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AN INFINITE DIMENSIONAL HODGE-TATE THEORY

BY

SHANKAR SEN (*)

RÉSUMÉ. — Soient \mathcal{G} le groupe de Galois absolu d'un corps p -adique K et \mathcal{R} une algèbre de Banach sur K . Étant donné un homomorphisme continu $\rho : \mathcal{G} \rightarrow \mathcal{R}^*$ (\mathcal{R}^* = groupe des unités de \mathcal{R}) on construit un «opérateur» canonique $\varphi \in \mathcal{R} \widehat{\otimes}_K \mathbb{C}$ qui détermine la \mathbb{C} -extension de ρ à isomorphisme local près ($\widehat{\otimes}$ = produit tensoriel complet, \mathbb{C} = complétion d'une fermeture algébrique de K). Si \mathcal{R} est un anneau de matrices sur un anneau de séries entières convenable, l'opérateur φ permet d'étudier la structure de Hodge-Tate de familles de représentations de \mathcal{G} de dimension finie.

ABSTRACT. — Let \mathcal{G} be the absolute Galois group of a p -adic field K and \mathcal{R} a Banach algebra over K . Given a continuous homomorphism $\rho : \mathcal{G} \rightarrow \mathcal{R}^*$ (\mathcal{R}^* = units of \mathcal{R}) we construct a canonical «operator» $\varphi \in \mathcal{R} \widehat{\otimes}_K \mathbb{C}$ which determines the \mathbb{C} -extension of ρ upto local isomorphism ($\widehat{\otimes}$ = complete tensor product, and \mathbb{C} = completion of an algebraic closure of K). If \mathcal{R} is a matrix ring over a suitable power series ring one obtains information about the variation of the Hodge-Tate structure in families of finite-dimensional representations of \mathcal{G} .

Introduction

In this paper we are concerned with a two-step generalization of the p -adic analogue of the Hodge decomposition for complex cohomology which TATE discovered in connection with his theory of p -divisible groups [10].

Suppose given a continuous homomorphism $\psi : \mathcal{G} \rightarrow \text{Aut}_K V$ where \mathcal{G} is the absolute Galois group of the p -adic field K and V is a finite-dimensional K -vector space. Let \mathbb{C} be the completion of an algebraic closure of K , X the \mathbb{C} -vector space $V \otimes_K \mathbb{C}$ on which \mathcal{G} acts diagonally, and χ the p -cyclotomic character of \mathcal{G} . Recall that ψ (more precisely (V, ψ)) is

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said to have a Hodge-Tate decomposition if $X = \bigoplus_{i \in \mathbb{Z}} X(i)$, where $X(i)$ is the \mathbb{C} -subspace of X spanned by all $x \in X$ such that $\sigma(x) = \chi^i(\sigma)x$ for all $\sigma \in \mathcal{G}$. (TATE showed that for any ψ the sum is direct, so that only finitely many of the $X(i)$'s are non-zero.) In [7] we constructed, for arbitrary ψ , a canonical operator φ on X which has the eigenvalue i and corresponding eigenspace $X(i)$ whenever the latter is non-zero. Then φ , or rather its primary subspaces, may be regarded as giving a generalized Hodge-Tate decomposition for any ψ .

Following a suggestion of MAZUR it is natural to try to generalize the construction of φ . He had in mind particularly the case of representations over p -adic power series rings or their quotients (more on this below). Here we deal with something in principle still more general (though as yet we know of no new examples to exploit this generality). Given a continuous homomorphism $\psi : \mathcal{G} \rightarrow \mathcal{R}^*$ where \mathcal{R}^* is the group of units of a Banach algebra \mathcal{R} over K , we define a canonical element $\varphi \in \mathcal{B} = \mathcal{R} \widehat{\otimes} \mathbb{C}$ (where $\widehat{\otimes}$ denotes the complete tensor product over K) which agrees with the « φ » of the previous paragraph in case $\mathcal{R} = \text{End}_K V$ with V as above. The «operator» φ (in fact its conjugacy class in \mathcal{B}) determines the semi-linear \mathbb{C} -extension of ψ upto local isomorphism («local» means «on an open subgroup of \mathcal{G} »). Moreover it has natural functorial properties with respect to Banach algebra homomorphisms, and, though we don't make it explicit, tensor products. Further, the conjugacy class of φ in \mathcal{B} is defined over some finite extension K' of K (i.e., φ is conjugate to an element of $\mathcal{R} \otimes_K K'$) providing a comparatively down-to-earth invariant of ψ (defined over K if $\mathcal{R} = \text{End}_K V$). This invariant may be seen as taking one a step towards classifying the representation ψ .

Let D be any quotient of a power series ring in finitely many variables over \mathbb{Z}_p (for simplicity). Then a (continuous) representation $\psi_0 : \mathcal{G} \rightarrow \text{GL}_n(D)$ is of the sort obtained by restricting Mazur's universal deformations ([5]; see also [4], [6]) to the local Galois group. Note that ψ_0 is equivalent to a p -adic analytic family of representations in $\text{GL}_n(\mathbb{Z}_p)$. MAZUR proposed that φ should be defined for ψ_0 , for this would imply that the (generalized) Hodge-Tate structure varies analytically in such families, giving information also about the variation of the image of the Galois group. In [8] the construction of φ was carried out to the point needed to settle positively the question of analytic variation, though we worked, as here, not directly with ψ_0 but with the composed representation ψ obtained by imbedding $\text{GL}_n(D)$ in \mathcal{R}^* for a suitable Banach algebra \mathcal{R} . Our present results go beyond those of [8] in several respects. First, as mentioned already, φ is defined here for any \mathcal{R} (which is then specialized for the application to ψ_0), while in [8] \mathcal{R} has to be of a particular type (although

some of the key lemmas of [8] are general, and are used here). Secondly, in [8] the φ obtained is not shown to be canonical, whereas here we show it is the unique element satisfying the appropriate conditions. Finally, the \mathbb{C} -extension of \mathcal{R} used in [8] does not have the right technical property (the action of \mathcal{G} is not «sufficiently» continuous). Here the \mathbb{C} -extension \mathcal{B} of \mathcal{R} , which is just the complete tensor product, not only corrects the mistake in [8] but even strengthens the result.

0. Notation

The following is a partial list of the notation used, having to do mainly with fields and Galois groups :

K = a p -adic field, i.e., a local field of characteristic 0 with a perfect residue field of characteristic $p \neq 0$.

π = a uniformizing parameter of K .

A = the ring of integers of K (it is a complete discrete valuation ring).

\bar{K} = an algebraic closure of K .

\mathcal{G} = the Galois group of \bar{K}/K .

\mathbb{C} = the completion of \bar{K} with respect to the absolute value which extends that of K . This absolute value extends canonically to \mathbb{C} and is denoted by $|\cdot|$.

$\chi : \mathcal{G} \rightarrow \mathbb{Q}_p^*$ = a continuous infinitely ramified character, i.e., there exists an infinitely ramified extension of K whose Galois group is mapped isomorphically into \mathbb{Q}_p^* by χ . The canonical choice of χ is the p -power cyclotomic character.

$\mathcal{H} = \text{Ker } \chi$.

Γ = the Galois group of $\bar{K}^{\mathcal{H}}/K$ (superscripts denote invariants).

Γ_0 = the maximal pro- p -subgroup of Γ , so that $\Gamma_0 \xrightarrow{\sim} \mathbb{Z}_p$.

$\Gamma_n = \Gamma_0^{p^n}$.

\mathcal{G}' (resp. \mathcal{G}'') = an open subgroup of \mathcal{G} (satisfying further conditions depending on the context).

\mathcal{H}' (resp. \mathcal{H}'') = $\mathcal{H} \cap \mathcal{G}'$ (resp. $\mathcal{H} \cap \mathcal{G}''$).

Γ' (resp. Γ'') = $\mathcal{G}'/\mathcal{H}'$ (resp. $\mathcal{G}''/\mathcal{H}''$).

Γ'_0 (resp. Γ''_0) = the maximal pro- p -subgroup of Γ' (resp. Γ'').

Γ'_n (resp. Γ''_n) = $(\Gamma'_0)^{p^n}$ (resp. $(\Gamma''_0)^{p^n}$).

\mathcal{G}_n (resp. $\mathcal{G}'_n, \mathcal{G}''_n$) = inverse image in \mathcal{G} (resp. $\mathcal{G}', \mathcal{G}''$) of Γ (resp. Γ'_n, Γ''_n).

1. Complete tensor products

This section is devoted to elementary results, some of which appear in the literature, but are still collected here in a form that suits us for later use in this paper, or perhaps elsewhere.

1.1. Normed linear spaces. — Let V be a normed linear space over K satisfying, as do all the normed spaces we consider, the non-archimedean axiom $|x + y| \leq \max\{|x|, |y|\}$. Let V_0 be the subset of V consisting of all x such that $|x| \leq 1$ (the unit ball in other normed linear spaces will be denoted similarly). Then V_0 is an A -module. Let $\{v_i\}_{i \in I}$ be a set of elements of V such that the image set $\{\bar{v}_i\}$ is a basis of $V_0/\pi V_0$ over the residue field $A/\pi A$ (we omit the index set I when confusion is not likely to result). We shall call $\{v_i\}$ a «basis» of V . Such a «basis» has the following properties :

(1) The subspace of V spanned by $\{v_i\}$ is dense in V .

(2) Every element v of V has a unique expression $v = \sum c_i v_i$, where $c_i \in K$ and $c_i \rightarrow 0$ with respect to the filter of complements of finite subsets in I . (The equality of course means that the sum converges to v in the topology of V . By abuse of notation we shall write $c_i \rightarrow 0$ as $i \rightarrow \infty$). For a given v , only countably many of the c_i 's are non-zero.

(3) The completion \widehat{V} of V can be identified with the K -vector space, with the obvious operations, of all expressions of the form $\sum c_i v_i$, where $c_i \in K$ and $c_i \rightarrow 0$ as $i \rightarrow \infty$. The norm $\|v\| = \sup |c_i|$ on \widehat{V} (and V), where $v = \sum c_i v_i$, is equivalent to the norm of \widehat{V} induced by the original norm of V .

(4) If W is a second normed linear space over K then $\mathcal{L}(V, W)$, the space of continuous linear maps from V to W , can be identified with the space of families $(w_i)_{i \in I}$, $w_i \in W$ such that $\sup |w_i| < \infty$. The usual norm on $\mathcal{L}(V, W)$ is equivalent to the norm $|(w_i)| = \sup |w_i|$.

The above facts, notation apart, are the content of [2, I.23, ex. 7]. One has further :

(5) If $|\cdot|$ is the norm of V , and $\|\cdot\|$ the modified norm defined in (3), which we shall call *basal*, then :

$$\|x\| = |\pi|^n \iff |\pi|^{n+1} < |x| \leq |\pi|^n.$$

Thus to each norm $|\cdot|$ is associated a unique basal norm $\|\cdot\|$ independently of the choice of «basis».

Putting $y = \pi^{-n}x$, and $y = \sum c_i v_i$, this follows from :

$$\begin{aligned} \|y\| = 1 &\Leftrightarrow \sup |c_i| = 1 \\ &\Leftrightarrow y \in V_0 \text{ and } y \notin \pi V_0 \\ &\Leftrightarrow 1 \geq |y| > |\pi|. \end{aligned}$$

(6) Suppose V is complete and let W be a closed subspace. Then W has a topological complement in V .

This is [2, I.26, ex. 11]. It follows from the fact that the natural map $W_0/\pi W_0 \rightarrow V_0/\pi V_0$ is an injection so that one may choose a «basis» of W and extend it to one of V .

1.2. Lattices. — An element x of an A -module M will be called *divisible* if it is divisible by π in M , *infinitely divisible* if it is divisible by all powers of π in M .

An A -module M will be called a *lattice* if it is torsion-free and contains no infinitely divisible elements. If V is any K -vector space then an A -submodule M of V will be called a *lattice in V* if M is a lattice and if $KM = V$. A finitely generated A -module is a lattice if and only if it is free. If V is finite-dimensional then a lattice in V is any A -submodule generated by a basis of V . (See [1, chap. VII, §4.1.]) In the infinite-dimensional case these last two statements are not true. For example an infinite product of copies of A is a lattice but it is not free, since the product is complete in the π -adic topology but an infinite direct sum is not.

All vector spaces are assumed to be vector space over K unless the context clearly indicates otherwise. Tensor products of vector spaces will be taken over K , unless otherwise indicated, and the subscript K will be omitted. Tensor products of two A -modules, unless both K -vector spaces, are taken over A , and again the subscript is omitted.

If V is a normed linear space it is clear that V_0 is a lattice in V . Conversely, given a lattice M in a vector space V one can define a norm on V as follows. For $x \in V$, define $\mathcal{O}(x)$, the order of x with respect to M , by $\mathcal{O}(x) =$ the largest integer r such that $\pi^{-r}x \in M$. Since M is a lattice in V it follows that for each $x \neq 0$ the integer $\mathcal{O}(x)$ does exist. Setting $|x| = |\pi|^{\mathcal{O}(x)}$ gives a non-archimedean norm on V , which we shall say is defined by the lattice M , and it is clear that V_0 , the unit ball with respect to this norm, is equal to M . The topology induced on M by this norm is the usual π -adic topology. It is also evident that the basal norm in (5) is the same as the norm on V defined by the lattice V_0 .

We shall need the following fact about the tensor product of lattices :

(7) Let M and N be lattices in the vector spaces V and W respectively. Then $M \otimes N$ (identified with its image) is a lattice in $V \otimes W$.

To prove (7) one uses the standard properties of tensor products over Prüfer rings in general [3, chap. VII, § 4] or principal ideal domains in particular [1, chap. I, § 2.4]. The main property needed, apart from general consequences of flatness, is that over such rings a module is flat if and only if it is torsion-free. Since the tensor product of flat modules is flat it follows immediately that $M \otimes N$ is flat, hence torsion-free. This implies that the natural map from $M \otimes N$ to $V \otimes W$ is injective, and, identifying $M \otimes N$ with its image, that $K(M \otimes N) = V \otimes W$ (using [1, chap. II, § 2.7, prop. 18]).

It remains to be shown that $M \otimes N$ has no non-zero infinitely divisible elements. This can be done by reducing to the finite rank case as follows. Given $x \in M \otimes N$ let V' and W' be finite dimensional subspaces of V and W such that $x \in V' \otimes W'$. Let $M' = M \cap V'$, $N' = N \cap W'$. Then M/M' and N/N' , being submodules of the vector spaces V/V' and W/W' , are torsion-free, hence flat. Again because the product of flat modules is flat, one sees that

$$\begin{aligned} (M \otimes N') / (M' \otimes N') &\xrightarrow{\sim} M/M' \otimes N', \\ (M \otimes N) / (M \otimes N') &\xrightarrow{\sim} M \otimes N/N' \end{aligned}$$

are flat. Then because an extension of a flat module by a flat module is flat, or more simply by replacing «flat» by «torsion-free», one sees that $(M \otimes N) / (M' \otimes N')$ is torsion-free. It follows that x is infinitely divisible in $M \otimes N$ if and only if it is infinitely divisible in $M' \otimes N'$. Since the latter module is free (because it is torsion-free of finite rank) x cannot be infinitely divisible if $x \neq 0$. Thus $M \otimes N$ is a lattice in $V \otimes W$, proving (7).

1.3. Tensor products of normed spaces.—Let V and W be normed linear spaces. We define several norms (product norms) on $V \otimes W$ which turn out to be equivalent and thus may be used to define the «same» (as topological vector space) complete tensor product. This leaves open the question of whether there is more than one natural equivalence class of norms on $V \otimes W$ as in the Archimedean case.

(A) Let $|\cdot|_1$ be the norm on $V \otimes W$ defined by the lattice $V_0 \otimes W_0$ as in § 1.2. This is stable under equivalences in the sense that if the norms on V and W are replaced by equivalent norms, then $|\cdot|_1$ is also replaced by an equivalent norm. For suppose V_0^* and W_0^* are the unit balls for the second pair of norms. Then the equivalence of norms corresponds to

relations of the form $\pi^a V_0 \subset V_0^* \subset \pi^b V_0$ and $\pi^c W_0 \subset W_0^* \subset \pi^d W_0$ for some integers a, b, c, d , which in turn imply

$$\pi^{a+c} V_0 \otimes W_0 \subset V_0^* \otimes W_0^* \subset \pi^{b+d} V_0 \otimes W_0$$

and the equivalence of the product norms.

Consider the special case in which the norms on V and W are defined by V_0 and W_0 as in § 1.2. Then $|v \otimes w|_1 = |v| \cdot |w|$ for all $v \in V, w \in W$. To see this let V' and W' be the one-dimensional subspaces of V and W spanned by v and w respectively. Let $V'_0 = V_0 \cap V', W'_0 = W_0 \cap W'$ and suppose V'_0 and W'_0 are generated by $\pi^a v$ and $\pi^b w$, which means $|v| = |\pi|^{-a}, |w| = |\pi|^{-b}$. Then $x = \pi^{a+b}(v \otimes w)$ generates $V'_0 \otimes W'_0$. By the argument given above in proving (7) x is not divisible in $V_0 \otimes W_0$, so that $|x| = 1$ and $|v \otimes w|_1 = |v| \cdot |w|$.

(B) Choose «bases» $\{v_i\}$ and $\{w_j\}$ of V and W respectively (where as usual we have suppressed the index sets I and J). Define Y to be the normed linear space (with the natural operations on the coordinates a_{ij}) of all expressions of the form $\sum a_{ij}(v_i, w_j)$, where $a_{ij} \in K$, and $a_{ij} \rightarrow 0$ as $i \rightarrow \infty$ or $j \rightarrow \infty$, with the norm of such an expression being $\sup |a_{ij}|$. Then $\{(v_i, w_j)\}$ is clearly a «basis» of Y . Define a (continuous) bilinear pairing $V \times W \rightarrow Y$ by putting $(v, w) \mapsto \sum a_i b_j (v_i, w_j)$, where $v = \sum a_i v_i$ and $w = \sum b_j w_j$.

This pairing induces a map $f : V \otimes W \rightarrow Y$, taking $v_i \otimes w_j$ to (v_i, w_j) , which is in fact injective. To see this injectivity it is enough to realize that the restriction $f : V_0 \otimes W_0 \rightarrow Y_0$ is injective. For the latter, tensor with the residue field, which gives a map $\bar{f} : V_0/\pi V_0 \otimes W_0/\pi W_0 \rightarrow Y_0/\pi Y_0$, and this map is injective, actually an isomorphism, because it carries $\bar{v}_i \otimes \bar{w}_j$ to (\bar{v}_i, \bar{w}_j) , or a basis of the first space onto a basis of the second in a 1-1 way (the bars denote the image (mod π)). Identify $V_0/\pi V_0 \otimes W_0/\pi W_0$ with $(V_0 \otimes W_0)/\pi(V_0 \otimes W_0)$. Since $V_0 \otimes W_0$ is a lattice by (7) above, we may replace x by $y = \pi^a x$ where y is an indivisible element of $V_0 \otimes W_0$. Then $\bar{y} \neq 0$ (where \bar{y} is the image of y (mod π)), which implies $\bar{f}(\bar{y}) \neq 0$, hence $f(y) \neq 0$ and so finally $f(x) \neq 0$, which shows f is injective.

Let $|\cdot|_2$ be the norm on $V \otimes W$ induced, *via* the injection f , by the norm of Y . A priori it depends on the choice of «bases» in V and W but in fact $|\cdot|_2$ is the same as the norm $|\cdot|_1$, defined in (A). If y is an indivisible element of $V_0 \otimes W_0$ then from what we saw above, $f(y)$ is an indivisible element of Y_0 , which implies $|y|_2 = 1$, while from the definition of $|\cdot|_1$ it is clear that $|y|_1 = 1$. Since every $x \in V \otimes W$ is a scalar times such an y , it follows that $|x|_1 = |x|_2$ for all x .

NOTE. — Since f is 1-1 we may without confusion write $v_i \otimes w_j$ for (v_i, w_j) .

(C) An asymmetric variant of (B) will be convenient for technical reasons. For this let Z be the normed linear space of all expressions of the form $\sum [v_i, w_i]$ where $\{v_i\}$ is a «basis» of V (let us say the same as in (B)) and $w_i \in \widehat{W}$ (where \widehat{W} is the completion of W) with $w_i \rightarrow 0$ as $i \rightarrow \infty$. Addition and scalar multiplication are defined by addition and scalar multiplication of the «coefficients» w_i while the norm of the above expansion is taken to be $\sup |w_i|$. One defines a (continuous) pairing $V \times W \rightarrow Z$ by putting $(v, w) \mapsto \sum [v_i, a_i w]$ where $v = \sum a_i v_i$ (i.e., take $w_i = a_i w$). This defines a map $g : V \otimes W \rightarrow Z$ which is injective because g is carried to f under an obvious identification of Z with Y : if $\{w_j\}$ is the «basis» of W chosen in (B), map Z to Y by sending $\sum [v_i, w_i]$ to $\sum a_{ij}(v_i, w_j)$ where $w_i = \sum a_{ij} w_j$. This clearly carries the pairing of this paragraph to the one in (B), hence g to f . The norm on Z need not agree with that of Y under this identification, but they are equivalent because the norm of W is equivalent to the basal norm given by $\{w_j\}$ (by (3) or (5) of § 1.1) and thus using $|w_i|$ or $\sup_j |a_{ij}|$ must give equivalent norms on $Y = Z$.

Define $|\cdot|_3$ on $V \otimes W$ via the imbedding g in Z . From what we have seen $|\cdot|_3$ is equivalent to $|\cdot|_1 = |\cdot|_2$, but unlike the latter it depends on the choice of a «basis» of V , thus $|\cdot|_3$ is really one of an infinite number of possible norms from which there is in general no natural way to pick one.

NOTE. — Because g is 1-1 even after replacing W by \widehat{W} it is possible to write $v_i \otimes w_i$ for $[v_i, w_i]$ without confusion.

1.4. The completion. — Define the complete tensor product of V and W to be the completion, denoted $V \widehat{\otimes} W$, of $V \otimes W$ with respect to any of the norms defined above. Since all the norms are equivalent, this determines $V \widehat{\otimes} W$ up to canonical isomorphism as a topological vector space. Although the norm in (A) or (B) is also uniquely defined, it will usually be more convenient to select instead one of those described in (C). The resulting ambiguity in the structure of $V \widehat{\otimes} W$ as a normed linear space is perhaps a blemish, but it will have no serious consequences.

Since Y and Z are both complete spaces, and the image of $V \otimes W$, under f and g respectively, is dense in each, either may be identified with $V \widehat{\otimes} W$ (via f or g which depend on choice of «bases»). In making such an identification we shall select the corresponding norm on $V \widehat{\otimes} W$. Finally, note that $V \widehat{\otimes} W = \widehat{V} \widehat{\otimes} \widehat{W}$. (Here \widehat{V} , \widehat{W} are of course the completions of V and W and equality means that the natural map from the left hand space

to the one on the right is an isomorphism. That this is the case follows from the fact that both sides may be identified with Y .)

1.5. Normed algebras. — Let F be a non-archimedean normed field — we shall only need the case where F is a subfield of \mathbb{C} . By a normed F -algebra we shall mean an F -algebra which is a normed linear space over F satisfying as always the non-archimedean axiom (\mathcal{N}) as well as the multiplicative axiom (\mathcal{M}) : $|xy| \leq |x| \cdot |y|$. If $F = K$ we shall usually omit the field.

Let \mathcal{R} be such an algebra. Then the basal norm on \mathcal{R} also satisfies (\mathcal{M}). For let $\{r_i\}$ be a «basis» of \mathcal{R} , and $x = \sum a_i r_i, y = \sum b_i r_i$ be two elements and their expansions as in (2). Because of the axiom (\mathcal{N}), the product xy can be obtained by formal multiplication of the expansions of x and y . Also, if $\| \cdot \|$ is the basal norm then for all i, k (running through the same index set) $\|r_i r_k\| \leq 1$ because $|r_i r_k| \leq |r_i| \cdot |r_k| \leq 1$. Then one has :

$$\|xy\| = \left\| \sum a_i b_k r_i r_k \right\| \leq \sup \|a_i b_k r_i r_k\| \leq \sup |a_i b_k| = \|x\| \cdot \|y\|.$$

Consider next two normed algebras \mathcal{R} and \mathcal{S} and their tensor product $\mathcal{R} \otimes \mathcal{S}$ equipped with one of the product norms discussed above. It may be seen that this product norm also satisfies axiom (\mathcal{M}). In the symmetric case, i.e., where the product norm is defined as in (A) or (B), this follows easily from the fact that $\mathcal{R}_0 \otimes \mathcal{S}_0 = (\mathcal{R} \otimes \mathcal{S})_0$ is closed under multiplication : given x and y , let $x' = \pi^a x$ and $y' = \pi^b y$ be indivisible elements of $\mathcal{R}_0 \otimes \mathcal{S}_0$ so that $|x'| = |y'| = 1$. But $x'y' \in \mathcal{R}_0 \otimes \mathcal{S}_0$ which shows $|x'y'| \leq 1$. This implies $|xy| \leq |x| \cdot |y|$.

In the unsymmetric case the proof is formally the same as for the norm $\| \cdot \|$ above. Let $x = \sum r_i \otimes s_i, y = \sum r_i \otimes s'_i$, with $s_i, s'_i \in \mathcal{S}$. Using the fact that $\|r_i r_k\| \leq 1$ one sees :

$$\|xy\| = \left\| \sum r_i r_k \otimes s_i s'_k \right\| \leq \sup \|r_i r_k \otimes s_i s'_k\| \leq \sup |s_i s'_k| \leq \|x\| \cdot \|y\|.$$

Passing to the completion we see that $\widehat{\mathcal{R} \otimes \mathcal{S}}$ is a complete normed algebra, or Banach algebra, with respect to either type of norm (symmetric or unsymmetric case).

1.6. Base extension. — The case which we particularly need is the one in which \mathcal{S} is a normed field extension of K (with a norm extending that of K). Then $\mathcal{R} \otimes \mathcal{S}$ and $\widehat{\mathcal{R} \otimes \mathcal{S}}$ are normed \mathcal{S} -algebras, i.e., satisfy also the axiom $|sx| = |s| \cdot |x|$ for all $s \in \mathcal{S}$ provided we choose the product norm as in (C) — as we shall unless otherwise stated. (Multiplication by

elements of \mathcal{S} is defined in the obvious way *via* its copy $1 \otimes \mathcal{S}$ inside the algebra. This is the same as the action on the second component given by $s(\sum r_i \otimes s_i) = \sum r_i \otimes s s_i$ from which the property $|sx| = |s| \cdot |x|$ follows immediately.) Notice that if \mathcal{R} is complete and \mathcal{S} is of finite degree over K then $\mathcal{R} \otimes \mathcal{S} = \mathcal{R} \widehat{\otimes} \mathcal{S}$ or, in other words, $\mathcal{R} \otimes \mathcal{S}$ is already complete.

1.7. An example. — Let V be a complete normed linear space and \mathcal{R} the algebra of bounded operators on V with the usual norm. Choose a «basis» $\{v_i\}$ of V . With respect to this «basis» \mathcal{R} can be identified with the ring of (in general infinite) matrices over K with bounded entries indexed by $I \times I$ in which, further, the entries approach 0 along the columns (or rows if one prefers).

Similarly let \mathcal{B} be the algebra of all \mathbb{C} -linear bounded operators on $V \widehat{\otimes} \mathbb{C}$. Then with respect to the \mathbb{C} -«basis» $\{v_i\}$, \mathcal{B} is the ring of matrices over \mathbb{C} , of the same «size» as before, satisfying the same conditions on the entries as does \mathcal{R} . Then $\mathcal{R} \widehat{\otimes} \mathbb{C}$ is a subalgebra of \mathcal{B} but not equal to it unless I is finite. For $b = (b_{ik}) \in \mathcal{B}$, with $i, k \in I$, we know that $b \in \mathcal{R} \widehat{\otimes} \mathbb{C}$ if and only if for any «basis» $\{c_j\}$ of \mathbb{C} one has $b = \sum r_j \otimes c_j$, with $r_j \in \mathcal{R}$ such that $r_j \rightarrow 0$ as $j \rightarrow \infty$. This condition means that $\sup_{i,k} |r_{ijk}| \rightarrow 0$ as $j \rightarrow \infty$ where $b_{ik} = \sum_j r_{jik} \otimes c_j$ with $r_{jik} \in K$, while for a general element b of \mathcal{B} we only know that $r_{jik} \rightarrow 0$ for every fixed pair i, k . This makes it clear that \mathcal{B} is larger than $\mathcal{R} \widehat{\otimes} \mathbb{C}$ in general. Incidentally the condition on the b_{ik} 's for b to be in $\mathcal{R} \widehat{\otimes} \mathbb{C}$ can be stated without reference to a «basis» of \mathbb{C} : for $b \in \mathcal{B}$, if M is the A -submodule of \mathbb{C} generated by the b_{ik} 's, then $b \in \mathcal{R} \widehat{\otimes} \mathbb{C}$ if and only if, for every n , $M/(M \cap \pi^n \mathcal{O}_{\mathbb{C}})$, where $\mathcal{O}_{\mathbb{C}}$ denotes the ring of integers of \mathbb{C} , is a finitely generated A -module. See § 2.6 for another concrete example of complete tensor products.

2. Galois representations and associated operators

By a (linear) Galois representation ψ we mean a continuous homomorphism $\psi : \mathcal{G} \rightarrow \mathcal{R}^*$ where \mathcal{R}^* is the group of units of a Banach K -algebra \mathcal{R} , and \mathcal{G} and \mathcal{R}^* are given the Krull topology and the norm topology respectively. As a potential example consider the case where \mathcal{G} acts on the Banach space V by continuous (not necessarily norm-preserving) automorphisms. (As before K is the field of scalars unless otherwise stated.) If \mathcal{R} is the algebra of bounded linear operators on V one obtains a homomorphism $\psi : \mathcal{G} \rightarrow \mathcal{R}^*$. However the required continuity of ψ , essential in most of what follows, places a strong restriction on the Galois action which is not satisfied in many of the infinite-dimensional examples which occur naturally (for instance the standard action on the completion

of an infinite Galois extension of K). Two representations ψ_1 and ψ_2 with values in \mathcal{R} are considered equivalent if they differ only by conjugation in \mathcal{R}^* , i.e., if there is $r \in \mathcal{R}^*$ such that $\psi_2(\sigma) = r^{-1}\psi_1(\sigma)r$ for all $\sigma \in \mathcal{G}$.

Extending scalars to \mathbb{C} by tensoring, and composing ψ with the natural map $\mathcal{R} \rightarrow \mathcal{B} = \mathcal{R} \widehat{\otimes} \mathbb{C}$, one obtains a semi-linear \mathbb{C} -representation $\psi_{\mathbb{C}} : \mathcal{G} \rightarrow \mathcal{B}^*$ (where $*$ again denotes units) which we call the *semi-linearization* of ψ . More generally we shall consider semi-linear \mathbb{C} -representations (with values in some Banach \mathbb{C} -algebra \mathcal{B}) which do not necessarily arise from some linear representation ψ . Our main object is to study the category of such representations, or at least their «germs» or «local isomorphism classes». Information about these yields at least indirectly information about linear representations. For example any invariant of $\psi_{\mathbb{C}}$, notably the operator φ constructed below, is of course automatically an invariant of ψ .

2.1. Base extension and Galois action. — Let F be a subfield of \mathbb{C} , containing K , stable under the action of \mathcal{G} . Then \mathcal{G} acts on $\mathcal{R} \otimes F$ (as before tensor products are taken over K unless otherwise indicated) *via* its action on F , and this action extends to $\mathcal{R}_F = \mathcal{R} \widehat{\otimes} F$ by continuity. The immediately following are properties of the action to be naturally expected, except for the last which is more technical but important for the sequel.

- (1) \mathcal{G} preserves the norm of \mathcal{R}_F .
- (2) The function $\mathcal{G} \times \mathcal{R}_F \rightarrow \mathcal{R}_F$ given by the action of \mathcal{G} is continuous (for the Krull topology of \mathcal{G} and the norm topology of \mathcal{R}_F).
- (3) Let U be a subgroup of \mathcal{G} . Then :
 - (a) $(\mathcal{R} \otimes \mathbb{C})^U = \mathcal{R} \otimes \mathbb{C}^U$ and
 - (b) $(\mathcal{R} \widehat{\otimes} \mathbb{C})^U = \mathcal{R} \widehat{\otimes} \mathbb{C}^U$ (where the superscript denotes invariants).

(4) Let \mathcal{G}' be any open subgroup of \mathcal{G} . Then (using also the notation of §0) the action of Γ' on $\widehat{\mathcal{R}}'_\infty = (\mathcal{R}_{\mathbb{C}})^{\mathcal{H}'}$ satisfies the following two conditions :

- (a) For each n , $\widehat{\mathcal{R}}'_\infty = \mathcal{R}'_n \oplus (\gamma'_n - 1)\widehat{\mathcal{R}}'_\infty$ where $\mathcal{R}'_n = (\widehat{\mathcal{R}}'_\infty)^{\Gamma'_n}$ and γ'_n is a topological generator of Γ'_n .
- (b) The inverse operators $(\gamma'_n - 1)^{-1}$ on $(\gamma'_n - 1)\widehat{\mathcal{R}}'_\infty$ satisfy a uniform (i.e., independent of n) bound d (which may however depend on \mathcal{G}' ; one necessarily has $d \geq 1$).

Though (3a) is obvious we give a proof because the others can be

patterned after it. Let $\{r_\alpha\}$ be a basis, in the ordinary sense, for the K -vector space \mathcal{R} . Then $\mathcal{R} \otimes \mathbb{C} = \bigoplus_\alpha \mathbb{C}(r_\alpha \otimes 1)$ is a direct sum of copies of \mathbb{C} . This description is compatible with the action of \mathcal{G} , which immediately implies (3a).

For the other properties we have to replace the purely algebraic expansion just used by a topological one of the sort we have already considered several times. Thus let $\{r_i\}$ be a «basis» of \mathcal{R} , and taking F to be complete without loss of generality, we may write $\mathcal{R} \widehat{\otimes} F = \widehat{\bigoplus}_i F(r_i \otimes 1)$ where the notation on the right indicates a topological complete direct sum, i.e., one in which we allow all infinite expansions $\sum r_i \otimes f_i$, $f_i \in F$ such that $f_i \rightarrow 0$ as $i \rightarrow \infty$. Once again the expansion is compatible with the action of \mathcal{G} , i.e., \mathcal{G} acts on the elements f_i .

Then (1) clearly holds if we choose the norm on $\mathcal{R} \widehat{\otimes} F$ according to (C) of § 1.3 as we have agreed to do, i.e., if $|\sum r_i \otimes f_i| = \sup |f_i|$. For (2) it is enough to combine the continuity of the map $\mathcal{G} \times F \rightarrow F$ given by $(\sigma, f) \mapsto \sigma(f)$ with the fact that for each $x = \sum r_i \otimes f_i$ the coefficients $f_i \rightarrow 0$ as $i \rightarrow \infty$. The uniqueness of the expansion for each x , and its compatibility with Galois action, imply (3b). Property (4) is known to hold for the special case $\mathcal{R} = K$ and $\mathcal{R}_{\mathbb{C}} = \mathbb{C}$ when it is a consequence of propositions 6 and 7 of [10] (see also prop. 2 of [7]). The general case is deduced from this by once again using the expansion for $\mathcal{R}_{\mathbb{C}}$ together with its Galois compatibility, as well as lemma 3 of [8]. This last lemma implies that in the special case the projections given by the decomposition of (4a) are bounded operators, with bounds independent of n . This is the fact needed to extend (4) to the general case.

NOTE :

(a) The innocent seeming property (2) breaks down for the action of \mathcal{G} on $\prod F_i$, an infinite product of copies of F , with the norm $|\prod f_i| = \sup |f_i|$, if F is an infinite extension of K . This is because the function $(\sigma, f) \mapsto \sigma(f)$ where $\sigma \in \mathcal{G}$, $f \in F$, is continuous, but not uniformly continuous in the variable σ . Thus (2) depends crucially on the condition $f_i \rightarrow 0$ as $i \rightarrow \infty$.

(b) If we put $\mathcal{R}'_\infty = \cup_n \mathcal{R}'_n$ then it is easy to show, by referring back to the special case $\mathcal{R} = K$, that \mathcal{R}'_∞ is dense in $(\mathcal{R}_{\mathbb{C}})^{\mathcal{H}'}$. This is the justification for the notation $\widehat{\mathcal{R}}'_\infty$ for the latter.

(5) The projection operators given by the decomposition of (4a) are bounded, with bounds independent of n .

This is a consequence of (4) and [8, lemma 3] just as much in the general case as in the special case needed in the proof above. Also see the LEMMA in § 2.4.

If \mathcal{B} is a Banach algebra over \mathbb{C} on which \mathcal{G} acts by (\mathbb{C} -semi-linear) ring automorphisms, we shall say that \mathcal{G} acts *properly* on \mathcal{B} if the conditions (1), (2) and (4) above are satisfied (with \mathcal{B} instead of \mathcal{R}_F or $\mathcal{R}_{\mathbb{C}}$). Then (5) too can be seen to hold. Although $\mathcal{B} = \mathcal{R} \widehat{\otimes} \mathbb{C}$ is the case we are mainly interested in, some twisted versions of this (where the Galois group acts differently, in a way to be described later), also seem to satisfy these conditions. Since that is sufficient for the main construction below, a general definition appeared desirable.

2.2. Semi-linear representations. — Let \mathcal{B} be a Banach \mathbb{C} -algebra on which \mathcal{G} acts properly. By a *semi-linear representation* u of \mathcal{G} in \mathcal{B} we mean a continuous function $u : \mathcal{G} \rightarrow \mathcal{B}^*$, with values in the units of \mathcal{B} , and satisfying the condition of a 1-cocycle, namely : $u(\sigma\tau) = u(\sigma)\sigma(u(\tau))$, for $\sigma, \tau \in \mathcal{G}$. If $\psi : \mathcal{G} \rightarrow \mathcal{R}^*$ is a linear representation then of course its semi-linearization $\psi_{\mathbb{C}}$ trivially satisfies this condition.

If u_1 and u_2 are two semi-linear representations with values in \mathcal{B} we shall say they are *isomorphic* ($u_1 \xrightarrow{\sim} u_2$) if there is a $b \in \mathcal{B}^*$ such that $u_2(\sigma) = b^{-1}u_1(\sigma)\sigma(b)$ for all $\sigma \in \mathcal{G}$ (i.e., the cocycles u_1 and u_2 are cohomologous) and *locally isomorphic* ($u_1 \approx u_2$) if the condition is satisfied for some $b \in \mathcal{B}^*$ and all $\sigma \in \mathcal{G}'$ where \mathcal{G}' is some open subgroup of \mathcal{G} . The last notion obviously extends to the case where u_1 and u_2 are only locally defined, i.e., on (not necessarily identical) open subgroups of \mathcal{G} . It is clear that the equivalence of linear representations implies the isomorphism of their semi-linearizations.

This definition of isomorphism has the following rationale. Let X be a \mathcal{B} -module on which \mathcal{G} operates semi-linearly, i.e., so that $\sigma(bx) = \sigma(b)\sigma(x)$ for $b \in \mathcal{B}$, $x \in X$. Using the cocycle (i.e., semi-linear representation) u we may define a new action of \mathcal{G} on X by $\sigma[x] = u(\sigma)\sigma(x)$. Two such twisted actions, given by the cocycles u_1 and u_2 , are isomorphic under an element of \mathcal{B}^* if u_1 and u_2 are isomorphic (i.e., cohomologous). If \mathcal{B} acts faithfully on X then the latter is even a necessary condition for the actions to be isomorphic by an element of \mathcal{B}^* (note it is not a question of \mathcal{B} -isomorphisms). The most natural example of the situation discussed would be to take $X = V \widehat{\otimes} \mathbb{C}$ where V is a normed K -space, with \mathcal{G} operating *via* its action on \mathbb{C} . Then if \mathcal{B} is the ring of all bounded operators on X the notion of «isomorphic under an element of \mathcal{B}^* » simply becomes that of a topological isomorphism. Unfortunately this example is not one we can proceed very far with because \mathcal{G} does not act properly on \mathcal{B} if V is infinite

dimensional (property (2) of the definition breaks down. See note (a) above and the discussion at the end of § 1).

One may define a twisted action of \mathcal{G} on \mathcal{B} by the rule

$$\sigma[b] = u(\sigma)\sigma(b)u(\sigma)^{-1}$$

and this is compatible with the action defined on X in the sense that $\sigma[bx] = \sigma[b]\sigma[x]$. It is this type of twisting that was referred to at the end of § 2.1.

Our object is to classify, in some sense, the local isomorphism classes of semi-linear representations, which in turn may be seen as a step toward the classification of equivalence classes of linear representations.

2.3. The operator φ .—Given a semi-linear representation $u : \mathcal{G} \rightarrow \mathcal{B}^*$ we define a canonical element $\varphi(u) \in \mathcal{B}$, or simply φ if u is understood, which, as we show later, carries non-trivial information about u . We refer to φ as an «operator» even if \mathcal{B} is not explicitly given as an algebra of operators.

Let \mathcal{G}' be an open normal subgroup of \mathcal{G} such that $u(\mathcal{G}') \subset \mathcal{B}^1$ where \mathcal{B}^1 is the subgroup of \mathcal{B}^* consisting of principal units, i.e.,

$$\mathcal{B}^1 = \{x \in \mathcal{B}; |1 - x| < 1\}.$$

It is clear that \mathcal{G}' exists by the continuity of u and that \mathcal{B}^1 is in fact a subgroup (inverses are given by geometric series which converge by the completeness of \mathcal{B}). We use the notation of §0 in addition to that introduced since. By [8, prop. 1] we know that $H^1(\mathcal{H}', \mathcal{B}^1) = 1$. (In [8] the proposition is stated for \mathcal{H} , but it holds equally well for \mathcal{H}' .) Thus there exists $b \in \mathcal{B}^1$ such that if $v(\sigma) = b^{-1}u(\sigma)\sigma(b)$ for $\sigma \in \mathcal{G}$ then $v(\sigma) = 1$ for all $\sigma \in \mathcal{H}'$. The cocycle v is also continuous because by property (2) of proper action the function $\sigma \mapsto \sigma(b)$ is continuous.

A simple calculation shows that v may be viewed also as a cocycle on \mathcal{G}/\mathcal{H}' with values in $\mathcal{B}^{\mathcal{H}'}$ (we write v also for this induced cocycle). For suppose $\sigma \in \mathcal{H}'$, $\tau \in \mathcal{G}$. Then $v(\tau\sigma) = v(\tau)\tau(v(\sigma)) = v(\tau)$ since $v(\sigma) = 1$. This shows v is a cocycle on \mathcal{G}/\mathcal{H}' . One also has

$$v(\sigma\tau) = v(\sigma)\sigma(v(\tau)) = \sigma(v(\tau))$$

since $v(\sigma) = 1$. But $v(\sigma\tau) = v(\tau(\tau^{-1}\sigma\tau)) = v(\tau)$ so we see that $v(\tau)$ is fixed by σ , and thus $v(\tau) \in \mathcal{B}^{\mathcal{H}'}$ since σ was arbitrary. (This is just the

standard exactness of the inflation-restriction sequence for 1-cohomology for the groups $\mathcal{H}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}'$ acting on non-commutative modules.)

Let $\widehat{K}'_\infty = \mathcal{C}^{\mathcal{H}'}$, $\mathcal{B}'_\infty = \mathcal{B}^{\mathcal{H}'}$. Then because of the assumption of proper action (particularly property (4) above) the \widehat{K}'_∞ -Banach algebra \mathcal{B}'_∞ satisfies the conditions of [8, prop. 3] (over the field $K' = \overline{K}^{\mathcal{G}'}$ instead of K , with $\Gamma' = \mathcal{G}'/\mathcal{H}'$ instead of Γ , etc.). Thus one sees that for large enough n there is a continuous homomorphism $\rho : \Gamma'_n \rightarrow \mathcal{B}'_n^*$, where $\mathcal{B}'_n = (\mathcal{B}'_\infty)^{\Gamma'_n}$, such that ρ is cohomologous to the restriction of v to Γ'_n . (We also write ρ for the composed map $\mathcal{G}'_n \rightarrow \Gamma'_n \xrightarrow{\rho} \mathcal{B}'_n^*$.)

Retracing our steps we see that u restricted to \mathcal{G}'_n is cohomologous to ρ . Thus there exists $m \in \mathcal{B}^*$ such that for all $\sigma \in \mathcal{G}'_n$ one has $m^{-1}u(\sigma)\sigma(m) = \rho(\sigma)$. Define

$$\varphi = \lim_{\sigma \rightarrow 1} \frac{m \log \rho(\sigma) m^{-1}}{\log \chi(\sigma)}$$

where «log» is the p -adic logarithm. Note that φ depends on χ ; but as the latter is fixed we shall omit it in the notation. There are two points which have to be verified : that the limit exists (existence of φ) and that φ depends only on u and not on the choice of m and ρ (uniqueness of φ).

Since ρ and χ factor through Γ'_n we may take σ to be in Γ'_n . Because Γ'_n is isomorphic to \mathbb{Z}_p , taking a topological generator γ of it we may write, for σ close to 1, $\sigma = \gamma^a$ where $a \in \mathbb{Z}_p$. Clearly the exponent a can be cancelled from the numerator and denominator of the quotient in the definition of φ , thus showing that this quotient is constant for σ close to 1. (The normalizing factor $\log \chi(\sigma)$ is introduced just for this purpose.) Roughly speaking φ is, up to conjugation, an «infinitesimal generator» of the one parameter subgroup of \mathcal{B}^* given by the image of ρ .

REMARK. — In [8] the continuity requirement of condition (2) of proper action is not explicitly mentioned as a property of the Galois action on the Banach algebras considered there, but it is tacitly used. This in fact is the source of the mistake referred to in the Introduction, and which will be discussed further towards the end of this paper.

2.4. Uniqueness of φ . — Suppose (m_1, ρ_1) and (m_2, ρ_2) are two pairs of the same sort as (m, ρ) used to define φ . We shall say that such pairs, or the homomorphisms appearing in them, are *associated with u* . There are thus (for $i = 1, 2$) open subgroups \mathcal{G}_i of \mathcal{G} , such that $\rho_i : \mathcal{G}_i \rightarrow \mathcal{B}^{*\mathcal{G}_i}$ are continuous homomorphisms and, for all $\sigma \in \mathcal{G}_i$, one has $m_i^{-1}u(\sigma)\sigma(m_i) = \rho_i(\sigma)$. To simplify put $\mathcal{G}' = \mathcal{G}_1 \cap \mathcal{G}_2$, $\mathcal{H}' = \mathcal{H} \cap \mathcal{G}'$, $\Gamma' = \mathcal{G}'/\mathcal{H}'$, etc. as in §0. We must show that substituting either pair

for (m, ρ) in the definition leads to the same operator φ . A major part of this consists in showing that a homomorphism ρ associated with u is essentially unique up to conjugation, or more precisely that, restricted to \mathcal{G}' , ρ_1 and ρ_2 agree after conjugation.

Property (4) of proper action implies the following, with the same constant $d \geq 1$ as before, and for any $x \in \mathcal{B}'_\infty$:

LEMMA.

- (a) $(\gamma'_n - 1)x \equiv 0 \pmod{\epsilon} \Rightarrow x \equiv c \pmod{d\epsilon}$ for some $c \in \mathcal{B}'_n$.
- (b) If $(\gamma'_n - 1)x \equiv c \pmod{\epsilon}$ where $c \in \mathcal{B}'_n$, then $(\gamma'_n - 1)x \equiv 0 \pmod{d\epsilon}$.

(Here the notation $a \equiv b \pmod{\epsilon}$ for a real number $\epsilon > 0$ means $|a - b| \leq \epsilon$.)

NOTE. — Part (b) of the Lemma is the same as (5) of § 2.1 together with the definite bound d . The proof given here is more direct.

Proof of lemma. — By $(\gamma'_n - 1)^{-1}$ we mean the inverse of $(\gamma'_n - 1)$ on $(\gamma'_n - 1)\mathcal{B}'_\infty$. For (a) put $(\gamma'_n - 1)x = (\gamma'_n - 1)y$ with $y \in (\gamma'_n - 1)\mathcal{B}'_\infty$. Since $(\gamma'_n - 1)y \equiv 0 \pmod{\epsilon}$, applying $(\gamma'_n - 1)^{-1}$, which has the bound d , shows that $|y| < d\epsilon$. Since $(\gamma'_n - 1)$ kills $x - y = c$, it is clear that c is as required. For (b) notice that $(\gamma'_n - 1)x \equiv c \pmod{\epsilon} \Rightarrow (\gamma'_n - 1)^2x \equiv 0 \pmod{\epsilon}$. Once again, applying $(\gamma'_n - 1)^{-1}$ shows $(\gamma'_n - 1)x \equiv 0 \pmod{d\epsilon}$.

Returning to the situation above, we see that, for $\sigma \in \mathcal{G}'$,

$$u(\sigma) = m_1\rho_1(\sigma)\sigma(m_1)^{-1} = m_2\rho_2(\sigma)\sigma(m_2)^{-1}$$

or $\rho_2(\sigma) = x^{-1}\rho_1(\sigma)\sigma(x)$ where $x = m_1^{-1}m_2$. Note that $x \in \mathcal{B}'_\infty$ since ρ_i is trivial on \mathcal{H}' . Put $\rho_1(\sigma)^{-1} = 1 + a_1(\sigma)$, $\rho_2(\sigma) = 1 + a_2(\sigma)$ and let n be large enough (such an n exists by the continuity of the ρ'_i s) so that for all $\sigma \in \Gamma'_n$ one has $|a_i(\sigma)| < \epsilon < 1/d^2$ where d is the constant of the LEMMA. Multiplying one of the m'_i s by an element of K if necessary (this does not spoil any of our assumptions), we may assume $|x| \leq 1$. We then have, for $\sigma = \gamma'_n$:

$$\gamma'_n(x) = \rho_1(\gamma'_n)^{-1}x\rho_2(\gamma'_n) = (1 + a_1(\gamma'_n))x(1 + a_2(\gamma'_n)),$$

$$(*) \quad \therefore (\gamma'_n - 1)x = a_1(\gamma'_n)x + xa_2(\gamma'_n) + a_1(\gamma'_n)xa_2(\gamma'_n).$$

Then $(\gamma'_n - 1)x \equiv 0 \pmod{\epsilon}$ implies by the LEMMA (a) that $x \equiv c_1 \pmod{d\epsilon}$ with $c_1 \in \mathcal{B}'_n$. Substituting c_1 for x in the right hand side of (*) shows $(\gamma'_n - 1)x \equiv c_2 = a_1(\gamma'_n)c_1 + c_1a_2(\gamma'_n) \pmod{d\epsilon^2}$. Since $c_2 \in \mathcal{B}'_n$,

by the LEMMA (b) we see that $(\gamma'_n - 1)x \equiv 0 \pmod{d^2\epsilon^2}$. From this point on the argument can be repeated, with $d^2\epsilon^2$ instead of ϵ . Because $d^2\epsilon < 1$, we see that $d^2\epsilon^2 < \epsilon$. If we put $\epsilon_1 = \epsilon$, $\epsilon_2 = d^2\epsilon_1^2$, $\epsilon_3 = d^2\epsilon_2^2$, etc., we see that $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$, so that by continued repetition we obtain $(\gamma'_n - 1)x = 0$.

From this it follows that for $\sigma \in \mathcal{G}'_n$,

$$\rho_2(\sigma) = x^{-1}\rho_1(\sigma)x = m_2^{-1}m_1\rho_1(\sigma)m_1^{-1}m_2$$

and thus $m_2\rho_2(\sigma)m_2^{-1} = m_1\rho_1(\sigma)m_1^{-1}$. This implies that using (m_2, ρ_2) for (m, ρ) in the definition of φ gives the same result as using (m_1, ρ_1) , which concludes the proof that φ depends only on u .

Finally note that if the norm on \mathcal{B} is replaced by an equivalent norm (satisfying the requirements of proper action) this has no effect on the operator φ . This is because the requirements for (m, ρ) to be associated to u depend only on the structure of \mathcal{B} as a topological ring (more precisely, topological \mathbb{C} -algebra). From this one also sees that φ behaves in a natural way under continuous homomorphisms, because these carry pairs associated with u to pairs associated with the composed representations. Thus if \mathcal{B}_1 and \mathcal{B}_2 are two Banach \mathbb{C} -algebras on which \mathcal{G} acts properly, $u : \mathcal{G} \rightarrow \mathcal{B}_1^*$ is a semi-linear representation, and $f : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a continuous \mathbb{C} -algebra homomorphism commuting with the action of \mathcal{G} , then one has $\varphi(f \cdot u) = f(\varphi(u))$.

REMARK. — By analogy with the finite-dimensional case if $u = \psi_{\mathbb{C}}$ is the semi-linearization of a linear representation $\psi : \mathcal{G} \rightarrow \mathcal{R}^*$ one may expect a connection between φ and the image of the Galois group $\psi(\mathcal{G})$. If the same relation were to hold in this case it would be this : if $K = \mathbb{Q}_p$ and I is the inertia subgroup of \mathcal{G} then the Lie algebra of $\psi(I)$ is the « support » of φ , i.e., it is the smallest \mathbb{Q}_p -subspace of \mathcal{R} which, tensored with \mathbb{C} , contains φ (see [9], [7]). The proof should depend on the ramification theory of the extension with the infinite-dimensional p -adic Lie group $\psi(\mathcal{G})$ as Galois group.

2.5. Local isomorphism classes. — Consider semi-linear representations which are defined on open subgroups of \mathcal{G} but do not necessarily extend to \mathcal{G} . In this subsection we use this looser definition. Given two such representations $u_i : \mathcal{G}_i \rightarrow \mathcal{B}^*$, for $i = 1, 2$, recall that they are said to be locally isomorphic ($u_1 \approx u_2$) if they are isomorphic when restricted to some open subgroup \mathcal{G}_3 of $\mathcal{G}_1 \cap \mathcal{G}_2$. In other words there should be an $x \in \mathcal{B}^*$ such that for all $\sigma \in \mathcal{G}_3$ one has $u_2(\sigma) = x^{-1}u_1(\sigma)\sigma(x)$. Denote the local isomorphism class of u by $[u]$.

Two elements b_1, b_2 of \mathcal{B} will be called *conjugates* ($b_1 \sim b_2$) if for some $x \in \mathcal{B}^*$ one has $b_2 = xb_1x^{-1}$. Denote the conjugacy class of any $b \in \mathcal{B}$ by $\{b\}$. Also call b *admissible* if there is some conjugate of b which belongs to $\mathcal{B}^{\mathcal{G}'}$ for some open subgroup \mathcal{G}' of \mathcal{G} .

If \mathcal{B} is of the form $\mathcal{R} \widehat{\otimes} \mathbb{C}$ and K is locally compact (this may not be essential; perhaps linear compactness can be used instead of compactness) then there is a correspondence between local isomorphism classes of semi-linear representations in \mathcal{B} on the one hand, and conjugacy classes of admissible elements of \mathcal{B} on the other, as follows :

(1) The map $u \mapsto \varphi(u)$ maps the set of semi-linear representations with values in \mathcal{B} onto the set of admissible elements of \mathcal{B} .

(2) If u_1 and u_2 are semi-linear representations in \mathcal{B} then :

$$u_1 \approx u_2 \iff \varphi(u_1) \sim \varphi(u_2).$$

(3) The mapping $u \mapsto \varphi(u)$ establishes a 1-1 correspondence between the local isomorphism classes $[u]$ of semi-linear representations in \mathcal{B} and the conjugacy classes $\{\alpha\}$ of admissible elements α of \mathcal{B} .

Clearly (3) follows from (1) and (2). To prove (1) note first that $\varphi(u)$ is admissible. Next let α be an admissible element of \mathcal{B} and suppose $\alpha \sim a$ where $a \in \mathcal{B}^{\mathcal{G}'}$ for some open subgroup \mathcal{G}' of \mathcal{G} . Define

$$\rho(\sigma) = \exp(a \log \chi(\sigma))$$

(where \exp and \log are of course the p -adic exponential and logarithm) and let \mathcal{G}'' be an open subgroup of \mathcal{G}' such that the exponential converges for all $\sigma \in \mathcal{G}''$. It is clear that $\varphi(\rho) = a$. Suppose $\alpha = xax^{-1}$ with $x \in \mathcal{B}^*$. Put $u(\sigma) = x^{-1}\rho(\sigma)\sigma(x)$ for $\sigma \in \mathcal{G}''$. Then (x, ρ) is a pair associated with u , and we have :

$$\varphi(u) = \lim_{\sigma \rightarrow 1} \frac{x \log \rho(\sigma) x^{-1}}{\log \chi(\sigma)} = xax^{-1}.$$

This proves (1).

To prove (2), suppose $u_1 \approx u_2$. Let ρ be associated with u_1 . Then $u_1 \approx \rho$, so that also $u_2 \approx \rho$, i.e., ρ is also associated with u_2 . From the definition of the operators $\varphi(u_i)$ we know that, for σ close to 1, one has

$$\varphi(u_1) \sim \frac{\log \rho(\sigma)}{\log \chi(\sigma)} \quad \text{and} \quad \varphi(u_2) \sim \frac{\log \rho(\sigma)}{\log \chi(\sigma)},$$

which implies $\varphi(u_1) \sim \varphi(u_2)$.

Conversely, suppose $\varphi(u_1) \sim \varphi(u_2)$. Let ρ_1 and ρ_2 be associated with u_1 and u_2 respectively. Then, again for σ sufficiently close to 1, and some $x \in \mathcal{B}^*$, $x \log \rho_1(\sigma)x^{-1} = \log \rho_2(\sigma)$ or $x\rho_1(\sigma)x^{-1} = \rho_2(\sigma)$ (since the exponential series can be applied for σ close to 1). We need the following :

LEMMA. — *Let K be locally compact, \mathcal{R} a Banach K -algebra, and $\mathcal{B} = \mathcal{R} \widehat{\otimes} \mathbb{C}$. If $b_1, b_2 \in \mathcal{B}^{\mathcal{G}'}$ where \mathcal{G}' is an open subgroup of \mathcal{G} , and $xb_1x^{-1} = b_2$ for some $x \in \mathcal{B}^*$, then there is some $a \in (\mathcal{B}^*)^{\mathcal{G}''}$, where \mathcal{G}'' is an open subgroup of \mathcal{G}' , such that $ab_1a^{-1} = b_2$.*

Proof of Lemma. — Because $\mathcal{R} \otimes \overline{K}$ is dense in $\mathcal{R} \widehat{\otimes} \mathbb{C} = \mathcal{B}$ we can find $x_0 \in \mathcal{B}^{\mathcal{G}''}$ for some open subgroup \mathcal{G}'' of \mathcal{G}' with x_0 as close to x as one pleases. Replacing x by xx_0^{-1} , b_1 by $x_0b_1x_0^{-1}$ we may assume $x \equiv 1 \pmod{\epsilon}$ where ϵ is arbitrarily small (but fixed), and $b_1, b_2 \in \mathcal{B}^{\mathcal{G}''}$.

Let $\{x_n\}$ be a sequence in $\mathcal{R} \otimes \overline{K}$ with $x_n \rightarrow x$. If $F = \mathbb{C}^{\mathcal{H}''}$ (see § 0 for notation used in this proof), let F_n be a finite extension of F contained in \mathbb{C} such that $x_n \in \mathcal{R} \otimes F_n$. Let T_n be the trace for F_n/F which operates also on $\mathcal{R} \otimes F_n$ in the obvious way, with values in $\mathcal{R} \otimes F \subset \mathcal{B}^{\mathcal{H}''}$. Let T be the projection $\mathcal{B}^{\mathcal{H}''} \rightarrow \mathcal{B}^{\mathcal{G}''}$ given by the decomposition in (4a), § 2.1 and known to be bounded by (5), § 2.1. (In applying (4a) we may assume without loss of generality that $\mathcal{G}'' = \mathcal{G}_0''$ so that $\mathcal{B}^{\mathcal{G}''} = (\mathcal{B}^{\mathcal{H}''})^{\Gamma_0''}$.) By [10, prop. 9] (or [7, prop. 1]) one sees easily that for any $\delta > 1$ one may choose $\theta_n \in F_n$ such that $|\theta_n| < \delta$ and $T_n(\theta_n) = 1$.

Define $a_n \in \mathcal{B}^{\mathcal{G}''}$ by $a_n = T(T_n(\theta_n x_n))$. From $xb_1 = b_2x$ it follows, on replacing x by x_n , multiplying by θ_n , and applying $T \cdot T_n$ that $(a_n b_1 - b_2 a_n) \rightarrow 0$ because $x_n \rightarrow x$. To complete the proof of the LEMMA we need two more steps, first to show that a suitable subsequence of the a_n 's converges to an element a , and secondly that a is invertible. (The multipliers θ_n are needed to guarantee the second fact.) Actually we do this in the reverse order.

By adjusting the real constant ϵ above, while $\delta > 1$ has any value, we can ensure that a_n is close to 1, hence invertible. For by choosing ϵ small enough we can make $\theta_n x$ as close to θ_n as we wish, or for large n , $\theta_n x_n$ close to θ_n . This implies that $T_n(\theta_n x_n)$ is arbitrarily close to 1 since $T_n(\theta_n) = 1$, and by the boundedness of T in turn this implies that a_n is close to 1 (one merely needs $|a_n - 1| < 1$) hence invertible. Any limit a of a subsequence of the a_n 's then has the same property.

Finally, to construct a proceed as follows. Let $x = \sum r_i \otimes c_i$ where, as usual $\{r_i\}$ is a «basis» of \mathcal{R} , $c_i \in \mathbb{C}$, and $c_i \rightarrow 0$ as $i \rightarrow \infty$. Then if $x_n = \sum r_i \otimes c_{in}$, with $c_{in} \in F_n$ we may assume $|c_{in}| \leq |c_i|$. Because $T \cdot T_n$ has a bound independent of n , and $|\theta_n| < \delta$ where δ too is independent of n , we see that there is a constant k independent of

n such that $|a_{in}| < k|c_i|$ where $a_n = \sum r_i \otimes a_{in}$. (Of course $a_{in} \in K$ is given by $a_{in} = T(T_n(\theta_n c_{in}))$.) Since $k|c_i| \rightarrow 0$ as $i \rightarrow \infty$, we can choose integers m_i , independent of n , such that $m_i \rightarrow \infty$ as $i \rightarrow \infty$, with $a_{in} \in \pi^{m_i} A$. In other words $a_n \in \prod_i (r_i \otimes \pi^{m_i} A)$, where the product is direct. Since this product is compact in the product topology we can choose a convergent subsequence of the a_n 's which converges in this topology. But because $\pi^{m_i} A \rightarrow 0$ as $i \rightarrow \infty$, this subsequence converges also in the norm topology (convergence in the product topology does not of course imply convergence in the norm in general). Its limit a has the required properties.

Returning to the proof of (2), let \mathcal{G}' be an open subgroup of \mathcal{G} (possibly smaller than the \mathcal{G}' chosen before) such that ρ_1, ρ_2 are defined on \mathcal{G}' , have values in $\mathcal{B}^{\mathcal{G}'}$, and satisfy $x\rho_1(\sigma)x^{-1} = \rho_2(\sigma)$ for all $\sigma \in \mathcal{G}'$. Let γ' be a generator of some open subgroup of Γ' . Then applying the LEMMA with $b_1 = \rho_1(\gamma')$ and $b_2 = \rho_2(\gamma')$ shows that there exists $a \in \mathcal{B}^{\mathcal{G}''}$, for an open subgroup \mathcal{G}'' of \mathcal{G} , such that $a\rho_1(\sigma)a^{-1} = \rho_2(\sigma)$ for all σ in the subgroup of Γ' generated by γ' . This implies $\rho_1 \approx \rho_2$ and thus $u_1 \approx u_2$, which proves (2).

REMARKS. — In the finite-dimensional case, if \mathcal{R} is a full matrix algebra, these results can be strengthened in various ways (see [7]). For example, if representations are defined on all of \mathcal{G} their local isomorphism implies their isomorphism. For another, if u is a semi-linear representation defined on all of \mathcal{G} then $\varphi(u)$ is conjugate to an element of \mathcal{R} . Since the problem of classifying conjugacy classes in \mathcal{R} has a well-known solution, this means one has rather good control of the isomorphism classes of semi-linear representations. Although this probably breaks down in the infinite-dimensional case, one may know for some specific u that $\varphi(u)$ is conjugate to something in \mathcal{R} , and the conjugacy class problem in \mathcal{R} may be accessible.

2.6. The power series case. — Motivated by the work of HIDA, MAZUR has developed a general theory of p -adic liftings with restricted ramification (e.g. unramified outside p) of a given (global) Galois representation over a finite field of characteristic p (see [5], also [4], [6]). He constructs a complete Noetherian local ring, which we denote by D , and a Galois representation by matrices over D (the universal deformation of the given representation) which satisfy a universal mapping property for such liftings. These universal deformations, and representations intimately related to them, provide (on restriction to local Galois groups) the only infinite-dimensional examples known so far of the sort of (linear) representation we have been considering.

The universal deformation ring D is isomorphic to a quotient of a power series ring over \mathcal{O}_E , the ring of integers of a finite extension E of \mathbb{Q}_p . Thus $D \xrightarrow{\sim} \mathcal{O}_E[[T_1, \dots, T_\ell]]/I$. We may assume our ground field K contains E . Suppose $\psi_0 : \mathcal{G} \rightarrow GL_n(D)$ is the local restriction of some universal deformation. To carry out our construction of φ we must replace $GL_n(D)$ by \mathcal{R}^* , the units of some Banach algebra \mathcal{R} over K . For this define a norm $|\cdot|_k$ on $\mathbb{C}[[T]] = \mathbb{C}[[T_1, \dots, T_\ell]]$ for an ℓ -tuple $k = (k_1, \dots, k_\ell)$ of real constants, $0 < k_i < 1$. Define $|f(T)|_k$ by putting $|aT_1^{r_1} \dots T_\ell^{r_\ell}| = |a|k_1^{r_1} \dots k_\ell^{r_\ell}$ for monomials, and extending it to general $f(T)$ by taking the sup over all monomials occurring in $f(T)$. The norm $|\cdot|_k$ can of course be infinite. So we consider the subring $\mathbb{C}\langle T \rangle_k$ of $\mathbb{C}[[T]]$ consisting of those $f(T)$ for which $|a_r T^r| \rightarrow 0$ as $r \rightarrow \infty$ where $a_r T^r$ is the monomial in $f(T)$ corresponding to the exponents $r = (r_1, \dots, r_\ell)$. Let $F\langle T \rangle_k = F[[T]] \cap \mathbb{C}\langle T \rangle_k$ for closed subfields F of \mathbb{C} . This is a Banach algebra over F with the norm $|\cdot|_k$.

NOTE. — In [8] we introduced the slightly different subring $\mathbb{C}\{T\}_k$ consisting of those $f(T)$ for which $|f(T)|_k < \infty$. Unfortunately \mathcal{G} does not operate properly on this ring — property (2) breaks down (see § 2.1, particularly the note), and thus the construction of φ (or Φ_T in [8]) cannot be carried out using this ring. This error in [8] was pointed out to us by Emily PETRIE.

Let \mathcal{R} be defined as follows. Let J be the closure as a topological ideal of the image of I under the natural inclusion $\mathcal{O}_E[[T]] \hookrightarrow K\langle T \rangle_k$. If $\mathcal{P} = K\langle T \rangle_k/J$, let \mathcal{R} be the $n \times n$ matrix algebra over \mathcal{P} with the usual sup norm. Then denoting by ψ the composed map $\mathcal{G} \xrightarrow{\psi_0} GL_n D \rightarrow \mathcal{R}^*$ (here the second arrow is induced by the natural map $M_n(D) \rightarrow \mathcal{R}$), we can carry out our construction of $\varphi(\psi_{\mathbb{C}}) \in \mathcal{R} \widehat{\otimes} \mathbb{C}$.

Now $K\langle T \rangle_k \widehat{\otimes} \mathbb{C}$ can be identified with $\mathbb{C}\langle T \rangle_k$. Since the exact sequence

$$0 \rightarrow J \longrightarrow K\langle T \rangle_k \longrightarrow \mathcal{P} \rightarrow 0$$

splits topologically (for the vector space structure — forgetting the multiplication) by § 1.1, (6), taking complete tensor products with \mathbb{C} also yields an exact sequence. This means that $\mathcal{P} \widehat{\otimes} \mathbb{C}$ is a quotient of $\mathbb{C}\langle T \rangle_k$, and $\varphi(\psi_{\mathbb{C}})$ may be viewed as a matrix whose entries are determined by (the images of) power series in $\mathbb{C}\langle T \rangle_k$. This provides all that was required by the construction of Φ_T in [8], not to mention the extra fact of the uniqueness of φ which we have established.

Note finally that in case D is isomorphic to a power series ring, i.e., $I = 0$, the so-called smooth, or unobstructed case, we have

$$\mathcal{R} \widehat{\otimes} \mathbb{C} \xrightarrow{\sim} \mathbb{C}\langle T \rangle_k,$$

which is an integral domain. Since $\varphi(\psi_{\mathbb{C}})$ is a matrix over this ring it makes sense to speak of φ as being semi-simple (or non-semi-simple) regarded as a matrix over the field of fractions. Thus the universal deformations are divided into two classes. If φ is semi-simple, say, it is still possible that some specializations of ψ_0 should have non-semi-simple Hodge-Tate structure, but this happens rarely : on a dense open set of the parameter space the specializations have semi-simple Hodge-Tate structure. Similarly in the opposite case most specializations are non-semi-simple. Thus we may label the two cases by saying ψ_0 is *generically* semi-simple (non-semi-simple) according as $\varphi(\psi_{\mathbb{C}})$ is semi-simple (non-semi-simple). The special cases treated by MAZUR in the latter part of [5] are all easily seen (given his results) to be generically semi-simple.

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