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MULTIPLICITY ESTIMATES AND THE PRODUCT THEOREM

PAR

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RÉSUMÉ. — Nous établissons un lemme de zéros pour certaines variétés munies d'une action de groupe. Ceci généralise des résultats de D.W. Masser, G. Wüstholz et P. Phillipon. La démonstration utilise de manière systématique la théorie de l'intersection suivant en cela une suggestion de D. Bertrand.

ABSTRACT. — We prove a zero estimate for certain varieties endowed with a group action. The result generalizes theorems of D.W. Masser, G. Wüstholz and P. Phillipon. It makes systematic use of intersection theory as suggested by D. Bertrand.

0. Introduction

Since NESTERENKO's work in the 70's [N], zero estimates have been of central importance in transcendence theory. For an overview of some of the main results, see BERTRAND's Bourbaki Seminar [B] and its bibliography. More recently, similar methods were employed by FALTINGS [F] in his paper on rational points of subvarieties of abelian varieties. It is the purpose of this note to give a unified treatment of these results from the algebraic geometric point of view.

We will derive a slight generalization of the zero estimates of [P1] and [W1], [W2], extending these results from commutative algebraic groups to certain projective varieties endowed with a group action. We will also show how these methods naturally yield a more general version of Faltings' product theorem (see [F, thm 3.1]). This is not surprising since FALTINGS himself remarks that, « The proof [of the product theorem] uses

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methods similar to the zero estimates (NESTERENKO, MASSER-WÜSTHOLZ) of transcendental number theory [F, p. 549]. »

Our proof of the zero estimates differs from those of PHILIPPON, MASSER-WÜSTHOLZ, and WÜSTHOLZ in that it makes systematic use of intersection theory. In particular the main tool is a refined version of Bézout's theorem due to FULTON and LAZARSFELD. Note that this was already suggested by BERTRAND ([B, p. 26 and p. 29]), though it was from S. LANG that the author received the idea of replacing Proposition 3.3 of [P1] with an intersection theoretic result. Other than this, we freely borrow from both [P1] and [W2] and, wherever possible, try to point out the interconnections between them.

To state the main result, we need to fix some notation. For $1 \leq i \leq m$, let $X_i \subset \mathbb{P}^{n_i}$ be a projectively normal variety defined over $\mathbb{C}$. Let $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ with projective coordinate ring R and let

$$X = \prod_{i=1}^{m} X_i \hookrightarrow \mathbb{P}.$$

Let $\mathcal{O}_{X_i}(1)$ be the pull-back to X_i of the invertible sheaf $\mathcal{O}_{\mathbb{P}^{n_i}}(1)$. Let $\pi_i : X \to X_i$ denote the projection to the i^{th} factor and for positive integers $d_1, \ldots, d_m$ write

$$\mathcal{O}_X(d_1, \ldots, d_m) = \bigotimes_{i=1}^{m} \pi_i^* \mathcal{O}_{X_i}(d_i).$$

We will abbreviate $d = (d_1, \ldots, d_m)$. For $V \subset \mathbb{P}$ let $\deg_d V$ be the degree of V computed with respect to $\mathcal{O}_X(d_1, \ldots, d_m)$. Suppose that each X_i is furnished with a group action $\phi_i : G_i \times X_i \to X_i$ and let

$$\phi : G \times X \longrightarrow X$$

be the product group action. For $g \in G$ and $x \in X$ we will usually write $g(x) = \phi(g, x)$. The group law on G will be written multiplicatively. Denote by $\pi_2 : G \times X \to X$ the projection to the second factor and let e be the identity element of G.

DEFINITION 0.1. — Let $c = (c_1, \ldots, c_m)$ be an m-tuple of positive integers (depending only on the embedding $X \hookrightarrow \mathbb{P}$) such that

$$\pi_2^* \mathcal{O}_X(c_1, \ldots, c_m) \otimes \phi^* \mathcal{O}_X(-1, \ldots, -1)$$

is generated by global sections on $U \times X$ for some open affine subset $U \subset G$ containing the identity. If $d = (d_1, \cdots, d_m)$ is another m-tuple then let $cd = (c_1 d_1, \ldots, c_m d_m)$.

Let $\psi : \mathbb{C}^d \to G$ denote an analytic group homomorphism with $A = \psi(\mathbb{C}^d)$. For $g \in G$ and $x \in X$ define the analytic map

$$\psi_{g,x} : \mathbb{C}^d \longrightarrow X \quad \text{by} \quad z \mapsto \big[\psi(z) \cdot g\big](x).$$

The order of a multihomogeneous polynomial P along A at $g(x)$ can be defined as in PHILIPPON [P1, p. 357] and MASSER-WÜSTHOLZ [MW2, p. 234] :

DEFINITION 0.2. — Given a multihomogeneous $P \in R$, let the *order of P along A* at $g(x)$ be the order of the analytic function $P[\psi_{g,x}(z)]$ at $z = 0$.

Let $S \subset G$ be a finite subset with $e \in S$. As in [P1] write

$$S_n = \Big\{ \sum_{i=1}^{n} g_i \,;\ g_i \in S \Big\}.$$

Also let $Gx = \{g(x)\,;\ g \in G\}$ and

$$\deg_d(Gx) = \deg_d(\overline{Gx})$$

where $\overline{Gx}$ is the Zariski closure of Gx.

The following is a slight generalization of [P1, Thm 2.1] :

THEOREM 0.3. — *Let $x \in X$, $T \in \mathbb{N}$ and let $P \in R$ be a multihomogeneous polynomial of multidegree d. Assume that G is connected and commutative and that $\overline{Gx}$ is projectively normal of dimension n. Suppose P vanishes to order $\geq nT+1$ along A at $S_n(x)$. Then either P vanishes on all of Gx or there exists a proper connected algebraic subgroup $H \subset G$ and an element $g \in G$ such that P vanishes along $(g' \cdot g)Hx$ for all $g' \in S$ and*

$$\binom{T + \operatorname{codim}(Ax \cap Hx, Ax)}{\operatorname{codim}(Ax \cap Hx, Ax)} \times \operatorname{card}\big((S + Hx)/Hx\big) \deg_d Hx \leq \deg_{cd} Gx. \tag{0.3.1}$$

Under certain further hypotheses, THEOREM 0.3 can be extended to the case when G is non-commutative. No effort has been made to estimate the constants c_i appearing in THEOREM 0.3. We do, however, address the problem of obtaining a projectively normal embedding of X. Using techniques of BERTRAM, EIN, and LAZARSFELD (see [BEL], [EL]), we give a quick effective construction of projectively normal embeddings of certain commutative group compactifications, obtaining results similar to those of KNOP and LANGE [KL1].

The outline of the paper is as follows.

In § 1 we recall the prerequisites from intersection theory. The results presented here have been well known to intersection theorists for some ten years, but there seems to be no universally accepted method or reference so we have tried to give a unified account. We also prove a transversality result needed in the proof of the zero estimates. The end of this section is devoted to some remarks on commutative group compactifications.

In § 2 we show how to estimate lengths of primary ideals in terms of certain differential operators. This idea is originally due to WÜSTHOLZ (cf. [W2, Lemma 3]). The main theorems of the first two sections, THEOREMS 1.1 and 2.9, imply the product theorem (THEOREM 5.2) and, except for some technical lemmas proved in § 3, THEOREM 0.3 as well.

In § 3 we make some technical remarks about differential operators and fill in a small gap in PROPOSITION 1 of [W2].

Finally, in § 4 and § 5 we prove THEOREM 0.3 and the product theorem; the proofs follow PHILIPPON [P1] and FALTINGS [F] respectively and contain no new ideas.

1. Intersection Theory

The main tool used in the product theorem and zero estimates is a refined Bézout theorem. Before presenting this, we need to fix some notation. As in the introduction, let

$$\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$$

and let R be the corresponding projective coordinate ring :

$$R = k\big[X_{1,0}, \ldots, X_{1,n_1}, \ldots, X_{m,0}, \ldots, X_{m,n_m}\big].$$

We will assume that k is of characteristic zero. If $I \subset R$ is a multihomogeneous ideal, then denote by $V(I)$ the subscheme of $\mathbb{P}$ determined by I. Given a subscheme $X \subset \mathbb{P}$ and an irreducible component V of X, let $\ell_V(X)$ denote the length of $\mathcal{O}_{V,X}$. Note that if $X = V(I)$ then $\ell_V(X) = \ell(\mathcal{O}_{V,\mathbb{P}}/I)$. All intersections will be in the scheme theoretic sense (cf. [Fu, App. B 2.3]). If $f : X \to Y$ is a morphism of schemes let $f(X) \subset Y$ denote the *scheme theoretic* image (cf. [H, p. 93, 3.11 (d)]). If Y is a scheme, denote by $[Y]$ the associated cycle (cf. [Fu, section 1.5]).

Both PHILIPPON [P1, Prop. 3.3] and FALTINGS [F, Prop. 2.3] consider the problem of controlling the degree of the zero scheme of an arbitrarily large number of multihomogenous polynomials. For a treatment of this type of problem from the intersection theoretic viewpoint, one can refer to [Fu], [F-L] or [P-V], [S-V], [V]. The relationship between the approaches of FULTON-LAZARSFELD and VOGEL et al. is made explicit in [G, especially 3.6–3.10]. For an older treatment of the multihomogeneous Bézout theorem, one can refer to [Wa]. We have the following general result :

THEOREM 1.1. — *Let $S \subset H^0(\mathcal{O}(d_1,\dots,d_m))$ be a collection of multihomogeneous forms of multidegree d and let $J = (S) \subset R$ be the multihomogeneous ideal generated by S. Let X be a pure dimensional subscheme of $\mathbb{P}$ and let Y_j be the irreducible components of $X \cap V(J)$. Then*

$$\sum_{\mathcal{O}_{Y_j,X}\ \mathrm{CM}} \ell_{Y_j}\big(X \cap V(J)\big) \cdot \deg_d Y_j \le \deg_d X, \tag{1.1.1}$$

where the sum is over those Y_j such that $\mathcal{O}_{Y_j,X}$ is Cohen-Macaulay.

Proof. — First, following FALTINGS [F, Prop. 2.3], let

$$t = \max_j \big\{\operatorname{codim}(Y_j, X)\big\}$$

and choose $Q_1,\dots,Q_t \in J$ of multidegree d so that each Y_j is an irreducible component of $V(Q_1,\dots,Q_t) \cap X$; this can be done by taking[1] generic linear combinations of generators for J. Let $I = (Q_1,\dots,Q_t)$. Since $I \subset J$ it follows that

$$\ell_{Y_j}\big(X \cap V(J)\big) \le \ell_{Y_j}\big(X \cap V(I)\big) \tag{1.1.2}$$

We want to apply [Fu, Example 12.3.7] (obtained also by PATIL and VOGEL [P-V]; see [V, Cor. 2.28]), but there is a minor complication caused by the fact that the intersection takes place in a multiprojective space.[2] So let $\mathbb{P} \xrightarrow{i} \mathbb{P}^N$ be the Segre embedding determined by a basis for $H^0(\mathcal{O}(d_1,\dots,d_m))$. Let $X' = i(X)$ and similarly let $Y'_j = i(Y_j)$. Choose

$$L_\alpha \in H^0\big(\mathcal{O}_{\mathbb{P}^N}(1)\big) \quad \text{such that} \quad i^* L_\alpha = Q_\alpha$$

[1] The technique of taking general members of a linear system in order to obtain Bézout type results dates back to van der WAERDEN (cf. [Wa, especially p. 769]) A similar method is used in [Br] and plays a particularly central role in Vogel's approach to Bézout's theorem (cf. [G, pp. 199-201] and [S-V]).

[2] The refined Bézout's theorem remains true in this more general setting but does not give the desired inequality for degrees with respect to $\mathcal{O}(d_1,\dots,d_m)$ unless all $d_i = 1$. In particular, if $T_{\mathbb{P}}$ denotes the tangent bundle of $\mathbb{P}$, one needs $T_{\mathbb{P}} \otimes \mathcal{O}(-d_1,\dots,-d_m)$ to be generated by global sections (cf. [Fu, Cor. 12.2]); alternatively, one can use [Fu, Example 12.3.3] together with Example 12.3.7.

and set $I' = (L_1, \ldots, L_t)$. For a subscheme $W \subset \mathbb{P}^N$ we denote by $\deg W$ the degree computed with respect to $\mathcal{O}_{\mathbb{P}^N}(1)$. Since the hypersurfaces $V(L_\alpha)$ are all Cohen-Macaulay, an application of [Fu, Example 12.3.7] to the schemes $X, V(L_1), V(L_2), \ldots, V(L_t)$ yields

$$(1.1.3) \qquad \sum_{\mathcal{O}_{Y'_j,X'}\ \mathrm{CM}} \ell_{Y'_j}\big(X' \cap V(I')\big) \cdot \deg Y'_j \le \deg X' \cdot \prod_{\alpha=1}^{t} \deg V(L_\alpha) = \deg X'.$$

Note that $\ell_{Y'_j}(X', \cap V(I')) = \ell_{Y_j}(X \cap V(I))$ which is clear since $i^* I' = I$. But $\deg i(V) = \deg_d V$ for any $V \subset \mathbb{P}$ and so (1.1.1) follows immediately from (1.1.2) and (1.1.3).

REMARK 1.2. — Using [Fu, Example 12.3.1] (cf. also [La2, Chap. 3, Lemma 3.5]) one can avoid taking generic linear combinations of generators for the ideal J in the proof of THEOREM 1.1. The problem here is that if $P_1, \ldots, P_r$ are an arbitrary set of generators for J then the intersection class $V(P_1) \cdots V(P_r) \cdot X$ (always defined as in [Fu] section 8.1) would be 0 if $r > \dim X$. In order to remedy this situation, one reduces by a Segre embedding to an intersection of hyperplanes $H_1, \ldots H_r$ with a projective variety $X \subset \mathbb{P}^n$. Next choose (cf. [Fu, Example 12.3.1]) a suitably large positive integer N and a linear projection $\pi : \mathbb{P}^N \backslash L \to \mathbb{P}^n$. Then consider the cones $\pi^{-1}(X)$ and $\pi^{-1}(H_i)$ and their closures $\overline{X}$ and $\overline{H}_i$ in $\mathbb{P}^N$. Since

$$\dim \overline{X} \cap \overline{H}_1 \cap \cdots \cap \overline{H}_r \ge \dim X + (N - n - r),$$

when $N \ge n + r - \dim X$ one can apply [Fu, Example 12.3.7], giving

$$(1.2.1) \qquad \sum_{\mathcal{O}_{\overline{Y}_j,\overline{X}}\ \mathrm{CM}} \ell_{\overline{Y}_j}\big(\overline{X} \cap V(\bar{J})\big) \cdot \deg \overline{Y}_j \le \deg \overline{X},$$

where $\bar{J} = \pi^* J$ and $\overline{Y}_j$ are the irreducible components of $V(\bar{J}) \cap \overline{X}$. But if $\{Z_i\}$ are the irreducible components of $V(J) \cap X$ then $\{\overline{Z}_i\} \subset \{Y_j\}$. Note that $\mathcal{O}_{\overline{Z}_i,\overline{X}}$ is Cohen-Macaulay if and only if $\mathcal{O}_{Z_i,X}$ is Cohen-Macaulay and that

$$(1.2.2) \qquad \ell_{\overline{Z}_i}\big(\overline{X} \cap V(\bar{J})\big) = \ell_{Z_i}\big(X \cap V(J)\big) \quad \text{for all } i.$$

Finally $\deg \overline{Z}_i = \deg Z_i$ and (1.1.1) follows[3] directly from (1.2.1) and (1.2.2).

[3] I would like to thank D. BERTRAND for suggesting this alternative method of proof for THEOREM 1.1.

REMARK 1.3. — We show here briefly how THEOREM 1.1 implies PROPOSITION 3.3 of [P1]. PHILIPPON states his results in terms of primary ideals and Hilbert polynomials. Using [Fu, Example 2.5.2], this can be translated into the language of degrees of schemes. We will abuse notation slightly, however, because Philippon considers non-equidimensional schemes $X \subset \mathbb{P}$ and the degree $\deg_d X$ only counts those components of maximal dimension. So if

$$X = \bigcup_{i=0}^{\dim X} X_i$$

where X_i is the union of irreducible components of X of dimension i, then let

$$\deg_d X = \sum_{i=0}^{\dim X} \deg_d X_i.$$

Let $X \subset \mathbb{P}$ be a subscheme and let $U \subset X$ be a Zariski open subset which is Cohen-Macaulay (with the induced scheme structure from X). Let $J = (P_1, \dots, P_r)$ be an ideal generated by multihomogeneous forms P_i of multidegree $\leq d$. Let Y_j be the irreducible components of $X \cap V(J)$. Then the two parts of Proposition 3.3 in [P1] can be expressed as follows :

$$\sum_j \deg_d Y_j \leq \deg_d X_{\mathrm{red}}\,, \tag{1.3.1}$$

$$\sum_{Y_j \cap U \neq \phi} \ell_{Y_j}\bigl(X \cap V(J)\bigr) \cdot \deg_d Y_j \leq \deg_d X. \tag{1.3.2}$$

In (1.3.1), X_{red} denotes the reduced scheme corresponding to X. The only new ingredients in this proposition are that X is no longer assumed to be pure dimensional and that the multihomogeneous forms are only assumed to be of multidegree $\leq d$ and not necessarily $= d$. One quickly reduces, however, to the case where all P_i have multidegree d by multiplying P_i by a set of forms of multidegree $(d - \deg P_i)$ which generate a projectively irrelevant ideal. We first show how to derive (1.3.1) from THEOREM 1.1. Note that a slightly stronger version of inequality (1.1.1) actually holds. In particular, using the trivial lower bound $\bigl(e_{Y_j}(V(J) \cap X)\bigr)_{\mathbb{P}} \geq 1$ for the Samuel intersection multiplicity (cf. [Fu, Example 4.3.4] as well as [Fu, Example 12.2.9] for this particular application) the proof of THEOREM 1.1 shows that

$$\sum_{\mathcal{O}_{Y_j,X} \text{ not CM}} \deg_d Y_j \;+ \sum_{\mathcal{O}_{Y_j,X} \text{ CM}} \ell_{Y_j}\bigl(X \cap V(J)\bigr) \cdot \deg_d Y_j \leq \deg_d X. \tag{1.3.3}$$

Decompose $X_{\text{red}} = \bigcup_{i=0}^{\dim X} X_i$ into pure dimensional components. Then (1.3.1) follows by applying (1.3.3) to X_i and summing over $0 \leq i \leq \dim X$.

Next we derive (1.3.2) from THEOREM 1.1. If X is pure dimensional then (1.3.2) is an immediate consequence of (1.1.1). In the general case, simply note that if $X_{\text{red}} = \bigcup_{i=0}^{\dim X} X_i$ as before, then all points of $X_i \cap X_j$ for $i \neq j$ are *not* Cohen-Macaulay. But then THEOREM 1.1 applies to X_i (with some scheme structure restricting to the induced scheme structure on $U \cap X_i$) for each $0 \leq i \leq \dim X$. Since each Y_j in the sum of (1.3.2) is contained in a unique X_i, (1.3.2) follows by summing these inequalities.

Note that THEOREM 1.1 is slightly stronger than PROPOSITION 3.3 of Philippon since it gives some minimal information on those components Y_j where $\mathcal{O}_{Y_j,X}$ is not Cohen-Macaulay. BROWNAWELL [Br] gave a simplified proof of Philippon's result (1.3.2) several years ago using only the classical version of Bézout's theorem, while A. HIRSCHOWITZ did the same for (1.3.1) (see [B, Prop. 3]). One can also find (1.3.1) in [Fu, Example 8.4.5] and [V, Cor. 2.26]. Finally, both (1.3.1) and (1.3.2) are proven in [G, Example 1.7] using ideas of BROWNAWELL, FULTON, and VOGEL.

Proposition 2.3 of [F] is closely related both to THEOREM 1.1 and to the corresponding result of Philippon. We state it here for convenience of the reader.

COROLLARY 1.4. (FALTINGS). — *Let J be as in Theorem* 1.1 *and let Z_j be the irreducible components of $V(J)$ of codimension t in $\mathbb{P}$. Let $\mathcal{L}_i = \pi_i^*\mathcal{O}(1)$ where $\pi_i : \mathbb{P} \to \mathbb{P}^{n_i}$ is the projection to the i^{th} factor. For any m-tuple $e = (e_1, \ldots, e_m)$ with $\sum e_i = t$, let V_e denote a generic cycle representing $c_1(\mathcal{L}_1)^{e_1} \cap \ldots \cap c_1(\mathcal{L}_m)^{e_m}$. Then*

$$\sum_{Z_j} \ell_{Z_j}\big(V(J)\big) \cdot \deg(Z_j \cdot V_e) \leq \deg_d V_e.$$

This follows directly from THEOREM 1.1 taking $X = V_e$ (cf. [Fu, Example 8.4.8] for why $\ell_{Z_j}(V(J)) = \ell_Q(V_e \cap V(J))$ where $Q \in Z_j \cap V_e$). To give a proof of the Proposition in the language of [Fu] without using the refined Bézout theorem, one first reduces via a Segre embedding to intersecting a smooth projective variety $V \subset \mathbb{P}^N$ of dimension r with a set of r hyperplanes $H_1, \ldots, H_r$ of degree d. If $\{Q_j\}$ are the isolated points of the intersection $H_1 \cap \cdots \cap H_r \cap V$ then, using [Fu, Def. 7.1] for the intersection multiplicity $i(Q_j, H_1 \cdots H_r \cdot V\,;\,\mathbb{P}^N)$, COROLLARY 1.4 reduces to

$$\sum_{Z_j} i\big(Q_j, H_1 \cdots H_r \cdot V; \mathbb{P}^N\big) \leq \deg V.$$

This follows from Bézout's theorem [Fu, Thm 8.4] since the points $\{Q_j\}$ are proper components of the intersection. Alternatively, one can proceed inductively as in Faltings; at each step, throw away all components which will have excess dimension at the following step and then use [Fu, Example 7.1.8 and Example 7.1.10] to compute the intersection multiplicities. Using the same type of argument, one easily deduces that if X and J are as in THEOREM 1.1 and Z_j are the irreducible components of the intersection $X \cap V(J)$ of codimension t then

$$\sum_{\mathcal{O}_{Z_j,X}\ \mathrm{CM}} \ell_{Z_j}\big(X \cap V(J)\big) \cdot \deg_d Z_j \leq \deg_d X. \tag{1.4.1}$$

It should be emphasized that (1.4.1) is enough for the application to zero estimates since this only requires information about components of the *same* dimension.

We need a couple more results from intersection theory more closely related to the specific set up of zero estimates on group varieties. Consider the following situation : let G be an arbitrary connected algebraic group. Suppose G acts on a projective variety X, i.e. there is a morphism $\phi : G \times X \to X$ satisfying

(1) $\phi(e,x) = x$ for all $x \in X$ where e is the identity element of G,

(2) $\phi\big[g, \phi(g',x)\big] = \phi(g \cdot g', x)$ for all $g, g' \in G$ and all $x \in X$.

We will normally write $\phi(g,x) = g(x)$.

We begin with a transversality result which is behind Lemme 4.6 in [P1] and which is essential in order to complete the proof of Proposition 1 in [W2] (see REMARK 3.9 below). It is closely related to a result of KLEIMAN [K] (cf. [Fu, Appendix B 9.2] and [H, III, Thm 10.8]). If $X, Y \subset W$ are two subvarieties of an algebraic variety and $\{Z_i\}$ are the irreducible components of intersection, let $A = \mathcal{O}_{Z_i,Y}$ and let J be the ideal in A generated by the ideal of X in W. We say that X and Y intersect generically transversally if J is the maximal ideal of A.

THEOREM 1.5. — *Suppose G is a connected algebraic group acting on a quasi-projective variety X, with both defined over an algebraically closed field of characteristic zero. Let $V \subset X$ be an irreducible subvariety. Then there exists a non-empty Zariski open subset $U \subset V$ such that $Gu \cap V$ is generically transverse for all $u \in U$.*

Proof. — We copy the proof from [H, III, Thm 10.8] with a few modifications necessary for this setting. Let $U \subset V$ be a non-empty Zariski open subset which is non-singular with the induced scheme structure. Define $Y := (\phi(G \times U))_{\mathrm{red}}$ and consider the dominant morphism $\phi : G \times U \to Y$. Shrinking U if necessary we can assume that ϕ is smooth;

this follows as in Hartshorne by generic smoothness [H, III, Cor. 10.7] or by the fact that G acts transitively on the fibres $\phi^{-1}(y)$. Construct the fibred square as in [H] :

$$\begin{tikzcd} W \arrow[r, "\phi'"] \arrow[d, "j"'] & U \arrow[d, "i"] \\ G \times U \arrow[r, "\phi"] \arrow[d, "\pi_2"'] & Y \\ U & \end{tikzcd}$$

Here $\pi_2 : G \times U \to U$ is the projection to the second factor and $i : U \to Y$ the natural inclusion. Let $q : W \to U$ denote the composition $\pi_2 \cdot j$. Since W is non-singular, applying generic smoothness to q shows that, shrinking U if necessary, q is smooth and hence $q^{-1}(u)$ is smooth for $u \in U$. But it is easy to verify that this is possible only when $Gu \cap U$, and hence, $Gu \cap V$ is generically transverse.

REMARK 1.6. — We will need to apply THEOREM 1.5 in the case where $A \subset G(\mathbb{C})$ is an analytic subgroup of a connected group variety G, defined by a coherent analytic ideal sheaf $\mathcal{I}$. This can be done essentially as above. Let $p : W \to G$ denote the composition $\pi_1 \cdot j$. Then replace W in the above commutative diagram by the complex analytic subspace $\mathcal{C}$ defined by the inverse image ideal sheaf Im : $p^*\mathcal{I} \to \mathcal{O}_W$ (cf. [G-R, p. 19]). We know that $\mathcal{C}$ is reduced because p is smooth. By [G-R, p. 117] there exists a non-empty (analytic) open subset $\mathcal{U} \subset \mathcal{C}$ which is a complex manifold (with the induced structure of complex space). Instead of generic smoothness, one applies its analytic analogue, Sard's theorem ([G-G, Thm 1.12]), to $q : \mathcal{U} \to U$.

COROLLARY 1.7. — *Suppose G acts on X and $V \subset X$ is an irreducible subscheme with*

$$[V] = m[V_{\mathrm{red}}].$$

Let Z be an irreducible component of $Gv \cap V$ for generic $v \in V$. Then $m = \ell_Z(Gv \cap V)$.

Proof. — Consider $Y := \big(\phi(G,V)\big)_{\mathrm{red}} \subset X$. We claim that for generic $v \in V$ the intersection $Gv \cap V$ is proper in Y. This follows either from the proof of THEOREM 1.5 or by considering $\phi : G \times V \to Y$ since the fibre over $g(v)$ has dimension

$$\dim Gv \cap V + \dim G - \dim Gv.$$

If $Y_{\mathrm{reg}} \subset Y$ denotes the set of regular points, then for generic choice of $v \in V$, the intersection $Gv \cap Y_{\mathrm{reg}}$ is not empty. Moreover $V \cap Y_{\mathrm{reg}}$ is not empty since for any smooth point $g(v) \in Y$, $v = g^{-1}g(v)$ is also a smooth point of Y. Also no irreducible component of the intersection $Gv \cap V$ will be contained in $Y_{\mathrm{sing}} = Y \setminus Y_{\mathrm{reg}}$. Since intersection products are local ([Fu, Example 6.2.5] and see p. 137 for a discussion closely related to this particular application) it will suffice to work on Y_{reg}. We will continue to denote by V (resp. Gv) the intersection $V \cap Y_{\mathrm{reg}}$ (resp. $Gv \cap Y_{\mathrm{reg}}$). Since Gv meets V generically transversally by THEOREM 1.5, and since intersection products commute with the cycle map ([Fu, Example 6.2.1]), it follows that $i(Z, Gv \cdot V; Y_{\mathrm{reg}}) = m$. Thus it remains to show that

$$i\big(Z, Gv \cdot V; Y_{\mathrm{reg}}\big) = \ell_Z(Gv \cap V).$$

This follows from [Fu], Proposition 7.1 or Proposition 8.2 because V (in fact any irreducible projective scheme) is generically Cohen-Macaulay and Gv is regular. The fact that a projective scheme always has an open subset of Cohen-Macaulay points is perhaps most convincingly seen by intersecting with generic hypersurfaces $\{V(f_i)\}$; at each stage f_i will form part of a system of regular paramaters away from the closed subscheme of embedded components.

Finally we need an easy result on $\deg g(V)$ for $V \subset X$. This is given in [Mo, Lemme 2] and a weaker version appears as Lemma 4.5 in [P1]. We give a proof here as another application of intersection theory.

LEMMA 1.8. — *Suppose a connected algebraic group G acts on a projective variety X. Then for all $g \in G$ and all $V \subset X$*

$$\deg V = \deg g(V)$$

where the degree is computed with respect to any ample line bundle on X.

Proof. — In the special case when G is affine, the cycles $[V]$ and $[g(V)]$ are rationally equivalent and hence numerically equivalent. This follows from [Fu, Example 10.1.7] since G is rationally connected (and in fact rational). In the general case when G is no longer necessarily affine, the cycles $[g(V)]$ and $[V]$ are algebraically equivalent and hence numerically equivalent. The algebraic equivalence of $[g(V)]$ and $[V]$ follows from [Fu, Def. 10.3] by considering $\phi : X \times G \to X$ given by $\phi(x, g) = g(x)$: consider the cycle

$$\big[\phi^{-1}(V)\big] \in A_k(X \times G), \quad k = \dim V + \dim G$$

and observe that $[\phi^{-1}(V)]_g = [g^{-1}(V)]$.

In the proofs of zero estimates and of the product theorem we will need to assume that a (normal) projective variety X with group action by G is given a fixed projectively normal embedding. KNOP and LANGE [KL1], [KL2] have given an effective construction of such embeddings in certain cases. We here sketch an alternative more abstract method which uses results of BERTRAM, EIN, and LAZARSFELD [BEL], [EL].

The assumption of projectively normality is required for the following reason. Given an embedding $X \hookrightarrow \mathbb{P}^N$ let $\mathcal{O}_X(1)$ be the associated very ample line bundle on X. Both the multiplicity estimates and the product theorem are based on taking «derivatives» of sections of $\mathcal{O}_X(n)$. As we will see in the next section this means associating to each $\sigma \in H^0(\mathcal{O}_X(n))$ a family $\mathcal{F}_\sigma \subset H^0(\mathcal{O}_X(cn))$ for some positive integer c. In order to apply THEOREM 1.1 the map $H^0(\mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(\mathcal{O}_X(n))$ must to be surjective so that $\mathcal{F}_\sigma$ is a set of hyperpsurface sections in the given projective embedding. Now if $X \subset \mathbb{P}^N$ is any subscheme (not necessarily normal) then the ideal sheaf $\mathcal{I}_X$ is coherent [H, II, 5.9] and one has the standard exact sequence

$$0 \to \mathcal{I}_X(n) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(n) \longrightarrow \mathcal{O}_X(n) \to 0,$$

for any integer n. But it is well known [H, III 5.2 (b)] that there exists n_0 such that $H^1(\mathcal{I}_X(n)) = 0$ for all $n \geq n_0$, and hence

$$H^0\big(\mathcal{O}_{\mathbb{P}^N}(n)\big) \longrightarrow H^0\big(\mathcal{O}_X(n)\big) \to 0$$

is exact for $n \geq n_0$. The trouble with this argument is that n_0 is not easy to compute in general; one encounters the same problem when asking for an embedding to be projectively normal. Thus for example LANGE [L, p. 262] assumes that his equivariant compactification is furnished with a projectively normal embedding. Without this assumption, his effective bound on the degree of homogeneous polynomials representing translations is not valid. But the analysis of Knop and Lange in [KL1] (especially Thm 6.4) makes the choice of n_0 effective for the projective embeddings used in transcendence theory.

Recent work of BERTRAM, EIN, and LAZARSFELD [BEL] gives, under suitable hypotheses, effective bounds on the size of n_0. More precisely, suppose X is a smooth complex projective variety of dimension d and let K_X denote the canonical bundle on X. If D is a very ample line bundle and E is a numerically effective line bundle (recall that a Cartier divisor D on a variety X is said to be *numerically effective* or *nef* if the intersection number satisfies $D \cdot C \geq 0$ for all integral curves $C \subset X$), then the adjoint bundle $K_X \otimes D^{\otimes k} \otimes E$ defines a projectively normal embedding for $k \geq d+1$ provided that it is very ample ([BEL, Prop. 6]; see also [EL, Thm 1]). This result does not always apply directly in our situation, however, because X is not necessarily smooth or even normal. Under certain hypotheses on A and X one can lift to a desingularization $\widetilde{X}$ and use Proposition 6 of [BEL] in order to obtain projective normality of X embedded by $K_X \otimes D^{\otimes k} \otimes E$. To this end, it would be nice to have a version of this result which only requires D to be ample but as of yet this is only a conjecture (cf. [EL, Conjecture 4.2]).

Since the equivariant compactifications considered in transcendence theory (cf. [KL1, section 6]) are smooth one can use the results of [BEL]. Let X be a compactification of a connected commutative algebraic group as constructed in section 6 of [KL1] and let D be a very ample Cartier divisor on X. We claim that $-K_X$ is generated by global sections and hence is *nef.* Taking $E = -K_X$ in Proposition 6 of [BEL] shows that $D^{\otimes k}$ gives a projective embedding for any $k \geq \dim X + 1$. To see that $-K_X$ is *nef*, consider the construction of X in [KL1]. Let G be a connected commutative group variety over $\mathbb{C}$. There is a canonical exact sequence

$$0 \to L \longrightarrow G \longrightarrow A \to 0,$$

where L is a linear group and A is an abelian variety. There is a (non-canonical) splitting

$$L \cong \prod_1^r G_a \times \prod_1^s G_m,$$

where G_a is the additive group and G_m is the multiplicative group. Knop and Lange consider compactifications of G as an L-bundle over A. To each equivariant compactification V of L they associate an equivariant compactification of X. In transcendence theory, one is interested in compactifications of the form $\prod \mathbb{P}^{n_i}$ where $\sum n_i = r + s$ and where the immersion $L \hookrightarrow \prod \mathbb{P}^{n_i}$ respects the given product structure on L. Fix such an immersion and let X denote the corresponding equivariant compactification of G. Let

$$\pi : X \longrightarrow A$$

denote the canonical projection. Each invertible sheaf $\mathcal{O}_{\mathbb{P}^{n_i}}(1)$ gives rise to an invertible sheaf $\mathcal{L}_i$ on X. A theorem of Knop and Lange [KL2, Thm 2.1] says that

$$\mathrm{Pic}_L(X) \cong \pi^*\mathrm{Pic}A \otimes \bigoplus_i \mathbb{Z}$$

where $\mathrm{Pic}_L(X)$ denotes those invertible sheaves on X which admit an L-action and where each copy of $\mathbb{Z}$ in the direct sum corresponds to the sheaf $\mathcal{L}_i$. Since K_X is admits an L-action, $-K_X = \pi^*\mathcal{L} + \mathcal{M}$ where $\mathcal{M} \cong \sum a_i\mathcal{L}_i$. Intersecting with curves C in the fibres of π shows that $\mathcal{M}$ is generated by global sections. On the other hand, [Fu, Example 12.2.1] shows that $-K_X|_G$ is generated by global sections which forces $\mathcal{L}$ to be generated by global sections.

Using this same argument and Theorem 1 of [EL] shows that if D is a very ample divisor on X, then the embedding given by $(\dim X + 2)D$ is cut out by quadrics. This is related to [KL1, Thm 6.4]. The result of Knop and Lange, however, is sharper applied to the specific line bundles which they consider; in particular, they do not need to assume that D is very ample. On the other hand, the approach sketched here gives a potentially more general result applying to *any* very ample line bundle on X, including those which may not admit an L-action.

2. Length estimates

We need to estimate lengths of primary ideals in terms of differential operators as in [W2, Lemma 3] and [P1, Prop. 4.7]. Multiplicity estimates occur in a similar fashion in Faltings' product theorem. As before, X denotes a projective variety with group action by a connected algebraic group G given by $\phi : G \times X \to X$. From this point on, we will assume for simplicity that all varieties and morphisms are defined over $\mathbb{C}$. Assume also that X is normal and fix a projective embedding $X \xrightarrow{i} \mathbb{P}^N$ in which X is projectively normal. Let R denote the projective coordinate ring of $\mathbb{P}^N$ and let $\mathcal{L} = i^*\mathcal{O}(1)$. Fix a very ample line bundle $\mathcal{M}$ on G. We will define translation operators and differential operators and verify that they satisfy certain basic properties (cf. [P1, section 4.1], [MW1, sections 2–3], and [MW2, sections 2–3] for similar considerations). In the applications, X will be a product of varieties with group actions; it is not difficult to extend all of the definitions and lemmas of this section to the multihomogeneous case.

Translation operators can be defined in a completely natural fashion in the context of schemes. In particular if $V \subset X$ is a subscheme then for $g \in G$ recall that $t_g(V) \subset X$ denotes the scheme theoretic image. If $V = V(I)$ then we will often write $t_g(I) \subset R$ for a homogeneous ideal defining $t_g(V)$. Theorem 1.1 requires an explicit estimate on the degree of polynomials generating a particular choice of $t_g(I)$. To obtain this estimate, choose a positive integer a such that $\mathcal{L}^{\otimes a} \otimes t^*_{g^{-1}}\mathcal{L}^{\otimes -1}$ is generated by global sections; there is no difficulty choosing a uniformly in g (cf. the main theorem of [L]) as we will show later when defining differential operators. Choose a basis $\{\gamma_i\}$ for $H^0(X, \mathcal{L}^{\otimes a} \otimes t^*_{g^{-1}}\mathcal{L}^{\otimes -1})$. Then there are morphisms

$$\mathcal{O}_X(1) \xrightarrow{t_{g,i}} \mathcal{O}_X(a)$$

given by $\sigma \mapsto t^*_{g^{-1}}\sigma \otimes \gamma_i$. One can extend $t_{g,i}$ to a map

$$H^0\big(\mathcal{O}_X(n)\big) \xrightarrow{t_{g,i}} H^0\big(\mathcal{O}_X(an)\big)$$

in the natural manner respecting the ring structure on

$$R_X = \bigoplus_{i=0}^{\infty} H^0\big(\mathcal{O}_X(i)\big).$$

If $X = V(I)$ and $P_1, \ldots, P_r$ are homogeneous generators of I of degree $\leq n$ then $\{t_{g,i}(P_j)\}$ are homogeneous generators of $t_g(I)$ of degree $\leq an$. Since homogeneous ideals defining projective varieties will occur frequently in what follows and since these are only determined up to a projectively irrelevant ideal, we make the convention that for homogeneous ideals $I, J \subset R$, $I \simeq J$ if and only if $V(I) \simeq V(J)$.

REMARK 2.1. — Suppose $X \subset \mathbb{P}^N$ is defined by a homogeneous ideal $I \subset R$. Then PHILIPPON [P1, Def. 4.1 and 4.2] defines $t_g(I)$ to be the ideal in R_X generated by $t_{g,i}(I)$ for all i. Our more geometric formulation of Philippon's definition makes it clear that $t_g(I)$ is independent of the choice of a up to a projectively irrelevant ideal and it is also independent of the choice of basis for $H^0(X, \mathcal{L}^{\otimes a} \otimes t^*_{g^{-1}}\mathcal{L}^{\otimes -1})$. It is also clear that $t_{g'}[t_g(I)] \simeq t_{g'\cdot g}(I)$. MASSER-WÜSTHOLZ take a slightly different approach using «contracted extensions» of ideals (cf. [MW1, p. 492–498] and [MW2, p. 237–240] for the details).

Next we define differential operators. This is done similarly to translations except that we work on $G \times X$. All sheaves and cohomology groups will be on $G \times X$ until specified otherwise. Choose positive integers b, c such that

$$\mathcal{F}_{b,c} := \pi_1^*\mathcal{M}^{\otimes b} \otimes \pi_2^*\mathcal{L}^{\otimes c} \otimes \phi^*\mathcal{L}^{\otimes -1} \tag{2.2}$$

is generated by global sections[4] where π_1 and π_2 denote the projections of $G \times X$ to the first and second factor respectively. Choose global

[4] This shows, by restricting $\mathcal{F}_{b,c\,|\,g\times X}$, that in REMARK 2.1 above a can be chosen uniformly for all translation operators t_g. However, in general there is *not* an explicit effective bound on a as is given by [L] in the special case where X is an equivariant compactification of a commutative group variety and $\phi : G\times X \to X$ is the group action.

sections $\{\gamma_i\}$ which generate $\mathcal{F}_{b,c}$; note that $H^0(\mathcal{F}_{b,c})$ may be infinite dimensional since G is only quasi-projective. Precisely as above, define

$$\eta_i : H^0\big(\mathcal{O}_X(n)\big) \longrightarrow H^0\big(\phi^*\mathcal{O}_X(n) \otimes \mathcal{F}_{b,c}^{\otimes n}\big) = H^0\big(\pi_1^*\mathcal{M}^{\otimes nb} \otimes \pi_2^*\mathcal{L}^{\otimes nc}\big). \tag{2.3}$$

Let $A \subset G$ be a d-parameter analytic subgroup given by an analytic homomorphism

$$\psi : \mathbb{C}^d \longrightarrow G.$$

Let $z_1, \dots, z_d$ denote the standard coordinate system on $\mathbb{C}^d$. Let $B(0,\epsilon)$ denote the ball of radius ϵ centered at the origin in $\mathbb{C}^d$ and let $\mathcal{H}(B(0,\epsilon))$ denote the ring of holomorphic functions on $B(0,\epsilon)$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is a d-tuple of non-negative integers, then let

$$\Delta^\alpha = \frac{\partial^{|\alpha|}}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial z_d^{\alpha_d}}.$$

DEFINITION 2.4. — Suppose $\sigma \in H^0(\pi_1^*\mathcal{M}^{\otimes\beta} \otimes \pi_2^*\mathcal{L}^{\otimes\gamma})$ for $\beta, \gamma > 0$. Take a local trivialization of $\pi_1^*\mathcal{M}^{\otimes\beta}$ along $U \times X$ where $U \subset G$ is some small affine open subset containing e. If $U = \operatorname{Spec} B$ then $\sigma_U \in B \otimes_{\mathbb{C}} H^0(\pi_2^*\mathcal{O}_X(\gamma))$ where σ_U is the image of σ under the trivialization [L, Lemma 1]. Since $e \in U$, we can further restrict σ_U to $\psi(B(0,\epsilon)) \times X \subset U \times X$. Composing with ψ on the first factor gives $\sigma_U(z) \in \mathcal{H}(B(0,\epsilon)) \otimes H^0(\pi_2^*\mathcal{O}_X(\gamma))$. Then

$$D^\alpha\sigma := \Delta^\alpha\big[\sigma_U(z)\big]_{|z=0} \in H^0\big(\mathcal{O}_X(\gamma)\big).$$

It is easy to check that (up to multiplication by a non-zero constant) $D^\alpha\sigma$ does not depend on the choice of U or on the local trivialization of $\pi_1^*\mathcal{M}^{\otimes\beta}$. For future reference, write $D_i(\sigma) = \partial/\partial z_i[\sigma_U(z)]_{|z=0}$ so that

$$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}.$$

DEFINITION 2.5. — Suppose $P \in R$ is a homogeneous polynomial of degree n considered as a global section of $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(n)$. Then

$$D^\alpha P := \big(\{D^\alpha[\eta_i(P)]\}\big)$$

where η_i run over all homomorphisms defined in (2.3).

DEFINITION 2.6. — Let $I \subset R_X$ be a homogeneous ideal and D^α a differential operator. Then write

$$D^\alpha(I) = \big(\{D^\beta P\,;\ P \in I \text{ and } 0 \le \beta \le \alpha\}\big),$$

where $\beta = (\beta_1, \dots, \beta_d) \le (\alpha_1, \dots, \alpha_d) = \alpha$ if and only if $\beta_i \le \alpha_i$ for all i.

REMARK 2.7. — One can verify, using the Leibnitz formula, that if $I = (P_1, \ldots, P_r)$ for homogeneous polynomials P_i then

$$D^\alpha(I) \simeq \left(\left\{D^\beta P_i\,;\ i = 1, \ldots r,\ 0 \leq \beta \leq \alpha\right\}\right).$$

We now proceed to estimate the length of primary ideals in terms of differential operators. This requires a constant which is closely related to the constants σ_r and ρ_r of WÜSTHOLZ [W2, p. 473–474 and p. 479]. Suppose $V \subset X$ is a subvariety. Let

(2.8) $\qquad c_V := \operatorname{codim}[Av \cap V, Av] \quad$ where $v \in V$ is a general point.

Also given a subset $S \subset \{\partial/\partial z_1, \ldots, \partial/\partial z_d\}$ let

$$\mathcal{D}_S^T = \left\{D^\alpha\,;\ |\alpha| \leq T \text{ and } \alpha_i \neq 0 \Rightarrow \frac{\partial}{\partial z_i} \in S\right\}.$$

Thus $\mathcal{D}_S^T$ is just the differential operators of weight $\leq T$ in S. Without a suffix

$$\mathcal{D}^T = \left\{D^\alpha\,;\ |\alpha| \leq T\right\}.$$

If $I \subset R_X$ is a homogeneous ideal then let

$$\mathcal{D}_S^T(I) = \left(\{D^\alpha(I)\}\,;\ D^\alpha \in \mathcal{D}_S^T\right).$$

For $v \in V$, define the analytic map $\psi_v : \mathbb{C}^d \to X$ by $z \mapsto \psi(z) \cdot v$. Also define the complex analytic space

$$Y_v := \psi_v^{-1}(V) \cap B(0, \epsilon).$$

The following result is implicitly stated by FALTINGS in the proof of the Product Theorem (see [F, Thm 3.1]) for the special case when $G = \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$, $X = \mathbb{P}$, and $A = \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$; it generalizes Lemma 3 of [W2].

THEOREM 2.9. — *Suppose $\mathcal{P}_V \subset R$ is a homogeneous prime ideal defining a subvariety V and $I \subset R$ is $\mathcal{P}_V$-primary. For generic $v \in V$ let $\{\partial/\partial z_i\}_{i \in S}$ be a basis of derivations transversal to Y_v at 0 so that $|S| = c_V$. Let δ be the cardinality of the set C of transversal differential operators which map I into $\mathcal{P}_V$:*

$$C = \left\{\alpha\,;\ \alpha_i \neq 0 \Rightarrow \partial/\partial z_i \in S \text{ and } D^\alpha(I) \subset \mathcal{P}_V\right\}$$

Then $\ell_V[V(I)] \geq \delta + 1$.

Proof. — Choose a small affine open subset $U \subset X$ so that $U \cap V$ is a smooth complete intersection in U. Suppose $\operatorname{codim}(V,X) = r$ and $Q_1, \ldots, Q_r \in \mathcal{P}_V$ are homogeneous polynomials which generate $I(V \cap U)$. Order the coordinates of $\mathbb{C}^d$ so that $S = \{1, \ldots, c_V\}$; note that $c_V \leq r$. Let δ_{ij} denote the Kronecker delta. By THEOREM 1.5 and REMARK 1.6, there exists $v \in V \cap U$ such that $Av \cap V$ is transverse at v. This means that for ϵ sufficiently small $\psi_v^*(\mathcal{P}_V)$ generates $i(Y_v)$ where $i(Y_v)$ denotes the ideal sheaf of Y_v in $\mathcal{O}_{B(0,\epsilon)}$ (cf. [G-R, p. 77]). Consequently, we can assume (taking linear combinations of the Q_i if necessary) that for some fixed dehomogenization

$$\frac{\partial}{\partial z_j} Q_i\big[\psi_v(z)\big]_{|z=0} = \delta_{ij}, \quad \text{for } 1 \leq i,j \leq c_V. \tag{2.9.1}$$

Given $\alpha \in C$ let $Q^\alpha = \prod_{i=1}^{c_V} Q_i^{\alpha_i}$. We claim that

$$P = \sum_{\alpha \in C} a_\alpha Q^\alpha \notin I \quad \text{unless all } a_\alpha = 0. \tag{2.9.2}$$

Since I is homogeneous, it suffices to verify (2.9.2) in the case when P is homogeneous of degree κ. Suppose $a_\beta \neq 0$. Then it follows from (2.9.1) that

$$\Delta^\beta P[\psi_v(z)]_{|z=0} \neq 0.$$

DEFINITION 2.6 then implies that $D^\beta P \not\subset \mathcal{P}_V$ and thus (2.9.2) holds. Denoting by «ˆ» the completion of a local ring,

$$\widehat{\mathcal{O}}_{V,X} \simeq k(V)[[Q_1, \ldots, Q_r]].$$

But [Ma, p. 63 Remark 2 and Thm 8.11]

$$\mathcal{O}_{V,X}/I \simeq \widehat{\big(\mathcal{O}_{V,X}/I\big)} \simeq \widehat{\mathcal{O}}_{V,X}/I\widehat{\mathcal{O}}_{V,X}, \tag{2.9.3}$$

and by [Ma, p. 63 (3)], $\sum_{\alpha \in C} a_\alpha Q^\alpha \notin I\widehat{\mathcal{O}}_{V,X}$ unless all $a_\alpha = 0$. It is then an easy exercise to check, using (2.9.2) and (2.9.3), that $\ell(\widehat{\mathcal{O}}_{V,X}/I\widehat{\mathcal{O}}_{V,X}) \geq \delta + 1$ and this concludes the proof of THEOREM 2.9.

COROLLARY 2.10. — *Suppose $\mathcal{P}_V \subset R$ is a homogeneous prime ideal defining $V \subset \mathbb{P}^N$ and suppose I is $\mathcal{P}_V$-primary. If $\mathcal{D}^T I \subset \mathcal{P}_V$ then*

$$\ell_V\big[V(I)\big] \geq \binom{c_V + T}{c_V}.$$

This is a trivial consequence of THEOREM 2.9; it follows simply from counting the number of differential operators in $\mathcal{D}_S^T$, where S is as in the statement of THEOREM 2.9.

REMARK 2.11. — It is essential in some of the applications to have THEOREM 2.9 instead of COROLLARY 2.10. In diophantine approximation one is interested in polynomials with large «index» at a given point. But the index is a weighted version of the order and so it is not sufficient to have a multiplicity estimate which only considers the order of a function at a point. This will be clear in the application to the product theorem.

3. Lemmas about Differential Operators

In this section we gather together lemmas about the functorial behavior of differential operators which will be needed in the next section. The end of the section discusses how our differential operators are related to those defined by WÜSTHOLZ [W2] and how THEOREM 1.5 relates to Proposition 1 in [W2].

LEMMA 3.1. — *Let $I \subset R_X$ be a homogeneous ideal and D^α a differential operator. Then $D^\alpha(I)$ is independent of the choice of b and c in* (2.2) *and of the choice $\{\gamma_i\}$ of global sections generating $\mathcal{F}_{b,c}$.*

This can be seen as in [P1, Prop. 4.3] by using the derivation properties of D^α. We briefly sketch a proof in our set-up. First, *assuming* independence of the choice of $\{\gamma_i\}$, we show that $D^\alpha(I) \simeq (D^\alpha)'(I)$ where $(D^\alpha)'(I)$ is defined via $\{\gamma_i'\} \subset H^0(G \times X, \mathcal{F}_{b',c'})$ with $b' > b$ and $c' > c$. Choose $\gamma_i' = \gamma_j \otimes \delta_k$ where $\{\gamma_j\}$ is the choice of generating sections for $\mathcal{F}_{b,c}$ and $\{\delta_k\}$ is a set of sections which generate

$$\pi_1^* \mathcal{M}^{\otimes b'-b} \otimes \pi_2^* \mathcal{L}^{\otimes c'-c} \cong \mathcal{F}_{b',c'} \otimes \mathcal{F}_{b,c}^{\otimes -1}.$$

Let $s \in H^0(\mathcal{O}_X(1))$ and let $\sigma_i' = \phi^* s \otimes \gamma_i'$ and $\sigma_j = \phi^* s \otimes \gamma_j$. For all $\beta \leq \alpha$,

$$(3.1.1)\quad D^\beta \sigma_i' = D^\beta(\sigma_j \otimes \delta_k) = D^\beta \sigma_j \otimes (\delta_{k|e\times X}) + \cdots + \sigma_{j|e\times X} \otimes D^\beta \delta_k,$$

where the intermediate terms are given by the Leibnitz formula and are of the form

$$\left[(D^{\beta_1}\sigma_j) \otimes (D^{\beta_2}\delta_k)\right], \quad \beta_1 + \beta_2 = \beta.$$

But as k varies, $\delta_{k|e\times X}$ generate a projectively irrelevant ideal and the rest of the terms in (3.1.1) are in $D^\alpha(I)$. This verifies independence of b, c for differential operators applied to generators of $\mathcal{O}_X(1)$. The general case follows from this since the operators D^α are determined by their values on $H^0(\mathcal{O}_X(1))$ by the Leibnitz formula.

Next we show that $D^\alpha(I)$ is independent of the choice $\{\gamma_i\}$ of global sections generating $\mathcal{F}_{b,c}$. Let $\gamma \in H^0(\mathcal{F}_{b,c})$ be an arbitrary global section.

Then it suffices to show that adding γ to the set $\{\gamma_i\}$ does not change the scheme defined by $D^\alpha(I)$. This is a local condition which we verify on an open affine cover of X. Since $\{\gamma_i\}$ generate $\mathcal{F}_{b,c}$ the restrictions $\{\gamma_{i|e\times X}\}$ generate $\mathcal{F}_{b,c|e\times X} = \mathcal{O}_X(c-1)$. So if $U_i \subset X$ is the complement $X \setminus Z(\gamma_{i|e\times X})$ of the zero locus of $\gamma_{i|e\times X}$ then $\bigcup U_i = X$. It is clear that if ϵ is sufficiently small then for all i, γ_i does not vanish on $B(0,\epsilon)\times U_i$. In particular the rational function γ/γ_i has no poles along $B(0,\epsilon)\times U_i$. Now choose a global section $s \in H^0(\mathcal{O}_X(1))$. Then for all $\beta \leq \alpha$,

$$\begin{aligned} D^\beta(\phi^* s\otimes\gamma)_{|0\times U_i} &= D^\beta\Big(\phi^* s\otimes\gamma_i\cdot\frac{\gamma}{\gamma_i}\Big)_{|0\times U_i} \\ &= \frac{\gamma}{\gamma_i}\big[D^\beta(\phi^* s\otimes\gamma_i)\big]_{|0\times U_i} \\ &\qquad + \cdots + (\phi^* s\otimes\gamma_i)\Big(D^\beta\Big(\frac{\gamma}{\gamma_i}\Big)\Big)_{|0\times U_i}, \end{aligned}$$

where the intermediate terms are, as above, of the form

$$\Big[\Big(D^{\beta_1}\Big(\frac{\gamma}{\gamma_i}\Big)\Big)\cdot\big(D^{\beta_2}(\phi^* s\otimes\gamma_i)\big)\Big]_{|0\times U_i}, \quad \beta_1+\beta_2=\beta.$$

Since $D^{\beta_1}(\gamma/\gamma_i)_{|0\times U_i}$ is computed on $B(0,\epsilon)\times U_i$ and since γ/γ_i has no poles on $B(0,\epsilon)\times U_i$ the result follows for sections of $\mathcal{O}_X(1)$. The argument for sections of $\mathcal{O}_X(n)$ is precisely the same except that now $\gamma \in H^0(\mathcal{F}_{b,c}^{\otimes n})$ and γ_i is replaced by $\bigotimes_{j=1}^n \gamma_j$ where $\gamma_j \in H^0(\mathcal{F}_{b,c})$.

REMARK 3.2. — LEMMA 3.1 can be rephrased in the terminology of MASSER-WÜSTHOLZ [MW1, Lemma 1] and PHILIPPON [P1, 4.1]. One wants to represent the morphism $\phi : G\times X \to X$ in terms of homogeneous polynomials in a fixed set of coordinates on X and G. A collection $\{P_i\}$ of homogeneous polynomials representing ϕ is called *complete* if for any $g\times x \in G\times X$ there exists P_i representing ϕ in a neighborhood of $g\times x$. The homogeneous polynomial $\phi^* P\otimes\gamma$ only represents ϕ off of $Z(\gamma)$. Thus if $P_i = \phi^* P\otimes\gamma_i$ then $\{P_i\}$ is a complete collection if and only if $\{\gamma_i\}$ generate $\mathcal{F}_{b,c}$. LEMMA 3.1 shows that our definition of derivatives does not depend on the choice of a complete collection of homogeneous polynomials representing ϕ.

Now we must show that the differential operators and translation operators satisfy certain compatibility conditions as in [P1, Prop. 4.3].

LEMMA 3.3. — *Let $I \subset R_X$ be a homogeneous ideal and let D^{α_1} and D^{α_2} be two differential operators. Then*

$$D^{\alpha_1}\big[D^{\alpha_2}(I)\big] \simeq D^{\alpha_1+\alpha_2}(I).$$

Proof. — We will verify the lemma when $I = (P)$ is the ideal generated by a single homogeneous polynomial $P \in \mathcal{O}_X(n)$; the general case then follows easily. Consider the following commutative diagram :

$$\begin{tikzcd} & G \times G \times X \arrow[dl, "\theta"'] \arrow[d, "\mu"] \arrow[dr, "\mathrm{id}\times\phi"] & \\ G \times X \arrow[r, "\phi"'] & X & G \times X \arrow[l, "\phi"] \end{tikzcd}$$

Here $\theta = m \times \mathrm{id}$ where $m : G \times G \to G$ is the group law. Denote by z the coordinates on $\mathbb{C}^d$ for the first factor G and by z' the coordinates for $\mathbb{C}^d$ on the second G. As in LEMMA 3.1 it suffices to prove the result locally. So let $\xi = \{\xi_i\}$ denote local coordinates on some small affine open subset $U \subset X$. The commutative diagram gives two power series representations for $(z, z', \xi) \mapsto \mu(z, z', \xi)$ and we equate coefficients of $z^{\alpha_1} z'^{\alpha_2}$. Using the fact that

$$\theta(z, z', \xi) = \big(\psi(z) + \psi(z'), \xi\big) = \big(\psi(z + z'), \xi\big),$$

it follows that the coefficient of $z^{\alpha_1} z'^{\alpha_2}$ in the power series coming from $\mu = \phi \cdot \theta$ gives a local generator for $D^{\alpha_1+\alpha_2}(P)$. On the other hand, one verifies that the coefficient of $z^{\alpha_1} z'^{\alpha_2}$ in the expansion for $\mu = \phi \cdot (\mathrm{id} \times \phi)$ gives a local generator for $D^{\alpha_1}[D^{\alpha_2}(I)]$.

REMARK 3.4.—Note that in the proof of LEMMA 3.3 it is essential that ψ is an analytic group homomorphism. Most of the set-up for zero estimates applies at least abstractly to an arbitrary analytic map $\psi : \mathbb{C}^d \to G$.

In the following Lemma we see why commutativity is important in zero estimates which makes it natural for MASSER-WÜSTHOLZ and PHILIPPON to work on commutative algebraic groups.

LEMMA 3.5. — *Suppose $g \in G$ commutes with $A = \psi(\mathbb{C}^d)$. Then $t_g[D^\alpha(I)] \simeq D^\alpha[t_g(I)]$ where D^α is an arbitrary differential operator.*

Proof. — We will use the commutative diagram from LEMMA 3.3 and once more will prove the result when $I = (P)$ with P homogeneous of degree n, the general case following easily. Let

$$\mathcal{F}_{a,b,c} = \pi_1^* \mathcal{M}^{\otimes a} \otimes \pi_2^* \mathcal{M}^{\otimes b} \otimes \pi_3^* \mathcal{L}^{\otimes c}.$$

Fix a, b, c so that $\mathcal{F}_{a,b,c}$ is generated by global sections and choose generating sections

$$\{\gamma_i\} \subset H^0\big(G \times G \times X, \mathcal{F}_{a,b,c}^{\otimes n}\big).$$

Let

$$\sigma_i = \mu^* P \otimes \gamma_i \in H^0\big(G \times G \times X, \mathcal{L}^{\otimes n} \otimes \mathcal{F}_{a,b,c}^{\otimes n}\big).$$

One verifies using the definitions that

$$t_g\big[D^\alpha(P)\big] \simeq \big(\big\{D^\beta(\sigma_{i|A\times g\times X})\big\}_{\beta\leq\alpha}\big),$$
$$D^\alpha\big[t_g(P)\big] \simeq \big(\big\{D^\beta(\sigma_{i|g\times A\times X})\big\}_{\beta\leq\alpha}\big).$$

But by the commutativity assumption, $\mu_{|A\times g\times X} = \mu_{|g\times A\times X}$. Thus an application of REMARK 3.2 (which, to be rigorous, only applies to $\bar{A}\times\bar{A}\times X$ where $\bar{A}$ is the Zariski closure of A in G since the methods are only valid in the algebraic category) concludes the proof of the lemma.

REMARK 3.6. — As is clear from the proof of LEMMA 3.5, we can apply a translation operator and a differential operator simultaneously by working on $G\times G\times X$. Moreover one can choose a, b, c such that $\mathcal{F}_{a,b,c}$ is generated by global sections with c the constant appearing in (2.2). The importance of this is that if one applies first a differential operator and then a translation operator to a homogeneous polynomial of degree n then one obtains an ideal generated by forms of degree c^2n whereas applying the operators simultaneously gives an ideal generated by forms of degree cn.

Finally we will show how, in certain special cases, our definition of differential operators following Philippon corresponds with the more intrinsic definition of WÜSTHOLZ in [W2]. Let $H = \bar{A}$ denote the Zariski closure of A in G. Then H is a commutative group variety [La1, p. 173]. Assume that for some $x \in X$ the orbit map $H \to X$ by $h \mapsto h(x)$ is an immersion. Composing with $X \hookrightarrow \mathbb{P}^N$ and taking the Zariski closure gives an equivariant compactification of H denoted by H_x. Assume that H_x is projectively normal. WÜSTHOLZ [W2, p. 477–478] defines differential operators on an affine coordinate ring $k[H]$ associated to an equivariant compactification. We sketch this here for the convenience of the reader. Let T_eH denote the tangent space to H at the origin and $\exp : T_eH \to H$ the exponential map. Also let $f : T_eH \to H_x \hookrightarrow \mathbb{P}^N$ denote the composition of exp with the natural inclusion $H \hookrightarrow \mathbb{P}^N$. Identify $\mathbb{C}^d$ with a linear subspace of T_eH. Let $U \subset \mathbb{P}^N$ denote the open affine subset of $\mathbb{P}^N$ given by the complement of the hyperplane $X_0 = 0$ and let $\xi_1, \dots, \xi_N$ denote the affine coordinates on U. Assume that $H_x \cap U$ is not empty. WÜSTHOLZ [W2, p. 478] defines

$$F_j(z) = \xi_j \cdot f : T_eH \longrightarrow \mathbb{C}.$$

By [Wal, Prop. 1.2.3] $\partial/\partial z_i(F_j) = H_{ij}(F_1,\ldots,F_N)$ where H_{ij} is a polynomial. WÜSTHOLZ defines

$$\frac{\partial}{\partial z_i}\xi_j = H_{ij}(\xi_1,\ldots,\xi_n)$$

and extends $\partial/\partial z_i$ to a derivation on $k[\xi_1,\ldots,\xi_N]$, factoring through the ideal $I(U\cap H_x)$ defining $U\cap H_x$. If R_X is the projective coordinate ring of X and R_{H_x} is the projective coordinate ring of H_x for our fixed embedding in $\mathbb{P}^N$ then there is a natural surjection $R_X \xrightarrow{p} R_{H_x}$. If $I\subset R_X$ is a homogeneous ideal, then denote by I_{H_x} the image $p(I)$ and by $I_{H_x\cap U}$ the corresponding dehomogenization. Note that $V(I_{H_x}) = V(I)\cap H_x$. If $I\subset R_X$ is a homogeneous ideal, and D^α a differential operator then define the *Wüstholz differential operator* D^α_{w} as follows :

$$D^\alpha_{\mathrm{w}}(I) = \Big(I_{H_x\cap U}, \Big\{\frac{\partial^{|\beta|}_{\mathrm{w}}}{\partial_{\mathrm{w}} z^\beta}(P)\Big\}\ ;\ P\in I_{H_x\cap U},\ \beta\le\alpha\Big),$$

where the derivative $\partial^{|\beta|}_{\mathrm{w}}/\partial_{\mathrm{w}} z^\beta$ is taken using WÜSTHOLZ' definition. We will now show that WÜSTHOLZ' definition of differential operators and DEFINITION 2.6 are compatible.

LEMMA 3.7. — *Suppose $I\subset R_X$ is a homogeneous ideal. Let $U\subset X$ be the open subset given by the complement of the hyperplane section $X_0 = 0$. Let D^α be a differential operator. If $H_x\cap U$ is not empty then $D^\alpha(I)_{(U\cap H_x)} = D^\alpha_{\mathrm{w}}(I_{(U\cap H_x)})$.*

Proof. — First note that derivatives as defined in DEFINITIONS 2.5 and 2.6 commute with restriction to subvarieties so that

$$D^\alpha(I)_{H_x\cap U} = D^\alpha(I_{H_x\cap U}).$$

This reduces the question to H_x. Consider the following commutative diagram :

$$\begin{tikzcd} \mathbb{C}^d\times T_eH \arrow[r, "\mathrm{add}"] \arrow[d, "\psi\times\exp"'] \arrow[dr, "\mu"] & T_eH \arrow[d, "f"] \\ H\times H_x \arrow[r, "\phi"'] & H_x \end{tikzcd}$$

Let z' denote analytic coordinates on T_eH. Note that if $G_j = \xi_j\cdot\mu$: $\mathbb{C}^d\times T_eH\to\mathbb{C}$ then

$$\frac{\partial}{\partial z_i}\big[G_j(z,z')\big]_{|z=0} = H_{ij}(F_1,\ldots,F_N).$$

On the other hand, DEFINITION 2.5 and REMARK 2.7 give

$$D_i^1(X_j) = \big(\{D_i^1(\phi^* X_j \otimes \eta_k)\}\big)\,.$$

Composing with exp on the second factor and using the commutative diagram shows that on $U \cap H_x$

$$\big(\phi^* X_j \otimes \eta_k\big)(z, z') = G_j(z, z') \cdot \eta_k(z, z')$$

and $\eta_k(z, z')$ have no common zeroes. It follows that

$$\big(D_i^1 X_j\big)_{U \cap H_x} = \big(H_{ij}(\xi_1, \dots, \xi_n)\big)(J)$$

where $J \subset k[\xi_1, \dots, \xi_n]$ is an ideal with no common zeroes. An application of Hilbert's Nullstellensatz concludes the proof in the special case when $I = (X_j)$ is generated by a coordinate function and $D^\alpha = D_i^1$. The general case follows by LEMMA 3.3 and the Leibnitz formula.

REMARK 3.8. — One can prove COROLLARY 2.10 by reducing to [W2, Lemma 3] in the special case when LEMMA 3.7 applies. So assume that for generic $v \in V$, H_v is an equivariant compactification of H. Choose a non-empty open affine subset $U \subset H_x$ given by the complement of a hyperplane section in the given projective embedding. Let $Y \subset H_x \cap V$ be an irreducible component. By COROLLARY 1.7,

$$\ell\big(\mathcal{O}_{V,\mathbb{P}^N}/I\big) = \ell\big(\mathcal{O}_{Y,\mathbb{P}^N}/[I + I(H_x)]\big).$$

Since lengths are computed at the generic point,

$$\ell_Y\big(V[I + I(H_X)]\big) = \ell\big(\mathcal{O}_{Y \cap U, \mathbb{A}^N}/[(I + I(H_x)) \cap U]\big).$$

Then LEMMA 3.7 gives that $\partial_{\mathrm{w}}^T[(I + I(H_x)) \cap U] \subset I(Y \cap U)$ and Lemma 3 of [W2] applies.

REMARK 3.9. — We show here how THEOREM 1.5 is needed in the proof of Proposition 1 in [W2]. For the convenience of the reader, we first recall some definitions from [W2], suitably generalized to our setting. For $V \subset X$ let $I(V) = (P_1, \dots, P_r)$ be its homogeneous ideal and fix some $\eta \in H^0(G \times X, \mathcal{F}_{b,c})$ which does not vanish on $e \times V$. One defines (cf. [W2, p. 479])

$$\rho(V) = \operatorname{rank} \begin{bmatrix} D_1^1[\eta(P_1)](\operatorname{mod} I(V)) & \cdots & D_d^1[\eta(P_1)](\operatorname{mod} I(V)) \\ \cdots & & \cdots \\ D_1^1[\eta(P_r)](\operatorname{mod} I(V)) & \cdots & D_d^1[\eta(P_r)](\operatorname{mod} I(V)) \end{bmatrix}.$$

It is easy to verify that the definition is independent of the choice of P_i and also of η. Recall (2.8) : $c_V = \operatorname{codim}(Av \cap V)$ for generic $v \in V$ ($c_V = \min\{\tau(g(V))\,;\ g \in G\}$ in the terminology of [W2]; cf. p. 473). One clearly has, for sufficiently general $v \in V$ and a fixed dehomogenization of the P_i,

$$\rho(V) = \operatorname{rank} \begin{bmatrix} \dfrac{\partial}{\partial z_1} P_1[\psi_v(z)]_{|z=0} & \cdots & \dfrac{\partial}{\partial z_d} P_1[\psi_v(z)]_{|z=0} \\ \cdots & & \cdots \\ \dfrac{\partial}{\partial z_1} P_r[\psi_v(z)]_{|z=0} & \cdots & \dfrac{\partial}{\partial z_d} P_r[\psi_v(z)]_{|z=0} \end{bmatrix}.$$

From this it follows immediately that $c_V \geq \rho(V)$ and that equality holds if and only if $Av \cap V$ is transverse at v. Thus THEOREM 1.5 implies that $\rho(V) = c_V$ for all $V \subset X$ and this implies Proposition 1 of [W2]. The gap in the proof of WÜSTHOLZ occurs on p. 495 where he states

$$I\big[\exp^{-1}(V \cap A)\big]_\xi = \big(\phi_1, \ldots, \phi_l, M_1, \ldots, M_{n-d}\big)_\xi.$$

This is only true when the intersection $A \cap V$ is transverse at x.

4. Zero Estimates

In this section we prove THEOREM 0.3 and a generalization to deal with the case of non-commutative groups. So let $X = \prod_{i=1}^m X_i$ be as in the statement of THEOREM 0.3 with a product group action $\phi : G \times X \to X$. If $Y \subset X$ is a closed subscheme and $U \subset Y$ is quasi-projective, U is said to be an irreducible component of Y if $\overline{U}$ is an irreducible component of Y.

REMARK 4.1. — If g commutes with A then one can use LEMMA 3.5 to show that P vanishes along A at $g(x)$ to order $> n$ if and only if $t_g[D^\alpha(P)]$ vanishes at x for all $|\alpha| \leq n$. This is the content of [P1, Prop. 4.4].

Proof of Theorem 0.3. — Since the conclusion of the theorem only involves the orbit Gx one can replace X with $\overline{Gx}$ and assume that P is not identically zero on $\overline{Gx}$. Define ideals $I_\alpha \subset R_{\overline{Gx}}$ as in PHILIPPON, namely let $I_1 = (P)$ and

$$I_\alpha = \big(\{t_{g^{-1}}[\mathcal{D}^T(I_{\alpha-1})] : g \in S\}\big), \quad \text{for } 2 \leq \alpha \leq n+1.$$

Let $X_\alpha = V(I_\alpha)$. LEMMAS 3.3 and 3.5 and REMARK 4.1 show that $x \in X_{n+1}$ and hence X_{n+1} is non-empty. This gives rise to a chain

$$X_1 \supset X_2 \supset \cdots \supset X_{n+1}$$

with $\dim X_1 = n - 1$ and $\dim X_{n+1} \geq 0$ so there exists[5] r such that $\dim X_r = \dim X_{r+1}$. Let Y denote an irreducible component of X_r which is also an irreducible component of X_{r+1}. Since $t_{g^{-1}}[\mathcal{D}^T(I_r)] \subset I_{r+1}$ it follows that $Y \subset V(t_{g^{-1}}[\mathcal{D}^T(I_r)])$ or equivalently

$$g(Y) \subset V\big[\mathcal{D}^T(I_r)\big] \subset X_r, \quad \text{for all } g \in S. \tag{4.1.1}$$

It follows that $g(Y)$ is an irreducible component of X_r for all $g \in S$ since $g(Y) \subset X_r$ and $\dim g(Y) = \dim X_r$. Moreover both I_r and $\mathcal{D}^T(I_r)$ are $\mathcal{P}_{g(Y)}$-primary for all $g \in S$ so that COROLLARY 2.10 applies.

Let $\phi : G \to Gx$ denote the morphism determined by $g \mapsto g(x)$ and write

$$H := \big\{g \in G\,;\ g(Y) = Y\big\}.$$

Since $V(P) = X_1 \supset X_r \supset g(Y)$ for all $g \in S$, it is clear that P vanishes on $g(Hy)$ for all $g \in S$ and any $y \in Y$. To obtain (0.3.1) we will show that Hy as well as $g(Hy)$ can be cut out (with high multiplicity) by multihomogeneous polynomials of multidegree $\leq cd$ and then apply THEOREM 1.1. To this end, define

$$J := \big(\big\{t_{g^{-1}}(I_r)\,;\ g \in \phi^{-1}(Y)\big\}\big).$$

Hence

$$\begin{aligned} \big[V(J) \cap \phi(G)\big]_{\mathrm{red}} &= \big\{g(x) \in Gx\,;\ [\phi^{-1}(Y)\cdot g](x) \in X_r\big\} \\ &= \big\{g(x) \in Gx\,;\ g(Y) \subset X_r\big\}. \end{aligned} \tag{4.1.2}$$

Since $g(Y) \subset X_r$ for all $g \in S$, $g(Hx) \subset V(J)$ for all $g \in S$. Moreover we claim that $g(Hx)$ is an irreducible component of $V(J)$. This follows from the definition of H and the fact that $g(Y) \subset X_r$ is an irreducible component. By LEMMA 3.5,

$$\big[V[\mathcal{D}^T(J)] \cap \phi(G)\big]_{\mathrm{red}} = \big\{g(x) \in Gx\,;\ g(Y) \subset V(\mathcal{D}^T I_r)\big\}. \tag{4.1.3}$$

Equations (4.1.1) and (4.1.3) imply that $g(Hx)$ is an irreducible component of $V[\mathcal{D}^T(J)]$ for all $g \in S$. Let H' denote the connected component of H containing the origin. The number of distinct irreducible components of the form $g(H'x) \subset V(\mathcal{D}^T(J))$ is at least

$$\operatorname{card}\big((S + H'x)/H'x\big).$$

---

[5] Note that this part of the argument is formally equivalent with Faltings' use of the product theorem [F, Remark 3.4].

By LEMMA 1.8, $\deg_d g(H'x) = \deg_d H'x$ for all $g \in S$. REMARK 3.6 shows that, up to a projectively irrelevant ideal, J is generated by multihomogeneous polynomials of multidegree cd. Then THEOREM 1.1 (or 1.4.1) and COROLLARY 2.10 combine to give

$$\binom{T + \operatorname{codim}(Ax \cap H'x, Ax)}{\operatorname{codim}(Ax \cap H'x, Ax)} \operatorname{card}\big((S + H'x)/H'x\big) \deg_d H'x \le \deg_{cd} Gx.$$

THEOREM 0.3 can be extended under certain hypotheses to non-commutative groups. The only points in the proof of THEOREM 0.3 where commutativity of G is used are LEMMA 3.5 and equation (4.1.2). Suppose now that X is a compactification of an arbitrary connected algebraic group G such that the group action extends on both the right and the left (such compactifications always exist by a construction in [CH]); we call such a compactification *bi-equivariant*. In order to simplify notation, identify G with its natural inclusion in X. We let $\phi_r : G \times X \to X$ and $\phi_l : G \times X \to X$ denote action on the right and action on the left respectively. Use ϕ_l for the definition of derivatives and let c_1 denote the corresponding constant so that, in the notation of (2.2), $\mathcal{F}_{b_1,c_1} \otimes \phi_l^* \mathcal{L}^{\otimes -1}$ is generated by global sections. Similarly, let c_2 be a bound for the multidegree of a complete system of multihomogeneous polynomials representing ϕ_r, i.e. suppose $\mathcal{F}_{b_2,c_2} \otimes \phi_r^* \mathcal{L}^{\otimes -1}$ is generated by global sections. Denote by t_g^r translation on the right and t_g^l translation on the left. In order to use LEMMA 3.5 we need to assume that each element $g \in S$ commutes with A though it is *not* necessary for the elements of S to commute amongst themselves. For $V \subset G$ we will denote by $\deg_d V$ the corresponding degree of the Zariski closure of V in the given projective embedding.

THEOREM 4.2. — *Let X be a bi-equivariant compactification of a connected algebraic group of dimension n as above. Let $P \in H^0(\mathcal{O}(d_1, \dots, d_m))$ be a multihomogeneous polynomial of multidegree d. Suppose P vanishes to order $\ge nT + 1$ along A at S_n. Then either P is identically zero or there exists a proper connected algebraic subgroup $H \subset G$ and an element $g \in G$ such that P vanishes along $(g' \cdot g)H$ for all $g' \in S$ and*

$$\binom{T + \operatorname{codim}(A \cap H, A)}{\operatorname{codim}(A \cap H, A)} \cdot \operatorname{card}\big((S + H)/H\big) \cdot \deg_d H \le \deg_{c_1 c_2 d} G.$$

Proof. — The only difference in proof is in the definition of J since, as noted above, we used commutativity of G in order to guarantee that $g(Y) \subset V(J)$ for all $g \in S$. Here we make use of translation on the right.

So with notation as in the proof of THEOREM 0.3, let Y be such that $g(Y)$ is an irreducible component of both X_t and X_{t+1} for all $g \in S$. Let

$$J = \{t^r_{g^{-1}}(I_t) : g \in Y \cap G\}.$$

Then as before, $(V(J) \cap G)_{\mathrm{red}} = \{g \in G : g(Y \cap G) \subset X_t\}$. The multiplicity estimates (COROLLARY 2.10) apply as before and we only need to bound the degree of homogeneous polynomials generating J. But one sees as in LEMMA 3.5 that derivatives, which were defined via translation on the left, commute with translation on the right. REMARK 3.6 shows that the derivatives increase the degree of generators of a homogeneous ideal by a factor of c_1 while translation on the right further increases the degrees by c_2. This gives the required estimate.

REMARK 4.3. — PHILIPPON states [P2, p. 398] that one can choose $g = e \in G$ in THEOREM 0.3. In other words, the variety $Y \subset X_r$ constructed in the proof of THEOREM 0.3 can be assumed to pass through $e(x) = x$. Since $x \in X_{n+1}$ one can let Y_α be an irreducible component of X_α containing x and then there must exist r such that Y_r is an irreducible component of both X_r and X_{r+1} and $x \in Y_r$. Thus, taking H to be the stabilizer of Y_r, there is an algebraic subgroup H such that P vanishes on all translates $g(H)$ for $g \in S$. Unfortunately, inequality (0.3.1) may not hold with this choice of Y because the translates $g(Y)$ may not be *irreducible* components of X_r and consequently $g(H)$ may not be an irreducible component of $V(J)$.

One can, however, obtain some small amount of information about the point $g \in G$ relating to the set S where P vanishes to high order. Recall the chain

$$X_1 \supset X_2 \supset \cdots \supset X_{n+1}.$$

Let r be the smallest positive integer with $\dim X_r = \dim X_{r+1}$. Let $\{Y_j\}$ be the set of irreducible components of X_r of dimension $\dim X_r$ which are also irreducible components of X_{r+1}. If there exists some Y_j with $x \in Y_j$ then choose $Y = Y_j$. Otherwise let $U = \overline{Gx} \setminus \bigcup_j Y_j$ be the complement of the Y_j's in $\overline{Gx}$. Then consider the chain

$$X_1 \cap U \supset X_2 \cap U \supset \cdots \supset X_{n+1} \cap U.$$

Since $x \in X_{n+1} \cap U$ the same argument applies. Repeating this process, there must exist an open subset $U' \subset \overline{Gx}$ and an irreducible component $Z' \subset X_t \cap U'$ of dimension $\dim(X_t \cap U')$ which is also an irreducible component of $X_{t+1} \cap U'$ and with $x \in Z'$. Let Z denote the Zariski closure of Z' in $\overline{Gx}$. If $g(Z)$ is not an irreducible component of X_t for all $g \in X$ then there exists $g \in S$ such that $g(Z) \subset X \backslash U'$; this means that $g(Z) \subset V_\alpha$ for some V_α which is an irreducible component of $X_{r'}$ and $X_{r'+1}$ for some r'. Clearly V_α contains $g(x)$ for some $g \in S$ and $\dim V_\alpha > \dim Z \geq 0$. Repeat the same argument with V_α. This cannot go on forever since the dimension increases at each step. In fact this shows that there exists Y such that $g(Y)$ is an irreducible component of both X_r and X_{r+1} for some r and for all $g \in S$ and such that $Y \cap S_{n-1}(X)$ is not empty. In the special case when $S \subset G$ is a finite subgroup, it follows that one can always choose $g = e$.

5. The Product Theorem

We will extend the Product Theorem of Faltings to arbitrary commutative algebraic groups. Let $G = \prod_{i=1}^m G_i$ be a product of connected commutative algebraic groups and let $X \hookrightarrow \mathbb{P}$ be an equivariant, projectively normal compactification respecting the product structure on G. Let $c_i = \deg X_i$ and let $\delta_i = \dim X_i$. Let

$$\psi : \prod_{i=1}^{m} T_e G_i \longrightarrow G$$

be the exponential map. Since G is commutative, ψ is an analytic homomorphism. Let $\{z_{ij}\}_{j=1}^{\delta_i}$ denote coordinates on $T_e G_i$. Definitions 2.4–2.6 apply in this setting giving differential operators D_{ij} on the projective coordinate ring R_X. Note that it is clear that these differential operators preserve the product structure on X in the sense that $D_{ij}(P) = (0)$ if $P \in \mathcal{O}_X(d_1, \ldots, d_m)$ with $d_i = 0$.

We need to introduce the notion of the *index* of a polynomial at a point. Let

$$S_i = \{D_\alpha \,;\ \alpha_{jk} \neq 0 \Rightarrow j = i\}.$$

Thus S_i is the set of differential operator on the i^{th} factor G_i. Thus any differential operator can be written uniquely as $D = \prod_{i=1}^m D_i$ with $D_i \in S_i$.

Definition 5.1. — Let $P \in H^0(\mathcal{O}(d_1, \ldots, d_m))$ be a multiform of multidegree d and let $x \in X$ be a closed point. Then define the index $\operatorname{ind}(x, P)$ of P at x as follows :

$$\operatorname{ind}(x, P) = \min \Big\{ \sum_{i=1}^{m} \frac{|D_i|}{d_i} \,;\ x \notin V[D(P)] \text{ where } D = \prod_{i=1}^{m} D_i \Big\}.$$

Thus $\operatorname{ind}(x, P) = 0$ if and only if $P(x) \neq 0$ and similarly $\operatorname{ind}(x, P) = \infty$ if and only if P vanishes identically on X. Faltings considers the case where $G_i = \mathbb{A}^{n_i}$ with $X_i = \mathbb{P}^{n_i}$. In this instance, $\operatorname{ind}(x, P) \leq m$ (provided of course that P does not vanish identically on X). This is not true in the more general setting, but it is still the case that $\operatorname{ind}(x, P) \leq \sum_{i=1}^{m} c_i$. We define sets of operators $\mathcal{D}^\sigma$ for $\sigma \in \mathbb{R}$ by analogy with $\mathcal{D}^T$ from the previous section :

$$\mathcal{D}^\sigma = \Big\{ D = \prod_{i=1}^{m} D_i \, ; \ \sum_{i=1}^{m} \frac{|D_i|}{d_i} \leq \sigma \Big\}.$$

We will need a rough estimate on the cardinality $|\mathcal{D}^\sigma|$. Assuming $d_i > 1/\sigma$, let

$$S = \Big\{ D = \prod_{i=1}^{m} D_i \, ; \ \frac{|D_i|}{d_i} \leq \frac{1}{m\sigma} \Big\}.$$

From the definitions it is clear that $S \subset \mathcal{D}^\sigma$. Hence

$$|\mathcal{D}^\sigma| \geq c(\sigma) \cdot \prod_{i=1}^{m} d_i^{\delta_i} \ + \ \text{lower order terms},$$

where $c(\sigma)$ is a constant depending only on σ. Finally, for positive real numbers a, b, write $a \gg b$ if $a/b \geq r$ for some «large» positive real number r. The following generalizes Proposition 3.1 of [F] :

THEOREM 5.2. — *Given $\epsilon > 0$ there exist constants $r(\epsilon, X)$ and $c(\epsilon, X)$ depending only on ϵ and the fixed projective embedding of X satisfying the following property. Let $J \subset R$ be a multihomogeneous ideal generated by forms of multidegree d. Suppose Z is an irreducible component of $V(J) \cap X$ which is also an irreducible component of $V[\mathcal{D}^\epsilon(J)] \cap X$ and suppose $\mathcal{O}_{Z,X}$ is Cohen-Macaulay (e.g. $Z \cap G \neq \phi$). If $d_i/d_{i+1} \geq r(\epsilon, X)$ for $1 \leq i \leq m-1$ then $Z = Z_1 \times \cdots \times Z_m$ is a product subvariety of X with* $\deg Z_i \leq c(\epsilon, X)$ *for all i.*

Proof. — The proof resembles that of THEOREM 0.3. Choose

$$I = \big(P_1, \ldots, P_t\big) \subset J$$

where $(P_1, \ldots, P_t)$ is a set of multihomogeneous polynomials of multi-degree d which cut out all components of $V(J) \cap X$ of codimension $\leq t = \operatorname{codim}(Z, X)$. Then THEOREM 2.9 gives an estimate for $\ell_Z[X \cap V(I)]$ and THEOREM 1.1 (or COROLLARY 1.4) bounds the degree of $V(I) \cdot X$. If Z were *not* a product subvariety then the requirement $d_i/d_{i+1} \gg 0$ forces the length estimate to be larger than the bound on the degree of the intersection.

For the proof, recall the notation of COROLLARY 1.4. Let $\mathcal{L}_i = \pi_i^* \mathcal{O}(1)$ where $\pi_i : \mathbb{P} \to \mathbb{P}^{n_i}$ is the projection to the i^{th} factor. For any m-tuple $e = (e_1, \ldots, e_m)$ with $\sum e_i = \dim Z$, let V_e denote a generic cycle representing $c_1(\mathcal{L}_1)^{e_1} \cap \ldots \cap c_1(\mathcal{L}_m)^{e_m}$. Write

$$s_i : \mathbb{P} \longrightarrow \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i},$$

$$t_i : \mathbb{P} \longrightarrow \mathbb{P}^{n_{m-i+1}} \times \cdots \times \mathbb{P}^{n_m}.$$

Set $a_i = \dim s_i(Z) - \dim s_{i-1}(Z)$ for $2 \le i \le m$ and $a_1 = \dim s_1(Z)$ and similarly let $b_m = \dim t_1(Z)$ and $b_j = \dim t_{m-j+1}(Z) - \dim t_{m-j}(Z)$ for $1 \le j \le m-1$. With these definitions Z is a product subvariety if and only if $a_i = b_i$ for all i.

Suppose $\{Q_i\}_{i=1}^{\deg Z \cdot V_e}$ are the points of intersection of Z and V_e. Since V_e is generic $\mathcal{O}_{Q_i, X \cap V_e}$ is Cohen-Macaulay and applying THEOREM 1.1 to $V_e \cap X$ gives

$$\sum_{i=1}^{\deg Z \cdot V_e} \ell_{Q_i}\big((V_e \cap X) \cap V(I)\big) \le \deg_d (V_e \cap X) \le \prod_{i=1}^{m} c_i d_i^{\delta_i - e_i}. \tag{5.2.1}$$

By [Fu, Example 8.4.8], it follows that

$$\ell_{Q_i}\big((V_e \cap X) \cap V(I)\big) = \ell_Z\big(X \cap V(I)\big). \tag{5.2.2}$$

Combining (5.2.1) and (5.2.2) gives

$$\ell_Z\big(X \cap V(I)\big) \cdot [\deg (Z \cdot V_e)] \le \prod_{i=1}^{m} c_i d_i^{\delta_i - e_i}. \tag{5.2.3}$$

We want to minimize the right hand side (5.2.3) without making $\deg (Z \cdot V_e) = 0$. Since $d_1 \gg d_2 \gg \cdots \gg d_m$ this means that we want to choose $e_i = \delta_i - b_i$. On the other hand, as to the left hand side of (5.2.3), by THEOREM 2.9 $\ell_Z(X \cap V(I)) \ge |\mathcal{D}_Z^\epsilon|$ where $\mathcal{D}_Z^\epsilon$ denotes the differential operators transverse to Z of weight $\le \epsilon$. One can estimate $|\mathcal{D}_Z^\epsilon|$ by making a specific choice of basis for differentials transversal to Z. Since $d_1 \gg d_2 \gg \cdots \gg d_m$ and since we want to maximize $|\mathcal{D}_Z^\epsilon|$ the best choice is to take a_i derivations in S_i. Arguing as above (assume without loss of generality that $d_m \gg 1/\epsilon$ by taking a power of P since the index is invariant under this operation)

$$|\mathcal{D}_Z^\epsilon| \ge c(\epsilon) \prod_{i=1}^{m} d_i^{a_i} + \text{ lower order terms.}$$

Putting everything together, inequality (5.2.3) now becomes

$$c(\epsilon)\prod_{i=1}^{m} d_i^{a_i} \leq \prod_{i=1}^{m} c_i \prod_{i=1}^{m} d_i^{b_i}. \tag{5.2.4}$$

But by the choice of a_i and b_i

$$\prod_{i=1}^{m} d_i^{a_i-b_i} \geq \min\left\{\frac{d_i}{d_{i+1}}\right\}, \tag{5.2.5}$$

unless $a_i = b_i$ for all i. Thus as soon as

$$r(\epsilon, X) > \frac{c_1\cdots c_m}{c(\epsilon)}$$

inequalities (5.2.4) and (5.2.5) show that $a_i = b_i$ for all i and hence Z is a product subvariety. Since Z is a product subvariety, (5.2.3) together with the estimate on $\ell(X \cap V(I))$ shows that one can take $c(\epsilon, X) = r(\epsilon, X)$.

REMARK 5.3. — The proof of the Product Theorem works in a slightly more general setting. In particular, it suffices to assume that X_i is an equivariant compactification of a homogeneous space V_i with group action by a connected commutative algebraic group G_i. The point is simply that in (5.2.3) the length along Z must be proportional to the degree which requires the maximal possible dimension of differential operators transverse to Z.

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