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STRICTLY ERGODIC, UNIFORM POSITIVE ENTROPY MODELS

PAR

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RÉSUMÉ. — Le théorème suivant, de type Jewett-Krieger est démontré. Soit (Ω, m, T) un processus ergodique d'entropie positive; il existe alors un système dynamique strictement ergodique (X, μ, T), d'entropie uniformément positive, isomorphe à (Ω, m, T). D'autres résultats concernant les extensions et les couplages des systèmes d'entropie uniformément positive sont donnés.

ABSTRACT. — The following Jewett-Krieger type theorem is proved : given any ergodic process (Ω, m, T) of positive entropy, there exists a strictly ergodic, uniform positive entropy (u.p.e.), dynamical system (X, μ, T) which is measure theoretically isomorphic to (Ω, m, T). Other results about extensions and joinings of u.p.e. systems are given.

1. Introduction

The class of uniform positive entropy (u.p.e.) dynamical systems was introduced in [B1], as one candidate for an analogue in topological dynamics of the basic notion of a K-process in ergodic theory. In particular every non-trivial factor of a u.p.e. dynamical system has positive topological entropy. The precise definition of u.p.e. is as follows. Let (X, T) be a dynamical system; an open cover $\mathcal{U} = \{U, V\}$ of X is called a *standard cover* if both U and V are none-dense in X. The system (X, T) has *uniform positive entropy* (u.p.e.) if for every standard cover $\mathcal{U}$ of X, the topological entropy $h(\mathcal{U}, T) > 0$. Further developments of the theory of u.p.e. systems were obtained in [B2] and [B-L].

[B2] concludes with the question : do there exist non-trivial minimal u.p.e. dynamical systems? We prove here the following :

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THEOREM A. — *Let (X,T) be a dynamical system. Suppose there exists a T-invariant probability measure μ on X with* $\mathrm{Supp}(\mu) = X$, *such that the process (X,μ,T) is a K-process; then (X,T) is u.p.e.*

Combining this result with the representation theorem of JEWETT and KRIEGER, (see [D-G-S]) we immediately get a wealth of examples of strictly ergodic (hence a fortiori, minimal) u.p.e. dynamical systems. Another corollary of THEOREM A, (COROLLARY 3.2) is obtained when one recalls the construction in [W1], of a universal minimal dynamical system (X,T) with the property that for every measure theoretical ergodic process (Ω,m,T), there exists an invariant probability measure μ on X such that (X,μ,T) is isomorphic to (Ω,m,T). By THEOREM A, (X,T) is u.p.e. and it therefore serves as a minimal u.p.e. model for *every* ergodic process.

Let us call an extension of dynamical systems $(X,T) \xrightarrow{\pi} (Y,T)$, *solid* if whenever $A \subset X$ is closed and $\pi[A] = Y$, then there exists a sequence n_i with $\lim T^{n_i}A = X$, (in the space 2^X of closed subsets of X, with respect to the Hausdorff topology). It is easy to check that an almost 1-1 extension of minimal dynamical systems is solid.

The extension $(X,T) \xrightarrow{\pi} (Y,T)$ is called *weakly solid* if whenever $W \subset X \times X$ is a closed $T \times T$-invariant subset with $\pi \times \pi[W] = Y \times Y$, then $W = X \times X$. As we shall see (PROPOSITION 4.1), every proximal extension of minimal systems is weakly solid and every solid extension is weakly solid.

THEOREM B. — *If (Y,T) is a minimal u.p.e. dynamical system and $(X,T) \xrightarrow{\pi} (Y,T)$ is a weakly solid extension with (X,T) minimal, then (X,T) is u.p.e.*

From THEOREMS A and B, using theorems of WEISS, [W2] and FURSTENBERG and WEISS, [F-W], we now deduce :

THEOREM C. — *Given an arbitrary ergodic process (Ω,m,T) of positive entropy, there exists a strictly ergodic, uniform positive entropy dynamical system (X,T) with invariant measure μ, such that the processes (Ω,m,T) and (X,μ,T) are measure theoretically isomorphic.*

Using a generic construction of skew product dynamical systems we further use THEOREMS A and B to get

THEOREM D. — *There exist two minimal u.p.e. dynamical systems and a minimal joining of the two, which is not even weakly mixing (hence, a fortiori, not u.p.e.)*

In section 2 we recall the definitions of entropy pairs and u.p.e. systems and list the basic results obtained in [B1] and [B2] for such systems. THEOREMS A, B, C and D are proved in sections three, four, five and six respectively. We use standard notation and terminology of ergodic theory and topological dynamics. To avoid confusion however; we call a measure preserving transformation on a Borel space $(\Omega, \mathcal{B}, m, T)$ a *process* (and omit the σ-algebra $\mathcal{B}$), and a topological transformation or a flow (i.e. a pair (X, T) where X is a compact metric space and T a self-homeomorphism of X) a *dynamical system* (or just system). With only a few exceptions we denote by T the acting transformation in every process or system we consider.

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2. Entropy pairs and u.p.e. systems

Let (X, T) be a dynamical system; an open cover $\mathcal{U} = \{U, V\}$ of X is called a *standard cover* if both U and V are none-dense in X. (X, T) has *uniform positive entropy (u.p.e.)* if for every standard cover $\mathcal{U}$ of X, the topological entropy $h(\mathcal{U}, T) > 0$. A pair $(x, x') \in X \times X$ is an *entropy pair* if for every standard cover $\mathcal{U}$ with $x \in \operatorname{int}(U^c)$ and $x' \in \operatorname{int}(V^c)$, $h(\mathcal{U}, T) > 0$.

Denote the set of entropy pairs by $E = E_X$. Let $\Delta = \{(x, x) : x \in X\}$. The following facts are proved in [B1], [B2].

1) $h(X, T) > 0$ implies $E \neq \emptyset$,
2) (X, T) is u.p.e. iff $E = X \times X \backslash \Delta$,
3) $\bar{E} \subset E \cup \Delta$,
4) E (hence also $\bar{E}$) is $T \times T$ invariant,
5) if $(X, T) \xrightarrow{\pi} (Y, T)$ is a homomorphism then $\pi \times \pi[\bar{E}_X] = \bar{E}_Y$,
6) u.p.e. implies weak mixing,
7) every non-trivial factor of a u.p.e. system has positive topological entropy.

As a warm up, we next give a slightly different proof of 1) than that given in [B2].

PROPOSITION 1.1. — *If $h(X, T) > 0$, then $E \neq \emptyset$.*

Proof. — Assume on the contrary that $h(X, T) > 0$ and $E = \emptyset$. Let

$\delta > 0$ be given and for every

$$(x, x') \in K_\delta = \big\{(z, z') \in X \times X : d(z, z') \geq \delta\big\},$$

let $\mathcal{U} = \{U, V\}$ be a standard cover with $x \in \operatorname{int}(U^c)$, $x' \in \operatorname{int}(V^c)$ and $h(\mathcal{U}, T) = 0$. Choose $\epsilon > 0$ with $B_\epsilon(x) \subset \operatorname{int}(U^c)$ and $B_\epsilon(x') \subset \operatorname{int}(V^c)$ and let

$$U(x, x') = \big(\overline{B}_\epsilon(x)\big)^c, \quad V(x, x') = \big(\overline{B}_\epsilon(x')\big)^c.$$

Then $\mathcal{U}(x, x') := \{U(x, x'), V(x, x')\} \prec \mathcal{U}$ and therefore

$$h(\mathcal{U}(x, x'), T) \leq h(\mathcal{U}, T) = 0.$$

Let $\{(x_i, x'_i)\}_{i=1}^k$ be a finite set in K_δ such that $\{B_i \times B'_i\}_{i=1}^k$ is a cover of K_δ, where $B_i = B_{\epsilon(x_i, x'_i)}(x_i)$ and $B'_i = B_{\epsilon(x_i, x'_i)}(x'_i)$.

Now let $\mathcal{U}_i = \big\{U_i = (\overline{B}_i)^c,\ V_i = (\overline{B}'_i)^c\big\}$, $1 \leq i \leq k$ and $\mathcal{V} = \bigvee_{i=1}^k \mathcal{U}_i$. Then :

$$h(\mathcal{V}, T) \leq \sum_{i=1}^k h(\mathcal{U}_i, T) = 0.$$

We show next that :

$$\max\big\{\operatorname{diam}(W) : W \in \mathcal{V}\big\} \leq \delta.$$

Suppose $W = W_1 \cap W_2 \cap \cdots \cap W_k \in \bigvee_{i=1}^k \mathcal{U}_i$ is non-empty, where for each i, $W_i = U_i$ or V_i. Let $x \in W$; if $(x, x') \in K_\delta$ then there exists i with $(x, x') \in B_i \times B'_i$. Hence $x \notin U_i$ so that $x \in W_i = V_i$. However $x' \in B'_i$ implies $x' \notin V_i$ hence $x' \notin W$. This proves our claim and since $\delta > 0$ is arbitrary, we deduce that $h(X, T) = 0$. This contradicts our assumption and the proof is complete. $\square$

REMARKS

1) This proof yields the following result : if (X, T) is expansive with a constant $\delta > 0$ then $h(X, T) > 0$ implies $E \cap K_\delta \neq \emptyset$.

2) Using a similar argument, applied to sets of the form

$$K_\delta^\epsilon = \big\{(x, x') : \delta \leq d(x, x') \leq \epsilon\big\},$$

one can show that $h(X, T) > 0$ implies $\overline{E} \cap \Delta \neq \emptyset$.

3. K implies u.p.e.

As usual we let, for a probability vector $(p_1, p_2, \dots, p_n)$,

$$H(p_1, p_2, \dots, p_n) = -\sum p_i \log p_i.$$

PROPOSITION 3.1. — *Let (X,T) be a dynamical system, μ an ergodic T-invariant probability measure on X which is positive on non-empty open sets. Let $\mathcal{U} = \{U,V\}$ be a standard cover of X and let $A = U^c$, $B = V^c$, $C = U \cap V$. Let $\mathcal{P} = \{A,B,C\}$ be the corresponding partition and put $\alpha = \mu(A)$, $\beta = \mu(B)$, $\gamma = \mu(C)$ and $(\alpha, \beta > 0)$. Then with $h = h_\mu(\mathcal{P},T)$ and $h' = h(\mathcal{U},T)$ we have :*

$$h \leq h' + H(\gamma, 1-\gamma).$$

Proof. — Let $\frac{1}{2} > \epsilon > 0$ be given. The ergodic theorem implies the existence of a positive integer n_1 such that for $n \geq n_1$ the set $F = F_n$ of $(\mathcal{P}, n, \epsilon)$-*good* points in X (i.e. those points x, whose atom or «name» in the partition $\mathcal{P}_0^{n-1} = \mathcal{P} \vee T\mathcal{P} \cdots \vee T^{n-1}\mathcal{P}$, has up to ϵ the right frequency of letters A, B, C), has measure $> 1-\epsilon$.

For any n let $N_n = N(n,\epsilon)$ be the minimal cardinality of a set of $\mathcal{P}_0^{n-1}$ names sufficient to cover all but a set of measure 2ϵ of X. Then, by the Shannon-MacMillan theorem (see e.g. [R, p. 72]) :

$$h = \lim_{n\to\infty} \frac{1}{n} \log N(n,\epsilon).$$

Put :

$$\mathcal{G} = \big\{G \in \mathcal{P}_0^{n-1} : \mathrm{e}^{-n(h+\epsilon)} < \mu(G) < \mathrm{e}^{-n(h-\epsilon)}\big\},$$

$$\mathcal{G}' = \big\{G \in \mathcal{G} : G \cap F \neq \emptyset\big\},$$

$$\mathcal{G}'' = \mathcal{G} \backslash \mathcal{G}'.$$

Then clearly $\mu\big(\bigcup \mathcal{G}''\big) < \epsilon$ and again by the Shannon-MacMillan theorem, for n large enough $\mu\big(\bigcup \mathcal{G}\big) > 1-\epsilon$, whence $\mu\big(\bigcup \mathcal{G}'\big) > 1-2\epsilon$. If we let $N' = N'_n =$ cardinality of $\mathcal{G}'$, then clearly $N'_n \geq N_n$. Thus we have :

$$h = \lim \frac{1}{n} \log N_n \leq \underline{\lim} \frac{1}{n} \log N'_n.$$

We may therefore choose n_2 such that for $n \geq n_2$:

$$(*) \qquad \mathrm{e}^{n(h-\epsilon)} < N'_n.$$

Finally, choose n_3 with the property that for $n \geq n_3$

$$\mathrm{e}^{n(h'-\epsilon)} \leq \#\mathcal{W} \leq \mathrm{e}^{n(h'+\epsilon)},$$

where, $\mathcal{W}$ is a subcover of $\mathcal{U}_0^{n-1}$ of minimal cardinality. Take $n \geq \max\{n_1, n_2, n_3\}$. Given $G \in \mathcal{G}'$ there exists $W \in \mathcal{W}$ for which $F \cap G \cap W$ is not empty. Write

$$W = W_0 \cap T^{-1}W_1 \cap \cdots \cap T^{-n+1}W_{n-1},$$
$$G = D_0 \cap T^{-1}D_1 \cap \cdots \cap T^{-n+1}D_{n-1},$$

where $W_j \in \{U, V\}$, $D_j \in \{A, B, C\} = \mathcal{P}$, $0 \leq j \leq n-1$. Let $x \in F \cap G \cap W$, then since x is in F, there are q C's among the D_j's where $n(\gamma - \epsilon) \leq q \leq n(\gamma + \epsilon)$. These q C's can appear anywhere, but once we know which of the D_j's are C's, at any other position, the appearance of either A or B is determined by reading whether W_j is $U = A^c$ or $V = B^c$. Thus we get $\binom{n}{n(\gamma\pm\epsilon)}$ as an upper bound on the number of names in $\mathcal{G}'$ intersecting W. We now deduce :

$$N' \leq \#\mathcal{W} \cdot \binom{n}{n(\gamma \pm \epsilon)} \leq \mathrm{e}^{n(h'+\epsilon)} \cdot \binom{n}{n(\gamma \pm \epsilon)}.$$

Use Stirling's formula to get

$$\binom{n}{n(\gamma \pm \epsilon)} = \frac{n!}{(n(\gamma \pm \epsilon))!\,(n(1-(\gamma \pm \epsilon)))!} \leq K\sqrt{n}\,\mathrm{e}^{nH(\gamma\pm\epsilon, 1-(\gamma\pm\epsilon))},$$

hence :

$$(**) \qquad N' \leq K\sqrt{n}\,\mathrm{e}^{n[H(\gamma\pm\epsilon,1-(\gamma\pm\epsilon))+h'+\epsilon]}.$$

From (*) and (**) we get

$$\mathrm{e}^{n(h-\epsilon)} \leq K\sqrt{n}\,\mathrm{e}^{n[H(\gamma\pm\epsilon,1-(\gamma\pm\epsilon))+h'+\epsilon]},$$

hence :

$$h - \epsilon \leq \frac{\log K\sqrt{n}}{n} + H\big(\gamma \pm \epsilon, 1 - (\gamma \pm \epsilon)\big) + h' + \epsilon.$$

Finally let first $n \to \infty$ and then $\epsilon \to 0$ to get $h \leq H(\gamma, 1-\gamma) + h'$. $\square$

A proof of theorem A. — Keeping notations as in the previous proposition, we now assume that (X, μ, T) is a K-process. For every integer $m > 0$

$$H(\alpha, \beta, \gamma) \geq h(\mathcal{P}, T^m) = H\Big(\mathcal{P} \mid \bigvee_{k=1}^{\infty} T^{-mk}\mathcal{P}\Big)$$

$$\geq H\Big(\mathcal{P} \mid \bigvee_{k=m}^{\infty} T^{-k}\mathcal{P}\Big) \xrightarrow[m\to\infty]{} H(\mathcal{P} \mid \tau),$$

where $\tau = \bigcap_{m=1}^{\infty} \bigvee_{k=m}^{\infty} T^{-k}\mathcal{P}$ is the tail field, which by assumption is trivial. Thus :

$$H(\mathcal{P} \mid \tau) = H(\mathcal{P}) = H(\alpha, \beta, \gamma).$$

Therefore given $\epsilon > 0$, there exists an integer m with $h(\mathcal{P}, T^m) > H(\mathcal{P}) - \epsilon$. Since the dynamical system (X, T^m) is u.p.e. if and only if (X, T) is u.p.e., we now assume for convenience's sake and with no loss of generality, that $h = h(\mathcal{P}, T) > H(\alpha, \beta, \gamma) - \epsilon$.

We have, by PROPOSITION 3.1,

$$H(\alpha, \beta, \gamma) - \epsilon \leq h \leq h' + H(\gamma, 1 - \gamma).$$

Now if (X, T) is not u.p.e., there exists a standard $\mathcal{U}$ for which $h' = h_{\text{top}}(\mathcal{U}, T) = 0$ and since ϵ is arbitrary, we get

$$H(\alpha, \beta, \gamma) \leq H(\gamma, 1 - \gamma),$$

which is absurd since α and β are positive. This completes the proof of THEOREM A. $\square$

REMARK. — Using Jewett-Krieger theorem we can, given an arbitrary K-process (Ω, m, T), find a strictly ergodic dynamical system (X, μ, T) which is measure theoretically isomorphic to (Ω, m, T). Applying theorem A we now deduce that (X, μ, T) is also u.p.e. This answers F. BLANCHARD's question about the existence of minimal u.p.e. systems.

Of course it is natural to ask whether every minimal (or strictly ergodic) u.p.e. dynamical system necessarily admits an invariant measure with respect to which it is a K-process. We shall see in section 5 that there are many examples of strictly ergodic u.p.e. dynamical systems which do not admit such measures.

Another corollary of THEOREM A is the existence of a universal minimal u.p.e. system, in the following sense.

COROLLARY 3.2. — *There exists a minimal u.p.e. dynamical system (X,T) with the property that for every ergodic process (Ω,m,T), there is an invariant probability measure μ on X such that the processes (X,μ,T) and (Ω,m,T) are measure theoretically isomorphic.*

Proof. — In [W1] a minimal system (X,T) is constructed which has the property stated in the corollary. Since for some μ, (X,μ,T) is a K-process, THEOREM A implies that (X,T) is also u.p.e. $\square$

4. Weakly solid extensions preserve u.p.e.

PROPOSITION 4.1.

1) *A minimal almost* 1-1 *extension is solid.*
2) *A minimal proximal extension is weakly solid.*
3) *A solid extension is weakly solid.*

Proof.

1) In fact minimal almost 1-1 extensions are characterized by a much stronger property which is purely topological : if $(X,T) \xrightarrow{\pi} (Y,T)$ is a homomorphism of minimal systems then π is almost 1-1 if and only if whenever $A \subset X$ is a closed subset with $\pi[A] = Y$ then $A = X$.

Suppose π is almost 1-1, then the sets $Y_0 = \{y \in Y : |\pi^{-1}(y)| = 1\}$ and $\pi^{-1}[Y_0] = X_0$ are dense G_δ subsets of X and Y respectively. (It is for the density of X_0 that we need the minimality.) Thus if $\pi[A] = Y$ then necessarily $X_0 \subset A$ and $A = X$, since A is closed.

Conversely, assume our condition holds and let $V \neq \emptyset$ be an open set in X. Then $A = V^c \neq X$ is closed, and our condition implies that $A \cap \pi^{-1}(y) = \emptyset$ for some $y \in Y$; i.e. $\pi^{-1}(y) \subset V$. Now the map $\pi^{-1} : Y \to 2^X$ is upper-semi-continuous and therefore has a dense G_δ subset $Y_0 \subset Y$ of continuity points. If $y_0 \in Y_0, x_0 \in \pi^{-1}(y_0)$, and $V_n \searrow \{x_0\}$ is a decreasing sequence of open balls around x_0, then there exists a sequence $\{y_n\} \subset Y$ with $\pi^{-1}(y_n) \subset V_n$. Clearly $y_n \to y_0$ and the continuity of π^{-1} at y_0 yields :

$$\{x_0\} = \lim \pi^{-1}(y_n) = \pi^{-1}(y_0).$$

Thus $|\pi^{-1}(y)| = 1$ for every $y \in Y_0$ and π is almost 1-1.

2) Let $(X,T) \xrightarrow{\pi} (Y,T)$ be a proximal extension of minimal systems. Let $W \subset X \times X$ be a closed $T \times T$-invariant set with $\pi \times \pi[W] = Y \times Y$. Given $n \in \mathbb{Z}$, let x_0 be some point in X. By assumption there exists

a point $(x_1, x_2) \in W$ with $\pi \times \pi(x_1, x_2) = \pi \times \pi(x_0, T^n x_0)$. Since π is proximal, there exists a unique minimal set in the orbit closure of (x_1, x_2) in $X \times X$ and this has to be $\Delta_n = \{(x, T^n x) : x \in X\}$. Thus $\Delta_n \subset W$, and since $\bigcup_{n\in\mathbb{Z}} \Delta_n$ is dense in $X \times X$, we deduce $W = X \times X$.

3) Let $(X,T) \xrightarrow{\pi} (Y,T)$ be a solid extension of minimal systems, and let $W \subset X \times X$ be a closed $T \times T$-invariant set with $\pi \times \pi[W] = Y \times Y$. For each $y' \in Y$, let :

$$W_{y'} = \{x \in X : \exists (x, x') \in W,\ \pi(x') = y'\}.$$

Denoting by p_1 the projection on the first component, we have

$$W_{y'} = p_1\big[(\pi \times \pi)^{-1}[Y \times \{y'\}] \cap W\big],$$

and by our assumption $\pi[W_{y'}] = Y$. Since π is solid, there exists a sequence n_i with $\lim T^{n_i} W_{y'} = X$. Without loss of generality we may assume that $y'' = \lim T^{n_i} y'$ exists. Choose m_i such that $\lim T^{m_i} y'' = y'$ and then, iterating T^{n_i} and T^{m_i}, choose a sequence k_i for which $\lim T^{k_i} y' = y'$ and $\lim T^{k_i} W_{y'} = X$.

Since W is $T \times T$-invariant, we have

$$X = \lim T^{k_i} W_{y'} = \lim W_{T^{k_i} y'} \subset W_{y'},$$

whence $W_{y'} = X$. Thus given $y' \in Y$ and $x \in X$ there exists $x' \in X$ such that $\pi x' = y'$ and $(x, x') \in W$. Fixing x we see that

$$W^x = \{x' \in X : (x, x') \in W\}$$

satisfies $\pi[W^x] = Y$. Using solidity again, choose a sequence ℓ_i with $\lim T^{\ell_i} x = x$ and $\lim T^{\ell_i} W^x = X$. Then also

$$X = \lim T^{\ell_i} W^x = \lim W^{T^{\ell_i} x} \subset W^x,$$

and $W^x = X$. Since x was arbitrary we get $W = X \times X$. □

A proof of theorem B. — Denote by E_X and E_Y the sets of entropy pairs in $X \times X$ and $Y \times Y$ respectively; then by assumption $\overline{E}_Y = Y \times Y$ and hence :

$$\pi \times \pi\big[\overline{E}_X\big] = \overline{E}_Y = Y \times Y.$$

Since $\overline{E}_X$ is closed and $T \times T$-invariant and since π is weakly solid, we have $\overline{E}_X = X \times X$ and the proof is complete. □

COROLLARY 4.2. — *The class of minimal u.p.e. dynamical systems is closed under minimal proximal extensions.*

Proof. — This corollary follows immediately from THEOREM B and from PROPOSITION 4.1 □

5. The representation theorem

Our main tool in proving THEOREM C is the following construction of FURSTENBERG and WEISS, [F-W].

THEOREM 5.1. — *Let (Y,T) be a non-periodic dynamical system and let $(\overline{X},T) \xrightarrow{\bar\pi} (Y,T)$ be an extension of (Y,T), where $(\overline{X},T)$ is topologically transitive. Then there exist an almost* 1-1 *extension $(X,T) \xrightarrow{\pi} (Y,T)$, a Borel subset $\overline{X}_0 \subset \overline{X}$ and a Borel measurable map $\overline{X}_0 \xrightarrow{\theta} X$ satisfying :*

1) $\theta T = T\theta$,
2) $\pi\theta = \bar\pi$,
3) *θ is a Borel isomorphism of $\overline{X}_0$ onto its image X_0 in X, and*
(4) *$\mu(\overline{X}_0) = 1$ for any T-invariant measure μ on $\overline{X}$.*

REMARK. — It is important to note that from the proof of THEOREM 5.1 given in [F-W], one can deduce some additional information on the structure of the space of the T-invariant measures on X. In particular it can be seen that if $(\overline{X},T)$ is uniquely ergodic, one can construct (X,T) as above with the further property that it is itself uniquely (hence strictly) ergodic.

We are now ready to prove THEOREM C.

A proof of theorem C. — We start with an ergodic process (Ω,m,T) with $h_\mu(T) > 0$. By Sinai's theorem, a factor map

$$(\Omega,m,T) \xrightarrow{\phi} (\Omega',m',T)$$

exists where the process (Ω',m',T) is Bernoulli and in particular a K-process. Using a relative version of Jewett-Krieger theorem, [W2], we can find a (continuous) homomorphism of strictly ergodic dynamical systems $(\overline{X},\bar\mu,T) \xrightarrow{\pi} (Y,\nu,T)$ such that the diagram

$$\begin{array}{ccc}(\Omega,m,T) & \xrightarrow{\quad\phi\quad} & (\Omega',m',T)\\ \updownarrow & & \updownarrow\\ (\overline{X},\bar\mu,T) & \xrightarrow{\quad\bar\pi\quad} & (Y,\nu,T)\end{array}$$

is commutative and the double edged arrows denote measure theoretical isomorphisms. By THEOREM A then, the system (Y,ν,T) is u.p.e.

Next use THEOREM 5.1 (and the remark that follows it) to construct a commutative diagram

$$\begin{tikzcd} (X,\mu,T) \arrow[dr, "\pi"'] & & (\overline{X},\bar{\mu},T) \arrow[ll, "\theta"'] \arrow[dl, "\bar{\pi}"] \\ & (Y,\nu,T) & \end{tikzcd}$$

where (X,μ,T) is strictly ergodic, π is an almost 1-1 extension and θ, defined on a full-measure Borel subset $\overline{X}_0 \subset X$ is a Borel isomorphism of $\overline{X}_0$ onto its image $X_0 \subset X$.

Finally we use THEOREM B to deduce that (X,μ,T) is a strictly ergodic u.p.e. dynamical system which is measure theoretically isomorphic to the original process (Ω,m,T). $\square$

REMARKS

1) By the variational principle, a strictly ergodic, u.p.e. dynamical system (X,μ,T) satisfies $h_\mu(T) > 0$. Thus, in fact, THEOREM C gives a necessary and sufficient condition for an ergodic process (Ω,m,T) to possess a strictly ergodic u.p.e. model; namely that it has positive (measure theoretical) entropy.

2) In [L], E. LEHRER proved a version of the Jewett-Krieger theorem which provides a topologically mixing, strictly ergodic model for every ergodic process. By COROLLARY 3.2 ,every ergodic process possesses a minimal,topologically mixing, u.p.e. model. Is it true that every minimal u.p.e. system is necessarily topologically mixing? (In [B2] there are examples of (non-minimal) u.p.e. systems which are not topologically mixing.)

COROLLARY 5.2. — *There exists a strictly ergodic, u.p.e. system (X,μ,T) which, as a process, is not K.*

Proof. — Let (Y,ν,T) be some strictly ergodic system which is a K-process. Let (Z,T) be an irrational rotation of the circle and let λ denote Lebesgue's measure on Z. Put

$$(\overline{X},\bar{\mu},T) = (Y \times Z, \nu \times \lambda, T \times T)$$

and let $\bar{\pi} : \overline{X} \to Y$ be the projection. Now apply THEOREM 5.1 to obtain the diagram

$$\begin{tikzcd}(X,\mu,T) \arrow[dr, "\pi"'] & & (\bar{X},\bar{\mu},T) \arrow[ll, "\theta"'] \arrow[dl, "\bar{\pi}"] \\ & (Y,\nu,T) & \end{tikzcd}$$

(X,μ,T) is then a strictly ergodic, u.p.e. system which is measure theoretically isomorphic with $(Y\times Z,\nu\times\lambda,T\times T)$ and hence not a K-process. $\square$

6. Joinings

In [B2] the question whether the product of two u.p.e. systems is also u.p.e. is posed. (Remark that the analogous question is open even for the topological mixing property; there however, FURSTENBERG has shown that if two systems are weakly mixing and at least one of them is minimal then their product is also weakly mixing, [F].) Here we consider the related question about the nature of joinings of two minimal u.p.e. systems. We show that there exist two minimal u.p.e. systems and a minimal joining of the two (i.e. a minimal subset of the product system) which is not even weakly mixing. In fact using the machinery developed in [G-W] and [G] and the basic idea of [P], we only have to draw some straightforward conclusions concerning the u.p.e. case.

A proof of Theorem D. — We recall the following setup from [G]. (Z,σ) is an arbitrary (metric) minimal dynamical system; Y a compact metric space. The space of self-homeomorphisms of Y equipped with the topology of uniform convergence of homeomorphisms and their inverses, is a polish topological group, denoted by $\mathcal{H}(Y)$. We assume the existence of a pathwise connected, closed subgroup $\mathcal{G}$ of $\mathcal{H}(Y)$ which acts minimally on Y.

Let $X = Z\times Y$ and let $X \xrightarrow{\pi} Z$ be the projection. With every continuous map $z\mapsto g_z$ of Z into $\mathcal{G}$, associate a homeomorphism G of X onto itself given by :

$$G(z,y) = (z, g_z y).$$

Identify σ with the map $\sigma\times \mathrm{id}_Y$ and put :

$$\mathcal{S}_{\mathcal{G}}(\sigma) = \big\{G^{-1}\circ\sigma\circ G : G \text{ as above}\big\}.$$

Since every element of $\mathcal{S}_{\mathcal{G}}(\sigma)$ has the form

$$G^{-1}\circ\sigma\circ G = \big(\sigma z, g_{\sigma z}^{-1} g_z y\big),$$

it follows that every $T \in \bar{\mathcal{S}}_{\mathcal{G}}(\sigma)$ has the form

$$T(z,y) = (\sigma z, h_z y),$$

for some continuous map $z \to h_z$ of Z into $\mathcal{G}$.

The following result is a stronger version of proposition 1.1 in [G]. A proof can be obtained by a slight reinforcement of the proof of that proposition.

THEOREM 6.1. — *Let (Z,σ) be a metric minimal dynamical system, Y a compact metric space and $\mathcal{G}$ a pathwise connected subgroup of $\mathcal{H}(Y)$ such that $(Y,\mathcal{G})$ is minimal. Then there exists a dense G_δ subset $\mathcal{R}$ of $\bar{\mathcal{S}}_{\mathcal{G}}(\sigma)$ such that for every $T \in \mathcal{R}$ the dynamical system (X,T) is minimal and the extension $(X,T) \xrightarrow{\pi} (Z,\sigma)$ is solid.*

Now let (Z,σ) be a fixed metric minimal u.p.e. system. Put

$$Y = K = \{y \in \mathbb{C} : |y| = 1\}$$

and let $\mathcal{G} = K$, acting on itself as a group of translations. Then clearly $(Y,\mathcal{G})$ is minimal and THEOREM 6.1 applies. If $T \in \bar{\mathcal{S}}_{\mathcal{G}}(\sigma)$ has the form $T(z,y) = (\sigma z, h_z y)$ where $z \to h_z$ is a continuous map from $K = Z$ to itself, let :

$$\widehat{T}(z,y) = \big(\sigma z, (-1)h_z y\big).$$

Let $\mathcal{R}$ be the subset of $\bar{\mathcal{S}}_{\mathcal{G}}(\sigma)$ given by THEOREM 6.1. It is easy to see that for $T \in \mathcal{R}$ (since (X,T) is weakly mixing) the system $(X,\widehat{T})$ is also minimal. Since $\mathcal{R}$ is a residual subset of $\bar{\mathcal{S}}_{\mathcal{G}}(\sigma)$, so is $\widehat{\mathcal{R}}$ and we can choose $T \in \mathcal{R} \cap \widehat{\mathcal{R}}$. Then the systems (X,T) and $(X,\widehat{T})$ are minimal and u.p.e.

Let now $V \subset X \times X$ be the set :

$$V = \big\{\big((z,y),(z,\pm y)\big) : (z,y) \in X\big\}.$$

Then clearly V is $T \times \widehat{T}$-invariant. Moreover the function $\phi : V \to K$,

$$\phi\big((z,y),(z,y')\big) = y^{-1}y'$$

is an eigenfunction of $(V, T \times \widehat{T})$ with eigenvalue -1. From this we easily deduce that $(V, T \times \widehat{T})$ is isomorphic to the product system $(X \times \{1,-1\}, T \times \text{flip})$, which is minimal and non-weakly-mixing. Thus $(V, T \times \widehat{T})$ provides an example of a non u.p.e. minimal joining of the minimal u.p.e. systems (X,T) and $(X,\widehat{T})$. This completes the proof of THEOREM D. $\square$

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