{ord}}$. To each subset

$$\{s_1, \ldots, s_m\} \subset \{1, 2, \ldots, n\},$$

we associate a partition $n = n_1 + \ldots + n_m$ by $n_1 = s_1$, $n_j = s_j - s_{j-1}$ $(2 \leq j \leq m-1)$ and $t_m = n - s_m$. Then for the standard parabolic subgroup P associated to G_1, the standard Levi component M_P is given by

$$M_P(A) = \big\{\mathrm{diag}(A_1, \ldots, A_m) \in \mathrm{SL}_n(A)\,;\ A_j \in \mathrm{GL}_{n_j}(A)\ (j = 1, \ldots, m)\big\}$$

for any algebra A. Then $P = M_P U_P$ for the unipotent radical U_P of P. We write $M_P^{(1)}$ for the derived group of M_P. Thus

$$M_P^{(1)} = \mathrm{SL}(n_1) \times \cdots \times \mathrm{SL}(n_m).$$

We define a torus T_P by:

$$T_P(A) = \Big\{\mathrm{diag}(a_1 1_{n_1}, \ldots, a_{m-1} 1_{n_{m-1}}) \in \mathrm{GL}_{n-n_{m-1}}(A)\,;\\ a_j \in A^\times\ (j = 1, \ldots, m-1)\Big\}.$$

Then T_P is the center of M'_P given by:

$$M'_P(A) = \Big\{\mathrm{diag}(A_1, \ldots, A_{m-1}) \in \mathrm{GL}_{n-n_m}(A)\,;\\ A_j \in \mathrm{GL}_{n_j}(A)\ (j = 1, \ldots, m-1)\Big\}.$$

For each algebraic group H, we write $Z(H)$ for its center. In particular, we write $T_M = Z(M)$. Then, the projection of M_P into M' composed with the determinant map

$$\mathrm{diag}(A_1, \ldots, A_{m-1}) \longmapsto (\det(A_1) 1_{n_1}, \ldots, \det(A_{m-1}) 1_{n_{m-1}})$$

induces an isogeny $i : T_M \to T_P$.

Sometimes it is necessary to consider the standard parabolic subgroup of GL(n) associated to a standard parabolic subgroup P of SL(n). In that case, we write it as $\mathbb{P}$; therefore, $\mathbb{P} \cap \mathrm{SL}(n) = P$. Similarly, we write $\mathbb{T}$ for the standard torus of GL(n).

2. Local Hecke algebras

In this section, we shall determine the structure of abstract Hecke algebras made of double cosets of Γ_0-type open compact subgroups S (with respect to a standard parabolic subgroup) of SL(n) with coefficients in a non-archimedean local field. The result for open compact subgroups of GL(n) is well known. However we need to deal with double cosets $S\xi S$ for open compact subgroups S of SL(n) and ξ in GL(n). Even in this situation, we can determine the Hecke algebra in an elementary manner.

Let P be a proper standard parabolic subgroup of GL(n) associated with the partition $n = n_1 + n_2 + \ldots + n_m$. Let $\mathcal{V}$ be a discrete valuation ring finite flat over $\mathbb{Z}_p$ for a rational prime p. Let ϖ be a prime element of $\mathcal{V}$ and we write $\mathfrak{m} = \varpi\mathcal{V}$ and $\kappa = \mathcal{V}/\mathfrak{m}$. We write v for the valuation with $v(\varpi) = 1$.

In this section, we indicate $P(\mathcal{V})$ simply by P. Let:

$$D = \Big\{\mathrm{diag}(\varpi^{e_1}1_{n_1}, \ldots, \varpi^{e_m}1_{n_m}) \in \mathrm{GL}_n(\mathcal{F})\,; \quad \mathcal{V} \supset \varpi^{e_1}\mathcal{V} \supset \varpi^{e_2}\mathcal{V} \supset \cdots \supset \varpi^{e_m}\mathcal{V} \neq 0\Big\}. \tag{2.1}$$

When we emphasize the dependence on P, we write D_P instead of D. We then consider

$$\Delta_\infty = PDP \subset \mathrm{GL}_n(\mathcal{F})$$

for the field $\mathcal{F}$ of fractions of $\mathcal{V}$. For $\xi \in D$, we consider the double coset $P\xi P$. Then we decompose for a subset X of P:

$$P = \coprod_{\eta \in X} (\xi^{-1}P\xi \cap P)\eta.$$

Then multiplying by $\xi^{-1}P\xi$ from the left, we get:

$$\xi^{-1}P\xi P = \coprod_{\eta\in X} \xi^{-1}P\xi\eta \Longleftrightarrow P\xi P = \coprod_{\eta\in X} P\xi\eta.$$

In this way, we get a coset decomposition of $P\xi P$. We write

$$\xi = \mathrm{diag}(a_1 1_{n_1}, a_2 1_{n_2}, \ldots, a_m 1_{n_m}).$$

To describe the group $\xi^{-1}P\xi \cap P$, we write an $n \times n$ matrix A as (A_{ij}) for $n_i \times n_j$ blocks A_{ij}. Then we see:

$$\xi^{-1}P\xi \cap P = \{\gamma = (\gamma_{ij}) \in P\,;\ \gamma_{ij} \in a_i^{-1}a_j M_{n_i \times n_j}(\mathcal{V}) \text{ for all } j > i\}.$$

Then, choosing X to be a subset of P consisting unipotent matrices, such that

$$X \ni \eta \longmapsto (\eta_{ij} \mod a_i^{-1}a_j\mathcal{V})_{j>i} \in \bigoplus_{j>i} M_{n_i \times n_j}(\mathcal{V}/a_i^{-1}a_j\mathcal{V})$$

is a bijection, we see that X gives a complete representative set for $(\xi^{-1}P\xi \cap P)\backslash P$. In other words, to identify η from the product $\xi\eta$, we just need to look at $a_i^{-1}(\xi\eta)_{ij} \mod a_i^{-1}a_j M_{n_i\times n_j}(\mathcal{V})$. In particular,

$$\big(P : (\xi^{-1}P\xi \cap P)\big) = |\kappa|^{[\xi]},$$

where

$$[\xi] = \sum_{j>i} v(a_i^{-1}a_j)n_i n_j.$$

Then for ζ and $\xi \in D$, it is obvious that

$$[\xi\zeta] = [\xi] + [\zeta].$$

Thus writing $\deg(P\xi P)$ for $\#(P\backslash P\xi P)$, we see that

$$\deg(P\xi\zeta P) = \deg(P\xi P)\deg(P\zeta P).$$

Since $P\xi P\zeta P \supset P\xi\zeta P$, if we can show that $\deg(P\xi P\zeta P) = \deg(P\xi\zeta P)$, we will see that $P\xi P\zeta P = P\xi\zeta P$. Writing

$$\zeta = \operatorname{diag}(b_1 1_{n_1}, b_2 1_{n_2}, \ldots, b_m 1_{n_m}),$$

we take a complete representative set Z for $(\zeta^{-1}P\zeta \cap P)\backslash P$ made of unipotent elements. By definition, we know that:

$$P\xi P\zeta P = \bigcup_{\alpha \in X}\bigcup_{\beta \in Z} P\xi\alpha\zeta\beta.$$

Thus $\deg(P\xi P\zeta P) \le \deg(P\xi P)\deg(P\zeta P) = \deg(P\xi\zeta P)$. This shows the desired equality $\deg(P\xi P\zeta P) = \deg(P\xi\zeta P)$. Thus we see:

PROPOSITION 2.1. — *We have*:

(2.2a) $\Delta_\infty = PDP$ *is a semi-group*;

(2.2b) $P\xi P\zeta P = P\xi\zeta P = P\zeta P\xi P$ *and* $P\xi P \cdot P\zeta P = P\xi\zeta P = P\zeta P \cdot P\xi P$ *for* $\xi,\zeta \in D$ *where the latter identity is the identity in the Hecke algebra of* P *relative to the semi-group* Δ_∞.

Let $\mathcal{R}(P;\Delta_\infty)$ be the space of locally constant P-bi-invariant functions (with values in $\mathbb{Z}$) compactly supported on Δ_∞. The space $\mathcal{R}(P;\Delta_\infty)$ has a natural structure of algebra under the convolution product under the Haar measure μ with $\mu(P) = 1$. We write $P\xi P \in \mathcal{R}(P;\Delta_\infty)$ for the characteristic function of the open set $P\xi P$. For each $s > 0$, we write

$$\xi_s = \begin{pmatrix} 1_{n-s} & 0 \\ 0 & \varpi 1_s \end{pmatrix} \in \Delta_\infty.$$

For $j = 1, \ldots, m$, let:

$$s_j = n_{m-j+1} + n_{m-j+2} + \cdots + n_m.$$

Then we write $T_{s_j}(\varpi)$ for $P\xi_{s_j}P$. We have:

COROLLARY 2.1. — *We have an algebra isomorphism*

$$\mathcal{R}(P;\Delta_\infty) \cong \mathbb{Z}[T_1, \ldots, T_m]$$

given by $T_{s_j}(\varpi) \mapsto T_j$.

Let M be the standard Levi-part of P. Then:

$$M(A) = \{\mathrm{diag}(x_i) \in P(A) \mid x_i \in \mathrm{GL}_{n_i}(A)\}.$$

We write $\pi : P \to M$ for the natural projection, and we put:

$$P^{(1)} = M^{(1)}U_P = \{x \in P \mid \pi(x) = \mathrm{diag}(x_i) \text{ with } x_i \in \mathrm{SL}_{n_i}(A) \text{ for all i.}\}$$

Then $P/P^{(1)} \cong T_P$ *via* $\det : M \to T_P$ given by:

$$\det(\mathrm{diag}(x_i)) = \mathrm{diag}(\det(x_1)1_{n_1}, \ldots, \det(x_m)1_{n_m}).$$

Then we conclude from COROLLARY 2.1.

COROLLARY 2.2. — *We have an algebra isomorphism*:

$$\mathcal{R}(P^{(1)}(\mathcal{V});\Delta_\infty) \cong \mathbb{Z}[T_P(\mathcal{V})][T_1, \ldots, T_m]$$

given by $T_{s_j}(\varpi) \mapsto T_j$ *and* $P^{(1)}(\mathcal{V})uP^{(1)}(\mathcal{V}) \mapsto [\det(\pi(u))]$ *for* $u \in \mathbb{P}(\mathcal{V})$, *where* $[t]$ *is the group element* t *in the group algebra* $\mathbb{Z}[T_P(\mathcal{V})]$.

We consider the following subgroups:

$$(2.3a)\quad \begin{cases} I_\alpha = I_{P,\alpha}(\mathcal{V}) = \{\gamma \in \mathrm{SL}_n(\mathcal{V})\,;\ \gamma \bmod p^\alpha \in P(\mathcal{V}/p^\alpha)\}, \\ I_\alpha^{(1)}(\mathcal{V}) = \{\gamma \in \mathrm{SL}_n(\mathcal{V})\,;\ \gamma \bmod p^\alpha \in P^{(1)}(\mathcal{V}/p^\alpha)\}\,; \end{cases}$$

$$(2.3b)\quad I_{\mathbb{P},\alpha}(\mathcal{V}) = \{\gamma \in \mathrm{GL}_n(\mathcal{V})\,;\ \gamma \bmod p^\alpha \in \mathbb{P}(\mathcal{V}/p^\alpha)\}.$$

We write I_P for $I_{P,1}$. Let C be an open compact subgroup of $\mathrm{SL}_n(\mathcal{V})$ such that:

(P) *C contains $I_\alpha^{(1)}$ and C is contained in I_α.*

Put

$$\Delta_C(\mathcal{V}) = I_{P,\alpha}(\mathcal{V}) D_P(\mathcal{V}) I_{P,\alpha}(\mathcal{V}).$$

Then by the same argument as above, we get

$$C\xi C = \coprod_{\eta \in X} C\xi\eta,$$

where X is the complete representative set for $(\xi^{-1}P\xi \cap P)\backslash P$ already specified. Note that $U(\mathcal{V}) \subset I_\alpha^{(1)} \subset I_\alpha$. This in particular shows:

PROPOSITION 2.2. — *Suppose* (P). *Then we have*

(2.4a) *$\Delta_C(\mathcal{V})$ is a semi-group;*

(2.4b) *$C\xi C \cdot C\zeta C = C\xi\zeta C = C\zeta C \cdot C\xi C$ for $\xi,\zeta \in D$;*

(2.4c) *$\mathcal{R}(C,\Delta_C(\mathcal{V})) \cong \mathcal{R}(P;\Delta_\infty) \cong \mathbb{Z}[I_\alpha/C][T_1,T_2,\dots,T_m]$ by $T_{s_j}(\varpi) \mapsto T_j$, where $T_{s_j}(\varpi) = C\xi_{s_j}C$.*

Actually the argument which proves PROPOSITION 2.2 yields a bit more general result. Take a normal subgroup C of I_α such that

$$I_\alpha \supset C \supset \{x \in \mathrm{SL}_n(r_p)\,;\ x \bmod p^\alpha \in U_p(r/p^\alpha r)\}.$$

In this case, the assertions (2.4a), (2.4c) are valid without any change, although the Hecke ring can be non-commutative.

3. Flag varieties

Let I be the set of all field embeddings of F into an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Fixing an embedding i_p of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ for an algebraic closure $\overline{\mathbb{Q}}_p$ containing K, we regard I as the set of embeddings of F into $\overline{\mathbb{Q}}_p$. Then we assume that $\mathcal{O}$ contains r^σ for all $\sigma \in I$. For each parabolic subgroup P of SL(n), we define a p-adic Lie group I_P and $I_{\mathbb{P}}$ by

$$(3.1)\quad \begin{cases} I_{P,\alpha} = \{x \in \mathrm{SL}_n(r_p)\,;\ x \mod p^\alpha \in P(r/p^\alpha r)\}, \quad I_p = I_{P,1}, \\ I_{\mathbb{P},\alpha} = \{x \in \mathrm{GL}_n(r_p)\,;\ x \mod p^\alpha \in \mathbb{P}(r/p^\alpha r)\}, \end{cases}$$

where $r_p = r \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If we write

$$r_p = \prod \mathcal{V}$$

for valuation rings $\mathcal{V}$, then the above group $I_{P,\alpha}$ is the product of the groups $I_{P,\alpha}(\mathcal{V})$ introduced in the previous section. Let Φ be a congruence subgroup of $G_1(\mathbb{Q})$. In this section and the following sections, we study Φ-modules of continuous functions on the space $Y_p = I_P/U_p(r_p)$, which is naturally a left I_P-module and the right $M_P(r_p)$-module because $M_P \cong P/U_P$. To describe such spaces in terms of flags, we put for any $\mathbb{Z}_p$-algebras A, $L = L(A) = A^n$ viewed as the space of column vectors. Then we put:

$$(3.2)\quad \begin{cases} y_p(A) = \{(L_i)_{i=1,\dots,m} \mid L = L_m,\ L_i \supset L_{i-1}, \\ \qquad\qquad L_0 = \{0\},\ L_i/L_{i-1} \cong A^{n_i} \text{ for } i = 1, 2, \dots, m\}; \\ \mathcal{Y}_P(A) = \{(L_i, \tau_i)_{i=1,\dots,m} \mid (L_i) \in y_p(A), \\ \qquad\qquad \tau_i : A^{n_i} \cong L_i/L_{i-1} \text{ for } i = 1, 2, \dots, m\}. \end{cases}$$

Then $\mathrm{GL}_n(A)$ acts on $y_P(A)$ and $\mathcal{Y}_P(A)$ from the left by

$$g(L_i, \tau_i) = (gL_i, g \circ \tau_i).$$

Writing

$$s_m = n,\ s_{m-1} = n - n_1, \dots, s_{m-i} = n - \sum_{1 \le j \le i} n_j,$$

we define the origin O of $y_P(A)$ by the standard filtration

$$\mathrm{St}_i = \{{}^t(x_1, \dots, x_{s_i}, 0, \dots, 0)\,;\ x_i \in A\}$$

and the origin (O, id) of $\mathcal{Y}_P(A)$ by $(\mathrm{St}_i, \mathrm{id}_i)$ for the identity

$$\mathrm{id}_i : A^{n_i} \cong L_i/L_{i-1}.$$

Then it is easy to see that the stabilizer of the origin is $P(A)$ and $U_P(A)$, respectively. Thus we see:

$$y_P(A) \cong \mathrm{GL}_n(A)/\mathbb{P}(A) \quad \text{and} \quad \mathcal{Y}_P(A) \cong \mathrm{GL}_n(A)/U_P(A).$$

Since $\mathbb{P}$ normalizes U_P, we have a natural right action on $\mathrm{GL}_n(A)/U_P(A)$ of

$$x = (x_1, \dots, x_m) \in M_{\mathbb{P}}(A) = \mathbb{P}(A)/U_P(A) = \prod_j \mathrm{GL}(n_j).$$

This action can be written in terms of $\mathcal{Y}_P$ as $(L_i, \tau_i)x = (L_i, \tau_i \circ x_i)$. When

$$A = \varprojlim_{\alpha} A/p^\alpha A$$

(we call such a ring a *p-adic ring*), we look at a p-adic open set of $\mathcal{Y}_P(A)$ given by

$$\widehat{\mathcal{Y}}_\alpha(A) = \big\{(L_i, \tau_i) \in \mathcal{Y}_P(A) \mid (L_i/P^\alpha L_i) = O \text{ in } y_P(A/P^\alpha A)\big\}.$$

Note that $\widehat{\mathcal{Y}}_\alpha$ is the « α-th » formal neighborhood of the fiber over O of the special fiber of P/U (over r_p), and we have

$$\widehat{\mathcal{Y}}_\alpha(r_p) \cong I_{\mathbb{P},\alpha}/U_P(r_P).$$

The reduction map $(L_i, \tau_i) \mapsto (L_i/p^\beta L_i, \tau_i \bmod p^\beta)$ induces a projection:

$$Y_{P,\alpha} = I_{P,\alpha}/U_P(r_p) \subset \widehat{\mathcal{Y}}_\alpha(r_p) \longrightarrow \widehat{\mathcal{Y}}_\alpha(r/p^\beta r).$$

For $A = r/p^\beta r$, we write $Y_{P,\alpha}(A)$ for the image of $Y_{P,\alpha}$ in $\widehat{\mathcal{Y}}_\alpha(A)$. Then

$$Y_{P,\alpha} = \varprojlim_{\beta} Y_{P,\alpha}(r/p^\beta r),$$

and $Y_{P,\alpha}$ is a left $I_{P,\alpha}$-set and a right $M_P(r_p)$-set. If an A-free module M of rank s is a direct summand of an A-module N of rank $n = s+t$, then the exterior product $\bigwedge^n N$ is canonically isomorphic to $(\bigwedge^s M) \otimes_A (\bigwedge^t N/M)$. Using this fact, we can define $Y_{P,\alpha}(A)$ in $\widehat{\mathcal{Y}}_\alpha(A)$ by

$$\begin{aligned} Y_{P,\alpha}(A) = \Big\{(L_i, \tau_i) \mid \bigotimes_{1 \le i \le m} \Big(\bigwedge^{n} \tau_i\Big) : A = \bigotimes_i \Big(\bigwedge^{n_i} A^{n_i}\Big) \\ \longrightarrow \bigotimes_i \Big(\bigwedge^{n_i} L_i/L_{i-1}\Big) = \bigwedge^{n} L = A \text{ is a identity}\Big\}. \end{aligned}$$

Then this definition coincides with the previous one when $A = r_p$ and $r/p^\alpha r$. We now want to extend the left $I_{P,\alpha}$-action on $Y_{P,\alpha}(A)$ for $A = r_p$ and $r/p^\alpha r$ to an action of the following semi-group $\Delta_{P,\alpha}$. For that, we choose a prime element $\varpi_\mathfrak{p}$ of $r_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ dividing p. Let Σ be the set of prime ideal in r dividing p. For each Σ-tuple $e = (e_\mathfrak{p})$ of integers, we write $\varpi^e = (\varpi_\mathfrak{p}^{e_\mathfrak{p}})_{\mathfrak{p}\in\Sigma}$. Then we define

$$\Delta_\alpha = \Delta_{P,\alpha} = I_{P,\alpha} D_P I_{P,\alpha}$$

with

$$D_P = \big\{\mathrm{diag}(\varpi^{e_1} 1_{n_1}, \ldots, \varpi^{e_m} 1_{n_m}) \mid r_p \supset \varpi^{e_1} r_p \supset \varpi^{e_2} r_p \supset \cdots \supset \varpi^{e_m} r_p\big\}.$$

Since y_p is a projective scheme, we have

$$y_P(r_p) = y_P(F_p) = \mathrm{GL}_n(F_p)/P(F_p).$$

This identification is given by

$$y_P(r_p) \ni (L_i) \mapsto (L_i \otimes_\mathbb{Z} \mathbb{Q}_p) \in y_p(F_p).$$

Thus we have a natural left action of $\Delta_{P,\alpha} \subset G(\mathbb{Q}_p) = \mathrm{GL}_n(F_p)$ on $y_p(r_p)$ given by:

$$(L_i) \longmapsto \big(g(L_i \otimes_\mathbb{Z} \mathbb{Q}_p) \cap L(r_p)\big).$$

Abusing the notation, we write

$$(\delta L_i) = \big(\delta(L_i \otimes_\mathbb{Z} \mathbb{Q}_p) \cap L(r_p)\big) \quad \text{for} \quad \delta \in \Delta_{P,\alpha}.$$

Then we can write

$$\delta = udu' \quad \text{for} \quad u, u' \in I_{P,\alpha} \text{ and } d \in D_P.$$

We write $e_i(\delta) = e_i$ and $a_i = \varpi^{e_i}$ when $d = \mathrm{diag}(\varpi^{e_1} 1_{n_1}, \ldots, \varpi^{e_m} 1_{n_m})$. For $(L_i, \tau_i) \in Y_{P,\alpha}, L_i$ is generated by L_{i-1} and vectors whose first $n_1 + \ldots + n_{i-1}$ entries all vanish. Thus we can canonically identify L_i/L_{i-1} and dL_i/dL_{i-1}, although $(dL_i)_i$ may be distinct from (L_i). From this it is clear that d induces the multiplication by a_i on

$$L_i/L_{i-1} = dL_i/dL_{i-1}.$$

Since u and u' induces isomorphisms

$$[u]_i : u'L_i/u'L_{i-1} \cong uu'L_i/uu'L_{i-1},$$
$$[u']_i : L_i/L_{i-1} \cong u'L_i/u'L_{i-1},$$

respectively, we know that the morphism $[\delta]_i : L_i/L_{i-1} \to \delta L_i/\delta L_{i-1}$ is divisible by a_i, and

$$\varpi^{-e_i(\delta)}[\delta]_i : L_i/L_{i-1} \longrightarrow \delta L_i/\delta L_{i-1}$$

is a surjective isomorphism. Moreover

$$\bigotimes_i \Big(\bigwedge^{n_i} \varpi^{-e_i(\delta)}[\delta]_i\Big) : A = \bigotimes_i \Big(\bigwedge^{n_i} L_i/L_{i-1}\Big) \longrightarrow \bigotimes_i \Big(\bigwedge^{n_i} \delta L_i/\delta L_{i-1}\Big) = A$$

is the identity map. Thus we can define the action of $\Delta_{P,\alpha}$ on $Y_{P,\alpha}$ by

$$(L_i, \tau_i) \longmapsto \big(\delta L_i, \varpi^{-e_i(\delta)}[\delta]_i \circ \tau_i\big).$$

Under the expression $Y_{P,\alpha} = I_{P,\alpha}/U_P(r_p)$, δ acts on $Y_{P,\alpha}$ by conjugation.

Let $t = t_j = n_1 + n_2 + \ldots + n_j$ and $s_j = n - t_j$. Write

$$\xi_j = \begin{pmatrix} 1_t & 0 \\ 0 & p1_s \end{pmatrix} \quad \text{for} \quad s = s_j.$$

Let $(L_i, \tau_i) \in Y_{P,1}$. Since $L_j/pL_j = \mathrm{St}_j$, if $x_1, \ldots, x_t$ is a base of L_j, the first $t \times t$ block u of the $n \times t$ matrix $(x_1, \ldots, x_t)$ is an element in $\mathrm{GL}_t(r_p)$. Thus $\xi_j^\alpha L_j$ is generated by $(\xi_j^\alpha x_1, \ldots, \xi_j^\alpha x_t)$ whose first $t \times t$ block is equal to u and the bottom $s \times t$ block is divisible by p^α. Thus $\xi_j^\alpha L_j \bmod p^\alpha$ coincides with St_j. Since ξ_j fixes (St_i), we know that $\prod_j \xi_j^\alpha(L_i)$ is standard modulo p^α. Thus we have proven the following fact:

LEMMA 3.1. — *Let $\xi = \prod_{1 \le j \le m} \xi_j$. Then ξ^α contracts $\widehat{\mathcal{Y}}_\beta(r/p^\alpha r)$ and $Y_{P,\beta}(r/p^\alpha r)$ to the origin O for $\beta \ge 1$. In other words, $\xi^\alpha(L_i) = O$ for all $(L_i) \in \widehat{\mathcal{Y}}_\beta(r/p^\alpha r)$ and $Y_{P,\beta}(r/p^\alpha r)$.*

4. Independence of weight

We keep the notation introduced in the previous section. Let Φ be a congruence subgroup of $G_1(\mathbb{Q})$ and S be the closure of Φ in $G_1(A^{(\infty)})$. We assume that

$$S = \prod_{\ell} S_\ell \quad \text{and} \quad I_B \subset S_p \subset I_P, \tag{S}$$

where S_ℓ be the closure of Φ in $G_1(\mathbb{Q}_\ell)$ under the ℓ-adic topology for each prime ℓ. We write $\pi : P \longrightarrow M_P$ for the projection. Let H be a closed subgroup of $M_P(r_p)$ and H_α be the image of H in $M_P(r/p^\alpha r)$. Then associated to this parabolic subgroup P, we define several congruence subgroups of $G_1(\mathbb{Q})$:

$$\begin{cases} \Phi_{0,P}(p^\alpha) = \{\gamma \in \Phi\,;\ \gamma_p \mod p^\alpha \in P(r/p^\alpha r)\}, \\ \Phi_{H,P}(p^\alpha) = \{\gamma \in \Phi_{0,P}(p^\alpha)\,;\ \pi(\gamma_p \mod p^\alpha) \in H_\alpha\}. \end{cases} \tag{4.1}$$

When $H = M_P^{(1)}(r_p)$ (resp. $H = \{1\}$), we simply write $\Phi_{\mathrm{SL},P}(p^\alpha)$ (resp. $\Phi_{1,P}(p^\alpha)$) for $\Phi_{H,P}(p^\alpha)$.

If L is a left module over a semi-group generated by $\xi \in G(\mathbb{Q})$ and Φ, we can define the Hecke operator T acting on $H^q(\Phi, L)$ associated to the double coset $\Phi\xi\Phi$ (cf. [H2, § 1.10]). Let $S_{H,P}(p^\alpha)$ (resp. S) be the closure of $\Phi_{H,P}(p^\alpha)$ (resp. Φ) in $G_1(A^{(\infty)})$, which is an open compact subgroup of $G_1(A^{(\infty)})$. We consider for a partition $n = t + s$

$$x = \begin{pmatrix} p^{-s}1_t & 0 \\ 0 & p^t 1_s \end{pmatrix} \in \mathrm{SL}_n(F_p) = G_1(\mathbb{Q}_p).$$

Then by (SA), we can find $\xi \in G_1(\mathbb{Q})$ such that $\xi \equiv x \mod S_{1,P}(p^\alpha)$. Because of the finiteness of the class group of the central torus $Z = Z(G)$ of G, we know that

$$(p)\quad \begin{cases} \textit{there exist a positive integer } h \textit{ and a global element } \varpi \in Z(\mathbb{Q}) \\ \textit{such that } \varpi^{-1}p^h \in r_p^\times \textit{ in } G(\mathbb{Q}_p),\ \varpi^{-1}p^h \equiv 1 \mod p^\alpha \textit{ and } \varpi \\ \textit{is in a open compact subgroup of } Z(\mathbb{A}^{(\infty p)}) \textit{ outside } p \textit{ (that is,} \\ \varpi \textit{ is a unit outside } p\textit{),} \end{cases}$$

where r is the integer ring of F, $r_p = r \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and we view $r_p^\times$ as the maximal open compact subgroup of $Z(\mathbb{Q}_p)$. Then we put:

$$\xi'_s = \varpi^s \xi^h.$$

Thus ξ'_s satisfies for each $\alpha > 0$:

$$\begin{cases} \xi'_{s,p} \equiv \begin{pmatrix} 1_t & 0 \\ 0 & \varpi^s p^{t\,h} 1_s \end{pmatrix} \bmod S_{1,P}(p^\alpha), \\ \xi'_{s,\ell} S_\ell = S_\ell \xi'_{s,\ell} \quad \text{for all } \ell \neq p, \end{cases}$$

where S_ℓ is the ℓ-component of S. By modifying ξ'_s by a unit, we can find $\xi_s \in G(\mathbb{Q})$ such that for a suitable positive $j = (j, j, \ldots, j) \in \mathbb{Z}^\Sigma$

$$(\mathrm{T}p_s) \qquad \begin{cases} \xi_{s,p} \equiv \begin{pmatrix} 1_t & 0 \\ 0 & \varpi^j 1_s \end{pmatrix} \bmod S_{1,P}(p^\alpha), \\ \xi_{s,\ell} S_\ell = S_\ell \xi_{s,\ell} \quad \text{for all } \ell \neq p. \end{cases}$$

Because of the density of Φ in S, we can embed the Hecke algebra with respect to the group $\Phi_{H,P}(p^\alpha)$ and the semi-group generated by

$$\Big\{ \xi_{s_k} \mid s_k = \sum_{k<i\leq m} n_i \quad \text{for } k = 1, \ldots, m-1 \Big\}$$

isomorphically into $\bigotimes_{\mathfrak{p}|p} \mathcal{R}(C_\mathfrak{p}, \Delta_{C_\mathfrak{p}})$ for $C_\mathfrak{p} = S_{H,P}(p^\alpha)_\mathfrak{p}$ in the previous section, where $\mathfrak{p}$ runs over prime factors of p in r and $S_{H,P}(p^\alpha)_p$ is the closure of $\Phi_{H,P}(p^\alpha)_p$ in $\mathrm{GL}_n(r_\mathfrak{p})$ for the local ring $r_\mathfrak{p}$ of r_p at $\mathfrak{p}$. We define the operator $T_s(\varpi^{j\gamma})$ by the action of the double coset $\Phi_{H,P}(p^\alpha)\xi_s^\gamma \Phi_{H,P}(p^\alpha)$ on $H^i(\Phi_{H,P}(p^\alpha), L)$ for L. Since $T_s(\varpi^{j\gamma}) = T_s(\varpi^j)^\gamma$ as seen in the previous section, the idempotent e_P attached to $\prod_k T_{s_k}(\varpi^j)$ (see [H2, § 1.11] and (2.4) in the text) is well defined independent of the choice of ξ_s.

Let A be a ring. For any A-module L and an ideal $\mathfrak{a}$ in A, we write

$$L[\mathfrak{a}] = \big\{ m \in L \mid \alpha m = 0 \text{ for all } \alpha \in \mathfrak{a} \big\}.$$

Write $\mathcal{C}(T;T')$ for the space of continuous functions on T with values in T' for two topological spaces T and T'. For each $\mathcal{O}$-module R, we let $g \in M_P(r_p)$ act on $\phi \in \mathcal{C}(Y_{P,\alpha}(A);R)$ by

$$g\phi(y) = \phi(yg) \qquad \text{for } A = r_p \text{ and } r/p^\alpha r.$$

We say that R is of *finite corank* if its Pontryagin dual is an $\mathcal{O}$-module of finite type. If R is of finite corank, $\mathcal{C}(Y_{P,\alpha}(A);R)$ becomes a left $\mathcal{O}[[M_P(r_p)]]$-module. We consider $\mathcal{C}(Y_{P,\alpha}(A);R)$ as a left Δ_P^{-1}-module (resp. a right Δ_P-module) in the following manner:

$$(b\phi)(x) = \phi(b^{-1}x) \quad \big(\text{resp. } \phi \mid b(x) = \phi(bx)\big).$$

Then $\mathcal{C}(Y_{P,\alpha}(A);R)[\mathfrak{a}]$ for each ideal a in $\mathcal{O}[[M_P(r_p)]]$ is a left Δ^{-1}-module (since the action of $M_P(r_p)$ and Δ commutes each other).

We can think of the dual version of $\mathcal{C}(Y_{P,\alpha}(A);K/\mathcal{O})$. The Pontryagin dual module of $\mathcal{C}(Y_{P,\alpha}(A);K/\mathcal{O})$ is isomorphic to the space of p-adic measures $\mathcal{M}(Y_{P,\alpha}(A);\mathcal{O})$ on $Y_{P,\alpha}(A)$ with values in $\mathcal{O}$. Let ${}^tU_P$ be the opposite unipotent subgroup. For an additive subgroup X of A, we write ${}^tU_P(X)$ for the subgroup of ${}^tU_P(A)$ made of matrices whose lower triangular entries are in X. Then we know from the Iwahori decomposition $I_{P,\alpha} = {}^tU_P(p^\alpha r_p)M_P(r_p)U_P(r_p)$ that

$$\mathcal{M}(Y_{P,\alpha};\mathcal{O}) \cong \mathcal{O}[[{}^tU_P(p^\alpha r_p)]]\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[M_P(r_p)]],$$

where

$$\mathcal{O}[[{}^tU_P(p^\alpha r_p)]] = \varprojlim_{\beta} \mathcal{O}\big[{}^tU_P(p^\alpha r_p/p^\beta r_p)\big].$$

Let $\mathcal{Z}$ be the symmetric space of G_1; thus, $\mathcal{Z} = G_1(\mathbb{R})/C_{\infty+}$ for a maximal compact subgroup $C_{\infty+}$. Then for any Φ-module L and a subgroup Γ of Φ, we write $\underline{L}$ the sheaf of locally constant sections on $Y = Y(\Gamma) = \Gamma\backslash\mathcal{Z}$ with values in $\mathcal{L} = \Gamma\backslash(\mathcal{Z}\times L)$, where $\gamma \in \Gamma$ acts on $\mathcal{Z}\times L$ by $\gamma(z,m) = (\gamma z, \gamma m)$. That is, we put the discrete topology on L (disregarding any inherent topology on L) and for any open set U of Y, $\underline{L}(U)$ is the space of continuous sections over U with values in $\mathcal{L}$. Note that

$$Y(S) = G_1(\mathbb{Q})\backslash G_1(\mathbb{A})/SC_{\infty+}$$

in the introduction is isomorphic to $Y(\Phi)$. When we deal with sheaf cohomology groups, we suppose that

(TF) Φ *is torsion-free.*

Then we define $H^q_c(\Gamma, L)$ by the compactly supported sheaf cohomology group $H^q_c(Y, \underline{L})$ for any subgroup Γ of Φ. Suppose now that L is either discrete or compact. Let L^* be the Pontryagin dual of L. Write

$$d = \dim_R \mathcal{Z}.$$

Then $H^q(\Gamma, L)$ and $H^{d-q}_c(\Gamma, L^*)$ are mutually Pontryagin dual by the cup product. We have for each injective system $\{L_\alpha\}$ of Γ-modules

$$H^q(\Gamma, L) = \varinjlim_{\alpha} H^q(\Gamma, L_\alpha) \quad \text{and} \quad H^q_c(\Gamma, L) = \varinjlim_{\alpha} H^q_c(\Gamma, L_\alpha). \tag{4.2}$$

To see (4.2) for compactly supported cohomology groups, we note that the Borel-Serre compactification of $\Gamma\backslash\mathcal{Z}$, which is a manifold with corner by construction, is topologically equivalent to a manifold with boundary (see Appendix of [BS] by A. DOUADY and L. HÉRAULT). Then we may use the argument in [H2, § 1.8] to show (4.2). Then $\Delta = \Delta_\alpha$ acts naturally on $Y_{P,\alpha}(r/p^\beta r)$ and hence $\mathcal{C}(Y_{P,\alpha}(r/p^\beta r);R)$ is naturally a left Δ^{-1}-module. Note that

$$\mathcal{LC}(Y_{P,\alpha};R) = \varinjlim_{\beta} \mathcal{C}\big(Y_{P,\alpha}(r/p^\beta r);R\big)$$

is the space of locally constant functions on $Y_{P,\alpha}$. In particular,

$$\mathcal{LC}(Y_{P,\alpha};R) = \mathcal{C}(Y_{P,\alpha};R)$$

if R is a discrete module. Thus, writing H^q_* for any one of H^q_c and H^q, we see

$$\text{(4.3a)} \qquad H^q_*\big(\Gamma, \mathcal{LC}(Y_{P,\alpha};R)\big) = \varinjlim_{\beta} H^q_*\big(\Gamma, \mathcal{C}(Y_{P,\alpha}(r/p^\beta r);R)\big),$$

$$\text{(4.3b)} \quad H^q_*\big(\Gamma, \mathcal{C}(Y_{P,\alpha};p^{-\gamma}\mathcal{O}/\mathcal{O})\big) = \varinjlim_{\beta} H^q_*\big(\Gamma, \mathcal{C}(Y_{P,\alpha}(r/p^\beta r);p^{-\gamma}\mathcal{O}/\mathcal{O})\big)$$

for $\gamma = 1, 2, \dots, \infty$. We call a $(\Delta_\alpha^{-1}, \mathcal{O})$-module *admissible* if it is an injective limit of $(\Delta_\alpha^{-1}, \mathcal{O})$-modules of finite corank. As seen in [H2, § 1.11], we know that

$$\text{(4.4)} \quad \begin{cases} \textit{There is an idempotent } e_P \textit{ acting on } H^q_*(\Gamma, L) \textit{ for } \Gamma = \Phi_{H,P}(p^\alpha) \\ \textit{such that } \prod_{1\le k\le m-1} T_{s_k}(\varpi^j) \textit{ is an automorphism on } e_P H^q_*(\Gamma, L) \\ \textit{and topologically nilpotent on } (1-e_P)H^q_*(\Gamma, L), \end{cases}$$

where L is an admissible $(\Delta_\alpha^{-1}, \mathcal{O})$-module. We now consider an exact sequence of Δ_α^{-1}-modules

$$0 \to K_{\alpha,\beta} \to \mathcal{C}(Y_{P,1}(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O}) \xrightarrow{\ \pi\ } \mathcal{C}(M_P(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O}) \to 0,$$

where the last map is given by $\phi((L_i, \tau_o)) \mapsto ((\tau_i) \mapsto \phi((\mathrm{St}_i, \tau_i)))$. Thus we have the corresponding cohomology exact sequence for each subgroup Γ of $\Phi_{0,P}(p^\alpha) \subset \Delta_\alpha^{-1}$:

$$\begin{aligned} H^q_*(\Gamma, K_{\alpha,\beta}) &\to H^q_*\big(\Gamma, \mathcal{C}(Y_{P,1}(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O})\big) \\ &\to H^q_*\big(\Gamma, \mathcal{C}(M_P(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O})\big) \to H^{q+1}_*(\Gamma, K_{\alpha,\beta}). \end{aligned}$$

We now recall the definition of the operator

$$[\Gamma\xi\Gamma] : H^q_*(\Gamma;L) \longrightarrow H^q_*(\Gamma, L)$$

for $\xi \in G(\mathbb{Q})$ given in [H3] for a $\langle \Gamma, \xi^{-1} \rangle$-module L, where $\langle \Gamma, \xi^{-1} \rangle$ is the semi-group generated by Γ and ξ^{-1} in $G(\mathbb{Q})$. We define

$$[\Gamma\xi\Gamma] = \mathrm{Tr}_{\Gamma/\psi'} \circ [\xi] \circ \mathrm{res}_{\Gamma/\psi},$$

where $\psi' = \xi\psi\xi^{-1}$ for $\psi = \Gamma \cap \xi^{-1}\Gamma\xi$, $\mathrm{Tr}_{\Gamma/\psi'}$ is the transfer map and $\mathrm{res}_{\Gamma/\psi}$ is the restriction map. The morphism

$$[\xi] : H^q_*(\psi; L) \to H^q_*(\psi', L)$$

for the module L is given as follows. Let u be a homogeneous i-cocycle of $\psi = \Gamma \cap \xi^{-1}\Gamma\xi$ with values in L. Then we define the action of $[\xi]$ on u by

$$[\xi]u(\gamma_0, \dots, \gamma_i) = \xi^{-1}u(\xi\gamma_0\xi^{-1}, \dots, \xi\gamma_i\xi^{-1}).$$

The operator $[\xi]$ gives rise to a well defined operator between the cohomology groups as above. We may consider the closed subgroup $\mathbb{G}_1$ of $G(A)$ generated by $G_1(\mathbb{A})$ and $G(\mathbb{Q})$. For each open compact subgroup S of $G_1(\mathbb{A}^{(\infty)})$, we consider another expression $Y(S) = G(\mathbb{Q})\backslash\mathbb{G}_1/SC_{\infty+}$ of the modular manifold $Y = \Phi\backslash\mathcal{Z}$. Then the sheaf $\underline{L}$ for an admissible S_p-module L can be realized as a sheaf of locally constant sections over Y to $G(\mathbb{Q})\backslash\mathbb{G}_1 \times L/SC_{\infty+}$, where the action of $\gamma \in G(\mathbb{Q})$ and $s \in SC_{\infty+}$ is given by

$$\gamma(g, \ell)s = (\gamma g s, s_p^{-1}\ell).$$

Under this circumstance, for $\xi \in G(\mathbb{Q})$, we have an expression:

$$[\Gamma\xi\Gamma] = \mathrm{Tr} \circ [\xi] \circ \mathrm{res} \quad \text{on } H^q_*\big(Y(S), \underline{L}\big),$$

where $\mathrm{res} : H^q_*(Y(S), \underline{L}) \to H^q_*(Y(S \cap \xi^{-1}S\xi), \underline{L})$ is the restriction map and Tr is the trace map from $H^q_*(Y(\xi S\xi^{-1} \cap S), \underline{L})$ into $H^q_*(Y(S), \underline{L})$. We say that the Hecke operator $S\xi S$ is *supported on a semi-group* X in $G(\mathbb{Q}_p)$ if $(S\xi S)_p \subset X$.

The operator

$$\prod_{1 \le k \le m-1} T_{s_k}(\varpi^{j\alpha})$$

is given by $[\Gamma\xi\Gamma]$ for $\xi = \prod_{1 \le k \le m-1} \xi^{\alpha}_{s_k}$ in (Tp_s). Suppose that u has values in $K_{\alpha,\beta}$. Then

$$u\big(\gamma_0, \dots, \gamma_i : (\mathrm{St}_i, \tau_i)\big) = 0$$

for all $(\tau_i) \in M_P(r/p^\alpha r)$ and all $(\gamma_0, \dots, \gamma_i) \in \Gamma^{i+1}$. By LEMMA 3.1, ξ contracts $Y_{P,1}(r/p^\alpha r)$ to $\{(\mathrm{St}_i, \tau_i)\,;\ (\tau_i) \in M_P(r/p^\alpha r)\}$. Thus we have

$$\begin{aligned}(4.5a)\quad [\xi]u\big(\gamma_0, \dots, \gamma_i; (L_i, \tau_i)\big) &= u\big(\xi\gamma_0\xi^{-1}, \dots, \xi\gamma_i\xi^{-1}, \xi(L_i, \tau_i)\big)\\ &= u\big(\xi\gamma_0\xi^{-1}, \dots, \xi\gamma_i\xi^{-1}; (\mathrm{St}, \tau_i)\big) = 0.\end{aligned}$$

This shows

$$(4.5b)\qquad \prod_{1 \le k \le m-1} T_{s_k}(\varpi^{j\alpha}) \ \text{ kills } \ H^q(\Gamma, K_{\alpha,\beta}) \text{ for every } \alpha > 0.$$

In particular, we have for $\Gamma \subset \Phi_{0,P}(p^\alpha)$

$$\begin{aligned}(4.6)\qquad & e_P H^q\big(\Gamma, \mathcal{C}(Y_{P,1}(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \cong e_P H^q\big(\Gamma, \mathcal{C}(M_P(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})\big).\end{aligned}$$

Hereafter we write $e_P H^q$ as $H^q_{P\text{-}n\cdot\mathrm{ord}}$. The isomorphism (4.6) commutes with Hecke operators $[\Gamma\eta\Gamma]$ supported on Δ_P, because by choosing a suitable representative η in $[\Gamma\eta\Gamma]$, we may assume that the action of η on the modules $\mathcal{C}(Y_{P,1}(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})$ and $\mathcal{C}(M_P(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})$ commutes with the projection π. We also write

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\infty), \mathcal{C}(Y_{P,1}; p^{-\beta}\mathcal{O}/\mathcal{O})\big)$$

(where $\beta = 1, 2, \dots, \infty$ and $p^{-\infty}\mathcal{O}/\mathcal{O} = K/\mathcal{O}$) for

$$\begin{aligned}&\varinjlim_\alpha H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha), \mathcal{C}(Y_{P,1}; p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \cong \varinjlim_\alpha H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha), \mathcal{C}(Y_{P,1}(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})\big).\end{aligned}$$

Then we have:

PROPOSITION 4.1. — *Assume* (GL) *and* (S). *Then we have a canonical isomorphism of Hecke modules*:

$$\begin{aligned}&H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\infty), \mathcal{C}(Y_{P,1}; p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \cong \varinjlim_\alpha H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha), \mathcal{C}(M_P(r/p^\alpha r); p^{-\beta}\mathcal{O}/\mathcal{O})\big),\end{aligned}$$

where the injective limit is taken with respect to the restriction maps.

Here the terminology «morphism of Hecke modules» means that the morphism is compatible with Hecke operators supported on Δ. Note that:

$$\mathcal{C}\big(M_P(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O}\big) \cong \operatorname{ind}_{\Phi_{1,P}(p^\alpha)}^{\Phi_{0,P}(p^\alpha)} p^{-\beta}\mathcal{O}/\mathcal{O}.$$

Thus by Shapiro's lemma, we have:

$$\begin{aligned}(4.7\text{a})\qquad H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha),\mathcal{C}(M_P(r/p^\alpha r);p^{-\beta}\mathcal{O}/\mathcal{O})\big)&\\ \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{1,P}(p^\alpha),p^{-\beta}\mathcal{O}/\mathcal{O}\big).&\end{aligned}$$

This isomorphism is an isomorphism of Hecke modules.

We now take $\xi' = \prod_{1\le k\le m-1}\xi_{s_k}$ as in (Tp_s) for $\gamma \ge \alpha > 0$ and put $\xi = \xi'^{\gamma-\alpha}$. We consider the following diagram for each $\langle\Phi_{0,P}(p^\alpha),\xi^{-1}\rangle$-module L:

$$(4.7\text{b})\qquad \begin{array}{ccc} H^q\big(\Phi_{0,P}(p^\alpha),L\big) & \longrightarrow & H^q\big(\Phi_{0,P}(p^\gamma),L\big)\\ \Big\downarrow T_1 & \swarrow T_2 & \Big\downarrow T_3\\ H^q\big(\Phi_{0,P}(p^\alpha),L\big) & \longrightarrow & H^q\big(\Phi_{0,P}(p^\gamma),L\big) \end{array}$$

where

$$\begin{aligned}T_1 &= \big[\Phi_{0,P}(p^\alpha)\xi\Phi_{0,P}(p^\alpha)\big],\\ T_2 &= \big[\Phi_{0,P}(p^\gamma)\xi\Phi_{0,P}(p^\alpha)\big],\\ T_3 &= \big[\Phi_{0,P}(p^\gamma)\xi\Phi_{0,P}(p^\gamma)\big].\end{aligned}$$

The commutativity of the above diagram follows from the argument in Section 2. The key point is

$$\Phi_{0,P}(p^\gamma)\backslash\Phi_{0,P}(p^\gamma)\xi\Phi_{0,P}(p^\alpha) \cong \Phi_{0,P}(p^\alpha)\backslash\Phi_{0,P}(p^\alpha)\xi\Phi_{0,P}(p^\alpha)$$

which follows from (SA) combined with the explicit computation of coset decomposition given in Section 2 above PROPOSITION 2.2.

Using the diagram (4.7b), we conclude for any admissible Δ_α^{-1}-module L:

$$(4.7\text{c})\qquad H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha),L\big) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\gamma),L\big)\quad \text{for } \gamma\ge\alpha.$$

We put:

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{1,P}(p^\infty),K/\mathcal{O}\big) \cong \varinjlim_{\alpha} H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{1,P}(p^\alpha),K/\mathcal{O}\big).$$

Then we have:

THEOREM 4.1. — *Assume* (S) *and* (GL). *Then we have a canonical isomorphism of Hecke modules*:

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p),\mathcal{C}(Y_{P,1};K/\mathcal{O})\big) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{1,P}(p^\infty),K/\mathcal{O}\big).$$

5. Control Theorem

We start with a general lemma for our later use. Let $\mathfrak{g}$ be a topological group isomorphic to a product of a finite group μ and a $\mathbb{Z}_p$-free module W of finite rank m. For any $(\mathfrak{g}, A)$-module X and an A-valued character ε of $\mathfrak{g}$, we write $X[\varepsilon]$ for

$$\{m \in X\ ;\ \gamma m = \varepsilon(\gamma)m \text{ for } \gamma \in \mathfrak{g}\}.$$

Similarly for each A-module X and $T_i \in A$ $(i = 1,\dots,m)$ or an ideal $\mathfrak{a} \subset A$ of a ring A, we write

$$X[T_1,\dots,T_m] = \{x \in X\ ;\ T_i x = 0 \ \text{ for } \ i = 1,\dots,m\},$$

$$X[\mathfrak{a}] = \{x \in X\ ;\ ax = 0 \ \text{ for } \ a \in \mathfrak{a}\}.$$

Let $\mathbf{\Lambda} = \mathcal{O}[[\mathfrak{g}]]$, $\Lambda = \mathcal{O}[[W]]$ and χ be a continuous character of $\mathfrak{g}$ into $\mathcal{O}^\times$. Then χ induces an algebra homomorphism $\chi : \mathbf{\Lambda} \to \mathcal{O}$. Since Λ is a regular local ring, $\mathcal{P} \cap V$ for $\mathcal{P} = \mathrm{Ker}(\chi)$ in $\mathbf{\Lambda}$ is generated by a regular sequence $(T_1,\dots,T_m)$.

LEMMA 5.1. — *Let* $\mathfrak{C}$ *be an abelian subcategory of the category of discrete* $\mathbf{\Lambda}$*-modules and* $\mathcal{C}$ *be an object of* $\mathfrak{C}$. *Let* $L \mapsto H^\bullet(L)$ *be a cohomology functor* (*transforming short exact sequences into long ones*) *defined on* $\mathfrak{C}$ *with values in the category of discrete* $\mathbf{\Lambda}$*-modules. Suppose the following four conditions*:

(i) $\mathcal{C}$, $\mathcal{C}[\chi]$ *and* $\mathcal{C}[T_1,\dots,T_j]$ *for* $j = 1,\dots,m$ *are objects in* $\mathfrak{C}$;

(ii) *For each* $j = 1,\dots,m$, *the multiplication by* T_j *is surjective on* $\mathcal{C}[T_1,\dots,T_{j-1}]$ (*this in particular implies that* $x \mapsto T_1 x$ *is surjective on* $\mathcal{C}$);

(iii) $|H^q(\mathcal{C}[\chi])| < \infty$ *for all* $0 \le q < r$ *for an integer* r;

(iv) *The Pontryagin dual module of* $H^q(\mathcal{C}[\chi])$ *is of finite type over* $\mathcal{O}$ *for all* $0 \le q \le r$.

Then the natural morphism

$$H^r\big(\mathcal{C}[\chi]\big) \longrightarrow H^r(\mathcal{C})[\chi]$$

has finite kernel and cokernel. Further if $H^r(\mathcal{C}[T_1,\dots,T_m]) = 0$ *for all* $0 \le q < r$, *then*

$$H^r\big(\mathcal{C}[T_1,\dots,T_m]\big) \cong H^r(\mathcal{C})[T_1,\dots,T_m].$$

Proof. — By the hypothesis, we have a short exact sequence:

$$0 \to \mathcal{C}[T_1,\dots,T_j] \longrightarrow \mathcal{C}[T_1,\dots,T_{j-1}] \xrightarrow{T_j} \mathcal{C}[T_1,\dots,T_{j-1}] \to 0.$$

Thus we get another short exact sequence from the long cohomology exact sequence:

$$0 \to H^{q-1}\big(\mathcal{C}[T_1,\dots,T_{j-1}]\big) \otimes \Lambda/T_j\Lambda \longrightarrow H^q\big(\mathcal{C}[T_1,\dots,T_j]\big) \longrightarrow H^q\big(\mathcal{C}[T_1,\dots,T_{j-1}]\big)[T_j] \to 0.$$

Take the Pontryagin dual of the above short exact sequence

(*) $$0 \to E_{j-1}/T_jE_{j-1} \longrightarrow E_j \longrightarrow N_{j-1}[T_j] \to 0,$$

where $E_j = H^q(\mathcal{C}[T_1,\dots,T_j])^*$ and $N_j = H^{q-1}(\mathcal{C}[T_1,\dots,T_j])^*$. Here the « $*$ » indicates the Pontryagin dual module. As for the last assertion, what we need to show is the dual version:

$$E_m \cong E_0/(T_1,\dots,T_m)E_0.$$

The above sequence is valid even if $q = 0$. In this case, $N_{j-1} = 0$, and the assertion is evident from the above short exact sequences. Now we are going to show the assertion by induction on q. By topological Nakayama's lemma, we see from the above short exact sequence that for $j = m$, E_{m-1} is of finite type over $\Lambda/(T_1,\dots,T_{m-1})$, because E_m is of finite type over $\mathcal{O} = \Lambda/(T_1,\dots,T_m)$. Then by induction on $(m-j)$, we find the finiteness of E_j for all j. Now, we suppose that

$$H^q\big(\mathcal{C}[T_1,\dots,T_m]\big) = 0 \quad \text{for} \quad 0 \le q < r.$$

If $N_j = 0$ for all j, then we have $E_j \cong E_{j-1}/T_jE_{j-1}$. If moreover $E_m = 0$, by Nakayama's lemma, we conclude $E_j = 0$ for all j. Thus by induction on q, we get $N_j = 0$ for $q = r$. Therefore we obtain the exact control for E_j. Now, we suppose that $|H^q(\mathcal{C}[\chi])|$ is finite for $0 \le q < r$. Then localizing at $\mathcal{P}$, one gets

(**) $$H^q\big(\mathcal{C}[\chi]\big)^*_{\mathcal{P}} = 0 \quad \text{for } 0 \le q < r.$$

Note that

$$\mathbf{\Lambda}_{\mathcal{P}} = \Lambda_{\mathcal{P}\cap\Lambda}.$$

Then, in $\mathbf{\Lambda}_{\mathcal{P}}$, $\mathcal{P}\mathbf{\Lambda}_{\mathcal{P}}$ is generated by $T_1, \ldots, T_m$, and (**) implies that $H^q(\mathcal{C}[T_1, \ldots, T_m])^*_{\mathcal{P}} = 0$. In fact, for characters ω of μ, we have an exact sequence:

$$0 \to L \longrightarrow \bigoplus_{\omega} \mathcal{C}[\chi\omega] \longrightarrow \mathcal{C}[T_1, \ldots, T_m] \longrightarrow L' \to 0$$

with p-torsion L and L'. Thus we have a morphism:

$$\bigoplus_{\omega} H^q\big(\mathcal{C}[\xi\omega]\big) \longrightarrow H^q\big(\mathcal{C}[T_1, \ldots, T_m]\big)$$

with finite kernel and p-torsion cokernel. This shows that

$$H^q\big(\mathcal{C}[T_1, \ldots, T_m]\big)^*_{\mathcal{P}} \cong H^q\big(\mathcal{C}[\chi]\big)^*_{\mathcal{P}}.$$

Then the argument as above applied to

$$(***) \qquad 0 \to E_{j-1,\mathcal{P}}/T_j E_{j-1,\mathcal{P}} \longrightarrow E_{j,\mathcal{P}} \longrightarrow N_{j-1,\mathcal{P}}[T_j] \to 0$$

shows that $E_{m,\mathcal{P}} = E_{0,\mathcal{P}}/\mathcal{P}E_{0,\mathcal{P}}$ for $q = r$. Thus we have

$$\begin{aligned} H^r\big(\mathcal{C}[\chi]\big)^* \otimes_{\mathcal{O}} K = H^r\big(\mathcal{C}[\chi]\big)^*_{\mathcal{P}} &= E_{m,\mathcal{P}} \cong E_{0,\mathcal{P}}/\mathcal{P}E_{0,\mathcal{P}} \\ &\cong (E_0/\mathcal{P}E_0)_{\mathcal{P}} = \big(H^r(\mathcal{C})[\chi]\big)^* \otimes_{\mathcal{O}} K. \end{aligned}$$

Therefore the natural homomorphism

$$\iota^* : \big(H^r(\mathcal{C})[\chi]\big)^* \longrightarrow H^r\big(\mathcal{C}[\chi]\big)^*$$

has kernel and cokernel killed by sufficiently large power of p. Since the two sides of the above homomorphism are of finite type over $\mathbf{\Lambda}$, ι^* is of finite kernel and cokernel. This shows the result (see [H2, 1.16 b]).

Now we study the control theorem for the standard parabolic subgroup P. For that, we need to look at a $\Delta_{P,\alpha}^{-1}$-module a bit different from $\mathcal{C}(Y_{P,\alpha}; K/\mathcal{O})$. We now fix a finite dimensional polynomial representation

$$\rho : M_P(r_p) \longrightarrow \mathrm{GL}_d(\mathcal{O}) \quad \text{acting on } V(\rho;\mathcal{O}) = \mathcal{O}^d,$$

and suppose that ρ is absolutely irreducible after extending scalar to K. Here the word «polynomial» means that ρ is induced by a morphism of algebraic groups from $\mathrm{Res}_{r/\mathbb{Z}} M_P$ into $\mathrm{GL}(d)$ defined over $\mathcal{O}$. Note that the restriction of ρ to $M_P(r_p)$ remains absolutely irreducible. We write ω for the central character of ρ on $\mathrm{Res}_{F/\mathbb{Q}} T_M$ $(= \mathrm{Res}_{F/\mathbb{Q}} Z(M))$. For any torus $T_{/\mathbb{Z}_p}$ and a p-adic ring

$$A = \varprojlim_{\alpha} A/p^\alpha A,$$

we call a character χ of $T(A)$ *arithmetic* if it agrees with an algebraic character in $X(T) = \mathrm{Hom}_{\mathrm{alg\cdot gr}}(T, \mathbb{G}_m)$ on a neighborhood of the identity. We define a morphism of algebraic groups $\det : M' \to T_P$ by

$$\det(g_1, \ldots, g_{m-1}) = \big(\det(g_1), \ldots, \det(g_{m-1})\big) \in T_P,$$

where $g_i \in \mathrm{SL}(n_i)$. As before, we write $i : T_M \to T_P$ for the composite of the projection: $M_P \xrightarrow{\text{proj}} M'$ and $M' \xrightarrow{\det} T_P$. For a continuous character $\chi : T_P(r_p) \to \mathcal{O}^\times$, we define the twisted representation $\rho \otimes \chi$ by

$$\rho \otimes \chi(g)v = \chi\big(\det(g)\big)\rho(g)v \quad \text{for } v \in V(\rho;\mathcal{O}).$$

If χ is arithmetic as a character of $T_P(r_p) = \mathrm{Res}_{r/\mathbb{Z}} T_P(\mathbb{Z}_p)$, then there exists $0 < \alpha_0 \in \mathbb{Z}$ such that $\rho \otimes \chi$ induces for every $\alpha \geq \alpha_0$ a representation

$$M_P(r/p^\alpha r) \longrightarrow \mathrm{End}\big(V(\rho;\mathcal{O}) \otimes_{\mathcal{O}} p^{-\alpha}\mathcal{O}/\mathcal{O}\big),$$

which is again denoted by $\rho \otimes \chi$. For any admissible $\mathcal{O}$-module R, we just put

$$V(\rho \otimes \chi; R) = V(\rho \otimes \chi;\mathcal{O}) \otimes_{\mathcal{O}} R.$$

We then consider

$$\mathcal{C}(Y_{P,\alpha}, \rho; R) = \big\{\phi : Y_{P,\alpha} \to V(\rho;R) \mid \phi(x\gamma^{-1}) = \rho(\gamma)\phi(x) \text{ for } \gamma \in M_P^{(1)}(r_p)\big\}.$$

Note here that $\mathcal{C}(Y_{P,1}, \rho; R) = \mathcal{C}(Y_{P,1}, \rho \otimes \chi; R)$. Naturally $\delta \in \Delta_P^{-1}$ acts on $\mathcal{C}(Y_P, \rho; R)$ *via* $\delta\phi(y) = \phi(\delta^{-1}y)$. We consider

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}(Y_{P,1}, \rho; R)\big).$$

We have an exact sequence of $\Phi_{0,P}(p^\alpha)$-modules:

$$0 \to K_\alpha \to \mathcal{C}\big(Y_{P,1}(r/p^\alpha r); p^{-\alpha}\mathcal{O}/\mathcal{O}\big) \xrightarrow{\pi} \mathrm{ind}_{M^{(1)}}^{M} V(\rho; p^{-\alpha}\mathcal{O}/\mathcal{O}) \to 0,$$

$$\phi \mapsto \big(\pi(\phi)(\tau_i) = \phi(\mathrm{St}_i, \tau_i)\big)$$

where $V = V(\rho; p^{-\alpha}\mathcal{O}/\mathcal{O})$ and

$$\mathrm{ind}_{M^{(1)}}^{M} V = \big\{\phi : M_P(r/p^\alpha) \to V \,;\, \phi(x\gamma^{-1}) = \gamma\phi(x) \text{ for } \gamma \in M_P^{(1)}(r/p^\alpha r)\big\}.$$

Then:

$$\mathrm{ind}_{M^{(1)}}^{M} V\big(\rho; p^{-\alpha}\mathcal{O}/\mathcal{O})\big) = \mathrm{ind}_{\Phi_{\mathrm{SL},P}^{(p^\alpha)}}^{\Phi_{0,P}^{(p^\alpha)}} V(\rho; p^{-\alpha}\mathcal{O}/\mathcal{O}).$$

From this, similarly to THEOREM 4.1, we have:

PROPOSITION 5.1. — *Suppose* (GL) *and* (S). *Then we have a canonical isomorphism of Hecke modules*:

$$\iota: H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{0,P}(p),\mathcal{C}_{P,\rho}) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{SL,P}(p^\infty),V_\rho) = \varinjlim_{\alpha} H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{SL,P}(p^\alpha),V(\rho;p^{-\alpha}\mathcal{O}/\mathcal{O})\big),$$

where we write $\mathcal{C}_{P,\rho}$ (*resp.* V_ρ) *for* $\mathcal{C}(Y_{P,1},\rho;K/\mathcal{O})$ (*resp.* $V(\rho;K/\mathcal{O})$) *to simplify the notation.*

Since $\Phi_{0,P}(p^\alpha)/\Phi_{\mathrm{SL},P}(p^\alpha) \cong T_P(\mathbb{Z}/p^\alpha\mathbb{Z})$ *via* det, the left-hand side of the isomorphism in THEOREM 5.1 is naturally a $T_P(r_p)$-module. That is, choosing $\gamma \in \Phi_{0,P}(p^\alpha)$ which gives the image of $z \in T_P(r_p)$ in $\Phi_{H,P}(p^\alpha)/\Phi_{\mathrm{SL},P}(p^\alpha)$, the action of $z \in T_P(r/p^\alpha r)$ on each cocycle $u:\Phi_{\mathrm{SL},P}(p^\alpha)^{q+1} \to V(\rho;p^{-\alpha}\mathcal{O}/\mathcal{O})$ is given by

$$zu(\gamma_0,\dots,\gamma_q) = \gamma^{-1}u(\gamma^{-1}\gamma_0\gamma,\dots,\gamma^{-1}\gamma_q\gamma).$$

On the other hand, the action of $z \in T_M(r_p)$ on $\mathcal{C}_{P,\rho}$ given by $z\phi(y) = \phi(yz^{-1})$ induces the action of $T_M(r_p)$ on $H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{0,P}(p),\mathcal{C}_{P,\rho})$. Since π commutes with the action of T_M and T_P (*via* the isogeny $i:T_M \to T_P$ defined in the introduction), we see

$$\iota(zc) = i(z)\,\iota(c) \quad \text{for } z \in T_M(r_p). \tag{5.1}$$

Let η be a continuous character of $T_M(r_p)$ with values in $\mathcal{O}^\times$. We consider

$$\mathcal{C}(Y_{P,\alpha},\rho;R)[\eta] = \big\{\phi \in \mathcal{C}(Y_{P,\alpha},\rho;R)\,;\ \phi(xw^{-1}) = \eta(w)\phi(x) \text{ for } w \in T_M(r_p)\big\}.$$

Naturally Δ_P^{-1} (where $\Delta_P = \Delta_{P,1}$) acts on $\mathcal{C}(Y_{P,1},\rho;R)[\eta]$ similarly to $\mathcal{C}(Y_{P,1},\rho;R)$. Let

$$H = T_M(r_p)M_P^{(1)}(r_p)$$

in $M_P(r_p)$, and let χ be an arithmetic character of $T_P(r_p)$ with values in $\mathcal{O}$. We consider

$$V(\rho_\chi;R) = \mathrm{ind}_H^{\mathrm{GL}} V(\rho\otimes\chi;R) = \big\{\phi: M_P(r_p) \to V(\rho;R)\,;\ \phi(gh^{-1}) = \rho\otimes\chi(h)\phi(g) \text{ for } h \in H\big\},$$

where the above ϕ is supposed to be continuous. Then by Shapiro's lemma, we have:

$$H^q\big(\Phi_{H,P}(p^\alpha),V(\rho\otimes\chi;R)\big) \cong H^q\big(\Phi_{0,P}(p^\alpha),V(\rho_\chi;R)\big).$$

Note that the original ρ is an irreducible representation of $M_P(r_p)$. Then by the Frobenius reciprocity law, we have

$$V(\rho_\chi;K) \cong \bigoplus_\eta V(\rho\otimes\eta;K)$$

if K is sufficiently large, where η runs all characters of $T_P(r_p)$ such that $\eta=\chi$ on $i(T_M(r_p))\subset T_P(r_p)$. Note that

$$I_P/U_P(r_p)\cong Y_P = Y_{P,1}.$$

We now show that we can embed $\operatorname{Ind}_{P^0}^{G_1} V(\rho;K)$ into $\mathcal{C}(Y_P;K)[\omega]$ as a I_P-module, where $P^0=\operatorname{Res}_{F_P/\mathbb{Q}_P} P$ which is regarded as a subgroup of G_1. Such an embedding is unique up to unit multiples. To see this, we write simply V for the simple P^0-module $V(\rho;K)$. Then V is a P^0-module in the sense of algebraic geometry (*cf.* [J, I.2]). Let $\mathcal{L}(V)$ be the sheaf on the Grassmannian G_1/P^0 associated with V, that is $(G_1\times V)/P^0$. Then

$$H^0\big(G_1/P^0,\mathcal{L}(V)\big)\cong \operatorname{Ind}_{P^0}^{G_1} V.$$

We suppose that $\operatorname{Ind}_{P^0}^{G_1} V\neq\{0\}$. In this case, we call ρ *dominant.* The representation ρ is dominant if and only if the highest weight of ρ with respect to the Borel subgroup $B\cap M_P$ as the representation of M_P is dominant as a character of T with respect to B ([J, § 2.16 and Rem. p. 199]). Let ${}^tP$ be the opposite parabolic subgroup and ${}^tU$ be its unipotent radical. Then as shown in [J, Rem., p. 199], if $H^0({}^tU^0,\operatorname{Ind}_{P^0}^{G_1} V)\neq 0$, then

$$H^0({}^tU^0,\operatorname{Ind}_{P^0}^{G_1} V)\cong V \quad \text{as } P^0\text{-module},$$

where ${}^tU^0=\operatorname{Res}_{F_P/\mathbb{Q}_P}{}^tU\subset G_1$. Moreover $\operatorname{soc}_{G_1}\operatorname{Ind}_{P^0}^{G_1} V$ is simple, where «soc» indicate the socle. Since G_1 is reductive and K is of characteristic 0, $\operatorname{Ind}_{P^0}^{G_1} V$ is a semi-simple G_1-module. Thus

$$\operatorname{soc}_{G_1}\operatorname{Ind}_{P^0}^{G_1} V=\operatorname{Ind}_{P^0}^{G_1} V.$$

That is, $\operatorname{Ind}_{P^0}^{G_1} V$ is simple. Since Y_P is an open subset under the p-adic topology of $G_1(\mathbb{Q}_p)/U^0(\mathbb{Q}_p)$. Thus Y_P is Zariski dense in the algebraic variety G_1/U^0. In particular,

$$\operatorname{Ind}_{P^0}^{G_1} V=\big\{\phi G_1\to V\,;\ \phi(g\gamma^{-1})=\gamma\phi(g) \text{ for } \gamma\in M_P,\ \phi\colon \text{polynomial}\big\}$$

is embedded into $\mathcal{C}(Y_{P,\rho};K)[\omega]$ through the restriction of functions on G_1/U_P to Y_P. Since $L(\rho;K)=\operatorname{Ind}_{P^0}^{G_1} V$ is stable under the action of Δ^{-1}

on $\mathcal{C}(Y_{P,\rho};K)$, we hereafter consider $L(\rho;K)$ as a Δ^{-1}-module. This action may be a bit different, by a character, from the action of $G(\mathbb{Q}_p)$ *via* ρ. Let χ be an arithmetic character of $T_P(r_p)$ with values in $\mathcal{O}^\times$. Now we write $\chi = \chi_0\varepsilon$ with $\chi_0 \in X(\mathrm{Res}_{F/\mathbb{Q}} T_P)$ and $\varepsilon : T_P(r/p^\alpha r) \to \mathcal{O}^\times$ of finite order. We suppose that $\rho \otimes \chi_0$ is dominant (in this case, we say χ is *dominant with respect* to ρ). Then replacing ρ by $\rho \otimes \chi_0$, we know that we can realize

$$L(\rho \otimes \chi_0;K) = \mathrm{Ind}_{P^0}^{G_1} V(\rho \otimes \chi_0;K) \ \text{ in } \ \mathcal{C}(Y_P, \rho;K)[\omega_{\chi_0}]$$

as a Δ^{-1}-module, where $\omega_\chi = \omega(\chi \circ i)$ and $\chi \circ i$ is the composite of

$$i : T_M \subset M_P \xrightarrow{\text{proj}} \mathrm{GL}(t) \xrightarrow{\det} T_P \quad \text{and } \chi.$$

We put

$$L(\rho \otimes \chi_0;\mathcal{O}) = L(\rho \otimes \chi_0;K) \cap \mathcal{C}(Y_P, \rho;\mathcal{O}).$$

We define $\varepsilon_Y : I_P(p^\alpha) \to \mathcal{O}^\times$ by $\varepsilon\big(\det\big(\pi(g \bmod p^\alpha)\big)\big)$, where $\pi : P \to M'$ is the projection. Then it is easy to see that ε_Y factors through $Y_{P,\alpha}$. We regard $L(\rho \otimes \chi_0;K)$ as a subspace of $\mathcal{C}^\alpha_{P,\rho}(K) = \mathcal{C}(Y_{P,\alpha}, \rho;K)$ restricting polynomial functions to Y_P^α. Then

$$L(\rho \otimes \chi;\mathcal{O}) = \varepsilon_Y L(\rho \otimes \chi_0;K) \cap \mathcal{C}^\alpha_{P,\rho}(\mathcal{O})$$

is stable under $\Delta^{-1}_{P,\alpha}$. We then define

$$L(\rho \otimes \chi;R) = L(\rho \otimes \chi;\mathcal{O}) \otimes_{\mathcal{O}} R$$

and write simply $L(\rho \otimes \chi)$ for $L(\rho \otimes \chi;K/\mathcal{O})$. We consider

$$\mathcal{C}(Y_P, \rho;A)[\chi] = \big\{\phi \in \mathcal{C}(Y_P, \rho;A)\,; \ \phi(zw^{-1}) = \chi(w)\phi(y) \text{ for } w \in T_P(r_p)\big\}. \tag{5.2}$$

Naturally Δ_P^{-1} acts on $\mathcal{C}(Y_P, \rho;A)[\chi]$. We put $\omega_\chi = \omega(\chi \circ i)$ for the central character ω of ρ. Now we want to prove:

THEOREM 5.1. — *Suppose* (GL) *and* (S). *Let* $H = T_M(r_p)M_P^{(1)}(r_p)$. *Then for each arithmetic character* $\chi : T_P(r_p) \to \mathcal{O}^\times$ *dominant with respect to* ρ *and for all* $q \geq 0$, *we have a canonical isomorphism of Hecke modules*:

$$\iota_\chi : H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{\mathrm{SL}\,,P}(p^\infty), L(\rho \otimes \chi)\big) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_0(p)n, \mathcal{C}_{P,\rho}\big),$$

where $\iota_\chi(\omega_\chi(z)i(z)c) = z\iota_\chi(c)$ *for* $z \in T_M(r_p)$. *Moreover suppose that* χ *is an arithmetic character in* $X(\mathrm{Res}_{F/\mathbb{Q}} T_P)$ *dominant with respect to* ρ *such that it is algebraic on* $\{x \in T_P(r_p); x \equiv 1 \bmod p^\alpha\}$ *for* $\alpha \geq 1$. *Then the above isomorphism induces an isogeny for* $\omega_\chi = \omega(\chi \circ i)$

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi)\big) \longrightarrow H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\infty), \mathcal{C}_{P,\rho}\big)[\omega_\chi].$$

Proof. — We consider the following exact sequence:

$$(*) \qquad 0 \to K_\beta \longrightarrow L(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O}) \xrightarrow{\ \pi\ } V(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O}) \to 0$$

given by $\pi(\phi)(\tau_i) = \phi((\mathrm{St}, \tau_i))$. Viewing $L(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})$ as a space of functions on Y_P with values in a finite set $V(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})$ we see that every element in $L(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})$ factors through $Y_P(r/p^{\alpha'}r)$ for a sufficiently large $\alpha' \geq \max(\alpha,\beta)$. Then (*) is an exact sequence of $\Delta^{-1}_{\alpha'}$-modules. Since the action of $\xi^{\alpha'}$ for ξ as in the proof of THEOREM 4.1 contracts $Y_P(r/p^{\alpha'}r)$ to $\{(\mathrm{St},\tau_i)\,;\ (\tau_i) \in M_P(r/p^{\alpha'}r)\}$ (LEMMA 3.1), by the same argument which proves THEOREM 4.1, we get, for all $\beta \geq \alpha$

$$\begin{aligned} &H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \equiv H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^{\alpha'}), L(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \equiv H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^{\alpha'}), V(\rho\otimes\chi; p^{-\beta}\mathcal{O}/\mathcal{O})\big)\\ &\qquad \equiv H_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^{\beta}), V(\rho_\chi; p^{-\beta}\mathcal{O}/\mathcal{O})\big). \end{aligned}$$

Thus we have used twice (4.7c). This implies:

$$\text{(a)} \qquad \begin{aligned} &H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi)\big)\\ &\qquad \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^{\beta}), V(\rho\otimes\chi; K/\mathcal{O})\big)\\ &\qquad \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^{\infty}), V(\rho\otimes\chi; K/\mathcal{O})\big). \end{aligned}$$

Similarly we look at the exact sequence of Δ^{-1}_β-modules

$$0 \to \mathcal{K}_\beta \longrightarrow \mathcal{C}(Y_P(r/p^\beta r), \rho; K/\mathcal{O})[\omega_\chi] \xrightarrow{\ \pi\ } V(\rho_\chi; K/\mathcal{O}) \to 0.$$

Since

$$\pi(\phi)(xz^{-1}) = \omega_\chi(z)\pi(\phi)(x) \qquad \text{for } z \in T_M(r_p),$$

$$\pi(\phi)(xg^{-1}) = \rho(g)\pi(\phi)(x) \qquad \text{for } g \in M_P^{(1)}(r_p),$$

we know that

$$\pi(\phi)(xh^{-1}) = \rho\otimes\chi(h)\pi(\phi)(x) \quad \text{for } h \in T_M(r_p)M_P^{(1)}(r_p).$$

This shows that $\pi(\phi) \in V(\rho_\chi : K/\mathcal{O})$ if H contains

$$\{x \in M_P(r_p)\,;\ x \equiv 1 \mod p^\beta\}.$$

We get

$$\text{(b)}\qquad \begin{aligned} &H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}_{P,_\rho}[\omega_\chi]\big)\\ &\quad\cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\alpha), \mathcal{C}_{P,_\rho}[\omega_\chi]\big) \quad \text{(cf. (4.7c))}\\ &\quad\cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p^\infty), V(\rho_\chi; K/\mathcal{O})\big)\\ &\quad\cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\infty), V(\rho\otimes\chi; K/\mathcal{O})\big). \end{aligned}$$

The last isomorphism in the above identity follows from Shapiro's lemma. Thus combining these two isomorphisms (a) and (b), we get

$$\text{(5.3a)}\qquad H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}_{P,_\rho}[\omega_\chi]\big) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi)\big).$$

Replacing in the above argument $\Phi_{H,P}(p^\alpha)$ by $\Phi_{\mathrm{SL},P}(p^\alpha)$ and $\mathcal{C}_{P,_\rho}[\omega_\chi]$ by $\mathcal{C}_{P,_\rho}$, we get

$$\text{(5.3b)}\qquad \iota_\chi : H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{\mathrm{SL},P}(p^\infty), L(\rho\otimes\chi)\big) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}_{P,_\rho}\big).$$

Since π in (*) is a morphism of $\Phi_{0,P}(p^\beta)$-modules, we conclude that

$$\iota_\chi(\omega_\chi(z)i(z)c) = z\iota_\chi(c) \quad \text{for } z \in T_M(r_p).$$

This shows that ι_χ induces the last morphism in the theorem. Now we suppose that

$$H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi; K)) = 0 \qquad \text{for all } 0 \le q < r.$$

Then we conclude from LEMMA 5.1 that the natural map

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}_{P,_\rho}[\omega_\chi]\big) \longrightarrow H^r_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(p), \mathcal{C}_{P,_\rho}[\omega_\chi]\big)$$

is an isogeny. We need to check the four assumptions of LEMMA 5.1. We take as $\mathfrak{C}$ the category of discrete admissible $(\Delta_P^{-1}, T_M(r_p))$-modules, and $\mathfrak{g} = T_M(r_p)$. We decompose $T_M(r_p) = W \times \mu$ as in LEMMA 5.1. Then the assumptions (i) is clear. The functor H^q is given by

$$H^q(X) = H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{0,P}(p^\alpha), X).$$

Then H^q has long exact sequences attached to short ones. The assumption (iv) follows from (5.3a). From the exact sequence:

$$0 \to L(\rho \otimes \chi; \mathcal{O}) \longrightarrow L(\rho \otimes \chi; K) \longrightarrow L(\rho \otimes \chi) \to 0,$$

we get another exact sequence for $\Gamma = \Phi_{H,P}(p^\alpha)$:

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Gamma, L(\rho \otimes \chi; K)\big)\big) \longrightarrow H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Gamma, L(\rho \otimes \chi)\big) \longrightarrow H^{q+1}_{P\text{-}n\cdot\mathrm{ord}}\big(\Gamma, L(\rho \otimes \chi; \mathcal{O})\big)$$

Thus from the assumption that

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho \otimes \chi; K)\big) = 0 \quad \text{for all } 0 \le q < r,$$

we conclude the finiteness of $H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi))$ for all $0 \le q < r$. Then the assumption (iii) of LEMMA 5.1 holds again by (5.3a). We now check the assumption (ii). Note that Y_P is isomorphic to a disjoint union of k copies of ${}^tU(pr_p) \times W \times M_P^{(1)}(r_p)$ for some $0 < k \in \mathbb{Z}$ just as W-sets. Thus the Pontryagin dual module $\mathcal{C}^*$ of $\mathcal{C} = \mathcal{C}(Y_P, \rho; K/\mathcal{O})$ is isomorphic to

$$\Lambda^k \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[{}^tU(pr_p)]] \otimes_{\mathcal{O}} V^*$$

for the Pontryagin dual V^* of $V(\rho; K/\mathcal{O})$. Since $T_1, \dots, T_m$ is a regular sequence, the multiplication by T_j is injective on $\Lambda/(T_1, \dots, T_{j-1})$ and hence injective on

$$(\Lambda/(T_1, \dots, T_{j-1}))^k \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[{}^tU(p\,r_p)]] \otimes_{\mathcal{O}} V^*.$$

The statement dual to this is actually the assumption (ii). Therefore we get the last assertion from LEMMA 5.1.

Now we have the control theorem with respect to $T_M(r_p)$. Since the image of $T_M(r_p)$ in $T_P(r_p)$ is of finite index, by the Hochschild-Serre spectral sequence, we get the control theorem with respect to $T_P(r_p)$. We now identify *via* ι_χ

$$H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{\mathrm{SL},P}(p^\infty), L(\rho \otimes \chi)) \cong H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{\mathrm{SL},P}(p^\infty), L(\rho)),$$

and we take the standard action of $T_P(r_p)$ to be the action of the right-hand side. Then we have proven the following result:

THEOREM 5.2. — *Suppose* (GL) *and* (S), *and let* χ *be a continuous character of* $T_P(r_p)$ *with values in* $\mathcal{O}^\times$. *Suppose that*

(i) χ *coincides with an algebraic character in* $X(\mathrm{Res}_{r_p/\mathbb{Z}_p} T_P)$ *on* $\{x \in T_P(\mathbb{Z}_p); x \equiv 1 \bmod p^\alpha\}$,

(ii) χ *is dominant with respect to* ρ, *and*

(iii) $H^q_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{H,P}(p^\alpha), L(\rho \otimes \chi; K)) = 0$ *for* $0 \le q < r$.

Then we have a canonical morphism of Hecke modules:

$$H^r_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{0,P}(p^\alpha), L(\rho\otimes\chi)) \longrightarrow H^r_{P\text{-}n\cdot\mathrm{ord}}(\Phi_{\mathrm{SL},P}(p^\infty, L(\rho))[\chi]$$

with finite kernel and cokernel.

COROLLARY 5.1. — *The Pontryagin dual module of*

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{\mathrm{SL},P}(p^\infty), L(\rho)\big)$$

is of finite type over $\mathcal{O}[[T_M(r_p)]]$ *if* ρ *is dominant.*

Let W be the torsion-free part of $T_M(r_p)$. Therefore $T_M(r_p) = W \times \mu$ for a finite group μ and a $\mathbb{Z}_p$-free module W of finite rank. We can state a stronger control theorem using W in place of $T_M(r_p)$:

COROLLARY 5.2. — *Let* $H = W \cdot M_P^{(1)}(r_p)$. *Suppose that*

$$H^q_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi)\big) = 0 \quad \textit{for all } 0 \le q < r.$$

If χ *is algebraic on* $\{x \in T_P(\mathbb{Z}_p); x \equiv 1 \bmod p^\alpha\}$ $(\alpha > 0)$ *and is dominant with respect to* ρ, *then we have for the prime ideal* $P_\chi = \mathrm{Ker}(\chi)$ *in* Λ

$$H^r_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi)\big) \cong H^r_{P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{\mathrm{SL},P}(p^\infty), L(\rho)\big)[P_\chi].$$

We omit the proof, since it is basically the same as the proof of THEOREM 5.1 and follows directly from LEMMA 5.1.

When $P = B$, the standard Borel subgroup, the isogeny $i : T_M \to T_B$ is actually an isomorphism. Thus THEOREM 1.1 and 1.2 follows from the result in Sections 4 and 5.

6. Further generalization

We can generalize our result to the following cases in an obvious manner:

Case J. *The ordinary part associated with* $\{P_{\mathfrak{p}} \in \mathcal{P}\,;\ \mathfrak{p} \in J\}$ *for a subset* J *of prime ideals in* F *dividing* p.

Since the argument is completely parallel to the case already treated, we just state the result in Case J without proof when J is made of one prime ideal $\mathfrak{p}$ of r dividing p and leave the formulation in the general case to the reader. This has been worked out by Brad Wilson [W] when $n = 2$ and F is totally real. Let $n = n_1 + \ldots + n_m$ be the partition of n by positive integers n_i into m parts. We write P for the standard parabolic subgroup of SL(n) associated to $(n_1, \ldots, n_m)$. Then we associate to P subsets

$$\Xi = \Big\{n_m, n_{m-1} + n_m, \ldots, \sum_{j=i}^{m} n_j, \ldots, n_2 + \cdots + n_m\Big\}$$

$$^t\Xi = \Big\{n_1, n_1 + n_2, \ldots, \sum_{j=1}^{i} n_j, \ldots, n_1 + \cdots + n_{m-1}\Big\}$$

of $\{1, 2, \ldots, n-1\}$. We write

$$\mathcal{V} = r_{\mathfrak{p}} = \varprojlim_{\alpha} r/\mathfrak{p}^{\alpha}.$$

Let $\mathcal{F}$ be the field of fractions of $\mathcal{V}$ and $\varpi_{\mathfrak{p}}$ be a prime element of $\mathcal{V}$. We suppose that $\mathcal{O}$ is a $\mathcal{V}$-algebra. In the previous section, we studied the operator $\Pi_{s\in\Xi} T_s(\varpi)^j$ for some $j > 0$ and e_P (see (Tp$_s$)). Of course the idempotent e_P does not depends on the choice of j. We can work out a similar result for

$$\mathbb{T} = \prod_{s\in\Xi} T_s(\varpi_{\mathfrak{p}}^j)$$

for a suitable $j > 0$ and the idempotent $\mathbf{e}_P$ attached to $\mathbb{T}$. The operator $T_s(\varpi_{\mathfrak{p}}^j)$ is the action of $\Phi\xi\Phi$ for $\Xi \in G(\mathbb{Q})$ close to $\begin{pmatrix} 1_t & 0 \\ 0 & \varpi_{\mathfrak{p}}^j 1_s \end{pmatrix}$. We will specify ξ more precisely later. Let Φ be a congruence subgroup of $G_1(\mathbb{Q})$ satisfying

$$(\mathrm{S}_{\mathfrak{p}}) \qquad S_p = \prod_{\mathcal{P}|p} S_{\mathcal{P}} \quad \text{for } S_{\mathcal{P}} \subset \mathrm{SL}_n(r_{\mathcal{P}}) \text{ and } I_{P,\mathfrak{p}} \supset S_{\mathcal{P}} \supset I_{B,\mathfrak{p}},$$

where $S_{\mathcal{P}}$ is the $\mathcal{P}$-adic closure of Φ, and $I_{\mathcal{P},\mathfrak{p}} = I_{P,1,\mathfrak{p}}$ for

$$I_{P,\alpha,\mathcal{P}} = \big\{x \in \mathrm{SL}_n(\mathcal{V})\,;\ x \mod \mathfrak{p}^{\alpha} \in P(\mathcal{V})\big\}.$$

We can suppose as explained in Section 4 (see ($\mathrm{T}\mathfrak{p}_s$)) that

$$(\mathrm{T}\mathfrak{p}_s)\quad \begin{cases} \textit{there exist } \theta \in \mathfrak{p} \textit{ and } \xi \in G(\mathbb{Q}) \textit{ such that } \xi_{s,\mathfrak{p}} \equiv \begin{pmatrix} 1_t & 0 \\ 0 & \theta 1_s \end{pmatrix} \\ \bmod \mathfrak{p}^\alpha M_n(\mathcal{V}),\ \xi_{s,\ell} S_\ell = S_\ell \xi_{s,\ell} \textit{ for } \ell \neq p,\ \xi_{s,\mathcal{P}} S_{\mathcal{P}} = S_{\mathcal{P}} \xi_{s,\mathcal{P}}, \\ \textit{and } \xi_{s,\mathcal{P}} \in \mathrm{GL}_n(r_{\mathcal{P}}) \textit{ for all prime ideals } \mathcal{P} \neq \mathfrak{p} \textit{ and } \mathcal{P} \mid p. \end{cases}$$

where $S_{\mathcal{P}}$ (resp. $\xi_{\mathcal{P}}$) is the $\mathcal{P}$-adic closure of Φ in $\mathrm{GL}_n(F_{\mathcal{P}})$ (resp. the image of ξ in $GL_n(F_{\mathcal{P}})$). Let H be a closed subgroup of $M_P(\mathcal{V})$ and H_α be the image of H in $M_P(r/\mathfrak{p}^\alpha)$. We put

$$(6.1)\quad \begin{cases} \Delta^\alpha_{\mathfrak{p},P} = \Delta_C(\mathcal{V}) \quad \text{for } C = S_0(\mathfrak{p}^\alpha)_\mathfrak{p} \text{ and } \Delta = \Delta_{\mathfrak{p},P} = \Delta^1_{\mathfrak{p},P}, \\ \Phi_{0,P}(\mathfrak{p}^\alpha) = \{\gamma \in \Phi\,;\ \gamma_\mathfrak{p} \bmod \mathfrak{p}^\alpha \in P(r/\mathfrak{p}^\alpha)\} \\ \Phi_{\mathrm{SL},P}(\mathfrak{p}^\alpha) = \{\gamma \in \Phi_{0,P}(\mathfrak{p}^\alpha)\,;\ \pi(\gamma_\mathfrak{p}) \bmod \mathfrak{p}^\alpha \in M_P^{(1)}(r/\mathfrak{p}^\alpha)\}, \\ \Phi_{H,P}(\mathfrak{p}^\alpha) = \{\gamma \in \Phi_{0,P}(\mathfrak{p}^\alpha)\,;\ \pi(\gamma_\mathfrak{p}) \bmod \mathfrak{p}^\alpha \in H_\alpha\}, \end{cases}$$

where $\gamma_\mathfrak{p}$ is the image of γ in $\mathrm{GL}_n(\mathcal{V})$ *via* (GL), $S_0(\mathfrak{p}^\alpha)_\mathfrak{p}$ is the closure of $\Phi_0(\mathfrak{p}^\alpha)_\mathfrak{p}$ in $\mathrm{GL}_n(\mathcal{V})$ and $\pi : P \to M_P$ is the projection. We define

$$Y_{P,\mathfrak{p}} = I_{P,\mathfrak{p}}/U_P(\mathcal{V}), \quad Y^\alpha_{P,\mathfrak{p}} = I_{P,\alpha,\mathfrak{p}}/U_P(\mathcal{V}).$$

We put for each $\mathcal{O}$-module R and $\gamma \in M_P^{(1)}(r_\mathfrak{p})$

$$(6.2\mathrm{a})\quad \begin{cases} \mathcal{C}_{P,\mathfrak{p},\rho}(R) = \{\phi \in \mathcal{C}(Y_{P,\mathfrak{p}}; V(\rho_\mathfrak{p};R))\,;\ \phi(x\gamma^{-1}) = \rho_\mathfrak{p}(\gamma)\phi(x)\}, \\ \mathcal{C}^\alpha_{P,\mathfrak{p},\rho}(R) = \{\phi \in \mathcal{C}(Y^\alpha_{P,\mathfrak{p}}; V(\rho_\mathfrak{p};R))\,;\ \phi(x\gamma^{-1}) = \rho_\mathfrak{p}(\gamma)\phi(x)\}, \end{cases}$$

where $V(\rho_\mathfrak{p};R) = V(\rho_\mathfrak{p};\mathcal{O}) \otimes_\mathcal{O} R$, and

$$\rho_\mathfrak{p} : M_P(\mathcal{V}) \longrightarrow \mathrm{End}_\mathcal{O}\big(V(\rho_\mathfrak{p};\mathcal{O})\big) \qquad (V(\rho_\mathfrak{p};\mathcal{O}) \cong \mathcal{O}^d)$$

is a polynomial representation: $\mathrm{Res}_{\mathcal{V}/\mathbb{Z}_p} M_P \to \mathrm{GL}(d)_{/\mathcal{O}}$. We suppose that $V(\rho_\mathfrak{p};\mathcal{O})$ is a absolutely irreducible after extending scalar to K. We write $\mathcal{C}_{P,\mathfrak{p},\rho}$ (resp. $\mathcal{C}^\alpha_{P,\mathfrak{p},\rho}$) for $\mathcal{C}_{P,\mathfrak{p},\rho}(K/\mathcal{O})$ (resp. $\mathcal{C}^\alpha_{P,\mathfrak{p},\rho}(K/\mathcal{O})$). Let $L(\rho^{(\mathfrak{p})};\mathcal{O})$ be an $(\mathcal{O}, \mathrm{SL}_n(r^{(\mathfrak{p})}))$-module, where we have decomposed $r_p = r_\mathfrak{p} \times r^{(\mathfrak{p})}$ as a ring product. Of course, we suppose that $L(\rho^{(\mathfrak{p})};\mathcal{O})$ is of finite rank over $\mathcal{O}$. When we put:

$$\begin{aligned} L(\rho_\mathfrak{p};K) &= \mathrm{Ind}_{P^0}^{\mathrm{SL}(n)^0} V(\rho_\mathfrak{p};K) \\ &= \{\phi : \mathrm{SL}(n)^0/U^0 \to V(\rho_\mathfrak{p};K)\,; \\ &\qquad\qquad \phi : \text{polynomial},\ \phi(y\gamma^{-1}) = \gamma\phi(y)\ (\gamma \in M_P^{(1)})\}, \end{aligned}$$

where $\mathrm{SL}(n)^0 = \mathrm{Res}_{\mathcal{F}/\mathbb{Q}_p} \mathrm{SL}(n)$, $P^0 = \mathrm{Res}_{\mathcal{F}/\mathbb{Q}_p} P$ and $U^0 = \mathrm{Res}_{\mathcal{F}/\mathbb{Q}_p} U_P$. When $L(\rho_{\mathfrak{p}};K) \neq 0$, it is absolutely irreducible. For an arithmetic $\chi : T_M(r_p) \to \mathcal{O}^\times$, we take an algebraic $\chi_0 \in X(\mathrm{Res}_{\mathcal{V}/\mathbb{Z}_p} T_P)$ such that $\varepsilon = \chi\chi_0^{-1}$ factors through $\mathcal{V}/\mathfrak{p}^\alpha$ for some $\alpha > 0$, an put inside $\mathcal{C}^\alpha_{P,\mathfrak{p},\rho}(K)$

$$L(\rho_{\mathfrak{p}} \otimes \chi;K) = \{\varepsilon_Y \phi;\ \phi \in L(\rho_{\mathfrak{p}} \otimes \chi_0;K\},$$

where ε_Y is defined in the same manner as in Section 5. Here we have regarded $L(\rho_{\mathfrak{p}} \otimes \chi_0;K)$ as a subspace of $\mathcal{C}^\alpha_{P,\mathfrak{p},\rho}(K)$ considering the polynomial in $L(\rho_{\mathfrak{p}} \otimes \chi_0;K)$ as a function on $Y^\alpha_{P,\mathfrak{p}}$. Then $L(\rho_{\mathfrak{p}} \otimes \chi;K)$ is stable under $\Delta^\alpha_{\mathfrak{p},P}$. We call χ *dominant with respect* to $\rho_{\mathfrak{p}}$ if $L(\rho_{\mathfrak{p}} \otimes \chi;K) \neq 0$. We put:

$$L(\rho_{\mathfrak{p}} \otimes \chi;\mathcal{O}) = L(\rho_{\mathfrak{p}} \otimes \chi;K) \cap \mathcal{C}^\alpha_{P,\mathfrak{p},\rho}(\mathcal{O}),$$
$$L(\rho_{\mathfrak{p}} \otimes \chi) = L(\rho_{\mathfrak{p}} \otimes \chi;\mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O}.$$

Then we further put:

$$\text{(6.2b)} \quad \begin{cases} L(\rho \otimes \chi;\mathcal{O}) = L(\rho_{\mathfrak{p}} \otimes \chi;\mathcal{O}) \otimes_{\mathcal{O}} L(\rho^{(\mathfrak{p})};\mathcal{O}), \\ L(\rho \otimes \chi;R) = L(\rho \otimes \chi;\mathcal{O}) \otimes_{\mathcal{O}} R, L(\rho \otimes \chi) = L(\rho \otimes \chi;K/\mathcal{O}). \end{cases}$$

Let $\boldsymbol{e}_P$ be the ordinary projector associated to $\mathbb{T}$. Then, we have, writing $H^q_{*,P\text{-}n\cdot\mathrm{ord}}$ for $\boldsymbol{e}_P H^q_*$,

THEOREM 6.1. — *Assume* (GL), *and* ($\mathrm{S}_{\mathfrak{p}}$) *Then we have a canonical isomorphism of Hecke modules for all q and for all arithmetic character χ of $T_P(r_{\mathfrak{p}})$ dominant with respect to $\rho_{\mathfrak{p}}$:*

$$\iota_\chi : H^q_{*,P\text{-}n\cdot\mathrm{ord}}(\Phi_{\mathrm{SL},P}(\mathfrak{p}^\infty), L(\rho \otimes \chi)) \cong H^q_{*,P\text{-}n\cdot\mathrm{ord}}(\Phi_{0,P}(\mathfrak{p}), \mathcal{C}_{P,\mathfrak{p},\rho} \otimes_{\mathcal{O}} L(\rho \otimes \chi;\mathcal{O})),$$

where ι_χ satisfies

$$\omega_\chi(z)\iota_\chi(i(z)c) = z\iota_\chi(c) \quad \text{for } z \in T_M(r_{\mathfrak{p}}),$$
$$\omega_\chi = \omega(\chi \circ i) \quad \text{for the isogeny } i : T_M \to T_P.$$

Moreover let $H = T_M(r_{\mathfrak{p}})M^{(1)}_P(r_{\mathfrak{p}})$ and ω be the central character of $\rho_{\mathfrak{p}}$, and suppose that

(i) *χ coincides with a character in $X(\mathrm{Res}_{\mathcal{V}/\mathbb{Z}_p} T_P)$ on*

$$\{x \in T_P(r_{\mathfrak{p}}); x \equiv 1 \mod \mathfrak{p}^\alpha\},$$

(ii) χ *is a dominant with respect to* $\rho_{\mathfrak{p}}$, *and*

(iii) $H^q_{*,s-\mathrm{ord}}\big(\Phi_{H,P}(p^\alpha), L(\rho\otimes\chi;K)\big) = 0$ *for* $0 \le q < r$.

Then we have a canonical morphism of Hecke modules:

$$H^r_{*,P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{0,P}(\mathfrak{p}^\alpha), L(\rho\otimes\chi)\big) \longrightarrow H^r_{*,P\text{-}n\cdot\mathrm{ord}}\big(\Phi_{\mathrm{SL},P}(\mathfrak{p}^\infty); L(\rho;K/\mathcal{O})\big)[\chi]]$$

with finite kernel an cokernel.

7. Exact control theorem for Borel subgroups

In the previous sections, we have proven a control theorem up to finite error. Here we like to get an exact control theorem for the standard Borel subgroup. We can formulate our result for general parabolic subgroup P, but it is not so much transparent because in that case, T_M may not be isomorphic to T_P by the morphism i. For example, when $n = 1+s$ and P is the maximal parabolic subgroup associated to this partition, we see that

$$T_M \cong \big\{(a,b) \in \mathbb{G}_m \times \mathbb{G}_m\ ;\ a = b^{-s}\big\}$$

is not isomorphic to $T_P \cong \mathbb{G}_m$ by the first projection. This is the reason why we assume here that $P = B$. To get exact control, we need to pay some price to bring the theorem in exact form. Namely we need to exclude finite number of primes.

We now state a version of LEMMA 5.1. Recall that $\mathfrak{g}$ is a topological group isomorphic to a product of a finite group μ and a $\mathbb{Z}_p$-free module W of finite rank m. Let $\mathbf{\Lambda} = \mathcal{O}[[\mathfrak{g}]]$, $\Lambda = \mathcal{O}[[W]]$ and χ be a continuous character of $\mathfrak{g}$ into $\mathcal{O}^\times$. Then χ induces an algebra homomorphism $\chi:\mathbf{\Lambda} \to \mathcal{O}$ Let $R = R_\chi$ be the local ring of $\mathbf{\Lambda}$ through which χ factors. Let $T_0, T_1, \ldots, T_m$ be a regular sequence in the maximal ideal $\mathcal{M}$ of Λ.

LEMMA 7.1. — *Let* $\mathfrak{C}$ *be an abelian subcategory of the category of discrete* R_χ*-modules and* C *be an object of* $\mathfrak{C}$. *Let* $M \mapsto H^\bullet(M)$ *be a cohomology functor* (*transforming short exact sequences into long ones*) *defined on* $\mathfrak{C}$ *with values in the category of discrete* R_χ*-modules. Suppose the following four conditions*:

(i) $\mathcal{C}$, $\mathcal{C}[\chi]$ *and* $\mathcal{C}[T_0, \ldots, T_j]$ *for* $j = 0, \ldots, m$ *are objects in* $\mathfrak{C}$;

(ii) *For each* $j = 0, \ldots, m$, *the multiplication by* T_j *is surjective on* $\mathcal{C}[T_0, \ldots, T_{j-1}]$ (*this in particular implies that* $x \mapsto T_1 x$ *is surjective on* $\mathcal{C}$);

(iii) $H^q(\mathcal{C}[T_0, \ldots, T_m]) = 0$ *for all* $0 \le q < r$;

(iv) *The Pontryagin dual module of* $H^q(\mathcal{C}[T_0, \ldots, T_m])$ *is of finite type over* $\mathcal{O}$ *for all* $0 \le q \le r$.

Then we have a canonical isomorphism:

$$H^r\big(\mathcal{C}[T_0,\dots,T_m]\big) \cong H^r(\mathcal{C})[T_0,\dots,T_m].$$

This lemma can be proven in exactly the same manner as in the proof of LEMMA 5.1; so, we omit the proof.

We return to the notation in Section 5. Thus we take an open compact subgroup S of $G_1(\mathbb{A}^{(\infty)})$ such that $S = S_p \times S^{(p)}$. We write $\Phi = S \cap G_1(\mathbb{Q})$ and assume

$$\text{(F)} \qquad S_p = G_1(\mathbb{Z}_p) = \iota^{-1}\big(\mathrm{SL}_n(r_p)\big),$$

where r is the integer ring of F and $r_p = r \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let B be the standard Borel subgroup of SL(n). We put:

$$S_0(p^\alpha) = \big\{x \in S\,;\ x \mod p^\alpha \in B(r/p^\alpha r)\big\},$$

$$S_1(p^\alpha) = \big\{x \in S\,;\ x \mod p^\alpha \in U_B(r/p^\alpha r)\big\}.$$

We put $Y = S_0(p)/U_B(r_p)$. We fix a dominant character χ of $\mathrm{Res}_{F/\mathbb{Q}} T$. Then we consider the Δ^{-1}-module $\mathcal{C}(Y;K/\mathcal{O})$ for the semi-group

$$\Delta = S_0(p)_p D_B S_0(p)_p.$$

Let $R = R_\chi$ be the local ring of $\mathbf{\Lambda} = \mathcal{O}[[T(r_p)]]$ through which χ factors. Write 1_χ for the idempotent of R_χ. We take as $\mathcal{C}$ in the lemma the module $\mathcal{C}_\chi = 1_\chi \mathcal{C}(Y;K/\mathcal{O})$. We write m for the maximal ideal of $\mathcal{O}$.

THEOREM 7.1. — *Suppose* (GL), (F) *and* (S). *Let* $\chi \in X(\mathrm{Res}_{F/\mathbb{Q}} T)$ *be a regular dominant character. Then there exists a positive integer* δ *depending on* χ *such that for all primes* p *outside* δ, *the exact control theorem holds for* $\mathcal{C}_\chi$. *In other words, suppose that* p *is prime to* δ *and* $H^q(\Phi,L(\chi;\mathcal{C})) = 0$ *for* $0 \le q < r$, *then for all arithmetic character* ψ *of* $T(r_p)$ *such that* $\psi \equiv \chi \bmod \mathfrak{m}$, *we have*

$$\text{(7.1a)} \quad H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p^\alpha),L(\psi)\big) \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p),\mathcal{C}_\chi\big)[\psi] = 0 \quad \text{if } q < r;$$

$$\text{(7.1b)} \quad H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p^\alpha),L(\psi)\big) \cong H^r_{n\cdot\mathrm{ord}}\big(\Phi_0(p),\mathcal{C}_\chi\big)[\psi].$$

Proof. — We apply the above lemma to the functor

$$M \longmapsto H^q(M) = H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), M\big)$$

on the category of admissible (Δ^{-1}, R)-modules. Here the «admissibility» means that M is a injective limit of discrete (Δ^{-1}, R)-modules whose Pontryagin dual is of finite type over $\mathcal{O}$. Write $\mathcal{C} = \mathcal{C}_\chi$. As checked in the proof of THEOREM 5.1, this choice of $\mathcal{C}$ satisfies the assumption (i), (ii) and (iv) of LEMMA 7.1 for any algebraic χ. It is easy to see that $H^0(M) = 0$ for all objects M in the category because $T_s(\varpi^j)$ acts on $H^0(M)$ by its degree which is a non-trivial p-power. Let $T_0, \dots, T_m$ be a regular sequence. Then we have an exact sequence:

$$0 \to \mathcal{C}[T_0, \dots, T_j] \longrightarrow \mathcal{C}[T_0, \dots, T_{j-1}] \xrightarrow{T_j} \mathcal{C}[T_0, \dots, T_{j-1}] \to 0.$$

We write E_j (resp. N_j) for the Pontryagin dual of $H^q(\mathcal{C}[T_0, \dots, T_j])$ (resp. $H^{q-1}(\mathcal{C}[T_0, \dots, T_j])$). Then by the cohomology sequence associated to the above short exact sequence, we get another short exact sequence:

$$(*) \qquad 0 \to E_{j-1}/T_j E_{j-1} \longrightarrow E_j \longrightarrow N_{j-1}[T_j] \to 0.$$

Now we assume that $N_j = 0$ for all j. Then by the above sequence, we get the exact control:

$$(**) \qquad E_j \cong E_{j-1}/T_j E_{j-1}.$$

Since we know that $N_j = 0$ for $q = 0$, we have the control as above for $q = 0$. Since we have proven for $q = 0$, $E_m = H^0(\mathcal{C}[T_0, \dots, T_m])^* = 0$, we know $E_j = 0$ for all j by Nakayama's lemma. Thus we can apply again our argument to $q = 1$ and get (**) again for $q = 1$. Now we see from THEOREM 5.1 that

$$H^q\big(\mathcal{C}(Y; K/\mathcal{O})\big) \cong H^q_{n\text{-ord}}\big(\Phi_1(p^\infty), L(\chi)\big).$$

Let μ be the maximal torsion subgroup of $T(r_p)$. Then the set Ξ of all primes p such that $p \mid \#\mu$ is finite. Let $d = \prod_{p \in \Xi} p$. Then if p is prime to d, we know that

$$H^q(\mathcal{C}) \cong 1_\chi H^q_{n\text{-ord}}\big(\Phi_1(p^\infty), L(\chi)\big).$$

Let $Y_0 = \mathrm{SL}_n(r_p)/U_B(r_p)$, and define

$$L_0(\chi; O) = \big\{\phi \in H^0(X_{/\mathcal{O}}, \mathcal{O}_X)\,;\ \phi(xz^{-1}) = \chi(z)\phi(x) \text{ for } z \in T(r_p)\big\},$$

where X is the scheme $\mathrm{Res}_{r/\mathbb{Z}}(\mathrm{SL}_n / U_B)$ defined over $\mathcal{O}$ and $\mathcal{O}_X$ is its structure sheaf. We view $L_0(\chi; O)$ as a space of functions on Y_0 with values in $\mathcal{O}$. By definition,

$$L_0(\chi; \mathcal{O}) = \bigoplus_\eta \mathcal{O} v_\eta$$

for weight vectors v_η belonging weights η occurring on $L(\chi;K)$ (see [J, I.2.11]). We put

$$L_0(\chi) = L_0(\chi;\mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O}, \quad L_0(\chi;A) = L_0(\chi;\mathcal{O}) \otimes_{\mathcal{O}} A.$$

Then $L_0(\chi;A)$ is a Φ-module. We also have a natural action ρ_χ of Δ^{-1} on $L_0(\chi;K)$ Since $P = B$, we know that $i : T_M \cong T_P$. Since $D_B \subset \mathbb{T}(F_p)$, for $\xi \in D_B$, we can think of $\chi(\pi(\xi))$ for the projection $\pi : \mathbb{T} \to T_P$. Let

$$\Delta_0 = \mathrm{SL}_n(r_p) D\, \mathrm{SL}_n(r_p).$$

Then Δ_0 is a semi-group and acts on $L_0(\chi;K)$.

LEMMA 7.2. — *Let $\omega_1, \ldots, \omega_m$ be the fundamental dominant weights of T with respect to (G_1,B). If $\chi = \sum_i a_i\omega_i$ with $a_i > 0$, then for $\xi \in D_B$, $\chi(\pi(\xi))^{-1}\rho_\chi(\xi^{-1})$ preserves $L_0(\chi;\mathcal{O})$. In particular, $L_0(\chi)$ is stable under Δ_0 by the action induced by $\xi \mapsto \chi(\pi(\xi))^{-1}\rho_\chi(\xi^{-1})$. Moreover for the idempotent e_0 attached to the double coset action of $\Phi\xi\Phi$ with $\xi = \prod_{1\le s\le n-1} \xi_s$ for ξ_s in $(\mathrm{T}p_s)$, we have , if $\omega = \sum_i a_i\omega_i$ with $a_i > 0$,*

$$H^q_{n\text{-}\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big) \cong e_0 H^q\big(\Phi, L_0(\chi)\big).$$

We first prove the theorem admitting the lemma. Since χ is regular, we know that $\chi = \sum_i a_i\omega_i$ with $a_i > 0$. The lemma implies

$$H^q_{n\text{-}\mathrm{ord}}\big(\Phi_0(p), L(\chi;\mathcal{O})\big) \cong e_0 H^q\big(\Phi, L_0(\chi,\mathcal{O})\big).$$

From the exact sequence:

$$0 \to L_0(\chi;\mathcal{O}) \longrightarrow L_0(\chi;K) \longrightarrow L_0(\chi) \to 0,$$

we get another sequence:

$$H^q_{n\text{-}\mathrm{ord}}(\Phi, L_0(\chi;K)) \longrightarrow H^q_{n\text{-}\mathrm{ord}}(\Phi, L_0(\chi)) \\ \longrightarrow H^{q+1}_{n\text{-}\mathrm{ord}}\big(\Phi, L_0(\chi;\mathcal{O})\big) \longrightarrow H^{q+1}_{n\text{-}\mathrm{ord}}\big(\Phi, L_0(\chi;K)\big).$$

Here we have written $H^q_{n\text{-}\mathrm{ord}}$ also for $e_0 H^q$ by abusing the notation. Therefore the maximal finite quotient of $H^q_{n\text{-}\mathrm{ord}}(\Phi, L_0(\chi))$ is isomorphic to the p-torsion part $H^{q+1}_{n\text{-}\mathrm{ord}}(\Phi, L_0(\chi;\mathcal{O}))[p^\infty]$ of $H^{q+1}_{n\text{-}\mathrm{ord}}(\Phi, L_0(\chi;\mathcal{O}))$. Let $\mathcal{O}_0$ be an integral domain dense in $\mathcal{O}$ which is finite over $\mathbb{Z}$. We may suppose that G is split over $\mathcal{O}_0$. Note that $\mathcal{O}$ is flat over $\mathcal{O}_0$. Thus:

$$H^{q+1}_{n\text{-}\mathrm{ord}}\big(\Phi, L_0(\chi;\mathcal{O})\big) = \varprojlim_\alpha H^{q+1}_{n\text{-}\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O}_0/p^\alpha\mathcal{O}_0)\big).$$

We have an exact sequence:

$$0 \to H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O}_0)\big) \otimes_{\mathbb{Z}} \mathbb{Z}/p^\alpha\mathbb{Z} \to H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi; \mathcal{O}_0/p^\alpha\mathcal{O}_0)\big) \to H^{q+2}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi; \mathcal{O}_0)\big)[p^\alpha] \to 0.$$

This shows that:

$$H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O}_0)\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \varprojlim_{\alpha} H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O}_0/p^\alpha\mathcal{O}_0)\big).$$

Thus $H^{q+1}_{n\cdot\mathrm{ord}}(\Phi, L_0(\chi, \mathcal{O}_0))$ is dense in $H^{q+1}_{n\cdot\mathrm{ord}}(\Phi, L_0(\chi, \mathcal{O}))$. In particular, we have:

$$H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O})\big)[p^\infty] \cong H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi, \mathcal{O}_0)\big)[p^\infty].$$

We write δ_q for the order of the torsion part of $H^{q+1}(\Phi, L_0(\chi; \mathcal{O}_0))$. Then we define a positive integer

$$\delta = d \prod_{1 \le q \le r} \delta_q.$$

Now suppose

(A$_1$) $\qquad$ *p is prime to δ.*

We also suppose that $H^q(\Phi, L(\chi; \mathbb{C})) = 0$ for $0 \le q < r$. Then for $0 \le q < r$, we see

$$H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi; K)\big) = 0.$$

Then the condition (A$_1$) implies for $0 \le q < r$

$$H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big) \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi)\big) \cong H^{q+1}_{n\cdot\mathrm{ord}}\big(\Phi, L_0(\chi; \mathcal{O})\big)[p^\infty] = \{0\}$$

and

$$H^r_{n\cdot\mathrm{ord}}(\Phi_0(p), L_0(\chi)) \text{ is } p\text{-divisible.}$$

From the exact sequence

$$0 \to L(\chi)[\pi] \longrightarrow L(\chi) \xrightarrow{\ \pi\ } L(\chi) \to 0$$

for a prime element π of $\mathcal{O}$, we get:

$$0 \to H^{q-1}_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big) \otimes \mathcal{O}/\pi\mathcal{O} \longrightarrow H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)[\pi]\big) \longrightarrow H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big)[\pi] \to 0.$$

This shows by LEMMA 7.1 that if

(V$_1$) $\qquad$ $$H^{q-1}_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big) = 0,$$

the control for $H^q_{n\cdot\mathrm{ord}}$ works well for the maximal ideal $\mathcal{M}$ of Λ.

That is

$$
\begin{aligned}
&H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)[T_0]\big)\big) \\
&\qquad \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), L(\chi)\big)[T_0] \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}\big)[\mathcal{M}],
\end{aligned}
$$

where we have chosen T_0 to be a prime element of $\mathcal{O}$. Thus:

$$H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}[\mathcal{M}]\big) \cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}\big)[\mathcal{M}].$$

In particular, since (V_1) is true for all $q \le r$, we know

(V_2) $$H^q_{n\cdot\mathrm{ord}}(\Phi_0(p), \mathcal{C})[\mathcal{M}] = 0 \quad \text{for all} \quad q < r.$$

Now we shall show from (V_2) that, for all j, for all $q < r$ and all regular sequence $T_0, \dots, T_m$ generating $\mathcal{M}$, we have

$$H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}[T_0, \dots, T_j]\big) = 0.$$

From the exact sequence

$$
\begin{aligned}
0 \to &H^{q-1}\big(\mathcal{C}[T_0, \dots, T_{j-1}]\big) \otimes \Lambda/T_j\Lambda \\
&\longrightarrow H^q\big(\mathcal{C}[T_0, \dots, T_j]\big) \longrightarrow H^q\big(\mathcal{C}[T_0, \dots, T_{j-1}]\big)[T_j] \to 0,
\end{aligned}
$$

supposing

(V_3) $$H^{q-1}(\mathcal{C}[T_0, \dots, T_j]) = 0 \quad \text{for all } j,$$

we conclude by induction on $(m-j)$ that

$$H^q\big(\mathcal{C}[T_0, \dots, T_j]\big) = H^q(\mathcal{C})[T_0, \dots, T_j]$$

for every regular sequence $T_0, \dots, T_m$ generating $\mathcal{M}$. Since we know the assumption (V_3) is true for $q = 0$ and (V_1) holds for $q \le r$, we conclude (V_3) for $q = 2$. Repeating this argument through induction on q, we conclude the control theorem:

$$H^q\big(\mathcal{C}[T_0, \dots, T_j]\big) \cong H^q(\mathcal{C})[T_0, \dots, T_j] \quad \text{for } q \le r$$

and the vanishing

$$H^q\big(\mathcal{C}[T_0, \dots, T_j]\big) = 0 \quad \text{for } q < r.$$

We apply this to a regular sequence $(T_1, \dots, T_m)$ generating $\mathrm{Ker}(\psi) \cap \Lambda$, we have:

$$
\begin{aligned}
H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}\big)[\psi] &\cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p), \mathcal{C}[\psi]\big) \\
&\cong H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p^\alpha), L(\psi)\big) \quad \text{for } q \le r, \\
H^q_{n\cdot\mathrm{ord}}\big(\Phi_0(p^\alpha), L(\psi)\big) &= 0 \qquad\qquad\qquad\qquad \text{if } q < r.
\end{aligned}
$$

This finishes the proof.

COROLLARY 7.1. — *Let the assumption be as in Theorem* 7.1. *If p is prime to δ, then $1_\chi H^r_{n\cdot\mathrm{ord}}(\Phi_1(p^\infty), L(\psi;\mathcal{O}))$ is $\mathcal{O}$-free.*

Proof. — We consider the exact sequence:

$$0 \to L(\psi;\mathcal{O}) \longrightarrow L(\psi;K) \longrightarrow L(\psi) \to 0.$$

From this, we get:

$$0 \to H^{r-1}_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\psi;\mathcal{O})\big) \otimes K/\mathcal{O} \longrightarrow H^{r-1}_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\psi)\big) \longrightarrow H^r_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\psi)\big)[p^\infty] \to 0.$$

We have

$$1_\xi H^r_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\psi;\mathcal{O})\big)[p^\infty] \cong 1_\chi H^{r-1}_{n\cdot\mathrm{ord}}\big(\Phi_1(p^\infty), L(\psi)\big) = 0.$$

Thus $1_\chi H^r_{P,n-\mathrm{ord}}(\Phi_1(p^\infty), L(\psi;\mathcal{O}))$ is $\mathcal{O}$-free

Let us prove LEMMA 7.2. We consider $\mathcal{V} = r_\mathfrak{p}$, $Y_0 = \mathrm{SL}_n(\mathcal{V})/U(\mathcal{V})$ and $Y = S_\mathfrak{p}/U(\mathcal{V})$ for

$$S_\mathfrak{p} = \big\{x \in \mathrm{SL}_n(\mathcal{V})\,;\; x \mod \varpi \in B(\mathcal{V}/\varpi\mathcal{V})\big\}$$

for a prime element ϖ of $\mathcal{V}$. Let $\chi_\mathfrak{p}$ (resp. $\chi^{(\mathfrak{p})}$ be the $\mathfrak{p}$-part (resp. prime-to-$\mathfrak{p}$ part) of χ. We then define $L_0(\chi_\mathfrak{p}) = L_0(\chi_\mathfrak{p};\mathcal{O}) \otimes_\mathcal{O} K/\mathcal{O}$ for

$$L_0(\chi_\mathfrak{p};\mathcal{O}) = H^0(X, \mathcal{O}_X)[\chi_\mathfrak{p}] \quad \text{for} \quad X = \mathrm{Res}_{r_\mathfrak{p}/z_\mathfrak{p}} \mathrm{SL}_n / U_B.$$

Let Φ be a congruence subgroup with closure S in $G_1(\mathbb{A}^{(\infty)})$ satisfying $S = S_p \times S^{(p)}$ and $S_p = \prod_{\mathcal{P}|p} S_\mathcal{P}$ with $\mathrm{SL}_n(r_\mathcal{P}) \supset S_\mathcal{P} \supset I_{B,\mathcal{P}}$. Let $e_\mathfrak{p}$ (resp. $e_{0,\mathfrak{p}}$) be the idempotent associated to $\prod_s T_s(\varpi)$ for $\Phi_0(\mathfrak{p})$ (resp. Φ). For any $\mathcal{O}$-lattice L in $L(\chi^{(\mathfrak{p})};K)$ stable under $S^{(\mathfrak{p})}$, what we need to prove is:

(i) $L_0(\chi_\mathfrak{p};\mathcal{O})$ is stable under $\chi(\pi(\xi))^{-1}\rho_\chi(\xi^{-1})$ for $\xi = \xi_s$ in $(T_{\mathfrak{p}_s})$;

(ii) $e_\mathfrak{p} H^q(\Phi_0(\mathfrak{p}), L(\chi_\mathfrak{p}) \otimes L) \cong e_{0,\mathfrak{p}} H^q(\Phi, L_0(\chi_\mathfrak{p}) \otimes L)$.

For simplicity, we hereafter write $L(\chi)$ (resp. $L_0(\chi)$, resp. $L(\chi;K)$) for $L(\chi_\mathfrak{p}) \otimes L$ (resp. $L_0(\chi_\mathfrak{p}) \otimes L$, resp. $L_0(\chi_\mathfrak{p};K) \otimes L$.

As we have already remarked, $L_0(\chi;\mathcal{O}) = \bigoplus_\eta \mathcal{O}v_\eta$ for weight vectors η for T. For any $d \in D$, we have

$$\big|\chi(\pi(d))\big|_p \geq \big|\eta(\pi(d))\big|_p$$

for any other weight η occurring in $L(\chi;K)$ because χ is the highest weight. Then the assertion (i) is obvious from this.

Put:

$$H = H_P = \{x \in SL_n(\mathcal{V})\,;\ x \bmod \varpi \in P(\mathcal{V}/\varpi\mathcal{V})\}$$

Note that if P is a proper (standard) maximal parabolic subgroup,

$$\mathrm{SL}(n) = \coprod_{w\in\{1,\tau\}} PwP \quad \text{for } \tau = (\delta_{n+1-i,j})_{i,j}.$$

This shows:

$$(*) \qquad \mathrm{SL}_n(\mathcal{V}) = \coprod_{w\in\{1,\tau\}} H_P w H_P = \coprod_{w\in\{1,\tau\}} \Big(\coprod_{u\in H/w^{-1}Hw\cap H} Hwu \Big).$$

Then for $\xi_s = \mathrm{diag}(1_t, \varpi 1_s)$ with a prime element ϖ of $\mathcal{V}$, we have:

$$H = \mathrm{SL}_n(\mathcal{V}) \cap (\xi\tau)^{-1}\,\mathrm{SL}_n(\mathcal{V})(\xi\tau) \quad \text{for } P = P_t.$$

Since $\mathrm{SL}_n(\mathcal{V})\xi_s\,\mathrm{SL}_n(\mathcal{V}) = \mathrm{SL}_n(\mathcal{V})\xi_s\tau\,\mathrm{SL}_n(\mathcal{V})$, this shows that for $P = P_t$

$$(\xi_s\tau)^{-1}\,\mathrm{SL}_n(\mathcal{V})(\xi_s\tau)\,\mathrm{SL}_n(\mathcal{V}) = \coprod_{w\in\{1,\tau\}} \Big(\coprod_{u\in H/w^{-1}Hw\cap H} (\xi_s\tau)^{-1}\,\mathrm{SL}_n(\mathcal{V})\xi_s\tau wu \Big)$$

which implies

$$\mathrm{SL}_n(\mathcal{V})\xi_s\,\mathrm{SL}_n(\mathcal{V}) = \coprod_{w\in\{1,\tau\}} \Big(\coprod_{u\in S/(\tau w)^{-1}S\tau w\cap S} \mathrm{SL}_n(\mathcal{V})\xi_s wu \Big).$$

Now we prove (ii). As we have seen already, the evaluation at O

$$\iota : L(\chi; K/\mathcal{O})[\pi] \longrightarrow L\big(\chi^{(\mathfrak{p})}; \mathcal{O}/\pi\mathcal{O}\big)(\chi)$$

is a morphism of $\Phi_0(\mathfrak{p})$-modules, where

$$L(\chi^{(\mathfrak{p})}; \mathcal{O}/\pi\mathcal{O})(\chi)$$

is the $L(\chi^{(\mathfrak{p})}; \mathcal{O}/\pi\mathcal{O})$ on which $\Phi_0(\mathfrak{p})/\Phi_1(\mathfrak{p}) = T(r/\mathfrak{p})$ acts *via* $\chi_{\mathfrak{p}}$. We may assume that

$$\omega_i(\mathrm{diag}(t_1, \ldots, t_n)) = \prod_{1\le j\le i} t_j \qquad \text{for } i = 1, \ldots, n-1.$$

Then if we write

$$\chi(\mathrm{diag}(t_1,\dots,t_n)) = \prod_{1\le i\le n-1} t_i^{j_i},$$

the assumption $a_i > 0$ implies $j_1 > j_2 > \ldots > j_{n-1}$. Thus for each cocycle c with values in $L_0(\chi;\mathcal{O})$, we claim that

$$(**)\qquad \begin{cases} (\omega w u)^{-1} c(*,\dots,*;O) = c(*,\dots,*;\xi w u O) \equiv 0 \mod \chi(\pi(\xi)) \\ \hfill \text{if } w = 1, \\ (\xi w u)^{-1} c(*,\dots,*;O) = c(*,\dots,*;\xi w u O) \equiv 0 \mod \chi(\pi(\xi))\varpi \\ \hfill \text{if } w = \tau. \end{cases}$$

Since u can be chosen to be unipotent upper triangular, it fixes O. Thus we need to look at $\xi w O$. The evaluation at O sends vector in $L(\chi;K)$ to its coefficient of the highest weight vector with respect to B. Thus for $d \in D$,

$$\rho_\chi(d^{-1})f(O) = f(dO) = \chi\big(\pi(d)\big)f(O).$$

Thus the evaluation at wO send it to the coefficient in a weight-vector associated to the conjugate $t \mapsto \lambda(t) = \chi(\pi(\tau^{-1}t\tau))$; that is,

$$\big(\rho_\chi(d^{-1})f\big)(\tau O) = \lambda(d) f(\tau O).$$

It is easy to see

$$\big|\lambda(\xi_s)\big|_p < \big|\chi\big(\pi(\xi)\big)\big|_p,$$

because χ is the highest, $\xi = \mathrm{diag}(1_t, \varpi 1_s)$ and $j_1 > j_2 > \cdots > j_{n-1}$. This shows the claim. The claim implies that modulo π, $T_s(\varpi)$ of level 1 at $\mathfrak{p}$ coincides with $T_s(\varpi)$ of level $\mathfrak{p}$. Thus the morphism ι induces a morphism

$$\begin{aligned} \iota_* \circ \mathrm{res} : e_{0,\mathfrak{p}} H^q\big(\Phi, L_0(\chi)[\mathfrak{p}]\big) &\longrightarrow e_{\mathfrak{p}} H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)[\mathfrak{p}]\big) \\ &\longrightarrow e_{\mathfrak{p}} H^q\big(\Phi_0(\mathfrak{p}), L(\chi^{(\mathfrak{p})};\mathcal{O}/\pi\mathcal{O})(\chi)\big). \end{aligned}$$

Thus we have a commutative diagram:

$$\begin{array}{ccc} e_{0,\mathfrak{p}} H^q\big(\Phi, L_0(\chi)[\mathfrak{p}]\big) & \longrightarrow & e_{\mathfrak{p}} H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)[\mathfrak{p}]\big) \\ \big\downarrow & & \iota_* \big\| \wr \\ e_{\mathfrak{p}} H^q\big(\Phi_0(\mathfrak{p}), L(\chi^{(\mathfrak{p})};\mathcal{O}/\pi\mathcal{O})\big) & = & e_{\mathfrak{p}} H^q\big(\Phi_0(\mathfrak{p}), L(\chi^{(\mathfrak{p})};\mathcal{O}/\pi\mathcal{O})(\chi)\big). \end{array}$$

Since $\mathrm{Tr} \circ \mathrm{res}$ is a multiplication by the index $(\Phi : \Phi_0(\mathfrak{p}))$ which is prime to p, we know that the upper horizointal arrow is an inclusion. Note that ι_* is a surjective isomorphism by the argument in the proof of THEOREM 5.1 applied to the exact sequence:

$$0 \to K_1 \longrightarrow L_0(\chi)[\mathfrak{p}] \longrightarrow L(\chi^{(\mathfrak{p})}; \mathcal{O}/\pi\mathcal{O})(\chi) \to 0$$

in place of (*) there.

Thus we conclude

$$\mathrm{res} \circ e_{0,\mathfrak{p}} = e_{\mathfrak{p}} \circ \mathrm{res}\,.$$

Then, again by the fact that $\mathrm{Tr} \circ \mathrm{res} = (\Phi : \Phi_0(\mathfrak{p}))$, we conclude that the upper horizontal arrow is a surjective isomorphism. In particular, the restriction map res of $H^q(\Phi, L_0(\chi)[\mathfrak{p}])$ into $H^q(\Phi_0(\mathfrak{p}), L_0(\chi)[\mathfrak{p}])$ satisfies

$$\mathrm{res} \circ T_0 = T \circ \mathrm{res} \quad \text{for } T = \big[\Phi_0(\mathfrak{p})\xi\Phi_0(\mathfrak{p})\big] \text{ and } T_0 = [\Phi\xi\Phi].$$

Anyway we have another commutative diagram:

$$\begin{array}{ccc} e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)[\mathfrak{p}]\big) & \overset{\simeq}{=\!=\!=} & e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)[\mathfrak{p}]\big) \\ \big\downarrow & & \big\downarrow \\ e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)[\mathfrak{p}]\big) & \longrightarrow & e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)\big)[\mathfrak{p}] \end{array}$$

Since vertical arrows are all surjective, again by $\mathrm{Tr} \circ \mathrm{res} = (\Phi : \Phi_0(\mathfrak{p}))$, lower horizontal arrow is a surjective isomorphism. Then by Nakayama's lemma, we see

$$e_{\mathfrak{p}} \circ \mathrm{res} : e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)\big)[\mathfrak{p}^\alpha] \longrightarrow e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)\big)[\mathfrak{p}^\alpha],$$

$$e_{0,\mathfrak{p}} \circ \mathrm{Tr} : e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)\big)[\mathfrak{p}^\alpha] \longrightarrow e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)\big)[\mathfrak{p}^\alpha]$$

are both injective. Thus $(e_{\mathfrak{p}} \circ \mathrm{res}) \circ (e_{0,\mathfrak{p}} \circ \mathrm{Tr})$ is an injective endomorphism on the finite set $e_{\mathfrak{p}}H^q(\Phi_0(\mathfrak{p}), L_0(\chi))[\mathfrak{p}^\alpha]$ and therefore is an automorphism. Similarly $(e_{0,\mathfrak{p}}\,\mathrm{Tr}) \circ (e_{\mathfrak{p}} \circ \mathrm{res})$ is an automorphism of $(e_{0,\mathfrak{p}}H^q(\Phi, L_0(\chi))[\mathfrak{p}^\alpha]$ for all α. This shows

$$e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)\big) \cong e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)\big).$$

We have another commutative diagram using (4.7c):

$$\begin{array}{ccc} e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)[\mathfrak{p}^\alpha]\big) & \longrightarrow & e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L(\chi)[\mathfrak{p}^\alpha]\big) \\ \iota_*\Big\|\wr & & \iota_*\Big\|\wr \\ e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi^{(p)};\mathcal{O}/\pi^\alpha\mathcal{O})(\chi)\big) & = & e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L(\chi^{(p)};\mathcal{O}/\pi^\alpha\mathcal{O})(\chi)\big). \end{array}$$

This shows:

$$e_{0,\mathfrak{p}}H^q\big(\Phi, L_0(\chi)\big) \cong e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L_0(\chi)\big) \cong e_{\mathfrak{p}}H^q\big(\Phi_0(\mathfrak{p}), L(\chi)\big),$$

which finishes the proof.

8. The bottom degree

Now we clarify the bottom degree $r = r(\chi)$ with respect to B. We write $\mathfrak{s}$ for the Lie algebra for $G_1(\mathbb{R})$ and $C_{\infty+}$ for the standard maximal compact subgroup of $G_1(\mathbb{R})$. First suppose that

(Cpt) $\quad G_1(\mathbb{Q})\backslash G_1(\mathbb{A})$ *is compact for the adèle ring* $\mathbb{A}$ *of* $\mathbb{Q}$.

Let Π_∞ be that set of irreducible admissible representations of $G_1(\mathbb{R})$. We write $m(\pi;\Gamma)$ for the multiplicity of π occurring discretely on $L_2(\Gamma\backslash G_1(\mathbb{R}))$, where Γ is a congruence subgroup of $G_1(\mathbb{Q})$. Then by [MM] and [BW, V.3.2] (see also [C, § 3.5]), we know for each polynomial representation L of $G_1(\mathbb{R})$ over $\mathbb{C}$

$$H^i(\Gamma, L) \cong \bigoplus_{\pi\in\Pi_\infty} H^i\big(\mathfrak{s}, C_{\infty+}; H(\pi)\otimes \widetilde{L}\big)^{m(\pi,\Gamma)}, \tag{8.1}$$

where $H(\pi)$ is the space of smooth vectors of π and $\widetilde{L}$ is the contragredient of L. Now we suppose the following two conditions:

(D) $\quad G(A) = (R\otimes_{\mathbb{Z}}A)^\times$ *for a maximal order* R *of a central simple algebra* D *over* F*;*

(GL(∞)) $\quad G(\mathbb{R}) \cong \mathrm{GL}_n(F_\infty)$ *for* $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$.

The condition (D) and (Cpt) combined imply that D is a division algebra. By (GL(∞)), we have the determinant map

$$\det : G(\mathbb{R}) \longrightarrow F_\infty^\times.$$

Then we put

$$G_0 = \det^{-1}(T_\infty) = \{x \in G_0(\mathbb{R})\,;\ \det(x) \in \mathbb{T}_\infty\},$$

where

$$\mathbb{T}_\infty = \{y \in F_\infty^\times\,;\ |x^\sigma| = 1 \text{ for all field embeddings } \sigma : F \to \mathbb{C}\}.$$

Let $\mathfrak{g}$ be the Lie subalgebra of G_0 in the Lie algebra of $G(\mathbb{R})$ and C_∞ be the maximal compact subgroup of $G(\mathbb{R})$ containing $C_{\infty+}$. We can extend the action of $G_1(\mathbb{R})$ on L to $G(\mathbb{R})$ (which is analytic but may not be a polynomial representation of G). Let ∞ be the set of archimedean places of F. Then we decompose

$$L = \bigotimes_{v\in\infty} L_v$$

for irreducible representations L_v. Now, there is at most one ∞-dimensional tempered irreducible unitary representation

$$\pi_L = \bigotimes_{v\in\infty} \pi_{L_v}$$

of $G(\mathbb{R})$ such that $H^i(\mathfrak{g}, C_\infty; H(\pi_L) \otimes \widetilde{L}) \neq 0$ for some i (see [C, § 3.5]). In this case, we call π_L *cohomological.* This π_L has the same infinitesimal character with L (if such π_L exists). The infinitesimal character characterizes π_{L_v} if v is complex because $\mathbb{C}^\times$ is connected. If v is real, for a given infinitesimal character, there can be several if there exist such irreducible representations. One of them is π_{L_v} (see [C, Lemma 3.14] for details). By [C, Lemma 3.14, p. 114], under (SL(∞)), we know that for cohomological π_L

$$H^i(\mathfrak{g}, C_\infty; H(\pi_{L_v}) \otimes \widetilde{L}_v) \cong \begin{cases} \bigwedge^{i-(n(n-1)/2)} \mathbb{C}^{n-1} & \text{if } v \text{ is complex,} \\ \bigwedge^{i-(n^2/4)} \mathbb{C}^{(n/2)-1} & \text{if } n \text{ is even and } v \text{ is real,} \\ \bigwedge^{i-[n/2][(n/2)]+1} \mathbb{C}^{[n/2]} & \text{if } n \text{ is odd and } v \text{ is real.} \end{cases}$$

Here we use the convention that $\bigwedge^0 V = \mathbb{C}$ and $\bigwedge^1 V = V$ for any complex vector space V. As explained in the proof of Lemma 4.9, p. 144 of [C], π_L is the unique irreducible generic representation with non-trivial $H^i(\mathfrak{g}, C_\infty(\pi_{L_v}) \otimes \widetilde{L}_v)$ for some i. It is also known that:

(8.3) *If $L = L(\chi)$ for a dominant character χ of the standard maximal torus $\mathbb{T}$ of $\mathrm{GL}(n)$ and $\chi^{-\tau}|_T \neq \chi|_T$ for $\chi^{-\tau}(t) = \chi(\tau t^{-1}\tau^{-1})$ with $\tau = (\delta_{n+1-i,j})_{i,j} \in W$, there is no cohomological π_L, where $T = \mathbb{T} \cap \mathrm{SL}(n)$ and W is the Wyle group of $\mathbb{T}$.*

We now assume:

(JL) $\begin{cases} \textit{The global Jacquet-Langlands correspondence compatible with} \\ \textit{the local correspondence holds for } G \textit{ and } \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(n). \end{cases}$

The local Jacquet-Langlands correspondence is known by [DKV] and [R]. The existence of global correspondence is known under a certain ramification condition (see [AC, Thm B] and [C1, Thm 3.3]). Under this assumption, $H^i(\Gamma, L)$ is embedded in $H^i_{\mathrm{sq}}(\Gamma', L)$ for a suitable congruence subgroup Γ' of

$$\mathrm{SL}_n(F) \subset G_1(\mathbb{R})\big(\mathrm{GL}(\infty)\big),$$

where the latter cohomology group is the square integrable cohomology group [B] and [C, § 3.5]. On the other hand, we know from [B] that

$$H^i_{\mathrm{sq}}(\Gamma', L) \cong \bigoplus_{\pi\in\Pi_\infty} H^i\big(\mathfrak{s}, C_{\infty+}; H(\pi)\otimes \widetilde{L}\big)^{m(\pi,\Gamma')}, \tag{8.4}$$

where $\mathfrak{s}$ is the Lie algebra of $G_1(\mathbb{R})$. Note that

$$H^i\big(\mathfrak{s}, C_{\infty+}; H(\pi)\otimes\widetilde{L}\big) \cong \mathrm{Hom}_{C_{\infty+}}\big(\textstyle\bigwedge^i\mathfrak{p}, H(\pi)\otimes\widetilde{L}\big),$$

$$H^i\big(\mathfrak{g}, C_{\infty}; H(\pi)\otimes\widetilde{L}\big) \cong \mathrm{Hom}_{C_{\infty}}\big(\textstyle\bigwedge^i\mathfrak{p}', H(\pi)\otimes\widetilde{L}\big),$$

where $\mathfrak{s} = \mathfrak{c}_+ \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{c}\oplus\mathfrak{p}'$ are the Cartan decomposition for the Lie algebras $\mathfrak{c}_+$ and $\mathfrak{c}$ of $C_{\infty+}$ and C_∞. Thus we can extend any representation π on $H(\pi)$ of $G_1(\mathbb{R})$ to $E = \mathbb{T}_\infty G_1(\mathbb{R})$ choosing a central character. We write π^0 for the extension. Note that

$$G_0/E \cong C_\infty/\mathbb{T}_\infty C_{\infty+}$$

is an abelian group of type $((2,2,\ldots,2)$. If $\mathrm{Hom}_{C_{\infty+}}(\bigwedge^i\mathfrak{p}, H(\pi)\otimes\widetilde{L}) \neq 0$, choosing the central character of π^0 suitably, we see:

$$\mathrm{Hom}_{C_{\infty+}}\big(\textstyle\bigwedge^i\mathfrak{p}, H(\pi)\otimes\widetilde{L}\big) \cong \mathrm{Hom}_{C_\infty}\big(\textstyle\bigwedge^i\mathfrak{p}', \big(\mathrm{Ind}_{G_1}^{G_0} H(\pi)\big)\otimes\widetilde{L}\big).$$

Thus as long as π is tempered,

$$H^i\big(\mathfrak{s}, C_{\infty+}; H(\pi)\otimes\widetilde{L}\big) = 0 \quad\text{if}\quad i\le r_0,$$

where

$$r_0 = \begin{cases} \frac14 r_1 n^2 + \frac12 r_2 n(n-1) & \text{if } n \text{ is even } r,\\ r_1\big[\frac12 n\big]\big[(\frac12 n)+1\big] + \frac12 r_2 n(n-1) & \text{if } n \text{ is odd.}\end{cases} \tag{8.5}$$

If π with $m(\pi;\Gamma') > 0$ is cuspidal, then it is generic and tempered at ∞ by [C, Lemma 4.9]. If π with $m(\pi;\Gamma') > 0$ is not cuspidal, by the determination of the discrete spectrum of GL(n) [MW], its extension to GL(n) is of the form

$$\pi_0 \boxplus \pi_0 \otimes \mid \quad \mid_A^1 \boxplus \cdots \boxplus \otimes \mid \quad \mid_A^{a-1}$$

under the notation of [C, (1.5), p. 85], where $n = am$ and π_0 is an irreducible cuspidal representation of GL(m). Here we lifted first $\pi \in \Pi_\infty$ to an automorphic representation of $\mathrm{SL}_n(F_A)$ by the strong approximation theorem. The local component of π at each finite place v is, by Langlands' classification, of the form

$$\rho_1 \boxplus \rho_2 \boxplus \cdots \boxplus \rho_j$$

with square integrable representations ρ_i of $\mathrm{GL}_{n_i}(F_v)$, where

$$n_1 + n_2 + \cdots + n_j = n$$

(see [C, Chapter 1]). Since D is a division algebra by (Cpt), writing

$$D \otimes_F F_v \cong M_b(D_v)$$

for a division algebra D_v of degree d^2 over the local field F_v, we conclude from [R, Thm 5.8] that d divides n_i for $i = 1, 2, \ldots, j$. Since D is globally a division algebra, we can find v such that m is not divisible by d. Thus the representation of the form

$$\pi_0 \boxplus \pi_0 \otimes \mid \quad \mid_A^1 \boxplus \cdots \boxplus \pi_0 \otimes \mid \quad \mid_A^{a-1}$$

cannot be in the image of the Jacquet-Langlands correspondence except when $a = n$ and $m = 1$. In the exceptional case, the Langlands quotient is the trivial representation [C, p. 84]. Thus it is killed by the nearly ordinary projector e. Thus under (Cpt), (JL), (D) and (GL(∞)), the bottom degree $r(\chi)$ is given by r_0 or ∞ depending on χ. By (8.3), if $\xi^{-\tau}|_T \neq \chi|_T$, $r(\chi) = \infty$. This combined with COROLLARY 7.1 shows:

THEOREM 8.1. — *Let the assumption be as in Theorem* 7.1 *Assume* (GL), (S), (Cpt), (D), (GL(∞)) *and* (JL). *Then* $r(\chi) = r_0$ *or* ∞ *depending on* χ *and* Φ. *If* $\chi^{-\tau}|_T \neq \chi$, *then* $r(\chi) = \infty$ *and for almost all prime* p, *we have*

$$H^q_{n\text{-ord}}\big(\Phi_0(p), L(\chi)\big) = H^q_{n\text{-ord}}\big(\Phi_0(p), L(\chi;\mathcal{O})\big) = 0.$$

If (GL(∞)) is not satisfied, the number r depends on the shape of $G(\mathbb{R})$ but can be determined similarly. In place of (D) and GL(∞), we now assume that for a totally imaginary quadratic extension L over a totally real field F:

$$\text{(U)}\qquad \begin{cases} G = \mathrm{Res}_{F/\mathbb{Q}}\, U \textit{ for a unitary group } U \textit{ on a} \\ \textit{hermitian space } H \textit{ over the } CM \textit{ field } L. \end{cases}$$

In this case, F is the maximal totally real field of L, and (GL) implies that all prime factors of p in F split in L and $G_1 = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SU}$ for the special unitary group SU of H. Let $\mathfrak{a}$ be the set of all archimedean places of F, and write (p_v, q_v) for the signature of $H \otimes_{F_v} \mathbb{R}$. Then it is known by [BW, V.3.4] that under (Cpt), the bottom degree r satisfies:

$$r \geq \sum_{v \in \mathfrak{a}} \min(p_v, q_v) = \mathrm{rank}_{\mathbb{R}}\, G_1(R). \tag{8.6}$$

For some specific unitary group, a much stronger result is known [C2]. For example, if L is imaginary quadratic, U is associated to a central division algebra over L with an involution of second kind, $p = 1$ and $q + 1$ is a prime, we conclude from [C2] that $r \geq q$. Thus in this case, the control theorem holds for the middle cohomology. The study of the geometry of the p-adic locus of χ in $\mathrm{Hom}_{\mathrm{conti}}(T(r_p), \overline{\mathbb{Q}}_p^{\times})$ with a given bottom degree is an interesting problem for a given algebraic group G satisfying (GL).

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