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RINGS OF DIFFERENTIAL OPERATORS OVER RATIONAL AFFINE CURVES

BY

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RÉSUMÉ. — Soit X une courbe algébrique irréductible sur $\mathbb{C}$ dont la normalisée est la droite affine et telle sur le morphisme de normalisation est injectif. Soit $D(X)$ l'anneau des opérateurs différentiels sur X. Nous étudions un invariant pour l'anneau $D(X)$ des opérateurs différentiels sur X, noté $\operatorname{codim} D(X)$. En particulier, nous montrons que $D(X) \cong D(Y)$ implique $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$. Cela permet de distinguer dans certains cas les anneaux d'opérateurs différentiels de courbes non-isomorphes. En outre, nous décrivons les sous-algèbres ad-nilpotentes maximales de $D(X)$. Nous montrons que si B est une sous-algèbre ad-nilpotente maximales de $D(X)$, alors B est un sous-anneau de type fini d'un $\mathbb{C}[b]$ où b désigne un élément du corps des fractions de $D(X)$; de plus, la clôture intégrale de B est $\mathbb{C}[b]$.

ABSTRACT. — Let X be an irreducible algebraic curve over the complex numbers such that its normalization is the affine line, and the normalization map is injective. Let $D(X)$ denote its ring of differential operators. We find an invariant for $D(X)$ denoted as $\operatorname{codim} D(X)$. In particular, we show that $D(X) \cong D(Y)$ implies $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$. This allows us to distinguish certain rings of differential operators of non-isomorphic curves. We also describe the maximal ad-nilpotent subalgebras of $D(X)$. We show that if B is a maximal ad-nilpotent subalgebra of $D(X)$, then B is a finitely generated subring of $\mathbb{C}[b]$ for some element b of the quotient field of $D(X)$ and the integral closure of B is $\mathbb{C}[b]$.

1. Introduction

Let X and Y be irreducible algebraic curves over the complex numbers, $\mathbb{C}$. Let $D(X)$ and $D(Y)$ denote their ring of differential operators, respectively. (For definition see [9]). This paper is motivated by the following open question.[†] Does $D(X) \cong D(Y)$ imply that $X \cong Y$? Let $\widetilde{X}$ denote

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† G. LETZTER has now found nonisomorphic curves X and Y with isomorphic rings of differential operators (see [4]).

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the normalization of X. MAKAR–LIMANOV [5] shows that the set of ad-nilpotent elements $N(X)$ is exactly $O(X)$ whenever $O(X)$ is not a subring of a polynomial ring in one variable over $\mathbb{C}$. He thus answers the question affirmatively for these curves. Let $\mathbb{A}^1$ denote the affine line. PERKINS [8] extends this result showing that $D(X) \cong D(Y)$ implies $X \cong Y$ whenever $\widetilde{X} \neq \mathbb{A}^1$, or $\widetilde{X} = \mathbb{A}^1$ but the normalization map $\pi : \widetilde{X} \to X$ is not injective. Thus, in the paper, we are interested in curves X such that $\widetilde{X} \cong \mathbb{A}^1$ and $\pi : \widetilde{X} \to X$ is injective. STAFFORD [10] shows the conjecture holds the following two examples of such curves : when X is the affine line $\mathbb{A}^1$, or when X is the cubic cusp $y^2 = x^3$.

For the remainder of the paper, assume that X is a curve such that its normalization is isomorphic to the affine line $\mathbb{A}^1$ with an injective normalization map. We may therefore assume that the coordinate ring of X, denoted $O(X)$, is a subring of a polynomial ring in one variable $\mathbb{C}[x]$ such that the integral closure of $O(X)$, written $\widetilde{O(X)}$, is equal to $C[x]$. Furthermore $D(X)$ is a subring of $\mathbb{C}(x)[\partial]$ where $[\partial, x] = 1$. Here ∂ is just $\partial/\partial x$ and the element $f_n(x)\partial^n + \cdots + f_0(x)$ of $D(X)$ sends $g(x) \in O(X)$ to $f_n(x)g^{(n)}(x) + \cdots + f_0(x)g(x)$ where $g^{(n)}(x)$ denotes the n^{th} derivative of $g(x)$.

PERKINS studies rings that satisfy these conditions in [8]. He shows that in many cases, $D(X)$ contains maximal commutative ad-nilpotent subalgebras not isomorphic to $O(X)$. Thus, for these curves, the set $N(X)$ of ad-nilpotent elements does not determine $O(X)$.

In this paper, we obtain an invariant for $D(X)$ and a nice description of the maximal ad-nilpotent subalgebras of $D(X)$. Set $T = \mathbb{C}(x)[\partial]$ and set $\partial\text{-deg}\, w = n$ where $w = f_n(x)\partial^n + \cdots + f_0(x)$ is an element of T. Define a filtration on T by $T_i = \{w \in T \mid \partial\text{-deg}\, w \leq i\}$ and hence on any subring R of T by $R_i = R \cap T_i$. (Note that this is the same filtration on $D(X)$ as the one defined by the order of the differential operator.) We may form the associated graded ring $\partial\text{-gr}\, R = \bigoplus R_i/R_{i-1}$. We define $\operatorname{codim} R$ to be equal to $\dim_{\mathbb{C}} \partial\text{-gr}\, \mathbb{C}[x,\partial]/\partial\text{-gr}\, R$ for those subrings R of T such that $\partial\text{-gr}\, R \subset \partial\text{-gr}\, \mathbb{C}[x,\partial]$.

Now assume that both X and Y are affine curves with normalization equal to the affine line and injective normalization map. By [9], both $\partial\text{-gr}\, D(X)$ and $\partial\text{-gr}\, D(Y)$ are subrings of $\partial\text{-gr}\, \mathbb{C}[x,\partial]$ and $\operatorname{codim} D(X)$ and $\operatorname{codim} D(Y)$ are finite numbers.

Our main results are :

THEOREM. — *Suppose that B is a maximal* ad-*nilpotent subalgebra of $D(X)$. Then there exists elements x' and ∂' in the quotient field of*

$\mathbb{C}(x)[\partial]$ such that $[\partial', x'] = 1$, $D(X)$ is a subring of $\mathbb{C}(x')[\partial']$, $D(X) \cap \mathbb{C}(x') = B$, and the integral closure of B is $\mathbb{C}[x']$. Furthermore, ∂'-$\mathrm{gr}\, D(X)$ is a subring of ∂'-$\mathrm{gr}\, \mathbb{C}[x', \partial']$ and

$$\dim_{\mathbb{C}} \partial'\text{-}\mathrm{gr}\, \mathbb{C}[x', \partial'] / \partial'\text{-}\mathrm{gr}\, D(X) = \operatorname{codim} D(X).$$

COROLLARY. — *If $D(X) \cong D(Y)$, then* $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$.

This result permits one to distinguish many rings of differential operators. For example, set $O(X_n) = \mathbb{C} + x^n \mathbb{C}[x]$. Then it will follow from the COROLLARY, that $D(X_n) \cong D(X_m)$ implies that $n = m$.

2. Graded Algebras of $D(X)$

In this section, α and β are nonnegative real numbers with $\alpha + \beta > 0$. Define valuations $V_{\alpha,\beta}$ on $\mathbb{C}(x)[\partial]$ as follows. Set

$$V_{\alpha,\beta}\Big(w_n(x)\partial^n + w_{n-1}(x)\partial^{n-1} + \cdots + w_0(x)\Big)$$

equal to $\max\{\alpha d_m + \beta m \mid n \geq m \geq 0\}$ where $d_m = \deg(w_n(x))$. This extends the notion of valuations introduced by DIXMIER in [2] for the Weyl algebra. For each valuation $V_{\alpha,\beta}$ we may define a filtration of $\mathbb{C}(x)[\partial]$, and hence on any subring R of $\mathbb{C}(x)[\partial]$ as follows. Recall that $T = \mathbb{C}(x)[\partial]$. Set $T_i = \{z \in T \mid V_{\alpha,\beta}(z) \leq i\}$ and $R_i = R \cap T_i$. We may then define the associated graded algebra $\mathrm{gr}_{\alpha,\beta} R = \bigoplus R_i / R_{i-1}$. Now the commutator $[x^i\partial^j, x^k\partial^\ell] = (kj - i\ell)x^{i+k-1}\partial^{j+\ell-1} +$ terms with x-degree less than $i + k - 1$ and ∂-degree less than $j + \ell - 1$. Therefore $V_{\alpha,\beta}([x^i\partial^j, x^k\partial^\ell]) < \alpha(i + k) + \beta(\ell + j)$. It follows that $\mathrm{gr}_{\alpha,\beta}(\mathbb{C}(x)[\partial])$ is a commutative algebra.

Note that when $\alpha = 0$ and β is positive, then the filtration defined by $V_{0,\beta}$ on $D(X)$ is the same filtration on $D(X)$ as the one defined by ∂-deg in the introduction. We will write ∂-$\mathrm{gr}\, D(X)$ for $\mathrm{gr}_{0,\beta} D(X)$ and ∂-deg for $V_{0,\beta}$. Similarly, when $\beta = 0$ and α is positive the graded algebra determined by $V_{\alpha,0}$ is the same as x-$\mathrm{gr}\, R$ determined by x-deg defined in [8].

Set $\mathrm{gr}_{\alpha,\beta}\, x = x$ and $\mathrm{gr}_{\alpha,\beta}\, \partial = y$. Since $D(\widetilde{X})$ is just the first Weyl algebra, A_1, we have that ∂-$\mathrm{gr}\, D(\widetilde{X}) = \mathbb{C}[x, y]$ where ∂-$\mathrm{gr}\, x = x$ and ∂-$\mathrm{gr}\, \partial = y$. By [9, Proposition 3.11], it follows that ∂-$\mathrm{gr}\, D(X)$ is a subring of $\mathbb{C}[x, y]$ and by [8, Lemma 2.3], x-$\mathrm{gr}\, D(X)$ is also a subring of $\mathbb{C}[x, y]$. In the following lemma, we extend this to other gradings.

LEMMA 2.1. — *Let R be a subring of $\mathbb{C}(x)[\partial]$ such that ∂-$\mathrm{gr}\, R \subset \mathbb{C}[x, y]$. Then the graded algebra $\mathrm{gr}_{\alpha,\beta} R$ is a subring of $\mathbb{C}[x, y]$.*

Proof. — If $\alpha = 0$ then $\mathrm{gr}_{\alpha,\beta} R = \partial\text{-}\mathrm{gr}\, R$. So we may assume that α is positive. Let w be a typical element of $D(X)$. Write $w = g_m(x)\partial^m + \cdots + g_0(x)$ where $g_i(x) \in \mathbb{C}(x)$ for $0 \le i \le m$. Set degree of $g_i(x)$ equal to d_i for $0 \le i \le m$. Since $\partial\text{-}\mathrm{gr}\, R \subset \mathbb{C}[x,y]$, it follows that $g_m(x) \subset \mathbb{C}[x]$ and thus $d_m \ge 0$. Set $N = V_{\alpha,\beta}(w)$. By the definition of $V_{\alpha,\beta}$, it follows that $N = \max\{d_i\alpha + i\beta \mid 0 \le i \le m\}$. Hence $\mathrm{gr}_{\alpha,\beta}(w) = \sum_{0\le s\le m} \gamma_s x^{d_s} y^s$ where $\gamma_s = 0$ if $V_{\alpha,\beta}(x^{d_s}\partial^s) < N$, and $\gamma_s x^{d_s}$ is the leading term of $g_s(x)$ if $V_{\alpha,\beta}(x^{d_s}\partial^s) = N$. We need to show that whenever $\gamma_s \ne 0$, we have $x^{d_s}y^s \in \mathbb{C}[x,y]$. In particular, since $0 \le s \le m$, we need to show that $d_s \ge 0$ whenever $\gamma_s \ne 0$. Now $N = V_{\alpha,\beta}(w) \ge V_{\alpha,\beta}(g_m(x)\partial^m) = d_m\alpha + m\beta$. Hence $d_s\alpha + s\beta \ge d_m\alpha + m\beta$. Recall that $m \ge s, d_m \ge 0$, and that α is positive. It follows that $d_s \ge d_m \ge 0$. The lemma now follows.

Define a linear map $\phi : \mathbb{C}(x)[\partial] \to \mathbb{C}[x,\partial]$ as follows. Suppose that $w = g_m(x)\partial^m + \cdots + g_0(x)$ is an element of $\mathbb{C}(x)[\partial]$. For each i such that $1 \le i \le m$, there exists a unique polynomial $f_i(x)$ such that $\deg(g_i(x) - f_i(x)) < 0$. Set

$$\phi(w) = f_m(x)\partial^m + \cdots + f_0(x).$$

Now consider two rational functions $g_1(x)$ and $g_2(x)$ such that $\phi(g_1(x)) = f_1(x)$ and $\phi(g_2(x)) = f_2(x)$. Then clearly

$$\deg\Big(\lambda_1 g_1(x) + \lambda_2 g_2(x) - (\lambda_1 f_1(x) + \lambda_2 f_2(x)\Big) < 0 \quad \text{and}$$
$$\phi(\lambda_1 g_1(x) + \lambda_2 g_2(x)) = \lambda_1 f_1(x) + \lambda_2 f_2(x).$$

It follows that ϕ is a well defined linear map from $\mathbb{C}(x)[\partial]$ to $\mathbb{C}[x,\partial]$.

COROLLARY 2.2. — *Let R be a subring of $\mathbb{C}(x)[\partial]$ such that $\partial\text{-}\mathrm{gr}\, R \subset \mathbb{C}[x,y]$. If w is an element of R, then $\mathrm{gr}_{\alpha,\beta}\,\phi(w) = \mathrm{gr}_{\alpha,\beta}(w)$.*

Proof. — This is clear since $\mathrm{gr}_{\alpha,\beta}(w - \phi(w))$ does not contain any monomials $x^{d_s}y^s$ with $d_s \ge 0$.

Remark 2.3. — Note that $\phi(R)$ is a linear subspace of the first Weyl algebra $A_1 = \mathbb{C}[x,\partial]$, but, generally speaking, is not a subalgebra. Nevertheless α,β gradings are defined on $\phi(R)$ and $\mathrm{gr}_{\alpha,\beta}\,\phi(R) = \mathrm{gr}_{\alpha,\beta} R$. Now

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-}\mathrm{gr}\, D(X) < \infty \quad ([9, 3.12]) \text{ and}$$
$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/x\text{-}\mathrm{gr}\, D(X) < \infty \quad ([8, \text{Lemma } 2.5])$$

In the next proposition, we will show that these two finite numbers are equal. We will later show that this codimension is an invariant for $D(X)$.

PROPOSITION 2.4. — *Suppose that R is a subring of $\mathbb{C}(x)[\partial]$ such that $\partial\text{-gr}\, R \subset \mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R < \infty$. Then $\mathrm{gr}_{\alpha,\beta} R$ is a subring of $\mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\mathrm{gr}_{\alpha,\beta} R = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R$.*

Using COROLLARY 2.2 and REMARK 2.3, we may replace R by $\phi(R)$ and prove the following.

PROPOSITION 2.4′. — *Suppose that R' is a linear subspace of the Weyl algebra $\mathbb{C}[x,\partial]$ and that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R' < \infty$. Then $\mathrm{gr}_{\alpha,\beta} R'$ is a linear subspace of $\mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\mathrm{gr}_{\alpha,\beta} R' = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R'$.*

Before proving PROPOSITION 2.4′, we need some additional notation and lemmas. Set, for $i \geq 0$,

$$\begin{aligned} E_i &= \mathbb{C}[x] + \mathbb{C}[x]y + \cdots + \mathbb{C}[x]y^i \quad \text{and} \\ B_i &= \{w \in R' \mid \partial\text{-gr}\, w \in E_i\}. \end{aligned}$$

Note that $\bigcup_{i \geq 0} B_i = R'$. Set $E = \bigcup_{i \geq 0} E_i = \mathbb{C}[x,y]$.

In PROPOSITION 2.4′, we assume that $\dim_{\mathbb{C}} E/\partial\text{-gr}\, R' < \infty$. Since $\partial\text{-gr}\, w \in E_i$ if and only if $w \in B_i$ for any $w \in R'$, it follows that $\dim_{\mathbb{C}} E_i/\partial\text{-gr}\, B_i < \infty$ for all $i \geq 0$, and that there exists an $N > 0$ such that $\dim_{\mathbb{C}} E_i/\partial\text{-gr}\, B_i = \dim_{\mathbb{C}} E/\partial\text{-gr}\, R'$ for all $i \geq N$. Hence for each $i \geq 0$, there exists an integer $M_i \geq -1$ such that for each $m > M_i$ there exists a monic polynomial $p_{i,m}(x)$ of degree m in $\mathbb{C}[x]$ such that $p_{i,m}(x)y^i$ is an element of $\partial\text{-gr}\, B_i$. Furthermore, for $i \geq N$, we may assume that $M_i = -1$.

We have the following lemmas.

LEMMA 2.5

Suppose that R' satisfies the conditions of PROPOSITION 2.4′. *Suppose that $w = \big(\alpha x^d + f_{i+1}(x)\big)\partial^{i+1} + \cdots + f_0(x)$ is an element of B_{i+1} where $\alpha \in \mathbb{C} - \{0\}$ and $\deg f_{i+1}(x) < d$. Then there exists a $w' \in B_{i+1}$ such that $w' = (\alpha x^d + g_{i+1}(x))\partial^{i+1} + g_i(x)\partial^i + \cdots + g_0(x)$ and $\deg g_k(x) \leq M_k$ for each k such that $i+1 \geq k \geq 0$.*

Proof. — Let us use the following induction. Set $w_{-1} = w$. Suppose that

$$\begin{aligned} w_k = \big(ax^d + g_{i+1}(x)\big)\partial^{i+1} + \cdots + g_{i-k}(x)\partial^{i-k} \\ + f_{i-k-1}(x)\partial^{i-k-1} + \cdots + f_0(x), \end{aligned}$$

where $\deg g_j(x) \leq M_j$, is defined. There exists $b \in B_{i-k-1}$ such that $\partial\text{-gr}\, b = (f_{i-k-1} - g_{i-k-1})y^{i-k-1}$ where $\deg g_{i-k-1} \leq M_{i-k-1}$ by the paragraph preceding the lemma. So we can define w_{k+1} as $w_k - b$, and w' as w_i.

Let P_i be the set of positive integers m such that there exists a nonzero polynomial $q_{i,m}(x)$ of degree m in $\mathbb{C}[x]$ with $q_{i,m}(x)y^i \in \partial\text{-gr}\, R'$. Note that if n is an integer such that $n > M_i$, then $n \in P_i$. By LEMMA 2.5, it now follows that for each $m \in P_i$ there exists a monic polynomial $p_{i,m}(x)$ of degree $m \in \mathbb{C}[x]$ such that $b_{i,m} = p_{i,m}(x)\partial^i + g_{i-1}(x)\partial^{i-1} + \cdots + g_0(x)$ is an element of B_i with $\deg g_k(x) \leq M_k$ for $i-1 \geq k \geq 0$. Furthermore, for $i \geq N$, we may assume that $p_{i,m}(x) = x^m$. Note that the set

$$\{b_{i,m} \mid i \geq 0 \text{ and } m \in P_i\}$$

forms a basis for R' over $\mathbb{C}$, and

$$\{p_{i,m}(x)y^i \mid i \geq 0 \text{ and } m \in P_i\}$$

forms a basis for $\partial\text{-gr}\, R'$ over $\mathbb{C}$. Thus if $w \in R'$, with $\partial\text{-gr}\, w = f(x)y^i$, then for $i > k \geq 0$, there exist $f_k(x) \in \mathbb{C}[x]$ with $\deg f_k(x) \leq M_k$, such that $f(x)\partial^i + f_{i-1}(x)\partial^{i-1} + \cdots + f_0(x)$ is an element of R'.

Set $M = \max\{M_k \mid N > k \geq 0\}$. Then we may assume that $b_{i,m} = p_{i,m}(x)\partial^i + w_{i,m}$ with $\partial\text{-deg}\, w_{i,m} < \min(i, N)$ and $x\text{-deg}\, w_{i,m} \leq M$.

LEMMA 2.6

Assume that R' satisfies the conditions of PROPOSITION 2.4$'$. *For each $m \geq 0$, there exists a positive integer S_m such that for all $i \geq S_m$, there is an element $c_{i,m}$ in R' of the form $p_{i,m}(x)\partial^i + t_{i,m}$ with $\deg p_{i,m}(x) = m$ and $\partial\text{-deg}\, t_{i,m} < i$ and $x\text{-deg}\, t_{i,m} \leq m$. If $m > M$ we may set $S_m = 0$.*

Proof. — If $m > M$, then we may take $c_{i,m} = b_{i,m}$. So we may assume that $m \leq M$. Consider the subset $\{b_{i,m} = p_{i,m}(x)\partial^i + w_{i,m} \mid i \geq 0\}$ of R'. Let $E_{M,N} = \{r \in E \mid x\text{-deg}\, r \leq M \text{ and } y\text{-deg}\, r \leq N\}$, and let V be the vector space spanned by $\{w_{i,m} \mid i \geq 0\}$. Set $W = \{x\text{-gr}\, w \mid w \in V\} \cap E$. Note that W is a subspace of $E_{M,N}$. It is clear that $E_{M,N}$ and hence W is a finite dimensional subspace of E. So there is an $S_m > 0$ such that W is spanned by a subset of

$$\Big\{x\text{-gr}\, w \mid w \text{ is in the span of the set } \{w_{i,m} \mid S_m \geq i \geq 0\}\Big\}.$$

It follows that for $i > S_m$, there exist complex numbers $\alpha_{k,m}$ for $S_m \geq k \geq 0$ such that

$$x\text{-deg}\Big(w_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} w_{k,m}\Big) < 0 \quad \text{and}$$

$$\partial\text{-deg}\Big(w_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} w_{k,m}\Big) < 0.$$

We may now set $c_{i,m} = b_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} b_{k,m}$.

The next corollary follows immediately from LEMMA 2.6.

COROLLARY 2.7. — *We have* $\dim_{\mathbb{C}} \mathbb{C}[x,y]/x\text{-}\mathrm{gr}\, R' < \infty$.

LEMMA 2.8
Let W be a linear subspace of A_1. Then $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \mathrm{gr}_{\alpha,\beta} W$.

Proof. — Suppose that W is a vector space and that

$$\{W_i \mid i \text{ is an integer }\}$$

is a filtration for W such that the vector spaces $W_i = 0$ for $i < 0$ and $W = \bigcup_{i\geq 0} W_i$. Then clearly W and $\bigoplus W_i/W_{i-1}$ are isomorphic as vector spaces. Hence $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \bigoplus W_i/W_{i-1}$. In particular if W is a linear subspace of A_1, then $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \mathrm{gr}_{\alpha,\beta} W$.

We are now ready to prove PROPOSITION 2.4′.

Proof of PROPOSITION 2.4′. — Note that R' is a linear subspace of $\mathbb{C}[x,\partial]$. Hence, it follows from the definition of $\mathrm{gr}_{\alpha,\beta} R'$ that $\mathrm{gr}_{\alpha,\beta} R'$ is a linear subspace of $\mathrm{gr}_{\alpha,\beta} \mathbb{C}[x,\partial]$. Thus we only need to prove the statement about dimensions.

Set $V_n = \{x^i y^j \mid \alpha i + \beta j \leq n\}$ for all $n \geq 0$. Note that each V_n has finite dimension and that $\bigcup_{n\geq 0} V_n = \mathbb{C}[x,y]$. Set $W_n = \{w \in R' \mid \mathrm{gr}_{\alpha,\beta} w \in V_n\}$. Since $\mathrm{gr}_{\alpha,\beta} R' \subset \mathbb{C}[x,y]$, we have that $\bigcup_{n\geq 0} W_n = R'$. Suppose that $w \in W_n$. We can write $w = p(x)\partial^k + c$ for some $p(x) \in \mathbb{C}[x]$ and $k \geq 0$ such that $\partial\text{-}\deg(c) < k$ and $\alpha \deg p(x) + \beta k \leq n$. So $\partial\text{-}\mathrm{gr}\, w = p(x)y^k$ is also in V_n. Thus $\partial\text{-}\mathrm{gr}\, W_n \subset V_n$ for all $n \geq 0$.

Set $L = \alpha M + \beta N$. We will show that $\partial\text{-}\mathrm{gr}\, W_n = \partial\text{-}\mathrm{gr}\, R' \cap V_n$ for all $n \geq L$. Since $\partial\text{-}\mathrm{gr}\, W_n \subset V_n$, it is clear that $\partial\text{-}\mathrm{gr}\, W_n \subset \partial\text{-}\mathrm{gr}\, R' \cap V_n$. Suppose $\partial\text{-}\mathrm{gr}\, w = p(x)y^j$ is an element of $\partial\text{-}\mathrm{gr}\, R' \cap V_n$. So $\alpha \deg p(x) + \beta j \leq n$. By LEMMA 2.5, we may find in R' an element $w = p(x)\partial^j + g_N(x)\partial^N + \cdots + g_0(x)$ and $\deg g_k(x) \leq M_k$ for each k such that $N \geq k \geq 0$. Now

$$V_{\alpha,\beta}(g_N(x)\partial^N + \cdots + g_0(x)) \leq \alpha M + \beta N = L.$$

Hence $V_{\alpha,\beta}(w) \leq \max\{\alpha \deg p(x) + \beta j, L\}$. If $\alpha \deg p(x) + \beta j > L$, then $V_{\alpha,\beta}(w) = \alpha \deg p(x) + \beta j \leq n$ since $p(x)y^j$ is an element of V_n. Hence $w \in W_n$. If $\alpha \deg p(x) + \beta j \leq L$, then $V_{\alpha,\beta}(w) \leq L \leq n$, hence again $w \in W_n$. Therefore $\partial\text{-}\mathrm{gr}\, W_n = \partial\text{-}\mathrm{gr}\, R' \cap V_n$ for all $n \geq L$.

Since W_n is a linear subspace of $\mathbb{C}[x,\partial]$, by LEMMA 2.8, we have that

$$\dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \partial\text{-}\mathrm{gr}\, W_n \text{ and}$$
$$\dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \mathrm{gr}_{\alpha,\beta} W_n.$$

Furthermore, for all $n \geq L$, we have that $\dim_{\mathbb{C}} \partial\text{-gr}\, R' \cap V_n = \dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \operatorname{gr}_{\alpha,\beta} W_n$. Since $\dim_{\mathbb{C}} V_n$ is finite, it follows that $\dim_{\mathbb{C}} V_n/\partial\text{-gr}\, R' \cap V_n = \dim_{\mathbb{C}} V_n/\operatorname{gr}_{\alpha,\beta} W_n$ for all $n \geq L$. Clearly

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R' = \lim_{n\to\infty} \dim_{\mathbb{C}} V_n/\partial\text{-gr}\, R' \cap V_n \quad \text{and}$$
$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\operatorname{gr}_{\alpha,\beta} R' = \lim_{n\to\infty} \dim_{\mathbb{C}} V_n/\operatorname{gr}_{\alpha,\beta} W_n.$$

Therefore $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R' = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\operatorname{gr}_{\alpha,\beta} R'$.

By COROLLARY 2.7, we have that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/x\text{-gr}\, R' < \infty$. So we may apply the first part of the proof with x replaced by ∂ and vice versa to show that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/x\text{-gr}\, R' = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\operatorname{gr}_{\alpha,\beta} R'$ which completes the proof of PROPOSITION 2.4′ and therefore of PROPOSITION 2.4.

Recall that $\operatorname{codim} R$ is defined to be $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial\text{-gr}\, R$. PROPOSITION 2.4 implies that $\operatorname{codim} R = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\operatorname{gr}_{\alpha,\beta} R$ for any two nonnegative not both zero real numbers α and β. We will eventually show that $\operatorname{codim} R$ is an invariant of R.

3. Ad-Nilpotent subalgebras of $D(X)$

Suppose that $D(X) \cong D(Y)$. Then $D(X)$ contains a maximal commutative ad-nilpotent subalgebra isomorphic to $O(Y)$. So it is interesting to understand the maximal commutative ad-nilpotent subalgebras of $D(X)$. Let D denote the quotient field of the first Weyl algebra, A_1. In this section, we show that if B is a maximal commutative ad-nilpotent subalgebra of $D(X)$, then there exists an element $b \in D$ such that B is a subring of $\mathbb{C}[b]$.

LEMMA 3.1. — *Suppose that R is a subalgebra of D so that the quotient ring of R is D, and that u is an element of $D - \mathbb{C}$ that acts* ad*-nilpotently on R. Then there exists a $v \in D$ such that $[u,v] = 1$. Furthermore, for any $v \in D$ such that $[u,v] = 1$, we have $R \subset C_D(u)[v]$ where $C_D(u)$ denotes the centralizer of u in D.*

Proof. — Define $R_0 = C_D(u)$ and $R_i = \{z \in D \mid [z,u] \in R_{i-1}\}$.

Now $R \subset \bigcup_{i\geq 0} R_i$ since u acts ad-nilpotently on R. Let a be a nonzero element of $R_1 - R_0$. (Note that $R_1 - R_0$ is nonempty since $u \notin \mathbb{C}$ and $\mathbb{C}$ is the center of R.) Then $0 \neq [u,a] = b \in R_0$. So $[u, b^{-1}a] = b^{-1}[u,a] = 1$. Set $v = b^{-1}a$.

Clearly $R_0 \subset C_D(u)$. We will show by induction on i that

$$R_i \subset C_D(u)v^i + \cdots + C_D(u) \quad \text{for all } i \geq 0.$$

Assume that $R_{i-1} \subset C_D(u)v^{i-1} + \cdots + C_D(u)$ and choose $z \in R_i$. Then $[z, u] \in R_{i-1}$, hence $[z, u] = \sum_{0 \le m \le i-1} f_m(u)v^m$. Then

$$\Big[z - \sum_{0 \le m \le i-1} f_m(u)\frac{v^{m+1}}{m+1}, u\Big] = 0.$$

Hence $z - \sum_{0 \le m \le i-1} f_m(u)v^{m+1}/(m+1) \in C_D(u)$. Therefore

$$z \in C_D(u)v^i + \cdots + C_D(u).$$

We may define the graded algebra $v\text{-gr}\, C_D(u)[v]$ by setting $v\text{-gr}\, a = u_i w^i$ where $a = u_i v^i + \cdots + u_0$ is an element of $C_D(u)[v]$ with $u_k \in C_D(u)$ for $i \ge k \ge 0$.

We will show that $C_D(u)$ is in fact a rational function field in one variable.

The next lemma is well known. See for example [3, Corollary 3.2].

LEMMA 3.2. — *If $f \in D - \mathbb{C}$ then $C_D(f)$ is commutative.*

LEMMA 3.3. — *If $u \in D$ acts* ad*-nilpotently on R, where R is a subalgebra of D such that the quotient ring of R is D, then there exists $z \in D$ such that $C_D(u)$ is isomorphic to a rational function field $\mathbb{C}(z)$.*

Proof. — Let us call an element $a \in D$ ad-nilpotent if it acts ad-nilpotently on some subalgebra $R(a)$ of D such that the quotient ring of $R(a)$ is D. By LEMMA 3.1, there exists an element $v \in D$ such that $[v, u] = 1$ and $D = C_D(u)(v)$.

We will first assume that there exists an ad-nilpotent element a of D with $v\text{-deg}\, a \ne 0$. Now for each element $c \in C_D(u)$, there exists elements $c_1 = c_1(c)$ and $c_2 = c_2(c)$ in $R(a)$ such that $c = c_1 c_2^{-1}$. It is clear that $v\text{-gr}\, a$ acts nilpotently by Poisson bracket action on $v\text{-gr}\, c_1$ and $v\text{-gr}\, c_2$. Let $v\text{-gr}\, a = a_0 w^n$, $v\text{-gr}\, c_1 = c_{1,0} w^m$, and $v\text{-gr}\, c_2 = c_{2,0} w^m$. (Since $c \in C_D(u)$, it is clear that $v\text{-deg}\, c_1 = v\text{-deg}\, c_2$.)

By the same arguments as in [5, Lemma 7], there exists an element b in the algebraic closure of $C_D(u)$ such that $c_{1,0} w^m = (a_0 w^n)^{m/n} p_1(b)$ and $c_{2,0} w^m = (a_0 w^n)^{m/n} p_2(b)$ where $p_1(b)$ and $p_2(b)$ are polynomials.

Since $v\text{-deg}\, c = 0$, we have that $c = c_1 c_2^{-1} = c_{1,0} c_{2,0}^{-1} = p_1(b)(p_2(b))^{-1}$. Therefore $C_D(u) \subset \mathbb{C}(b)$. By Luroth's theorem, $C_D(u)$ is isomorphic to a field of rational functions in one variable.

Now assume that $v\text{-deg}\, a = 0$ for all ad-nilpotent elements. Consider the standard generators x and ∂ for D. These are ad-nilpotent elements of D since they act ad-nilpotently on $\mathbb{C}[x, \partial]$. Therefore $1 = [\partial, x]$ has negative v-degree which is impossible.

4. Codim is an invariant of $D(X)$

In this section $R = D(X)$ for a curve X satisfying the conditions of the introduction. Suppose that u and v are elements of D with commutator $[v,u] = 1$ such that $D(X) \subset \mathbb{C}(u)[v]$ and $v\text{-gr}\, D(X)$ is a subring of the polynomial ring in two generators, $u = v\text{-gr}\, u$ and $w = v\text{-gr}\, v$. We may define $\operatorname{codim}_{u,v} D(X)$ as $\dim_{\mathbb{C}} \mathbb{C}[u,w]/v\text{-gr}\, D(X)$. In this section, we will show that $\operatorname{codim}_{u,v} D(X) = \operatorname{codim} D(X)$. So codim does not depend on the embedding of $D(X)$ inside of $\mathbb{C}(x)[\partial]$.

Note that $u\text{-gr}\, \mathbb{C}[u,v]$ and $v\text{-gr}\, \mathbb{C}[u,v]$ are isomorphic polynomial rings. We will identify these isomorphic rings and thus write $u\text{-gr}\, u = v\text{-gr}\, u = u$ and $u\text{-gr}\, v = v\text{-gr}\, v = w$.

LEMMA 4.1. — *Suppose that $R \subset \mathbb{C}(u)[v] \subset D$, where $[v,u] = 1$, such that the quotient ring of R is D, the graded algebra $v\text{-gr}\, R$ is a subset of $\mathbb{C}[u,w]$, and $\operatorname{codim}_{u,v} R$ is finite. Then there exist elements u' and v' of D such that $u\text{-gr}\, v' = w$ and $u\text{-gr}\, u' = -u$, the commutator $[u',v']$ is 1, and the ring R is a subring of $\mathbb{C}(v')[u']$. Moreover, there is an isomorphism from $u'\text{-gr}\, \mathbb{C}[u',v']$ to $u\text{-gr}\, \mathbb{C}[u,v]$ which restricts to an isomorphism from the graded algebra $u'\text{-gr}\, R$ to $u\text{-gr}\, R$, and $\operatorname{codim}_{v',u'} R = \operatorname{codim}_{u,v} R$.*

Proof. — Define subalgebras R_i of R for $i \geq 0$ as follows :

$$R_i = \{z \in R \mid u\text{-deg}(z) \leq 1\}.$$

(The following argument is similar to [8, Theorem 2.7].) Now

$$u\text{-gr}\,[f(v)u^i, g(v)] = u\text{-gr}\big(-if(v)g'(v)u^{i-1}\big) \quad \text{for } i \geq 0.$$

Also $u\text{-gr}\, R$ is a subset of $\mathbb{C}[u,w]$ by LEMMA 2.1. Hence, it is easy to see that R_0 is a maximal commutative ad-nilpotent subalgebra of R. Furthermore the map which sends z to $u\text{-gr}\, z$ is an isomorphism of R_0 to $u\text{-gr}\, R_0 = u\text{-gr}\, R \cap \mathbb{C}[w]$. By assumption, $\operatorname{codim}_{u,v} R < \infty$, hence $\dim_{\mathbb{C}} \mathbb{C}[w]/u\text{-gr}\, R_0 < \infty$. So the integral closure of $u\text{-gr}\, R_0$ is $\mathbb{C}[w]$, and thus the integral closure of R_0 is $\mathbb{C}[v']$ for some $v' \in D$ with $u\text{-gr}\, v' = w$ and $R_0 = R \cap \mathbb{C}[v']$ for some $v' \in D$ with $u\text{-gr}\, v' = w$ and $R_0 = R \cap \mathbb{C}[v']$. Note that $u\text{-gr}\, p(v') = p(w)$ for any polynomial $p(t) \in \mathbb{C}[t]$.

By LEMMA 3.3, $C_D(v')$ is a rational function field in one variable. Let us check that $C_D(v') = \mathbb{C}(v')$. Let $f \in C_D(v')$. Then $u\text{-deg}\, f = 0$, because otherwise $[v', f] \neq 0$, and $u\text{-gr}\, f = r(w)$ where $r(w) \in \mathbb{C}(w)$. Therefore $f = r(v') + f_1$ where $u\text{-deg}\, f_1 < 0$. But $f_1 \in C_D(u)$ and can not have a negative degree. Hence f_1 is 0. Now, according to LEMMA 3.1, there exists a $u' \in D$ such that $[u',v'] = 1$ and $R \subset \mathbb{C}(v')[u']$.

Suppose that $u\text{-gr}\, u' = f(w)u^i$. Since $u\text{-gr}\, v' = w$, we must have $u\text{-gr}[u', v'] = -if(w)u^{i-1}$ unless $i = 0$. If $i = 0$, then either $[u', v'] = 0$ or $u\text{-deg}[u', v'] < -1$. Since $[u', v'] = 1$, it follows that $i \neq 0$. Hence $-if(w)u^{i-1}$ must equal 1. Therefore $i = 1$ and $f(w) = -1$ and $u\text{-gr}\, u' = -u$.

Suppose that z is an element of $R \subset \mathbb{C}(v')[u']$. We may write $z = f(v')(u')^j + e$ where $u'\text{-deg}\, e < j$, and $f(v')$ is a polynomial, and $j \geq 0$. Since $u\text{-deg}\, v' = 0$ and $u\text{-deg}\, u' = 1$, we must have that $u\text{-deg}\, e < j$ and $u\text{-gr}\, z = u\text{-gr}\, f(v')(u')^j$. Since $u\text{-gr}\, f(v') = f(w)$ and $u\text{-gr}\, u' = -u$, it follows that $u\text{-gr}\, z = f(w)(-u)^j$. Hence the isomorphism from $u'\text{-gr}\, \mathbb{C}[u', v']$ to $u\text{-gr}\, \mathbb{C}[u, v]$ which sends $u'\text{-gr}\, u'$ to $u\text{-gr}\, u' = -u$ and $u'\text{-gr}\, v'$ to $u\text{-gr}\, v' = w$ restricts to an isomorphism from $u'\text{-gr}\, R$ to $u\text{-gr}\, R$. Since $\operatorname{codim}_{u,v} R$ is finite, by PROPOSITION 2.4, we have that $\operatorname{codim}_{u,v} R = \dim_{\mathbb{C}} \mathbb{C}[u, w]/u\text{-gr}\, R$. It follow immediately that $\operatorname{codim}_{u,v} R = \operatorname{codim}_{v',u'} R$.

For the next three lemmas, assume that R is a subring of $\mathbb{C}(u)[v] \subset D$, where u and v are elements of D whose commutator is 1, and that $v\text{-gr}\, R \subset \mathbb{C}[u, w]$ with $\operatorname{codim}_{u,v} R < \infty$. Write R_0 for the ad-nilpotent subalgebra $\{z \in R \mid u\text{-gr}\, z = 0\}$. We may define valuations $V_{\alpha,\beta}$ and corresponding graded algebras on R as in Section 1 using u and v instead of x and ∂. For example, $V_{\alpha,\beta}(u^i v^j) = \alpha i + \beta j$.

LEMMA 4.2. — *Suppose that r is an* ad*-nilpotent element of R that is not contained in $\mathbb{C}(u)$ and is not contained in R_0. Then there exist positive integers n and m and complex numbers λ and γ such that $u\text{-gr}\, r = (\lambda u)^n$ and $v\text{-gr}\, r = (\gamma w)^m$. Furthermore, $V_{m,n}(r) = mn$.*

Proof. — Since r is not an element of $\mathbb{C}(u)$ and is not an element of R_0, it follows that $u\text{-deg}\, r > 0$ and $v\text{-deg}\, r > 0$. We will argue as in [2, Lemma 8.7]. We may write

$$r = \sum_{i \geq 0, j \geq 0} \sigma_{i,j} u^i v^j + f_k(u) v^k + \cdots + f_0(u)$$

where $\deg f_j(u) < 0$ for $k \geq j \geq 0$. Clearly, $v\text{-deg}\, r > k$. Let n be the smallest nonnegative integer such that $\sigma_{j,0} = 0$ for all $j > n$. Let m be the smallest nonnegative integer such that $\sigma_{0,k} = 0$ for all $k > m$. We claim that $\sigma_{i,j} = 0$ for all pairs i, j such that $mi + nj > mn$.

Assume the claim is false. Then there exist positive real numbers α and β and a pair of positive integers i and j with $\sigma_{i,j} \neq 0$, such that $\mathrm{gr}_{\alpha,\beta}\, r = \sigma_{i,j} u^i w^j$. Without loss of generality, $\sigma_{i,j} = 1$. First assume $i \geq j$.

Now there exists a monic polynomial $p(t)$ such that $p(u) \in R$. Since both α and β are positive, we have that $\mathrm{gr}_{\alpha,\beta}\, p(u) = u^d$ where $d = \deg p(u)$. Note that $\mathrm{gr}_{\alpha,\beta}[r, p(u)] = dju^{i-1+d}w^{j-1}$. Suppose that

$$\mathrm{gr}_{\alpha,\beta}\,\mathrm{ad}_r^k\big(p(u)\big) = \alpha_k\, u^{k(i-1)+d}\, w^{k(j-1)}.$$

Then

$$\mathrm{gr}_{\alpha,\beta}\,\mathrm{ad}_r^{k+1}\big(p(u)\big) = \alpha_k\big[\big(k(i-1)+d\big)j - ik(j-1)\big]u^{(k+1)(i-1)+d}\, w^{(k+1)(j-1)}.$$

Now $(k(i-1)+d)j - ik(j-1) = (i-j)k + dj > 0$ for all $k \geq 0$ since $i \geq j$. This contradicts the fact that r is ad-nilpotent.

Now assume that $i < j$. Consider a nonconstant element $z \in R_0$. Recall that R_0 sits inside a polynomial algebra $\mathbb{C}[v']$ where $v' \in D$ where u-$\mathrm{gr}\, v' = w$. So $z = q(v')$ for some nonconstant polynomial $q(t)$. Since both α and β are positive, it follows that $\mathrm{gr}_{\alpha,\beta}\, z = w^k$ where $k = \deg q(t)$. The argument now follows as in the preceding paragraph.

We have shown that $\sigma_{i,j} = 0$ for all pairs of positive integers i and j such $mi + nj > nm$. In particular, u-$\mathrm{gr}\, r = \sigma_{n,0}u^n$, and v-$\mathrm{gr}\, r = \sigma_{0,m}w^m$, and $V_{m,n}(r) = mn$.

LEMMA 4.3. — *Suppose that r is an* ad*-nilpotent element of R that is not contained in $\mathbb{C}(u)$ and is not contained in R_0. Set $n = u$-$\deg r$ and $m = v$-$\deg r$. Then one of the following two statements hold where $\lambda, \lambda', \gamma, \gamma'$ are elements of $\mathbb{C}$, and i is an integer such that $n \geq i \geq 0$.*

(1) *If $n \geq m$, then n is a multiple of m and*

$$\mathrm{gr}_{m,n}\, r = \big((\lambda u)^{n/m} + \gamma w\big)^m.$$

(2) *If $m > n$, then m is a multiple of n and*

$$\mathrm{gr}_{n,m}\, r = \big(\lambda u + (\gamma w)^{m/n}\big)^n.$$

Proof. — By LEMMA 4.2, both n and m are positive. So there exist nonzero complex numbers σ_1 and σ_2 such that u-$\mathrm{gr}\, r = \sigma_1 u^n$ and v-$\mathrm{gr}\, r = \sigma_2 w^m$. Now by LEMMA 2.1, $\mathrm{gr}_{m,n} R \subset \mathbb{C}[u,w]$, and by PROPOSITION 2.4, $\dim_{\mathbb{C}} \mathbb{C}[u,w]/\mathrm{gr}_{m,n} R < \infty$. Hence we may apply the arguments of [2, Lemma 7.3] to the ad-nilpotent element r of R.

In the next lemma, we will show that $\mathrm{codim}\, R$ is independent of the choice of generator for $\mathbb{C}(u)$.

LEMMA 4.4. — *Suppose that u_1 and v_1 are elements of D whose commutator is 1 such that $\mathbb{C}(u) = \mathbb{C}(u_1)$, the ring R is a subring of $\mathbb{C}(u_1)[v_1] \subset D$, and that v_1-$\operatorname{gr} R \subset \mathbb{C}[u_1, w_1]$ with $\operatorname{codim}_{u_1,v_1} R < \infty$. Then $\operatorname{codim}_{u,v} R = \operatorname{codim}_{u_1,v_1} R$.*

Proof. — Set $B = R \cap \mathbb{C}(u) = R \cap \mathbb{C}[u]$. Since $\mathbb{C}(u_1) = \mathbb{C}(u)$ and v_1-$\operatorname{gr} R \subset \mathbb{C}[u_1, w_1]$, we have that $B = R \cap \mathbb{C}(u_1) = R \cap \mathbb{C}[u_1]$. By assumption, both $\operatorname{codim}_{u,v} R$ and $\operatorname{codim}_{u_1,v_1} R$ are finite. Hence both $\dim_{\mathbb{C}} \mathbb{C}[u]/B$ and $\dim_{\mathbb{C}} \mathbb{C}[u_1]/B$ are finite. Therefore the integral closure of B in $\mathbb{C}(u)$ is $\mathbb{C}[u]$ and is also $\mathbb{C}[u_1]$. So $\mathbb{C}[u] = \mathbb{C}[u_1]$ and there exist integers α and β such that $u = \alpha u_1 + \beta$. Since $[v_1, u_1] = 1$, we have that $[\alpha v - v_1, u] = 0$. So $v + g(u) = \alpha^{-1} v_1$ for some $g(u) \in \mathbb{C}(u)$. Set $v_2 = v + g(u)$. Note that $[v_2, u] = 1$ and $R \subset \mathbb{C}(u)[v_2]$. Now $f(u)v^i = f(u)(v_2 - g(u))^i$, hence v-$\operatorname{gr} R = v_2$-$\operatorname{gr} R$ and $\operatorname{codim}_{u,v} R = \operatorname{codim}_{u,v_2} R$. Without loss of generality, we may assume that $v = v_2$ and that $v = \alpha^{-1} v_1$. The isomorphism of $\mathbb{C}[u, w]$ to $\mathbb{C}[u_1, w_1]$ which sends u to αu_1 and w to $\alpha^{-1} w_1$ clearly induces an isomorphism from v-$\operatorname{gr} R$ to u-$\operatorname{gr} R$. The result now follows.

We are now ready to show that $\operatorname{codim} D(X)$ is an invariant of $D(X)$.

THEOREM 4.5. — *Suppose that X is an affine curve such that the normalization of X is the affine line, with the normalization map $\pi : \widetilde{X} \to X$ injective. Then for any pair of elements u and v in D, such that $[v, u] = 1$, the ring $D(X)$ is a subring of $\mathbb{C}(u)[v]$, and v-$\operatorname{gr} D(X)$ is a subring of the polynomial ring with generators v-$\operatorname{gr} u$ and v-$\operatorname{gr} v$, we have that $\operatorname{codim}_{u,v} D(X) = \operatorname{codim} D(X)$.*

Proof. — Now $D(X)$ is a subring of $\mathbb{C}(x)[\partial]$ and $\operatorname{codim} D(X) = \operatorname{codim}_{x,\partial} D(X)$. Assume that u and v are elements of D such that $[v, u] = 1$, the ring $D(X)$ is a subring of $\mathbb{C}(u)[v]$, and v-$\operatorname{gr} D(X)$ is a subring of the polynomial ring $\mathbb{C}[u, w]$ where v-$\operatorname{gr} u = u$ and v-$\operatorname{gr} v = w$. Let r be a nonconstant ad-nilpotent element of $D(X)$ contained inside $\mathbb{C}(u)$. Set x-$\deg r = n$ and ∂-$\deg r = m$. We will induct on $t = m + n$.

If $m = 0$, then r is an element of $\mathbb{C}(x)$ and the result now follows by LEMMA 4.4.

If $n = 0$, then r is an element of $\{z \in D(X) \mid x\text{-}\deg z = 0\}$, and the result follows from LEMMA 4.1 and LEMMA 4.4. Hence the theorem holds for $t = 0$.

So we may assume that both n and m are positive.

First assume that $n \geq m$. By LEMMA 4.3, n is a multiple of m and there exist elements λ, and γ of $\mathbb{C}$ such that $\operatorname{gr}_{m,n} r = ((\lambda x)^{n/m} + \gamma y)^m$. Hence

$$r = \big((\lambda x)^{n/m} + \gamma \partial\big)^m + c$$

where $V_{m,n}(c) < mn$ and x-deg $c < n$ and ∂-deg $c < m$. Set $\partial_1 = \partial - (\gamma)^{-1}(\lambda x)^{n/m}$ and $x_1 = x$. Note that $((\lambda x)^{n/m} + \gamma\partial)^m = (\gamma\partial_1)^m$. Furthermore $(\partial)^i = (\partial_1 + (\gamma)^{-1}(\lambda x_1)^{n/m})^i$. It follows that ∂_1-deg $c < m$ and ∂_1-deg $r = m$. Also x_1-deg $c \leq (m-1)n/m < n$. Since $r = (\gamma\partial_1)^m + c$, we have that x_1-deg $r < n$. By LEMMA 4.4, $\operatorname{codim}_{x_1,\partial_1} D(X) = \operatorname{codim}_{x,\partial} D(X)$. Now ∂_1-deg $r + x_1$-deg $r < t$, hence the result now follows by induction for this case.

Now assume that $n < m$. By LEMMA 4.1, there exist elements x_1 and ∂_1 in D such that $D(X) \subset \mathbb{C}(\partial_1)[x_1]$, $[x_1, \partial_1] = 1$, x_1-gr $\partial = x_1$-gr ∂_1, x_1-gr $x = -x_1$, x_1-gr $R \cong x$-gr R, and $\operatorname{codim}_{\partial_1,x_1} R = \operatorname{codim}_{x,\partial} R$. It follows that x_1-deg $r = x$-deg $r = n$. If ∂_1-deg $r < m$, then the proof follows by induction.

Otherwise ∂_1-deg $r \geq m > n$ and we may apply the methods used above repeatedly to find elements $\partial_2 = \partial_1$ and $x_2 = x_1 + g(\partial_1)$ where $g(\partial_1) \in \mathbb{C}(\partial_1)$ such that x_2-deg $r = n$ and ∂_2-deg $r < m$. The proof again follows by induction.

We are now able to obtain a nice description of the maximal ad-nilpotent subalgebras of $D(X)$.

COROLLARY 4.6. — *Suppose that X is an affine curve such that the normalization of X is the affine line, with the normalization map $\pi : \widetilde{X} \to X$ injective. Suppose that B is a maximal* ad*-nilpotent subalgebra of $D(X)$. Then there exists an element u in D such that B is a commutative finitely generated algebra with integral closure $\mathbb{C}[u]$ and the centralizer of B in $D(X)$ is the rational function field $\mathbb{C}(u)$.*

Proof. — By LEMMA 3.3 and LEMMA 3.4, there exists u in D such that $C_D(B) = \mathbb{C}(u)$ and $B \subset \mathbb{C}[u]$. By LEMMA 3.1, there exists v in D such that $D(X) \subset \mathbb{C}(u)[v]$. Recall that the set of ad-nilpotent elements of $D(X)$ is strictly larger than the maximal commutative ad-nilpotent subalgebra $O(X)$ of $D(X)$. Since B is commutative, B cannot contain all the ad-nilpotent elements of $D(X)$. Hence $D(X)$ contains an ad-nilpotent element s not contained in B. By [8, Lemma 1.7], v-gr $s = \lambda w^n$ for some $\lambda \in \mathbb{C}$ and $n > 0$. Since s acts ad-nilpotently on $D(X)$, it is clear that v-gr $D(X) \subset \mathbb{C}[u, w]$. By THEOREM 4.5, $\dim_{\mathbb{C}} \mathbb{C}[u]/B$ is finite hence the integral closure of B is $\mathbb{C}[u]$. By Eakin's theorem [6, Section 35], B is finitely generated.

The invariant $\operatorname{codim} D(X)$ can be used to distinguish rings of differential operators.

COROLLARY 4.7. — *Suppose that X and Y are both affine curves with normalization equal to the affine line and with injective normalization* *maps. If* $D(X) \cong D(Y)$, *then* $\operatorname{codim} D(X) = \operatorname{codim} D(Y)$.

Proof. — Consider both $D(X)$ and $D(Y)$ as subalgebras of $\mathbb{C}(x)[\partial]$ using the standard embedding. Let ϕ be an isomorphism which maps $D(Y)$ to $D(X)$. Set $u = \phi(x)$ and $v = \phi(\partial)$. Clearly u and v satisfy the conditions of THEOREM 4.5. Therefore $\operatorname{codim} D(Y) = \operatorname{codim}_{u,v} D(X) = \operatorname{codim} D(X)$.

5. Examples

In this section, we will consider two families of curves. We will calculate codimensions to show that their rings of differential operators are mutually nonisomorphic.

Recall that X is a monomial curve if $O(X)$ is generated by monomials x^k as an algebra over $\mathbb{C}$. Let Λ be the subset $\{k \mid x^k \in O(x)\}$ of the integers. Define the set $\Lambda - i$ to be $\{k - i \mid k \in \Lambda\}$ where i is an integer. MUSSON gives a complete description of $D(X)$ in [7]. In particular,

$$D(X) = \sum_{k \in \mathbb{Z}} x^k f_k(x\partial)\mathbb{C}[x\partial]$$

where

$$f_k(x\partial) = \prod_{\alpha \in \Lambda - (\Lambda - k)} (x\partial - \alpha).$$

Let X_n be the monomial curve with $O(X_n) = \mathbb{C} + x^n\mathbb{C}[x]$ as coordinate ring, where n is a positive integer. Then by the previous paragraph, we have

$$D(X_n) = \sum_{k \in \mathbb{Z}} x^k f_k(x\partial)\mathbb{C}[x\partial]$$

where the polynomial f_i is 1 for $i = 0$ and $i \geq n$; the polynomial f_i is $x\partial$ for $1 \leq i \leq n - 1$; the polynomial f_i is

$$(x\partial) \prod_{n-i > k \geq n} (x\partial - k) \quad \text{for} \quad -1 \geq i \geq -(n-1)$$

and the polynomial f_i is

$$(x\partial) \prod_{n \leq k < -i} (x\partial - k) \prod_{-i < k < n-i} (x\partial - k) \quad \text{for } i \leq -n.$$

Note that if $g(x\partial)$ is a monic polynomial in $\mathbb{C}[x\partial]$, then

$$\partial\text{-gr}\, g(x\partial) = x^d \partial^d \quad \text{where } d = \deg g(x\partial).$$

Hence ∂-gr $D(X_n) = \sum_{k\in\mathbb{Z}} g_k\mathbb{C}[xy]$ where

$$\begin{aligned} &g_0 = 1\,; \\ &g_i = x^{i+1}y \quad \text{for } 1 \le i \le n-1\,; && g_i = x^i \quad \text{for } i \ge n\,; \\ &g_i = xy^{i+1} \quad \text{for } -n+1 \le i \le -1\,; && g_i = y^i \quad \text{for } i \le -n. \end{aligned}$$

A basis for $\mathbb{C}[x,y]/\partial$-gr $D(X_n)$ is just $x, x^2, \ldots, x^{n-1}, y, y^2, \ldots, y^{n-1}$. Therefore codim $D(X_n) = 2(n-1)$. By COROLLARY 4.7, $D(X_n)$ is isomorphic to $D(X_m)$ if and only if $O(X_n) \cong O(X_m)$.

Now set $Y_{2n} = \mathbb{C} + \mathbb{C}x^2 + \cdots + \mathbb{C}x^{2n}\mathbb{C}[x]$ for $n \ge 1$. A similar calculation shows that codim $D(Y_{2n}) = n(n+1)$. Therefore $D(Y_{2n}) \cong D(Y_{2m})$ if and only if $O(Y_{2n}) \cong O(Y_{2m})$.

Consider just the curves X_4 and Y_4. Now $O(X_4) = \mathbb{C} + x^4\mathbb{C}[x]$ and $O(Y_4) = \mathbb{C} + \mathbb{C}x^2 + x^4\mathbb{C}[x]$. Clearly $O(X_4)$ is not isomorphic to $O(Y_4)$. But codim $D(X_4) =$ codim $D(Y_4) = 6$. Therefore codim does not distinguish between these two rings of differential operators. We should add that it has now been shown that $D(X_4)$ and $D(Y_4)$ are actually isomorphic rings even though $O(X_4)$ and $O(Y_4)$ are not isomorphic (see [4]).

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