BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 106 (1978), p. 3-24
<http://www.numdam.org/item?id=BSMF_1978__106__3_0>

Bull. Soc. math. France,
106, 1978, p. 3-24.

A HAHN-BANACH EXTENSION THEOREM FOR ANALYTIC MAPPINGS

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RÉSUMÉ. — Soient E un sous-espace vectoriel fermé d'un espace de Banach G, U un ouvert de E, et F un espace de Banach. On considère le problème du prolongement des applications analytiques de U à valeur dans F à un ouvert de G, et on trouve des conditions nécessaires et suffisantes pour l'existence de tels prolongements. Ces conditions entraînent l'existence d'une application linéaire continue de prolongement de E' à G' ce qui, à tour de rôle, se rapporte au théorème de Hahn-Banach vectoriel.

ABSTRACT. — Let E be a closed subspace of a Banach space G, let U be an open subset of E, and let F be another Banach space. The problem of extending analytic F-valued mappings defined on U to an open subset of G is discussed, and necessary and sufficient conditions are found for such extensions to exist. These conditions involve the existence of a continuous linear extension mapping of E' to G', which in turn is related to the Hahn-Banach theorem for linear transformations.

We consider the problem of extending an analytic mapping defined on an open subset U of a closed subspace E of a Banach space G to an analytic mapping defined on an open neighbourhood of U in G. Our general approach is to obtain extensions to the whole space G of polynomials defined on E, and then to use local Taylor series representations to extend analytic functions locally. It is necessary to show that the local extensions are "coherent in the overlaps". This can be done when one can define a linear and continuous extension mapping taking polynomials defined on E to their extensions defined on G, which in turn is closely related to the vector-valued Hahn-Banach property as studied by NACHBIN, LINDENSTRAUSS, and others.

The general question of extending analytic mappings on topological vector spaces was raised by DINEEN in [4]. He and other authors (HIRSCHOWITZ,

(1) The research of this author was supported in part by the Université de Nancy (France).

(2) The research of this author was supported by a post-doctoral fellowship from An Roinn Oideachais, Department of Education (Ireland).

NOVERRAZ et coll.) discussed the existence of an analytic extension of mappings defined on a dense (vector) subspace of a locally convex space. The case of extending from a closed subspace E of a Fréchet nuclear space G was studied by BOLAND who showed that every analytic function defined on E has an extension to an analytic function on G.

The main result (theorem 1.1) and its corollaries are presented in section 1, after the review of some necessary terminology. Briefly, this result states that given a pair of Banach spaces $E \subset G$, the existence of various types of holomorphic extensions is equivalent to the existence of a continuous linear extension mapping from E' to G'. The proof of this result relies on the important special case of extending analytic mappings from a Banach space to its second dual, which is discussed in section 2 where we prove our main result. In addition, section 2 contains several related results including an extension result for "nuclear" entire functions analogous to that of BOLAND, and several examples. Throughout, the case of complex analytic (or holomorphic) mappings is emphasized as the real analytic case (theorem 1.2) follows easily from it.

The authors wish to acknowledge helpful conversations with Philippe NOVERRAZ, Séan DINEEN, and Philip BOLAND concerning this work.

1. Main result

We recall some notation from [9] (*see* also [10]). All Banach spaces considered will be complex Banach spaces except when indicated otherwise. For all $n \in \mathbf{N}$, $L(^nE, F)$ (resp. $L_s(^nE, F)$) is the space of all continuous n-multilinear (resp. and symmetric) mappings from $E \times \ldots \times E$ into F normed by $A \mapsto \sup \| A(x_1, \ldots, x_n) \|$ where each x_i, $1 \leqslant i \leqslant n$, ranges over the unit ball of E. $(L(^0E, F) \equiv F)$. $\mathscr{P}(^nE, F)$, the *continuous n-homogeneous polynomials from E to F*, consists of the Banach space of mappings $\{ \hat{A} : x \in E \mapsto A(x, x, \ldots, x);\ A \in L_s(^nE, F) \}$, normed by $\hat{A} \to \sup \{ \| \hat{A}(x) \|;\ \| x \| \leqslant 1 \}$. Let $U \subset E$ be open and non-empty. A mapping $f : U \to E$ is called *holomorphic* if for each $x \in U$ there exists a power series $\sum_{k=0}^{\infty} \hat{P}_k(y-x)$, with $\hat{P}_k \in \mathscr{P}(^kE, F)$ for each $k \in \mathbf{N}$, which converges uniformly to $f(y)$ in a neighbourhood of x. Such a series is necessarily unique and for each k, $\hat{P}_k$, called the kth *Taylor series coefficient of f at x*, is denoted by $(1/k!)\, \hat{d}^k f(x)$. The space of all holomorphic mappings from U to F is denoted by $\mathscr{H}(U, F)$. The space spanned in $\mathscr{P}(^nE, \mathbf{C})$ by $\{ \varphi^n : x \in E \mapsto (\varphi(x))^n;\ \varphi \in E' \}$ is the space of continuous n-homogeneous polynomials of *finite type*, denoted $\mathcal{P}_f(^nE)$. The closure of $\mathcal{P}_f(^nE) \otimes F$ in $\mathcal{P}(^nE, F)$ is denoted by $\mathcal{P}_C(^nE, F)$. $\mathcal{P}_K(^nE, F)$ (resp. $\mathcal{P}_{WK}(^nE, F)$) is the closed subspace of $\mathcal{P}(^nE, F)$ consisting of those polynomials which map the unit ball in E into a relatively compact (resp. relatively weakly compact) subset of F. Notice that

$$\mathcal{P}_f(^nE) \otimes F \subset \mathcal{P}_C(^nE, F) \subset \mathcal{P}_K(^nE, F) \subset \mathcal{P}_{WK}(^nE, F) \subset \mathcal{P}(^nE, F).$$

In general, these inclusions are proper. For $\theta = C$, K, or WK, we let

$$\mathcal{H}_\theta(U, F) = \left\{ f \in \mathcal{H}(U, F);\ \text{for all } n \in \mathbf{N} \text{ and } x \in U,\ \frac{1}{n!}\hat{d}^n f(x) \in \mathcal{P}_\theta(^nE, F) \right\}.$$

The elements of $\mathcal{H}_\theta(U, F)$ are said to be of *type* θ. We remark that for connected U, $f \in \mathcal{H}_K(U, F)$ (resp. $\mathcal{H}_{WK}(U, F)$) if, and only if, f maps some ball in U into a compact (resp. weakly compact) set (*see* [1] and [13]). Similarly, we define $L_K(^nE, F)$ (resp. $L_{WK}(^nE, F)$) as the subspace of all $A \in L(^nE, F)$ mapping a neighbourhood of zero in E^n to a relatively (resp. weakly) compact set in F. Finally, we set

$$\mathcal{H}_b(E, F) = \{ f \in \mathcal{H}(E, F);\ f \text{ is bounded on bounded sets} \}$$

and

$$\mathcal{H}_{\theta b}(E, F) = \mathcal{H}_\theta(E, F) \cap \mathcal{H}_b(E, F), \qquad \text{for} \quad \theta = C, K, \quad \text{or} \quad WK.$$

$\mathcal{H}_b(E, F)$ and $\mathcal{H}_{\theta b}(E, F)$ are Fréchet spaces with the topology of uniform convergence on bounded sets.

DEFINITION. — *A Banach space F is called a $\mathcal{C}_\lambda$-space (or just a $\mathcal{C}$-space) if F is complemented in its second conjugate space F'' and there is a projection $\pi : F'' \to F$ onto F of norm $\leqslant \lambda$.*

Example 1. — Every conjugate space F' is a $\mathcal{C}_1$-space (the transpose of the inclusion $F \hookrightarrow F''$ projects F''' onto F' with norm 1).

Example 2. — $L^1(\mu, X)$ is a $\mathcal{C}_1$-space for X locally compact and μ σ-finite (restrict an element of $L^\infty(\mu, X)'$ to the subspace $C_b(X)$ and apply the Lebesgue-Radon-Nikodym theorem).

Example 3. — Every $\mathfrak{P}_\lambda$-space (in the sense of DAY [3]) is a $\mathcal{C}_\lambda$-space (*A* $\mathfrak{P}_\lambda$-space may be defined as a space which is complemented in every space that contains it with projection of norm $\leqslant \lambda$).

Example 4. — c_0, the Banach space of all complex null sequences, is not a $\mathcal{C}$-space.

THEOREM 1.1. — *Let G be a complex Banach space, and $E \subset G$ a closed subspace. Then the following conditions are equivalent:*

(1) *for every $\mathscr{C}$-space F, $U \subset E$ open and non-empty, and $f \in \mathscr{H}(U, F)$ there exists a $W \subset G$ open, and $\tilde{f} \in \mathscr{H}(W, F)$ such that $U \subset W$ and $\tilde{f}|_U = f$;*

(2) *for every $f \in \mathscr{H}(E, E'')$ there exists a $W \subset G$, open, and $\tilde{f} \in \mathscr{H}(W, E'')$ such that $E \subset W$ and $\tilde{f}|_E = f$;*

(3) *for every $\mathscr{C}$-space F, there exists a strict morphism*

$$T: \quad \mathscr{H}_b(E, F) \to \mathscr{H}_b(G, F)$$

such that $Tf|_E = f$ for all $f \in \mathscr{H}_b(E, F)$;

(4) *there exists a continuous linear mapping $T: \mathscr{H}_b(E, \mathbf{C}) \to \mathscr{H}_b(G, \mathbf{C})$ such that $Tf|_E = f$ for all $f \in \mathscr{H}_b(E, \mathbf{C})$;*

(5) *for all Banach spaces F, there exists a strict morphism*

$$T: \quad \mathscr{H}_{\theta b}(E, F) \to \mathscr{H}_{\theta b}(G, F)$$

such that $Tf|_E = f$ for all $f \in \mathscr{H}_{\theta b}(E, F)$ where $\theta = C$ (resp. $\theta = K$, $\theta = WK$);

(6) *there exists a continuous linear map $\varphi \in E' \to \tilde{\varphi} \in G'$ such that $\tilde{\varphi}|_E = \varphi$;*

(7) *there exists a continuous linear map $S : G \to E''$ such that $S|_E = \mathrm{Id}_E$.*

Furthermore the above conditions imply, and if E has the "bounded approximation property" (³) *are implied by, the following:*

(8) *for all Banach spaces F, $U \subset E$ open and non empty, and $f \in \mathscr{H}_\theta(U, F)$, there exists an open set $W \subset G$ and $\tilde{f} \in \mathscr{H}_\theta(W, F)$ such that $U \subset W$ and $\tilde{f}|_U = f$ where $\theta = C$ (resp. $\theta = K$, $\theta = WK$).*

The proof will be given in section 2.

Remarks.

(*a*) Conditions (1), (6) and (7) are complex analytic analogs of part of Lindenstrauss' theorem on the extension of compact operators (*see* [7], theorem 2.1). This relationship is emphasised further in corollary 1.3 (iii) below.

(*b*) The class of $\mathscr{C}$-spaces is the largest class of range spaces for which conditions (1) and (3) can hold in the following sense: if F is not a $\mathscr{C}$-space

(³) E has the "bounded approximation property" if for some constant $C > 0$, for any compact set $K \subset E$ and $\varepsilon > 0$, there is a $T \in E' \otimes E$ with $\| T \| \leqslant C$ and $\| Tx - x \| < \varepsilon$ for $x \in K$ (*see* for example P. NOVERRAZ [11]).

then there exists a G and a closed subspace E of G satisfying (6), but not (1) and (3) with range F. (To see this we let $E = F$ and $G = F''$. If $\mathrm{Id}_F \in \mathscr{H}_b(F, F)$ extends to a function f defined on an open set in F'' then $\pi(y) \equiv \hat{d}^1 f(0)(y)$ would give a projection from F'' onto F.)

(*c*) For every complex Banach space E, the pair $E \subset G \equiv E''$ always satisfies condition (6); thus holomorphic extensions of the type indicated in the theorem are always possible from a space E to its second conjugate E''. Conversely, in section 2, we will reduce the proof of the theorem to this special case which we prove directly.

(*d*) If E is complemented in G, then the holomorphic extensions of the type indicated in the theorem are trivially possible. For many spaces E, the converse is true as shown in the following corollary.

COROLLARY 1.1. — *Let G be a Banach space, and $E \subset G$ a closed subspace. If E is a $\mathscr{C}$-space then the equivalent conditions of theorem* 1.1 *hold if, and only if, E is complemented in G. In that case, we may drop the restriction that F be a $\mathscr{C}$-space in conditions* (1) *and* (3).

Proof. — If E is complemented in G, then it is clear that the conditions hold. Conversely, if the equivalent conditions of theorem 1.1 hold, (3) implies that $\mathrm{Id}_E \in \mathscr{H}_b(E, E)$ extends to some $\widetilde{\mathrm{Id}}_E \in \mathscr{H}_b(G, E)$. Reasoning as in remark (*b*) above, we conclude that E is complemented in G.

COROLLARY 1.2. — *Let G be a reflexive Banach space. The equivalent conditions of theorem* 1.1 *hold for every closed subspace $E \subset G$ if, and only if, G is isomorphic to a Hilbert space.*

Proof. — Every closed subspace of a reflexive space is reflexive and therefore a $\mathscr{C}_1$-space. So if the equivalent conditions of theorem 1.1 hold for every closed subspace $E \subset G$, corollary 1.1 implies every closed subspace of G is complemented in G. However, it is known [8] that every closed subspace of a space G is complemented if, and only if, it is isomorphic to a Hilbert space.

If G is isomorphic to a Hilbert space, the converse is trivial.

COROLLARY 1.3. — *Let E be a Banach space. Then the following statements are equivalent :*

(i) *the equivalent conditions of theorem* 1.1 *hold for every $G \supset E$;*

(ii) *there exists a $\mathfrak{P}_\lambda$-space $G \supset E$ for some $\lambda \geqslant 1$ for which the equivalent conditions of theorem* 1.1 *hold;*

(iii) *E'' is a $\mathfrak{P}_\lambda$-space for some $\lambda \geqslant 1$.*

Proof. — (i) $\Rightarrow$ (iii): Suppose E'' is a closed subspace of G. We want to show that E'' is complemented in G. Since $E \subset G$, (i) and condition (7) imply that there is a continuous linear map $S : G \rightarrow E''$ such that $S|_E = \mathrm{Id}_E$. Now $S|_{E''} : E'' \rightarrow E''$ is weak*-continuous, and E is weak*-dense in E''. Hence $S|_{E''} = \mathrm{Id}_{E''}$ so S is a projection onto E'', and E'' is complemented in G.

(iii) $\Rightarrow$ (ii): Condition (7) holds trivially for $G = E''$.

(ii) $\Rightarrow$ (i): Let G_1 be a $\mathfrak{P}_\lambda$-space satisfying (ii). Then by condition (7) there is a continuous linear map $S_1 : G_1 \rightarrow E''$ which extends the inclusion $E \hookrightarrow E''$. Now suppose G is any Banach space containing E as a closed subspace. Since G_1 is a $\mathfrak{P}_\lambda$-space there exists a continuous linear map $T : G \rightarrow G_1$ which extends the inclusion $E \rightarrow G_1$. Let $S = S_1 \circ T : G \rightarrow E''$. S satisfies condition (7) and so (i) holds.

Remark. — The extension of an entire function given in condition (1) of theorem 1.1 is not necessarily entire, but if it is also bounded on bounded subsets then it may be extended to an entire function as in condition (3). This is illustrated in the following proposition (*see* [4]).

PROPOSITION 1.1. — *$f \in \mathscr{H}(c_0, \mathbf{C})$ can be extended to an entire function on l_∞ if, and only if, $f \in \mathscr{H}_b(c_0, \mathbf{C})$.*

Proof. — Since $(c_0)'' = l_\infty$ the sufficiency follows from the equivalence of conditions (7) and (3). Conversely JOSEFSON [6] has shown that each $\tilde{f} \in \mathscr{H}(l_\infty, \mathbf{C})$ is bounded on each bounded set contained in c_0. That is $\tilde{f}|_{c_0} \in \mathscr{H}_b(c_0, \mathbf{C})$, hence the result.

Real case. — If E and F are real Banach spaces, we define spaces of polynomials $\mathscr{P}(^nE, F)$, $\mathscr{P}_{WK}(^nE, F)$, etc. in a way analogous to the complex case. Similarly, if $U \subset E$ is open, and $f : U \rightarrow F$ is a mapping, we say f is (*real*) *analytic* if f can be represented in a neighbourhood of each point in U by a convergent power series. We define spaces of (real) analytic mappings $\mathscr{A}(U, F)$, $\mathscr{A}_\theta(U, F)$, $\mathscr{A}_b(U, F)$, etc. in an obvious way. The following theorem is the real analog of theorem 1.1, and is proved in section 2.

THEOREM 1.2 (Real case). — *Let G be a real Banach space, and E a closed subspace. Then if the holomorphic $\mathscr{H}$ is replaced everywhere by the real analytic $\mathscr{A}$ in the conditions of theorem* 1.1, *the same implications and equivalences hold with F a real Banach space.*

2. Related results

In this section, we prove a lemma which shows how certain polynomial extension mappings give rise to holomorphic extension mappings. We apply it to the case of extending holomorphic maps from a Banach space E to its second conjugate space E'' and then use these results to prove theorems 1.1 and 1.2. The section concludes with some further results and comments.

Let us recall that if E and F are Banach spaces, $U \subset E$ is open and non-empty, $x \in U$ and $f \in \mathcal{H}(U, F)$, then the *radius of convergence of f at x* is

$$r_C(x, f) \equiv \frac{1}{\limsup_{n \in \mathbf{N}} \| (1/n!) \hat{d}^n f(x) \|^{1/n}},$$

and the *radius of boundedness of f at x*, $r_b(x, f)$, is the supremum of all $\rho > 0$ such that the ball of radius ρ centred at x is contained in U and f is bounded on it. It is shown in [9] (§ 7, proposition 2) that

$$r_b(x, f) = \min\{ r_C(x, f); \operatorname{dist}(x, E \setminus U)\}.$$

DEFINITION. — *Let $R : U \to (0, \infty]$ be a function satisfying $R(y) \leqslant \operatorname{dist}(y, E \setminus U)$ for all $y \in U$. Then we define*

$$\mathcal{H}(R;\, U, F) \equiv \{ f \in \mathcal{H}(U, F);\ r_b(y, f) \geqslant R(y) \text{ for all } y \in U \}$$

and

$$\mathcal{H}_\theta(R;\, U, F) \equiv \mathcal{H}(R;\, U, F) \cap \mathcal{H}_\theta(U, F) \qquad \text{for} \quad \theta = C, K, \text{ or } WK.$$

We note that for $U = E$ and $R \equiv \infty$, we have

$$\mathcal{H}(\infty;\, E, F) = \mathcal{H}_b(E; F) \qquad \text{and} \qquad \mathcal{H}_\theta(\infty;\, E, F) = \mathcal{H}_{\theta b}(E; F).$$

Also given any $f \in \mathcal{H}(U, F)$ there exists a function R such that $f \in \mathcal{H}(R;\, U, F)$, namely $R(y) = r_b(y, f)$ for all $y \in U$.

Given $m \in \mathbf{N}$, $P \in \mathcal{P}(^m E, F)$, $y \in E$, and $0 \leqslant k \leqslant m$ we recall that $(1/k!)\, \hat{d}^k P(y) \in \mathcal{P}(^k E, F)$ and (*see* [9]) if $P \in \mathcal{P}_\theta(^m E, F)$, $\theta = C, K$, or WK, then $(1/k!)\, \hat{d}^k P(y) \in \mathcal{P}_\theta(^k E, F)$. If A is the (unique) element of $L_s(^m E, F)$ such that $\hat{A} = P$ then

$$\frac{1}{k!} \hat{d}^k P(y)(x) = \binom{m}{k} A\, y^{m-k} x^k \quad \text{for all } x \in E,$$

where $A\,y^{m-k}\,x^k$ denotes

$$A(\overbrace{y, y, \ldots, y}^{m-k}, \overbrace{x, x, \ldots, x}^{k}).$$

In this notation, $A\,y^{m-k}$ is a continuous k-multilinear mapping on E.

We state and prove the following lemma for spaces of type $\theta = C$, K or WK, but it is obvious from the proof that it also holds for $\mathscr{H}(U, F)$, and we will use this fact.

LEMMA. — *Let G and F be Banach spaces, and E a closed subspace of G; and let* $\theta = C$, K, *or* WK. *Assume there exists a sequence of linear maps* $\{ T_n : \mathscr{P}_\theta(^nE, F) \to \mathscr{P}_\theta(^nG, F) \}_{n\in\mathbf{N}}$ *and a sequence of real numbers* $\geqslant 1$, $\{ a_n \}_{n\in\mathbf{N}}$ *such that all* k, $m \in \mathbf{N}$, $k \leqslant m$ *and* $P \in \mathscr{P}_\theta(^mE, F)$,

(i) $$\frac{1}{k!}\hat{d}^k(T_m P)(y) = T_k \frac{1}{k!}\hat{d}^k P(y) \quad \text{for all } y \in E;$$

(ii) $$\| T_m P \| \leqslant a_m \| P \|,$$

and

(iii) $$\alpha \equiv \limsup_{n\in\mathbf{N}} (a_n)^{1/n} < \infty.$$

Then for all $U \subset E$ *open and non-empty and all* $R : U \to (0, \infty]$ *such that* $R(y) \leqslant \operatorname{dist}(y, E \setminus U)$ *for all* $y \in U$, *there exists an open set W in G containing U and a linear mapping* $T : f \in \mathscr{H}_\theta(R; U, F) \mapsto Tf \in \mathscr{H}_\theta(W, F)$ *such that* $Tf|_U = f$ *for all* $f \in \mathscr{H}_\theta(R; U, F)$. *We may take W to be the set* $W = \{ x \in G;\ \alpha \| x - y \| < R(y) \text{ for some } y \in U \}$.

Furthermore, when $U = E$ *and* $R = \infty$, *we may take* $W = G$ *and* $T : \mathscr{H}_{\theta b}(E, F) \to \mathscr{H}_{\theta b}(G, F)$ *is a strict morphism.*

(Note: For $k = 0$, condition (i) implies that T_m is an extension mapping.)

Proof. — For each $y \in U$, let U_y denote the set $\{ x \in G;\ \alpha \| x - y \| < R(y) \}$. Clearly, $W = \bigcup_{y\in U} U_y$. Define $T_y f : U_y \to F$ for each $f \in \mathscr{H}_\theta(R; U, F)$ by

$$T_y f(x) \equiv \sum_{k=0}^{\infty} \left(T_k \frac{1}{k!} \hat{d}^k f(y) \right)(x - y) \qquad \text{for} \quad x \in U_y.$$

If $\alpha \| x - y \| \leqslant r < R(y)$ and $\hat{P}_k \equiv (1/k!)\,\hat{d}^k f(y)$, then:

$$\sum_{k=0}^{\infty} \| (T_k \hat{P}_k)(x - y) \| \leqslant \sum_{k=0}^{\infty} \| T_k \hat{P}_k \| \quad \| x - y \|^k \leqslant \sum_{k=0}^{\infty} a_k \| \hat{P}_k \| \left(\frac{1}{\alpha} r \right)^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{a_k}{\alpha^k} \right) (\| \hat{P}_k \|^{1/k} r)^k < \infty$$

since $\alpha = \limsup (a_k)^{1/k}$ and $r_C(y, f) \geqslant R(y) > r$. Therefore $T_y f$ is well defined and holomorphic on U_y and by lemma 1, paragraph 9 of [9] we have that $T_y f \in \mathscr{H}_\theta(U_y, F)$. Now suppose:

$$\text{(A)} \qquad T_y f\,|_{U_y \cap U_z} = T_z f\,|_{U_y \cap U_z} \quad \text{for all } y, z \in U$$

is satisfied. Then $Tf : W \to F$ may be defined by

$$Tf\,|_{U_y} = T_y f \qquad \text{and} \qquad Tf \in \mathscr{H}_\theta(W, F) \quad \text{for all } f \in \mathscr{H}_\theta(R; U, F).$$

Furthermore the mapping $T : f \in \mathscr{H}_\theta(R; U, F) \mapsto Tf \in \mathscr{H}_\theta(W, F)$ is clearly linear. So we will proceed to show that (A) is satisfied.

Let $U_y^3 = \{ x \in G;\ \alpha \| x - y \| < (1/3)\, R(y) \}$ for all $y \in U$ and suppose that:

$$\text{(B)} \qquad T_v f_{|\, U_v^3 \cap U_w^3} = T_w f_{|\, U_v^3 \cap U_w^3} \quad \text{all } v, w \in U$$

is satisfied. Then (A) is also satisfied for the following reasons:

If $U_y \cap U_z \neq \varnothing$ and $w = \lambda y + (1 - \lambda) z$, $0 \leqslant \lambda \leqslant 1$, then:

$$\| w - y \| + \| w - z \| = \| y - z \| \leqslant \| x - y \| + \| x - z \| < \frac{1}{\alpha}(R(y) + R(z)),$$

for all $x \in U_y \cap U_z$. So either $\alpha \| w - y \| < R(y)$ or $\alpha \| w - z \| < R(z)$. In either case, since $\alpha \geqslant 1$ and $R(\cdot) \leqslant \operatorname{dist}(\cdot, E \setminus U)$, we have that $w \in U \cap (U_y \cup U_z)$. Now if (B) holds then each Tf is well defined on the connected open set: $V = \bigcup_{0 \leqslant \lambda \leqslant 1} U^3_{\lambda y + (1-\lambda) z}$, and since $z, y \in V$ and $V \cap (U_y \cap U_z) \neq \varnothing$, (A) holds by uniqueness of analytic continuation.

To show (B) let $y, z \in U$ and $f \in \mathscr{H}_\theta(R; U, F)$ be given and suppose $x \in U_y^3 \cap U_z^3 \neq \varnothing$. We must show that $T_y f(x) = T_z f(x)$. Without loss of generality we may assume that $R(z) \leqslant R(y)$, and (by translation) that $z = 0$.

Let $\hat{P}_n = (1/n!)\, \hat{d}^n f(y)$ and $\hat{Q}_n = (1/n!)\, \hat{d}^n f(0)$, $n \in \mathbf{N}$. If $\hat{A}$ is an n-homogeneous polynomial, the Taylor series expansion of the non-homogeneous polynomial $v \mapsto \hat{A}(v - y)$ at 0 is

$$\hat{A}(v - y) = \sum_{m=0}^{n} \frac{1}{m!} (\hat{d}^m \hat{A}(-y))(v) \quad \text{for all } v.$$

Expanding the k-th partial Taylor series sum of f at y,

$$\tau_{k, f, y}(v) \equiv \sum_{n=0}^{k} \hat{P}_n(v - y) \quad \text{for all } v \in E,$$

in this way we have

$$\tau_{k,f,y}(v) = \textstyle\sum_{n=0}^{k}\left(\sum_{m=0}^{n}\dfrac{1}{m!}\hat{d}^m \hat{P}_n(-y)(v)\right)$$
$$= \textstyle\sum_{m=0}^{k}\dfrac{1}{m!}\hat{d}^m(\sum_{n=m}^{k}\hat{P}_n)(-y)(v).$$

Therefore, $\hat{d}^m(\tau_{k,f,y})(0) = \hat{d}^m(\sum_{n=m}^{k}\hat{P}_n)(-y)$, $m = 0\ 1, \ldots, k$. Expanding the polynomials $T_n\hat{P}_n(\cdot - y)$, $n = 0, \ldots, k$, and using hypothesis (i) we obtain:

$$\textstyle\sum_{n=0}^{k}(T_n\hat{P}_n)(v-y) = \sum_{n=0}^{k}\left(\sum_{m=0}^{n}\dfrac{1}{m!}\hat{d}^m(T_n\hat{P}_n)(-y)(v)\right)$$
$$= \textstyle\sum_{m=0}^{k}\dfrac{1}{m!}T_m\hat{d}^m(\sum_{n=m}^{k}\hat{P}_n)(-y)(v) \quad \text{for all } v \in G.$$

Substituting for $\hat{d}^m(\sum_{n=m}^{k}\hat{P}_n)(-y)$ we obtain:

$$\text{(C)} \quad \textstyle\sum_{n=0}^{k}(T_n\hat{P}_n)(v-y) = \sum_{m=0}^{k}\dfrac{1}{m!}T_m\hat{d}^m\tau_{k,f,y}(0)(v) \quad \text{for all } v \in G.$$

Since $\alpha\|x-y\| < (1/3)R(y)$ and $\alpha\|x\| < (1/3)R(0)$, and (by assumption) $R(0) \leqslant R(y)$ we can find real numbers λ, ρ, and σ such that

$$\alpha\|x\| < \lambda < \frac{1}{3}R(0), \quad \lambda < \rho,$$

$$\alpha\|x-y\| < \rho < \frac{1}{3}R(y) \qquad \text{and} \qquad 1 < \sigma < \frac{R(y)}{3\rho}.$$

By Cauchy's inequalities applied to $f - \tau_{k,f,y}$ at 0:

$$\left\|\hat{Q}_m - \frac{1}{m}\hat{d}^m\tau_{k,f,y}(0)\right\| \leqslant \lambda^{-m}\sup_{v\in E,\,\|v\|=\lambda}\|f(v) - \tau_{k,f,y}(v)\|$$

for all $m, k \in \mathbf{N}$.

Now $\|v\| = \lambda$ and $|\mu| \leqslant \sigma$ implies

$$\|\mu(v-y)\| \leqslant \sigma(\|v\| + \|x\| + \|x-y\|)$$
$$\leqslant \sigma(\lambda + \alpha\|x\| + \alpha\|x-y\|)$$
$$\leqslant 3\sigma\rho < R(y) \leqslant r_b(y, f).$$

Hence:

$$M = (\sigma-1)^{-1} \sup\{ \| f(y+\mu(v-y)) \| ; v\in E, \| v \| = \lambda, | \mu | = \sigma \} < \infty.$$

Now applying [9] (lemma 1, § 6), we have

$$\left\| \hat{Q}_m - \frac{1}{m!} \hat{d}^m \tau_{k,f,y}(0) \right\| \leqslant \lambda^{-m} \sigma^{-k} M \quad \text{for all } m, k \in \mathbf{N}.$$

Therefore, by the linearity of each T_m and the hypothesis on a_m, we obtain:

$$\text{(D)} \qquad \left\| T_m \hat{Q}_m - \frac{1}{m!} T_m \hat{d}^m \tau_{k,f,y}(0) \right\| \leqslant \lambda^{-m} \sigma^{-k} a_m M.$$

So for all $k \in \mathbf{N}$:

$$\| T_0 f(x) - T_y f(x) \| = \| \textstyle\sum_{m=0}^{\infty} T_m \hat{Q}_m(x) - \sum_{m=0}^{\infty} T_m \hat{P}_m(x-y) \|$$

(using (C)):

$$\leqslant \| \textstyle\sum_{m=k+1}^{\infty} T_m \hat{Q}_m(x) \| + \left\| \sum_{m=0}^{k} T_m \hat{Q}_m(x) - \sum_{m=0}^{k} \frac{1}{m!} T_m \hat{d}^m \tau_{k,f,y}(0)(x) \right\|$$

$$+ \| \textstyle\sum_{m=0}^{k} T_m \hat{P}_m(x-y) - \sum_{m=0}^{\infty} T_m \hat{P}_m(x-y) \|$$

$$\leqslant \textstyle\sum_{m=k+1}^{\infty} \| T_m \hat{Q}_m \| . \| x \|^m + \sum_{m=0}^{k} \left\| T_m \hat{Q}_m - \frac{1}{m!} T_m \hat{d}^m \tau_{k,f,y}(0) \right\| . \| x \|^m$$

$$+ \textstyle\sum_{m=k+1}^{\infty} \| T_m \hat{P}_m \| . \| x-y \|^m$$

(using (D)):

$$\leqslant \textstyle\sum_{m=k+1}^{\infty} \| \hat{Q}_m \| a_m \| x \|^m$$

$$+ \textstyle\sum_{m=0}^{k} \frac{a_m}{\sigma^k \lambda^m} M \| x \|^m + \sum_{m=k+1}^{\infty} \| \hat{P}_m \| a_m \| x-y \|^m.$$

Since $\limsup_m (\| \hat{Q}_m \| a_m \| x \|^m)^{1/m} \leqslant (1/r_C(0,f)) \alpha \| x \| < 1$, it follows that the first (and similarly the third) term above tends to 0 as $k \to \infty$. The second term is dominated by $(M/\sigma^k) \sum_{m=0}^{\infty} (a_m/\alpha^m) (\alpha \| x \| / \lambda)^n$. Since:

$$\limsup_m \left(\frac{a_m}{\alpha^m} \left(\frac{\alpha \| x \|}{\lambda} \right)^m \right)^{1/m} \leqslant \frac{\alpha \| x \|}{\lambda} < 1$$

the series is convergent and since $\sigma > 1$ the second term also tends to 0 as $k \to \infty$. Therefore $T_0 f(x) = T_y f(x)$ and (B) is established.

It only remains now to consider the case $U = E$ and $R \equiv \infty$. Let $n \geqslant 1$ be an integer, and suppose $x \in G$ with $\| x \| \leqslant n$. Then:

$$\| Tf(x) \| = \left\| \sum_{m=0}^{\infty} T_m \frac{1}{m!} \hat{d}^m f(0)(x) \right\| \leqslant \sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m f(0) \right\| a_m n^m$$

for all $f \in \mathcal{H}_\theta(\infty, E, F) = \mathcal{H}_{\theta b}(E, F)$. The Cauchy estimates imply that

$$\left\| \frac{1}{m!} \hat{d}^m f(0) \right\| \leqslant (2\alpha n)^{-m} \sup_{z \in E, \| z \| \leqslant 2\alpha n} \| f(z) \| \quad \text{for all } m \in \mathbf{N}.$$

So $\| Tf(x) \| \leqslant \sigma \sup_{\| z \| \leqslant 2\alpha n} \| f(z) \|$ where $\sigma = \sum_{m=0}^{\infty} (2\alpha n)^{-m} (a_m n^m) < \infty$. Hence $\sup_{D_n} \| Tf \| \leqslant \sigma \sup_{B_n} \| f \|$ for all $f \in \mathcal{H}_{\theta b}(E, F)$ where

$$D_n = \{ x \in G ; \| x \| \leqslant n \} \qquad \text{and} \qquad B_n = \{ z \in E ; \| z \| \leqslant 2\alpha n \}.$$

Since $\{ D_n \}_{n \in \mathbf{N}}$ is a fundamental sequence of bounded subsets of G, and each B_n is bounded, we see that $Tf \in \mathcal{H}_{\theta b}(G, F)$ for all $f \in \mathcal{H}_{\theta b}(E, F)$ and that $T : \mathcal{H}_{\theta b}(E, F) \to \mathcal{H}_{\theta b}(G, F)$ is continuous. On the other hand, restriction to E is obviously a continuous left inverse for T, so T is a strict morphism and the proof is complete.

PROPOSITION 2.1. — *If E and F are Banach spaces, then there exists a sequence of continuous linear mappings $\{ \Psi_n : L(^nE, F) \to L(^nE'', F'') \}_{n \in \mathbf{N}}$ such that ;*

1° $\| \Psi_n \| = 1$ *for all $n \in \mathbf{N}$;*

2° $\Psi_k(A y^{n-k}) = (\Psi_n A) y^{n-k}$ *for all $0 \leqslant k < n$, all $y \in E$ and all $A \in L(^nE, F)$, and*

3° $\Psi_k^\theta : L_\theta(^nE, F) \to L_\theta(^nE'', F) \subset L(^nE, F'')$ *for all $k \in \mathbf{N}$ where Ψ_k^θ is the restriction of Ψ_k to the subspace $L_\theta(^kE, F)$, and $\theta = K$ or WK.*

Proof. — First we assume that the proposition is true when $F = \mathbf{C}$. Then the mappings $\{ \Psi_n : L(^nE, \mathbf{C}) \to L(^nE'', \mathbf{C}) \}_{n \in \mathbf{N}}$ induce n-linear mappings $\zeta_n : (E'')^n \to (L(^nE, \mathbf{C}))'$ defined by

$$\zeta_n(x_1'', \ldots, x_n'') : \quad B \in L(^nE, \mathbf{C}) \mapsto (\Psi_n B)(x_1'', \ldots, x_n'') \in \mathbf{C}$$

for all $(x_1'', \ldots, x_n'') \in (E'')^n$ and $n \in \mathbf{N}$. The fact that $\{ \Psi_n \}_n$ satisfies property 2° implies:

$$(a) \qquad \zeta_k(x_1, \ldots, x_k)(B y^{n-k}) = \zeta_n y^{n-k}(x_1, \ldots, x_k)(B)$$

for all $k, n \in \mathbf{N}$, $0 \leqslant k < n$, $B \in L(^nE, \mathbf{C})$, $y \in E'$ and $x_1, \ldots, x_k \in E''$, and property 1° implies $\| \zeta_n \| = 1$.

Now we let F be any Banach space and use the sequence of mappings $\{ \zeta_n \}_{n \in \mathbf{N}}$ to construct mappings $\{ \Psi_n : L({}^nE, F) \to L({}^nE'', F'') \}_{n \in \mathbf{N}}$ satisfying the proposition. For fixed $n \in \mathbf{N}$, we identify (isometrically) $L({}^nE, F)$ with $L(\bigotimes_{i=1}^n E, F)$ using the projective tensor product topology. Under this identification $A \in L_\theta({}^nE, F)$ if, and only if, the associated mapping $A : \bigotimes_{i=1}^n E \to F$ is a compact (resp. weakly compact) mapping where $\theta = K$ (resp. $\theta = WK$). A'', the double transpose of a mapping $A \in L(\bigotimes {}^nE, F)$, is a continuous linear mapping from $(\bigotimes {}^nE)'' \simeq L({}^nE, \mathbf{C})'$ into F'' and if A is compact (resp. weakly compact) then A'' is also and has range in F (*see* [5]). Define Ψ_n by

$$\Psi_n : \quad A \in L({}^nE, F) \mapsto A'' \circ \zeta_n \in L({}^nE'', F''), \quad \text{for all } n \in \mathbf{N}.$$

$\| \Psi_n \| \leqslant 1$ since $\| A \| = \| A'' \|$ and $\| \zeta_n \| = 1$, and since the composition of a (weakly) compact mapping with the continuous n-linear mapping ζ_n is (weakly) compact, we have that $\Psi_n A \in L_\theta({}^nE'', F)$ whenever $A \in L_\theta({}^nE, F)$, where $\theta = K$ or WK.

For each $y \in E$, n, $k \in \mathbf{N}$ satisfying $0 \leqslant k \leqslant n$ and $A \in L({}^nE, F)$ we have have that $\varphi \circ A y^{n-k} \in L({}^kE, \mathbf{C})$ whenever $\varphi \in F'$, so by (a) when $(x_1, \ldots, x_k) \in (E'')^k$:

$$\zeta_k(x_1, \ldots, x_k)(\varphi \circ A y^{n-k}) = \zeta_n y^{n-k}(x_1, \ldots, x_k)(\varphi \circ A).$$

That is

$$\zeta_k(x_1, \ldots, x_k) \circ (A y^{n-k})' = \zeta_n y^{n-k}(x_1, \ldots, x_k) \circ A'.$$

Hence

$$\begin{aligned} &\Psi_k(A y^{n-k})(x_1, \ldots, x_k) \\ &\quad \equiv (A y^{n-k})''(\zeta_k(x_1, \ldots, x_k)) = A''(\zeta_n y^{n-k}(x_1, \ldots, x_k)) \\ &\quad \equiv (\Psi_n A) y^{n-k}(x_1, \ldots, x_k). \end{aligned}$$

Thus condition 2° holds for the sequence $\{ \Psi_n \}_n$. In particular, when $k = 0$, we see that the $\{ \Psi_n \}$ are extension mappings. Thus $\| \Psi_n \| = 1$ so 1° is satisfied, and we have already seen that 3° holds. Therefore, we need only consider the case $F = \mathbf{C}$.

First we notice that for all $n \in \mathbf{N}$, $L({}^nE'', \mathbf{C}) \simeq (\bigotimes_{i=1}^n E'')'$ is a conjugate space so, by example 1 of section 1, there is a natural norm 1 projection ρ_n of $L({}^nE'', \mathbf{C})''$ onto $L({}^nE'', \mathbf{C})$. Let Φ_n be the mapping

$$\Phi_n : \quad B \in L(E, L({}^nE'', \mathbf{C})) \mapsto \rho_n \circ B'' \in L(E'', L({}^nE'', \mathbf{C})),$$

where B'' is the double transpose, $B'' : E'' \to L(E'', \mathbf{C})''$. It is easy to see that $\| \Phi_n \| = 1$ and that Φ_n is an extension mapping.

Now we will define the sequence $\{ \Psi_n : L(^nE, \mathbf{C}) \to L(^nE'', \mathbf{C}) \}_{n \in \mathbf{N}}$ by induction. Let Ψ_0 be the identity on $L(^0E, \mathbf{C}) \equiv \mathbf{C} \equiv L(^0E'', \mathbf{C})$ and let $\Psi_1 = \Phi_0 : A \in L(E, \mathbf{C}) \mapsto A'' \in L(E'', \mathbf{C})$. Notice that

$$\Psi_0(A\,y^1) = A\,y = (\Psi_1 A)\,y^1$$

for all $y \in E$ since Φ_0 is an extension mapping. Suppose that for some $m \geqslant 1$, we have mappings $\Psi_n : L(^nE, \mathbf{C}) \to L(^nE'', \mathbf{C})$, $n = 0, 1, \ldots, m$, which satisfy 1° and 2° of the proposition. Let $\tilde{\Psi}_m$ be the mapping:

$$\tilde{\Psi}_m : \quad A \in L(^{m+1}E, \mathbf{C}) \mapsto \{ \tilde{\Psi}_m A : y \in E \mapsto \Psi_m(A\,y^1) \in L(^mE'', \mathbf{C}) \} \in L(E, L(^mE'', \mathbf{C})).$$

Clearly $\| \tilde{\Psi}_m \| = \| \Psi_m \| = 1$ so that

$$\Psi_{m+1} \equiv \Phi_m \circ \tilde{\Psi}_m : \quad L(^{m+1}E, \mathbf{C}) \to L(E'', L(^mE'', \mathbf{C})) \simeq L(^{m+1}E'', \mathbf{C})$$

is of norm 1 since the natural isomorphism is isometric. Using the fact that Φ_m is an extension mapping, it follows that, for all $A \in L(^{m+1}E, \mathbf{C})$ and $y \in E$,

$$(b) \qquad \Psi_m(A\,y^1) = (\tilde{\Psi}_m A)(y) = (\Phi_m \circ \tilde{\Psi}_m A)(y) = (\Psi_{m+1} A)\,y^1.$$

So using the fact that 2° holds for $n = m$ and (b), we have that for all $0 \leqslant k < m+1$ and all $y \in E$,

$$\Psi_k(A\,y^{m+1-k}) = \Psi_k((A\,y^1)\,y^{m-k}) = (\Psi_m(A\,y^1))\,y^{m-k}$$
$$= (\Psi_{m+1} A)\,y^1\,y^{m-k} = (\Psi_{m+1} A)\,y^{m+1-k}.$$

Therefore by induction there is a sequence of mappings $\{ \Psi_n \}_{n \in \mathbf{N}}$ satisfying 1° and 2°. We complete the proof by noting that every $\mathbf{C}$-valued continuous multilinear mapping is (weakly) compact, and thus 3° holds trivially.

COROLLARY 2.1. — *Let E be a Banach space, and F a $\mathscr{C}$-space. If $U \subset E$ is open and non-empty, and $f \in \mathscr{H}(U, F)$, then there exists a $W \subset E''$, open, and an $\tilde{f} \in \mathscr{H}(W, F)$ such that $U \subset W$ and $\tilde{f}|_U = f$. We may take W to be the set*

$$W = \{ x \in E'';\ e \| x - y \| < r_b(y, f) \text{ for some } y \in U \}.$$

Furthermore, there is a strict morphism $T : \mathscr{H}_b(E, F) \to \mathscr{H}_b(E'', F)$ such that $Tf|_E = f$ for all $f \in \mathscr{H}_b(E, F)$.

Proof. — The result follows immediately from the lemma once we set $R(y) = r_b(y, f)$ for all $y \in U$ and construct sequences

$$\{ T_n : \mathscr{P}(^nE, F) \to \mathscr{P}(^nE'', F) \}_{n \in \mathbf{N}} \qquad \text{and} \qquad \{ a_n \}_{n \in \mathbf{N}}$$

satisfying the conditions of the lemma with $\alpha = e$.

Let $n \in \mathbf{N}$ and let $\{\Psi_m\}_{m \in \mathbf{N}}$ be the sequence of proposition 2.1. It is easy to see that if $A \in L(^nE, F)$ is symmetric then $\Psi_n A \in L(^nE'', F'')$ is also symmetric. Thus the restriction of Ψ_n to the subspace $L_s(^nE, F)$ which we also denote by Ψ_n maps $L_s(^nE, F)$ into $L_s(^nE'', F'')$. Since F is a $\mathscr{C}$-space there is a projection $\pi : F'' \to F$. Let $\lambda = \|\pi\|$. For each $B \in L_s(^nE'', F'')$, $\pi \circ B \in L_s(^nE'', F)$ and the mapping $B \mapsto \pi \circ B$ is of norm λ.

For each Banach space X, the mapping $B \in L_s(^nX, F) \mapsto \hat{B} \in \mathscr{P}(^nX, F)$ is bijective with norm 1, and its inverse $\hat{B} \mapsto B$ is of norm $\leqslant n^n/n!$ (*see* [9]). Define $T_n : \mathscr{P}(^nE, F) \to \mathscr{P}(^nE'', F)$ for all $n \in \mathbf{N}$ by

$$T_n \hat{P} = \widehat{\pi \circ \Psi_n P} \quad \text{for all } \hat{P} \in \mathscr{P}(^nE, F).$$

Clearly $\|T_n\| \leqslant \lambda(n^n/n!)$. We also have that for each $m, k \in \mathbf{N}$, $0 \leqslant k < m$, each $y \in E$ and each $\hat{P} \in \mathscr{P}(^mE, F)$:

$$\begin{aligned} T_k\left(\frac{1}{k!}\hat{d}^k \hat{P}(y)\right) &= \widehat{\pi \circ \Psi_k \binom{m}{k} (P y^{m-k})} \\ &= \widehat{\binom{m}{k} \pi \circ (\Psi_m P)\, y^{m-k}} \\ &= \frac{1}{k!}\hat{d}^k (T_m \hat{P})(y). \end{aligned}$$

Finally we note that by Stirling's formula

$$\limsup_{n \in \mathbf{N}} \left(\lambda \frac{n^n}{n!}\right)^{1/n} = e < \infty.$$

Hence the mappings $\{T_n\}_{n \in \mathbf{N}}$ satisfy the lemma and the proof is complete.

COROLLARY 2.2. — *Let E and F be Banach spaces. If $U \subset E$ is open and non-empty and $f \in \mathscr{H}_\theta(U, F)$ then there exists a $W \subset E''$, open, and an $\tilde{f} \in \mathscr{H}_\theta(W, F)$ such that $U \subset W$ and $\tilde{f}|_U = f$, where $\theta = K$ (resp. $\theta = WK$). We may take W to be the set $W = \{x \in E''; e\|x-y\| < r_b(y, f)$ for some $y \in U\}$.*

Furthermore, there is a strict morphism $T : \mathscr{H}_{\theta b}(E, F) \to \mathscr{H}_{\theta b}(E'', F)$ such that $Tf|_E = f$ for all $f \in \mathscr{H}_{\theta b}(E, F)$, where $\theta = K$ (resp. $\theta = WK$).

Proof. — The proof is the same as for corollary 2.1, except that we define the sequence $\{ T_n : \mathscr{P}_\theta({}^nE, F) \to \mathscr{P}_\theta({}^nE'', F) \}_{n \in \mathbf{N}}$ by

$$T_n : \quad \hat{P} \in \mathscr{P}_\theta({}^nE, F) \mapsto \widehat{\Psi_n^\theta P} \in \mathscr{P}_\theta({}^nE'', F)$$

for all $n \in \mathbf{N}$ and $\theta = K$ or WK.

PROPOSITION 2.2. — *Let E and F be Banach spaces. If $U \subset E$ is open and non-empty and $f \in \mathscr{H}_C(U, F)$ then there exists an open set $W \subset E''$ and an $\tilde{f} \in \mathscr{H}_C(W, F)$ such that $U \subset W$ and $\tilde{f}|_U = f$. We may take W to be the set*

$$W = \{ x \in E'' ;\ \| x - y \| < r_b(y, f) \text{ for some } y \in U \}.$$

Furthermore, there is a strict morphism $T : \mathscr{H}_{Cb}(E, F) \to \mathscr{H}_{Cb}(E'', F)$ such that $Tf|_E = f$ for all $f \in \mathscr{H}_{Cb}(E, F)$.

Proof. — The subspace topology on E induced by the weak* topology $\sigma(E'', E)$ on E'' is just the weak topology on E. For each $n \in \mathbf{N}$, each $P \in \mathscr{P}_f({}^nE) \otimes F$ is obviously weakly continuous and, as E is $\sigma(E'', E)$ dense in E'', P has a unique extension $\tilde{P}$ by continuity. $\tilde{P} \in \mathscr{P}_f({}^nE'') \otimes F$ and the extension mapping $P \mapsto \tilde{P}$ is a linear isometry. Hence it can be extended to an isometry:

$$T_n : \quad \overline{\mathscr{P}_f({}^nE) \otimes F} \equiv \mathscr{P}_C({}^nE, F) \to \overline{\mathscr{P}_f({}^nE'') \otimes F} \equiv \mathscr{P}_C({}^nE'', F).$$

Let $\varphi \in E'$ and $x \in F$; then $\varphi^n \otimes x : y \in E \mapsto (\varphi(y))^n x \in F$ is an element of $\mathscr{P}_C({}^nE, F)$ and $T_n(\varphi^n \otimes x) = \tilde{\varphi}^n \otimes x$ where $\tilde{\varphi}$ is the extension of φ to E'' (by evaluation). Now, for all $y \in E$ and $k \in \mathbf{N}$, $0 \leqslant k < n$:

$$\frac{1}{k!} \hat{d}^k T_n(\varphi^n \otimes x)(y) = \binom{n}{k} (\tilde{\varphi}(y))^{n-k} \tilde{\varphi}^k \otimes x = T_k\left(\binom{n}{k} (\varphi(y))^{n-k} \varphi^k \otimes x \right)$$
$$= \frac{1}{k!} T_k \hat{d}^k (\varphi^n \otimes x)(y).$$

Since the operators T_k, T_n, and $P \mapsto (1/k!)\, \hat{d}^k P(y)$ are all continuous and linear and since the span of polynomials of the form $\varphi^n \otimes x$ is dense in $\mathscr{P}_C({}^nE, F)$ it follows that $(1/k!)\, \hat{d}^k T_n P(y) = (1/k!)\, T_k \hat{d}^k P(y)$ for all $n \in \mathbf{N}$, $y \in E$, $P \in \mathscr{P}_C({}^nE, F)$ and $k \in \mathbf{N}$, $0 \leqslant k < n$. Since $\| T_n \| = 1$ for all n, we complete the proof by applying the lemma.

Remark. — Corollary 2.2 could be applied to the case $\theta = C$, as well as to $\theta = K$, WK. However, the construction of proposition 2.2 allows extensions to larger open sets W in this case.

With the above results we may now easily prove the theorems of section 1.

Proof of theorem 1.1. – The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (7): Let $I : E \hookrightarrow E''$ be the canonical inclusion. Since $I \in \mathscr{H}(E, E'')$, (2) implies that there is an open set $W \subset G$, $E \subset W$, and a mapping $\tilde{I} \in \mathscr{H}(W, E'')$ extending I. Let $S = \hat{d}^1 \tilde{I}(0) \in L(G, E'')$. For each $x \in G$, $S(x)$ is then the directional derivative of $\tilde{I}$ at 0 in direction x. It is easy to check that $S|_E = I$ and so (7) is satisfied.

(5) $\Rightarrow$ (6): Each $\varphi \in E'$ belongs to $\mathscr{H}_{\theta b}(E, \mathbf{C})$ ($\theta = C, K$, or WK). Let T be the mapping of (5) and set $\tilde{\varphi} = \hat{d}^1 (T\varphi)(0) \in G'$. It is easy to see that the mapping $\varphi \in E' \mapsto \tilde{\varphi} \in G'$ satisfies (6).

(4) $\Rightarrow$ (6): $E' \subset \mathscr{H}_b(E, \mathbf{C})$ so we may reason as above.

(6) $\Rightarrow$ (7): Let $T : E' \to G'$ be a mapping satisfying (6), let $T' : G'' \to E''$ be its transpose and let $S : G \to E''$ be the restriction of T' to G. An elementary calculation shows that S satisfies (7).

(7) $\Rightarrow$ (1) and (3): Let S be a mapping satisfying (7) and let $U \subset E$, an open set, and $f \in \mathscr{H}(U, F)$ be given where F is a $\mathscr{C}$-space. By corollary 2.1, there is an open set $V \subset E''$ and an $f_1 \in \mathscr{H}(V, F)$ such that $U \subset V$ and $f_1|_U = f$, and there is a strict morphism $\tilde{T} : \mathscr{H}_b(E, F) \to \mathscr{H}_b(E'', F)$ such that $\tilde{T}g|_E = g$ for all $g \in \mathscr{H}_b(E, F)$. Let $W = S^{-1}(V)$ and let $\tilde{f} \equiv f_1 \circ S : W \to F$. As composition of a continuous linear mapping with a holomorphic mapping is holomorphic, we have $\tilde{f} \in \mathscr{H}(W, F)$ and it is trivial to see that (1) is satisfied. Similarly $T \equiv \tilde{T} \circ S : \mathscr{H}_b(E, F) \to \mathscr{H}_b(G, F)$ *is* a continuous linear mapping such that $Tg|_E = g$, for all $g \in \mathscr{H}_b(E, F)$. T is also a strict morphism as the restriction mapping $g \in \mathscr{H}_b(G, F) \to g|_E$ is a continuous left inverse for T, so (3) is satisfied.

(7) $\Rightarrow$ (8) and (5): We reason precisely as in the case above using corollary 2.2 for $\theta = K$ or WK and proposition 2.2 for $\theta = C$ in place of corollary 2.1 above. We need to note, however, that if $S : G \to E''$ is any continuous linear map and $P \in \mathscr{P}_\theta({}^n E'', F)$ then $P \circ S \in \mathscr{P}_\theta({}^n G, F)$ for all $n \in \mathbf{N}$, all Banach spaces F and $\theta = C, K$ or WK. Hence the composition of a continuous linear map with a holomorphic map of type θ is again holomorphic of type θ, ($\theta = C, K$, or WK).

It only remains to show that if E has the *bounded approximation property*, then (8) $\Rightarrow$ (7). By condition (8), if E has the *bounded approximation property* with constant C, and if $T : E \to F$ is a continuous linear finite rank operator, there is a holomorphic function $\tilde{T} : W \to F$ which extends T, where $E \subset W$, and W is open in G. Then $T_G = \hat{d}^1 \tilde{T}(0)$ is a linear extension of T to G, and as in [7] for some $m > 0$, independent of T and F, $\| T_G \| \leqslant m \| T \|$. Now, let $H \subset E$ be a finite dimensional subspace and let $\varepsilon > 0$. Choose $T \in E' \otimes E$, $\| T \| \leqslant C$, such that

$$\| Th - h \| < \varepsilon (h \in H, \| h \| \leqslant 1),$$

and let T_G be a linear extension of T with $\| T_G \| \leqslant m \| T \| \leqslant m\, C$. Thus, for each pair (H, ε), we obtain linear mappings T and T_G, as described above. Partially order the collection of such pairs by $(H, \varepsilon) > (H', \varepsilon')$ if $H \supset H'$ and $\varepsilon \leqslant \varepsilon'$, and consider the compact set $X = \prod_{y \in G} B_{E''}(0, m\, C \| y \|)$, where each closed ball $B_{E''}(0, m\, C \| y \|)$ has the $\sigma(E'', E')$ topology. For each (H, ε), we get a point $\theta_{(H, \varepsilon)}$ in X, given by $\theta_{(H, \varepsilon)}(y) = T_G(y)$. An argument similar to that described in [7] yields that the linear operator S corresponding to a limit point of the net $\{ \theta_{(H, \varepsilon)} \}$ is an extension of $\mathrm{id}\,|_E$ proving condition (7).

Q.E.D.

Proof of theorem 1.2. – Let E, G and F be real Banach spaces and let $E_{\mathbf{C}}$, $G_{\mathbf{C}}$ and $F_{\mathbf{C}}$ denote their complexifications: $E_{\mathbf{C}} = E \otimes_{\mathbf{R}} \mathbf{C}$, etc.

Let $U \subset E$ be open and non-empty, and let $f \in \mathscr{A}(U, F)$. f may be continued analytically to an open subset $U_f \subset E_{\mathbf{C}}$, $U \subset U_f$, and its analytic continuation $f_{\mathbf{C}}$ is an element of $\mathscr{H}(U_f, F_{\mathbf{C}})$. In particular, if $U = E$ and $f \in \mathscr{A}_b(E, F)$, then $E_f = E_{\mathbf{C}}$ and $f_{\mathbf{C}} \in \mathscr{H}_b(E_{\mathbf{C}}, F_{\mathbf{C}})$. Furthermore, using a result in [9], if $f \in \mathscr{A}_\theta(U, F)$ then $f_{\mathbf{C}} \in \mathscr{H}_\theta(U_f, F_{\mathbf{C}})$ when $\theta = C$, K, or WK.

Now suppose the mapping $S : G \to E''$ satisfies (7) of theorem 1.2. Then $S_{\mathbf{C}} : x + iy \in G_{\mathbf{C}} \mapsto S(x) + i\, S(y) \in E''_{\mathbf{C}}$ satisfies (7) of theorem 1.1. Hence if F is a $\mathscr{C}$-space there is a $W \subset G_{\mathbf{C}}$ open and an $\tilde{f}_{\mathbf{C}} \in \mathscr{H}(W, F_{\mathbf{C}})$ such that $U_f \subset W$ and $\tilde{f}_{\mathbf{C}}|_W = f_{\mathbf{C}}$. Let $V = W \cap G$ and let $\tilde{f} = \rho \circ \tilde{f}_{\mathbf{C}}$ where $\rho : F_{\mathbf{C}} \to F$ is the projection onto the "real part" of $F_{\mathbf{C}}$. Then $U \subset V$, V is open in G, $\tilde{f} \in \mathscr{A}(V, F)$ and $\tilde{f}|_U = f$. Hence (7) $\Rightarrow$ (1). If $T : \mathscr{H}_b(E_{\mathbf{C}}, F_{\mathbf{C}}) \to \mathscr{H}_b(G_{\mathbf{C}}, F_{\mathbf{C}})$ is the mapping of theorem 1.1 (3), with F a $\mathscr{C}$-space, then

$$T_{\mathbf{R}} : \; f \in \mathscr{A}_b(E, F) \to (\rho \circ T f_{\mathbf{C}})|_G \in \mathscr{A}_b(G; F)$$

satisfies (3). The implications (7) $\Rightarrow$ (5) and (7) $\Rightarrow$ (8) are proved similarly.

For all the remaining implications, we may argue as in the proof of theorem 1.1.

Remarks. – A particular case of theorem 1.1 gives conditions in which every element P of $\mathscr{P}(^n E; \mathbf{C})$ extends to an element $\tilde{P}$ of $\mathscr{P}(^n G; \mathbf{C})$. Unlike the linear case, it is not in general possible to ensure that $\| P \| = \| \tilde{P} \|$ for $n > 1$. We are grateful to R. M. SCHOTTENLOHER for allowing us to use the following example, illustrating this. Let $G = \mathbf{C}^3$, with the supremum norm, and let $E = \{ (x, y, z) \in G;\ x+y+z = 0 \}$ with basis vectors $v = (1, -1, 0)$ and $w = (1, 0, -1)$. Let $P \in \mathscr{P}(^2E; \mathbf{C})$ be defined by $P(\alpha v + \beta w) = \alpha^2 + \alpha\beta + \beta^2$. Then $\| P \| = 1$, but $\| \tilde{P} \| > 1$ for every extension $\tilde{P}$ of P to $\mathscr{P}(^2G; \mathbf{C})$. In fact, if $(x, y, z) \in E$, $\| (x, y, z) \|_\infty = 1$ with, say, $| x | = 1$, then

$$| P(x, y, z) | = | y^2 + yz + z^2 | = | y(y+z) + z^2 | = | P(\lambda x, \lambda y, \lambda z) |$$

for any complex number λ of modulus 1. Thus, $| P(x, y, z) | = | y + z^2 |$, where we may assume that $y+z = 1$, and it is routine to verify that $| y + z^2 | \leqslant 1$ in this case. The cases where $| y | = 1$ and $| z | = 1$ are treated in exactly the same manner. Now, let $u = (1, 0, 0)$, and let $\tilde{P} \in \mathscr{P}(^2G; \mathbf{C})$ be any extension of P,

$$\tilde{P}(\alpha v + \beta w + \gamma u) = \alpha^2 + \alpha\beta + \beta^2 + \delta_1 \gamma^2 + \delta_2 \alpha\gamma + \delta_3 \beta\gamma.$$

If $\| \tilde{P} \| = 1$, then $| \tilde{P}(1, 1, -1) |$, $| \tilde{P}(1, -1, 1) |$, $| \tilde{P}(i, -1, 1) |$, and $| \tilde{P}(i.1, -1) |$ are all no bigger than 1. The first two inequalities imply $| 1 + \delta_1 | \leqslant 1$ while the last two imply $| 1 - \delta_1 | \leqslant 1$. Hence $\delta_1 = 0$. Finally, $| \tilde{P}(1, 1, 1) | \leqslant 1$ and $| \tilde{P}(-1, 1, 1) | \leqslant 1$ respectively yield that $| 1 - (\delta_2 + \delta_3) | \leqslant 1/3$ and $| 3 - (\delta_2 + \delta_3) | \leqslant 1$, which is impossible. Thus, the norm of $\tilde{P}$ must be strictly larger than 1.

Theorem 1.1 characterizes pairs of spaces $E \subset G$ for which extensions of many vector valued holomorphic extensions exist (conditions (1), (3), (5) and (8)) and for which scalar valued extensions exist and are given by a continuous linear extension mapping (conditions (4) and (5)). The question remains whether given a pair of spaces $E \subset G$ not satisfying say (6) or (7), do extensions of scalar valued holomorphic functions exist, though not given by a continuous linear extension mapping? In general, the answer is no as illustrated in the following example. However, if we restrict the class of holomorphic functions to a sufficiently small class (nuclear bounded type described below) then the answer is yes for all pairs $E \subset G$ (theorem 2.1) and we may even allow vector values.

Example 2.1. — Let p be an integer $\geqslant 2$ and let $E = (L_p(\mu), \| \ \|_p)$. Then E can be isometrically embedded in a space G having the polynomial Dunford-Pettis Property (PDP). For example (*see* [12]), we may take $G = C(K)$, $K =$ the weakly compact unit ball of E'. When E is infinite dimensional, the function $P : f \in E \longrightarrow \int f^p \, d\mu$ is a p-homogeneous continuous polynomial on E which cannot be extended as a holomorphic function to any open set in G containing E.

Indeed, if $W \subset G$ were open, $E \subset W$, and there existed a $g \in \mathscr{H}(W, \mathbf{C})$ such that $g|_E = P$, then $\tilde{P} \equiv (1/p!)\, \hat{d}^p g(0)$ would be an element of $\mathscr{P}({}^pG, \mathbf{C})$ satisfying $\tilde{P}|_E = P$. But G has PDP, so $\tilde{P}$ would be weakly continuous, hence P would be weakly continuous on E which is clearly not the case.

BOLAND showed in [2] that every entire function defined on a closed subspace of the dual of a Fréchet-nuclear (DFN) space has an extension to an entire function on the larger space. Entire functions defined on a DFN space correspond in form to the "nuclear bounded type" of Banach spaces. Motivated by this fact we will prove a similar result in the class of Banach spaces for entire functions of "nuclear bounded type".

DEFINITION. — *Let E and F be Banach spaces and let $n \in \mathbf{N}$. The* nuclear norm $\| \ \|_N$ *is defined on $\mathscr{P}_f({}^nE) \otimes F$ by*

$$\|P\|_N \equiv \inf\{\textstyle\sum_{i=1}^r \|\varphi_i\|^n \|x_i\|;\ r \in \mathbf{N}, \text{ and } P = \sum_{i=1}^r \varphi_i^n \otimes x_i;\ \varphi_i \in E',\ x_i \in F\}.$$

The completion of $(\mathscr{P}_f({}^nE) \otimes F, \| \ \|_N)$ is denoted by $\mathscr{P}_N({}^nE, F)$ and can be identified algebraically with a subspace of $\mathscr{P}({}^nE, F)$.

$\mathscr{H}_{Nb}(E, F)$, *the space of* entire mappings of nuclear bounded type from E into F *is defined as follows: $f \in \mathscr{H}_{Nb}(E, F)$ if, and only if, $f \in \mathscr{H}(E, F)$, $\hat{d}^k f(0) \in \mathscr{P}_N({}^kE, F)$ for all $k \in \mathbf{N}$, and* $\limsup_{k \in \mathbf{N}} (\|(1/k!)\, \hat{d}^k f(0)\|_N)^{1/k} = 0$.

The sequence of norms

$$\| \ \|_m : \ f \in \mathscr{H}_{Nb}(E, F) \mapsto \textstyle\sum_{k=0}^{\infty} m^k \|(1/k!)\, \hat{d}^k f(0)\|_N$$

defines a Fréchet topology on $\mathscr{H}_{Nb}(E, F)$.

The proof of the following theorem is modelled on the method used by BOLAND [2].

THEOREM 2.1. — *Let G and F be Banach spaces and let E be a closed subspace of G. Then the restriction mapping:*

$$R : \ f \in \mathscr{H}_{Nb}(G, F) \mapsto f|_E \in \mathscr{H}_{Nb}(E, F)$$

is a strict morphism onto $\mathscr{H}_{Nb}(E, F)$.

Proof. — Let $\alpha > 0$, $n \in \mathbf{N}$ and let $B_\alpha = \{P \in \mathscr{P}_f({}^nE) \otimes F;\ \|P\|_N < \alpha\}$. For each $P \in B_\alpha$ there exists $r \in \mathbf{N}$, $\varphi_1, \ldots, \varphi_r \in E'$ and $x_1, \ldots, x_r \in F$ such that $P = \sum_{i=1}^r \varphi_i^n \otimes x_i$. By the Hahn-Banach theorem, there exists

$\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r \in G'$ such that $\tilde{\varphi}_i |_E = \varphi_i$ and $\| \tilde{\varphi}_i \| = \| \varphi_i \|$, $i = 1, \ldots, r$. It follows that there exists a $\tilde{P} \in \mathscr{P}_f(^nG) \otimes F$ such that $\tilde{P}|_E = P$ and $\| \tilde{P} \|_N = \| P \|_N$. Let

$$D_\alpha = \{ P \in \mathscr{P}_N(^nG, F);\ \| P \|_N < \alpha \}.$$

Then $R(D_\alpha) \supset B_\alpha$, so

$$\overline{R(D_\alpha)} \supset \{ P \in \mathscr{P}_N(^nE, F);\ \| P \|_N \leqslant \alpha \} = \bar{B}_\alpha.$$

It follows from the continuity of R and the completeness of $\mathscr{P}_N(^nE, F)$ (*see* [14], p. 76) that $R(D_{\alpha+\varepsilon}) \supset \bar{B}_\alpha$ for all α, $\varepsilon > 0$. Let $f \in \mathscr{H}_{Nb}(E, F)$ and let $\sum_{n=0}^{\infty} P_n$ be the Taylor series of f at 0. Then, for each $n \in \mathbf{N}$ there exists a $\tilde{P}_n \in \mathscr{P}_N(^nG, F)$ such that $\tilde{P}_n |_E = P_n$ and $\| \tilde{P}_n \|_N < \| P_n \|_N + n^{-n}$. Let $\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{P}_n(x)$. Since

$$\limsup_n (\| \tilde{P}_n \|_N)^{1/n} \leqslant \limsup_n \left(\| P_n \|_N^{1/n} + \frac{1}{n} \right) = 0,$$

it follows that $\tilde{f}(x)$ is defined for all $x \in G$, that $\tilde{f} \in \mathscr{H}_{Nb}(G, F)$ and that $\tilde{f}|_E = f$. Hence $R : \mathscr{H}_{Nb}(G, F) \to \mathscr{H}_{Nb}(E, F)$ is onto. It is also a continuous map between Fréchet spaces so it follows that R is a strict morphism onto.

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(Texte reçu le 16 mars 1977.)

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