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TWO REMARKS IN PRIME NUMBER THEORY

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RÉSUMÉ. — Nous considérons deux problèmes de théorie additive des nombres premiers. Le premier concerne une inégalité dont un cas particulier est le suivant : min $| p-q \alpha^a |$ (quand α est un nombre réel fixe plus grand que 1 ; p et q sont des nombres premiers, et le minimum est pris sur l'ensemble des entiers positifs inférieurs ou égaux à $4/\varepsilon$, où ε est un nombre fixe avec $0 < \varepsilon < 1/2$ est inférieur à p^ε pour une infinité de couples (p, q). Le second résultat montre que si $\theta > 7/72$, l'intervalle X, $X+X^\theta$ contient au moins $\gg X^\theta$ nombres distincts qui sont sommes de deux nombres premiers impairs.

SUMMARY. — Two questions in additive prime number theory are considered. First is an inequality of which a special case is this min $| p-q \alpha^a |$ (where α is a fixed real number exceeding 1 ; p, q are primes, and the minimum is over all positive integers a not exceeding $4/\varepsilon$ [$0 < \varepsilon < 1/2$, ε fixed)] is less than p^ε for an infinity of prime pairs (p, q). The second result is that if $\theta > 7/72$ then the interval X, $X+X^\theta$ containing at least $\gg X^\theta$ distinct numbers which are expressible as a sum of two odd primes.

1. Introduction

In this note, we consider two questions of an additive nature on prime numbers. Our results are as follows.

THEOREM 1. — *Let ε be a positive constant less than* 1, *and let N be any natural number exceeding* $2\,\varepsilon^{-1}$. *Let $\alpha_1, \ldots, \alpha_N$ be any given positive real numbers no two of which are equal. Then there exist two of the numbers α_j say β and γ such that the inequality*

$$|\beta p - \gamma q| < p^\varepsilon,$$

where p and q are required to be prime numbers has infinity of solutions in p, q.

By choosing $\alpha_1, \ldots, \alpha_N$ appropriately, we get the following corollary.

COROLLARY. — *Let α be a real number exceeding* 1 *and ε as before. Then there exists a natural number a satisfying* $1 \leqq a < 1+[2\,\varepsilon^{-1}]$, *such that the inequality*

$$|p - q\,\alpha^a| < p^\varepsilon,$$

where p and q are required to be prime numbers has infinity of solutions in p, q.

In other words, the inequality

$$\min_{a=1,2,\ldots,[2\varepsilon^{-1}]}\left|\alpha-\left(\frac{p}{q}\right)^{1/a}\right|<p^{-1+\varepsilon}$$

has infinity of solutions in prime pairs p, q.

THEOREM 2. — *Let* θ *be a constant exceeding* 7/72, $X > X_0(\theta)$ *and* $h = X^{\theta}$. *Let G denote the number of Goldbach numbers (a natural number n is said to be Golbach if there exist two odd prime numbers whose sum is n) in the interval* $[X, X+h]$. *Then G exceeds ch where c is an absolute positive constant.*

The weaker version $G \geqslant 1$ of theorem 2 is due to H. L. MONTGOMERY and R. C. VAUGHAN and was communicated to me by Professor H. L. MONTGOMERY in a letter to me a few years back, and is now published [2]. I am very much indebted to Professor H. L. MONTGOMERY both for his letter and for his preprint.

2. Proof of theorem 1

Let $H = X^{\varepsilon}$ and for $X \leqslant x \leqslant 2X$ and any positive constant α, put

$$f(\alpha, x) = \vartheta\left(\frac{x+[\alpha H]}{\alpha}\right)-\vartheta\left(\frac{x}{\alpha}\right),$$

where, for positive real u, we have written $\vartheta(u) = \sum_{p\leqslant u}\log p$. A simple application of the prime number theorem shows that

$$\sum_{X\leqslant n\leqslant 2X} f(\alpha, n) = HX(1+0(1))$$

and so

$$\sum_{X\leqslant n\leqslant 2X}\sum_{j=1}^{N} f(\alpha, n) = NXH(1+o(1)).$$

From this equality it follows that, for some integer n satisfying $X \leqslant n \leqslant 2X$,

$$\sum_{j=1}^{N} f(\alpha_j, n) \geqslant NH(1+o(1))$$

(it may be remarked that here actually equality holds for some n). In view of the inequality (note that $N > 2\varepsilon^{-1}$),

$$\pi(x)-\pi(x-y) \leqslant \frac{2y}{\log y}\left(1+\frac{8}{\log y}\right)$$

(this is a consequence of Selberg sieve ; the first result in this direction is due to G. H. HARDY and J. E. LITTLEWOOD who obtained a bigger constant in place of 2 by the use of the sieve method of V. BRUN) valid for all x, y satisfying $1 < y \leqslant x$ (*see* page 107 of [1]), it follows that there exist k_1 k_2 $(k_1 \neq k_2)$ for which $f(\alpha_{k_1}, n) \neq 0$ and $f(\alpha_{k_2}, n) \neq 0$. From these, it follows that there exist primes p, q satisfying

$$\frac{n}{\alpha_{k_1}} \leqslant p \leqslant \frac{n+[\alpha_{k_1} H]}{\alpha_{k_1}}, \qquad \frac{n}{\alpha_{k_2}} \leqslant q \leqslant \frac{n+[\alpha_{k_2} H]}{\alpha_{k_2}}$$

(note that $X \leqslant n \leqslant 2X$) and so

$$|p\alpha_{k_1} - q\alpha_{k_2}| \leqslant (\alpha_{k_1}+\alpha_{k_2}) H.$$

This is true for all X and, in particular, for $X = 2^M$ $(M = 1, 2, 3, \ldots)$. The pair (k_1, k_2) depends on M, but there are only finitely many $(\leqslant N^2)$ pairs and so there exists some pair say (1, 2) for simplicity, which is the same for an infinite subsequence of integers M. This proves Theorem 1.

3. Proof of Theorem 2

For integral h, x, and Y with $h \leqslant x^\varepsilon$ and $x^\varepsilon \leqslant Y \leqslant x/3$, consider the sum

$$\begin{aligned} S &= \textstyle\sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y))(\vartheta(y+h) - \vartheta(y)), \\ &= \textstyle\sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y))\,((\vartheta(y+h) - \vartheta(y) - h) + h), \\ &= h \textstyle\sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y)), \\ &\quad + O(\max_{Y \leqslant y \leqslant 2Y} (\vartheta(x+h-y) - \vartheta(x-y))\, Y^{1/2} \\ &\qquad \times (\textstyle\sum_{Y \leqslant y \leqslant 2Y} (\vartheta(y+h) - \vartheta(y) - h)^2)^{1/2}). \end{aligned}$$

The O-term is easily proved to be $O(h^2\, Y \exp(-(\log x)\, 1/6))$ provided $h \geqslant Y^{(1/6)+\varepsilon}$ (these results are due to A. SELBERG and M. N. HUXLEY, *see* [3]). The main term is easily seen to be

$$h \textstyle\sum_{x-2Y \leqslant n \leqslant x-2Y+h-1} (\vartheta(n+Y) - \vartheta(n)),$$

which is asymptotic to $h^2\, Y$ provided that $Y \geqslant x\,(7/12)+\varepsilon$ (these results are due to A. E. INGHAM and M. N. HUXLEY, *see* [3]). Thus we have following result.

LEMMA 1. — *If h, x, Y are integers with $Y \geqslant h \geqslant Y^{1/6+\varepsilon}$ $Y \geqslant x^{7/12+\varepsilon}$ and $Y \leqslant x/3$, then*

$$\sum_{y=Y}^{2Y}(\vartheta(x+h-y)-\vartheta(x-y))(\vartheta(y+h)-\vartheta(y)) = h^2\, Y(1+o(1)).$$

Next we record the following lemma.

LEMMA 2. — *There exists, under the conditions of lemma 1, an integer y_0 satisfying $Y \leqslant y_0 \leqslant 2\,Y$ such that*

$$(\vartheta(x+h-y_0)-\vartheta(x-y_0))(\vartheta(y_0+h)-\vartheta(y_0)) \geqslant h^2(1+o(1))$$

(*actually equality may be secured for a suitable y_0*).

Proof. — Follows from lemma 1.

LEMMA 3. — *Let $r(n)$ denote the number of solutions of the equation $n = p_1 + p_2$ where p_1 and p_2 are prime numbers satisfying $x - y_0 \leqslant p_1 \leqslant x+h-y_0$ and $y_0 \leqslant p_2 \leqslant y_0+h$. Then*

$$(\log x)^2 \sum_{n=x}^{x+2h} r(n) \geqslant h^2(1+o(1)).$$

Proof. — Follows from lemma 2.

LEMMA 4. — *We have*

$$r(n) \leqslant 16\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \times \prod_{2<p\mid n}\left(1+\frac{1}{p-2}\right)\frac{h}{(\log h)^2}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)$$

where the constant implied by the O-symbol is absolute.

Proof. — This is corollary 5.8.3 on page 179 of [1].

LEMMA 5. — *We have*

$$\sum_{n=x}^{x+2h}\prod_{2<p\mid n}\left(1+\frac{1}{p-2}\right)^2 \leqslant 2h(1+o(1))\prod_{p>2}\left(1+\frac{2p-3}{p(p-2)^2}\right).$$

Proof. — In the product on the left, the contributions from the primes $p > \log x$ are negligible, and so we may restrict to those $p \leqslant \log x$. Accordingly the left side is

$$< 2h+1+\sum_r \sum_{x\leqslant p_1 \ldots p_2 \leqslant x+2h}\prod_{j=1}^{r}(2(p_j-2)^{-1}+(p_j-2)^{-2}),$$

with $2 < p_1 < p_2 < \ldots < p_r \leqslant \log x$,

$$< 2h + \sum_r \left(\frac{2h}{p_1 \cdots p_r} + 2 \right) \prod_{j=1}^{r} (2(p_j - 2)^{-1} + (p_j - 2)^2)$$

with $2 < p_1 < p_2 < \ldots < p_r \leqslant \log x$, and this proves lemma 5.

We now fix $h = [Y^{(1/6)+\varepsilon}]$, $Y = [x^{(7/12)+\varepsilon}]$ and apply Hölder's inequality to the inequality of lemma 3 and use lemmas 4 and 5. We see that theorem 2 is proved with any positive constant c satisfying

$$1/2 > 16.\left(\frac{72}{5}\right)^2 c^{1/2} \prod_{p>2} \left((1 - (p-1)^{-2}) \left(1 + \frac{2p-3}{p(p-2)^2} \right) \right).$$

REFERENCES

[1] Halberstam (H.) and Richert (H. E.). — *Sieve Methods.* — London, New York, Academic Press, 1974 (*London mathematical Society Monographs*, 4).

[2] Montgomery (H. L.) and Vaughan (R. C.). — The exceptional set in Goldbach's problem, *Acta Arithm.*, Warszawa, t. 27, 1975, p. 353-370.

[3] Ramachandra (K.). — Some problems of analytic number theory, *Acta Arithm.*, Warszawa, t. 31, 1976, p. 313-324.

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