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ON MICROLOCAL b-FUNCTION

BY

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RÉSUMÉ. — Soit f un germe de fonction holomorphe en n variables. En utilisant des opérateurs différentiels microlocals, on introduit la notion de b-fonction microlocale $\tilde{b}_f(s)$ de f, et on démontre que $(s+1)\tilde{b}_f(s)$ coïncide avec la b-fonction (i.e. le polynôme de Bernstein) de f. Soient R_f les racines de $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$ et $m_\alpha(f)$ la multiplicité de $\alpha \in R_f$. On démontre $R_f \subset [\alpha_f, n-\alpha_f]$ et $m_\alpha(f) \le n - \alpha_f - \alpha + 1$ $(\le n - 2\alpha_f + 1)$. Le théorème de type Thom-Sebastiani pour b-fonction est aussi démontré sous une hypothèse raisonnable.

ABSTRACT. — Let f be a germ of holomorphic function of n variables. Using microlocal differential operators, we introduce the notion of microlocal b-function $\tilde{b}_f(s)$ of f, and show that $(s+1)\tilde{b}_f(s)$ coincides with the b-function (i.e. Bernstein polynomial) of f. Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$. Then we prove $R_f \subset [\alpha_f, n-\alpha_f]$ and $m_\alpha(f) \le n-\alpha_f-\alpha+1$ $(\le n-2\alpha_f+1)$. The Thom-Sebastiani type theorem for b-function is also proved under a reasonable hypothesis.

Introduction

Let f be a holomorphic function defined on a germ of complex manifold (X,x). The b-function (i.e., Bernstein polynomial) $b_f(s)$ of f is defined by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.1) \qquad b(s)f^s = Pf^{s+1} \quad \text{in } \mathcal{O}_{X,x}[f^{-1}][s]f^s$$

for $P \in \mathcal{D}_{X,x}[s]$. Let $\delta(t-f)$ denote the delta function on $X' := X \times \mathbb{C}$ with support $\{f = t\}$, where t is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t t$, f^s and $\delta(t-f)$ satisfy the same relation (see for example [8]). So f^s in (0.1) can be replaced by $\delta(t-f)$, and f^{s+1} by $t\delta(t-f)$. We define the

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microlocal b-function $\tilde{b}_f(s)$ by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.2) \qquad b(s)\delta(t-f) = P\partial_t^{-1}\delta(t-f) \quad \text{in } \mathcal{O}_{X,x}[\partial_t, \partial_t^{-1}]\delta(t-f)$$

for $P \in \mathcal{D}_{X,x}[\partial_t^{-1}, s]$. Here we can also allow for P a microdifferential operator [4], [6], [17] satisfying a condition on the degree of t and ∂_t (see (1.4)). We have :

PROPOSITION 0.3. — $b_f(s) = (s+1)\tilde{b}_f(s)$.

See (1.5). The microlocal b-function $\tilde{b}_f(s)$ is sometimes easier to treat than the b-function $b_f(s)$. Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$, and $n = \dim X$. Then, using the duality of filtered $\mathcal{D}$-Modules [15] and the theory of Hodge Modules [12], we prove

THEOREM 0.4. — $R_f \subset [\alpha_f, n - \alpha_f]$.

THEOREM 0.5. — $m_\alpha(f) \leq n - \alpha_f - \alpha + 1 \quad (\leq n - 2\alpha_f + 1)$.

See (2.8), (2.10).

The estimate (0.4) is optimal because $\max R_f = n - \alpha_f$ in the quasi-homogeneous isolated singularity case. See also remark after (2.8) below. Note that $R_f \subset \mathbb{Q}$ and $\alpha_f > 0$ by [4], and (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (with $\delta_{\alpha,1}$ Kronecker's delta) which is shown in [9] as a corollary of the relation with Deligne's vanishing cycle sheaf $\varphi_f\mathbb{C}_X$ [2] (see also [5]). This relation implies for example that $\exp(2\pi i\alpha)$ for $\alpha \in R_f$ are the eigenvalues of the monodromy on $\varphi_f\mathbb{C}_X$. But $\varphi_f\mathbb{C}_X$ cannot be replaced with the reduced cohomology of a Milnor fiber at x as in the isolated singularity case, because we have to take the Milnor fibration at several points of Sing $f^{-1}(0)$ even when we consider the b-function of f at x. See (2.12) below.

Let T_u and T_s denote respectively the unipotent and semisimple part of the monodromy T on $\varphi_f\mathbb{C}_X$. Let $\varphi_f^\alpha\mathbb{C}_X = \operatorname{Ker}(T_s - \exp(-2\pi i\alpha))$ (as a shifted perverse sheaf), and $N = \log T_u/2\pi i$. In the proof of (0.5), we get also :

PROPOSITION 0.6. — *We have* $N^{r+1} = 0$ *on* $\varphi_f^\alpha\mathbb{C}_X$ *for* $\alpha \in [\alpha_f, \alpha_f + 1)$ *and* $r = [n - \alpha_f - \alpha]$. *In particular,* $N^{r+1} = 0$ *on* $\varphi_f\mathbb{C}_X$ *for* $r = [n - 2\alpha_f]$.

For the proof of (0.4)–(0.6), we use the filtration V (similar to that in [5], [9]) defined on the $\mathcal{D}_{X,x}[t, \partial_t, \partial_t^{-1}]$-module $\widetilde{\mathcal{B}}_f$ generated by the delta function $\delta(t-f)$. Note that (0.3) may be viewed as an extension of Malgrange's result [8] to the nonisolated singularity case (see (1.7) below), and in the isolated singularity case, (0.4)–(0.6) can be deduced from results of [8], [19], [20] (and [18]) using an argument as in [14]. In the nondegenerate Newton boundary case [7], we get an estimate of α_f using the Newton polyhedron (see (3.3)). The idea of its proof is essentially same as [16].

Let g be a holomorphic function on a germ of complex manifold (Y, y). Let $Z = X \times Y, z = (x, y)$, and $h = f + g \in \mathcal{O}_{Z,z}$. We define R_g, R_h as above. Then we have :

PROPOSITION 0.7. — $R_f + R_g \subset R_h + \mathbb{Z}_{\leq 0}$, $R_h \subset R_f + R_g + \mathbb{Z}_{\geq 0}$.

THEOREM 0.8. — *Assume there is a holomorphic vector field ξ such that $\xi g = g$. Then we have $R_f + R_g = R_h$, and*

$$m_\gamma(h) = \max_{\alpha+\beta=\gamma} \{m_\alpha(f) + m_\beta(g) - 1\}.$$

See (4.3)–(4.4). Here $\mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}_{\leq 0}$) is the set of nonnegative (or nonpositive) integers. In the case where f and g have isolated singularities, (0.7)–(0.8) can be easily deduced from results of MALGRANGE [8], [10] (see (4.6) below), and (0.8) was first obtained by [21] in this case. Note that (0.8) is not true in general if the hypothesis is not satisfied. See (4.8) below.

1. Microlocal b-function

1.1. — Let X be a complex manifold of pure dimension n, and $x \in X$. Let $\mathcal{O} = \mathcal{O}_{X,x}, \mathcal{D} = \mathcal{D}_{X,x}$. We define rings $\mathcal{R}, \widetilde{\mathcal{R}}$ by

$$\mathcal{R} = \mathcal{D}[t, \partial_t], \quad \widetilde{\mathcal{R}} = \mathcal{D}[t, \partial_t, \partial_t^{-1}], \tag{1.1.1}$$

where t, ∂_t satisfy the relation $\partial_t t - t\partial_t = 1$, and $\mathcal{D}[t, \partial_t] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[t, \partial_t]$, etc. We define the filtration V on $\mathcal{R}, \widetilde{\mathcal{R}}$ by the differences of the degrees of t and ∂_t :

$$V^p\mathcal{R} = \sum_{i-j\geq p} \mathcal{D}t^i\partial_t^j \quad (\text{same for } \widetilde{\mathcal{R}}). \tag{1.1.2}$$

Then we have :

$$\begin{cases} V^p\mathcal{R} = t^pV^0\mathcal{R} = V^0\mathcal{R}t^p & (p > 0), \\ V^{-p}\mathcal{R} = \sum_{0\leq j\leq p} \partial_t^j V^0\mathcal{R} = \sum_{0\leq j\leq p} V^0\mathcal{R}\partial_t^j & (p > 0), \\ V^p\widetilde{\mathcal{R}} = \partial_t^{-p}V^0\widetilde{\mathcal{R}} = V^0\widetilde{\mathcal{R}}\partial_t^{-p}. \end{cases} \tag{1.1.3}$$

1.2. — Let $f \in \mathcal{O}$ such that $f(0) = 0$ and $f \neq 0$. Let

$$\mathcal{B}_f = \mathcal{O}[\partial_t]\delta(t-f), \quad \widetilde{\mathcal{B}}_f = \mathcal{O}[\partial_t, \partial_t^{-1}]\delta(t-f), \tag{1.2.1}$$

where $\mathcal{O}[\partial_t]\delta(t-f)$ is a free module of rank one over $\mathcal{O}[\partial_t]$ $(= \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])$ with a basis $\delta(t-f)$ (similarly for $\widetilde{\mathcal{B}}_f$). Here $\delta(t-f)$ denotes the delta function supported on $\{f = t\}$ (see remark below). We have a structure of $\mathcal{R}$-module and $\widetilde{\mathcal{R}}$-module on $\mathcal{B}_f$ and $\widetilde{\mathcal{B}}_f$ respectively by

$$\begin{cases} \xi\big(a\partial_t^i\delta(t-f)\big) = (\xi a)\partial_t^i\delta(t-f) - (\xi f)a\partial_t^{i+1}\delta(t-f), \\ t\big(a\partial_t^i\delta(t-f)\big) = fa\partial_t^i\delta(t-f) - ia\partial_t^{i-1}\delta(t-f) \end{cases} \tag{1.2.2}$$

for $a \in \mathcal{O}$ and $\xi \in \Theta_{X,x}$. We define a decreasing filtration G on $\mathcal{B}_f$, $\widetilde{\mathcal{B}}_f$ by

$$G^p\mathcal{B}_f = V^p\mathcal{R}\delta(t-f), \quad G^p\widetilde{\mathcal{B}}_f = V^p\widetilde{\mathcal{R}}\delta(t-f), \tag{1.2.3}$$

and an increasing filtration F by

$$F_p\mathcal{B}_f = \bigoplus_{0\le i\le p} \mathcal{O}\partial_t^i\delta(t-f), \quad F_p\widetilde{\mathcal{B}}_f = \bigoplus_{i\le p} \mathcal{O}\partial_t^i\delta(t-f) \tag{1.2.4}$$

Then we have :

$$\partial_t^i : G^p\widetilde{\mathcal{B}}_f \xrightarrow{\sim} G^{p-i}\widetilde{\mathcal{B}}_f, \quad \partial_t^i : F_p\widetilde{\mathcal{B}}_f \xrightarrow{\sim} F_{p+i}\widetilde{\mathcal{B}}_f, \tag{1.2.5}$$

$$\mathcal{D}_{X,x}[s](F_p\widetilde{\mathcal{B}}_f) \subset G^{-p}\widetilde{\mathcal{B}}_f. \tag{1.2.6}$$

Remark. — The $\mathcal{R}$-module $\mathcal{B}_f$ is identified with the germ at $(x, 0)$ of the direct image of $\mathcal{O}_X$ as $\mathcal{D}$-Module by the closed embedding i_f defined by the graph of f, where t is identified with the coordinate of $\mathbb{C}$. See [4] and [17].

1.3 Definition. — The *b-function* $b_f(s)$ (resp. *microlocal b-function* $\tilde{b}_f(s)$) is defined by the minimal polynomial of the action of $s := -\partial_t t$ on $\mathrm{Gr}_G^0\mathcal{B}_f$ (resp. $\mathrm{Gr}_G^0\widetilde{\mathcal{B}}_f$).

Remark. — Since $\mathrm{Gr}_V^0\mathcal{R} = \mathrm{Gr}_V^0\widetilde{\mathcal{R}} = \mathcal{D}[s]$, $b_f(s)$ (resp. $\tilde{b}_f(s)$) is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$b(s)\delta(t-f) = P\delta(t-f) \tag{1.3.1}$$

for $P \in V^1\mathcal{R}$ (resp. $V^1\widetilde{\mathcal{R}}$). For $b_f(s)$, we may assume $P = tQ$ with $Q \in \mathcal{D}[s]$ using (1.1.3) and (1.2.2). So the above definition coincides with the usual definition of b-function (i.e., Bernstein polynomial), because $\delta(t-f)$ and f^s satisfy the same relation (see [8]).

1.4. — Let $X' = X \times \mathbb{C}$, and $\mathcal{E}$ the germs of microlocal differential operators at $p := (x, 0; 0, \mathrm{d}t) \in T^*X'$ (see [17], [4]). Let $\mathcal{C}_f$ be the microlocalization of the $\mathcal{D}_{X',x'}$-module $\mathcal{B}_f$ at $p \in T^*X'$ (see [4], [17]), where $x' = (x, 0)$. It is an $\mathcal{E}$-module, and we have an isomorphism

$$\mathcal{C}_f = \mathcal{O}\{\{\partial_t^{-1}\}\}[\partial_t]\delta(t-f), \tag{1.4.1}$$

where the $\mathcal{E}$-module structure is defined as in (1.2.2). Here $\mathcal{O}\{\{\partial_t^{-1}\}\}$ is defined by

$$\Big\{\sum_{i\geq 0} g_i\partial_t^{-i} : \sum_{i\geq 0}\frac{g_i t^i}{i\,!} \in \mathcal{O}_{X',x'}\Big\}. \tag{1.4.2}$$

We have the filtration V on $\mathcal{E}$ by the difference of the degrees of ∂_t and t as in (1.1.2), and define the filtrations G, F on $\mathcal{C}_f$ by

$$G^p\mathcal{C}_f = V^p\mathcal{E}\delta(t-f), \quad F_p\mathcal{C}_f = \mathcal{O}\{\{\partial_t^{-1}\}\}\partial_t^p\delta(t-f). \tag{1.4.3}$$

Let $b'(s)$ be the minimal polynomial of the action of s on $\mathrm{Gr}_G^0\,\mathcal{C}_f$. See also [6]. Then we have :

$$\tilde{b}_f(s) = b'(s). \tag{1.4.4}$$

In fact, it is enough to show the canonical isomorphism :

$$\mathrm{Gr}_G^0\,\widetilde{\mathcal{B}}_f \xrightarrow{\sim} \mathrm{Gr}_G^0\,\mathcal{C}_f. \tag{1.4.5}$$

We have $\mathrm{Gr}_p^F\,\widetilde{\mathcal{B}}_f = \mathrm{Gr}_p^F\,\mathcal{C}_f$, $F_0\mathcal{C}_f \subset G^0\mathcal{C}_f$ and (1.2.6). So the assertion is reduced to the isomorphism :

$$G^0\widetilde{\mathcal{B}}_f/F_0\widetilde{\mathcal{B}}_f \xrightarrow{\sim} G^0\mathcal{C}_f/F_0\mathcal{C}_f. \tag{1.4.6}$$

Both terms are identified with subspaces of $\mathcal{C}_f/F_0\mathcal{C}_f$ $(= \mathcal{O}[\partial_t]\partial_t\delta(t-f))$, and it is enough to show the surjectivity. Using local coordinates, we can check

$$V^0\mathcal{E} = \sum_{\nu,i}\mathcal{E}(0)\partial^\nu(t\partial_t)^i = \sum_{\nu,i}\partial^\nu(t\partial_t)^i\mathcal{E}(0), \tag{1.4.7}$$

where $\mathcal{E}(0)$ denotes the microdifferential operators of degree ≤ 0 (see [17], [4]), and ∂^ν is as in the proof of (1.6) below. So we get (1.4.6), because $\mathcal{E}(0)\delta(t-f) = F_0\mathcal{C}_f$.

1.5 Proof of 0.3. — We show first

$$(s+1)\tilde{b}_f(s) \mid b_f(s). \tag{1.5.1}$$

It is well known that $b_f(s)$ is divisible by $s+1$ (by substituting $s=-1$ to $b_f(s)f^s = Pf^{s+1}$). This can be verified also by restricting X to the complement of Sing $f^{-1}(0)_{\mathrm{red}}$. By (1.3.1) for $b_f(s)$, we get

$$(s+1)\Big(\frac{b_f(s)}{s+1} + \partial_t^{-1}Q\Big)\delta(t-f) = 0, \tag{1.5.2}$$

because $s+1 = -t\partial_t$, and $P = tQ$ for $Q \in \mathcal{D}[s]$. So the assertion is reduced to the injectivity of the action of t on $\widetilde{\mathcal{B}}_f$. We may replace $\widetilde{\mathcal{B}}_f$ by $\mathrm{Gr}_p^F \widetilde{\mathcal{B}}_f$, and the action of t on $\mathrm{Gr}_p^F \widetilde{\mathcal{B}}_f$ is the multiplication by f. Then the assertion is clear.

For the converse of (1.5.1), we use (1.3.1) for $\tilde{b}_f(s)$. By the next lemma, we may assume $P \in \partial_t^{-1}V^0\mathcal{R}$. So we get the assertion by multiplying $s+1 = -t\partial_t$.

LEMMA 1.6. — *With the above notation, we have*

$$\partial_t^{-1}V^0\widetilde{\mathcal{R}}\delta(t-f)\cap\mathcal{O}[\partial_t]\delta(t-f) = \partial_t^{-1}V^0\mathcal{R}\delta(t-f)\cap\mathcal{O}[\partial_t]\delta(t-f). \tag{1.6.1}$$

Proof. — Since $V^0\widetilde{\mathcal{R}} = (V^0\mathcal{R}\cap\partial_t\mathcal{R}) + \mathcal{D}_{X,x}[t,\partial_t^{-1}]$, it is enough to show

$$\partial_t^{-1}\mathcal{D}_{X,x}[t,\partial_t^{-1}]\delta(t-f)\cap\mathcal{O}[\partial_t]\delta(t-f) \subset \mathcal{D}_{X,x}\partial_t^{-1}\delta(t-f).$$

We have $\mathcal{D}_{X,x}[t,\partial_t^{-1}]\delta(t-f) = \mathcal{D}_{X,x}[\partial_t^{-1}]\delta(t-f)$ by (1.2.2). So the assertion is reduced to

$$\mathcal{D}_{X,x}\partial_t^{-j-1}\delta(t-f)\cap\mathcal{O}[\partial_t]\partial_t^{-j}\delta(t-f) \subset \mathcal{D}_{X,x}\partial_t^{-j}\delta(t-f)$$

by decreasing induction on $j > 0$. Let $(x_1,\dots,x_n)$ be a local coordinate system of X, and $\partial_i = \partial/\partial x_i, \partial^\nu = \prod_i \partial_i^{\nu_i}$ for $\nu = (\nu_1,\dots,\nu_n)$. Take $P = \sum_\nu a_\nu\partial^\nu \in \mathcal{D}_{X,x}$ such that

$$P\partial_t^{-j-1}\delta(t-f) \subset \mathcal{O}[\partial_t]\partial_t^{-j}\delta(t-f).$$

By (1.2.2), the condition is equivalent to $a_0 = 0$, and the assertion follows.

1.7 Remark. — Assume f has isolated singularity, and $n \geq 2$. Let L_f denote Brieskorn's module $\Omega^n_{X,x}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-2}_{X,x}$ (see [1]). Then it was shown by MALGRANGE [10] and PHAM [11] that L_f is a free A-module of rank μ, where $A = \mathbb{C}\{\{\partial_t^{-1}\}\}$, and μ is the Milnor number of f. MALGRANGE [8] also showed

$$(1.7.1)\qquad \frac{b_f(s)}{(s+1)} \text{ is the minimal polynomial of the action of } -\partial_t t \text{ on } \bar{L}_f/\partial_t^{-1}\bar{L}_f,$$

where $\bar{L}_f$ is the saturation of L_f (see (4.7) below). So (0.3) may be viewed as an extension of (1.7.1) to the nonisolated singularity case, because the Gauss-Manin system associated with a Milnor fibration does not provide enough information of b-function in general. See (2.12) below. Note that (0.4)–(0.6) can be easily deduced from (1.7.1) combined with [19], [20] (and [18]). See also [14].

2. Filtration V

2.1. — With the notation of paragraph 1, let V denote the filtration of Kashiwara [5] and Malgrange [9] on $\mathcal{B}_f$ indexed by $\mathbb{Q}$ (see also [12, (3.1)] and [13]). Here we index V decreasingly so that the action of $\partial_t t - \alpha$ on $\mathrm{Gr}_V^\alpha \mathcal{B}_f$ is nilpotent, where $\mathrm{Gr}_V^\alpha = V^\alpha/V^{>\alpha}$ with $V^{>\alpha} = \bigcup_{\beta>\alpha} V^\beta$. In particular, we have isomorphisms for $\alpha \neq 0$:

$$(2.1.1)\qquad \begin{cases} t : \mathrm{Gr}_V^\alpha \mathcal{B}_f \xrightarrow{\sim} \mathrm{Gr}_V^{\alpha+1} \mathcal{B}_f, \\ \partial_t : \mathrm{Gr}_V^{\alpha+1} \mathcal{B}_f \xrightarrow{\sim} \mathrm{Gr}_V^\alpha \mathcal{B}_f. \end{cases}$$

By negativity of the roots of b-function [4], we have :

$$(2.1.2)\qquad F_0\mathcal{B}_f \subset V^{>0}\mathcal{B}_f.$$

See (1.2.4) for $F_p\mathcal{B}_f$. We define the filtration V on $\widetilde{\mathcal{B}}_f$ by

$$(2.1.3)\qquad V^\alpha\widetilde{\mathcal{B}}_f = \begin{cases} V^\alpha\mathcal{B}_f + \mathcal{O}[\partial_t^{-1}]\partial_t^{-1}\delta(t-f) & \text{for } \alpha \leq 1, \\ \partial_t^{-j}V^{\alpha-j}\widetilde{\mathcal{B}}_f & \text{for } \alpha > 1,\ 0 < \alpha - j \leq 1. \end{cases}$$

Then we have filtered isomorphisms

$$(2.1.4)\qquad (\mathrm{Gr}_V^\alpha \mathcal{B}_f, F) \xrightarrow{\sim} (\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f, F) \quad \text{for } \alpha < 1.$$

LEMMA 2.2. — *For any $\alpha \in \mathbb{Q}$ and $j > 0$, we have isomorphisms :*

$$\partial_t^j : V^\alpha \widetilde{\mathcal{B}}_f \xrightarrow{\sim} V^{\alpha-j}\widetilde{\mathcal{B}}_f. \tag{2.2.1}$$

Proof. — It is enough to show the surjectivity of (2.2.1) for $0 < \alpha \leq 1$. Let $u \in V^{\alpha-j}\widetilde{\mathcal{B}}_f$. Since the action of ∂_t on $\widetilde{\mathcal{B}}_f$ is bijective, there exists uniquely $v \in \widetilde{\mathcal{B}}_f$ such that $u = \partial_t^j v$, and we have to show $v \in V^\alpha \widetilde{\mathcal{B}}_f$. Assume $v \in V^\beta \widetilde{\mathcal{B}}_f$ and $v \notin V^{>\beta}\widetilde{\mathcal{B}}_f$ for $\beta < \alpha \leq 1$. By (2.1.2)–(2.1.3), we have :

$$F_{-1}\widetilde{\mathcal{B}}_f \subset V^{>1}\widetilde{\mathcal{B}}_f. \tag{2.2.2}$$

So there exists $v' \in V^\beta B_f$ such that $\mathrm{Gr}_V v = \mathrm{Gr}_V v'$ in $\mathrm{Gr}_V^\beta \widetilde{\mathcal{B}}_f$. Then $\mathrm{Gr}_V \partial_t^j v \neq 0$ in $\mathrm{Gr}_V^{\beta-j} \widetilde{\mathcal{B}}_f$ by (2.1.1) and (2.1.4). This is contradiction.

REMARK. — By (1.2.5) (2.2.1), we have isomorphisms :

$$\partial_t^j : F_p V^\alpha \widetilde{\mathcal{B}}_f \xrightarrow{\sim} F_{p+j} V^{\alpha-j}\widetilde{\mathcal{B}}_f. \tag{2.2.3}$$

2.3. — We say that L is a *lattice* of $\widetilde{\mathcal{B}}_f$ if L is a finite $V^0\widetilde{\mathcal{R}}$-submodule of $\widetilde{\mathcal{B}}_f$, which generates $\widetilde{\mathcal{B}}_f$ over $\widetilde{\mathcal{R}}$. For two lattices L, L' of $\widetilde{\mathcal{B}}_f$, we have

$$L \subset \partial_t^j L' \quad \text{for} \quad j \gg 0, \tag{2.3.1}$$

because $\widetilde{\mathcal{R}} = \bigcup_j \partial_t^j V^0 \widetilde{\mathcal{R}}$ by (1.1.3). By the same argument as in [5], the filtration V on $\widetilde{\mathcal{B}}_f$ is uniquely characterized by the conditions :

(i) $V^j \widetilde{\mathcal{R}} V^\alpha \widetilde{\mathcal{B}}_f \subset V^{\alpha+j}\widetilde{\mathcal{B}}_f$,

(ii) $V^\alpha \widetilde{\mathcal{B}}_f$ are lattices of $\widetilde{\mathcal{B}}_f$,

(iii) $s + \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$,

(see also [12, (3.1.2)]). Here we assume that the filtration V is indexed by $\mathbb{Q}$ discretely (see [*loc. cit.*]).

For a lattice L of $\widetilde{\mathcal{B}}_f$, we define a filtration G on $\widetilde{\mathcal{B}}_f$ by $G^i \widetilde{\mathcal{B}}_f = \partial_t^{-i} L$, and the b-function $\tilde{b}_L(s)$ by the minimal polynomial of the action of s on $\mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f$. By (2.3.1), the induced filtration on $\mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f$ by V is a finite filtration, and $\tilde{b}_L(s)$ is the product of the minimal polynomial of s on each $\mathrm{Gr}_V^\alpha \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f = \mathrm{Gr}_G^0 \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$ (which is a power of $s + \alpha$), and hence $\tilde{b}_L(s)$ is nonzero. Note that, for a given number α_0, the b-function is determined by the induced filtration G on $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$ (with the action of s) for $\alpha_0 \le \alpha < \alpha_0 + 1$, using isomorphisms :

$$\partial_t^i : \mathrm{Gr}_G^0 \mathrm{Gr}_V^{\alpha+i} \widetilde{\mathcal{B}}_f \xrightarrow{\sim} \mathrm{Gr}_G^{-i} \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f. \tag{2.3.2}$$

For two lattices L, L' of $\widetilde{\mathcal{B}}_f$ such that $L \subset L'$, let R_L be the roots of $\tilde{b}_L(-s)$ (similarly for $R_{L'}$). Then

$$R_L \subset R_{L'} + \mathbb{Z}_{\ge 0}, \quad R_{L'} \subset R_L + \mathbb{Z}_{\le 0}, \tag{2.3.3}$$

where $\mathbb{Z}_{\ge 0}, \mathbb{Z}_{\le 0}$ are as in (0.7). In fact, setting $G'^i \widetilde{\mathcal{B}}_f = \partial_t^{-i} L'$, we have $G^i \subset G'^i$ on each $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$, and the assertion is checked using (2.3.2).

PROPOSITION 2.4. — *With the notation of* (2.1), *we have :*

$$\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = \mathcal{D}_{X,x}(F_p \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f) \quad \text{if} \quad F_{-p-1} \mathrm{Gr}_V^{n-\alpha} \widetilde{\mathcal{B}}_f = 0. \tag{2.4.1}$$

Proof. — Choosing a local coordinate system $(x_1, \dots, x_n)$, we have an involution of $\mathcal{D}_X$ such that $(\partial/\partial x_i)^* = -\partial/\partial x_i$, $(x_i)^* = x_i$, and $(PQ)^* = Q^* P^*$ (see [17]), and it identifies left and right $\mathcal{D}_X$-Modules. (For simplicity, we do not shift the filtration F in the transformation of left and right $\mathcal{D}_X$-Modules as in [13].) Let $\mathbb{D}$ denote the dual functor for filtered $\mathcal{D}$-Modules [12, § 2]. We define a filtration F on $\mathcal{O}_X$ (identified with a right $\mathcal{D}_X$-module ω_X) by $F_{-1}\mathcal{O}_X = 0, F_0\mathcal{O}_X = \mathcal{O}_X$. Then we have a natural duality isomorphism

$$\mathbb{D}(\mathcal{O}_X, F) = \big(\mathcal{O}_X, F[-n]\big), \tag{2.4.2}$$

which gives a polarization of Hodge Module (see remark 2.7 below), where $(F[m])_p = F_{p-m}$. (Note that $(\omega_X, F)[n]$ underlies the dualizing complex, and (ω_X, F) has weight $-n$.) Since $(\mathcal{B}_f, F)$ is identified with the direct image of $(\mathcal{O}_X, F)$ as filtered right $\mathcal{D}$-modules (see remark after (1.2)), we get

$$\begin{cases} \mathbb{D}(\mathrm{Gr}_V^\alpha \mathcal{B}_f, F) = \big(\mathrm{Gr}_V^{1-\alpha} B_f, F[1-n]\big) & \text{for } 0 < \alpha < 1, \\ \mathbb{D}(\mathrm{Gr}_V^0 B_f, F) = \big(\mathrm{Gr}_V^0 B_f, F[-n]\big), \end{cases} \tag{2.4.3}$$

by the duality for vanishing cycle functors [15]. (See also (2.7.2) and (2.7.5)–(2.7.6) below.) So we have

$$\mathbb{D}(\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f, F) = (\mathrm{Gr}_V^{n-\alpha} \widetilde{\mathcal{B}}_f, F) \quad \text{for any } \alpha, \tag{2.4.4}$$

by (2.1.4) (2.2.3), and the assertion is reduced to the following :

LEMMA 2.5. — *Let (M,F) be a holonomic filtered right $\mathcal{D}_X$-Module such that $\mathbb{D}(M,F)$ is a filtered $\mathcal{D}_X$-Module (i.e., M is holonomic and $\mathrm{Gr}^F M := \bigoplus_i \mathrm{Gr}_i^F M$ is coherent and Cohen-Macaulay over $\mathrm{Gr}^F \mathcal{D}_X$). Assume $F_{-p-1}\mathbb{D}M = 0$. Then :*

$$M = \mathcal{D}_X(F_p M). \tag{2.5.1}$$

Proof. — Let $\widetilde{\mathrm{DR}}(M,F)$ be as in the remark below. Then it is enough to show

$$\mathrm{Gr}_q^F \widetilde{\mathrm{DR}}(M,F) = 0 \quad \text{for } q > p, \tag{2.5.2}$$

because this implies $(\mathrm{Gr}_{q-1}^F M)\Theta_X = \mathrm{Gr}_q^F M$ (for $q > p$). We have

$$\widetilde{\mathrm{DR}}(M,F) = \mathbb{D}\big(\widetilde{\mathrm{DR}}(\mathbb{D}(M,F))\big) \tag{2.5.3}$$

by (2.6.5)–(2.6.6) below, and

$$\mathrm{Gr}_q^F \mathbb{D}\big(\widetilde{\mathrm{DR}}(\mathbb{D}(M,F))\big) = \mathbb{D}\,\mathrm{Gr}_{-q}^F\big(\widetilde{\mathrm{DR}}(\mathbb{D}(M,F))\big) \tag{2.5.4}$$

by (2.6.7). So it is zero for $q > p$, and the assertion follows.

2.6 Remark. — Let (M,F) be a filtered right $\mathcal{D}_X$-Module. The filtered differential complex $\widetilde{\mathrm{DR}}(M,F)$ associated with (M,F) is defined by

$$F_p\widetilde{\mathrm{DR}}(M)^i = F_{p+i}M \otimes \wedge^{-i}\Theta_X, \tag{2.6.1}$$

(see [12, § 2]), where Θ_X is the sheaf of holomorphic vector fields. The differential is defined like the Koszul complex associated with the action of $\partial/\partial x_i$ on M if we choose local coordinates. This induces an equivalence of categories

$$\widetilde{\mathrm{DR}}(M) : D^b_{\mathrm{coh}}F(\mathcal{D}_X) \xrightarrow{\sim} D^b_{\mathrm{coh}}F^f(\mathcal{O}_X, \mathrm{Diff}), \tag{2.6.2}$$

(see [12, 2.2.10]), where the right hand side is the derived category consisting of bounded coherent filtered differential complexes with finite filtration. We have the dual functor

$$\mathbb{D} : D^b_{\mathrm{coh}}F(\mathcal{D}_X) \longrightarrow D^b_{\mathrm{coh}}F(\mathcal{D}_X), \tag{2.6.3}$$

$$\mathbb{D} : D^b_{\mathrm{coh}}F^f(\mathcal{O}_X, \mathrm{Diff}) \longrightarrow D^b_{\mathrm{coh}}F^f(\mathcal{O}_X, \mathrm{Diff}), \tag{2.6.4}$$

such that

$$\widetilde{\mathrm{DR}} \circ \mathbb{D} = \mathbb{D} \circ \widetilde{\mathrm{DR}}, \tag{2.6.5}$$

$$\mathbb{D}^2 = \mathrm{id}, \tag{2.6.6}$$

(see [12], 2.4.5 and 2.4.11). By construction, we have

$$\mathrm{Gr}_i^F \mathbb{D}(L,F) = \mathbb{D}\,\mathrm{Gr}_{-i}^F(L,F) \tag{2.6.7}$$

for $(L,F) \in D^b_{\mathrm{coh}}F^f(\mathcal{O}_X, \mathrm{Diff})$, where $\mathbb{D}$ denotes also the dual functor for $\mathcal{O}_X$-Modules.

2.7 Remark. — Let $X' = X \times \mathbb{C}$ as in 1.4. Let (M,F) be a filtered right $\mathcal{D}_{X'}$-Module underlying a polarizable Hodge Module of weight n (see [12]). Then a polarization of Hodge Module induces an isomorphism :

$$\mathbb{D}(M,F) = \big(M, F[n]\big). \tag{2.7.1}$$

See [12, 5.2.10]. The nearby and vanishing cycle functors are defined by

$$\begin{cases} \psi_t(M,F) = \bigoplus_{-1\le\alpha<0} \mathrm{Gr}_\alpha^V\big(M,F[1]\big), \\ \varphi_{t,1}(M,F) = \mathrm{Gr}_0^V(M,F), \end{cases} \tag{2.7.2}$$

where t is the coordinate of $\mathbb{C}$, and V is the filtration of Kashiwara [5] and Malgrange [9] along $X \times \{0\}$ such that the action of $N := t\partial_t - \alpha$ on $\mathrm{Gr}_\alpha^V M$ is nilpotent locally on X. Here V is indexed increasingly, and we put $V^\alpha = V_{-\alpha}$. By [15, 1.6], we have the duality isomorphisms :

$$\psi_t\mathbb{D}(M,F) = \big(\mathbb{D}\psi_t(M,F)\big)(1), \tag{2.7.3}$$

$$\varphi_{t,1}\mathbb{D}(M,F) = \mathbb{D}\varphi_{t,1}(M,F). \tag{2.7.4}$$

Combined with (2.7.1), they imply the self duality :

$$\mathbb{D}\psi_t(M,F) = \psi_t(M,F)(n-1), \tag{2.7.5}$$

$$\mathbb{D}\varphi_{t,1}(M,F) = \varphi_{t,1}(M,F)(n). \tag{2.7.6}$$

Let W be the *monodromy filtration* of M associated with the action of N. This is uniquely characterized by the properties $NW_i \subset W_{i-2}$, $N^j : \mathrm{Gr}_j^W \xrightarrow{\sim} \mathrm{Gr}_{-j}^W$ $(j > 0)$. Then $W[n-1]$ (resp. W[n]) gives the *weight* filtration of mixed Hodge Modules on $\psi_t(M,F)$ (resp. $\varphi_{t,1}(M,F)$). Since N underlies a morphism of mixed Hodge Modules, N^j induces filtered isomorphisms

$$N^j : \mathrm{Gr}_j^W \psi_t(M,F) \xrightarrow{\sim} \mathrm{Gr}_{-j}^W \psi_t\big(M,F[-j]\big) \tag{2.7.7}$$

(same for $\varphi_{t,1}(M,F)$) by [12, 5.1.14]. We have the duality isomorphisms

$$\mathbb{D}\,\mathrm{Gr}_j^W \psi_t(M,F) = \mathrm{Gr}_{-j}^W \psi_t(M,F)(n-1), \tag{2.7.8}$$

$$\mathbb{D}\,\mathrm{Gr}_j^W \varphi_{t,1}(M,F) = \mathrm{Gr}_{-j}^W \varphi_{t,1}(M,F)(n), \tag{2.7.9}$$

because W is self dual. Note that these are used for the inductive definition of polarization in [12].

2.8 Proof of (0.4). — Since $G^1\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f \supset \mathcal{D}_{X,x}(F_{-1}\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f)$ by (1.2.6), it is enough to show $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = \mathcal{D}_{X,x}(F_{-1}\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f)$ for $\alpha > n - \alpha_f$ by (2.3). We have

$$F_0\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = G^0\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = 0 \quad \text{for} \quad \alpha < \alpha_f \tag{2.8.1}$$

by (1.2.6) and (2.3). So the assertion follows from (2.4) with $p = -1$.

REMARK. — We have $\max R_f = n - \alpha_f$ if f is quasihomogeneous and Sing $f^{-1}(0)$ is isolated. This follows for example from [8] together with Brieskorn's calculation of Gauss-Manin connection (unpublished). See also [13, (3.2.3)].

PROPOSITION 2.9. — *Let (M,F) be a filtered $\mathcal{D}_X$-Module with a morphism $N : (M,F) \to (M,F[-1])$. Let W be the monodromy filtration of M associated with the action of N. See* (2.7). *Assume*

$$N^j : F_p\,\mathrm{Gr}_j^W M \xrightarrow{\sim} F_{p+j}\,\mathrm{Gr}_{-j}^W M (j>0) \tag{2.9.1}$$

for any p, and there exist integers q, r such that, for any j :

$$F_{q-1}\,\mathrm{Gr}_j^W M = 0, \quad \mathrm{Gr}_j^W M = \mathcal{D}_X(F_{q+r}\,\mathrm{Gr}_j^W M). \tag{2.9.2}$$

Then $N^{r+1} = 0$ on M, and $N^{r-i} = 0$ on $M/\mathcal{D}_X[N](F_{q+i}M)$.

Proof. — We may assume $q = 0$ by replacing F with $F[-q]$. We apply (2.9.2) to $\mathrm{Gr}_{-j}^W M$, and get

$$\mathrm{Gr}_j^W M = \mathcal{D}_X(F_{r-j}\,\mathrm{Gr}_j^W M) \quad \text{for } j \geq 0, \tag{2.9.3}$$

using (2.9.1). In particular, $\mathrm{Gr}_j^W M = 0$ for $j > r$, and the first assertion follows. For the second assertion, it is enough to show the inclusion

$$W_{i-r}M \subset \mathcal{D}_X[N](F_iM) \tag{2.9.4}$$

and the surjectivity of

$$W_{r-i-1}M/W_{i-r}M \longrightarrow M/\mathcal{D}_X[N](F_iM), \tag{2.9.5}$$

because $N^{r-i} = 0$ on $W_{r-i-1}M/W_{i-r}M$. We have, by (2.9.3) :

$$\mathrm{Gr}_{-j}^W M = N^j \,\mathrm{Gr}_j^W \big(\mathcal{D}_X(F_iM)\big) \quad \text{for } j \geq r-i. \tag{2.9.6}$$

So (2.9.4) follows taking Gr_{-j}^W for $-j \leq i-r$. The surjectivity of (2.9.5) is equivalent to that of

$$\mathcal{D}_X[N](F_iM) \longrightarrow M/W_{r-i-1}M, \tag{2.9.7}$$

and follows from (2.9.3), taking Gr_j^W of (2.9.7) for $j \geq r-i$.

2.10 Proof of (0.5) and (0.6). — For (0.5), it is enough to show

$$N^{m+1} = 0 \quad \text{on} \quad \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f/\mathcal{D}_X[N](F_{-1}\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f) \tag{2.10.1}$$

for $m = [n-\alpha_f-\alpha]$ by (1.2.6), where $N = s+\alpha$. Take $\beta \in [\alpha_f, \alpha_f+1)$ such that $k := \alpha - \beta \in \mathbb{Z}$. By (2.2.3) and (2.8.1), we have $F_{-k-1}\,\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = 0$. Applying (2.9) to $(\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f, F), q = -k$ and $i = k-1$, it is enough to show

$$\mathrm{Gr}_j^W \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f = \mathcal{D}_X\big(F_m\,\mathrm{Gr}_j^W \mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f\big) \tag{2.10.2}$$

for m as above (i.e., (2.9.2) is satisfied for $r = [n-\alpha_f-\beta]$). Here the condition (2.9.1) is satisfied by (2.7.7). Furthermore, we have the duality

$$\mathbb{D}\,\mathrm{Gr}_j^W\big(\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f, F\big) = \mathrm{Gr}_{-j}^W\big(\mathrm{Gr}_V^{n-\alpha} \widetilde{\mathcal{B}}_f, F\big) \tag{2.10.3}$$

using (2.7.8)–(2.7.9). We have $F_{-p-1}\,\mathrm{Gr}_V^{n-\alpha} \widetilde{\mathcal{B}}_f = 0$ for $p = m$ by (2.2.3) and (2.8.1), because $n-\alpha-p-1 < \alpha_f$. So (2.10.2) follows from (2.5).

For (0.6), let $\alpha = \beta \in [\alpha_f, \alpha_f+1)$. Then the assertion follows from (2.9) using the remark below.

REMARK. — Let $\varphi_f\mathcal{O}_X = \bigoplus_{0<\alpha\le 1}\operatorname{Gr}_V^\alpha\widetilde{\mathcal{B}}_f$ as in (2.7.2). By Kashiwara [5] and Malgrange [9], we have an isomorphism

$$\mathrm{DR}_X(\varphi_f\mathcal{O}_X) = \varphi_f\mathbb{C}_X[n-1] \tag{2.10.4}$$

such that the action of $\exp(2\pi is)$ on the left hand side corresponds to the monodromy T on the right hand side, where DR_X is the de Rham functor [*loc. cit.*], and $\varphi_f\mathbb{C}_X$ is Deligne's vanishing cycle sheaf complex [2].

2.11 Remark. — We can consider $b_f(s)$ at each point y of $Y := \operatorname{Sing} f^{-1}(0)$, and $m_\alpha(f)$ determines a function $m_\alpha(f,y)$ on Y. By definition $m_\alpha(f,y)$ is upper semicontinuous.

Let $\mathcal{S} = \{S_j\}$ be a Whitney stratification of Y such that $\mathcal{H}^i\varphi_f\mathbb{C}_X{}_{|S_j}$ are local systems (e.g., a Whitney stratification satisfying Thom's A_f-condition). Then, for a subquotient K of $\varphi_f\mathbb{C}_X$ (as a shifted perverse sheaf), $\mathcal{H}^iK_{|S_j}$ are also local systems. Applying this to $\mathrm{DR}_X(\operatorname{Gr}_G^k\operatorname{Gr}_V^\alpha\widetilde{\mathcal{B}}_f)$, we see that the restriction of $m_\alpha(f,y)$ to S_j is locally constant (in particular, $m_\alpha(f,y)$ is a constructible function).

Furthermore, at $y \in S_j$, THEOREMS (0.4)–(0.5) hold with n replaced by $(n-r)$, where $r = \dim S_j$. In fact, it is enough to show that (2.4.1) holds with $F_p\operatorname{Gr}_V^\alpha\widetilde{\mathcal{B}}_f$ replaced by $F_{p-r}\operatorname{Gr}_V^\alpha\widetilde{\mathcal{B}}_f$ (or equivalently, $F_{-p-1}\operatorname{Gr}_V^{n-\alpha}\widetilde{\mathcal{B}}_f$ by $F_{-p-1}\operatorname{Gr}_V^{n-r-\alpha}\widetilde{\mathcal{B}}_f$, using (2.2.3)). This can be checked by restricting to a smooth submanifold Z of X, which intersects S_j transversally (at a general point y of S_j), because the restriction to Z is noncharacteristic, and is given by the tensor of $\mathcal{O}_Z$.

2.12 Remark. — Let $E(\varphi_f\mathbb{C}_X,T)$ be the eigenvalues of the action of the monodromy T on $\varphi_f\mathbb{C}_X$ (as shifted perverse sheaf), where X is restricted to a sufficiently small neighborhood of x. Then we have

$$\exp(2\pi iR_f) = E(\varphi_f\mathbb{C}_X,T) \tag{2.12.1}$$

by (2.3) and (2.10.4). See [9]. (Note that T is defined over $\mathbb{Z}$, and that $E(\varphi_f\mathbb{C}_X,T) = E(\varphi_f\mathbb{C}_X,T^{-1})$.)

Let $X(f,y)$ denote a Milnor fiber of a Milnor fibration defined around $y \in Y$, and define $E(\widetilde{H}^i(X(f,y),\mathbb{C}),T)$ as above. Then we have an isomorphism

$$\mathcal{H}^i(\varphi_f\mathbb{C}_X)_y = \widetilde{H}^i\big(X(f,y),\mathbb{C}\big), \tag{2.12.2}$$

and we get

$$\exp(2\pi iR_f) = \bigcup_{i,j} E(\widetilde{H}^i\big(X(f,y_j),\mathbb{C}),T\big) \tag{2.12.3}$$

for $y_j \in S_j$ with $\mathcal{S} = \{S_j\}$ as in (2.11), where S_j are assumed connected. But

$$\exp(2\pi i R_f) = \bigcup_i E(\widetilde{H}^i(X(f,x),\mathbb{C}),T) \tag{2.12.4}$$

is not true. For example, let $f = xy^3$ on $\mathbb{C}^2$. Then $X(f,0) \simeq \mathbb{C}^*$, and $\bigcup_i E(\widetilde{H}^i(X(f,0),\mathbb{C}),T) = \{1\}$. But $\tilde{b}_f(s) = (s+\frac{1}{3})(s+\frac{2}{3})(s+1)$.

3. Nondegenerate Newton boundary

3.1. — Let $(x_1,\dots,x_n)$ be a local coordinate system around $x \in X$ so that $\mathcal{O} = \mathbb{C}\{x\}$ $(:= \mathbb{C}\{x_1,\dots,x_n\})$. We have a Taylor expansion $f = \sum_\nu a_\nu x^\nu$, where $\nu = (\nu_1,\dots,\nu_n)$ and $x^\nu = \prod x_i^{\nu_i}$. Let $\Gamma_+(f)$ be the convex full of $\nu + (\mathbb{R}_{\geq 0})^n$ for $a_\nu \neq 0$. We define $f_\sigma = \sum_{\nu\in\sigma} a_\nu x^\nu$ for a face σ of $\Gamma_+(f)$. We say that f has *nondegenerate* Newton boundary with respect to the coordinate system [7], if $\partial_i f_\sigma$ $(1 \leq i \leq n)$ have no common zero in $(\mathbb{C}^*)^n$ for any compact face σ of $\Gamma_+(f)$, where $\partial_i = \partial/\partial x_i$. For a face σ of $\Gamma_+(f)$, let $C(\sigma)$ denote the closure of the cone over σ, and $C(\sigma)^\circ = C(\sigma) \setminus \sum_{\tau<\sigma} C(\tau)$, where $\tau < \sigma$ means that τ is a face of σ. Let A_σ denote the $\mathbb{C}$-subalgebra of $\mathbb{C}\{x\}$ generated topologically by x^ν for $\nu \in C(\sigma)$, and B_σ the ideal generated by x^ν for $\nu \in C(\sigma)^\circ$. By 6.4 in [7], f has nondegenerate Newton boundary if and only if

$$\dim_{\mathbb{C}} A_\sigma / \sum_i x_i(\partial_i f_\sigma)A_\sigma < \infty \tag{3.1.1}$$

for any compact face σ. (In fact, if $\partial_i f_\sigma$ $(1 \leq i \leq n)$ have no common zero in $(\mathbb{C}^*)^n$, we have $x^\nu \in \sum_i x_i(\partial_i f_\sigma)\mathbb{C}[x]$ for some ν, and then $x^\nu \in \sum_i x_i(\partial_i f_\sigma)A_\sigma$ by replacing ν.)

For an $(n-1)$-dimensional face σ of $\Gamma_+(f)$, let ℓ_σ denote the linear function whose restriction to σ is one. We define a function $\alpha : \mathbb{N}^n \to \mathbb{Q}$ by $\alpha(\nu) = \min\{\ell_\sigma(\nu)\}$, and $\alpha : \mathcal{O} \to \mathbb{Q}$ by $\alpha(\sum c_\nu x^\nu) = \min\{\alpha(\nu) : c_\nu \neq 0\}$. This induces a filtration V on $\mathcal{O}$ by $V^\alpha\mathcal{O} = \{g \in \mathcal{O} : \alpha(g) \geq \alpha\}$.

PROPOSITION 3.2. — *Assume f has nondegenerate Newton boundary with respect to the coordinate system. Then $V^\alpha\widetilde{\mathcal{B}}_f$ is generated over $\mathcal{D}_{X,x}[\partial_t^{-1},s]$ by $x^\nu\partial_t^i\delta(t-f)$ for $\alpha(\nu+\mathbf{1}) - i \geq \alpha$, where $\mathbf{1} = (1,\dots,1)$.*

Proof. — It is enough to show that the filtration V defined by the above condition satisfies the condition of filtration V in (2.3). The argument is essentially same as [12, 3.6] and [16, (3.3)]. For an $(n-1)$-dimensional face σ, let $\{c_{\sigma,i}\}$ be the coefficients of ℓ_σ, and $\xi_\sigma = \sum_i c_{\sigma,i} x_i \partial_i$ so that $\xi_\sigma f_\tau = f_\tau$ for $\tau < \sigma$. Then we have :

$$(3.2.1)\quad \sum_i c_{\sigma,i}\partial_i x_i\big(x^\nu \delta(t-f)\big) = \ell_\sigma(\nu+\mathbf{1})x^\nu\delta(t-f) - (\xi_\sigma f)\partial_t x^\nu \delta(t-f).$$

We have $\ell_\sigma(\nu+e_i) > \ell_\sigma(\nu)$ if $c_{\sigma,i} \neq 0$. So we can check the nilpotence of the action of $s+\alpha$ on $\mathrm{Gr}_V^\alpha \widetilde{\mathcal{B}}_f$ by induction on $m(\nu) := \#\{\sigma : \ell_\sigma(\nu) = \alpha(\nu)\}$, and it remains to show that $V^\alpha \widetilde{\mathcal{B}}_f$ is finitely generated over $\mathcal{D}_{X,x}[\partial_t^{-1}, s]$. Let $x = x_1 \cdots x_n$. By (1.2.2), the assertion is reduced to the surjectivity of

$$(3.2.2)\qquad \sum_i x_i(\partial_i f) : \bigoplus_i V^\alpha(x\mathcal{O}) \longrightarrow V^{\alpha+1}(x\mathcal{O}) \quad \text{for } \alpha \gg 1.$$

Since $V^\alpha(x\mathcal{O})$ is finitely generated over $\mathcal{O}$, we may replace $V^\alpha(x\mathcal{O})$, $V^{\alpha+1}(x\mathcal{O})$ by $\mathrm{Gr}_V^\alpha(x\mathcal{O})$ and $\mathrm{Gr}_V^{\alpha+1}(x\mathcal{O})$ respectively, using Nakayama's lemma. Taking the graduation of the filtration induced by $m(\nu)$, these terms are further replaced by $(B_\sigma \cap x\mathbb{C}[x])^\alpha, (B_\sigma \cap x\mathbb{C}[x])^{\alpha+1}$ (where the superscript α denotes the degree α part), and f by f_σ. Here we may assume that σ is not contained in the coordinate hyperplanes of $\mathbb{R}^n$. Since A_σ is nœtherian, we can replace $B_\sigma \cap x\mathbb{C}[x]$ by A_σ. So the assertion follows from hypothesis if σ is compact. In the noncompact case, let

$$I(\sigma) = \{i : \sigma + e_i \subset \sigma\}, \quad H(\sigma) = \sum_{i \in I(\sigma)} \mathbb{R}_{\geq 0} e_i,$$

where $e_i \in \mathbb{R}^n$ is the i-th unit vector (i.e. its j-th component is 1 for $j = i$, and 0 otherwise). Then $H(\sigma) + C(\sigma) \subset C(\sigma)$ (in particular, $H(\sigma) \subset C(\sigma)$) and σ is the union of $\tau + H(\sigma)$ for τ compact faces of σ. We define subsets of $H(\sigma)$ by :

$$U^\beta H(\sigma) = \Big\{\sum r_i e_i : \sum r_i \geq \beta\Big\},$$
$$U^{>\beta} H(\sigma) = \Big\{\sum r_i e_i : \sum r_i > \beta\Big\}.$$

Let $U^\beta C(\sigma) = U^\beta H(\sigma) + C(\sigma)$, and $U^\beta A_\sigma$ the ideal of A_σ generated by x^ν for $\nu \in U^\beta C(\sigma)$ (similarly for $U^{>\beta}C(\sigma)$ and $U^{>\beta}A_\sigma$). By Nakayama's lemma, the assertion is reduced to the surjectivity of

$$(3.2.3)\qquad \sum_i x_i(\partial_i f_\sigma) : \bigoplus_i \mathrm{Gr}_U^\beta(A_\sigma)^\alpha \longrightarrow \mathrm{Gr}_U^\beta(A_\sigma)^{\alpha+1} \quad \text{for } \alpha \gg 1.$$

Let $\partial U^\beta H(\sigma) = U^\beta H(\sigma) \setminus U^{>\beta} H(\sigma)$ (similarly for $\partial U^\beta C(\sigma)$). Then $(\partial U^\beta H(\sigma) + \partial U^0 C(\sigma)) \cap \mathbb{Z}^n$ is covered by a finite number of parallel translates of $\partial U^0C(\sigma)\cap\mathbb{Z}^n$ (using a partition of $\partial U^0C(\sigma)$). So $\mathrm{Gr}_U^\beta(A_\sigma)$ is finitely generated over $\mathrm{Gr}_U^0(A_\sigma)$, and we can restrict to the case $\beta=0$. Then the assertion is reduced to the σ compact case by the same argument as above (using the filtration induced by $m(\nu)$), because $\mathrm{Gr}_U^0(A_\sigma)$ is the sum of A_τ for τ compact faces of σ. So the assertion follows.

COROLLARY 3.3. — *We have* $\alpha_f\ge 1/t$ *for* $(t,\dots,t)\in\partial\Gamma_+(f)$.

REMARK. — In the isolated singularity case, it is known that the equality holds by [3], [16] (and [20] in the case $\alpha_f\le 1$) combined with [8].

4. Thom-Sebastiani type theorem

4.1. — Let Y be a complex manifold, $y\in Y$, and $g\in\mathcal{O}_{Y,y}$. Let $Z=X\times Y, z=(x,y)$, and $h=f+g\in\mathcal{O}_{Z,z}$. We define $\widetilde{\mathcal{B}}_g,\widetilde{\mathcal{B}}_h$ as in (1.2). Then we have a short exact sequence

$$0\to\widetilde{\mathcal{B}}_f\boxtimes\widetilde{\mathcal{B}}_g\xrightarrow{\ \iota\ }\widetilde{\mathcal{B}}_f\boxtimes\widetilde{\mathcal{B}}_g\xrightarrow{\ \eta\ }\widetilde{\mathcal{B}}_h\to 0 \tag{4.1.1}$$

with ι,η defined by

$$\begin{aligned}\iota\big(a\partial_t^i\delta(t-f)\otimes b\partial_t^j\delta(t-g)\big)&=a\partial_t^{i+1}\delta(t-f)\otimes b\partial_t^j\delta(t-g)\\&\qquad-a\partial_t^i\delta(t-f)\otimes b\partial_t^{j+1}\delta(t-g),\\ \eta\big(a\partial_t^i\delta(t-f)\otimes b\partial_t^j\delta(t-g)\big)&=ab\partial_t^{i+j}\delta(t-h)\end{aligned}$$

for $a\in\mathcal{O}_{X,x}, b\in\mathcal{O}_{Y,y}$. Here the external product $M\boxtimes N$ for an $\mathcal{O}_{X,x}$-module M and an $\mathcal{O}_{Y,y}$-module N is defined by

$$\mathcal{O}_{Z,z}\otimes_{\mathcal{O}_{X,x}\otimes_{\mathbb{C}}\mathcal{O}_{Y,y}}(M\otimes_{\mathbb{C}}N)\quad\big(=(\mathcal{O}_{Z,z}\otimes_{\mathcal{O}_{X,x}}M)\otimes_{\mathcal{O}_{Y,y}}N\big). \tag{4.1.2}$$

It is an exact functor for both factors (using the second expression) and commutes with inductive limit. By definition, we have

$$\begin{cases}\partial_t\eta(u\otimes v)=\eta(\partial_tu\otimes v)=\eta(u\otimes\partial_tv),\\ t\eta(u\otimes v)=\eta(tu\otimes v)+\eta(u\otimes tv),\\ P\eta(u\otimes v)=\eta(Pu\otimes v),\quad Q\eta(u\otimes v)=\eta(u\otimes Qv),\end{cases} \tag{4.1.3}$$

for $u\in\widetilde{\mathcal{B}}_f, v\in\widetilde{\mathcal{B}}_g, P\in\mathcal{D}_{X,x}, Q\in\mathcal{D}_{Y,y}$. In particular, we have :

$$s\eta(u\otimes v)=\eta(su\otimes v)+\eta(u\otimes sv). \tag{4.1.4}$$

We define a filtration G on $\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g$ by

$$G^k(\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g) = \sum_{i+j=k} G^i\widetilde{\mathcal{B}}_f \boxtimes G^j\widetilde{\mathcal{B}}_g, \tag{4.1.5}$$

and a filtration G' on $\widetilde{\mathcal{B}}_h$ by $G'^k\widetilde{\mathcal{B}}_h = \eta G^k(\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g)$. By Lemma (4.2) below, we have :

$$\mathrm{Gr}_G^k(\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g) = \bigoplus_{i+j=k} \mathrm{Gr}_G^i \widetilde{\mathcal{B}}_f \boxtimes \mathrm{Gr}_G^j \widetilde{\mathcal{B}}_g. \tag{4.1.6}$$

Then $\mathrm{Gr}_G \iota : \mathrm{Gr}_G^{k+1}(\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g) \to \mathrm{Gr}_G^k(\widetilde{\mathcal{B}}_f \boxtimes \widetilde{\mathcal{B}}_g)$ is injective (i.e., ι is strictly injective), and we get an isomorphism

$$\mathrm{Gr}_G \eta : \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f \boxtimes \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_g \xrightarrow{\sim} \mathrm{Gr}_{G'}^0(\widetilde{\mathcal{B}}_h) \tag{4.1.7}$$

by taking the graduation of (4.1.1). Furthermore, the action of s on the right hand side corresponds to that of $s \boxtimes \mathrm{id} + \mathrm{id} \boxtimes s$ on the left.

LEMMA 4.2. — *For an $\mathcal{O}_{X,x}$-module M and an $\mathcal{O}_{Y,y}$-module N with an exhaustive filtration G, we define a filtration G on $M \boxtimes N$ as in* (4.1.5). *Then* (4.1.6) *holds with $\widetilde{\mathcal{B}}_f, \widetilde{\mathcal{B}}_g$ replaced by M, N.*

Proof. — Since the external product is exact, we can replace M, N by G^pM, G^pN, considering inductive systems $(G^{-p}M, F), (G^{-p}N, F)$. So we may assume $G^pM = M, G^pN = N$ for $p \ll 0$. Then the summation in (4.1.6) is a finite direct sum, and we get the assertion taking the graduation of the filtration G on M, because $G^k(\mathrm{Gr}_G^i M \boxtimes N) = \mathrm{Gr}_G^i M \boxtimes G^{k-i}N$.

4.3 Proof of (0.7). — By (1.2.5) (4.1.3), we have

$$G'^k\widetilde{\mathcal{B}}_h = \eta(G^i\widetilde{\mathcal{B}}_f \boxtimes G^{k-i}\widetilde{\mathcal{B}}_g). \tag{4.3.1}$$

By [4], $G^0\mathcal{B}_f = \mathcal{D}_{X,x}[s]\delta(t-f)$ (resp. $G^0\widetilde{\mathcal{B}}_f = \sum_{i\geq 0} \partial_t^{-i} G^0\mathcal{B}_f$) is finite over $\mathcal{D}_{X,x}$ (resp. over $\mathcal{D}_{X,x}[\partial_t^{-1}]$). So we get

$$G'^k\widetilde{\mathcal{B}}_h \text{ are lattices of } \widetilde{\mathcal{B}}_h \quad \text{(see (2.3))}, \tag{4.3.2}$$

$$G'^k\widetilde{\mathcal{B}}_h \supset G^k\widetilde{\mathcal{B}}_h, \tag{4.3.3}$$

using (4.1.3). Then the assertion follows from (2.3).

4.4 Proof of (0.8). — Since $s\delta(t-g) = \xi\delta(t-g)$, we have

$$G^0\widetilde{\mathcal{B}}_g = \mathcal{D}_{Y,y}[\partial_t^{-1}]\delta(t-g),$$

and, by (4.1.4),

$$\text{(4.4.1)}\quad \eta\big(s^i\delta(t-f)\otimes\delta(t-g)\big) = s\eta\big(s^{i-1}\delta(t-f)\otimes\delta(t-g)\big) - \xi h\big(s^{i-1}\delta(t-f)\otimes\delta(t-g)\big).$$

So we get the equality :

$$\text{(4.4.2)}\qquad G'^k\widetilde{\mathcal{B}}_h = G^k\widetilde{\mathcal{B}}_h.$$

Taking Gr_V of (4.1.7), we have an isomorphism

$$\text{(4.4.3)}\qquad \bigoplus_{\alpha+\beta=\gamma} \mathrm{Gr}_V^\alpha \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f \boxtimes \mathrm{Gr}_V^\beta \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_g = \mathrm{Gr}_V^\gamma \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_h$$

by (4.2), because $\mathrm{Gr}_V^\alpha \mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f$ is identified with the α-eigenspace of $\mathrm{Gr}_G^0 \widetilde{\mathcal{B}}_f$ by the action of $-s$. So the assertion follows.

4.5 Remark. — The short exact sequence (4.1.1) is due to a discussion with J. STEENBRINK in 1987 at MPI. It is used to prove the Thom-Sebastiani type theorem for the vanishing cycles of filtered regular holonomic $\mathcal{D}$-Modules. This subject will be treated in a joint paper with him.

4.6 Remark. — In the isolated singularity case, MALGRANGE [10] showed essentially the natural isomorphism

$$\text{(4.6.1)}\qquad L_h = L_f \otimes_A L_g,$$

with the notation of (1.7) and (4.7) below. Using this and (1.7.1), we can easily check (0.7-8) in the isolated singularity case. This also gives an example such that (0.8) does not hold in the non quasi-homogeneous singularity case. See (4.8) below.

4.7 Remark. — In this paragraph, we denote by $\mathcal{E}$ the ring of microdifferential operators of one variable $\mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}[\partial_t]$, and let $\mathcal{E}(0) = \mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$ the subring of microdifferential operators of order ≤ 0. See [4], [17]. We define subrings of $\mathcal{E}$ by

$$K = \mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t], \quad A = \mathbb{C}\{\{\partial_t^{-1}\}\}.$$

Let M be a regular holonomic $\mathcal{E}$-module. An $\mathcal{E}(0)$-submodule L of M is called a *lattice* if it is finite over $\mathcal{E}(0)$ and generates M over $\mathcal{E}$. The *saturation* $\bar{L}$ of L is defined by

$$\bar{L} = \sum_{i\geq 0}(t\partial_t)^i L. \tag{4.7.1}$$

Note that $\bar{L}$ is also a lattice of M by regularity.

Let M_j $(j = 1, 2)$ be two regular holonomic $\mathcal{E}$-modules, and L_j a lattice of M_j. Let

$$M = M_1 \otimes_K M_2, \quad L = L_1 \otimes_A L_2. \tag{4.7.2}$$

Then M is a regular holonomic $\mathcal{E}$-module, and L is a lattice of M, where the action of t on M is defined by

$$t(u \otimes v) = tu \otimes v + u \otimes tv \quad \text{for } u \in M_1,\ v \in M_2. \tag{4.7.3}$$

However, we have

$$\bar{L} \neq \bar{L}_1 \otimes_A \bar{L}_2 \tag{4.7.4}$$

in general. For example, consider the case $M_1 = M_2, L_1 = L_2$, and L_j has a generator e_1, e_2 over A such that $\partial_t te_1 = e_1 + \partial_t e_2, \partial_t te_2 = 2e_2$. Then $\bar{L}$ is generated over A by $e_1 \otimes e_1$, $\partial_t(e_1 \otimes e_2 + e_2 \otimes e_1)$, $\partial_t^2(e_2 \otimes e_2)$ and $e_1 \otimes e_2$, and $\bar{L}_j$ by e_1 and $\partial_t e_2$.

4.8 Example. — Consider the singularity of type $T_{p,q,r}$:

$$f = x^p + y^q + z^r + xyz \quad \text{for } p^{-1} + q^{-1} + r^{-1} < 1.$$

Then L_f is generated over A by e, e' and $e_{a,i}$ $(1 \leq a \leq 3,\ 0 < i < p_a)$ such that

$$\partial_t te = e + \partial_t e', \quad \partial_t te' = 2e', \quad \partial_t te_{a,i} = (1 + i/p_a)e_{a,i},$$

where $p_1 = p$, $p_2 = q$, $p_3 = r$. This can be checked for example using [14, 3.4]. In particular, we get by (1.7.1) :

$$\tilde{b}_f(s) = (s+1)^2 \prod_{0<i<p} (s + 1 + i/p) \quad \text{if } p = q = r. \tag{4.8.1}$$

Let $h = f + g$ as in (4.1). Assume f, g singularities of type $T_{p,p,p}$ and $T_{q,q,q}$ respectively, and $(p,q) = 1$. Then

$$(4.8.2)\quad \tilde{b}_h(s) = (s+2)^3(s+3) \prod_{\substack{0<i<p \\ 0<j<q}} (s+2+i/p+j/q) \prod_{0<i<p} (s+2+i/p)^2 \prod_{0<j<q} (s+2+j/q)^2.$$

This gives a counter example to (0.8) in the non quasi-homogeneous case.

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