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SMOOTHNESS AND IRREDUCIBILITY OF VARIETIES OF PLANE CURVES WITH NODES AND CUSPS

BY

EUGENII SHUSTIN (*)

RÉSUMÉ. — Soit $V(d, m, k)$ la variété des courbes projectives planes irréductibles de degré d n'ayant pour singularités que m nodes et k cusps. Nous montrons que $V(d, m, k)$ est non vide, lisse et irréductible quand $m + 2k < \alpha d^2$ où α est une constante absolue explicite. Cette inégalité est optimale quant à l'exposant de d

ABSTRACT. — Let $V(d, m, k)$ be the variety of plane projective irreducible curves of degree d with m nodes and k cusps as their only singularities. We prove that $V(d, m, k)$ is non-empty, non-singular and irreducible when $m + 2k < \alpha d^2$, where α is some absolute explicit constant. This estimate is optimal with respect to the exponent of d

0. Introduction

In the present article we deal with plane projective algebraic curves over an algebraically closed field of characteristic 0.

It is well-known that the variety of irreducible curves of a given degree with a given number of nodes is non-singular [9], irreducible [2], and that each germ of this variety is a transversal intersection of germs of equisingular strata corresponding to all singular points [9] (from now on, speaking of a variety with the last property, we shall write T-*variety*, or *variety with property* T).

Our goal is a similar result for curves with nodes and ordinary cusps. Let $V(d, m, k)$ denote the set of irreducible curves of degree d with m nodes and k cusps as their only singularities.

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- It is known (see [5], [6]) that $V(d,m,k)=\emptyset$ if

$$\frac{9}{8}m+2k>\frac{5}{8}d^2.$$

- On the other hand (see [13]), $V(d,m,k)\neq\emptyset$ when

$$m+2k\leq\frac{1}{2}d^2+\mathrm{O}(d). \tag{0.1}$$

Our result is

THEOREM 0.2.

- *If* $m+2k\leq\alpha_0 d^2$, *where*

$$\alpha_0=\frac{7-\sqrt{13}}{81}\approx 0.0419, \tag{0.3}$$

then $V(d,m,k)$ *is a non-empty non-singular* T*-variety of dimension*

$$\tfrac{1}{2}d(d+3)-m-2k.$$

- *If* $m+2k\leq\alpha_1 d^2$, *where*

$$\alpha_1=\frac{2}{225}\approx 0.0089, \tag{0.4}$$

then $V(d,m,k)$ *is irreducible.*

Let us make some comments.

First, (0.3) implies (0.1), and then $V(d,m,k)\neq\emptyset$.

Let $\mathbb{P}^N$, with $N=\frac{1}{2}d(d+3)$, be the space of plane curves of degree d. Let z be a singular point of $F\in\mathbb{P}^N$. It is well-known (see [2, [9]) that :

(1) if z is a node then the germ at F of the variety of curves $\Phi\in\mathbb{P}^N$, having a node in some neighbourhood of z, is smooth, has codimension 1, and its tangent space is open in $\{\Phi\in\mathbb{P}^N\mid z\in\Phi\}$;

(2) if z is a cusp then the germ at F of the variety of curves $\Phi\in\mathbb{P}^N$, having a cusp in some neighbourhood of z, is smooth, has codimension 2, and its tangent space is open in $\big\{\Phi\in\mathbb{P}^N\mid(\Phi\cdot F)(z)\geq 3\big\}$ (here and further on the notation $(F\cdot G)(z)$ means the intersection number of the curves F,G at the point z).

Hence the property T implies the smoothness of $V(d,m,k)$ and the expected value of its dimension given in THEOREM 0.2. Further, it is well-known [15] that $V(d,m,k)$ is a non-singular T-variety, when

$$k<3d. \tag{0.5}$$

Generalizations of this fact to arbitrary singularities, given in [1], [12] are based — in fact — on the same idea. The following conditions are sufficient for the smoothness of $V(d,m,k)$ and the property T :

$$m=0, \quad 2k<\frac{(7-\sqrt{13})d^2}{81}\approx 0.0418\,d^2 \qquad \text{(see [10], [11])},$$

and for the irreducibility of $V(d,m,k)$:

$$m+2k<\frac{3}{2}d \qquad \text{(see [10], [11])},$$

$$k\le 3 \qquad \text{(see [7])},$$

$$\tfrac{1}{2}(d^2-4d+1)\le m\le \tfrac{1}{2}(d^2-3d+2) \quad \text{(see [8])}.$$

The main idea of our proof is as follows. We have to prove the property T and the irreducibility for $V(d,m,k)$. To any curve $F\in V(d,m,k)$ with nodes $z_1,\dots,z_m$ and cusps $w_1,\dots,w_k$ we assign two linear systems of curves of degree n :

$$\Lambda_1(n,F)=\{\Phi\mid z_1,\dots,z_m\in\Phi,\ (\Phi\cdot F)(w_i)\ge 3,\ i=1,\dots,k\},$$

$$\Lambda_2(n,F)=\{\Phi\mid z_1,\dots,z_m\in \operatorname{Sing}(\Phi),\ (\Phi\cdot F)(w_i)\ge 6,\ i=1,\dots,k\}.$$

First we show the non-speciality of $\Lambda_1(d,F)$ for any $F\in V(d,m,k)$, which means according to the Riemann-Roch theorem that

$$\dim\Lambda_1(d,F)=\tfrac{1}{2}d(d+3)-m-2k. \tag{0.6}$$

On the other hand, $\Lambda_1(d,F)$ is the intersection of the tangent spaces to germs of equisingular strata at F in the space of curves of degree d, and (0.6) gives us the transversality of this intersection, or the desired property T. Then we show that, for any F from some open dense subset $U\subset V(d,m,k)$, the system $\Lambda_2(d,F)$ is non-special. That implies the irreducibility. Indeed, first we show that an open dense subset of $\Lambda_2(d,F)$ is contained in $V(d,m,k)$; more precisely, it consists of curves of degree d having m nodes in a fixed position and k cusps in a fixed position with fixed tangents. Then from the non-speciality we derive that $\dim\Lambda_2(d,F)=\text{const}$, $F\in U$, and that conditions imposed by fixed singular points on curves of degree d are independent. Afterwards we represent U as an open dense subset of the space of some linear bundle, whose fibres are $\Lambda_2(d,F)$, $F\in U$, and whose base is an open dense subset of $\operatorname{Sym}^m(\mathbb{P}^2)\times\operatorname{Sym}^k(P(T\mathbb{P}^2))$, where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane.

The text is divided into five parts : in section 1 there are some preliminary notions and results; in section 2 we present examples of reducible varieties or ones without property T; in section 3 we construct irreducible curves in $\Lambda_1(n, F)$, where $n < d$; in section 4 we prove the property T; and in section 5 — the irreducibility.

1. Preliminaries

Here we shall recall some notions and well-known classical results [3], [14], and also present some simple technical results needed below. Namely, we introduce a certain class of linear systems of plane curves and show how to compute their dimensions by means of linear series on curves.

Let $\Sigma = \bigoplus_{t\geq 0} \Sigma(t)$ be the graded ring of polynomials in three homogeneous variables over the base field. We think of the space of plane curves of degree t as the projectivization $P(\Sigma(t))$. A linear system of plane curves of degree t is a subspace of $P(\Sigma(t))$. Let

$$I = \bigoplus_{t\geq 0} I(t) \subset \Sigma$$

be a homogeneous ideal, defining a zero-dimensional subscheme $Z \subset \mathbb{P}^2$. This ideal determines a sequence of linear systems $\Lambda(t) = P(I(t))$, $t \geq 1$. Denote this class of linear systems by $\mathcal{C}$. In other words, these are linear systems defined by linear conditions associated to finite many base points. It is well-known [3] that

$$\dim \Lambda(t) = \dim P\big(\Sigma(t)\big) - \deg Z + i\big(\Lambda(t)\big), \tag{1.1}$$

where $i(\Lambda(t)) \geq 0$ is called the *speciality index* of $\Lambda(t)$. If $i(\Lambda(t)) = 0$ then the linear system $\Lambda(t)$ is called *non-special.* For a given ideal I, $\Lambda(t)$ is non-special when t is big enough (see [3]).

PROPOSITION 1.2. — *Let $\Lambda(t), \Lambda'(t)$ belong to $\mathcal{C}$, and, for all $t \geq 1$,*

$$\Lambda(t) \subset \Lambda'(t).$$

If, for some $n \geq 1$, the system $\Lambda(n)$ is non-special then $\Lambda'(n)$ is non-special.

Proof. — The systems $\Lambda(t), \Lambda'(t)$ are non-special for t big enough. Take a straight line $L \in P(\Sigma(1))$ not intersecting the zero-dimensional schemes Z, Z' associated to our linear systems. Let us embed the space $P(\Sigma(n))$ into $P(\Sigma(t))$, multiplying by L^{t-n}. Then :

$$\Lambda(n) = \Lambda(t) \cap P\big(\Sigma(n)\big), \quad \Lambda'(n) = \Lambda'(t) \cap P\big(\Sigma(n)\big).$$

So, the non-speciality of $\Lambda(n)$ means the transversality of the intersection of $\Lambda(t)$ and $P(\Sigma(n))$ in $P(\Sigma(t))$. But this implies that $\Lambda'(t)$ and $P(\Sigma(n))$ intersect transversally in $P(\Sigma(t))$, hence

$$\operatorname{codim}\big(\Lambda'(n), P(\Sigma(n))\big) = \operatorname{codim}\big(\Lambda'(t), P(\Sigma(t))\big) = \deg Z',$$

what is equivalent to the desired non-speciality. ∎

From now on, divisor will always mean an effective Cartier divisor on a curve.

Let F be a reduced plane curve. For any divisor D on F and any component $H \subset F$ the symbol $D_{|H}$ means the restriction of D on H. For any curve G the symbol G_F means the formal expression $\sum n(P)\cdot P$, where the sum is taken over all local branches of F and $n(P)$ is the intersection number of P and G. If F, G have no common components, G_F is the divisor on F cut out by G, otherwise we admit infinite coefficients in the above expression.

By $D(F)$ we denote the double point divisor of the curve F. We omit its exact definition (see, for example, [14]), but only list the properties used in the sequel.

PROPOSITION 1.3 (see [14]).

(1) *The divisor $D(F)$ can be expressed as*

$$D(F) = \sum n(P) \cdot P,$$

where P runs through all the local branches of F centered at singular points, and the coefficients $n(P)$ are positive integers. In particular, $n(P) = 1$ for both branches centered at a node, and $n(P) = 2$ for a branch centered at a cusp.

(2) *Let z be a singular point of the curve F and a non-singular point of some curve G, then*

(i) *for any singular local branch P of F centered at z,*

$$(G \cdot P)(z) \leq n(P) + 1,$$

(ii) *for any pair P_1, P_2 of local branches of F centered at z,*

$$(G \cdot P_1) \leq n(P_1), \quad \text{or} \quad (G \cdot P_2) \leq n(P_2).$$

(3) *If F is an irreducible curve of degree d and geometric genus $g(F)$ then :*

$$\deg D(F) = d(d-3) + 2 - 2g(F).$$

(4) *If a reduced curve G has no common components with F then :*

$$D(FG)_{|F} = D(F) + G_F.$$

For any divisor D on F, the symbol $\mathcal{L}_F(n, D)$ denotes the linear system of plane curves of degree n

$$\{\Phi \mid \Phi_F \geq D + D(F)\}.$$

It is clear from the definition and PROPOSITION 1.3 that $\Lambda_1(n, F)$, $\Lambda_2(n, F)$ belong to this class. Also these systems belong to $\mathcal{C}$.

THEOREM 1.4 (Brill-Nœther (see [14])). — *If F is irreducible then curves from $\mathcal{L}_F(n, D)$ cut out on F the linear series $|nL_F - D - D(F)|$, where L is a general straight line.*

THEOREM 1.5 (Nœther (see [14])). — *Let $F_1, \ldots, F_k$ be different irreducible curves of degrees $n_1, \ldots, n_k$, and $F = F_1 \cdots F_k$,* deg $F = d$. *Then :*

$$(1.6) \qquad \mathcal{L}_F(n, D) = \sum_{i=1}^{k} \mathcal{L}_{F_i}\big(n + n_i - d, D_{|F_i}\big) \cdot F_1 \cdots F_{i-1} F_{i+1} \cdots F_k.$$

THEOREM 1.7 (Riemann-Roch for curves (see [3], [14])). — *For any divisor D on an irreducible curve F the dimension of the linear series $|D|$ is*

$$\dim |D| = \deg D - g(F) + i(D),$$

where $i(D)$ is non-negative. If $\deg D > 2g(F) - 2$ *then* $i(D) = 0$.

PROPOSITION 1.8. — *For any reduced curve F of degree $d \leq n$,*

$$(1.9) \qquad \dim \mathcal{L}_F(n, D) \geq \tfrac{1}{2} n(n+3) - \tfrac{1}{2} \deg D(F) - \deg D.$$

The non-speciality of $\mathcal{L}_F(n, D)$ is equivalent to the equality in (1.9).

Proof. — Assume that F is irreducible. Representing $\mathcal{L}_F(n, D)$ as the span of $|nL_F - D - D(F)|$ and $F \cdot P(\Sigma(n-d))$, we obtain

$$\dim \mathcal{L}_F(n, D) = \dim \big| nL_F - D - D(F) \big| + \dim \Sigma(n-d),$$

hence according to THEOREM 1.7 and PROPOSITION 1.3,

$$\begin{aligned}
\dim \mathcal{L}_F(n, D) &\geq nd - \deg D - \deg D(F) - g(F) \\
&\qquad\qquad + \tfrac{1}{2}(n-d+1)(n-d+2) \\
&= nd - \tfrac{1}{2} d(d-3) - 1 - \deg D - \tfrac{1}{2} \deg D(F) \\
&\qquad\qquad + \tfrac{1}{2}(n-d+1)(n-d+2),
\end{aligned}$$

which is equivalent to (1.9). Also we obtain that the equality in (1.9) means $i(D) = 0$. Therefore, for all $t \geq d$,

$$\text{codim}(\mathcal{L}_F(t, D), P(\Sigma(t))) = \tfrac{1}{2} \deg D(F) + \deg D. \tag{1.10}$$

On the other hand, for t big enough, $\mathcal{L}_F(t, D)$ is non-special. Comparing this with (1.1) and (1.10), we get that the equality in (1.9) means the non-speciality of $\mathcal{L}_F(n, D)$.

If F is reducible, combine the previous computation with (1.6).

PROPOSITION 1.11. — *Let $F \in V(d, m, k)$.*

(1) *If $G \in \Lambda_1(n, F)$ is reduced, then there is a divisor D on G of degree $\leq m + 2k$ such that, for all $t \geq 1$,*

$$\Lambda_1(t, F) \supset \mathcal{L}_G(t, D). \tag{1.12}$$

(2) *Let $G \in \Lambda_1(n, F)$ be irreducible, let S be a subset of* $\text{Sing}(F)$, *and let H be a reduced curve containing S but not G. Let $\Lambda_3(t, F, S)$ be a linear system of curves $\Phi \in \Lambda_1(t, F)$ such that $S \subset$* $\text{Sing}(\Phi)$, *and Φ meets F at each cusp from S with multiplicity ≥ 5. Then there is a divisor D on GH such that*

$$\deg D_{|G} \leq m + 2k, \quad \deg D_{|K} \leq \text{card}(S \cap K),$$

for each component $K \subset H$, and, for all $t \geq 1$,

$$\Lambda_3(t, F, S) \supset \mathcal{L}_{GH}(t, D).$$

Proof. — We will construct the divisor $D = \sum n(P) \cdot P$ explicitly.

(1) We have to find a divisor D on G such that any curve from $\mathcal{L}_G(t, D)$ goes through each node and each cusp of F, and intersects a tangent line to F at any cusp with multiplicity ≥ 2.

Let z be a node of F. Since $G \in \Lambda_1(n, F)$, then G goes through z. If G is non-singular at z we can put $n(P) = 1$ for the local branch P of G centered at z. If G is singular at z then we can put $n(P) = 0$ for all local branches of G centered at z, because in this case, according to PROPOSITION 1.3, curves from $\mathcal{L}_G(t, 0)$ go through z.

Let z be a cusp of F. Analogously, G goes through z. If G is non-singular at z, then the local branch P of G at z is tangent to the tangent line L to the curve F at z. Put $n(P) = 2$. Now, since any curve from $\mathcal{L}_G(p, D)$ intersects P with multiplicity ≥ 2, the same holds for L. If G is singular at z, then either there is a singular local branch P of G centered at z, or there are at least two local branches P_1, P_2 of G centered at z. In the first case we put $n(P) = 2$, in the second case we put $n(P_1) = n(P_2) = 1$. According to PROPOSITION 1.3 any curve from $\mathcal{L}_G(t, D)$ is singular at z, and thereby intersects L with multiplicity ≥ 2.

(2) We can obtain the second statement easily by combining the previous arguments with the Nœther theorem. □

2. Non-transversality and reducibility

The upper bounds in the sufficient conditions (0.3), (0.4) are the best possible as far as the exponent of d is concerned. The slightly modified classical examples [15] presented below give an upper bound for the allowable coefficient of d^2 in (0.3), (0.4).

THEOREM 2.1. — *The set $V(6p, 0, 6p^2)$ is reducible if $p = 1, 2$, and has components with different dimensions if $p \geq 3$.*

Proof. — The case $p = 1$ is well-known [15]. Let $p \geq 2$. It is easy to see that the curves

$$H = F_{2p}^3 + G_{3p}^2$$

belong to $V(6p, 0, 6p^2)$, where F_{2p}, G_{3p} are general curves of degrees $2p, 3p$ respectively. A simple computation gives us :

$$\begin{aligned} \dim\{H \in V(6p, 0, 6p^2) \mid H = F_{2p}^3 + G_{3p}^2\} \\ = \tfrac{1}{2}6p(6p + 3) - 12p^2 + \tfrac{1}{2}(p - 1)(p - 2). \end{aligned} \tag{2.2}$$

According to [13], for $p \geq 2$, there is a component of $V(6p, 0, 6p^2)$ with dimension :

$$\tfrac{1}{2}6p(6p + 3) - 12p^2.$$

If $p \geq 3$ we obtain at least two components of $V(6p, 0, 6p^2)$ with different dimensions.

Let $p = 2$. According to (0.5), $V(12, 0, 24)$ is a T-variety, and hence has dimension 42. According to (2.2) curves $H = F_4^3 + G_6^2$ form a component $\widetilde{V}$ of $V(12, 0, 24)$. Assume that $\widetilde{V} = V(12, 0, 24)$.

Let J be an irreducible curve of degree 12 with 28 cusps constructed in [13]. Since $V(12, 0, 28)$ is a T-variety (see (0.5)), we can smooth out any four cusps of J, preserving the others, by means of a variation of J in the space $P(\Sigma(12))$. Indeed, since all 28 equisingular strata intersect transversally at J, we can leave four of them by moving J along the intersection of the others. So we obtain that J belongs to the closure of $\widetilde{V}$, and hence to any set s_{24} of 24 cusps of J there correspond a quartic F_4 and a sextic G_6, passing through s_{24}. Distinct 24-tuples of cusps correspond to distinct quartics, because, according to Bézout's theorem, a quartic cannot contain more than 24 cusps of J. On the other hand, two 24-tuples s_{24}, s'_{24} with 23 common cusps give quartics F_4, F'_4 with 23 common points. Therefore F_4, F'_4 have a common component C_i of degree i =1, 2, or 3. If $i = 3$ then $F_4 = C_3C_1$, $F'_4 = C_3C'_1$. Since C_3 passes through at most 18 cusps of J, then the straight lines C_1, C'_1 have at least 5 common points, that means they coincide. The cases i =1 or 2 lead analogously to contradictions, which prove that $V(12, 0, 24)$ is reducible.

THEOREM 2.3. — *The set* $V(7p-3, 0, 6p^2)$ *contains a component without property* T *when* $p \geq 3$.

Proof. — Obviously, the curve $H = A_{p-3}F_{2p}^3 + B_{p-3}G_{3p}^2$ belongs to $V(7p-3, 0, 6p^2)$, if A_{p-3}, F_{2p}, B_{p-3}, G_{3p} are general curves of degrees $p-3$, $2p$, $p-3$, $3p$ respectively. The property T is equivalent to the non-speciality of $\Lambda_1(7p-3, H)$. From THEOREM 1.5 it is not difficult to deduce that

$$\Lambda_1(7p-3, H) = \{\Phi \mid \Phi = R_{3p-3}F_{2p}^2 + S_{4p-3}G_{3p}\}$$

with arbitrary curves R_{3p-3}, S_{4p-3} of degrees $3p-3, 4p-3$. Further, a trivial computation gives

$$\dim \Lambda_1(7p-3, H) = \tfrac{1}{2}(7p-3)\cdot 7p - 12p^2 + 1,$$

that means $\Lambda_1(7p-3, H)$ is special.

COROLLARY 2.4. — *The allowable coefficient at* d^2 *in the right hand side of* (0.3) *cannot exceed* $\frac{12}{49}$, *and in the right hand side of* (0.4) *cannot exceed* $\frac{1}{3}$.

3. Main lemma

LEMMA 3.1. — *For any curve* $F \in V(d, m, k)$ *and real* $\alpha \geq (m+2k)/d^2$, *there is an irreducible curve* $\Phi \in \Lambda_1(n, F)$, *where* $n = \left[(\sqrt{2\alpha} + 2/3)d\right]$.

Proof. — Let $z_1, \dots, z_m$ be the nodes of F, and let $w_1, \dots, w_k$ be the cusps of F. Let h be the minimal integer such that $\Lambda_1(h, F) \neq \emptyset$. Then, $\Lambda_1(h-1, F) = \emptyset$ implies

$$m + 2k > \tfrac{1}{2}(h-1)(h+2),$$

and hence

$$h < \sqrt{2(m+2k)} \leq \sqrt{2\alpha}\, d. \tag{3.2}$$

Take a general curve $H \in \Lambda_1(h, F)$. Assume that $H = H_1^{i_1} \cdots H_r^{i_r}$, where $H_1, \ldots, H_r$ are irreducible components of degrees $h_1, ..., h_r$ respectively. Since h is minimal,

$$\max\{i_1, \ldots, i_r\} \leq 2.$$

We shall construct the curve Φ as follows. First we will construct, for each $s = 1, \ldots, r$, a curve C_s of degree $\ell_s \leq i_s h_s + \frac{2}{3}d$ such that C_s does not contain H_s and the curve

$$R_s \overset{\text{def}}{=} H_1^{i_1} \cdots H_{s-1}^{i_{s-1}} C_s H_{s+1}^{i_{s+1}} \cdots H_r^{i_r} \tag{3.3}$$

belongs to $\Lambda_1(h + \ell_s - i_s h_s, F)$. After that we obtain the desired curve Φ in the form

$$G_0 H + G_1 R_1 + \cdots + G_r R_r,$$

where $G_0, G_1, \ldots, G_r$ are generic curves of suitable degrees.

The rest of the proof is divided into five steps : in steps 1, 2, 3 and 4 we construct the curves $C_1, \ldots, C_r$, in the fifth step we construct the curve Φ.

Let us do the construction of C_1. Let H_1 pass through $z_1, \ldots, z_p$, $w_1, \ldots, w_q$ and meet F at $w_{q+1}, \ldots, w_{q+t}$ with multiplicities ≥ 3. Let $\deg H_1 = h_1$. The Bézout theorem gives :

$$2p + 2q + 3t \leq h_1 d. \tag{3.4}$$

Step 1. — Assume $i_1 = 1$. Let us find a curve C_1 passing through $z_1, \ldots, z_p, w_1, \ldots, w_q$, meeting F at $w_{q+1}, \ldots, w_{q+t}$ with multiplicities ≥ 3, and not containing H_1. This can be done under the following sufficient condition on $\ell_1 = \deg C_1$

$$\tfrac{1}{2}\ell_1(\ell_1 + 3) - \tfrac{1}{2}(\ell_1 - h_1)(\ell_1 - h_1 + 3) > p + q + 2t,$$

which is equivalent to

$$\ell_1 > \tfrac{1}{2}h_1 - \tfrac{3}{2} + \frac{p + q + 2t}{h_1},$$

and, using (3.4), we can take

$$\ell_1 \leq \tfrac{1}{2}h_1 + \frac{p + q + 2t}{h_1} \leq \tfrac{1}{2}h_1 + \frac{p + q + 2t}{2p + 2q + 3t}d \leq \tfrac{1}{2}h_1 + \tfrac{2}{3}d. \tag{3.5}$$

Step 2. — From now on assume $i_1 = 2$. First we look for a curve C_1' of degree ℓ_1' passing through $z_1, \ldots, z_p$, meeting F at $w_{q+1}, \ldots, w_{q+t}$ with multiplicities ≥ 3 and not containing H_1. As in the first step we have the following sufficient condition for the existence of such a curve

$$\ell_1' > \tfrac{1}{2}h_1 - \tfrac{3}{2} + \frac{p+2t}{h_1},$$

and then we can take

$$\ell_1' \leq \tfrac{1}{2}h_1 + \frac{p+2t}{h_1}. \tag{3.6}$$

Step 3. — Assume that there is a curve $H_1' \neq H_1$ of degree $h_1' \leq h_1$, passing through $w_1, \ldots, w_q$, and that h_1' is the minimal such degree. This minimality implies :

$$h_1'^{\,2} \leq 2q. \tag{3.7}$$

Then we put $C_1 = C_1'(H_1')^2$. According to (3.6)

$$\ell_1 = \deg C_1 \leq \tfrac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h_1'.$$

Hence, using (3.4), (3.7) and the initial assumption $h_1' \leq h_1$, it is easy to compute

$$\begin{aligned} \ell_1 &\leq \tfrac{1}{2}h_1 + \frac{p+q+2t}{2p+2q+3t}d - \frac{q}{h_1} + 2h_1' \\ &\leq \tfrac{1}{2}h_1 + \tfrac{2}{3}d - \frac{q}{h_1} + 2h_1' \leq \tfrac{2}{3}d + 2h_1. \end{aligned} \tag{3.8}$$

Step 4. — Now assume that H_1 is the unique curve of degree $\leq h_1$, passing through $w_1, \ldots, w_q$. In particular, that means

$$\tfrac{1}{2}h_1(h_1+3) \leq q. \tag{3.9}$$

Let C_1' be the curve of degree ℓ_1' constructed in step 2. Consider two situations.

• Assume first $q \leq h_1^2$. Then $2q < 2h_1(2h_1+3)/2$. That means the set M of curves of degree $2h_1$, meeting F at $w_1, \ldots, w_q$ with multiplicities ≥ 3, is infinite. The only curve in M containing H_1 is H_1^2. Now take $H_1' \in M$ different from H_1^2, and put $C_1 = C_1'H_1'$. Here :

$$\ell_1 = \deg C_1 \leq \tfrac{1}{2}h_1 + \frac{p+2t}{h_1} + 2h_1.$$

Finally, from (3.4), (3.9) we get :

$$\ell_1 \leq \tfrac{2}{3}d + 2h_1. \tag{3.10}$$

- Now assume :

$$q \geq h_1^2 + 1. \tag{3.11}$$

Introduce the integer

$$h_1' = \max\{\nu \in \mathbb{N} \mid h_1(\nu - h_1) + 1 \leq q\}.$$

According to (3.11), $h_1' \geq 2h_1$. Denote by M the set of curves of degree h_1' meeting F with multiplicity ≥ 3 at each point $w_1, \dots, w_\pi$, where $\pi = h_1(h_1' - h_1) + 1$. If $G \in M$ contains H_1 then $G = G_1 H_1$, and G_1 goes through $w_1, \dots, w_\pi$, because

$$(G_1 \cdot F)(w_i) = (G \cdot F)(w_i) - (H_1 \cdot F)(w_i) \geq 3 - 2 = 1, \ i = 1, \dots, \pi.$$

Hence G_1 contains H_1, because these curves meet at $\pi > \deg H_1 \cdot \deg G_1$ points. Then there is a curve $H_1' \in M$ not containing H_1 as component, because the sufficient condition for this existence is

$$\tfrac{1}{2}h_1'(h_1' + 3) - 2(h_1(h_1' - h_1) + 1) > \tfrac{1}{2}(h_1' - 2h_1)(h_1' - 2h_1 + 3),$$

which is equivalent to

$$3h_1 > 2.$$

Finally, let H_1'' be a curve of the smallest degree h_1'' meeting F at w_i, $i = h_1(h_1' - h_1) + 2, \dots, q$, with multiplicities ≥ 3. By definition of h_1', the number of these points is $\leq h_1 - 1$. If $h_1 \leq 3$ then, obviously, $h_1'' \leq h_1 - 1$. If $h_1 \geq 4$, since

$$\tfrac{1}{2}(h_1'' - 1)(h_1'' + 2) \leq 2h_1 - 2,$$

we have :

$$h_1'' \leq \tfrac{5}{6}h_1. \tag{3.12}$$

The last inequality is true in the case $h_1 \leq 3$ too. In particular, H_1'' does not contain H_1. Now put $C_1 = C_1' H_1' H_1''$. Here we have from (3.4), (3.6), and (3.12)

$$\begin{aligned} \ell_1 = \deg C_1 &\leq \tfrac{1}{2}h_1 + \frac{p + 2t}{h_1} + h_1' + h_1'' \\ &= \tfrac{1}{2}h_1 + \frac{p + \frac{4}{3}q + 2t}{h_1} + h_1' + h_1'' - \frac{4q}{3h_1} \\ &\leq \tfrac{1}{2}h_1 + \tfrac{2}{3}d + h_1' + h_1'' - \tfrac{4}{3}(h_1' - h_1) \\ &\leq \tfrac{8}{3}h_1 + \tfrac{2}{3}d - \tfrac{1}{3}h_1' \leq 2h_1 + \tfrac{2}{3}d, \end{aligned} \tag{3.13}$$

because $h_1' \geq 2h_1$ as it was mentioned above.

Step 5. — Inequalities (3.2), (3.5), (3.8), (3.10), (3.13) and the definitions of n and α imply that the degree of any curve R_j, $j = 1, \dots, r$, defined by (3.3), is less than n. Now consider the linear system

$$\lambda_0 G_0 H + \lambda_1 G_1 R_1 + \cdots + \lambda_r G_r R_r \,, \quad (\lambda_0, \dots, \lambda_r) \in \mathbb{P}^r, \tag{3.14}$$

where $G_0, \dots, G_r$ are generic curves of *positive* degrees $n - h$, $n - \deg R_1$, $\dots, n - \deg R_r$ respectively. According to the construction of $H, C_1, \dots, C_r$, this is a subsystem of $\Lambda_1(n, F)$. Also note that the curves $G_0, \dots, G_r$ do not go through base points of our linear system. Then we obtain immediately from the Bertini theorem (see [3], [14]) that a generic member in the linear system (3.14) is reduced and irreducible. Thereby we can take this generic member as the desired curve Φ.

4. The property T

Let $F \in V(d, m, k)$. As said above, the property T and the smoothness of $V(d, m, k)$ at F follow from :

PROPOSITION 4.1. — *Under condition* (0.3) *the linear system* $\Lambda_1(d, F)$ *is non-special.*

Proof. — According to LEMMA 3.1, under condition (0.3) there is an irreducible curve $\Phi \in \Lambda_1(n, F)$, $n = [\sqrt{2\alpha_0} + \frac{2}{3} d]$. According to PROPOSITION 1.11 there is a divisor D on Φ of degree

$$\deg D \le m + 2k \tag{4.2}$$

such that

$$\Lambda_1(p, F) \supset \mathcal{L}_\Phi(p, D), \quad p \ge 1.$$

Therefore, according to PROPOSITION 1.2, it is enough to establish the non-speciality of $\mathcal{L}_\Phi(d, D)$, which will follow from

$$\deg\big(G_\Phi - D(\Phi) - D\big) > 2g(\Phi) - 2 \tag{4.3}$$

where $G \in \mathcal{L}_\Phi(d, D)$, and $g(\Phi)$ is the geometric genus of Φ. Indeed, we have by PROPOSITION 1.3 and (4.2) :

$$\begin{aligned} \deg(G_\Phi - D(\Phi) - D) &= nd - n(n-3) - 2 + 2g(\Phi) - \deg D \\ &\ge n(d - n + 3) - (m + 2k) + 2g(\Phi) - 2 \\ &> n(d - n + 3) - \alpha_0 d^2 + 2g(\Phi) - 2. \end{aligned} \tag{4.4}$$

Since $n = \big[(\sqrt{2\alpha_0} + \frac{2}{3})d\big]$ and α_0 is the positive root of the equation

$$\big(\sqrt{2\alpha} + \tfrac{2}{3}\big)\big(\tfrac{1}{3} - \sqrt{2\alpha}\big) = \alpha,$$

hence

$$n(d-n+3) \geq \alpha_0 d^2.$$

Then (4.4) implies (4.3) and completes the proof.

5. Irreducibility

We prove the irreducibility along the plan mentioned in introduction.

PROPOSITION 5.1. — *For any $F \in V(d,m,k)$, the intersection of $V(d,m,k)$ with $\Lambda_2(d,F)$ contains an open dense subset of $\Lambda_2(d,F)$, and consists exactly of curves from $V(d,m,k)$ with the same nodes, and the same cusps with the same tangents as F.*

Proof. — Since $F \in \Lambda_2(d,F)$ and any curve $G \in \Lambda_2(d,F)$ is singular at $z_1,\dots,z_m$, then the Bertini theorem implies that almost all curves in $\Lambda_2(d,F)$ have nodes at $z_1,\dots,z_m$, and are non-singular outside $\mathrm{Sing}(F)$. Consider the cusp $w_1 \in F$. In some affine neighbourhood of w_1 we fix an affine coordinate system (x,y) such that $w_1 = (0;0)$, and $y=0$ is a tangent to F at w_1. Then in this neighbourhood, F is defined by polynomial

$$Ay^2 + Bx^3 + \sum_{2i+3j>6} A_{ij}x^iy^j, \quad AB \neq 0, \tag{5.2}$$

and can be locally parametrized analytically (see [3], [14]) by

$$x = \tau^2, \quad y = \lambda\tau^3 + \mathrm{O}(\tau^4). \tag{5.3}$$

To determine $(G\cdot F)(w_1)$ we plug (5.3) into the affine equation

$$\sum a_{ij}x^iy^j = 0$$

of G, and then compute the order of vanishing at $\tau = 0$ (see [14]). Thus we obtain for curves $G \in \Lambda_2(d,F)$ that :

$$a_{00} = a_{01} = a_{10} = a_{11} = a_{20} = 0\ .$$

Since $F \in \Lambda_2(d,F)$, almost all curves in $\Lambda_2(d,F)$ have affine equations like (5.2), that means they have a cusp at w_1 with the tangent $y=0$. Now to complete the proof we should note that any curve $G \in V(d,m,k)$ with nodes $z_1,\dots,z_m$, cusps $w_1,\dots,w_k$ and tangents $T_{w_1}F,\dots,T_{w_k}F$, belongs to $\Lambda_2(d,F)$.

PROPOSITION 5.4. — *Under condition* (0.4), $V(d,m,k)$ *contains an open dense subset* $\widetilde{V}$ *consisting of curves* F *with non-special linear system* $\Lambda_2(d,F)$.

REMARK 5.5. — Even under condition (0.4), $V(d,m,k)$ may contain curves G with special system $\Lambda_2(d,G)$. This holds for $V(2p,3p,0)$, $p \geq 3$ (see [12]).

Proof of Proposition 5.4. — It is enough to prove the statement for any irreducible component V_0 of $V(d,m,k)$. For any curve $G \in V_0$, let $\mathcal{H}(G)$ denote the linear system of curves of the smallest degree h, passing through $\mathrm{Sing}(G)$. Evidently, any $H \in \mathcal{H}(G)$ is reduced, and

$$h = \deg H \leq \sqrt{2(m+k)} < \sqrt{2\alpha_1}\, d. \tag{5.6}$$

Denote by V_1 the set of curves $G \in V_0$ with maximal h. This is an open dense irreducible subset of V_0 (see [3]). Now denote by V_2 the set of curves $G \in V_1$ with minimal $\dim \mathcal{H}(G)$. Similarly this is an open dense irreducible subset of V_1. Then

$$W = \bigcup_{G \in V_2} \mathcal{H}(G)$$

is irreducible as the image in $P(\Sigma(h))$ of the space of a projective bundle with base V_2 and fibres $\mathcal{H}(G)$, $G \in V_2$. Denote by W_0 the set of curves $H \in W$ with minimal number of irreducible components. It is irreducible, open and dense in W. In particular, that means all the curves $H \in W_0$ determine the same sequence of degrees of their components (up to permutation). Moreover, if H runs through W_0, then any of its irreducible components K runs through some irreducible set

$$W(K) \subset P(\Sigma(\deg K)).$$

For $H \in W_0 \cap \mathcal{H}(G)$ define $N(H)$ to be $\sum N(K,H)$, where K runs through all components of H, and $N(K,H) = \mathrm{card}(K \cap \mathrm{Sing}(G))$. Denote by W_1 the set of curves $H \in W_0$ with minimal $N(H)$. First, it is an open dense irreducible subset of W_0, and, second, for any component K of $H \in W_1$,

$$N(K,H) = \min_{K' \in W(K)} N(K',H').$$

At last, introduce

$$V_3 = \{G \in V_2 \mid \mathcal{H}(G) \cap W_1 \neq \emptyset\}.$$

According to the above construction, this is an open dense subset of V_0. Now we will show that V_3 contains an open subset consisting of curves satisfying the conditions of PROPOSITION 5.4, in three steps.

Step 1. — Fix $G \in V_3$ and $H \in \mathcal{H}(G) \cap W_1$. Show that any component K of H of degree δ contains at most $\frac{1}{2}\delta(\delta+3)$ points from $\mathrm{Sing}(G)$.

Indeed, let K contain $\frac{1}{2}\delta(\delta+3)+1$ points from $\mathrm{Sing}(G)$. Denote the set of these points by S, and consider the linear system $\Lambda_3(d,G,S)$. Let us take an irreducible curve $\Phi \in \Lambda_1(n,G)$, $n = [(\frac{2}{3} + \sqrt{2\alpha_1})d]$, from LEMMA 3.1. According to PROPOSITION 1.11, for all $t \geq 1$,

$$\Lambda_3(t,G,S) \supset \mathcal{L}_{\Phi K}(t,D),$$

where

$$\deg D_{|\Phi} \leq m + 2k, \quad \deg D_{|K} \leq \tfrac{1}{2}\delta(\delta+3)+1. \tag{5.7}$$

We shall show that $\mathcal{L}_{\Phi K}(d,D)$ is non-special. According to PROPOSITION 1.8 and arguments from its proof, this is equivalent to

$$i\big((d-\delta)L_\Phi - D(\Phi) - D_{|\Phi}\big) = i\big((d-n)L_K - D(K) - D_{|K}\big) = 0.$$

According to THEOREM 1.7 these equalities follow from :

$$(d-\delta)n - \deg D(\Phi) - \deg D_{|\Phi} > 2g(\Phi) - 2, \tag{5.8}$$

$$\delta(d-n) - \deg D(K) - \deg D_{|K} > 2g(K) - 2. \tag{5.9}$$

According to PROPOSITION 1.3 and (5.7), inequality (5.8) follows from :

$$m + 2k < n(d - \delta - n + 3).$$

This inequality can be easily deduced from the definition of n and (5.6), because $\alpha_1 = \frac{2}{225}$ satisfies the inequality :

$$\alpha_1 < \big(\tfrac{2}{3} + \sqrt{2\alpha_1}\big)\big(\tfrac{1}{3} - 2\sqrt{2\alpha_1}\big).$$

Analogously, by PROPOSITION 1.3 and (5.7) the inequality (5.9) follows from :

$$d - n \geq \tfrac{3}{2}\delta.$$

This can be easily deduced from the definition of n and (5.6), because α_1 is the root of the equation

$$1 - \big(\tfrac{2}{3} + \sqrt{2\alpha}\big) = \tfrac{3}{2}\sqrt{2\alpha}.$$

So, according to PROPOSITION 1.2, $\Lambda_3(d,G,S)$ is non-special, and according to PROPOSITIONS 1.3 and 1.8

$$\dim \Lambda_3(d,G,S) = \tfrac{1}{2}d(d+3) - m - 2k - 2 \cdot \mathrm{card}\, S. \tag{5.10}$$

If G runs through V_3, then S runs through some subset of

$$\mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

thereby defining a morphism

$$\nu : \widetilde{V}_3 \longrightarrow \mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2),$$

where $\widetilde{V}_3$ is the finite covering of V_3 corresponding to different choices of $\frac{1}{2}\delta(\delta+3)+1$ points from the set $\mathrm{Sing}(G) \cap K$ (which might contain more than $\frac{1}{2}\delta(\delta+3)+1$ points). The tangent space to the fibre $\nu^{-1}(S)$ at the point $(G,S) \in \widetilde{V}_3$ is contained in $\Lambda_3(d,G,S)$ (see [2], [12]). Therefore (5.10) implies :

$$\dim \nu^{-1}(S) = \dim V_3 - 2\big(\tfrac{1}{2}\delta(\delta+3)+1\big),$$

hence $\nu(\widetilde{V}_3)$ is dense in $\mathrm{Sym}^{\delta(\delta+3)/2+1}(\mathbb{P}^2)$. Therefore there is a curve $G \in V_3$ such that, for any set S of $\frac{1}{2}\delta(\delta+3)+1$ points in $\mathrm{Sing}(G) \cap K$, no more than $\frac{1}{2}\delta(\delta+3)$ points of S lie on a curve of degree δ. But this contradicts the definition of the set V_3 and the initial assumption that $K \cap \mathrm{Sing}(G)$ contains more than $\frac{1}{2}\delta(\delta+3)$ points, and thus completes the proof.

Step 2. — Consider the linear system $\Lambda_3(d,G,\mathrm{Sing}(G))$. As in the previous step, for any curve $H \in \mathcal{H}(G) \cap W_1$ and all $t \geq 1$ we have

$$\Lambda_3\big(t,G,\mathrm{Sing}(G)\big) \supset \mathcal{L}_{\Phi H}(t,D),$$

where D satisfies (5.7) for any component $K \subset H$, hence $\Lambda_3(d,G,\mathrm{Sing}(G))$ is non-special.

Step 3. — As in the first step, the non-speciality of $\Lambda_3(d,G,\mathrm{Sing}(G))$, $G \in V_3$, implies that the image of V_3 by the morphism

$$\mu : V_3 \to \mathrm{Sym}^{m+k}(\mathbb{P}^2), \quad \mu(G) = \mathrm{Sing}(G),$$

contains an open dense subset U of $\mathrm{Sym}^{m+k}(\mathbb{P}^2)$. According to [4], under condition (0.4), for any Z from some open subset $U' \subset U$ the linear system $\Lambda(d,Z)$ of curves of degree d, having multiplicity ≥ 3 at each point $z \in Z$, is non-special. It is easy to see that for any $G \in V(d,m,k)$ and $t \geq 1$:

$$\Lambda_2(t,G) \supset \Lambda\big(t,\mathrm{Sing}(G)\big).$$

Therefore $\Lambda_2(d,F)$ is non-special for any curve $F \in \mu^{-1}(U') \cap V_3$. ∎

Now we can finish the proof of the irreducibility of $V(d,m,k)$, showing that $\widetilde{V}$ is irreducible. To any curve $F \in \widetilde{V}$ we assign the set $\mathrm{Sing}(F)$ and the set of tangents at its cusps. Thereby we obtain a morphism

$$\pi : \widetilde{V} \longrightarrow \mathrm{Sym}^m(\mathbb{P}^2) \times \mathrm{Sym}^k\big(P(T\mathbb{P}^2)\big),$$

where $P(T\mathbb{P}^2)$ is the projectivization of the tangent bundle of the plane. According to PROPOSITION 5.1 any fibre of π is an open subset of some linear system $\Lambda_2(d,F)$, $F \in \widetilde{V}$, hence is irreducible. The non-speciality of these linear systems, PROPOSITIONS 1.2 and 1.8 imply immediately that all the fibres have the same dimension :

$$\tfrac{1}{2}\,d(d+3) - 3m - 5k = \dim \widetilde{V} - \dim\big(\mathrm{Sym}^m(\mathbb{P}^2) \times \mathrm{Sym}^k(P(T\mathbb{P}^2))\big).$$

Finally, this equality means that $\pi(\widetilde{V})$ is dense in

$$\mathrm{Sym}^m(\mathbb{P}^2) \times \mathrm{Sym}^k\big(P(T\mathbb{P}^2)\big),$$

hence is irreducible. This completes the proof.

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