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WEIERSTRASS POINTS ON TRIGONAL CURVES OF GENUS FIVE

BY

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RÉSUMÉ. — On calcule la dimension des variétés des courbes trigonales de genre cinq ayant un point de Weierstrass de type donné et on démontre que génériquement une telle courbe a un seul point de Weierstrass qui n'est pas normal.

On démontre aussi que le groupe de monodromie des points de Weierstrass sur les courbes trigonales de genre cinq est le groupe symétrique.

ABSTRACT. — We determine the dimension of the varieties of trigonal curves of genus five with a Weierstrass point of given type and we show that generically such a curve has only one non-normal Weierstrass point.

We prove also that the monodromy of Weierstrass points on trigonal curves of genus five is the full symmetric group.

Introduction

Let S be the surface of degree three in $\mathbb{P}^4(\mathbb{C})$ image of $\mathbb{P}^2(\mathbb{C})$ under the rational map given by the conics through a fixed point x. The surface S is isomorphic to the blow-up of $\mathbb{P}^2(\mathbb{C})$ at x (in the notations of [3] S is the rational normal scroll $S_{1,2}$). It what follows we work always over $\mathbb{C}$.

We fix the following notations:

π, the projection map from S onto $\mathbb{P}^2$;

R, the lines of $\mathbb{P}^2$;

H, the hyperplane section of S;

E, the exceptional divisor of S;

L, the lines of the ruling of S;

$D = \pi * R$;

L_z, the unique line of the ruling passing through the point z of S.

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We have

$$\text{Pic}(S)=\mathscr{Z}.H\oplus\mathscr{Z}.E;$$
$$H=2D-E:$$
$$L=D-E$$

and the canonical divisor K_S of S is :

$$K_S=-3D+E.$$

We consider the complete linear system $|C|$ on X where $C=5D-2E$ is the proper transform of a quintic passing twice through x.

The smooth elements of this system are canonical curves of genus five with a g_3^1 given by the ruling of S. On the other hand the generic trigonal curve of genus five lies on such a scroll ([8], [12]). It's well known that the g_3^1 is unique (*see* for example [7], Ex. 5.5, page 348).

We are interested at the Weierstrass points of the curves C. Recall that if C is a smooth curve of genus g and p is a point of C we say that the number a is a gap value at p if $h^0(a.p)=h^0((a-1).p)$. To each p is thus associated the sequence $1=a_1<a_2\ldots<a_g<2g$ of gap values and p is called regular if the sequence is $1,2,3,\ldots,g$ otherwise *a* Weierstrass point. The weight of such a point is

$$W(p)=\sum_i(a_i-i)$$

and the only points with weight one are those with the sequence $1,2,3,\ldots,g-1,g+1$. They are called normal Weierstrass points. The total weight of C is the sum of the weights of its points and this number is $W=(g-1)g(g+1)$. It's known that the generic Riemann surface of genus $g\geqslant 3$ has only normal Weierstrass points (*see* [4], Chap. 2 or [10]).

Sequence	Weight
12345	0
13579	10
	hyperelliptic
12457	4
12458	5
12367	4
12356	2
12357	3
12359	5
12346	1
	normal
12347	2
12348	3
12349	4

The gap sequences which *a priori* can occur on curves of genus five are as follows:

The hyperelliptic sequence never occurs on smooth curves $C \in X$, since they are canonical.

This paper together with [3] is part of a general research program about the following problems:

(*a*) existence (and non-existence) of Weierstrass points with a given gap sequence (or having a given weight) on trigonal curves;

(*b*) dimension of the variety of trigonal curves with at least one Weierstrass point of a given type or weight;

(*c*) generic behaviour of trigonal curves with a Weierstrass point of a given type i. e. to see if generically such a curve has only one non-normal Weierstrass point;

(*d*) monodromy of the Weierstrass points on trigonal curves.

To this purpose let's put $X = |C| \simeq \mathbb{P}^{17}$ and define

$$Z = \{(z, H) \quad \text{such that} \quad z \in S \cap H\} \subseteq S \times \mathbb{P}^{4*},$$

where $\mathbb{P}^{4*}$ is the set of hyperplanes of $\mathbb{P}^4$.

The variety Z is smooth, irreducible, of dimension five.

Define then

$$I = \{(C, z, H) \quad \text{such that} \quad m_z(C.H) \geqslant 5\} \subseteq X \times Z,$$

where $m_z(C.H)$ is the intersection multiplicity of C with H at z.

If Ψ: $I \to X$ is the projection and C is a smooth element of X then (C, z, H) belongs to I exactly when z is a Weierstrass point of C and H is a section of $\mathcal{O}(K - 5z)$, where K is the canonical divisor of C (*see* [3]).

In this paper we prove:

1. The generic curve C of X has only normal Weierstrass points. In particular the map Ψ: $I \to X$ has degree $120 = 4.5.6$.

2. I has only one component of maximal dimension (seventeen) mapping over X and the monodromy group is the full symmetric group on 120 elements.

3. The generic element of the subvariety of X consisting of the curves with at least one Weierstrass point with a given sequence of gaps (different from 12346) has only one non-normal Weierstrass point.

In particular we get.

4. The dimensions of the subvarieties of X defined in (3) and their rationality.

We recall that two curves C and C' of X give the same point in the moduli space $\mathcal{M}_5$ of curves of genus five if and only if there is an automorphism φ of S such that

$$\varphi(E)=E \qquad \text{and} \qquad \varphi(C)=C'.$$

In fact all these automorphisms are projective since the curves are canonically embedded and the surface S must be sent to itself since it is the surface of trisecants of C (resp. C') (*see* [7], page 348).

Finally E must be sent to itself because it is the only line with self-intersection -1 and we get a six-dimensional group isomorphic to the group of automorphisms of $\mathbb{P}^2$ fixing a point. In particular the image of X in $\mathcal{M}_5$ is an irreducible subvariety of dimension 11 and in general the dimension in $\mathcal{M}_5$ of the images of the subvarieties of X under consideration is just gotten subtracting six from the original dimension.

We recall also that in the plane model of a curve C of X the unique g_3^1 is just given by the lines through x and the dual g_5^2 is given by the lines of $\mathbb{P}^2$.

In the following table we collect the results about the dimensions of subvarieties of X (resp. $\mathcal{M}_5$) consisting of curves with a Weierstrass point p with given gap sequence.

The computations in the last column refer to the present paper.

The dimension 9 relative to the sequence 12367 (these curves are all trigonals) contradicts the second part of Rauch's result [10] which is not correct as already pointed out by several authors.

The dimension 10 relative to the sequence 12357 is due to H. Pinkham and it's based on the following three steps:

(1) Using a result of DELIGNE ([9], page 75) or BUCHWEITZ ([2], Theorems II.2.1 and II.2.2) one computes that the dimension of the

versal deformation space of the semigroup ring corresponding to the semigroup of $\mathcal{N}$ generated by 4, 6, 9, 11 is 12.

(2) Another computation ([10], II.1.5) shows that $\dim_{n\geq 0} \oplus T^1(n)=1$ and then using theorem 13.9 of [9] one shows that the variety of curves having a point with sequence 12357 has dimension ≥ 10 in moduli.

(3) Now use Arbarello's result [1]: $W_{4,5}$ has dimension 11 and is irreducible; by Rim-Vitully generically the gap sequence is 12356. Thus the locus where the gap sequence is 12357 is a proper subvariety.

The author would like to thank H. Pinkham for very helpful comments and for providing the computation relative to the sequence 12357, J. Harris

Gap sequence	Dimension in moduli space of all curves with a Weierstrass point with given gap sequence	Dimension in moduli space of all trigonal curves with a Weierstrass point with given gap sequence
13579	9	∅
12457	10	10
($g_3^1=\|3p\|$)	(Rim-Vitulli [11])	(Sec. 2, Cor. 2)
12458	9	9
($g_3^1=\|3p\|$)		(Sec. 2, Cor. 2)
12367	9	9
($g_5^2=\|K-5p\|$)		(Sec. 3, Cor. 3′)
12356	11	10
	(Rim-Vitulli [11])	(Sec. 3, Cor. 3)
12357	10	∅
	(Computation due to H. Pinkham)	(Sec. 3, Lemma 2)
12359	9	∅
	(Rim-Vitulli [11])	(Sec. 3, Lemma 2)
12346	12	11
		(Sec. 1, Prop. 2)
12347	11	10
	(Pinkham [9])	(Sec. 4, Cor. 4)
12348	10	9
	(Pinkham [9])	(Sec. 4, Cor. 4)
12349	9	8
	(Pinkham [9])	(Sec. 4, Cor. 4)

for suggesting originally this area of research and G. P. Pirola for useful conversations during the preparation of this paper.

1. Components of I and monodromy

With the notations we introduced before we can prove the following.

PROPOSITION 1. — *I has only one component of maximal dimension* 17 *mapping over X.*

Proof. — Let's look at the hyperplane section H of S. We can have:

(A) H is smooth: in this case H is a rational normal curve in $\mathbb{P}^3$;

(B) $H=L+D$: the projection $\pi(H)\subseteq\mathbb{P}^2$ consists of two distinct lines meeting at a point different from x;

(C) $H=E+L_1+L_2$: the projection $\pi(H)$ consists of two distinct lines meeting at x;

(D) $H=E+2L$ and $\pi(H)$ consists of two coincident lines containing x.

If H is smooth, $|C|$ cuts on $H\simeq\mathbb{P}^1$ the complete linear system of degree 8. In fact if

$$\rho:\ |C|\to|C|_{|H},$$

in the restriction map, $\mathrm{Ker}(\rho)\approx|3D-E|$ and

$$9=\dim|C|-\dim\mathrm{Ker}(\rho)=h^0(H,\ \mathcal{O}\ (C_{|H})).$$

In the same way $|C|$ cuts on $D\simeq\mathbb{P}^1$ the complete linear system of degree $D.C=5$ and cuts on L the complete system of degree 3.

We study now the projection map $\Phi: I\to Z$, and denote by $p=(z,\ H)$ a point of Z and by $X_p\subseteq X$ the set $\Psi(\Phi^{-1}(p))$.

The variety Z is the union of the following subvarieties:

$$A=\{p=(z,\ H)\text{ s. t. } H\text{ is smooth}\};$$

$$B_1=\{p=(z,\ L+D)\text{ s. t. } z\notin L,\ z\in D\};$$

$$B_2=\{p=(z,\ L+D)\text{ s. t. } z\in L,\ z\notin D\};$$

$$B_3=\{p=(z,\ L+D)\text{ s. t. } z=L.D\};$$

$$C_1=\{p=(z,\ E+L_1+L_2)\text{ s. t. } z\in E,\ z\notin L_i,\ i=1,\ 2\};$$

$$C_2=\{p=(z,\ E+L_1+L_2)\text{ s. t. } z\in L_i,\ z\notin E\};$$

$$C_3=\{p=(z, E+L_1+L_2) \text{ s. t. } z=L_i.E\};$$
$$D_1=\{p=(z, E+2L) \text{ s. t. } z\in E, z\notin L\};$$
$$D_2=\{p=(z, E+2L) \text{ s. t. } z\notin E, z\in L\};$$
$$D_3=\{p=(z, E+2L) \text{ s. t. } z=E.L\};$$

If $p\in A\cup B_1$, clearly $X_p\simeq\mathbb{P}^{12}$ because $|C|$ cuts on H and D complete systems.

If $p\in B_2\cup C_2$, X_p is contained in the proper subvariety X_1 on X consisting of singular curves because $m_z(C.H)\geqslant 5$ only when $L\subseteq C$.

By the same argument if $p\in C_1\cup D_1$, $X_p\subseteq X_1$ (in this case $C\in X_p$ when $E\subseteq C$).

Let now consider *a* point $p\in B_3$. Since L and D are transversal at z a non-singular C can have $m_z(C.H)\geqslant 5$ only when $m_z(C.\mathrm{H})=1$ and $m_z(C.D)\geqslant 4$.

It follows that $X_p\subseteq\mathbb{P}^{13}\cup X_1$ where

$$\mathbb{P}^{13}=\{C\in X \text{ s. t. } m_z(C.D)\geqslant 4\}.$$

On the other hand $\dim(B_3)=3$ sothat $\Phi^{-1}(B_3)$ cannot give rise to a component of dimension 17 mapping over X.

When $p\in C_3$, say $z=L_1.E$, a curve C not containing E or L_1 belongs X_p only when $m_z(C.L_1)=3$ and $m_z(C.E)=2$.

Since E and L_1 are transversal, the curve C is singular at z and $X_p\subseteq X_1$.

We are left with two more cases: $p\in D_2$ and $p\in D_3$.

In the first case $m_z(C.H)=2.m_z(C.L)$ hence $C\in X_p$ when C contains L or when z is a point of total ramification for the g_3^1 of C.

If we call X_2 the proper subvariety of X consisting of the curves C having a point of total ramification, we have $X_p^{;} X_1\cup X_2$.

Finally if $p\in D_3$, a curve C belongs to X_p when one of the following happens:

$$L\subseteq C;$$
$$E\subseteq C;$$
$$m_z(C.E)=2 \quad \text{and} \quad m_z(C.L)\geqslant 2;$$
$$m_z(C.E)=1 \quad \text{and} \quad m_z(C.L)\geqslant 2.$$

Only in the last case C can be non-singular and the non-singular curves C belonging to X_p are contained in a $\mathbb{P}^{15} \subseteq X$.

Since $\dim(D_3)=1$, as before we do not get component of dimension 17 different from

$$Y = \text{closure of } \Phi^{-1}(A \cup B_1).$$

Q.E.D.

As in [3] or [6] it follows by standard arguments:

COROLLARY 1. — *The monodromy of* $\Psi: Y \to X$ *is transitive.*

We want to prove now:

PROPOSITION 2. — *The generic curve* $C \in X$ *has only normal Weierstrass points.*

Recall that a Weierstrass point is normal if the gap sequence is 1, 2, 3, 4, 6 i. e. if its weight is one [3], [4], [5].

A point z on a smooth curve C is not normal when:

(1) $$h^0(K-4.z) \geqslant 2,$$

or

(2) $$h^0(K-6.z) > 0 \qquad (\textit{see } [3]).$$

Following [3], we define $\tilde{I} \subset I$ as

$$\tilde{I} = \{(C, z, H) \text{ s. t. } m_z(C.H) \geqslant 6\}.$$

If $p \in A \cup B_1$, it's easy to see that $\Phi^{-1}(p) \cap \tilde{I} \approx \mathbb{P}^{11}$.

It follows $\tilde{I} \cap Y$ is a proper subvariety of Y and the generic C doesn't contain points of type (2).

In order to prove that the generic C doesn't contain points of type (1) define $I' \subseteq X \times Z$ as

$$I' = \{(C, z, H) \text{ s. t. } m_z(C.H) \geqslant 4\}$$

and call Ψ' and Φ' the projections over X and Z.

The inverse image of a point $C \in X$ under the map has positive dimension exactly when C has points of type (1) (or C is singular) [3].

If every curve $C \in X$ has points of type (1), then I' would have at least two components of dimension greater or equal to 18 mapping over X (*see* [3]).

We can exclude that this is the case looking at the fibers of Φ'.

If $X'_p \subseteq X$ is the set $\Psi'(\Phi'^{-1}(p))$ for any $p=(z, H)$ in Z, we can easily check that:

if $p \in A \cup B_1$, $X'_p \approx \mathbb{P}^{13}$;

if $p \in B_2 \cup C_2 \cup C_1 \cup D_1$ then $X'_p \subseteq X_1$;

if $p \in D_2$ then $X'_p \approx \mathbb{P}^{15}$.

When $p \in B_3$ a non-singular C belongs to X'_p if $m_z(C.L)=3$ or $m_z(C.D) \geqslant 3$ so that $X'_p \subseteq X_1 \cup X_2 \cup \mathbb{P}^{14}$ where

$$\mathbb{P}^{14} = \{ C \in X \text{ s. t. } m_z(C.D) \geqslant 3 \}.$$

In the same way we see that when $p \in C_3$ then $X'_p \subseteq X_1 \cup X_2$ and when $p \in D_3$ then $X'_p \subseteq \mathbb{P}^{15} \cup \mathbb{P}^{15} \cup X_1$ where the two projective spaces of dimension 15 are given by the curves C s. t. $m_z(C.E) \geqslant 2$ and by those s. t. $m_z(C.L) \geqslant 2$.

Since $\dim(D_2)=2$; $\dim(B_3)=3$ and $\dim(D_3)=1$, there is only one component of dimension 18 mapping over X.

Q.E.D.

Since we know that the generic $C \in X$ has only normal Weierstrass points, we can improve the statement of the Corollary to Proposition 1 showing.

PROPOSITION 3. — *The monodromy group is twice transitive.*

Proof. — We denote by X_3 the proper subvariety of X consisting of curves with at least one non-normal Weierstrass point. Then we fix a point $(C_0, z_0, H_0) \in Y$ such that:

z_0 does not belong to E;

H_0 is smooth;

C_0 does not belong to $X_1 \cup X_2 \cup X_3$ i. e. C_0 is smooth, has only normal Weierstrass points and does not have points of total ramification for the g^1_3.

The proposition will follow if we show that the stabilizer of (C_0, z_0, H_0) in the monodromy group acts transitively on the remaining points of $\Gamma = \Psi^{-1}(C_0)$ (*see* [3], [6]).

If X_0 is the set of $C \in X$ s. t. $m_z(C.H_0) \geqslant 5$, then $X_0 \approx \mathbb{P}^{12}$ and following [3] we call I_0 the closure in $X_0 \times Z$ of the complement of the set $\{(C, z_0, H_0) \text{ where } C \in X_0\}$.

By a standard argument [6] we want to trove that I_0 has only one component of dimension 12 mapping over X_0 looking at the fibers of the projection map $\varphi_0 : I_0 \to Z$.

From Proposition 1 we know that it's enough to consider points $p \in A \cup B_1 \cup B_3 \cup D_3$ since in all the other cases $X'_p \subseteq X_1 \cup X_2$.

If $p_0 = (z_0, H_0)$ and $p = (z, H)$, we notice also that we can suppose $z \neq z_0$ and $H \neq H_0$.

In fact if $z = z_0$ and H is smooth i. e. if $p \in A$ we have: $X_p \cap X_{p0} \subseteq X_1$ if H and H_0 are transversal at

$$z_0 = z \qquad \text{and} \qquad X_p \cap X_{p0} \subseteq X_1 \cap X_3$$

if H and H_0 are tangent at z_0 because a non-singular $C \in X_p \cap X_{p_0}$ has $h^0(K_C - 5z) \geqslant 2$ since both H and H_0 have a contact of order greater or equal to five with C ar z.

The same argument works when $p \in B_1 \cup B_3 \cup D_3$.

On the other hand if $H = H_0$ but $z \neq z_0$, $X_p \cap X_{p_0} \subseteq X_1$ because a smooth C has only 8 intersections with H.

From now on we consider points $p = (z, H)$ where $z \neq z_0$ and $H \neq H_0$. The set A is the union of the following sets:

$A^0 = \{ p = (z, H) \in A \text{ s. t. } z \notin H_0 \text{ and } z_0 \notin H \}$;

$A^1 = \{ p \in A \text{ s. t. } z \in H_0,\ z_0 \notin H \text{ or viceversa} \}$;

$A^2 = \{ p \in A \text{ s. t. } z \in H_0 \text{ and } z_0 \in H \}$;

we define also

B_1^0 (resp. B_3^0, D_3^0) as the set $\{ p \in B_1$ (resp. B_3, D_3) s. t. $z_0 \in H$ and $z \notin H_0 \}$;

B_1^1 (resp. B_3^1, D_3^1) as $\{ p \in B_1$, (resp. B_3, D_3) s. t. $z_0 \in L$, $z \notin D$ (resp. D, E) where $H = L + D$ (resp. $L + D$, $E + 2L$)$\}$;

B_1^2 (resp. B_3^2) as $\{ p \in B_1$ (resp. B_3) s. t. $z_0 \in D$, $z_0 \notin L$ where $H = L + D \}$;

B_1^3 as $\{ p \in B_1 \text{ s. t. } z_0 = L . D \text{ here } H = L + D \}$.

We show first that if $p \in A^0 \cup A^1 \cup B_1^0$ then $\Phi_0^{-1}(p) \approx \mathbb{P}^7$. If $p \in A^0$, the curves $C \in X$ of the form $H + R$ i. e. containing H as a component, form a linear system of projective dimension 8 and they satisfy the condition $m_z(C . H) \geqslant 5$. Since $z_0 \notin H$, such a curve has $m_z(C . H_0) \geqslant 5$ when $m_{z_0}(R . H_0) \geqslant 5$.

On the other hand $|R|$ cuts on H_0 the complete system of degree 5. It follows [3] that the requirement of having a 5-fold intersection with H_0 ar z_0 imposes 5 conditions on this linear system and that $X_p \cap X_{p0} \approx \mathbb{P}^7$.

The same argument works when $p \in B_1 \cup A_1$ (eventually exchanging the role of p and p_0). When $p \in D_3^0$ we know that the non-singular element of X_p are contained in the set

$$\{C \text{ s. t. } m_z(C.L) \geqslant 2\} \approx \mathbb{P}^{15}.$$

Using again curves C containing H we see that the non-singular elements of $X_p \cap X_{p0}$ are contained in a $\mathbb{P}^{10}$.

In the same way if $p \in B_3^0$ the non-singular elements of $X_p \cap X_{p0}$ are contained in a $\mathbb{P}^8$.

If $p \in B_1^1$ (resp. B_3^1) taking the linear system consisting of cruves $C \in X$ which contain D as a component we conclude as before that $X_p \cap X_{p0} \approx \mathbb{P}^7$ (resp. the non-singular elements of $X_p \cap X_{p0}$ are contained in a $\mathbb{P}^8$).

It's easy to see that if $p \in B_1^2 \cup B_1^3$, $X_p \cap X_{p0} \subseteq X_1$. In fact it consists of curves C containing D since $C.D \geqslant 6$.

Suppose now that $p \in D_3^1$: since $H_0.L=1$, z does not belong to H_0. In the linear system of curves $C=H_0+R$ those having a contact of order greater or equal to 2 with L at z form a subspace of codimension 2. It follows that the non-singular elements of $X_p \cap X_{p0}$ are contained in a $\mathbb{P}^{10}$.

We are left with two more cases $p \in A^2$ and $p \in B_3^2$.

In the first case choose three distinct points t_1, t_2, t_3 (resp. s_1, s_2, s_3) on N_0 (resp. on H) different from z and z_0 (resp. from z, z_0, t_1, t_2, t_3). In $X_p \cap X_{p0}$ the curves C containing t_1, t_2, t_3 and s_1, s_2, s_3 have codimension at most 6. But in fact they contain H_0 and H because they have at least nine intersections with each. Since these curves form a linear system of dimension two, $X_p \cap X_{p0}$ has dimension smaller or equal to eight.

Finally if $p \in B_3^2$, we know that the non-singular elements of $X_p \cap X_{p0}$ satisfy the following conditions:

(*a*) $m_{z_0}(C.H_0) \geqslant 5$;

(*b*) $m_z(C.D) \geqslant 4$.

Let's see that they are contained in a $\mathbb{P}^8$.

In fact if $z \notin H_0$, the curves C containing H_0 i. e. of the form H_0+R, form a linear system of projective dimension eight contained in $X_p \approx \mathbb{P}^{12}$.

Such a curve satisfies (b) when $m_z(R.D) \geqslant 4$ i. e. when R contains D. This means that if we want (b) to be satisfied the dimension drops by four on this linear system and *a fortiori* on X_p. On the other hand if $z \in H_0$, we can argue in the following way: choose three points P_1, P_2, P_3 on H_0 but not on H and a point Q on D but not on H_0.

The curves C which in addition to (a) and (b) contain P_1, P_2, P_3, Q must contain H_0 and D. In other words imposing four more conditions we get a $\mathbb{P}^4$ and the result follows.

Taking into account the dimensions of the fibers and the dimensions of the subsets of Z we defined before it's easy to check that I_0 has only one component mapping over X_0.

Q.E.D.

We will show in section 4 that in fact the monodromy groups is the full symmetric group.

2. Points of total ramification for the g_3^1.

In this section we want to characterize the points of total ramification for the g_3^1 as Weierstrass points.

LEMMA 1. — *Let C be a smooth element of X, a point z of C is a point of total ramification for the g_3^1 exactly when the gap sequence at z is* 12457 *or* 12458.

Proof. — Notice first that in the gap sequence of any point of a smooth C of X the first two values are always $1, 2$ since C is not hyperelliptic.

By Riemann-Roch.

$$h^0(K-3z)=3 \quad \text{if and only if } h^0(3z)=2.$$

Since the g_3^1 on C is unique (*see* for example [7]) we see that z is a point of total ramification for the g_3^1 when the first missing value in the gap sequence is 3.

This happens only when the sequences are 12457 or 12458.

Q.E.D.

We can easily distinguish the two cases: suppose first that z doesn't lie on E, the exceptional divisor of S.

We have:

$$h^0\,(K-z)=4;$$
$$h^0\,(K-2\,z)=3;$$
$$h^0\,(K-3\,z)=3.$$

Since $C.L_z=3\,z$, the sections of $(K-3\,z)$ are exactly the hyperplanes of the form L_z+D i. e. containing the line L_z. When the component D contains z the intersection multiplicity with C goes up by one and $h^0\,(K-4\,z)=2$.

Finally if $H=E+2\,L_z$ the intersection multiplicity at z is 6 and:

$$h^0\,(K-6\,z)=1;$$
$$h^0\,(K-7\,z)=0.$$

The sequence at z is then 12457. If $z\in E$ it's easy to check in a similar way that the sequence is 12458.

Let's now fix a point $z\in S$ and define

$$R_z=\{\,C\in X \text{ s. t. } m_z(C.L_z)\geqslant 3\,\}.$$

We have:

PROPOSITION 4. — *The generic element C of R_z is smooth, irreducible and the only point of total ramification of the g_3^1 on C is z.*

Proof. — The first statement is an elementary application of Bertini's theorem after checking that the base locus of R_z is z itself. For the second statement we show by dimension count that R_z cannot be covered by the union of the R_t's; $t\neq z$. If $t\in L_z$, then $R_t\cap R_z$ contains only singular curves since they contain L_z.

If $t\notin L_z$ let's show that $R_t\cap R_z$ is a projective space of dimension 11. In fact if we take the curves C containing L_z we see easily that those s. t. $m_t(C.L_t)\geqslant 3$ form a subspace of codimension three (they cut on L_t the complete system of degree three). Since z moves on S the proposition follows.

Q.E.D.

COROLLARY 2. — *The set of curves C having a point with gap sequence* 12457 *is a rational variety of dimension* 16 *and the set of those with a point whose gap sequence is* 12458 *is a rational variety of simension* 15.

Proof. — Define $J \subseteq S \times X$ as

$$J = \{ (C, z) \text{ s. t. } m_z(C . L_z) \geqslant 3 \}.$$

The variety J is rational of dimension 16. The corollary follows immediately from the previous proposition looking at the projection map $J \to X$.

If $R \subseteq X$ is the image of J under this map, the smooth elements of R form the set X_2 of curves with a point of total ramification for the g_3^1.

If $z \in E$ we get the statement about the curves having a point with gap sequence 12458.

Q.E.D.

We want to show now that a generic C with a point of total ramification z_0 has only normal Weierstrass points outside z_0.

Let z_0 be a point of S and R_{z_0} the set of curves C s. t. $m_{z_0}(C . L_{z_0}) \geqslant 3$.

As we saw in the proof of Proposition 2 of paragraph 1, *a* point z on a smooth C is not normal when:

(1) $$h^0(K - 4 . z) \geqslant 2$$

or:

(2) $$h^0(K - 6 . z) > 0.$$

In fact the only gap-sequence which gives a point of type (1) but not of type (1) is 12356.

Let's prove first:

PROPOSITION 5. — *The generic element C of R_{z_0} does not contain any point z different from z_0 such that $h^0(K - 6 . z) > 0$.*

Proof. — Let $\tilde{I}$ be as in Proposition 2 of paragraph 1. We want to see that the image of triples $(C, z, H) \in \tilde{I}$ where $z \neq z_0$ cannot cover R_{z_0}.

For any $p = (z, H) \in Z$ let $\tilde{X}_p \subseteq X$ be the set of curves C s. t. $(C, z, H) \in \tilde{I}$. From Proposition 1 of paragraph 1 we know that:

if $p \in A$; $\tilde{X}_p \simeq \mathbb{P}^{11}$;

if $p \in B_3$; $\tilde{X}_p \subseteq X_1 \cup \mathbb{P}^{12}$ where $\mathbb{P}^{12} = \{ C \text{ s. t. } m_z(C . D) \geqslant 5 \}$.

In the remaining cases $\tilde{X}_p \subseteq X_1 \cup X_2$. Since we already know that the generic $C \in R_{z_0}$ is smooth and doesn't have points of total ramification for the g_3^1 different from z_0 it suffices to consider $\tilde{X}_p$ when $p \in A \cup B_3$.

Notice also that we can suppose $z \notin R_{z_0}$: otherwise $\tilde{X}_p \cap R_{z_0} \subseteq X_1$ because it consists of curves C containing L_{z_0}. When $p = (z, H) \in A$ the intersection $\tilde{X}_p \cap R_{z_0}$ is isomorphic to $\mathbb{P}^8$: in fact the curves $C = L_{z_0} + C'$ form a subspace of codimension one in L_{z_0} *which intersects* $\tilde{X}_p$ in a $\mathbb{P}^8$ because C' cuts on H the complete system of degree seven.

When $p=(z, H)\in B_3$ we have $H=L_z+D$ and $z=L_z.D$.

If $z_0\in D$ the intersection $\tilde{X}_p\cap R_{z_0}$ consists of singular curves since they contain D.

If $z_0\notin D$ we take cuves of the form $C=D+F$: they are contained in $\tilde{X}_p$ and they belong to R_{z_0} when F contains R_{z_0}.

It follows that the smooth elements of $\tilde{X}_p\cap R_{z_0}$ are contained in a $\mathbb{P}^9$.

Since $\dim A=5$ and $\dim B=3$ the proposition follows.

Q.E.D.

We consider now points z s. t. $h^0(K-4z)\geqslant 2$ and we prove:

PROPOSITION 6. — *The generic C of R_{z_0} does not contain any point z different from z_0 such that $h^0(K-4z)\geqslant 2$.*

Proof. — Let I' be as in Proposition 2 of paragraph 1. We restrict to the inverse image I'_0 of R_{z_0} under the projection map onto X. As in Proposition 2 of paragraph 1, I'_0 has a component

$$\mathscr{T}=\text{closure of} \bigcup_{C\in R_{z_0},\, z\neq z_0}(C, z, H).$$

Since C moves in R_{z_0} there is also the component

$$\mathscr{L}=\text{closure of} \bigcup_{C\in R_{z_0},\, C \text{ smooth}} \mathbb{P}(H^0(C, \mathcal{O}(K_C-4z_0))).$$

These two components have both dimension fifteen and map onto R_{z_0}. If every $C\in R_{z_0}$ has a point $z\neq z_0$ s. t. $h^0(K_C-4z)\geqslant 2$, we would get a new component of dimension greater or equal to fifteen mapping over X from the union of $\mathbb{P}^0(C, \mathcal{O}(K-4z)))$ where $C\in R_{z_0}$, and $z\neq z_0$.

The map φ' from I'_0 to Z sends $\mathscr{L}$ in the subset $\{(z_0, H)$ where $H\in\mathbb{P}^{4*}\}$. It suffices then to show that the inverse images $\varphi'^{-1}(p)$, where $p=(z, H)$ with $z\neq z_0$, give only one component of dimension fifteen mapping over R_{z_0}.

For any $p\in Z$ we denote by X'_p the set $\psi'(\varphi'^{-1}(p))$ where ψ' is the projection map from I'_0 to X.

If $z\in L_{z_0}$ the set $X'_p\cap R_{z_0}$ consists of singular curves C since they have four intersections with L_{z_0}. So it suffices to consider points $p=(z, H)$

where $z \notin L_{z0}$. When we take a point p in $A \cup B_1$ the intersection $X'_p \cap L_{z0}$ is isomorphic to $\mathbb{P}^{10}$.

In fact the linear system of curves of R_{z0}, $C = L_{z0} + C'$ cuts on H (when $p \in A$) and on D (when $p \in B_1$) complete systems. If $p \in B_3$ we know that the non singular elements of X'_p are contained in the $\mathbb{P}^{14}$ of curves C s. t. $m_z(C.D) \geqslant 3$.

In case $z_0 \in L_z + D$ we consider the curves $C = D + F$: they form a linear system of projective dimension eleven contained in X'_p. Such a curve belongs to R_{z0} only when it contains L_{z0}. Counting dimensions it follows that the non-singular elements of $X'_p \cap R_{z0}$ are contained in a $\mathbb{P}^{11}$.

In case $z_0 \in D$ using the linear system of curves $C = L_{z0} + C'$ contained in R_{z0} and imposing the condition $m_z(C.D) \geqslant 3$; we conclude again that the non-singular elements of $X'_p \cap R_{z0}$ are contained in a $\mathbb{P}^{11}$.

When $p \in D_2$ the intersection $X'_p \cap R_{z0}$ has codimension two in R_{z0} as we can see taking again curves $C = L_{z0} + C'$ and imposing the condition $m_z(C.L) \geqslant 2$.

Finally when $p \in D_3$ we have to intersect R_{z0} with two distinct $\mathbb{P}^{15}$ given by curves C s. t. $m_z(C.E) \geqslant 2$ and by those s. t. $m_z(C.L) \geqslant 2$.

In the second case we see that the intersection with R_{z0} is isomorphic to $\mathbb{P}^{12}$ arguing as before.

For the curves C s. t. $m_z(C.E) \geqslant 2$ we have to distinguish two subcases: if $z_0 \in E$ the intersection with R_{z0} consists of singular curves since they contain E. If $z_0 \notin E$, the curves $C = E + N$ certainly satisfy the condition $m_z(C.E) \geqslant 2$ and if we impose the condition $m_{z0}(C.L_{z0}) \geqslant 3$ they must contain L_{z0}.

It follows that among the curves C s. t. $m_z(C.E) \geqslant 2$ those belonging to R_{z0} have codimension three.

In the remaining cases $X'_p \subseteq X_1 \cup X_2$ and we do not need to consider the intersection with R_{z0} as in Proposition 4.

Looking at the dimensions of the various subsets of Z we see that the only component of dimension fifteen (mapping over R_{z0}) comes from the inverse image of points $p \in A \cup B_1$.

Q.E.D.

3. Points with $h^0(K-4z)=2$

We consider in this paragraph points s. t. $h^0(K-4z)=2$. Since we already characterized the points of total ramification for the g_3^1 as those having sequences 12457 and 12458 we consider only those such that $h^0(K-3z)=2$ or equiva.ently such that 4 is the first missing value in the gap sequence.

We can prove.

LEMMA 2. — *Let C be a smooth element of X and z_0 a point of C such that $h^0(K-4z_0)=2$. If z_0 is not a point of total ramification for the g_3^1, the gap sequence of C at z_0 is* 12356 *or* 12367.

Proof. — Since $h^0(K-4z_0)=2$, there is a singular H such that $m_{z_0}(C.H)\geqslant 4$.

Suppose first that $z_0\in E$.

We have the following possibilities:

(*a*) $H=E+L+L'$, $z_0\notin L+L'$

(*b*) $H=E+L_{z_0}+L'$, $L'\neq L_{z_0}$

(*c*) $H=L_{z_0}+D$, $z_0\notin D$,

(*d*) $H=E+2L_{z_0}$.

In cases (*a*) and (*c*), the curve C should be singular against the hypothesis. Case (*b*) cannot occur because C is smooth and $m_{z_0}(C.L_{z_0})\leqslant 2$. Case (*d*) can be excluded because we should have

$$m_{z_0}(C.L_{z_0})=1 \qquad \text{and} \qquad m_{z_0}(C.E)=2$$

or:

$$m_{z_0}(C.L_{z_0})=2 \qquad \text{and} \qquad m_{z_0}(C.E)=1.$$

But in each case the sections of $\mathcal{O}(K-3z_0)$ are $E+L_{z_0}+L'$ where L' varies and $h^0(K-4z_0)=1$.

Suppose now $z_0\notin E$.

The possible configurations are:

(α) $H=E+L+L'$ and $z_0\in L$, $z_0\notin L$,

(β) $H=E+2L$ and $z_0\in L=L_{z_0}$,

(γ) $H=L+D$, $z_0\in L=L_{z_0}$, $z_0\notin D$,

(δ) $H=L+D$, $z_0=L.D$, $L=L_{z_0}$,

(ρ) $H=L+D$, $z_0\in D$, $z_0\notin L$.

The cases (α) and (γ) cannot occur because C should be singular. In (β) we must have $m_{z_0}(L.L_{z_0})\geqslant 2$, hence equality and the sections of $\mathcal{O}(K-3z_0)$ are $L_{z_0}+D$ where D varies and contains z_0. It follows $h^0(K-4z_0)=1$, the only section being $E+2L_{z_0}$. In case (δ) we must have $(C.D)_{z_0}\geqslant 3$. If $m_{z_0}(C.D)=3$ it's easy to see that $h^0(K-4z_0)=1$.

If $m_{z_0}(C.D)=4$ the only section of $\mathcal{O}(K-5z_0)$ is $D+L_{z_0}$ and $h^0(K-6z_0)=0$.

We get the sequence 12356.

If $m_{z_0}(C.D)=5$ then $h^0(K-5z_0)=2$ and $D+L_{z_0}$ is the only section of $(K-6z_0)$.

The sequence in this case is 12367.

In the last case since $m_{z_0}(C.D)$ is again 4 or 5 we conclude as before that the gap sequence is 12356 or 12367.

Q.E.D.

From the previous lemma we know that if z_0 is a point with gap sequence 12356 (resp. 12367) there is only one curve $D_0=\pi * R_0$ such that $m_{z_0}(C.D_0)=4$ (resp. 5).

It's clear then how we can construct curves $C\in X$ with such a point. We take a curve $D_0=\pi * R_0$, where R_0 is a line of $\mathbb{P}^2$ not containing x, and *a* point $z_0\in D_0$. Define

$$T_0=\{C\in X \text{ s. t. } m_{z_0}(C.D_0)\geqslant 4\}\simeq\mathbb{P}^{13}.$$

The generic element C of T_0 is smooth, $m_{z_0}(C.D_0)=4$, and z_0 is not a point of total ramification for the g_3^1 on C since L_{z_0} is transversal to D_0.

If $m_{z_0}(C.D_0)=4$ the sequence is 12356, if $m_{z_0}(C.D_0)=5$ the sequence is 12367.

We can be more precise about the points of total ramification for the g_3^1 on a generic $C\in T_0$:

PROPOSITION 7. — *The generic $C\in T_0$ does not contain any point of total ramification for the g_3^1.*

Proof. — Let z be a point of S different from z_0. We want to show that $R_z\cap T_0=\mathbb{P}^{10}$. If $z\notin D_0$ taking the curves $C\in T_0$ which contain D_0:

$$C=D_0+F,$$

we see that they belong to R_z only if they contain L_z. Counting dimensions we get the transversality of R_z and T_0 in X. If $z \in D_0$ we take the curves C of R_z containing L_z:

$$C = L_z + G.$$

Such a curve belongs to T_0 when $m_{z_0}(G . D_0) \geqslant 4$.

Since $|G|$ cuts on D_0 the complete system of degree four the proposition follows.

Q.E.D.

We can look now at the behaviour of the generic $C \in T_0$ from the point of view of the existence of non-normal Weierstrass points different from z_0.

Like in section 2 we prove first:

PROPOSITION 8. — *The generic $C \in T_0$ does not have points z (different from z_0) such that $h^0(K - 6z) > 0$.*

Proof. — As usual $\tilde{I} = \{(z, H, C) \text{ s. t. } m_z(C . H) \geqslant 6\} \subseteq Z \times X$.

For each $p = (z, H) \in Z$ we want to consider the curves of $\tilde{X}_p \cap T_0$ which are smooth and do not have points of total ramification for the g_3^1. Like in Proposition (5) it suffices to consider points $p \in A \cup B_3$.

If $p \subset A$, we can distinguish the following cases:

(*a*) $p = (z, H) \in A$ and $z_0 \notin H$,

(*b*) $p = (z, H) \in A$, $z_0 \in H$, $z \notin D_0$,

(*c*) $p = (z, H) \in A$, $z \in D_0$, $z_0 \in H$.

In cases (*a*) and (*b*) the intersection $\tilde{X}_p \cap T_0$ is a $\mathbb{P}^7$. In the first case it's easy to see that the curves C containing H belong to T_0 only when they contain also D_0. In case (*b*) the curves $C = D_0 + F$ belong to $\tilde{X}_p$ when $m_z(F . \mathrm{H}) \geqslant 6$. Since $|F|$ cuts on H the complete system of degree six, the statement about (*a*) and (*b*) is proved. In case (*c*) we choose two points t_1 and t_2 on H different from z and z_0, and a point y on D_0 different from z and z_0. It's immediate to check that a curve $C \in \tilde{X}_p \cap T_0$ contains t_1, t_2 and y only when it contains H and D_0. Such curves form a $\mathbb{P}^4$.

It follows that also in this case $\tilde{X} \cap T_0 \simeq \mathbb{P}^7$.

If $p = (z, H) \in B_3$ then $H = L + D$ and $z = L . D$. The non-singular curves of $\tilde{X}_p$ are contained in

$$\{\mathrm{C} \text{ s. t. } m_Z(C . D) \geqslant 5\} \simeq \mathbb{P}^{12},$$

so it suffices to intersect T_0 with this linear space.

We distinguish the following cases:

(α) $z_0 \notin H$,

(β) $z_0 \in D$,

(γ) $z_0 \in L$.

Arguing like in (*a*) we see that $T_0 \cap \mathbb{P}^{12} = \mathbb{P}^8$ in case (α). In case (β) $T_0 \cap \mathbb{P}^{12}$ contains only singular curves since they contain D having six intersections with it.

Finally if $z_0 \in L$, let y be the point of intersection of D with D_0 (which is supposed different from D because otherwise we would get only singular curves). Choose a point $t \notin H$, $t \in D_0$ and consider the curves of $T_0 \cap \mathbb{P}^{12}$ containing y and t.

They contain D_0 and D since they have six intersections with each and they form a linear system of dimension six.

It follows that $T_0 \cap \mathbb{P}^{12} = \mathbb{P}^8$.

Counting dimensions the proposition is proved.

Q.E.D.

Using lemma (2) it's immediate to prove also.

PROPOSITION 9. — *The generic $C \in T_0$ does not contain any point z different from z_0 such that $h^0(K-4z) \geq 2$.*

Proof. — If $D = \pi * R$ and $z \in D$ we have to show that T_0 cannot be covered by the T_z's where:

$$T_z = \{ C \in X \text{ s. t. } m_z(C.D) \geq 4 \},$$

where we let D and z vary ($z \neq z_0$).

Like in Proposition 8 we have to consider only the case $D \neq D_0$ because otherwise $T_z \cap T_0$ contains only singular curves.

For any couple (z, D) where $z \in D$ we have $z \notin D_0$ or $z_0 \notin D$.

Hence, as in proposition 5 (α)

$$T_z \cap T_0 \simeq \mathbb{P}^9$$

and the proposition follows.

Q.E.D.

Notice that we could prove the previous proposition along the lines of Proposition 6 of this section or Proposition 2 of Section 1, i. e. without using Lemma 2, but the proof would be longer. In those two propositions we used an argument which can be extended to more general cases i. e. for the family of trigonal curves of any odd genus lying on the same scroll we consider in the present paper and embedded in the appropriate projective space.

From the proposition we proved we get easily.

COROLLARY 3. — *The set of (smooth) curves C of X with a Weierstrass point whose gap sequence is* 12356 *or* 12367 *form a rational variety of dimension sixteen.*

Proof. — The couples (z, D) where $D=\pi * R$ is the pull-back of a line not containing x and $z\in D$, is a rational manifold of dimension three. We take then triples (C, z, D) such that $m_z(C.D)\geqslant 4$.

The Corollary follows immediately by projecting into X and using Propositions 4, 5 and 6.

Q.E.D.

If we define

$$T_0=\{C\in X \text{s. t.} \, m_{z_0}(C.D_0)\geqslant 5\}\simeq \mathbb{P}^{12},$$

it's easy to prove that the generic $C\in T_0$ is smooth and does not have non-normal Weierstrass points different from z_0 (proofs are almost identical to those of Propositions 7, 8 and 9 and left to the reader).

Hence we have.

COROLLARY 3'. — *The set of (smooth) curves C of X with a Weierstrass point whose gap sequence is* 12367 *form a rational variety of dimension fifteen.*

4. Points with $h^0(K-6z)=1$

In this section we want to describe the points with sequences 12347, 12348, 12349.

LEMMA 3. — *Suppose that on a smooth $C\in X$; $h^0(K-6z)=1$. The only section H of $H^0(K-6z)$ is a smooth curve of S if and only if the gap sequence at z is one of the following:* 12347, 12348, 12349.

Proof. — *A priori* the sequence could also be 12457, 12458 or 12367 but in these cases we know from Lemma 1 and 2 that the only section H of $H^0(K-6z)$ is singular.

Let's show then that if the sequence is one of those given in the lemma then H is smooth.

Suppose first H is singular outside z we can have:

(α) $H=E+L_1+L_2$; $z\in L_1$, $z\notin E+L_2$,

(β) $H=E+2L$; $z\in E$, $z\notin L$,

(γ) $H=E+L_1+L_2$; $z\in E$, $z\notin L_1+L_2$,

(ρ) $H=L+D$; $z\in L$, $z\notin D$,

(ζ) $H=L+D$; $z\in D$, $z\notin L$.

It's immediate to check that in each case the condition $m_z(C.H)\geqslant 6$ implies that C is reducible against the hypothesis that C is smooth.

Suppose now that H is singular at z.

The possibilities are:

(*a*) $H=E+L_1+L_2$; $z=H.L_1$,

(*b*) $H=E+2L$; $z=E.L$,

(*c*) $H=E+2L$; $z\in L$, $z\notin E$,

(*d*) $H=L+D$; $z=L.D$.

None of these cases can occure. In fact in case (*a*) the intersection multiplicity $m_z(C.H)$ can be at most 4 (when $m_z(C.L)=3$). In case (*b*) the point z is a point of total ramification for the g_3^1, otherwise $m_z(C.H)$ is at most 5.

In case (*c*), $m_z(C.H)=6$ only when z is a point of total ramification for the g_3^1, in the last case if D and C are transversal at z the intersection multiplicity $m_z(C.H)<6$, and if $m_z(C.D)=5$ the sequence at z is 12367 (lemma 2).

Q.E.D.

Define then:

$$V_0^8 \qquad (\text{resp. } V_0^7,\ V_0^6)$$

as:

$$V_0^8=\{\,C\in X \text{ s. t. } m_{z_0}(C.H_0)\geqslant 8\,\} \qquad (\text{resp.} \geqslant 7, 6),$$

where H_0 is smooth and $z_0\in H_0$.

Clearly $V_0^8\subseteq V_0^7\subseteq V_0^6$ and they are projective space of dimension 9, 10 and 11 respectively.

LEMMA 4. — *The generic element V_0^8 is smooth (hence the same is true for V_0^7, V_0^6).*

Proof. — By dimension count we see that not every element of V_0^8 contains H_0. It follows that the base locus of V_0^8 is $\{z_0\}$. In fact if $C \in V_0^8$ and it contains a point y of H_0 different from z_0 then it must contain H_0.

Let's show now that we can find $C \in V_0^8$ which is smooth at z_0.

If $z_0 = E.H_0$ we take for example

$$C = H_0 + H + D,$$

where $z_0 \notin H + D$.

If $z_0 \notin E$ we can take

$$C = H + L + 2D,$$

where $z_0 \notin L + 2D$.

By Bertini's theorem the Lemma follows.

(Notice that the generic C is irreducible hence connected because if not it would contain singular points.)

Q.E.D.

From Lemma 3 it follows immediately that if C is smooth the gap sequence at z_0 is

$$\begin{array}{ll} 12347 & \text{if } C \in V_0^8, \\ 12348 & \text{if } C \in V_0^7 - V_0^8, \\ 12349 & \text{if } C \in V_0^6 - V_0^7. \end{array}$$

As usual we want to see that the generic $C \in V_0^8$ (resp. V_0^7, V_0^6) has the good expected behaviour from the point of view of Weierstrass points.

PROPOSITION 10. — *The generic $C \in V_0^8$ does not have any point of total ramification for the g_3^1.*

Proof. — We already know that z_0 is not such a point. To show that V_0^8 cannot be covered by the R_z's where $z \neq z_0$ suppose first that $z \notin H_0$.

Taking the curves $C = H_0 + C'$ we see that they belong to R_z only when they contain L_z. It follows the $V_0^8 \cap R_z \simeq \mathbb{P}^6$.

If $z \in H_0$, then $z_0 \notin L_z$ and we can use the curves $C = L_z + G$. Since $|G|$ cuts on H_0 the complete system of degree we can check again that $V_0^8 \cap R_z \simeq \mathbb{P}^6$.

The proposition follows by dimension count.

Q.E.D.

The analogous statement is obviously valid for the generic $C \in V_0^7$ (resp. V_0^6) using the same proof or as corollary of the previous proposition.

We can prove now.

PROPOSITION 11. — *The generic $C \in V_0^8$ does not contain any points z different from z_0 such that $H_0(K-4z) \geqslant 2$.*

Proof. — Since we already know that the generic C does not have points of total ramification for the g_3^1 we have to exclude points with gap sequence 12356 or 12367.

From Lemma 2 we must consider couples $(z, D) = p$.

If $z \notin H_0$ taking curves $C = H_0 + C'$ we see that $m_z(C.D) \geqslant 4$ only if C' contains D. Counting dimensions we get:

$$V_0^8 \cap \{C \text{ s. t. } m_z(C.D) \geqslant 4\} \simeq \mathbb{P}^5.$$

If $z \in H_0$, taking a point $r \in D$, $r \notin H_0$ we check that a curve $C \in V_0^8$ contains r and $m_z(C.D) \geqslant 4$ only when $C \supseteq H_0 + D$.

It follows that also in this case

$$V_0^8 \cap \{C \text{ s. t. } m_z(C.D) \geqslant 4\} \simeq \mathbb{P}^5.$$

Q.E.D.

Again we have a similar statement for the generic $C \in V_0^7$ (resp. V_0^6) as a consequence of the previous proposition or proving it directly.

For example for the case of V_0^6 the only change is that if $z \in H_0$ we look at the curves C s. t.:

(i) $C \in V_0^6$;

(ii) $m_z(C.D) \geqslant 4$;

(iii) C contains three more points y_1, y_2, r where $r \in D$, $r \notin H_0$,

y_1 and y_2 lie on H_0, but not on D (and y_1 is the intersection of D with H_0 different from z in case $z_0 \notin D$).

Finally we prove.

PROPOSITION 12. — *The generic $C \in V_0^8$ does not contain any point $z \neq z_0$ such that $h^\circ(K-6z) > 0$.*

Proof. — Since we already know that the generic C doesn't have points of total ramification for the g_3^1 or points such that $h^\circ(K-4z) \geqslant 2$ we have to exclude points with sequences 12347; 12348; 12349. In other words we have to show that V_0^8 cannot be covered by the union of the sets $V_{(z,H)}^6 = \{C \in X \text{ s. t. } m_z(C.H) \geqslant 6\}$ where H is smooth and $z \in H$.

Notice that if $H=H_0$ the intersection $V_0^8 \cap V_{(z,H)}^6$ consists of singular curves C containing H_0.

If $H \neq H_0$ and $z \in H_0$ we get the same conclusion.

Finally if $H \neq H_0$ and $z \neq H_0$ taking the curves $C = \mathrm{H}_0 + \mathrm{C}'$ we see that (C' cuts on H the complete system of degree five and) those s. t. $m_z(C.H) \geqslant 6$ form a subspace of codimension six.

The result follows by dimension count.

Q.E.D.

Remarks. — (1) The same results is true for the generic $C \in V_0^7$ (resp. V_0^6).

(2) We could prove Proposition 12 along the lines of Proposition 6 i. e. without using explicitely the characterizations of the various Weierstrass points given in Lemma 1, 2, 3. This second proof is longer but can be applied in more general situations and will appear elsewhere.

COROLLARY 4. — *The variety X^6 (resp. X^7, X^8) of curves $C \in X$ having a point with gap sequence* 12347 (*resp.* 12348, 12349) *is rational of dimension* 16 (*resp.* 15, 14).

Proof. — It follows immediately from the fact the couples (z, H) form a variety of dimension five using Propositions 7, 8, 9.

Q.E.D.

5. Monodromy

From the Propositions 10, 11, 12 of the previous section we know that there is a point (C_0, z_0, H_0) of I such that:

(i) H_0 is smooth;

(ii) C_0 is smooth with z_0 as only non-normal Weierstrass point;

(iii) z_0 is a point with gap sequence 12347 (and weight 2). Since H_0 is smooth on a neighborhood U of (z_0, H_0) in Z the map $\varphi: I \to Z$ is a fibration with fiber $\mathbb{P}^{12}$. In particulier $\varphi^{-1}(U)$ is smooth and I is irreducible at (C_0, z_0, H_0). Since we already know that the monodromy group is twice transitive, using the Lemma in section II, 3 of [6] we conclude.

THEOREM. — *The monodomy group* $\mathrm{Mon}(\psi)$ *of* $\psi: I \to X$ *is the full symmetric group on* 120 *elements.*

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