BULLETIN DE LA S. M. F.

ALFONS VAN DAELE

A framework to study commutation problems

Bulletin de la S. M. F., tome 106 (1978), p. 289-309
<http://www.numdam.org/item?id=BSMF_1978__106__289_0>

Bull. Soc. math. France,
106, 1978, p. 289-309.

A FRAMEWORK TO STUDY COMMUTATION PROBLEMS

BY

ALFONS VAN DAELE

[Kath. Univ. Leuven]

RÉSUMÉ. — Soient A et B deux algèbres involutives d'opérateurs sur un espace hilbertien $\mathscr{H}$, telles que chacune d'elles soit contenue dans le commutant de l'autre. On énonce des conditions suffisantes sur A et B, en termes de certaines applications linéaires, $\eta : A \to \mathscr{H}$ et $\eta' : B \to \mathscr{H}$, pour que chacune de ces algèbres engendre le commutant de l'autre. Cette structure généralise d'une certaine façon celle d'une algèbre hilbertienne à gauche; elle permet de traiter le cas où l'algèbre de von Neumann et son commutant n'ont pas la même grandeur.

ABSTRACT. — If A and B are commuting *-algebras of operators on a Hilbert space $\mathscr{H}$, conditions on A and B are formulated in terms of linear maps $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ to ensure that A and B generate each others commutants. This structure generalizes the structure of a left Hilbert algebra in some sense. It makes it possible to deal with cases where the von Neumann algebra and its commutant are not necessarily of the same size.

1. Introduction

Very often in the theory of von Neumann algebras it is a problem to determine the commutant of a given von Neumann algebra. Probably, the most famous example to mention here is the commutation theorem for tensor products of von Neumann algebras. If M and N are von Neumann algebras acting in Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively, and if $M \otimes N$ is the von Neumann algebra on $\mathscr{H} \otimes \mathscr{K}$ generated by the operators $m \otimes n$ with $m \in M$ and $n \in N$, then the commutation Theorem for tensor products states that the commutant $(M \otimes N)'$ of the tensor product $M \otimes N$ is equal to the tensor product $M' \otimes N'$ of the commutants M' and N'. This Theorem was first proved in a number of special cases, but it was only until 1967 that TOMITA gave a proof in full generality using his theory of modular Hilbert algebras ([10], [12]).

Also, in other situations, the theory of modular or left Hilbert algebras has been a usefull tool to study commutation problems. Recently, it was used in the theory of crossed products of von Neumann algebras to determine the commutant $(M \otimes_\alpha G)'$ of the crossed product $M \otimes_\alpha G$ of a von Neumann algebra M with a (spatial) action α of a locally compact group G on M ([1], [2], [11]).

However it is not always possible to use the theory of left Hilbert algebras as such. Very often it is necessary to reduce the general case to a special case where it is possible to use left Hilbert algebra methods. The main reason for this is that the left von Neumann algebra $\mathcal{L}(\mathcal{A})$ of a left Hilbert algebra $\mathcal{A}$ is always "equal in size" with its commutant as $J \mathcal{L}(\mathcal{A}) J = \mathcal{L}(\mathcal{A})'$ where J is the canonical involution associated to $\mathcal{A}$. The reduction to the case where M and N have separating and cyclic vectors in the tensor product case is a good illustration of such a reduction [8].

Another fact we want to mention here is that in some cases it turned out that the full theory of left Hilbert algebras is not needed to solve commutation problems. Again a good example is provided by the tensor product case; there a number of more direct proofs have been obtained later. Recently, such a proof has been given by M. RIEFFEL and myself, using only bounded operators [5]. The proof is based upon the following result: Given two *-algebras A and B of operators on a Hilbert space $\mathcal{H}$, both of which contain the identity, and such that A and B commute, that is $A \subseteq B'$ (or equivalently $B \subseteq A'$); then if ω is vector in $\mathcal{H}$ cyclic for A, we have that A and B generate each others commutants, that is $A'' = B'$ if, and only if, $A_s\,\omega + i\,B_s\,\omega$ is dense in $\mathcal{H}$, where A_s and B_s denote the self-adjoint parts of A and B respectively.

The present paper deals with a generalization of this Theorem. It is formulated in a framework that also generalizes the structure of a left Hilbert algebra in some sense. We have called it a "commutation system" because we feel it is a powerful structure to solve commutation problems.

In section 2, we introduce commutation systems, and we prove the main theorem. In section 3, we treat the problem of associating commutation systems to a given von Neumann algebra. In section 4, we study "full commutation systems" behaving very much like full (achieved) left Hilbert algebras. Finally, in section 5, we give some examples. In a forthcoming paper with R. ROUSSEAU, we intend to apply our results to the theory of crossed products [7].

It should be mentioned here that also M. RIEFFEL has developed a structure for commutation problems [4], but there seems to be no real relationship between the two approaches except for the fact that in both cases it is possible to treat von Neumann algebras different in size from their commutants.

2. Commutation systems

Let A and B be *-algebras of operators on a Hilbert space $\mathcal{H}$, and suppose that A and B commute, that is that every operator in A commutes with every operator in B. We now want to introduce a structure in which it is possible to formulate conditions to ensure that A and B generate each others commutant.

A first quite natural condition is that A and B act non-degenerately on $\mathcal{H}$, that means that if $\xi \in \mathcal{H}$ is such that $a\xi = 0$ for all $a \in A$, then $\xi = 0$, and similarly for B. This is of course equivalent with the assumption that $A\mathcal{H}$ and $B\mathcal{H}$ are dense subsets of $\mathcal{H}$. By von Neumann's double commutant Theorem, we then have that the weak operator closure A^- of A equals the double commutant A'' of A, and similarly for B.

In the commutation Theorem of [5] mentioned in the introduction, we have in fact a very simple structure. There conditions are formulated in terms of a vector ω in $\mathcal{H}$. It is assumed that ω is cyclic for A (or B), and that $A_s\,\omega + i\,B_s\,\omega$ is dense in $\mathcal{H}$ where A_s and B_s denote the self-adjoint parts of A and B respectively.

However it is not always possible to find cyclic vectors and even if it is possible there might be no cyclic vector for which it is easy to check the conditions. Therefore we need something more general. Now a vector ω can be considered as a map $\eta : A \to \mathcal{H}$ by $\eta(a) = a\,\omega$, and similarly there is the map $\eta' : B \to \mathcal{H}$ defined by $\eta'(b) = b\,\omega$. The two maps are related by the formula $a\,\eta'(b) = b\,\eta(a)$ for all $a \in A$ and $b \in B$. On the other hand, if η and η' are such maps, and if they are continuous (with respect to one of the weaker topologies) they must come from a vector. Indeed, if they are continuous we may assume without restriction that A and B contain the identity. Then the relation $a\,\eta'(b) = b\,\eta(a)$ with $a = b = 1$ implies $\eta'(1) = \eta(1)$. Put $\omega = \eta(1)$, then using the Lemma 2.1 below we get $\eta(a) = \eta(a.1) = a\,\eta(1) = a\,\omega$ and similarly $\eta'(b) = b\,\omega$.

So there are good reasons to consider such maps as a generalization of the structure with a single vector. So as before let A and B be commuting *-algebras of operators acting non-degenerately on a Hilbert space $\mathscr{H}$. Furthermore let $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ be linear maps satisfying the relation

$$a\,\eta'(b) = b\,\eta(a) \quad \text{for all } a \in A,\ b \in B.$$

We will then formulate conditions on A and B in terms of the mappings η and η'.

We first give some easy but usefull results on the objects just defined.

2.1. Lemma. — *Let A, B, η and η' be as above. Then*:

(i) $\eta(a_1 a_2) = a_1 \eta(a_2)$ *for all* $a_1, a_2 \in A$,
$\eta'(b_1 b_2) = b_1 \eta'(b_2)$ *for all* $b_1, b_2 \in B$;

(ii) *real linear combinations of elements of the form* $\eta(a^* a)$ *with* $a \in A$ *are dense in* $\eta(A_s)$, *and similarly for* B;

(iii) $\eta(A_s)$ *and* $i\,\eta'(B_s)$ *are real orthogonal, that is* $\mathrm{Re}\,\langle \eta(a), i\,\eta'(b) \rangle = 0$ *whenever* $a \in A_s$ *and* $b \in B_s$ *or equivalently* $\langle \eta(a), \eta'(b) \rangle$ *is real for all* $a \in A_s$ *and* $b \in B_s$.

Proof:

(i) Let $b \in B$ and $a_1, a_2 \in A$. Then:

$$b\,\eta(a_1 a_2) = a_1 a_2\,\eta'(b) = a_1 b\,\eta(a_2) = b a_1\,\eta(a_2)$$

and as this holds for all $b \in B$ and B acts non-degenerately we get $\eta(a_1 a_2) = a_1 \eta(a_2)$. Similarly for B.

(ii) Let $\mathscr{K}$ denote the closed real subspace of $\mathscr{H}$ generated by vectors $\eta(a^* a)$ with $a \in A$. Let $a \in A_s$ and k a non-zero positive integer. Then trivially

$$a^k \eta(a) = \eta(a^k a) = \frac{1}{4}\eta((a^k + a)(a^k + a) - (a^k - a)(a^k - a)),$$

so that $a^k \eta(a) \in \mathscr{K}$. Then for every real polynomial p without constant term we get $p(a)\,\eta(a) \in \mathscr{K}$ and as the range projection e of a can be approximated by such polynomials of a we also get $e\,\eta(a) \in \mathscr{K}$. Now let $b \in B$ then:

$$be\,\eta(a) = eb\,\eta(a) = ea\,\eta'(b) = a\,\eta'(b) = b\,\eta(a)$$

and again this implies $e\,\eta(a) = \eta(a)$ so that $\eta(a) \in \mathscr{K}$. This proves the result.

(iii) If $a \in A$ and $b \in B_s$, we get

$$\begin{aligned}\langle \eta(a^* a), \eta'(b)\rangle &= \langle a^* \eta(a), \eta'(b)\rangle \\ &= \langle \eta(a), a\eta'(b)\rangle \\ &= \langle \eta(a), b\eta(a)\rangle\end{aligned}$$

which is real as b is self-adjoint. By (ii) then also $\langle \eta(a), \eta'(b)\rangle$ is real for all $a \in A_s$ and $b \in B_s$. This completes the proof.

Now, we want to impose conditions to ensure that A and B generate each others commutant. In view of what we have seen, it would be quite natural to impose something like $\eta(A_s) + i\,\eta'(B_s)$ being dense. However this condition appears to be to strong. Weaker conditions seem to be possible and will make it easier to find mappings η and η' for which the conditions can easily be checked (especially *see* [7]).

We introduce the following definition.

2.2. DEFINITION. — By a commutation system we mean a pair of commuting *-algebras A and B of operators acting non-degenerately on a Hilbert space $\mathcal{H}$, together with linear maps $\eta : A \to \mathcal{H}$ and $\eta' : B \to \mathcal{H}$ such that:

(i) $b\,\eta(a) = a\,\eta'(b)$ for all $a \in A$, $b \in B$;

(ii) the set $B\,\eta(A)$ of linear combinations of vectors $b\,\eta(a)$ is dense in $\mathcal{H}$;

(iii) the projection $[\eta(A)]$ onto the closure $\eta(A)^-$ of $\eta(A)$ belongs to the weak closure B^- of B, and similarly $[\eta'(B)] \in A^-$;

(iv) $\eta(A_s) + i\,\eta'(B_s)$ is dense in $\eta(A) + \eta'(B)$.

Remark that, by (i), $B\,\eta(A) = A\,\eta'(B)$ so that the formulation is completely symmetric in A and B. A condition like (ii) is of course necessary to avoid trivial cases like $\eta = \eta' = 0$. In fact, condition (ii) and condition (iv) are quite natural in view of earlier works [3]. Only condition (iii) is new but it turns out to be necessary for the proof of the main theorem below. However it is an easy one to verify and the condition follows anyway if A and B generate each others commutant. Indeed, by Lemma 2.1, $\eta(A)$ is invariant under A and so $[\eta(A)] \in A' = B'' = B^-$, and similarly for B.

2.3. THEOREM. — *If (A, B, η, η') is a commutation system, then A and B generate each others commutant.*

Proof. — Remark that we are in a situation where we can use Lemma 2.1. First, let $a \in A_s$, $b \in B$ and x self-adjoint in A', then:

$$\begin{aligned}\langle \eta(a), b^* x \eta'(b) \rangle &= \langle b \eta(a), x \eta'(b) \rangle \\ &= \langle a \eta'(b), x \eta'(b) \rangle \\ &= \langle xa \eta'(b), \eta'(b) \rangle.\end{aligned}$$

This expression is real as x and a are commuting self-adjoint operators.

Similarly, $\langle \eta'(b), a^* y \eta(a) \rangle$ is real for all $a \in A$, $b \in B_s$ and $y \in B'_s$. So $a^* y \eta(a) \perp i \eta'(B_s)$ with respect to the real scalar product. But, from $[\eta(A)] \in B''$, it follows that $\eta(A)^-$ is invariant for B', and so $a^* y \eta(a) \in \eta(A)^-$. As also $\eta(A_s) + i \eta'(B_s)$ is dense in $\eta(A) + \eta'(B)$, we get that

$$\eta(A)^- \subseteq (\eta(A_s) + i \eta'(B_s))^- = \eta(A_s)^- + i \eta'(B_s)^-.$$

So

$$a^* y \eta(a) \in \eta(A_s)^- + i \eta'(B_s)^- \qquad \text{and} \qquad a^* y \eta(a) \perp i \eta'(B_s).$$

Then $a^* y \eta(a) \in \eta(A_s)^-$. Combining the two results, we obtain that $\langle a^* y \eta(a), b^* x \eta'(b) \rangle$ is real for all $a \in A$, $b \in B$, $x \in A'_s$ and $y \in B'_s$.

Now

$$\begin{aligned}\langle a^* y \eta(a), b^* x \eta'(b) \rangle &= \langle by \eta(a), ax \eta'(b) \rangle = \langle yb \eta(a), xb \eta(a) \rangle \\ &= \langle xyb \eta(a), b \eta(a) \rangle.\end{aligned}$$

Because this is real, we get

$$\begin{aligned}\langle xyb \eta(a), b \eta(a) \rangle &= \langle b \eta(a), xyb \eta(a) \rangle \\ &= \langle yxb \eta(a), b \eta(a) \rangle.\end{aligned}$$

From this it is possible to obtain that $xy = yx$ by polarization and the fact that $B \eta(A)$ is dense in $\mathscr{H}$ (*see* e. g. [8]). Indeed consider first

$$\psi(a_1, a_2) = \langle (xy - yx) b \eta(a_1), b \eta(a_2) \rangle.$$

Then ψ is a sesquilinear form and $\psi(a, a) = 0$ for all a, hence $\psi = 0$. Next consider

$$\varphi(b_1, b_2) = \langle (xy - yx) b_1 \eta(a_1), b_2 \eta(a_2) \rangle.$$

Then also φ is sesquilinear and $\varphi(b, b) = 0$ for all b, hence $\varphi = 0$. By the density of $B \eta(A)$, it then follows that $xy = yx$ for all $x \in A'_s$ and $y \in B'_s$, and by linearity therefore also when $x \in A'$ and $y \in B'$. As A and B commute, this implies that $A' = B''$.

2.4. *Remark.* – In the proof of the previous Theorem, we have used that $[\eta(A)] \in B^-$, but we did not use the other condition $[\eta'(B)] \in A^-$. As we mentioned already this follows then from the conclusion $A' = B''$. Therefore to have a commutation system it is sufficient to check that $[\eta(A)] \in B^-$ as the other condition $[\eta'(B)] \in A^-$ will automatically follow.

In close connection with this remark we also have the following result.

2.5. THEOREM. – *Let A and B be commuting *-algebras of operators acting non-degenerately on a Hilbert space $\mathscr{H}$, and let $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ be linear maps such that:*

(i) $b\,\eta(a) = a\,\eta'(b)$ *for all* $a \in A$, $b \in B$;

(ii) $\eta(A_s) + i\,\eta'(B_s)$ *is dense in* $\mathscr{H}$.

Then (A, B, η, η') is a commutation system, in particular $A' = B''$.

Proof. – We always have that $\eta(A_s) + i\,\eta'(B_s) \subseteq \eta(A) + \eta'(B)$. Now as the weak closures of A and B contain the identity, we get

$$\eta(A) \subseteq (B\,\eta(A))^- \quad \text{and} \quad \eta'(B) \subseteq (A\,\eta'(B))^- = (B\,\eta(A))^-.$$

So $\eta(A) + \eta'(B) \subseteq (B\,\eta(A))^-$. So the density of $\eta(A_s) + i\,\eta'(B_s)$ implies the density of $B\,\eta(A)$. So conditions (i), (ii) and (iv) of definition 2.2 are fulfilled.

Now if we look back at the proof of Theorem 2.3, we see that condition (iii) was only used to obtain that

$$a^* y\,\eta(a) \in \eta(A_s)^- + i\,\eta'(B_s)^- \quad \text{for} \quad a \in A \quad \text{and} \quad y \in B'_s.$$

This is automatically fulfilled here as $\eta(A_s) + i\,\eta'(B_s)$ is dense. Therefore the proof of this Theorem goes through, and we get $A' = B''$ also here. But then we know that $[\eta(A)] \in B''$ and $[\eta'(B)] \in A''$ and so also condition (iii) of 2.2 is fulfilled. This completes the proof.

It was suggested to us by M. RIEFFEL that also in the general case condition (iii) might follow from the other conditions in 2.2. However we have not been able to show this, nor did we find a counter-example.

In sections 3 and 4, we will find some types of converses of the above commutation Theorems. Let us now finish this section by giving the relation with left Hilbert algebras.

2.6. PROPOSITION. – *Let $\mathscr{A}$ be a left Hilbert algebra. With the usual notation in left Hilbert algebra theory* [10], *let $A = \pi(\mathscr{A})$, $B = \pi'(\mathscr{A}')$, $\eta = \pi^{-1}$ and $\eta' = \pi'^{-1}$, then (A, B, η, η') is a commutation system with $\eta(A)^- = \eta'(B)^- = \mathscr{H}$. Conversely if (A, B, η, η') is a commutation* *system such that* $\eta(A)^- = \eta'(B)^- = \mathscr{H}$ *then* $\mathscr{A} = \eta(A)$ *can be made into a left Hilbert algebra by giving it the *-algebra structure from* A. *Moreover* $B \subseteq \pi'(\mathscr{A}')$ *and* $\pi = \eta^{-1}$ *and* $\pi' = \eta'^{-1}$.

Proof. – If $\mathscr{A}$ is a left Hilbert algebra it is well known that $A = \pi(\mathscr{A})$ and $B = \pi'(\mathscr{A}')$ are commuting *-algebras of operators acting non-degenerately on $\mathscr{H} = \mathscr{A}^-$. From the relation $\pi(\xi)\xi' = \pi'(\xi')\xi$ for all $\xi \in \mathscr{A}$ and $\xi' \in \mathscr{A}'$ and the fact that π and π' are non-degenerate it follows that π and π' are injective. Then we can define $\eta = \pi^{-1}$ and $\eta' = \pi'^{-1}$ and, rewriting the same relation, we get $b\,\eta(a) = a\,\eta'(b)$ for all $a \in A$ and $b \in B$. Now because $\eta(A) = \mathscr{A}$ and $\eta'(B) = \mathscr{A}'$ we have that $\eta(A)$ and $\eta'(B)$ are dense. Furthermore $\eta(A_s) + i\,\eta'(B_s) = \mathscr{A}_s + i\,\mathscr{A}'_s$ and this is dense as we know from left Hilbert algebra theory (*see* e. g. [6], Lemma 5.12). So, by Theorem 2.5, (A, B, η, η') is a commutation system. Conversely, let (A, B, η, η') be a commutation system such that $\eta(A)$ and $\eta'(B)$ are dense. From $a\,\eta'(b) = b\,\eta(a)$ for all $a \in A$ and $b \in B$, it follows that $\eta(a) = 0$ implies $a = 0$. Similarly $\eta'(b) = 0$ implies $b = 0$. So η and η' are injective. Then we can consider $\mathscr{A} = \eta(A)$ and equip $\mathscr{A}$ with the *-algebra structure it inherits from A. In particular

$$\pi(\eta(a_1))\,\eta(a_2) = \eta(a_1).\eta(a_2) = \eta(a_1 a_2) = a_1\,\eta(a_2),$$

so that $\pi(\eta(a_1)) = a_1$ and $\pi = \eta^{-1}$. Then the first three axioms of a left Hilbert algebra are easy to verify. Finally $\eta(A_s) \perp i\,\eta'(B_s)$ and $\mathscr{K} = \eta(A_s)^- = \mathscr{A}_s^-$ implies that $\eta'(B_s) \subseteq \mathscr{K}^\perp$. So $\eta'(B) \subseteq \mathscr{K}^\perp + i\,\mathscr{K}^\perp$ and as $\eta'(B)$ is also dense we have $\mathscr{K} \cap i\,\mathscr{K} = \{0\}$. This then proves the last condition to have a left Hilbert algebra [6].

Then $b\,\eta(a) = a\,\eta'(b)$ for all $a \in A$ and $b \in B$ implies that $b\,\xi = \pi(\xi)\,\eta'(b)$ and $b^*\,\xi = \pi(\xi)\,\eta'(b^*)$ for all $\xi \in \mathscr{A}$ so that $\eta'(b) \in \mathscr{A}'$ and $\pi'(\eta'(b)) = b$. Hence also $\pi' = \eta'^{-1}$ and $B \subseteq \pi'(\mathscr{A}')$.

3. Association of a commutation system to a von Neumann algebra

In this section, we start from a von Neumann algebra M acting in a Hilbert space and we will prove the existence of commutation systems (A, B, η, η') so that $A'' = M$ and hence $B'' = M'$. The situation is similar to the case of left Hilbert algebras. However there is one important difference: for any von Neumann algebra M there is a left Hilbert algebra $\mathscr{A}$ such that M is *isomorphic to* the left von Neumann algebra $\mathscr{L}(\mathscr{A})$ of $\mathscr{A}$, for commutation systems we actually get M *equal* to A''. This makes it possible to use our Theorem 2.3 in more general situations (for a good example, *see* [7]).

There are two different posibilities to associate commutation systems to a given von Neumann algebra.

So let M be a von Neumann algebra in a Hilbert space $\mathscr{H}$. In the first approach, we start from a maximal family of vectors $\{\xi_i\}_{i \in I}$ such that the projections $e_i = [MM'\xi_i]$ are mutually orthogonal. Here $[MM'\xi_i]$ denotes the projection onto the closed subspace generated by vectors of the form $xx'\xi_i$ with $x \in M$ and $x' \in M'$. Of course $e_i \in M \cap M'$ and $\sum_{i \in I} e_i = 1$. Define

$$A = \{x \in M;\ xe_i = 0 \text{ except for a finite number of indices } i\},$$
$$B = \{x' \in M';\ x'e_i = 0 \text{ except for a finite number of indices } i\}.$$

Remark that $\xi_i = e_i \xi_i$ so that $x\xi_i = 0$ for all but a finite number of indices when $x \in A$. Therefore we can define

$$\eta(a) = \sum_{i \in I} a\xi_i \qquad \text{when} \quad a \in A,$$
$$\eta'(b) = \sum_{i \in I} b\xi_i \qquad \text{when} \quad b \in B.$$

(The summation only runs over finite subsets of I.) We will show that (A, B, η, η') is a commutation system with $A'' = M$ and $B'' = M'$. We first prove the following Lemma.

3.1. LEMMA. – *With the notations of above, A and B are dense self-adjoint ideals in M and M' respectively.*

Proof. – Let $a_1, a_2 \in A$, choose finite subsets J_1 and J_2 of I such that $a_1 e_i = 0$ for all $i \in I \setminus J_1$ and $a_2 e_i = 0$ for all $i \in I \setminus J_2$. Then $(a_1 + a_2) e_i = 0$ for all $i \in I \setminus J$ when $J = J_1 \cup J_2$. Furthermore if $a \in A$ then $a^* e_i = e_i a^* = (ae_i)^*$ will be zero except for finite number of i. So A will be self-adjoint. Also $x \in M$ and $a \in A$ implies easily $xa \in A$ and it follows that A is a self-adjoint ideal. Similarly for B.

To prove the density observe that $e_J = \sum_{i \in J} e_i$ belongs to A as well as B for all finite subsets J of I, and that $e_J \to 1$ as J increases.

3.2. PROPOSITION. – *(A, B, η, η') is a commutation system.*

Proof. – From Lemma 3.1, we know already that A and B are commuting *-algebras of operators acting non-degenerately in $\mathscr{H}$. Clearly, η and η' are linear, and if $a \in A$ and $b \in B$, we trivially have

$$b\eta(a) = b\sum_{i \in I} a\xi_i = \sum_{i \in I} ba\xi_i = \sum_{i \in I} ab\xi_i$$
$$= a\eta'(b).$$

Now, if $x \in M$, then $xe_j \in A$ for all $j \in I$, and

$$\eta(xe_j) = \sum_{i \in I} xe_j \xi_i = \sum_{i \in I} xe_j e_i \xi_i$$
$$= xe_j \xi_j = x \xi_j.$$

If moreover $x' \in M'$, then $x' e_j \in B$ and $x' e_j \eta(xe_j) = x' e_j x \xi_j = x' x \xi_j$. Hence $B \eta(A)$ contains all vectors of the form $x' x \xi_j$ with $x' \in M$, $x \in M$ and $j \in I$, and such vectors span a dense subspace by assumption.

The only condition that remains to be checked is the density of $\eta(A_s) + i\eta'(B_s)$ in $\eta(A) + \eta'(B)$. Indeed here we know $A'' = B'$ so that automatically $[\eta(A)] \in B''$ and $[\eta'(B)] \in A''$.

Now, for any vector ω, we have that $M_s \omega + i M'_s \omega$ is dense in $M\omega + M'\omega$ ([5], [9]).

In particular, for every $a \in M$ and $j \in I$, we get that

$$a\xi_j \in (M_s \xi_j + i M'_s \xi_j)^-.$$

Now, if $x \in M_s$, then $xe_j \in A_s$ as e_j is central and $\eta(xe_j) = x\xi_j$ as we saw already. Similarly if $x' \in M_s$ then $x' e_j \in B_s$ and $\eta'(x' e_j) = x' \xi_j$. Therefore $M_s \xi_j + i M'_s \xi_j \subset \eta(A_s) + i\eta'(B_s)$, and we obtain that

$$a\xi_j \in (\eta(A_s) + i\eta'(B_s))^- \quad \text{for all } a \in M \text{ and } j \in I.$$

In general, $\eta(a) = \sum_{i \in I} a\xi_i$ for any $a \in A$ and as we only have a finite sum we get $\eta(a) \in (\eta(A_s) + i\eta'(B_s))^-$. Similarly $\eta'(b) \in (\eta(A_s) + i\eta'(B_s))^-$ for all $b \in B$. Therefore we have $\eta(A) + \eta'(B) \subset (\eta(A_s) + i\eta'(B_s))^-$ so that $\eta(A_s) + i\eta'(B_s)$ is dense in $\eta(A) + \eta'(B)$.

We conclude with the following result.

3.3. THEOREM. — *If M is a von Neumann algebra in $\mathscr{H}$, then there exist dense self-adjoint ideals A and B of M and M' respectively and linear mappings η and η' such that:*

(i) $a\eta'(b) = b\eta(a)$ *for all* $a \in A$, $b \in B$;

(ii) $B\eta(A)$ *is dense in* $\mathscr{H}$;

(iii) $\eta(A_s) + i\eta'(B_s)$ *is dense in* $\eta(A) + \eta'(B)$.

In particular, (A, B, η, η') *is a commutation system. If moreover the center of M is countably decomposable one can assume $A = M$, $B = M'$ and that there is a vector ω such that $\eta(a) = a\omega$ and that $\eta'(b) = b\omega$ for $a \in A$ and $b \in B$. So ω is a vector such that:*

(i) $MM'\omega$ *is dense;*

(ii) $M_s \omega + i M'_s \omega$ *is dense in* $M\omega + M'\omega$.

Proof. — The first part is essentially the content of the preceding Proposition. If $M \cap M'$ is countably decomposable the index set must be countable and we can normalize the vectors ξ_i such that $\sum_{i \in I} \| \xi_i \|^2 < \infty$. Then we put $\omega = \sum_{i \in I} \xi_i$, then $[MM' \omega] = \sum_{i \in I} e_i = 1$. Finally, $M_s \omega + i M'_s \omega$ is always dense in $M \omega + M' \omega$.

Now we come to the second approach. Here we start from a maximal family of vectors $\{ \xi_i \}_{i \in I}$ such that the families of projections

$$\{ e_i = [M' \xi_i] \}_{i \in I} \quad \text{and} \quad \{ e'_i = [M \xi_i] \}_{i \in I}$$

are both orthogonal families. Put

$$e = \sum_{i \in I} e_i \quad \text{and} \quad e' = \sum_{i \in I} e'_i,$$

then $(1-e)(1-e') = 0$ by maximality (*cf.* [9], Theorem 1.5). As before define

$$A = \{ x \in M;\ x e_i = 0 \text{ except for a finite number of indices } i \},$$
$$B = \{ x' \in M';\ x' e_i = 0 \text{ except for a finite number of indices } i \},$$
$$\eta(a) = \sum_{i \in I} a \xi_i \quad \text{if } a \in A,$$
$$\eta'(b) = \sum_{i \in I} b \xi_i \quad \text{if } b \in B.$$

Also here $\eta(a)$ and $\eta'(b)$ are well-defined as all sums are finite.

3.4. Proposition. — *A and B are dense left ideals in M and M' respectively, the maps η and η' are linear and satisfy:*

(i) $b\, \eta(a) = a\, \eta'(b)$ *for all* $a \in A$ *and* $b \in B$;

(ii) $\eta(A_s) + i\, \eta'(B_s)$ *is dense in* $\mathscr{H}$.

Proof. — As in Lemma 3.1, it is easy to show that A and B are left ideals in M and M' respectively. For any finite subset $J \subseteq I$, we have that $f_J = \sum_{i \in J} e_i + (1-e)$ is in A and also $f_J \to 1$ as J increases. Therefore as A is a left ideal, it must be dense. Similarly for B.

As before also $b\, \eta(a) = a\, \eta'(b)$ for all $a \in A$ and $b \in B$. It remains to show that $\eta(A_s) + i\, \eta'(B_s)$ is dense in $\mathscr{H}$. Put

$$f_J = 1 - e + \sum_{i \in J} e_i,$$
$$f'_J = 1 - e' + \sum_{i \in J} e'_i,$$
$$\xi_J = \sum_{i \in J} \xi_i$$

for any finite subset J of I.

Then $M_s \xi_J + i M'_s \xi_J$ is dense in $M \xi_J + M' \xi_J$. Let $x \in M$, then $x \xi_J \in (M_s \xi_J + i M'_s \xi_J)^-$ and if we apply $f_J f'_J$ we get

$$f_J x \xi_J = f_J x f'_J \xi_J = f_J f'_J x \xi_J \in (f_J f'_J M_s \xi_J + i f_J f'_J M'_s \xi_J)^-.$$

Now if $x \in M_s$ then $f_J\, x\, f_J \in A_s$ and $\eta\,(f_J\, x\, f_J) = f_J\, x\, \xi_J = f_J f_J'\, x\, \xi_J$. Therefore $f_J f_J'\, M_s\, \xi_J \in \eta\,(A_s)$ and similarly $f_J f_J'\, M_s'\, \xi_J \in \eta'\,(B_s)$.
So we get

$$f_J\, x\, \xi_J \in (\eta\,(A_s) + i\,\eta'\,(B_s))^-$$

for all $x \in M$ and finite J. Now if $x = x e_j$ and $j \in J$, we get

$$f_J\, x\, \xi_j \in (\eta\,(A_s) + i\,\eta'\,(B_s))^-$$

and in the limit as J increases we get $x\,\xi_j \in (\eta\,(A_s) + i\,\eta'\,(B_s))^-$. Similarly we get $x'\,\xi_j \in (\eta\,(A_s) + i\,\eta'\,(B_s))^-$ and by assumption $\mathscr{H}$ is spanned by the vectors $x\,\xi_j$ and $x'\,\xi_j$ with $x \in M$, $x' \in M'$ and j running through I.

Again we can conclude by the following result.

3.5. Theorem. — *If M is a von Neumann algebra in $\mathscr{H}$, then there are dense *-subalgebras A and B of M and M' respectively and linear mappings $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ such that:*

(i) *$b\,\eta\,(a) = a\,\eta'\,(b)$ for all $a \in A$, $b \in B$;*

(ii) *$\eta\,(A_s) + i\,\eta'\,(B_s)$ is dense in $\mathscr{H}$.*

In particular, (A, B, η, η') is a commutation system. If moreover either M or M' is countably decomposable one can take $A = M$ and $B = M'$ and then there is a vector ω such that $\eta\,(a) = a\,\omega$ and $\eta'\,(b) = b\,\omega$. So ω is a vector such that $M_s\,\omega + i\,M_s'\,\omega$ is dense (and so in particular $M\,\omega + M'\,\omega$ dense).

Proof. — With the notations of before consider $A^* \cap A$ and $B^* \cap B$ and consider the restrictions of η and η' to these sets. Then we can use Proportion 3.4 as $A^* \cap A \supseteq A^* A$ which is still dense and $(A^* \cap A)_s = A_s$.

Finally if M or M' is countably decomposable the set I is countable and again after normalizing, $\omega = \sum_{i \in I} \xi_i$ will be the desired vector.

The advantage of the second approach is that we get a commutation system such that $\eta\,(A_s) + i\,\eta'\,(B_s)$ is dense in the Hilbert space, while the advantage of the first one is that A and B are two-sided ideals.

4. Full commutation systems

In this section, we will define full commutation systems, and we will see how any commutation system can be enlarged to a full system. In [7], it turned out to be necessary to work with such systems.

Let A be *a* *-algebra of operators acting non-degenerately on a Hilbert space $\mathscr{H}$, and let η be a linear map from A to $\mathscr{H}$. To such a pair we associate a new pair $(\hat{A}, \hat{\eta})$ as follows.

4.1. NOTATION. — Let A and η be as above, then denote by $\hat{A}$ the set of operators $b \in A'$ such that there exist vectors ξ_1, ξ_2 in $\mathscr{H}$ satisfying

$$a\,\xi_1 = b\,\eta(a) \qquad \text{and} \qquad a\,\xi_2 = b^*\,\eta(a) \quad \text{for all } a \in A.$$

Because A acts non-degenerately the vectors ξ_1 and ξ_2 are uniquely defined by b. Therefore we can define a map $\hat{\eta} : \hat{A} \to \mathscr{H}$ by the relation

$$b\,\eta(a) = a\,\hat{\eta}(b) \quad \text{for all } a \in A,\ b \in \hat{A}.$$

4.2. LEMMA. — *$\hat{A}$ is a *-subalgebra of A', $\hat{\eta}$ is linear, and $\hat{\eta}(b_1 b_2) = b_1\,\hat{\eta}(b_2)$ for all $b_1, b_2 \in \hat{A}$.*

Proof. — Let $a \in A$, $b_1, b_2 \in \hat{A}$, then

$$a(\hat{\eta}(b_1)+\hat{\eta}(b_2)) = a\,\hat{\eta}(b_1)+a\,\hat{\eta}(b_2) = b_1\,\eta(a)+b_2\,\eta(a) = (b_1+b_2)\,\eta(a),$$

and similarly

$$a(\hat{\eta}(b_1^*)+\hat{\eta}(b_2^*)) = (b_1+b_2)^*\,\eta(a)$$

and it follows that $b_1+b_2 \in \hat{A}$ and that

$$\hat{\eta}(b_1+b_2) = \hat{\eta}(b_1)+\hat{\eta}(b_2).$$

Also :

$$ab_1\,\hat{\eta}(b_2) = b_1\,a\,\hat{\eta}(b_2) = b_1\,b_2\,\eta(a)$$

and

$$ab_2^*\,\hat{\eta}(b_1^*) = b_2^*\,b_1^*\,\eta(a) = (b_1\,b_2)^*\,\eta(a),$$

so that $b_1\,b_2 \in \hat{A}$ and $\hat{\eta}(b_1\,b_2) = b_1\,\hat{\eta}(b_2)$.

So we have associated a pair $(\hat{A}, \hat{\eta})$ to the pair (A, η). It would of course be nice to be able to repeat this procedure. This however can only be done when also $\hat{A}$ is non-degenerate. Unfortunately it seems difficult to impose easy conditions on the pair (A, η) to ensure that $\hat{A}$ acts non-degenerately.

Suppose however that we start from a pair of commuting *-algebras A and B acting non-degenerately on $\mathscr{H}$ and that we have given linear maps $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ such that $b\,\eta(a) = a\,\eta'(b)$ for all $a \in A$ and $b \in B$. Then of course $B \subset \hat{A}$ and $\hat{\eta} \mid B = \eta'$. In particular, $\hat{A}$ acts non-degenerately as B does.

Now if $\hat{A}$ is non-degenerate we can repeat the construction and consider the pair $(\hat{\hat{A}}, \hat{\hat{\eta}})$. We have the following Lemmas.

4.3. LEMMA. — *If* $\hat{A}$ *acts non-degenerately, then* $\hat{A}$ *is σ-weakly dense in* A' *so that in particular* $\hat{A}' = A''$. *Moreover we have that* $A \subseteq \hat{\hat{A}} \subseteq A''$ *and that* $\hat{\hat{\eta}} \mid A = \eta$.

Proof. — Let $\{ e_\lambda \}$ be a net in $\hat{A}$ such that $\| e_\lambda \| \leqslant 1$, and $e_\lambda \to 1$ strongly. Take $x \in A'$ and consider $x_\lambda = e_\lambda x e_\lambda$. Then x_λ converges strongly to x. To prove that $x_\lambda \in \hat{A}$, let $a \in A$ and consider

$$a e_\lambda x \hat{\eta}(e_\lambda) = e_\lambda x a \hat{\eta}(e_\lambda) = e_\lambda x e_\lambda \eta(a),$$
$$a e_\lambda^* x^* \hat{\eta}(e_\lambda^*) = (e_\lambda x e_\lambda)^* \eta(a),$$

so that $x_\lambda \in \hat{A}$.

Because $\hat{A}' = A''$, we clearly have that $A \subseteq \hat{\hat{A}} \subseteq A''$ and that $\hat{\hat{\eta}} \mid A = \eta$. Because $A \subseteq \hat{\hat{A}}$ and A is non-degenerate, this is also true for $\hat{\hat{A}}$, and we can repeat the operation once more. As was to be expected we get $(\hat{\hat{\hat{A}}}, \hat{\hat{\hat{\eta}}}) = (\hat{A}, \hat{\eta})$.

4.4. LEMMA. — $\hat{\hat{\hat{A}}} = \hat{A}$ *and* $\hat{\hat{\hat{\eta}}} = \hat{\eta}$.

Proof. — By Lemma 4.3 applied to $\hat{A}$, we know already that $\hat{A} \subseteq \hat{\hat{\hat{A}}}$ and that $\hat{\hat{\hat{\eta}}} \mid \hat{A} = \hat{\eta}$. So it remains to show that $\hat{\hat{\hat{A}}} \subset \hat{A}$. Therefore let $b \in \hat{\hat{\hat{A}}}$, then

$$a \hat{\hat{\hat{\eta}}}(b) = b \hat{\hat{\eta}}(a) \quad \text{and} \quad a \hat{\hat{\hat{\eta}}}(b^*) = b^* \hat{\hat{\eta}}(a) \quad \text{for all } a \in \hat{\hat{A}}.$$

As $\hat{\hat{A}} \supseteq A$ and $\hat{\hat{\eta}} \mid A = \eta$, we also get

$$a \hat{\hat{\hat{\eta}}}(b) = b \eta(a) \quad \text{and} \quad a \hat{\hat{\hat{\eta}}}(b^*) = b^* \eta(a) \quad \text{for all } a \in A$$

so that $b \in \hat{A}$.

We now come to some kind of converse of the commutation Theorem of section 2.

4.5. PROPOSITION. — *Let* A *and* B *be commuting *-algebras of operators acting non-degenerately on* $\mathscr{H}$ *and let* $\eta : A \to \mathscr{H}$ *and* $\eta' : B \to \mathscr{H}$ *be linear maps such that* $b\,\eta(a) = a\,\eta'(b)$ *for all* $a \in A$ *and* $b \in B$. *Suppose moreover that* $\hat{A} = B$ *and* $\hat{B} = A$, *then* $\eta(A_s) + i\,\eta'(B_s)$ *is dense in* $\eta(A) + \eta'(B)$.

Remark that under the above condition of course $\eta' = \hat{\eta}$ and $\eta = \hat{\eta}'$.

Proof. — Let $\xi \in \mathscr{H}$ be real orthogonal to $\eta(A_s) + i\,\eta'(B_s)$. Then $\operatorname{Re} \langle \xi, \eta(a) \rangle = 0$ and so $\langle \xi, \eta(a) \rangle + \langle \eta(a), \xi \rangle = 0$ for all $a \in A_s$.

Then as the scalar product is linear in the first term and conjugate linear in the second variable we can obtain

$$\langle \xi, \eta(a)\rangle + \langle \eta(a^*), \xi\rangle = 0 \quad \text{for all } a \in A$$

and similarly as also $\xi \perp i\,\eta'(B_s)$ we can obtain

$$\langle \xi, \eta'(b)\rangle - \langle \eta'(b^*), \xi\rangle = 0 \quad \text{for all } b \in B.$$

From the first relation with $a = a_1^* a_2$ and $a_1, a_2 \in A$ we get

$$\langle a_1 \xi, \eta(a_2)\rangle + \langle \eta(a_1), a_2 \xi\rangle = 0.$$

Then define $\mathscr{K}$ to be the closed subspace of $\mathscr{H} \oplus \mathscr{H}$ generated by the vectors $(a\,\xi, \eta(a))$ with $a \in A$. Denote linear operators on $\mathscr{H} \oplus \mathscr{H}$ by 2×2 matrices and let $P = \begin{pmatrix} p & r \\ r^* & q \end{pmatrix}$ be the projection on $\mathscr{K}$. Then $0 \leqslant p \leqslant 1$ and $0 \leqslant q \leqslant 1$ and as $\mathscr{K}$ is invariant for the diagonal action of A, we get that $p, q\ r \in A'$.

As $(a\,\xi, \eta(a)) \in \mathscr{K}$, we obtain $P(a\,\xi, \eta(a)) = (a\,\xi, \eta(a))$ and so

$$pa\,\xi + r\,\eta(a) = a\,\xi, \tag{1}$$

$$r^* a\,\xi + q\,\eta(a) = \eta(a) \quad \text{for all } a \in A. \tag{2}$$

As also $\langle a_1\,\xi, \eta(a_2)\rangle + \langle \eta(a_1), a_2\,\xi\rangle = 0$ for all $a_1, a_2 \in A$ we get that $(\eta(a), a\,\xi) \perp \mathscr{K}$ so that $P(\eta(a), a\,\xi) = 0$ and

$$p\,\eta(a) + ra\,\xi = 0, \tag{3}$$

$$r^*\eta(a) + qa\,\xi = 0 \quad \text{for all } a \in A. \tag{4}$$

The equations (1) to (4) can be rewritten as

$$a(1-p)\,\xi = r\,\eta(a), \tag{1'}$$

$$ar^*\,\xi = (1-q)\,\eta(a), \tag{2'}$$

$$ar\,\xi = -p\,\eta(a), \tag{3'}$$

$$aq\,\xi = -r^*\eta(a) \tag{4'}$$

and as $B = \hat{A}$ it follows that $r, p, 1-q \in B$ and that

$$\eta'(p) = -r\,\xi, \qquad \eta'(r) = (1-p)\,\xi,$$

$$\eta'(1-q) = r^*\,\xi, \qquad \eta'(r^*) = -q\,\xi.$$

Then we use that $\langle \xi, \eta'(b) \rangle - \langle \eta'(b^*), \xi \rangle = 0$ with $b = pr$. We get

$$\langle \xi, p(1-p)\xi \rangle + \langle r^* r \xi, \xi \rangle = 0,$$

which implies that $p(1-p)\xi = 0$ and $r\xi = 0$. Now, from the fact that P is a projection, we get $p^2 + rr^* = p$ so that $rr^*\xi = p(1-p)\xi = 0$ and also $r^*\xi = 0$. Then, for all $a \in A$, we get

$$r^* r \eta(a) = q(1-q)\eta(a) = qa\eta'(1-q) = qar^*\xi = 0$$

so that also $r^* r \eta(a) = 0$. Then $r\eta(a) = 0$ and by (1′) it follows that $a(1-p)\xi = 0$ for all a and hence $(1-p)\xi = 0$ and $p\xi = \xi$.

Finally

$$\langle \xi, \eta(a) \rangle = \langle p\xi, \eta(a) \rangle = \langle \xi, p\eta(a) \rangle = \langle \xi, a\eta'(p) \rangle = -\langle \xi, ar\xi \rangle = 0$$

so that $\xi \perp \eta(A)$.

By symmetry also $\xi \perp \eta'(B)$ and the result follows.

With the previous result, we came very close to commutation systems. We now introduce full commutation systems.

4.7. Definition. — A commutation system (A, B, η, η') is called full if $\hat{A} = B$ and $\hat{B} = A$.

4.8. Proposition. — *Let A and B be commuting *-algebras of operators acting non-degenerately on $\mathscr{H}$, and let $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ be linear maps such that:*

(i) *$b\eta(a) = a\eta'(b)$ for all $a \in A$, $b \in B$;*

(ii) *$B\eta(A)$ is dense;*

(iii) *$\hat{A} = B$ and $\hat{B} = A$;*

then (A, B, η, η') is a full commutation system.

Proof. — From the previous result, we know that $\eta(A_s) + i\eta'(B_s)$ is dense in $\eta(A) + \eta'(B)$. From $\hat{A} = B$ it also follows that $B'' = \hat{A}'' = A'$ by Lemma 4.3, so that $[\eta(A)] \in B''$ and $[\eta'(B)] \in A''$. Then all axioms are fulfilled.

4.9. Theorem. — *Let A and B be commuting *-algebras of operators acting non-degenerately on $\mathscr{H}$, and let $\eta : A \to \mathscr{H}$ and $\eta' : B \to \mathscr{H}$ be linear maps such that $b\eta(a) = a\eta'(b)$ for all $a \in A$ and $b \in B$ and such that $B\eta(A)$ is dense. Then (A, B, η, η') can be extended to a full commutation system, that is there is a full commutation system $(A_1, B_1, \eta_1, \eta'_1)$ such that $A \subseteq A_1$, $B \subseteq B_1$ and $\eta_1 \mid A = \eta$ and $\eta'_1 \mid B = \eta'$.*

Proof. — Consider $(\hat{\hat{A}}, \hat{A}, \hat{\hat{\eta}}, \hat{\eta})$ and apply the previous results.

One might think that also $(\hat{B}, \hat{A}, \hat{\eta}', \hat{\eta})$ is a full commutation system that extends (A, B, η, η'). In general, this will not be true since the relation $a\,\eta(b) = b\,\hat{\eta}'(a)$ may fail to hold for some $a \in \hat{B}$ and $b \in \hat{A}$. To see this take e. g. for A any von Neumann algebra in a Hilbert space $\mathscr{H}$ and assume that A has a cyclic vector ω but that A is not all of $\mathscr{B}(\mathscr{H})$. Define also $B = \mathbf{C}\,1$ and $\eta(a) = a\,\omega$ and $\eta'(b) = b\,\omega$ for all $a \in A$ and $b \in B$. Then the assumptions of the Theorem are fulfilled. A simple argument yields that $\hat{A} = A'$ with $\hat{\eta}(b) = b\,\omega$ for $b \in A'$ and $\hat{B} = \mathscr{B}(\mathscr{H})$ with $\hat{\eta}'(a) = a\,\omega$ for $a \in \mathscr{B}(\mathscr{H})$. Clearly $a\,\hat{\eta}(b) = b\,\eta'(a)$ will not hold for some $a \in \mathscr{B}(\mathscr{H})$ and $b \in A'$ as it will be possible to find such a and b not satisfying $ab\,\omega = ba\,\omega$.

We also remark that there is a relationship with the notion of full (= achieved) left Hilbert algebras. Indeed if we consider the commutation system $(A, B, \eta, \eta') = (\pi(\mathscr{A}), \pi'(\mathscr{A}'), \pi^{-1}, \pi'^{-1})$ associated to a left Hilbert algebra $\mathscr{A}$ as in Proposition 2.6 we can easily see that $B = \hat{A}$ since $\mathscr{A}'$ can be characterized as those vectors $\xi_1 \in \mathscr{H}$ such that there is a bounded operator b and another vector ξ_2 satisfying $b\,\xi = \pi(\xi)\,\xi_1$ and $b^*\,\xi = \pi(\xi)\,\xi_2$ for all $\xi \in \mathscr{A}$. Therefore also $\hat{B} = \pi(\mathscr{A}'')$ and we will have that $(\pi(\mathscr{A}), \pi'(\mathscr{A}'), \pi^{-1}, \pi'^{-1})$ is a full commutation system if, and only if, $\mathscr{A}$ is a full left Hilbert algebra.

We now finish this section by introducing the equivalent of left and right bounded elements in the theory of left Hilbert algebras.

So let (A, B, η, η') be a commutation system.

4.10. NOTATION:

$$\tilde{A} = \{ b \in A';\ \exists\, \xi \in \mathscr{H} \ \text{ such that } a\xi = b\,\eta(a) \ \text{ for all } a \in A \};$$

$$\tilde{B} = \{ a \in B';\ \exists\, \xi \in \mathscr{H} \ \text{ such that } b\xi = a\,\eta'(b) \ \text{ for all } b \in B \}.$$

Also define a mapping $\eta' : \tilde{A} \to \mathscr{H}$ by $a\,\eta'(b) = b\,\eta(a)$ for all $a \in A$ and $b \in \tilde{A}$, and a mapping $\eta : \tilde{B} \to \mathscr{H}$ by $b\,\eta(a) = a\,\eta'(b)$ for all $b \in B$ and $a \in \tilde{A}$. Clearly $B \subseteq \tilde{A}$ and $A \subseteq \tilde{B}$ and the notations η' and η are consistent as the new mappings are extensions of the old ones.

It is immediate from the definitions that $\hat{A} = \tilde{A} \cap \tilde{A}^*$ and $\hat{B} = \tilde{B} \cap \tilde{B}^*$.

4.11. PROPOSITION. — *$\tilde{B}$ and $\tilde{A}$ are σ-weakly dense left ideals in A'' and B'' respectively. If $x \in A''$ and $a \in \tilde{B}$ then $\eta(xa) = x\,\eta(a)$ and if $y \in B''$ and*

$b \in \tilde{A}$ then $\eta'(yb) = y\eta'(b)$. Moreover if (A, B, η, η') is a full commutation system, then $b\eta(a) = a\eta'(b)$ for all $a \in \tilde{B}$ and $b \in \tilde{A}$ and

$$\tilde{A} = \{ b \in A';\ \exists \xi \in \mathcal{H} \text{ such that } a\xi = b\eta(a) \text{ for all } a \in \tilde{B} \};$$
$$\tilde{B} = \{ a \in B';\ \exists \xi \in \mathcal{H} \text{ such that } b\xi = a\eta'(b) \text{ for all } b \in \tilde{A} \}.$$

Proof. — The proof of the first part of this Proposition can be obtained with methods similar to those used in Lemma 4.2. The density follows from the fact that $A \subseteq \tilde{B}$ and $B \subseteq \tilde{A}$. Suppose now that we have a full commutation system. Because $\tilde{A}$ is a left ideal in B'', we get $\tilde{A}^* \tilde{A} \subseteq \tilde{A} \cap \tilde{A}^* = \hat{A}$ and since our commutation system is assumed to be full so that $\hat{A} = B$ we get $\tilde{A}^* \tilde{A} \subseteq B$. We know that $b\eta(a) = a\eta'(b)$ for all $a \in \tilde{B}$ and $b \in B$. We apply this with $b = b_1^* b_2$ where $b_1, b_2 \in \tilde{A}$ and we get

$$b_1^* b_2 \eta(a) = a\eta'(b_1^* b_2) = ab_1^* \eta'(b_2) = b_1^* a\eta'(b_2).$$

Then as $\tilde{A}$ is non-degenerate we find $b_2 \eta(a) = a\eta'(b_2)$ for all $a \in \tilde{B}$ and $b_2 \in \tilde{A}$. It then follows from the definition that the last statements are correct. Several conclusions can be obtained from this result. There is one we especially want to mention here since it will be used in [7].

4.12. PROPOSITION. — *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$, then there exist σ-weakly dense left ideals N and $\tilde{N}$ of M and M' respectively, and linear maps $\eta : N \to \mathcal{H}$ and $\eta' : \tilde{N} \to \mathcal{H}$ such that:*

(i) *$b\eta(a) = a\eta'(b)$ for all $a \in N$ and $b \in \tilde{N}$;*

(ii) *$\eta(N_s) + i\eta'(\tilde{N}_s)$ is dense in $\mathcal{H}$;*

(iii) $\tilde{N} = \{ b \in M';\ \xi \in \mathcal{H} \text{ such that } a\xi = b\eta(a) \text{ for all } a \in N \};$

(iv) $N = \{ a \in M;\ \xi \in \mathcal{H} \text{ such that } b\xi = a\eta(b) \text{ for all } b \in \tilde{N} \}.$

Proof. — By Theorem 3.5 there exist a commutation system (A, B, η, η') such that $A'' = M$ and $B'' = M'$ and $\eta(A_s) + i\eta'(B_s)$ dense in $\mathcal{H}$. By Theorem 4.9, we can assume that (A, B, η, η') is a full system. Then if we put $\tilde{N} = \tilde{A}$ and $N = \tilde{B}$, we obtain the result from Proposition 4.11.

Remark that, by 4.11, it is notationally consistent to denote the two ideals by N and $\tilde{N}$. In fact $\tilde{\tilde{N}} = N$.

5. Applications

In this section, we want to apply the commutation theorem of section 2 to give new proofs of two well known results. In a forthcoming paper, we will apply the theory to give a new proof of the so called commutation Theorem for crossed products [7].

5.1 THEOREM. — *Let X be a locally compact Hausdorff space with a regular Borel measure μ. Let $\mathcal{H} = L_2(X, \mu)$ and let M be the von Neumann algebra on $\mathcal{H}$ generated by multiplications on $L_2(X, \mu)$ by continuous complex valued functions on X with compact support. Then $M = M'$.*

Proof. — Denote by $C_c(X)$ the space of continuous complex valued functions on X with compact support. For $f \in C_c(X)$ let m_f be multiplication by f on $L_2(X, \mu)$. Put $A = B = \{ m_f; f \in C_c(X) \}$ and define $\eta = \eta'$ by $\eta(m_f) = f$. Then trivially $a\,\eta'(b) = b\,\eta(a)$ for all $a \in A$ and $b \in B$. Also

$$\eta(A_s) + i\,\eta'(B_s) = \eta(A_s) + i\,\eta(A_s) = \eta(A)$$

which is dense and an application of Theorem 2.5 gives $A'' = B'$ proving the Theorem.

For our next example, we consider a locally compact group G with a left Haar measure ds. Define left and right translations in $L_2(G)$ as usual by $(\lambda_s f)(t) = f(s^{-1} t)$ and $(\rho_s f)(t) = \Delta(s)^{1/2} f(ts)$ where $s, t \in G$, $f \in L_2(G)$ and Δ is the modular function.

5.2. THEOREM. — *The commutant of the von Neumann algebra generated by the left translations $\{ \lambda_s; s \in G \}$ is the von Neumann algebra generated by the right translations $\{ \rho_t; t \in G \}$.*

Proof. — Let

$$A = \left\{ \int g(s)\lambda_s\, ds;\ g \in C_c(G) \right\},$$

$$B = \left\{ \int g(s)\rho_s\, ds;\ g \in C_c(G) \right\},$$

where the integrals are considered in the weak topology. It is well-known and easy to check that A and B are commuting *-algebras of operators in $L_2(G)$, that they act non-degenerately and that A'' is the von Neumann algebra generated by $\{ \lambda_s; s \in G \}$ and B'' the von Neumann algebra generated by $\{ \rho_s; s \in G \}$.

Define η on A and η' on B by

$$\eta\left(\int g(s)\lambda_s\, ds\right) = g,$$

$$\eta'\left(\int \Delta(s)^{-1/2} f(s^{-1})\rho_s\, ds\right) = f.$$

Then:

$$\left(\left(\int g(s)\lambda_s\,ds\right)f\right)(t)=\int g(s)(\lambda_s f)(t)\,ds$$
$$=\int g(s)f(s^{-1}t)\,ds$$
$$=\int g(ts)f(s^{-1})\,ds$$
$$=\int f(s^{-1})\Delta(s)^{-1/2}(\rho_s g)(t)\,ds$$
$$=\left(\left(\int f(s^{-1})\Delta(s)^{-1/2}\rho_s\,ds\right)g\right)(t).$$

This relation implies that η and η' are well defined and that $a\,\eta'(b)=b\,\eta(a)$ for all $a\in A$ and $b\in B$. We will now show that $\eta(A_s)+i\,\eta'(B_s)$ is dense. Now:

$$\left(\int g(s)\lambda_s\,ds\right)^*=\int \overline{g(s)}\lambda_{s^{-1}}\,ds=\int \Delta(s^{-1})\overline{g(s^{-1})}\lambda_s\,ds$$

and $\eta(A_s)$ consists of those functions $g\in C_c(G)$ satisfying

$$g(s)=\Delta(s^{-1})\overline{g(s^{-1})}.$$

Similarly

$$\left(\int \Delta(s)^{-1/2}f(s^{-1})\rho_s\,ds\right)^*=\int \Delta(s)^{-1/2}\overline{f(s^{-1})}\rho_{s^{-1}}\,ds$$
$$=\int \Delta(s^{-1})\Delta(s^{-1})^{-1/2}\overline{f(s)}\rho_s\,ds$$
$$=\int \Delta(s)^{-1/2}\overline{f(s)}\rho_s\,ds$$

and $\eta'(B_s)$ consists of those functions $f\in C_c(G)$ such that

$$f(s)=\overline{f(s^{-1})}.$$

Now any $f\in C_c(G)$ can be written as

$$f(s)=\left(\frac{1}{1+\Delta(s)}f(s)+\frac{\Delta(s)^{-1}}{1+\Delta(s^{-1})}\overline{f(s^{-1})}\right)$$
$$+i\frac{1}{i}\left(\frac{\Delta(s)}{1+\Delta(s)}f(s)-\frac{\Delta(s^{-1})}{1+\Delta(s^{-1})}\overline{f(s^{-1})}\right)$$

proving that $\eta(A_s)+i\,\eta'(B_s)$ is dense. Again an application of our Theorem 2.5 yields the result.

REFERENCES

[1] DIGERNES (T.). — Poids dual sur un produit croisé, *C. R. Acad. Sc. Paris*, t. 278, 1974, série A, p. 937-940.

[2] DIGERNES (T.). — *Duality for weights on covariant systems and its applications*, Thesis, University of California, Los Angeles, 1975.

[3] RIEFFEL (M.). — A commutation theorem and duality for free Bose fields, *Comm. math. Phys.*, t. 39, 1974, p. 153-164.

[4] RIEFFEL (M.). — Commutation theorems and generalized commutation relations, *Bull. Soc. math. France*, t. 104, 1976, p. 205-224.

[5] RIEFFEL (M.) and VAN DAELE (A.). — The commutation theorem for tensor products of von Neumann algebras, *Bull. London math. Soc.*, t. 7, 1975, p. 257-260.

[6] RIEFFEL (M.) and VAN DAELE (A.). — A bounded operator approach to Tomita-Takesaki theory, *Pacific. J. Math.*, t. 69, 1977, p. 187-221.

[7] ROUSSEAU (R.) and VAN DAELE (A.). — *Crossed products of commutation systems* (to appear).

[8] ROUSSEAU (R.), VAN DAELE (A.) and VANHEESWIJCK (L.). — A note on the commutation theorem for tensor products of von Neumann algebras, *Proc. Amer. math. Soc.*, t. 61, 1976, p. 179-180.

[9] ROUSSEAU (R.), VAN DAELE (A.) and VANHEESWIJCK (L.). — A necessary and sufficient condition for a von Neumann algebra to be in standard form, *J. London math. Soc.*, t. 15, 1977, p. 147-154.

[10] TAKESAKI (M.). — *Tomita's theory of modular Hilbert algebras and its applications.* — Berlin, Springer-Verlag, 1970 (*Lecture Notes in Mathematics*, 128).

[11] TAKESAKI (M.). — Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.*, Uppsala, t. 131, 1973, p. 249-310.

[12] TOMITA (M.). — Standard forms of von Neumann algebras, "*5th Functional analysis symposium of the mathematical society of Japan*, [1967. Sendai]". — Sendai, Tôhoku University, Mathematical Institute, 1967.

(Texte reçu le 16 mai 1977.)

Alfons VAN DAELE,
Departement Wiskunde,
Katholieke Universiteit Leuven,
Celestijnenlaan 200 B,
B-3030 Heverlee (Belgique).