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COMMUTATIVE SEMIGROUPS WHOSE LATTICE OF CONGRUENCES IS A CHAIN (*)

BY

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1. Introduction.

The structure of Γ-semigroups, the semigroups whose subsemigroups form a chain, was completely determined by the author [8], or afterwards by ŠEVRIN [5]. Analogously to this we present a problem : What are the semigroups whose congruence relations form a chain ? For convenience we give those a terminology :

DEFINITION. — *A semigroup S is called a Δ-semigroups if and only if the lattice of all congruences on S is a chain with respect to inclusion relation, that is, if ρ and σ are congruences on S, then exactly one of the following three holds·*

$$\rho \subset \sigma, \qquad \rho = \sigma, \qquad \sigma \subset \rho \quad (^1).$$

Basic examples of Δ-semigroups are all semigroups of order 2 and indecomposable semigroups [10], namely semigroups without proper congruences.

In this paper, we will study the structure of commutative Δ-semigroups, and eventually we shall have two important classes of commutative Δ-semigroups : quasicyclic groups and commutative nil-semigroups satisfying the divisibility chain condition. The first class will be equivalent to groups which are Γ-semigroups, the second one will be obtained by using the theory of structure of commutative archimedean semigroups, and it will be reduced to naturally totally ordered commutative archimedean semigroups with zero.

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(1) Throughout this paper, the notation « $\rho \subset \sigma$ » means « $\rho \subseteq \sigma$, but $\rho \neq \sigma$ ».

2. Basic Results.

In this section we state the basic results on (commutative) Δ-semigroups.

LEMMA 1. — *If S is a Δ-semigroup, then all the ideals of S form a chain, hence all the principal ideals of S form a chain.*

Proof. — If S is a Δ-semigroup, all Rees-congruences on S form a chain. Let ρ and σ be Rees-congruences modulo ideals I and J respectively [2]. Then $\rho \subset \sigma$ if and only if $I \subset J$. Therefore all the ideals, hence principal ideals, form a chain.

LEMMA 2. — *Every homomorphic image of a Δ-semigroup is a Δ-semigroup.*

Proof. — Let S be a semigroup and S' be a homomorphic image of S. Let f be the homomorphism $S \to S'$. Let ρ be the congruence on S induced by f. There is a one-to-one correspondence between the set of all congruences σ on S containing ρ and the set of all congruences σ' on S' in the following way :

$$x \sigma y \iff f(x) \sigma' f(y)$$

and $\rho \subseteq \sigma_1 \subset \sigma_2$ if and only if $\sigma'_1 \subset \sigma'_2$. Therefore if S is a Δ-semigroup then S/ρ, hence S' is a Δ-semigroup.

It is well known that any semigroup has a smallest semilattice-congruence [2], [9]. It is a natural way to consider the greatest semilattice-homomorphic image L (induced by the smallest semilattice congruence) of a Δ-semigroup S. By Lemma 2, L is also a Δ-semigroup.

LEMMA 3. — *A semilattice is a Δ-semigroup if and only if it is of order $\leqq 2$.*

Proof. — Let L be a semilattice of order $\geqq 2$. As usual we define x, $y \in L$, $x \leqq y$ by $x = yz$ for some $z \in L$. Let a, b be distinct elements of L and let

$$I_a = \{ x ; x \leqq a \}, \qquad I_b = \{ x ; x \leqq b \}.$$

Then I_a and I_b are ideals of L. Let ρ_a and ρ_b denote the Rees-congruences modulo the ideals I_a and I_b respectively. Since $I_a \neq I_b$, $\rho_a \neq \rho_b$. Suppose L is a Δ-semigroup. Then either $\rho_a \subset \rho_b$ or $\rho_b \subset \rho_a$. Hence either $I_a \subset I_b$ or $I_b \subset I_a$. For the first case, $a \in I_b$ namely $a < b$ (2); for the second $b \in I_a$, namely $b < a$. Therefore L is a chain.

(2) $a < b$ means $a \leqq b$, but $a \neq b$.

Suppose L is a chain containing at least three elements a, b, c, say $a < b < c$. Let

$$I^+ = \{ x; x \geqq b \}, \qquad I^- = \{ x; x \leqq b \},$$

where I^- is an ideal of L. We define congruences ρ^+ and ρ^- on L as follows :

$$x \rho^+ y \quad \text{if and only if either } x,\ y \in I^+ \quad \text{or} \quad x = y,$$
$$x \rho^- y \quad \text{if and only if either } x,\ y \in I^- \quad \text{or} \quad x = y.$$

Clearly, ρ^- is the Rees-congruence modulo I^-. It is obvious that ρ^+ is an equivalence, we may only show that $x \rho^+ y$ implies $xz \rho^+ yz$ for all $z \in L$. We assume x, $y \in I^+$. If $z \notin I^+$, then $xz = z = yz$. If $z \in I^+$, then xz, $yz \in I^+$, since I^+ is a subsemilattice of L. Now

$$a \rho^- b, \quad \text{but non } (a \rho^+ b),$$
$$c \rho^+ b, \quad \text{but non } (c \rho^- b).$$

Therefore $\rho^+ \nsubseteq \rho^-$ and $\rho^+ \nsupseteq \rho^-$. This is a contradiction to the assumption. Thus we have proved that L is a chain of order $\leqq 2$. The converse is obvious.

We know that every semigroup is a semilattice of s-indecomposable semigroups [11], [13]. An s-indecomposable semigroup is a semigroup which has no semilattice homomorphic image except trivial one (one-element semigroup).

Proposition 4. — *A Δ-semigroup S is either an s-indecomposable semigroup or the set union of two s-indecomposable semigroups*

$$S = S_0 \cup S_1,$$

where $S_0 S_1 \subseteq S_0$, $S_1 S_0 \subseteq S_0$, $S_0 \cap S_1 = \emptyset$, S_0, $S_1 \neq \emptyset$.

Let S be a commutative Δ-semigroup. Then S_0 and S_1 in Proposition 4 are commutative archimedean semigroups, that is, for a, $b \in S_i$ $(i = 0, 1)$ there are positive integers m, n and elements c, d of S_i $(i = 0, 1)$ such that (*cf.* [2], [7])

$$a^m = bc, \; b^n = ad.$$

3. Simple or 0-simple Δ-semigroups.

We will treat the special cases, commutative simple Δ-semigroups and commutative 0-simple Δ-semigroups. A commutative simple semigroup is an abelian group and a commutative 0-simple semigroup is an abelian group with zero adjoined.

As far as abelian groups are concerned, our problem is equivalent to the problem on abelian groups whose subgroups form chain.

Definition. — *Let p be a prime number. If a group G is the set union of a finite or infinite ascending chain of cyclic groups C_n of order p^n, that is,*

$$G = \bigcup_{n=1}^{\infty} C_n, \qquad C_1 \subset C_2 \subset \ldots \subset C_n \subset \ldots,$$

then G is called a p-quasicyclic group, or quasicyclic group if it is not necessary to specify p.

Remark. — This definition is originally due to Fuchs [3], [4], but in this paper it is understood that (p-) quasicyclic groups contain cyclic groups of order of prime power (p^n) as a special case.

A part of the following theorem was proved in more general case [8], but we state the proof here after suitable rearrangement. Commutativity is not assumed in (1.2). (1.3), (1.5) below.

Theorem 5. — *The following statements are equivalent :*

(1.1) *G is an abelian group which is a Δ-semigroup;*
(1.2) *G is a group in which all subgroups form a chain;*
(1.3) *For every two elements a and b of a group G, either $a = b^n$ or $b = a^n$ for some positive integer n;*
(1.4) *G is a p-quasicyclic group for some prime p;*
(1.5) *G is a group in which all subsemigroups form a chain.*

Proof. — $(1.1) \rightarrow (1.2)$ is obvious.

$(1.2) \rightarrow (1.3)$: Let G be a group satisfying (1.2). Then G is periodic and all cyclic subgroups form a chain, therefore we have (1.3).

$(1.3) \rightarrow (1.4)$: Immediately the periodicity of G follows from (1.3). Also it follows that all cyclic subgroups of G form a chain with respect to inclusion. Accordingly the order of every element, hence of every cyclic subgroup is a power of a same prime number p. Let $C(x)$ denote the cyclic subgroup generated by x. Let F_n be the set of all elements of order p^n in G.

We have a finite or infinite sequence $\{F_n\}$ and by the above remark

$$G = \bigcup_{n=1}^{\infty} F_n. \tag{2}$$

Let $x, y \in F_n$. By (1.3), either $x = y^m$ or $y = x^m$ for some $m > 0$. Assuming $x = y^m$, $C(x) \subseteq C(y)$. Since $|C(x)| = |C(y)| = p^n$, we have $C(x) = C(y)$. (The same for $y = x^m$.) Since the converse is obvious, $C(x) = C(y)$ if and only if x and y are in a same F_n. Choose one element a_n from each F_n. Then we have a finite or infinite sequence

(3) $$C(a_1) \subset C(a_2) \subset \ldots \subset C(a_n) \subset \ldots,$$

where $|C(a_n)| = p^n$ and $F_n \subset C(a_n)$.

By (2),

$$G = \bigcup_{n=1}^{\infty} C(a_n).$$

If the sequence (3) is finite, $G = C(a_n)$ for some n, that is, G is a cyclic subgroup of order p^n. Thus we have (1.4).

(1.4) → (1.5) : Let G be a p-quasicyclic group : $G = \bigcup_{n=1}^{\infty} C(a_n)$, where $C(a_n)$ is a cyclic group of order p^n. Let H be a subsemigroup of G, and let

$$H'_n = F_n \cap H,$$

where F_n has been defined above. $H = \bigcup_{n=1}^{\infty} H'_n$. Let $x \in H'_n$. By the definition of F_n, $C(a_n) = C(x) \subseteq H$. If the set $\{n_i;\ H'_{n_i} \neq \emptyset\}$ is infinite, then $H = G$; if the set is finite, and if n_m is its maximum, $H = C(a_{n_m})$. Consequently G has no proper subsemigroup, hence no proper subgroup except $C(a_n)$, $n = 1, 2, \ldots$ in (3). We have (1.5). Noting that G is abelian, we have proved also (1.4) → (1.1).

Finally (1.5) → (1.2) : It follows that G is periodic, therefore every subsemigroup is a subgroup. Hence we have (1.2). Thus we have proved that (1.1) through (1.5) are all equivalent.

Abelian groups with zero which are Δ-semigroups. — Let G be a group and G^0 be the group G with zero 0 adjoined. Let ρ be any congruence on G. A congruence ρ^0 on G^0 is associated with ρ as follows :

$a \rho^0 b$ if and only if either $a = b = 0$ or $a, b \in G$ and $a \rho b$.

The mapping $\rho \to \rho^0$ is a one-to-one; and $\rho \subset \sigma$ if and only if $\rho^0 \subset \sigma^0$. Let ω_G and ω_{G^0} denote the universal relations on G and G^0 respectively. We will prove that every congruence on G^0 is either ω_{G^0} or ρ^0, a congruence associated with ρ on G. Let σ be a congruence on G^0 such that

$$a \sigma 0 \quad \text{for some} \quad a \in G.$$

Multipliying the both sides by $a^{-1}x$, $x \in G^0$, we have

$$x \sigma 0 \quad \text{for all} \quad x \in G^0.$$

Therefore $\sigma = \omega_{G^0}$. Clearly $\omega^0 \subset \omega_{G^0}$.

Immediately we have :

PROPOSITION 6. — *A group G^0 with zero is a Δ-semigroup if and only if a group G is a Δ-semigroup.*

By Theorem 5, we have :

THEOREM 7. — *An abelian group G^0 with zero is a Δ-semigroup if and only if G is a p-quasicyclic group, p is arbitrary prime.*

4. Non-simple Δ-semigroups.

In this section, we will prove that if S is a Δ-semigroup and if S has a proper ideal I, then I can not be homomorphic onto a non-trivial group. We do not assume commutativity of S in this section.

The following lemma was obtained in [15].

LEMMA 8. — *Let I be an ideal of a semigroup S. If f is a homomorphism of I onto a non-trivial group G, then there is a homomorphism g of S onto G such that f is the restriction of g to I.*

THEOREM 9. — *If a semigroup S contains a proper ideal I and if S is a Δ-semigroup, then neither S nor I is homomorphic onto a non-trivial group.*

Proof. — Suppose there is a homomorphism f of S onto G, $f(S) = G$, $|G| > 1$. Since G contains no ideal except G, $f(I) = G$. Hence $|I| > 1$. Let ρ be the congruence on S induced by f. For each $a \in S \setminus I$, there is an element b in I such that $a \rho b$. On the other hand, let σ be the Rees-congruence on S modulo I. Then $a \rho b$, but non $(a \sigma b)$. Since $|G| > 1$, non $(x \rho y)$ for some $x, y \in I$, but $x \sigma y$. Thus $\rho \nsubseteq \sigma$ and $\rho \nsupseteq \sigma$, which is contradiction to the assumption. Therefore S is not homomorphic onto a group G, $|G| > 1$. Next, suppose that I is homomorphic onto G, $|G| > 1$. Then by Lemma 8 there is a homomorphism of S onto G. This leads to the same contradiction. Therefore I is not homomorphic onto G.

5. Commutative archimedean Δ-semigroups.

In this section, we will determine commutative archimedean Δ-semigroups. Since a commutative archimedean semigroup has at most one idempotent, we have three possible types :

(4.1) Commutative archimedean semigroup with zero;

(4.2) Commutative archimedean semigroup with non-zero idempotent;

(4.3) Commutative archimedean semigroup without idempotent.

The three types will be called *Type* 1, *Type* 2, *Type* 3 respectively. Let S be a commutative archimedean semigroup.

Type 1 : S is of Type 1 if and only if S has a zero 0 and for each $a \in S$ there is $n > 0$ such that $a^n = 0$. We will call a semigroup of Type 1 a commutative nil-semigroup.

Type 2 : If S is simple, it is an abelian group. If S is not simple, S is an ideal extension of an abelian group $G = Se$, by a commutative nil-semigroup, where e is the idempotent. This is also obtained as a special case of unipotent inversible semigroups [6]. (S is called inversible if each element has a right inverse element with respect to the idempotent.) By Lemma 8 we have :

LEMMA 10. — *If S is of Type 2, S is homomorphic onto a non-trivial abelian group G.*

Type 3 : We have the same result as in Type 2.

LEMMA 11. — *A commutative archimedean semigroup S without idempotent is homomorphic onto a non-trivial abelian group.*

Proof. — Let $a \in S$. Two relations η_a and ρ_a are define by

$$x \eta_a y \text{ if and only if } a^l x = a^l y \text{ for some positive integer } l;$$
$$x \rho_a y \text{ if and only if } a^m x = a^n y \text{ for some positive integers } m,n.$$

The two relations η_a and ρ_a are congruences on S. It is known [14], [16] that S/η_a is a commutative archimedean cancellative semigroup without idempotent, and S/ρ_a is a group. If $|S/\rho_a| = 1$, then S/η_a consists of

$$\bar{a}, \quad \bar{a}^2, \quad \ldots, \bar{a}^i, \quad \ldots,$$

where $\bar{a}$ denotes the η_a-class containing a (*cf.* [12]). In other words, S/η_a is isomorphic onto the semigroup of all positive integers with addition, and hence is homomorphic onto any finite cyclic group. Therefore S is still homomorphic onto a non-trivial abelian group even if S/ρ_a is trivial. The proof is completed.

THEOREM 12. — *If S is a commutative archimedean Δ-semigroup, then S is either an abelian group or a commutative nil-semigroup.*

Proof. — This theorem is an immediate consequence of Theorem 9, the facts (4.1), (4.2), (4.3), Lemma 10 and Lemma 11.

Thus the study in the present case is reduced to that of Type 1 which is a Δ-semigroup, since the groups (in Type 2) have been studied in paragraph 3.

Let S be a commutative nil-semigroup. In S we define a relation $|$ by

$$b \mid a \quad \text{if and only if either} \quad a = b \quad \text{or} \quad a = bx \text{ for some } x \in S.$$

Then $|$ is a partial ordering. Anti-symmetry follows from the fact that $a = ax$ implies $a = 0$. (The partial ordering $|$ is effective even if commutativity is not assumed.) The ordering $|$ is called the divisibility ordering.

THEOREM 13. — *Let S be a commutative nil-semigroup. The following statements are equivalent* :

(5.1) *S is a Δ-semigroup;*

(5.2) *The ideals form a chain with respect to inclusion;*

(5.3) *The principal ideals form a chain with respect to inclusion;*

(5.4) *The divisibility ordering is a chain.*

Proof. — (5.1) $\rightarrow$ (5.2) : Since the Rees-congruences form a chain, (5.2) immediately follows.

(5.2) $\rightarrow$ (5.3) : Obvious.

(5.3) $\rightarrow$ (5.4) : For any $a, b \in S$, $S^1 a \subseteq S^1 b$ [3], or $S^1 b \subseteq S^1 a$, hence we have proved either $a \mid b$ or $b \mid a$.

(5.4) $\rightarrow$ (5.1) : Suppose (5.4) holds. Let ρ be any congruence on S. We will prove that ρ is a Rees-congruence on S. If ρ is the equality relation ι, it is regarded as the Rees-congruence modulo $\{0\}$. So we assume $\rho \neq \iota$, and it is sufficient to prove

$$a \neq b, \qquad a \rho b \;\Rightarrow\; a \rho 0, \quad b \rho 0.$$

Suppose $a \neq b$, $a \rho b$. By (5.4), either $b = ax$ or $a = bx$ for some $x \in S$. Assume that $b = ax$. (The same argument for $a = bx$.) Then $a \rho ax$. Since S is a nilsemigroup, $x^n = 0$ for some $n > 0$. Accordingly,

$$a \rho ax \rho ax^2 \rho \ldots \rho ax^n = 0,$$

hence

$$a \rho b \rho 0.$$

Let $I = \{x; x \rho 0\}$. I is an ideal of S, and we have proved that a congruence ρ is the Rees-congruence modulo I. To prove (5.1), we may prove the ideals form a chain. Let I and J be ideals of S. Suppose $I \nsubseteq J$. There is an element $a \in I$, but $a \notin J$. Let x be any element of J. Clearly $a \neq x$. By (5.4), either $x = ay$ or $a = xu$ for some y, some u. If $a = xu$, $a \in J$ because $x \in J$ and J is an ideal. This is a contradiction with $a \notin J$. So $x = ay$. Since $a \in I$, we have $x \in I$. Thus we have proved $J \subseteq I$. The proof of (5.1) is completed.

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[3] $S^1 a = Sa \cup \{a\}$.

DEFINITION. — *If a semigroup S satisfies the condition* (5.4), *we say that S satisfies the divisibility chain condition.*

We conclude that S is a commutative Δ-nil-semigroup if and only if S is a commutative nil-semigroup which satisfies the divisibility chain condition.

DEFINITION. — *A semigroup D is called naturally totally ordered if and only if*

(6.1) *D is a semigroup;*
(6.2) *D is a totally ordered set ($\leqq$);*
(6.3) *$a \leqq b$ implies $ac \leqq bc$, $ca \leqq cb$ for all c;*
(6.4) *$a \leqq b$ implies $b \mid a$.*

A commutative nil-semigroup satisfying the divisibility chain condition is a naturally totally ordered commutative nil-semigroup. According to CLIFFORD [1], we have the following result :

PROPOSITION 14. — *Let R be the semigroup of all positive real numbers with addition. A naturally totally ordered commutative nil-semigroup S can be embedded into the Rees-factor semigroup R/I modulo I, where I is defined by either $\{x \in R; x > 1\}$ or $\{x \in R; x \geqq 1\}$, $\geqq$ is the usual order.*

Tully obtained in [17],

PROPOSITION 15. — *A naturally totally ordered commutative nil-semigroup is isomorphic with the intersection of the interval* (0, 1) *and some multiplicative subgroup of positive real numbers with either the interval* (0, 1/2), *or* (0, 1/2] *collapsed to a point.*

6. Commutative non-archimedean Δ-semigroups.

In this section, S denotes a commutative non-archimedean Δ-semigroup. According to Proposition 4 and the remark after that in paragraph 2,

$$S = S_0 \cup S_1, \tag{7}$$

where S_0 and S_1 are archimedean semigroups and S_0 is an ideal of S.

LEMMA 16. — *In* (7), *S_1 contains neither proper ideal nor zero, hence S_1 is an abelian group.*

Proof. — Suppose that S_1 has either a proper ideal or $\{0\}$, say denoted by I, $\{0\} \subseteq I \subset S_1$, $|S_1| > 1$. Then the set union $J = S_0 \cup I$ is an ideal of S. Let ρ denote the Rees-congruence on S modulo J and let σ be the congruence induced by the partition $S = S_0 \cup S_1$. For $x \in I$

and $y \in S_1 \setminus I$, $x \sigma y$, but non $(x \rho y)$, while, for $x \in I$ and $z \in S_0$, $x \rho z$ but non $(x \sigma z)$. Thus $\rho \nsubseteq \sigma$ and $\sigma \not\subset \rho$; S is not a Δ-semigroup. This is a contradiction. Therefore S_1 is simple, hence an abelian groupe since S is commutative.

LEMMA 17. — *In* (7), S_0 *is a commutative nil-semigroup. The zero of* S_0 *is a zero of* S.

Proof. — The commutative archimedean semigroup S_0 has one of the three types : Type 1, Type 2, Type 3 described in paragraph 5. By Lemmas 10, 11, S_0 of Type 1, or Type 2, is homomorphic onto a non-trivial group G, and then S would not be a Δ-semigroup by Theorem 9. Consequently S_0 has to be of Type 1. Let o be the zero of S_0. Since $ox \in S_0$ for all $x \in S$,

$$ox = (oo)x = o(ox) = o \quad \text{for all } x \in S.$$

That is, o is also a zero of S.

If S is a Δ-semigroup and if S_0 is trivial, then S is an abelian group S_0 with zero adjoined. We have already studied the Δ-semigroups of this kind in paragraph 3. So we assume $|S_0| > 1$.

LEMMA 18. — *If* $|S_0| > 1$, *then* S_1 *is a trivial group* $\{e\}$, *and* e *is the identity of* S.

Proof. — We define a relation π on $S = S_0 \cup S_1$ by

$$a \pi b \quad \text{if and only if } S^1 a = S^1 b.$$

π is a congruence on S. (Of course, we can define π on any commutative semigroup.) Since S_1 is a group by Lemma 16,

$$a \pi b \quad \text{for all } a, b \in S_1$$

and if $a \in S_0$ and $b \in S_1$, then non $(a \pi b)$. Lemma 17 tells us that S_0 contains o, which is a zero of S. Suppose $a \in S_0$ and $a \pi o$. Then $S^1 a = S^1 o$ implies $a = o$. On the other hand, let ρ be the Rees-congruence on S modulo S_0. Since $|S_0| > 1$, $\pi | S_0 \subset \rho | S_0$ by the fact mentioned above. ($\pi | S_0$ denotes the restriction of π to S_0.) In order that S be a Δ-semigroup, π must be comparable with ρ over S. Accordingly, $\pi | S_0 \subset \rho | S_0$ implies $\pi \subset \rho$ which leads to $\pi | S_1 \subseteq \rho | S_1$. Since $\rho | S_1$ is the equality relation, $\pi | S_1$ is also the equality relation, that is, $|S_1| = 1$, S_1 is a trivial group $\{e\}$.

It remains to prove that e is the identity of S. By Lemma 1, for all $x, y \in S$, either $S^1 x \subseteq S^1 y$ or $S^1 x \supseteq S^1 y$. Now $e \in S^1 e$ and if $a \in S_0$, $e \notin S^1 a$. Therefore $S^1 a \subset S^1 e$, which implies $a \in S^1 e$ for all $a \in S_0$. Therefore $S_0 \subset S^1 e$. Immediately we have $S_0 e = S_0$. Take any $x \in S_0$, $x = ye$ for some $y \in S_0$. Then

$$xe = (ye)e = y(ee) = ye = x \quad \text{for all } x \in S_0.$$

Combining this with $e^2 = e$, we have proved that e is the identity of S.

Thus we know that if S is a commutative non-archimedean Δ-semigroup, it is a commutative Δ-nil-semigroup with identity adjoined. To prove the converse, we consider the relationship between the congruences on S and S^1 where S is a commutative nil-semigroup, $S^1 = S \cup \{1\}$, 1 the identity of S^1.

Let ρ be a congruence on S. A congruence ρ^1 on S^1 is associated with ρ as follows :

$$x \rho^1 y \quad \text{if and only if either } x = y = 1 \quad \text{or} \quad x, y \in S \quad \text{and} \quad x \rho y.$$

It is easy to see that ρ^1 is a congruence on S^1, and that $\rho^1 \subset \xi^1$ if and only if $\rho \subset \xi$. Let σ be a congruence on S^1 such that $x \sigma 1$ for some $x \in S$. Since S is a nil-semigroup, $x^n = 0$ for some $n > 0$. Then $x \sigma 1$ implies $0 \sigma 1$, and it implies

$$0 \sigma y \quad \text{for all } y \in S^1.$$

Thus we have proved that $\sigma = \omega_{S^1}$, the universal relation on S^1. Consequently, every congruence ξ on S^1 is either ω_{S^1} or ρ^1 for some congruence ρ on S. If ω denotes the universal relation on S, $\omega^1 \subset \omega_{S^1}$.

Immediately, we have :

Lemma 19. — *Let S be a commutative nil-semigroup. S^1 is a Δ-semigroup if and only if S is a Δ-semigroup.*

By all the lemmas in this section and Theorem 7, we have :

Theorem 20. — *S is a commutative non-archimedean Δ-semigroup if and only if S is either :*

(8.1) *a p-quasicyclic group with* 0 *adjoined,*

or

(8.2) *a commutative nil-semigroup with identity adjoined satisfying the divisibility chain condition.*

Related to (8.2), we notice that if S satisfies the divisibility chain condition, S^1 also satisfies the same condition.

7. Conclusion.

Summarizing all the theorems and propositions obtained, we have :

Theorem 21. — *All the types of commutative Δ-semigroups are :*

(9.1) *Quasicyclic groups;*

(9.2) *Quasicyclic groups with zero adjoined;*

(9.3) *Commutative nil-semigroups satisfying the divisibility chain condition;*

(9.4) *The type* (9.3) *with identity element adjoined.*

We notice that cyclic groups of prime power are of Type (9.1) as mentioned in Remark in paragraph 3; all commutative semigroups of order 2 also belong to one of the above four types.

As far as Type (9.3) is concerned, CLIFFORD or TULLY's result gives its structure, but the author has been successful in another attack by means of the non-negative valued functions in [12]. The result will be published elsewhere.

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