CH. MEGIBBEN

On certain subsocles of a primary abelian group

Bulletin de la S. M. F., tome 97 (1969), p. 285-287
<http://www.numdam.org/item?id=BSMF_1969__97__285_0>

Bull. Soc. math. France,
97, 1969, p. 285 à 287.

ON CERTAIN SUBSOCLES OF A PRIMARY ABELIAN GROUP

BY

CHARLES MEGIBBEN.

Let G be a p-primary abelian group. Then $G[p] = \{x \in G : px = 0\}$ is called the socle of G and any subgroup of $G[p]$ will be referred to as a subsocle. In [3], the notion of a quasi-essential subsocle is introduced : A subsocle S is said to be quasi-essential if $G = H + K$, whenever H is a pure subgroup of G containing S, and K is maximal disjoint from S. Recall that K will be maximal disjoint from S if and only if $G[p] = K[p] \oplus S$ and K is a neat subgroup of G (that is, $pG \cap K = pK$). The purpose of this note is to prove the following proposition.

PROPOSITION. — *A subsocle S of G is quasi-essential if and only if either*

$$(1) \qquad S \subseteq G^1 = \bigcap_{n=1}^{\infty} p^n G$$

or

$$(2) \qquad (p^n G)[p] \supseteq S \supseteq (p^{n+1} G)[p]$$

for some nonnegative integer n.

That conditions (1) or (2) are sufficient is established in [3], but the converse is obtained there only when further conditions are placed either on S or G. Our basic tool will be the following lemma :

LEMMA. — *If $G = Zb \oplus Za \oplus H$, where $o(b) = p^i$ and $o(a) \geqq p^{i+2}$ and if S is a subsocle of G such that $S \subseteq Zb \oplus H$ and $S \cap Zb \neq 0$, then S is not quasi-essential.*

Proof. — Write $S = (Zb)[p] \oplus S_1$ with $S_1 \subseteq H$, and choose M maximal disjoint from S_1 with $a, b \in M$. Then M is a neat subgroup of G, and

$M = Za \oplus Zb \oplus M_1$, where $M_1 = M \cap H$. Let $M' = Z(b + pa) \oplus M_1$, and note that $G[p] = M'[p] \oplus S$. M' will therefore be maximal disjoint from S provided it is neat in G. But the neatness of M' is an easy consequence of the neatness of $Zb \oplus M_1$. To prove that S is not quasi-essential it suffices to show that $G \neq M' + (Zb \oplus H)$. But, in fact, $a \notin M' + (Zb \oplus H)$. For suppose $a = t(b + pa) + m_1 + sb + h$, where $m_1 \in M_1$, $h \in H$ and $t, s \in Z$. Then $m_1 + h \in H \cap (Za \oplus Zb) = 0$, and we have the absurd equation $(1 - pt)a = (t + s)b \in Za \cap Zb = 0$.

We shall require the notion of a *center of purity*: A subgroup H of G is said to be a center of purity if every subgroup maximal disjoint from H is pure in G. In [4], it is shown that a subsocle S of a p-group G is a center of purity if and only if either

$$\text{(i)} \qquad S \subseteq G^1$$

or

$$\text{(ii)} \qquad (p^n G)[p] \supseteq S \supseteq (p^{n+2} G)[p]$$

for some nonnegative integer n. Note the slight difference between (ii) and (2). In [3], it is actually proved that if a subsocle is both a center of purity and quasi-essential, then it satisfies (1) or (2). Consequently, we need only prove that every quasi-essential subsocle is a center of purity in order to establish our proposition.

Now if S supports a pure subgroup H (that is, $H[p] = S$) and if S is quasi-essential, then clearly $G = M \oplus H$ whenever M is maximal disjoint from S and, since direct summands are pure, S is a center of purity. The proof of our proposition thus reduces to showing that a quasi-essential subsocle that fails to support a pure subgroup is also a center of purity, or equivalently, that a subsocle S which neither supports a pure subgroup nor is a center of purity cannot be quasi-essential.

By a standard technique, we can construct a basic subgroup $B = A \oplus C$ of G where $C[p]$ is dense in S (relative to the subspace topology induced on $G[p]$ by the p-adic topology of G). Since S does not support a pure subgroup, $S \cap p^n G$ cannot be dense in $(p^n G)[p]$ for any n (*see* [2]). This fact forces A to be unbounded. But S is not a center of purity and therefore $S \nsubseteq G^1$. Hence there is a minimal nonnegative n such that $S \nsubseteq p^{n+1} G$. Then S has an element of height exactly n and, since $C[p]$ is dense in S, this element may be taken to be in C. Thus C has a cyclic direct summand Zb with $o(b) = p^{n+1}$. Recall that A is unbounded, and consequently has a cyclic summand Za with $o(a) = p^k \geq p^{n+3}$. Exploiting the purity of B and the fact that $C[p]$ is dense in S, one easily shows that $Za \cap (S + C + p^k G) = 0$. By Theorem 24.1 of [1], we then have a direct decomposition $G = Za \oplus M$, where $M \supseteq S + C$. But Zb is a pure subgroup of G and therefore $G = Za \oplus Zb \oplus H$, where $Zb \oplus H \supseteq S$ and $S \cap Zb \neq 0$. The conditions of our lemma are now satisfied, and we conclude that S is not quasi-essential.

REFERENCES.

[1] FUCHS (L.). — *Abelian groups.* — Budapest, Hungarian Academy of Sciences, 1958.
[2] HILL (P.) and MEGIBBEN (C.). — On primary groups with countable basic subgroups, *Trans. Amer. math. Soc.*, t. 124, 1966, p. 49-59.
[3] IRWIN (J.) and BENABDALLAH (K.). — On N-high subgroups of abelian groups, *Bull. Soc. math. France*, t. 96, 1968, p. 337-346.
[4] PIERCE (R. S.). — Centers of purity in abelian groups, *Pacific J. of Math.*, t. 13, 1963, p. 215-219.

(Texte reçu le 15 juillet 1969.)

Charles MEGIBBEN,
Department of Mathematics,
Vanderbilt University,
Nashville, Tenn. 37 214-U. S. A.
(États-Unis).