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NORMAL BUNDLE TO CURVES IN QUADRICS

BY

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RÉSUMÉ. — Dans cet article on démontre que le fibré normal à une courbe connexe, non singulière C d.'une quadratique est ample si et seulement si C n'est pas une droite. On donne aussi une application aux fonctions rationnelles formelles et donc au problème de la rigidité.

ABSTRACT. — In this paper we prove that the normal bundle to a nonsingular quadric is ample if and only if C is not a straight line. We give also applications to rational formal functions and therefore to the rigidity problem.

Introduction

In this paper we prove that the normal bundle $N_{C/Q}$ to a nonsingular curve C in a nonsingular quadric Q is ample if and only if C does not contain a straight line as a connected component. Similar results are well-known for a nonsingular subvariety contained in a projective space $\mathbb{P}_n$ because the tangent bundle to $\mathbb{P}_n$ is ample. A PAPANTONOPOULOU proved in [9] results of this kind for curves in grassmannians. In particular he proved this Theorem for $G(1,3)$, the grassmannian of lines in $\mathbb{P}_3$, which is a nonsingular quadric in $\mathbb{P}_5$. Therefore the Theorem is known if the quadric has dimension 4. His proof is different from our's.

The ampleness of the normal bundle has many well-known applications. R. HARTSHORNE proved under this assumptions a vanishing Theorem on formal schemes ([6], Thm. 4.1), some results on the cohomological dimension of a projective variety minus a subvariety ([5], chapt. 7), a Theorem on formal rational functions [5] which has application to the rigidity problem of embeddings. In PAPANTONOPOLOU's and our situation the usual application to formal rational functions and to the rigidity problem can be strenghted by using the notion of generating subspace and a Theorem by CHOW [1]. This is done in the second paragraph of this paper.

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1. Let k be an algebraically closed field with $\mathrm{ch}(k)=0$. In this paragraph every variety will be defined over k.

We want to prove the following.

THEOREM 1. — *Let Q be a nonsingular quadric hypersurface of $\mathbb{P}_n$ and C a nonsingular, connected curve contained in Q. The normal bundle $N_{C/Q}$ of C in Q is ample if and only if C is not a straight line.*

Let $X \subset Y \subset Z$ be algebraic varieties; $N_{X/Y}$ is the normal sheaf of X in Y and we have the exact sequence [E.G.A., IV, 16.2.7] :

$$0 \to N_{X/Y} \to N_{X/Z} \xrightarrow{r} N_{Y/Z|X}. \tag{1}$$

If the immersion of X into Y is regular, then $N_{X/Y}$ is locally free [E.G.A., IV, 19.9.8 (ii)] and the map r is surjective [E.G.A., IV, 16.9.13].

In the proof of the Theorem we distinguish two cases:

(*a*) C is not contained in a linear space contained in Q;

(*b*) C is contained in a linear space contained in Q.

(*a*) Let $Q_t \subset \mathbb{P}_{t+1}$ be an irreducible quadric and $\mathbb{P}_t$ a hyperplane such that $Q_{t-1} := Q_t \cap \mathbb{P}_t$ is irreducible; we have:

$$N_{Q_t}/\mathbb{P}_{t+1} \simeq \mathcal{O}_{Q_t}(2) \qquad \text{and} \qquad N_{Q_{t-1}/Q_t} \simeq \mathcal{O}_{Q_{t-1}}(1).$$

Let C be a nonsingular curve contained into Q_{t-1}. From the exact sequence (1) we obtain the exact sequence:

$$0 \to N_{C/Q_{t-1}} \xrightarrow{i} N_{C/\mathbb{P}_t} \xrightarrow{f_t} \mathcal{O}_C(2). \tag{2}$$

LEMMA 1. — *$N_{C/Q_{t-1}}$ is a locally free sheaf and i is an injection of vector bundles.*

Proof. — $N_{C/\mathbb{P}_t}$ is a locally free sheaf and therefore $N_{C/Q_{t-1}}$ is a torsion-free sheaf; $N_{C/Q_{t-1}}$ is a locally free sheaf because C is a nonsingular curve. Furthermore the injection $i : N_{C/Q_{t-1}} \to N_{C/\mathbb{P}_t}$ has a locally free cokernel because coker (i) is a subsheaf of $\mathcal{O}_C(2)$. Thus i is an injection of vector bundles. ■

LEMMA 2. — *Let $C \subset \mathbb{P}_t$ be a nonsingular, connected curve of degree d; let E be a vector bundle on C of rank r and degree h. If E is a quotient bundle of $T\mathbb{P}_{t|C}$, then $h \geqslant rd$ and if $h = rd$, then C is contained in a hyperplane.*

Proof. — We choose homogeneous coordinates $(z_0; \ldots; z_t)$ on $\mathbb{P}_t$.

We have an exact sequence on $\mathbb{P}_t$:

$$0 \to \mathcal{O}_{\mathbb{P}_t} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}_t}(1)^{t+1} \to T\,\mathbb{P}_t \to 0, \tag{3}$$

where $\alpha(f)=(fz_0, \ldots, fz_t)$.

From the restriction of (3) to C we obtain a diagram with exact row and column.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C \arrow[r, "\alpha"] & \mathcal{O}_C(1)^{t+1} \arrow[r, "\gamma"] & T\mathbb{P}_{t|C} \arrow[r] \arrow[d, "\delta"] & 0 \\ & & & E \arrow[d] & \\ & & & 0 & \end{tikzcd} \tag{4}$$

Let β be $\delta \circ \gamma : \mathcal{O}(1)^{t+1} \to \mathrm{E}$. We write $\mathrm{E}(s) : =\mathrm{E} \otimes_{\mathcal{O}_C} \mathcal{O}_C(s)$. From β we obtain a surjective map from $\mathcal{O}_C^{t+1}$ to $E(-1)$ and therefore a surjective map from $\Lambda^r \mathcal{O}^{t+1}$ to $\Lambda^r(E(-1))$.

This implies $h=\deg E \geqslant rd$.

Now suppose that we have $h=rd$. Any section of $E(-1)$ defines a subline bundle L of $E(-1)$ with $\deg L \geqslant 0$. As $E(-1)$ is generated by global sections, after $r-1$ steps we arrive at a line bundle M of degree d and quotient of E.

Therefore β induces a surjective map g from $\mathcal{O}(1)^{t+1}$ to M and thus a non zero element of:

$$\Gamma(C, Hom(\mathcal{O}_C(1)^{t+1}, M) \simeq \Gamma(C, M(-1))^{t+1}.$$

As $\deg M=d=\deg \mathcal{O}_C(1)$, we have

$$M \simeq \mathcal{O}_C(1) \qquad \text{and} \qquad \Gamma(C, Hom(\mathcal{O}_C(1)^{t+1}, M) \simeq k^{t+1}.$$

If g induces $t+1$ non zero constant $a_0, \ldots, a_t$, then C is contained in the hyperplane $\sum a_i z_i=0$. In fact, let $z=(z_0, \ldots, z_t)$ be a point of C. In the diagram (4) the image of the unitary section of $\mathcal{O}_C$ in $\mathcal{O}_C(1)^{t+1}$ is $(z_0, \ldots, z_t)$. As g factors through $T\mathbb{P}_{t|C}$, we obtain:

$$0=g((z_0, \ldots, z_t)=\sum a_i z_i.$$

This proves the Lemma. ■

Now we suppose that the nonsingular, connected curve $C \subset Q_{t-1}$ is not contained in the singular locus S of Q_{t-1}; then $C \cap S$ is a finite set.

Let g_t :

$$N_{C/Q_t} \to N_{Q_{t-1}/Q_t|C} \simeq \mathcal{O}_C(1)$$

and f_t :

$$N_{C/\mathbf{P}_t} \to N_{Q_{t-1}/\mathbf{P}_t|C} \simeq \mathcal{O}_C(2)$$

be the natural maps; g_t and f_t are surjective on $C/C \cap S$.

We define $M_t := Im\, f_t$ and $H_t := Im\, g_t$. As in Lemma 1, M_t and H_t are locally free sheaves on C because they are torsion-free sheaves. We recall that every morphism i between locally free sheaves on C which is injective on $C/C \cap S$ is injective because otherwise $Ker\, i$ would be a non zero locally free sheaf with support contained in $C \cap S$. A diagram of maps between locally free sheaves on C which is commutative on $C/C \cap S$ is commutative because C is reduced. Then we have a commutative diagram with exact rows and columns.

$$(5)\qquad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_{C/Q_{t-1}} \arrow[r] \arrow[d] & N_{C/\mathbf{P}_t} \arrow[r] \arrow[d] & M_t \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{C/Q_t} \arrow[r] \arrow[d] & N_{C/\mathbf{P}_{t+1}} \arrow[r] \arrow[d] & M_{t+1} \arrow[r] & 0 \\
0 \arrow[r] & H_t \arrow[r] \arrow[d] & \mathcal{O}_C(1) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

From Lemma 1 and the definitions we have

$$d \leqslant \deg M_t \leqslant \deg M_{t+1} \leqslant 2d.$$

Thus we obtain $\deg H_t = \deg M_t - \deg M_{t+1} + d$.

Therefore deg $H_t \geqslant 0$ and deg $H_t = 0$ i.e. H_t is not ample if and only if deg $M_t = d$ and deg $M_{t+1} = 2d$. From Lemma 2 it follows immediatly the following.

LEMMA 3. — *If* deg $M_t = d$, *then* C *is contained in a hyperplane of* $\mathbb{P}_t$. ■

We want to prove the following Proposition which in particular proves the Theorem 1 in case (a):

PROPOSITION 1. — *Let* C *be a nonsingular, connected curve contained in the irreducible quadric* $Q \subset \mathbb{P}_n$ *but not contained in a linear space contained in* Q; *then the normal bundle* $N_{C/Q}$ *is ample.*

Proof. — We have seen in Lemma 1 that $N_{C/Q}$ is a vector bundle. Let $\mathbb{P}_t$ be the linear space generated by C in $\mathbb{P}_n$. We define $Q_{t-1} = Q \cap \mathbb{P}_t$. By assumption Q_{t-1} is irreducible and C is not contained in the singular locus of Q_{t-1}.

We take a glance to diagram (5). It follows from Lemma 3 that deg $M_t > d$ because C generates $\mathbb{P}_t$. Then deg $M_t \leqslant$ deg M_{t+s} by the diagram (5). Thus deg $H_1 > 0$ for $t \leqslant l \leqslant n-1$. From the first exact column in diagram (5) we obtain that N_{C/Q_l} is ample if and only if $N_{C/Q_{l-1}}$ is ample. In a finite number of stemps we obtain that Proposition 1 is equivalent to the following.

LEMMA 4. — $N_{C/Q_{t-1}}$ *is an ample vector bundle.*

Proof of Lemma 4. — We use a criterion of ampleness proved by HARTSHORNE ([7], thm. 2.4). Let C be a nonsigular, complete curve and E a vector bundle over C; E is ample if and only if for every quotient bundle R of E we have deg $R > 0$.

Let R be a quotient bundle of $N_{C/Q_{t-1}}$ of rank $s > 0$. Let R_1 be the kernel of the surjective map $N_{C/Q_{t-1}} \to R$; from Lemma 1 it follows that R_1 injects as a vector bundle in $N_{C/\mathbb{P}_t}$.

Let E be the quotient bundle of $N_{C/\mathbb{P}_t}$ for the subbundle R_1.

We have rank $E = 1 + s \geqslant 2$ and:

$$\deg E = \deg N_{C/\mathbb{P}_t} - \deg R_1 = \deg N_{C/\mathbb{P}_t} - \deg N_{C/Q_{t-1}} + \deg R \leqq 2d + \deg R.$$

From Lemma 2 we have deg $R > 0$.

(b) Now let C be a nonsingular, connected curve contained in a linear space contained in Q.

Let Q be a nonsingular quadric. Every maximal linear subspace of Q has dimension s when dim Q is equal to $2s$ or $2s+1$.

PROPOSITION 2. — *Let* $\mathbb{P}_s$ *be a maximal linear space of* Q. *Then if* dim Q *is even* $N_{\mathbb{P}_s/Q}\simeq\Omega_{\mathbb{P}_s}(2)$ *and if dim* Q *is odd:*

$$N_{\mathbb{P}_s/Q}\simeq\Omega_{\mathbb{P}_s}(2)\oplus\mathcal{O}_{\mathbb{P}_s}(1).$$

Proof. — Let Q be a hypersurface of $\mathbb{P}(V)$ and W be a linear subspace of V with $\mathbb{P}_s=\mathbb{P}(W)$. Let $q: V\times V\to k$ a symmetric bilinear form which defines Q. We recall that $T\mathbb{P}(V)\simeq Hom(H, V/H)$ where $H:=\mathcal{O}_{\mathbb{P}(V)}(-1)$ is the tautological bundle. Let x be a point of Q and $y\in V/\{0\}$ be a point representing x. A vector $\bar{v}\in T\mathbb{P}(V)(-1)_x$ belongs to $TQ(-1)_x$ if and only if for any element $v\in V$ which represents $\bar{v}$ we have $q(v, y)=0$.

We define a symmetric bilinear form $B: TQ(-1)\times TQ(-1)\to\mathcal{O}_Q$ in the following way: if $\bar{v}, \bar{w}\in TQ(-1)_x$ and v, w represent $\bar{v}$, $\bar{w}$ we put $B(\bar{v}, \bar{w})=q(v, w)$. B is well-defined; in fact if v', w' represent $\bar{v}$ and $\bar{w}$ we have $v'=v+\lambda y$, $w'=w+\mu y$ for some λ, μ in k and $q(v', w')=q(v, w)$ because $q(v, y)=q(w, y)=q(y, y)=0$. B is non degenerate, In fact, let $\bar{v}$ be an element of $TQ(-1)_x$ represented by $v\in V$; $\bar{v}\neq 0$ if and only if v and y are not collinear i.e. if and only if the hyperplanes $L=\{l\in V: q(y, l)=0\}$ and $L'=\{l\in V: q(v, l)=0\}$ are different; let w be an element of L which does not belong to L'; for the element $\bar{w}\in TQ(-1)_x$ induced by w we have $B(\bar{v}, \bar{w})\neq 0$.

We consider the exact sequence:

$$0\to T\mathbb{P}_s(-1)\to TQ_{|\mathbb{P}_s}(-1)\to N_{\mathbb{P}_s/Q}(-1)\to 0. \tag{6}$$

Obviously $TQ_{|\mathbb{P}_s}(-1)\simeq TQ(-1)_{|\mathbb{P}_s}$ and then B induces a bilinear, symmetric, non degenerate form B' on $TQ_{|\mathbb{P}_s}(-1)$.

If $w, w'\in W$ represent elements of $T\mathbb{P}_s(-1)_x$, then $q(w, w')=0$ because $w+w'$ represents a point of Q if it is not zero. Therefore $T\mathbb{P}_s(-1)$ is a maximal isotropic subspace of $TQ_{|\mathbb{P}_s}(-1)$ for B' and B' induces a surjective map

$$h:\quad N_{\mathbb{P}_s/Q}(-1)\to(T\mathbb{P}_s(-1))^{\vee}\simeq\Omega_{\mathbb{P}_s}(1).$$

If dim Q is even, then h must be an isomorphism.

If dim Q is odd, $Ker(h)$ must be a trivial line bundle because $c_1(Ker(h))=0$ and the exact sequence induced:

$$0\to\mathcal{O}_{\mathbb{P}_s}(1)\to N_{\mathbb{P}_s/Q}\to\Omega_{\mathbb{P}}(2)\to 0$$

splits because $H^1(\mathbb{P}_s, T\mathbb{P}_s(-1))=0$. ■

Thus Theorem 1 follows in case (b) from the following.

PROPOSITION 3. — *Let C be a nonsingular, connected curve contained in $\mathbb{P}_s$. The vector bundle $\Omega_{\mathbb{P}_s}(2)_{|C}$ is ample if and only if C is not a straight line.*

Proof. — We prove this Proposition by induction by on s.

If $s=1$ the thesis is empty. Let s be greater than 1.

From Proposition 2 it follows that $\Omega_{\mathbb{P}_s}(2)$ is generated by global sections; in fact the tangent bundle to a nonsingular quadric is generated by global sections because the quadric is homogeneous. We suppose that $\Omega_{\mathbb{P}_s}(2)_{|C}$ is not ample.

By a criterion of GIESEKER-HARTSHORNE ([12], Prop. 2.1) $\mathcal{O}_C$ is a quotient line bundle of $\Omega_{\mathbb{P}_s}(2)_{|C}$. Dualizing we obtain an exact sequence:

$$0 \to \mathcal{O}_C(2) \to T\mathbb{P}_{s|C} \to E \to 0,$$

where E is a vector bundle of rank $s-1$ and degree $(s-1)d$. Lemma 1 proves that C is contained in a hyperplane $\mathbb{P}_{s-1} \subset \mathbb{P}_s$. The exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}_{s-1}}(1) \to \Omega_{\mathbb{P}_s}(2)_{|\mathbb{P}_{s-1}} \to \Omega_{\mathbb{P}_{s-1}}(2) \to 0$$

shows, by the inductive hypothesis, that C is a straight line.

If C is a straight line,

$$\deg(\Omega_{\mathbb{P}_s}(2)_{|C}) = s-1 < \operatorname{rank}(\Omega_{\mathbb{P}_s}(2)_{|C})$$

and therefore $\Omega_{\mathbb{P}_s}(2)$ is not ample.

2. In general the ampleness of the normal bundle of a subvariety has many interesting consequences:

(a) a Theorem about finiteness of cohomology on formal schemes ([6], thm. 4.1);

(b) cohomological dimension of a projective variety minus a subvariety [5];

(c) a Theorem on formal rational functions which has applications to the rigidity problem [5].

In our situation the usual consequences on formal rational and the rigidity problem can be strenghted by using the notion of generating subspace and a Theorem by CHOW [1].

In this paragraph we work over the field of complex numbers. We recall some definitions. Let X be a reduced and irreducible variety and Y a closed, reduced subvariety of X. The formal completion $Y \diagdown X$ of Y in X is the ringed space $(Y, \mathcal{O}_{Y\setminus X})$ with:

$$\mathcal{O}_{Y\setminus X} := \text{inv lim } \mathcal{O}_X / \mathcal{J}^{n+1} \mid Y$$

where $\mathcal{J}$ is the ideal sheaf of Y in X.

The set of formal rational functions $K(Y\setminus X)$ is the set of global sections of the sheaf of total quotients of $\mathcal{O}_{Y\setminus X}$; it is a field if Y is connected ([8], Lemma 1.4). $K(X)$, the field of rational functions of X, is in a natural way a subfield of $K(Y\setminus X)$. We say [8] that Y is $G-2$ in X if $K(Y\setminus X)$ is a finite module over $K(X)$. We say [8] that Y is $G-3$ in X if the natural injection of $K(X)$ in $K(Y\setminus X)$ is surjective. Now we suppose that X is nonsingular, Y complete, connected, of positive dimension and a locally complete intersection in X; if the normal bundle of Y in X is ample, then Y is $G-2$ in X ([5], Cor. 6.8).

Let X be a projective variety which is homogeneous under the action of an algebraic group G; let Y be a closed subvariety of X; let p be a point of Y.

We put $G_{p, Y} = \{ g \in G : gp \in Y \}$. The subgroup G_Y of G generated by $G_{p, Y}$ does not depend upon the choice of the point p.

We say [1] that Y is a generating subspace of X if $G_Y = G$.

We want to prove the following.

THEOREM 2. — *Let Y be a connected subvariety of a nonsingular homogeneous, projective variety X. If Y is a generating subspace of X and if $G-2$ in X; then Y is $G-3$ in X.*

Proof. — For a Theorem of GIESEKER ([3], thm. 4.3), since Y is $G-2$ in X, there exists an etale neighborhood (Y, W) of (Y, X) (i.e. an immersion of Y in a variety W and a regular map $p : W \to X$ which is etale at every point of Y and induces the identity on Y) such that Y is $G-3$ in W.

Let f be an element of $K(Y\setminus X)$; it induces an element f_1 of of $K(Y\setminus W)$ because p induces an isomorphism between $Y\setminus W$ and $Y\setminus X$ ([2], Lemma 4.5); as Y is $G-3$ in W, f_1 is induced by a rational function f_2 on W; in particular f_2 induces a meromorphic function on every complex neighborhood of Y in W. As p is etale on a neighborhood of Y and induces the identity on Y, there exist complex open neighborhoods U and U' of Y in W and X such that p maps U biholomorphically on U'. Therefore f_2 induces a meromorphic function f_3 on U'. As Y is a generating subspace of X, from a Theorem by CHOW ([1], Thm. 2) it follows that f_3 is induced by a rational function $g \in K(X)$.

It is obvious that g induces f. ■

COROLLARY. — *Every connected subspace of a nonsingular quadric Q which contains a nonsingular curve which is not a straight line, is $G-3$ in Q. In particular every connected nonsingular subvariety of dimension at least two of Q is $G-3$ in Q.*

Proof. — If $C \subset Q$ is $G-3$ in Q, then any connected subspace of Q, Y, containing C, is $G-3$ in Q. Therefore it is sufficient to prove that every connected nonsingular curve which is not a straight line is $G-3$ in Q. Such a curve is a generating subspace of Q; therefore the thesis follows from Theorem 1, the quoted Theorem of HARTSHORNE ([5], Cor. 6.8) and Theorem 2. ■

As any subspace of positive dimension of a grassmannian is a generating subspace, from PAPANTONOPOULOU's Theorem it follows a similar Corollary for a grassmannian.

It is well-known the rigidity of embeddings of $G-3$ subvarieties (*cf.* [2], Lemma 4.6, for example). Let X and W be reduced and irreducible varieties and Y and Z $G-3$ subvarieties respectively of X and W. Let f be a formal isomorphism between the formal completion of Y in X and the formal completion of Z in W. Then there exist Zariski open neighborhoods U and V of Y and Z and an isomorphism of U with V which induces f.

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