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THE RESOLVENT FOR A CONVOLUTION KERNEL SATISFYING THE DOMINATION PRINCIPLE

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ABSTRACT. — Let N be a convolution kernel on a locally compact abelian group. It is shown that if N satisfies the domination principle and is non-singular, then there exists a splitting $N = N_0 + N'$ of N in which N_0 is a resolvent kernel and N' is N-invariant. Furthermore, the singular part N' of N is either N_0-invariant or a N_0-potential of a N-invariant measure. These results simplify Theorems of M. ITÔ.

RÉSUMÉ. — Soit N un noyau de convolution dans un groupe abélien localement compact. Pour N satisfaisant au principe de domination et étant non singulier, on démontre qu'il existe une partition $N = N_0 + N'$ de N, où N_0 est un noyau à résolvante et N' est N-invariante. De plus, la partie singulière N' de N est ou bien N_0-invariante ou bien un N_0-potentiel d'une mesure N-invariante. Ces résultats simplifient des théorèmes de M. ITÔ.

Introduction

Let G be a locally compact abelian group and N a convolution kernel on G satisfying the domination principle. In [2], ITÔ introduced a family $(N_p)_{p>0}$ of convolution kernels, which in later papers ([3], [5]) turned out to be the resolvent family for the regular part N_0 of N. Some of the proofs in these papers are complicated, so it is of interest to give a simple and unified treatment of the resolvent and the regular part of N based entirely on the Riesz decomposition theorem, and this is the aim of the present paper.

A complete proof of the Riesz decomposition theorem was given in [7], which will be a prerequisite for the present paper. Less general versions of the Riesz decomposition Theorem appeared in [3] and [4], and the treatment in [3] assumes knowledge of the resolvent and the regular part.

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The idea behind our treatment is as follows:

For each $p > 0$, we have $p\,N+\varepsilon_0 \prec N$, so let $N = (p\,N+\varepsilon_0)\star N_p+\eta_p$ be the Riesz decomposition of N with respect to $p\,N+\varepsilon_0$ as sum of a $(p\,N+\varepsilon_0)$-potential, generated by a measure N_p, and a $(p\,N+\varepsilon_0)$-invariant measure η_p. The measures N_p and η_p are uniquely determined, and this leads to the resolvent equation for $(N_p)_{p>0}$. For p tending to zero, N_p increases to the regular part of N.

Preliminaries

In the following, G denotes an arbitrary locally compact abelian group, and N a convolution kernel on G satisfying the domination principle.

A positive measure ξ on G is called *N-excessive*, if N satisfies the relative domination principle with respect to ξ ($N \prec \xi$) (*cf.* [4], [7]). The set of N-excessive measures is a vaguely closed convex cone $E(N)$, which is infimum-stable, and every $\xi \in E(N)$ is the vague limit of an increasing net of N-potentials. For an open subset $\Omega \subseteq G$ and *a* measure $\xi \in E(N)$, the *reduced measure* ${}_NR_\xi^\Omega$ of ξ over Ω (with respect to N) is defined (*cf.* [7]) as

$$ {}_NR_\xi^\Omega = \inf\{\tau \in E(N);\ \tau \geqslant \xi \text{ in } \Omega\}. $$

We write R_ξ^Ω instead of ${}_NR_\xi^\Omega$ when N is clear from the context.

Let $\mathscr{V}$ denote the set of compact neighbourhoods of 0 in G. A measure $\xi \in E(N)$ is called *N-invariant* if $R_\xi^{\complement V} = \xi$ for all $V \in \mathscr{V}$. The set of N-invariant measures is a convex cone $I(N)$, closed under increasing limits.

Definition. — The singular part N' *of* N *is the limit* $N' = \lim_{V\uparrow G} R_N^{\complement V}$ *of the decreasing net* $(R_N^{\complement V})_{V\in\mathscr{V}}$, *when* $V \in \mathscr{V}$ *increases to* G.

The regular part N_0 *of* N *is* $N_0 = N-N'$. Note that $N_0 \geqslant 0$.

The convolution kernel N is called *singular* (resp. *non-singular*) if $N_0 = 0$ (resp. $N_0 \neq 0$).

The following Riesz decomposition theorem will be essential later (*cf.* [7]).

PROPOSITION 1. — *Suppose N is non-singular. Every $\xi \in E(N)$ has a decomposition*

$$ \xi = N\star\mu+\eta, \qquad \textit{where} \quad \eta \in I(N). $$

The invariant part η is uniquely determined, and the measure μ is uniquely determined if (and only if) N satisfies the principle of unicity of mass.

We shall use some alternative characterizations of N-excessive and N-invariant measures.

LEMMA 2 (ITÔ [4], LAUB [7]). — *Suppose N is non-singular. A positive measure ξ is N-invariant if (and only if) there exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures with compact support such that*

$$N \star \lambda_\alpha \uparrow \xi, \qquad \lambda_\alpha \to 0.$$

In Corollary 6 below the conclusion of Lemma 2 is shown to be valid also for singular kernels.

The following result is well-known and not difficult to establish.

LEMMA 3. — *Let $N = 1/a \sum_{n=0}^{\infty} \sigma^n$ be an elementary kernel $(a > 0)$. Then*

(i) $\xi \in E(N) \quad \Leftrightarrow \quad \sigma \star \xi \leqslant \xi$,

(ii) $\eta \in I(N) \quad \Leftrightarrow \quad \sigma \star \eta = \eta$.

For $c > 0$, the convolution kernel $N + c\,\varepsilon_0$ satisfies the domination principle and the principle of unicity of mass, where ε_0 denotes the Dirac measure at 0. Moreover, it is easily seen that if $N \prec \xi$ then $N + c\,\varepsilon_0 \prec \xi$, i. e. $E(N) \subseteq E(N + c\,\varepsilon_0)$. For $V \in \mathscr{V}$, we consequently have

$$_N R_N^{\complement V} \geqslant {}_{(N+c\varepsilon_0)} R_{N+c\varepsilon_0}^{\complement V},$$

so that $N + c\,\varepsilon_0$ is non-singular.

The following Lemma is an extension of [4] (Corollaire 1, p. 340); the hypothesis of N being non-singular is removed.

LEMMA 4. — *For $c > 0$, we have*

$$I(N) = I(N + c\,\varepsilon_0).$$

Proof. — Suppose first that $\eta \in I(N + c\,\varepsilon_0)$. By Lemma 2, there exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures such that $(N + c\,\varepsilon_0) \star \lambda_\alpha \uparrow \eta$ and $\lambda_\alpha \to 0$. Therefore, we have $\eta = \lim N \star \lambda_\alpha$ and then $\eta \in E(N)$. For $V \in \mathscr{V}$, we find

$$\eta = {}_{(N+c\varepsilon_0)} R_\eta^{\complement V} \leqslant {}_N R_\eta^{\complement V} \leqslant \eta.$$

hence $\eta \in I(N)$.

Suppose next that $\eta \in I(N)$. Then $\eta \in E(N) \subseteq E(N + c\,\varepsilon_0)$, and since $N + c\,\varepsilon_0$ is non-singular, η has a Riesz decomposition

$$\eta = (N + c\,\varepsilon_0) \star \nu_c + \eta_c,$$

where $\eta_c \in I(N + c\,\varepsilon_0) \subseteq I(N)$. We shall prove that $\nu_c = 0$.

Let $V \in \mathscr{V}$, and choose a net of positive measures $(\mu_\alpha)_{\alpha \in A}$ with compact support in $\complement V$ such that $N \star \mu_\alpha \uparrow R_N^{\complement V}$. Since $N \prec \eta$ the net $(\eta \star \mu_\alpha)_{\alpha \in A}$ is increasing and

$$\lim_A \eta \star \mu_\alpha \leqslant \eta.$$

Now choose $(\lambda_\beta)_{\beta \in B}$ such that $N \star \lambda_\beta \uparrow \eta$, and if N is non-singular with the additional property that $\lambda_\beta \to 0$. We then claim that

$$\lim_B (N - R_N^{\complement V}) \star \lambda_\beta = 0.$$

This is true, because if N is singular then $N = R_N^{\complement V}$, and if N is non-singular then $N - R_N^{\complement V}$ has compact support and $\lambda_\beta \to 0$. We then have

$$\begin{aligned} \eta &= \lim_B N \star \lambda_\beta = \lim_B R_N^{\complement V} \star \lambda_\beta \\ &= \lim_B (\lim_A N \star \mu_\alpha \star \lambda_\beta) \leqslant \lim_A \eta \star \mu_\alpha, \end{aligned}$$

hence $\eta = \lim_A \eta \star \mu_\alpha$.

Since $\eta_c \in I(N)$, we similarly find $\lim_A \eta_c \star \mu_\alpha = \eta_c$. If $\mu_{\complement V}$ denotes a vague accumulation point of $(\mu_\alpha)_{\alpha \in A}$, we may assume that $\mu_\alpha \to \mu_{\complement V}$, and since $N \star \mu_\alpha \leqslant N$ and $N \star \nu_c$ exists, Deny's convergence Lemma ([1], Lemma 5.2) shows that

$$\lim_A \mu_\alpha \star \nu_c = \mu_{\complement V} \star \nu_c.$$

If we convolve all terms in the Riesz decomposition of η with μ_α and go to the limit, we obtain

$$\eta = R_N^{\complement V} \star \nu_c + c\, \mu_{\complement V} \star \nu_c + \eta_c.$$

Finally, letting V increase to G, Deny's convergence Lemma shows that $\mu_{\complement V} \star \nu_c \to 0$ because $\operatorname{supp}(\mu_{\complement V}) \subseteq \overline{\complement V}$, and hence

$$\eta = N' \star \nu_c + \eta_c,$$

which compared to the original decomposition gives $\nu_c = 0$, so we have

$$\eta = \eta_c \in I(N + c\,\varepsilon_0).$$

As an application of Lemma 4, we prove the following result which will not be used in the sequel, but it might be of independent interest.

PROPOSITION 5. — *The following conditions about N are equivalent:*

(i) *N is singular.*

(ii) *$I(N) = E(N)$.*

(iii) *There exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures with compact support such that $N \star \lambda_\alpha \uparrow N$ and $\lambda_\alpha \to 0$.*

Proof:

(i) $\Rightarrow$ (ii): Let μ be a positive measure such that $N\star\mu$ exists. By Lemma 1.8 in [7], the net $R_{N\star\mu}^{\complement V}$ decreases to $N\star\mu$ as V increases to G, hence $R_{N\star\mu}^{\complement V} = N\star\mu$ for all $V\in\mathscr{V}$ so that $N\star\mu\in I(N)$. Since every measure $\xi\in E(N)$ is an increasing limit of potentials $N\star\mu$, we get $E(N)\subseteq I(N)$.

(ii) $\Rightarrow$ (iii): By Lemma 4, we have $N\in I(N+\varepsilon_0)$, so by Lemma 2 there exists a net $(\lambda_\alpha)_{\alpha\in A}$ such that $(N+\varepsilon_0)\star\lambda_\alpha\uparrow N$ and $\lambda_\alpha\to 0$. Therefore, $N\star\lambda_\alpha\to N$, and $(N\star\lambda_\alpha)_{\alpha\in A}$ is increasing because $N+\varepsilon_0 \prec N$.

(iii) $\Rightarrow$ (i): Let $V\in\mathscr{V}$, and suppose that $N\star\lambda_\alpha\uparrow N$ and $\lambda_\alpha\to 0$. Writing λ_α as sum of its restrictions $\lambda_\alpha \mid W$ and $\lambda_\alpha \mid \complement W$ to W and $\complement W$, where $W\in\mathscr{V}$ is a compact neighbourhood of V, we have $N\star(\lambda_\alpha \mid \complement W)\to N$. By the domination principle for measures, we find $R_N^{\complement V}\geqslant N\star(\lambda_\alpha \mid \complement W)$, so taking limits for $\alpha\in A$ we get $N\leqslant R_N^{\complement V}$, which proves (i).

COROLLARY 6. – *The conclusion of Lemma* 2 *is valid also for singular convolution kernels satisfying the domination principle.*

Proof. – In order to show that a N-invariant measure ξ is the limit of an increasing net $(N\star\lambda_\alpha)$ for which $\lambda_\alpha\to 0$ one proceeds like in (ii) $\Rightarrow$ (iii) above. In order to prove the converse one proceeds like in (iii) $\Rightarrow$ (i).

A family $(N_p)_{p>0}$ of convolution kernels is called a *resolvent* if

$$N_p = N_q+(q-p)\,N_p\star N_q \qquad \text{for} \quad p,\, q>0.$$

A convolution kernel N is called a *resolvent kernel* if there exists a resolvent $(N_p)_{p>0}$ such that $N=\lim_{p\to 0} N_p$.

A resolvent kernel N satisfies the domination principle, and for every $V\in\mathscr{V}$ there exists a balayaged measure $\varepsilon'_{\complement V}$ of ε_0 on $\complement V$ with respect to N such that $R_N^{\complement V} = N\star\varepsilon'_{\complement V}$ (*cf.* [4], § 3). Since $\operatorname{supp}\varepsilon'_{\complement V}\subseteq \overline{\complement V}$, we have $\lim_{V\uparrow G}\varepsilon'_{\complement V}=0$, and therefore $N'=\lim_{V\uparrow G} R_N^{\complement V}=0$ because of the dominated convergence property of a resolvent kernel (*cf.* [4] or [6]).

Suppose now that N is a non-zero resolvent kernel. KISHI showed in [6] that $\lim_{p\to\infty} p\,N_p$ exists and is the normalized Haar measure ω_K of a compact subgroup K of G. The group K is the periodicity group for N, i. e. $K=\{x\in G;\, N\star\varepsilon_x=N\}$.

If μ is a positive measure such that $N\star\mu=N$, it follows that $N_p\star\mu=N_p$ for all p, hence by the convergence Lemma of DENY that $\mu\star\omega_K=\omega_K$.

This shows that μ is a probability measure supported by K. In particular, every pseudo-period of N (i.e. a point $x \in G$ such that $N \star \varepsilon_x$ is proportional to N) is a period for N.

Denoting by $\mathscr{V}_K$ the set of compact neighbourhoods of K we consequently have $N \star \varepsilon'_{\complement V} \neq N$ for any $V \in \mathscr{V}_K$. This implies that the series $\sum_{n=0}^{\infty} (\varepsilon'_{\complement V})^n$ converges and the following formula holds

$$N = (N - N \star \varepsilon'_{\complement V}) \star \sum_{n=0}^{\infty} (\varepsilon'_{\complement V})^n, \qquad V \in \mathscr{V}_K.$$

Using this notation, the sets $E(N)$ and $I(N)$ can be characterized in the following way:

PROPOSITION 7. — *Let N be a non-zero resolvent kernel with resolvent $(N_p)_{p>0}$. Then*

(i) $\xi \in E(N) \Leftrightarrow \forall p > 0 : p N_p \star \xi \leqslant \xi$.

(ii) $\eta \in I(N) \Leftrightarrow \forall p > 0 \, (\exists p > 0) : p N_p \star \eta = \eta$.

(iii) $\xi \in E(N) \Leftrightarrow \forall V \in \mathscr{V}_K : \varepsilon'_{\complement V} \star \xi \leqslant \xi$ *and* $\omega_K \star \xi = \xi$.

(iv) $\eta \in I(N) \Leftrightarrow \forall V \in \mathscr{V}_K : \varepsilon'_{\complement V} \star \eta = \eta$ *and* $\omega_K \star \eta = \eta$.

The invariant part of $\xi \in E(N)$ is given as $\lim_{p \to 0} p N_p \star \xi$.

Proof:

(ii): If N is a resolvent kernel then $N + 1/p \, \varepsilon_0 = 1/p \sum_{n=0}^{\infty} (p N_p)^n$ is an elementary kernel for every $p > 0$ and hence Lemma 3 and 4 show that

$$\eta \in I(N) \quad \Leftrightarrow \quad \eta \in I\left(N + \frac{1}{p} \varepsilon_0\right) \quad \Leftrightarrow \quad p N_p \star \eta = \eta.$$

" (i) $\Rightarrow$ ": For $\xi \in E(N)$, Proposition 1 shows that $\xi = N \star \mu + \eta$, where $\eta \in I(N)$, and from this Riesz decomposition we obtain

$$p N_p \star \xi = p N_p \star N \star \mu + \eta \leqslant \xi.$$

" (i) $\Leftarrow$ ": From $p N_p \star \xi \leqslant \xi$ follows by Lemma 3 that $N + (1/p) \, \varepsilon_0 \prec \xi$. Letting p tend to infinity we find that $N \prec \xi$.

" (iii) $\Rightarrow$ ": The statement holds for N-potentials and hence for every N-excessive measure.

" (iii) $\Leftarrow$ ": Lemma 3 proves that ξ is excessive with respect to the elementary kernel $N_V = \sum_0^{\infty} (\varepsilon'_{\complement V})^n$, so there exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures such that $N_V \star \lambda_\alpha \uparrow \xi$. Using $N = (N - N \star \varepsilon'_{\complement V}) \star N_V$, we get

$$N \star \lambda_\alpha \uparrow \xi \star (N - N \star \varepsilon'_{\complement V}),$$

so that $\xi \star (N - N \star \varepsilon'_{\complement V}) \in E(N)$. Defining $a_V = (N - N \star \varepsilon'_{\complement V})(G)$, we have

$$\frac{1}{a_V}(N - N \star \varepsilon'_{\complement V}) \to \omega_K \quad \text{as } V \downarrow K,$$

which implies that

$$\xi = \xi \star \omega_K \in E(N).$$

"(iv) $\Rightarrow$": By Lemma 2 there exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures such that $N \star \lambda_\alpha \uparrow \eta$, $\lambda_\alpha \to 0$. Since $N - N \star \varepsilon'_{\complement V}$ has compact support, we get

$$\eta - \eta \star \varepsilon'_{\complement V} = \lim_A (N - N \star \varepsilon'_{\complement V}) \star \lambda_\alpha = 0.$$

"(iv) $\Leftarrow$": By (iii) $\eta \in E(N)$ and hence $\eta = N \star \mu + \zeta$, where $\zeta \in I(N)$, but since $\eta = \eta \star \varepsilon'_{\complement V} = (N \star \varepsilon'_{\complement V}) \star \mu + \zeta$, we get $\mu = 0$ and then $\eta \in I(N)$.

If $\xi \in E(N)$ has the Riesz decomposition $\xi = N \star \mu + \eta$, where $\eta \in I(N)$, we find $p N_p \star \xi = (N - N_p) \star \mu + \eta$, hence

$$\eta = \lim_{p \to 0} p N_p \star \xi.$$

Main result

THEOREM 8. — *Let N be a non-singular convolution kernel satisfying the domination principle. There exist a non-zero resolvent $(N_p)_{p>0}$ and a positive measure ν such that*

$$N = N_p \star (p N + \varepsilon_0 + \nu) \qquad \textit{for} \quad p > 0. \tag{1}$$

The resolvent kernel $\tilde{N} = \lim_{p \to 0} N_p$ exists, and denoting by K the compact periodicity group of $\tilde{N}$, the measure ν can be chosen such that $\nu \star \varepsilon_x = \nu$ for all $x \in K$ and $\nu \in I(N)$.

Proof. — Let $p > 0$ be fixed. Then $p N + \varepsilon_0$ is a non-singular convolution kernel satisfying the domination principle and $N \in E(p N + \varepsilon_0)$. By Proposition 1, there exist positive measures N_p and η_p such that

$$N = (p N + \varepsilon_p) \star N_p + \eta_p,$$

where $\eta_p \in I(p N + \varepsilon_0) = I(N)$. Furthermore, N_p and η_p are uniquely determined, N_p because $p N + \varepsilon_0$ satisfies the principle of unicity of mass.

For $q > p > 0$, we have

$$\begin{aligned} N &= (q N + \varepsilon_0) \star N_q + \eta_q = (q - p) N \star N_q + (p N + \varepsilon_0) \star N_q + \eta_q, \\ &= (q - p)((p N + \varepsilon_0) \star N_p + \eta_p) \star N_q + (p N + \varepsilon_0) \star N_q + \eta_q, \\ &= (p N + \varepsilon_0) \star (N_q + (q - p) N_p \star N_q) + \eta_q + (q - p) \eta_p \star N_q. \end{aligned}$$

The measure $\eta_q+(q-p)\,\eta_p\star N_q$ is N-invariant because both η_p and η_q are so (*cf.* [7]). By the unicity of the Riesz decomposition with respect to $p\,N+\varepsilon_0$, we conclude

$$(2)\qquad \begin{aligned} N_p &= N_q+(q-p)\,N_p\star N_q \qquad \text{for} \quad 0<p<q,\\ \eta_p &= \eta_q+(q-p)\,\eta_p\star N_q \qquad \text{for} \quad 0<p<q. \end{aligned}$$

This shows that $(N_p)_{p>0}$ is a resolvent family, and since $N_p \leqslant N$ for all p we get that $\tilde{N} = \lim_{p\to 0} N_p$ exists. The resolvent kernel $\tilde{N}$ is non-zero, because $\tilde{N} = 0$ would imply that $N \in I(p\,N+\varepsilon_0) = I(N)$, hence that N is singular. Moreover $N \geqslant \eta_p \geqslant \eta_q$ for $p < q$ so the limit $\eta_0 = \lim_{p\to 0}\eta_p$ exists and belongs to $I(N)$. From (2), we get

$$(3)\qquad \eta_0 = \eta_q+q\,N_q\star\eta_0 \qquad \text{for} \quad q>0,$$

which by Proposition 7 shows that η_0 is excessive with respect to the resolvent kernel $\tilde{N}$, but since from (3) $\lim_{q\to 0} q\,N_q\star\eta_0 = 0$, the $\tilde{N}$-invariant part of η_0 is 0. There exists consequently a positive measure ν such that $\eta_0 = \tilde{N}\star\nu$. The measure ν need not be uniquely determined. In fact, $\tilde{N}$ has a compact periodicity group K, and denoting the normalized Haar measure of K by ω_K, we have as well $\eta_0 = \tilde{N}\star(\omega_K\star\nu)$, so by replacing ν by $\omega_K\star\nu$, we may and will assume that ν is periodic with each $x\in K$ as period. In this case, ν is easily seen to be uniquely determined. From (3) follows

$$\eta_q = \tilde{N}\star\nu - q\,N_q\star\tilde{N}\star\nu = N_q\star\nu,$$

which implies (1).

Using $p\,N_p\star\tilde{N} \leqslant \tilde{N}$ and that $\tilde{N}\star\nu$ exists, the convergence Lemma of Deny implies that $\lim_{p\to\infty} p\,\eta_p = \omega_K\star\nu = \nu$, so $\nu\in E(N)$. Since $\eta_0 = \tilde{N}\star\nu\in I(N)$, it is easy to see that $\nu\in I(N)$, (*cf.* [7], Corollary 2.4).

Lemma 9. — *Let N_1 and N_2 be non-zero convolution kernels satisfying the domination principle and $N_1 \prec N_2$. Then*

$$I(N_1)\cap E(N_2)\subseteq I(N_2).$$

Proof. — The relation $\prec$ being transitive (*cf.* [4]), it follows that $E(N_2)\subseteq E(N_1)$. For $\eta\in I(N_1)\cap E(N_2)$ and $V\in\mathscr{V}$, we then have

$$\eta = {}_{N_1}R_\eta^{\complement V} \leqslant {}_{N_2}R_\eta^{\complement V} \leqslant \eta,$$

which proves that $\eta\in I(N_2)$.

THEOREM 10. — *Let N be a non-singular convolution kernel satisfying the domination principle, and let $(N_p)_{p>0}$ and ν be as in Theorem* 8.

Then we have $\lim_{p\to 0} N_p = N_0$ *and* $N_0 \prec N$, *where* N_0 *is the regular part of* N. *The Riesz decomposition of* N *with respect to* N_0 *is*

$$N = N_0 \star (\varepsilon_0 + \nu) + N^i \tag{4}$$

and

$$N^i \in I(N_0) \cap I(N).$$

The singular part N' *of* N *is* N*-invariant and given as*

$$N' = N_0 \star \nu + N^i. \tag{5}$$

Proof. — From Theorem 8 we know that the resolvent kernel $\tilde{N} = \lim_{p\to 0} N_p$ exists, and also that $p\, N_p \star N \leqslant N$, hence $\tilde{N} \prec N$. Furthermore, N has the $\tilde{N}$-invariant part $N^i = \lim_{p\to 0} p\, N_p \star N$, so by Lemma 9 N^i is also N-invariant. Letting $p \to 0$ in (1), we find

$$N = \tilde{N} \star (\varepsilon_0 + \nu) + N^i. \tag{6}$$

Since $\eta_0 = \tilde{N} \star \nu \in I(N)$, we have $\tilde{N} \star \nu + N^i \in I(N)$, so for $V \in \mathscr{V}$:

$$\tilde{N} \star \nu + N^i = {}_N R^{\complement V}_{\tilde{N}\star\nu+N^i} \leqslant {}_N R^{\complement V}_N,$$

which implies

$$\tilde{N} \star \nu + N^i \leqslant N', \tag{7}$$

where N' is the singular part of N.

For $V \in \mathscr{V}_K$, let $\varepsilon'_{\complement V}$ be a $\tilde{N}$-balayaged measure of ε_0 on $\complement V$. Then

$$\xi_V = N \star \varepsilon'_{\complement V} + (\tilde{N} - \tilde{N} \star \varepsilon'_{\complement V}) \star \nu \in E(N),$$

and using (6) and $\varepsilon'_{\complement V} \star N^i = N^i$, we find

$$\xi_V = \tilde{N} \star \varepsilon'_{\complement V} + \tilde{N} \star \nu + N^i.$$

In $\complement V$, we have $\tilde{N} \star \varepsilon'_{\complement V} = \tilde{N}$, hence $\xi_V = N$ in $\complement V$, so by the definition of reduced measure, we get ${}_N R^{\complement V}_N \leqslant \xi_V$. Letting V increase to G, we find using $\lim_{V\uparrow G} \tilde{N} \star \varepsilon'_{\complement V} = 0$ that

$$N' = \lim_{V\uparrow G}\, {}_N R^{\complement V}_N \leqslant \lim_{V\uparrow G} \xi_V = \tilde{N} \star \nu + N^i,$$

which combined with (7) yields $N' = \tilde{N} \star \nu + N^i$ and hence $N_0 = \tilde{N}$.

With the notation as in Theorem 8 and 10, we further have the following proposition.

PROPOSITION 11. — *If $N' = N_0 \star \nu + N^i$ is the Riesz decomposition of the singular part N' of N with respect to the regular part N_0 of N, then either ν or N^i is zero.*

Proof. — Suppose that $\nu \neq 0$. Since $\nu \in E(N)$ there exists a net $(\lambda_\alpha)_{\alpha \in A}$ of positive measures such that $N \star \lambda_\alpha \uparrow \nu$, and since $N_0 \star \nu$ exists, this shows that also $N \star N_0$ exists. Finally, since $N^i \leqslant N$ also $N^i \star N_0$ exists. Using $N^i \in I(N_0)$, it follows that

$$N^i = p N_p \star N^i \leqslant p N_0 \star N^i \quad \text{for all } p > 0,$$

and hence $N^i = 0$.

PROPOSITION 12. — *Let N be a non-singular convolution kernel with regular part N_0. Then N and N_0 have the same pseudo-periods. In particular, the group of pseudo-periods for a non-singular convolution kernel is compact.*

Proof. — Suppose that $N_0 \star \varepsilon_x = c\, N_0$. Since $N_0 \prec N$, it follows that $N \star \varepsilon_x = c\, N$. Conversely, if $N \star \varepsilon_x = c\, N$, then $N' \star \varepsilon_x = c\, N'$ because $N \prec N'$. Using $N = N_0 + N'$, we get $N_0 \star \varepsilon_x = c\, N_0$.

Let $V \in \mathscr{V}$ be fixed. For every open relatively compact set $\omega \subseteq G$ such that $V \subseteq \omega$, let $\mu_{\omega \setminus V}$ be a balayaged measure of ε_0 on $\omega \setminus V$ with respect to N such that ${}_N R_N^{\omega \setminus V} = N \star \mu_{\omega \setminus V}$. With this notation, we have the following result.

PROPOSITION 13.

(i) *Every accumulation point for the net $(\mu_{\omega \setminus V})_\omega$ as ω increases to G is a balayaged measure of ε_0 on $\complement V$ with respect to N_0.*

(ii) ${}_N R_N^{\complement V} = {}_{N_0} R_{N_0}^{\complement V} + N'$.

(iii) *If N satisfies the principle of unicity of mass* $\lim_{\omega \uparrow G} \mu_{\omega \setminus V}$ *exists and* ${}_{N_0} R_{N_0}^{\complement V} = N_0 \star \lim_{\omega \uparrow G} \mu_{\omega \setminus V}$.

Proof. — Since $N \star \mu_{\omega \setminus V} \leqslant N$, the net $(\mu_{\omega \setminus V})_\omega$ is vaguely bounded. Let $\mu_{\complement V}$ be an accumulation point and assume that $\mu_{\omega \setminus V} \to \mu_{\complement V}$ (For notational simplicity we do not write the subnet). From (1) follows

$$N \star \mu_{\omega \setminus V} = p N_p \star N \star \mu_{\omega \setminus V} + N_p \star \mu_{\omega \setminus V} + \nu \star N_p \star \mu_{\omega \setminus V}. \tag{8}$$

We have $N \star \mu_{\omega \setminus V} = {}_N R_N^{\omega \setminus V} \uparrow {}_N R_N^{\complement V}$ so the first term on the right-hand side increases to $p N_p \star {}_N R_N^{\complement V}$. Since $N_p \star N$ exists, Deny's convergence Lemma implies that

$$\lim_\omega N_p \star \mu_{\omega \setminus V} = N_p \star \mu_{\complement V}.$$

Finally, since $\nu \star N_p \in I(N)$, we have as in the proof of Lemma 4 that $\lim_\omega \nu \star N_p \star \mu_{\omega\setminus V} = \nu \star N_p$ so (8) leads to

$$ {}_N R_N^{\complement V} = p N_p \star {}_N R_N^{\complement V} + N_p \star (\mu_{\complement V} + \nu). \tag{9}$$

This shows that ${}_N R_N^{\complement V} \in E(N_0)$, and since $N' \leqslant {}_N R_N^{\complement V} \leqslant N$ the N_0-invariant part of ${}_N R_N^{\complement V}$ is equal to N^i which is the N_0-invariant part of N' as well as of N. Letting $p \to 0$ in (9), we get

$$ {}_N R_N^{\complement V} = N_0 \star (\mu_{\complement V} + \nu) + N^i = N_0 \star \mu_{\complement V} + N', $$

so it is clear that $\mu_{\complement V}$ is a balayaged measure of ε_0 on $\complement V$ with respect to N_0.

Let $\varepsilon'_{\complement V}$ be a balayaged measure of ε_0 on $\complement V$ with respect to N_0 such that ${}_{N_0} R_{N_0}^{\complement V} = N_0 \star \varepsilon'_{\complement V}$. Then $N_0 \star \varepsilon'_{\complement V} \leqslant N_0 \star \mu_{\complement V}$ and with the notation from the proof of Theorem 10, we have

$$ {}_N R_N^{\complement V} \leqslant \xi_V = N_0 \star \varepsilon'_{\complement V} + N', $$

hence

$$ {}_N R_N^{\complement V} = N_0 \star \mu_{\complement V} + N' \geqslant N_0 \star \varepsilon'_{\complement V} + N' \geqslant {}_N R_N^{\complement V}. \tag{10}$$

We shall finally prove (iii). When N satisfies the principle of unicity of mass, N and hence also N_0 have no pseudo-periods, so N_0 is a Hunt kernel. Therefore $\varepsilon'_{\complement V}$ is uniquely determined by the formula ${}_{N_0} R_{N_0}^{\complement V} = N_0 \star \varepsilon'_{\complement V}$, and every accumulation point $\mu_{\complement V}$ of $(\mu_{\omega\setminus V})_\omega$ is equal to $\varepsilon'_{\complement V}$. Therefore $\lim_\omega \mu_{\omega\setminus V} = \varepsilon'_{\complement V}$.

Remarks

1° The singular part of $N + c\,\varepsilon_0$ is equal to the singular part N' of N. In fact, for $V \in \mathscr{V}$, we have observed that

$$ {}_N R_N^{\complement V} \geqslant {}_{N+c\varepsilon_0} R_{N+c\varepsilon_0}^{\complement V}, $$

hence $N' \geqslant (N + c\,\varepsilon_0)'$. Since $N' \in I(N) = I(N + c\,\varepsilon_0)$, we also have

$$ N' = {}_{N+c\varepsilon_0} R_{N'}^{\complement V} \leqslant {}_{N+c\varepsilon_0} R_{N+c\varepsilon_0}^{\complement V}, $$

which shows that $N' \leqslant (N + c\,\varepsilon_0)'$.

2° Suppose that $N' = N_0 \star \nu$ where $\nu \neq 0$. If N is shift-bounded (i. e. the set $\{ N \star \varepsilon_x ; x \in G \}$ is vaguely bounded) then $N_0(G) < \infty$.

In fact, since $\nu \in E(N)$ there exists a non-zero measure $\lambda \geqslant 0$ such that $N \star \lambda \leqslant \nu$ and then

$$ N_0 \star N \star \lambda \leqslant N_0 \star \nu = N' \leqslant N. $$

The shift-boundedness of N implies that $N_0 \star \lambda\,(G) \leqslant 1$, hence $N_0\,(G) < \infty$.

If $N' = N_0 \star \nu$ with $\nu \neq 0$, and N is not shift-bounded, N_0 need not be of finite mass as the following example shows:

$$G = \mathbf{R},\ N = (1_{]0,\infty[}(x) + e^x)\, dx.$$

The regular part of N is the Heaviside kernel $1_{]0,\infty[}$ and $N' = \nu = e^x$.

If N is shift-bounded and $N_0(G) < \infty$, then N' is a N_0-potential, because $I(N_0)$ does not contain any shift-bounded non-zero measures.

REFERENCES

[1] DENY (J.). — Noyaux de convolution de Hunt et noyaux associés à une famille fondamentale, *Ann. Inst. Fourier*, Grenoble, t. 12, 1962, p. 643-667.

[2] ITÔ (M.). — Sur le principe de domination pour les noyaux de convolution, *Nagoya math. J.*, t. 50, 1973, p. 149-173.

[3] ITÔ (M.). — Caractérisation du principe de domination pour les noyaux de convolution non-bornés, *Nagoya math. J.*, t. 57, 1975, p. 167-197.

[4] ITÔ (M.). — Sur le principe relatif de domination pour les noyaux de convolution, *Hiroshima math. J.*, t. 5, 1975, p. 293-350.

[5] ITÔ (M.). — Une caractérisation du principe de domination pour les noyaux de convolution, *Japan. J. Math.*, t. 1, 1975, p. 5-35.

[6] KISHI (M.). — Positive idempotents on a locally compact abelian group, *Kodai math. Sem. Rep.*, t. 27, 1976, p. 181-187.

[7] LAUB (J.). — On unicity of the Riesz decomposition of an excessive measure, *Math. Scand.* t. 43, 1978, p. 141-156.