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The Lefschetz-Riemann-Roch formula

THE LEFSCHETZ-RIEMANN-ROCH FORMULA

BY

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This paper uses the results and some of the methods of the paper of
Borel and Serre on the Grothendieck-Riemann-Roch theorem ([0])
to obtain a fixed point formula for periodic automorphisms of non-
singular projective algebraic varieties over an algebraically closed field, k.

This formula is analogous to that of Atiyah, Singer and Segal [2]
(*see* also [5]) for elliptic differential operators over compact manifolds
with automorphisms of period n and for certain other automorphisms.
Both formulas depend heavily on the fact that the eigenvalues of certain
endomorphisms of vector bundles are locally constant. In [2], this is gua-
ranteed by hypotheses which force these eigenvalues to be n^{th} roots of
unity and considerations of continuity. In projective algebraic geometry,
the general principle that " all global functions are constant " gives this
fact. Thus the result of Atiyah, Singer and Segal can be extended
to non-periodic endomorphisms only if it is assumed that such endo-
morphisms have only isolated fixed points (*see* Atiyah and Bott [1]).
The results of this paper may be formulated under far weaker hypotheses
than periodicity, but, as is discussed in the last section, the proof of such
more general results is probably quite difficult. If k is taken to be the
field of complex numbers, the results of this paper are seen to overlap
those of Atiyah, Singer and Segal.

This formula is also analogous to that of Verdier [10] in the etale
cohomology of schemes. The Eichler trace formula ([4]) in the theory
of function fields is also somewhat analogous.

It has not yet been stated which cohomology theory is being used here.
It is the cohomology of coherent algebraic sheaves; however, as in [0],
it is better to work more generally with the direct image sheaves of
morphisms of varieties. Thus, let $y : Y \to Y$ and $x : X \to X$ be periodic
endomorphisms of non-singular projective algebraic varieties and let
$f : Y \to X$ be such that $fy = xf$.

Let $\psi : y^*F \to F$ be a morphism of coherent sheaves on Y. Then there are defined induced morphisms $R'f_!\psi : x^*R'f_*F \to R'f_*F$. ψ has a " Chern trace " which takes values in the Chow ring of the fixed point set of y tensored with a suitable field. I give a formula comparing the Chern trace of ψ with the alternating sum of the Chern traces of the $R'f_!\psi$. If F is locally free and if X is a point this looks like the formula of ATIYAH SINGER and SEGAL. On the other hand, if x, y and ψ are all identities, then my formula reduces to the Grothendieck-Riemann-Roch formula.

If k has characteristic $p \neq 0$, the word " periodic " has to be interpreted as meaning " periodic with period prime to p ". The formulation is also slightly more complicated : instead of dealing with the eigenvalues directly it is necessary to use a canonical (Teichmüller) lifting of them to the Witt ring of k. Thus, in the case when X is a point (in the above notation), a formula for a stronger invariant than the trace of the induced endomorphisms of the $H'[Y; F]$ is obtained.

In the first, three sections an appropriate formalism is set up. In the fourth, the embedding of a fixed component of a periodic endomorphism of a variety is dealt with. In the fifth, the " Lefschetz-Riemann-Roch " theorem is formulated. The proof is given in the next three sections. It consists of investigating three special cases which together give the general case. In the ninth, it is explained why the formalism of the first three sections is not dependent on the hypothesis of periodicity and then a few comments are made on the non-periodic case.

Although the algebraic geometry used is essentially that of [0], the references given are mostly to Grothendieck's " Éléments ", cited as [EGA]. The notation is that of [EGA].

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1. The functor M.

" *Variety* " will henceforth mean " projective non-singular (perhaps reducible) algebraic variety " except when the phrase " affine variety " is used. All varieties are over the fixed algebraically closed field k.

It is convenient to define a category, End, as follows : The objects of End are pairs, (X, x), consisting of a variety, X, and a periodic automorphism, x, of X. The period of x is required to be prime to the characteristic of k if this is not zero. An " empty variety " is allowed. The morphisms $(Y, y) \to (X, x)$ of End are the variety morphisms $f : Y \to X$ such that $fy = xf$. Note that the category of varieties is a full subcategory of End. The empty variety has only the identity automorphism and so is an initial object of End. In much of what follows,

the trivial special conventions required for this initial object will not be dealt with explicitly.

For $(X, x) \in \mathrm{End}$, a covering homomorphism is defined to be a homomorphism $\psi : x^*F \to F$, where F is a (coherent algebraic, as always) sheaf on X. A short exact sequence of covering homomorphisms is defined to be a commutative diagram in which the lower, and therefore the upper (as x^* is exact) row is exact :

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & x^*F_1 & \xrightarrow{x^*\alpha} & x^*F_2 & \xrightarrow{x^*\beta} & x^*F_3 & \longrightarrow & 0 \\
& & \downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_3} & & \\
0 & \longrightarrow & F_1 & \xrightarrow{\alpha} & F_2 & \xrightarrow{\beta} & F_3 & \longrightarrow & 0
\end{array}
$$

In fact, for a fixed (X, x), the class of covering homomorphisms forms an abelian category if, with the above notation, $\mathrm{Hom}(\psi_1, \psi_2)$ is defined to be the set of $\alpha : F_1 \to F_2$ such that $\psi_2 . x^*\alpha = \alpha . \psi_1$.

A cofunctor (= contravariant functor), M_1, from End to the category of abelian groups is defined thus : Let $(X, x) \in \mathrm{End}$. Consider the set of all (isomorphism classes of) covering homomorphisms $\psi : x^*F \to F$, where F is a locally free sheaf on X. Let $M_1(x)$ be the quotient of the free abelian group on this set by the subgroup generated by all elements of the form $[\psi_1] - [\psi_2] + [\psi_3]$ where ψ_1, ψ_2 and ψ_3 form a short exact sequence. If $f : (Y, y) \to (X, x)$ is a morphism of End, a homomorphism, $M_1(f)$, written $f^! : M(x) \to M(y)$, is induced by linearity from the rule $f^![\psi] = [f^!\psi]$, where, for $\psi : x^*F \to F$ a homomorphism of locally free sheaves on X, $f^!\psi$ is defined to be the composite of y^*f^*F "$=$" $f^*x^*F \xrightarrow{f^*\psi} f^*F$. (The "$=$" denotes the canonical isomorphism of functors arising from the equality $fy = xf$). It is easy to check that this rule is compatible with the relations and indeed defines a cofunctorial structure for M_1.

In fact, M_1 is a cofunctor from End to the category of commutative rings with unit. As this ring structure is used only on the subcategory of varieties, it suffices here to define the ring structure on $M_1(X) = M_1(\mathrm{id}_X)$ For $\psi_1 : F_1 \to F_1$ and $\psi_2 : F_2 \to F_2$ homomorphisms of locally free sheaves on X, set $[\psi_1] . [\psi_2] = [\psi_1 \otimes \psi_2] \in M_1(X)$. This rule induces by linearity the required ring structure. The unit of $M_1(X)$ is the class of the identity homomorphism of $\mathcal{O}_X$.

Let $\mathbf{Z}[k]$ denote the ring whose abelian group structure is free on the elements of k and whose multiplication is induced from that of k. The process of taking the eigenvalues of endomorphisms of finite rank k-modules gives an isomorphism : $M_1(pt) \approx \mathbf{Z}[k]$. M_1 is seen to be a cofunctor from End to the category of $\mathbf{Z}[k]$-algebras in the obvious way.

A (covariant) functor, M, from End to the category of abelian groups will now be defined. The reader is reminded that, if x is an automorphism of the variety X, the functors x^* and x_* are exact and that x^* is the

left adjoint of x_*. If $f : (Y, y) \to (X, x)$ is a morphism of End, there is a canonical natural transformation of functors $x^* f_* \to f_* y^*$ which is the composite of $x^* f_* \to x^* x_* f_* y_* y_*$ "$=$" $x^* x_* f_* y^* \to f_* y^*$, where the first and third factors are the adjunction morphisms and the second is the canonical isomorphism of functors. Hence there is induced a natural transformation of each derived functor : $x^* . R^l f_* \to R^l f_* . y^*$.

For $(X, x) \in$ End, let $M(x)$ be the quotient of the free abelian group on the set of all (isomorphism classes of) homomorphisms $\psi : x^* F \to F$, where F is a coherent sheaf on X, by the subgroup generated by all elements of the form $[\psi_1] - [\psi_2] + [\psi_3]$, whenever ψ_1, ψ_2 and ψ_3 form a short exact sequence. It may be checked that if $\psi_1, \ldots, \psi_n$ form an exact sequence of covering homomorphisms, the alternating sum of their images in $M(x)$ is zero. Now let $f : (Y, y) \to (X, x)$ be a morphism of End. Let $\Phi : y^* F \to F$ be a covering homomorphism on Y. Then I define covering homomorphisms $R^l f_! \Phi : x^* R^l f_* F \to R^l f_* F$ to be the composite of $x^* R^l f_* F \to R^l f_* y^* F \xrightarrow{R^l f_* \Phi} R^l f_* F$. Note that f is proper by proposition 4 (vi) of [0] and so each $R^l f_* F$ is coherent by [EGA], III, $3.2.1$. Then an abelian group homomorphism, $M(f)$, written $f_! : M(y) \to M(x)$, is defined by the rule $f_![\Phi] = \sum (-)^l . [R^l f_! \Phi] \in M(x)$.

Now let $(Z, z) \xrightarrow{g} (Y, y) \xrightarrow{f} (X, x)$ be two morphisms of End. I claim that, with the above notation, $(f . g)_! = f_! . g_! : M(z) \to M(x)$. This is proved by the well known spectral sequence :

$$E_2^{p, q} = R^p f_* . R^q g_* \implies R^*(f . g)_*.$$

Thus the functorial structure of M is established.

As for M_1, it is easily shown that $M(pt) \approx \mathbf{Z}[k]$ and that M is in fact a functor from End to the category of $\mathbf{Z}[k]$-modules. In the next section, it will be shown that $M(x)$ and $M_1(x)$ are canonically isomorphic. They are therefore analogous to the K and K_1 of [0].

2. Locally free resolutions.

LEMMA 2.1. — *Let $(X, x) \in$ End. Then there exists a projective space, P, and a diagonalisable linear automorphism, p, of P, with the same period as x, and a closed embedding $i : X \to P$ such that $ix = pi$.*

Proof. — Consider the cyclic group of order n generated by x acting on X. If H is a very ample invertible sheaf on X [i. e. $X \to \mathbf{P}((\Gamma_X H))$ is a projective embedding : see [EGA], II, 4, for the general theory used], consider $K = \bigotimes_{t=1}^{n} (x^t)^* H$. By [EGA], II, $4.4.9$, K is also very ample. The isomorphism $\beta : x^* K \to K$ induced by cyclic permutation of the

factors of the tensor product induces a linear automorphism of period n, denoted by p, of $P = \mathbf{P}((\Gamma_X K))$ which is such that $ix = pi$, where i is the canonical embedding. In view of the restriction imposed on the period, n, of x in the last section, p is necessarily diagonalisable.

LEMMA 2.2. — *Let* $(X, x) \in$ End. *Let* $\psi : x^* F \to F$ *be a covering homomorphism on* X, *with* F *a coherent sheaf. Then there exists a locally free sheaf,* L, *on* X *and a* $\neq \Phi : x^* L \to L$ *and a surjective* $\pi : L \to F$ *such that the following square commutes :*

$$
\begin{array}{ccc}
x^* L & \xrightarrow{\;x^* \pi\;} & x^* F \\
\Phi \downarrow & & \downarrow \psi \\
L & \xrightarrow{\;\pi\;} & F
\end{array}
$$

Proof. — Choose a projective embedding as in the proof of 2.1. After ψ has been tensored with a suitably high tensor power of the β of the proof of 2.1, by [EGA], III, 2.2.1, we may assume additionally that F is generated by its global sections. In this case, L may be taken to be the sum of copies of $\mathcal{O}_X$. Tensoring with the corresponding negative tensor power of β gives the result for the original ψ.

The following two lemmas, in which $(X, x) \in$ End, are trivial consequences of lemmas 8 and 9 of [0].

LEMMA 2.3. — *With the notation of the diagram of the previous section, if* F_2 *and* F_3 *are locally free,* F_1 *is locally free.*

LEMMA 2.4. — *Let* d *be the maximum of the dimension of the components of* X. *Let*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & x^* S & \longrightarrow & x^* L_p & \longrightarrow \ldots \longrightarrow & x^* L_0 & \longrightarrow & x^* F & \longrightarrow & 0 \\
& & \sigma \downarrow & & \Phi_p \downarrow & & \Phi_0 \downarrow & & \Psi \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & L_p & \longrightarrow \ldots \longrightarrow & L_0 & \longrightarrow & F & \longrightarrow & 0
\end{array}
$$

be an exact sequence of covering homomorphisms with each L_t *locally free. Then if* $p \geqq d - 1$, S *is locally free.*

There is, for $(X, x) \in$ End, an obvious canonical homomorphism $\varepsilon : M_1(x) \to M(x)$. The following theorem may be deduced from lemmas 2.2, 2.3 and 2.4 by a method formally identical to that used for the corresponding theorem for the functor K on pages 105-108 of [0]. (This proof is axiomatized by BASS, HELLER and SWAN in theorem 4 of [3].)

THEOREM 2.5. — *For* $(X, x) \in$ End, *the canonical homomorphism* $\varepsilon : M_1(x) \to M(x)$ *is an isomorphism.*

Henceforth M_1 and M will be identified and written M.

CONSTRUCTION 2.6. — Let Y be an irreducible variety, L a locally free sheaf on Y and $\psi : L \to L$ a homomorphism. Then the characteristic equation of ψ, which is first constructed locally, has coefficients which are global functions on Y and hence constant. Hence the roots of this equation are constant. Let c be one of them. Then

$$L_c = \operatorname{Ker}(\psi - c)^N, \qquad \text{for} \quad N > \operatorname{rank}(L),$$

is a subsheaf of L independent of N. It is clear that $L = \bigoplus_c L_c$. Also $\psi = \bigoplus_c \psi_c$, where each $\psi_c : L_c \to L_c$ is such that $\psi_c - c$ is nilpotent. As each L_c is a direct summand of L, it is locally free. If Y is a reducible variety, this construction may be carried out on each component and the same conclusion drawn.

LEMMA 2.7. — *Let F be a coherent sheaf on the variety Y and let $c \in k$. Let $\psi : F \to F$ be such that $\psi - c$ is nilpotent. Then $[\psi] = [c . \operatorname{id}_F] \in M(Y)$. Hence $M(Y)$ is generated by the images of the constant covering homomorphisms $c : L \to L$, with $c \in k$ and L locally free.*

Proof. — This is by induction on N, the least integer such that $(\psi - c)^N = 0$. For $N = 1$ the result is the hypothesis. For larger N, the two diagrams superimposed below, in which Ker and Coker are defined by the exactness of the rows, give the required inductive argument :

$$
\begin{array}{ccccccccc}
o & \longrightarrow & \operatorname{Ker} & \longrightarrow & F & \xrightarrow{\psi\,(-c)^{N-1}} & F & \longrightarrow & \operatorname{Coker} & \longrightarrow & o \\
& & {\scriptstyle c}\downarrow \ {\scriptstyle \psi\,|\,\operatorname{Ker}}\downarrow & & {\scriptstyle c}\downarrow \ {\scriptstyle \psi}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow & & \\
o & \longrightarrow & \operatorname{Ker} & \longrightarrow & F & \longrightarrow & F & \longrightarrow & \operatorname{Coker} & \longrightarrow & o
\end{array}
$$

The fourth sentence follows from the third sentence and theorem 2.5.

PROPOSITION 2.8. — *Let Y be a variety. Then there is a natural isomorphism $\alpha : M(Y) \to K(Y) \otimes \mathbf{Z}[k]$.*

Proof. — Although in [0] K is defined only for irreducible varieties, the extension to reducible varieties is clear. I construct an isomorphism $\alpha : M_1(Y) \to K(Y) \otimes \mathbf{Z}[k]$ and then use theorem 2.5. With the notation of 2.6, set $\alpha([\psi]) = \sum [L_c] \otimes [c]$. This extends by linearity to define a homomorphism α. Define a homomorphism $\gamma : K(Y) \otimes \mathbf{Z}[k] \to M_1(Y)$ by setting, for L a locally free sheaf on Y and $c \in k$, $\gamma([L] \otimes [c]) = [c . \operatorname{id}_L]$. Lemma 2.7 shows that $\gamma . \alpha$ is the identity, and it is clear that $\alpha . \gamma$ is the identity. The naturality of α, in both the covariant and contravariant senses, is clear.

3. Brauer trace and Chern character.

It is necessary to recall the basic facts about the Witt ring construction. If k has characteristic $p \neq 0$, then set $W(k) =$ its Witt ring in the sense of SERRE ([9], p. 45). There it is shown that $W(k)$ is a complete discrete valuation ring which is absolutely non-ramified and has k as its residue field. It is characterised by these properties. Its field of fractions, $FW(k)$, is of characteristic zero. According to page 44 of [9], there is a unique multiplicative (but not additive) map $w : k \to W(k)$ which sections the reduction $W(k) \to k$. Denote also by $w : k \to FW(k)$ the composition of the above w with the inclusion. If k has characteristic zero, set $W(k) = FW(k) = k$ and let w be the identity map.

Whatever the characteristic, w makes $W(k)$ into a $\mathbf{Z}[k]$-algebra. Define $M'(y)$ to be $M(y) \underset{\mathbf{Z}[k]}{\otimes} W(k)$. M' inherits the cofunctorial, functorial and ring structure of M; write, for $f : (Y, y) \to (X, x)$ a morphism of End, $f_! : M'(y) \to M'(x)$ for the covariantly induced map, and similarly $f^!$.

Proposition 2.8 shows that, for a variety Y, $M'(Y) \approx K(Y) \otimes W(k)$; this isomorphism is natural on the category of varieties.

If Y is an irreducible variety, and $i : pt \to Y$ is the inclusion of any point whilst $c : Y \to pt$ is the unique morphism, then $K(pt) \overset{c_!}{\to} K(Y) \overset{i_!}{\to} K(pt) = \mathbf{Z}$ shows that $\mathbf{Z}$ (the integers) is a direct summand of $K(Y)$, independently of i. Hence $K(Y) \approx \mathbf{Z} \oplus \tilde{K}(Y)$. The augmentation ideal, $\tilde{K}(Y)$, is nilpotent; this fact may be shown by use of the canonical filtrations of $K(Y)$ developed by JUSSILA [7]. If we define a decomposition $M'(Y) \approx W(k) \oplus \tilde{M}'(Y)$ in the analogous way, it follows that $\tilde{M}'(Y)$ is also nilpotent. Hence an element of $M'(Y)$ is a unit if and only if its projection onto the $W(k)$ summand is a unit in $W(k)$.

If ψ is an endomorphism of a finite rank k-vector space, it is convenient to define the *Brauer trace* of ψ, $\mathrm{B.tr.}\psi \in W(k) \subset FW(k)$ to be $\sum w(c_i)$, the sum being taken over the eigenvalues, c_i, of ψ taken with the appropriate multiplicity. If the characteristic of k is 0, this is the classical trace; if it is $p \neq 0$, the classical trace is the reduction mod p of the Brauer trace.

In [0], extensive use is made of the Chow ring of irreducible varities tensored with $\mathbf{Q}$, the field of rational numbers. It is clear that any other characteristic zero field will do instead of $\mathbf{Q}$ for the purposes of [0]; it is convenient to use $FW(k)$. For example, if Y is an irreducible non-singular projective variety, it has a Todd class, $\mathfrak{T}(Y)$, which is a unit in $A(Y) \otimes FW(k)$. The Chow ring functor and the Todd class may

trivially be extended to the category of (perhaps reducible) varieties. If $f: Y \to X$ is a morphism of varieties, I write

$$f^* : \quad A(Y) \otimes FW(k) \leftarrow A(X) \otimes FW(k)$$

and

$$f_* : \quad A(Y) \otimes FW(k) \to A(X) \otimes FW(k)$$

for the cofunctorially and functorially induced morphisms, the second being one of $FW(k)$-modules only.

If Y is a variety, a ring morphism, *Chern trace*, written ct, is defined to be composite of $M(Y) \approx K(Y) \otimes \mathbf{Z}[k] \xrightarrow{ch \otimes w} A(Y) \otimes FW(k)$, where ch is the Chern character morphism. It is natural in Y for the contravariantly induced morphisms. If $Y = pt$, the above Brauer trace is recovered. Similarly define a $W(k)$-algebra morphism, also called ct, $M'(Y) \to A(Y) \otimes FW(k)$.

In the next section, it will be shown that the fixed point set, written X^x, of the periodic endomorphism x of the variety X is also non-singular. We temporarily assume this result. Hence there is a morphism in End, $i_x : (X^x, \mathrm{id}) \to (X, x)$, where i_x is the embedding. The functor $A(-) \otimes FW(k)$ may now be extended to the category End as follows : for $(X, x) \in$ End, set

$$A(x) \otimes FW(k) = A(X^x) \otimes FW(k). \quad \text{If} f : (Y, y) \to (X, x)$$

is a morphism of End, there is induced a morphism $f_0 : Y^y \to X^x$. Define f^* to be f_0^* and f_* to be f_{0*}.

The concept of Chern trace may now be extended. For $(X, x) \in$ End, define $ct : M(x) [\text{or } M'(x)] \to A(x) \otimes FW(k) = A(X^x) \otimes FW(k)$ to be the composite of $i_x^! : M(x) \to M(X^x)$ with the ct defined above, and similarly for M'.

The reader may now verify that the following lemma is an immediate consequence of the standard properties of the Chow ring and the Chern character given in sections 5 and 6 of [0] :

LEMMA 3.1. — $A(-) \otimes FW(k)$ *is a cofunctor from* End *to the category of* $FW(k)$-*algebras. It is a functor from* End *to the category of* $FW(k)$-*modules. For* $f : (Y, y) \to (X, x)$ *a morphism of* End, *and for* $\xi \in M(x) [\text{or } M'(x)]$,

$$f^*(ct\xi) = ct(f^!\xi).$$

For $\eta \in A(y) \otimes FW(k)$ and $\xi \in A(x) \otimes FW(k)$,

$$f_*(\eta \cdot f^*\xi) = f_* \eta \cdot \xi.$$

4. The fixed components.

In this section, $x : X \to X$ is a periodic endomorphism of the variety X.

LEMMA 4.1. — *Each fixed component (i. e. component of the fixed point set) of x is non-singular.*

Proof. — The reader is referred to section 10 of [0] for the basic properties of local co-ordinates. Let $a \in X$ be such that $xa = a$. Choose an open affine irreducible neighbourhood U of dimension d of a such that $xU = U$ and such that there is a set of local co-ordinates $f_1, \ldots, f_d \in A(U)$ (the affine algebra of U) for a in U. By the usual averaging procedure, we may arrange things so that $f_t x = \alpha_t f_t$, where the $\alpha_t \in k$ are roots of unity and $\alpha_t \neq 1$ only for $1 \leq t \leq q$, for some $q \leq d$. It is well known (and easy to prove) that $f_1, \ldots, f_q$ form a set of local co-ordinates for the component of X^x containing a. Hence it is non-singular.

Now let Z be a fixed component of x. Let $i : Z \to X$ be the embedding. We will investigate $i^! i_! : M(Z) \to M(Z)$. By replacing X with the component containing Z, there will be no loss of generality in assuming that X is irreducible.

LEMMA 4.2. — *There is a canonically induced endomorphism, Φ, of the dual, $N^\vee$, of the normal bundle of Z in X. Hence there is an endomorphism, $\Lambda^t \Phi$, of each exterior power, $\Lambda^t N^\vee$, of $N^\vee$.*

Proof. — On page 120 of [0], an isomorphism $D : \mathfrak{I}(Z)/\mathfrak{I}(Z)^2 \to O_Z(N^\vee)$ of locally free sheaves on Z is set up, where $\mathfrak{I}(Z)$ denotes the sheaf of ideals of Z in X and $\mathcal{O}_Z(N^\vee)$ is the locally free sheaf associated with the vector bundle $N^\vee$. For each point $a \in Z$, find an open affine neighbourhood U and a set of local co-ordinates. $f_1, \ldots, f_d$, as in the proof of lemma 4.1. Then $D \mid U \cap Z$ is given by $f_t \mapsto df_t$, $t = 1, \ldots, q$. Define $\Phi \mid U \cap Z$ by $df_t \mapsto \alpha_t df_t$, $t = 1, \ldots, q$. These fit together to give the required global Φ.

LEMMA 4.3. — *Set*

$$\lambda_Z = \sum_{t=0}^{q} (-)^t [\Lambda^t \Phi] \in M(Z) \qquad \text{and} \qquad \lambda_Z' = \sum (-)^t [\Lambda^t \Phi] \in M'(Z).$$

Then λ_Z' is a unit in $M'(Z)$ and $ct(\lambda_Z) = ct(\lambda_Z')$ is a unit in $A(Z) \otimes FW(k)$.

Proof. — By construction 2.6, $N^\vee = \bigoplus_c L_c$ and $\Phi = \bigoplus_c \Phi_c$, with $\Phi_c : L_c \to L_c$ such that $\Phi_c - c$ is nilpotent. By the proofs of lemmas 4.1 and 4.2, each c that occurs is a root of unity but not unity. By the

criterion of the last section for an element of $M'(Z)$ to be a unit, and as λ'_Z has $W(k)$ component $\prod_1^q (1 - w(\alpha_t))$, λ'_Z is a unit. Hence $ct(\lambda'_Z)$ is a unit.

PROPOSITION 4.4. — *With the above notation, for $\xi \in M(Z)$, $i^! i_! \xi = \xi . \lambda_Z$.*

Proof. — By linearity and lemma 2.7, it suffices to prove this fact in the case when ξ is the image of the constant covering homomorphism $c : F \to F$, with $c \in k$ and F a locally free sheaf on Z.

The induced covering homomorphism $R^0 i_! c$ was defined in section 1 to be the composite of $x^* i_* F \; "=" \; x^* x_* i_* F \overset{adj}{\to} i_* F \overset{c}{\to} i_* F$. As i is an embedding, $R^t i_* = 0$ for $t \neq 0$. By lemmas 2.2 and 2.4, there exists an exact sequence of covering homomorphisms, with the L_t locally free sheaves on X :

(1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & x^* L_N & \longrightarrow & \ldots \longrightarrow & x^* L_0 & \longrightarrow & x^* i_* F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi^N} & & & \downarrow{\scriptstyle \Psi_0} & & \downarrow{\scriptstyle R^0 i_! c} & & \\
0 & \longrightarrow & L_N & \longrightarrow & \ldots \longrightarrow & L_0 & \longrightarrow & i_* F & \longrightarrow & 0
\end{array}$$

Now consider the diagram :

(2)
$$\begin{array}{ccccccc}
0 & \longrightarrow & i^* L_N & \longrightarrow \ldots \longrightarrow & i^* L_0 & \longrightarrow & 0 \\
 & & \parallel & & \parallel & & \\
 & & i^* x^* L_N & & i^* x^* L_0 & & \\
 & & \downarrow{\scriptstyle i^* \Psi_N} & & \downarrow{\scriptstyle i^* \Psi_0} & & \\
0 & \longrightarrow & i^* L_N & \longrightarrow \ldots \longrightarrow & i^* L_0 & \longrightarrow & 0
\end{array}$$

Now the rows of (2) have, by the proof of proposition 12 of [0] (p. 122), cohomology isomorphic to $F \otimes_Z \Lambda^t N^\vee$. So (2) defines certain endomorphisms $\omega_t : F \otimes \Lambda^t N^\vee \to F \otimes \Lambda^t N^\vee$. As $[R^0 i_! c] = \sum (-)^t [\Psi_t] \in M(x)$, $i^! i_! \xi = \sum (-)^t [\omega_t]$. It remains to show that $\omega_t = c . \Lambda^t \Phi$.

To do this, for each $a \in Z$ choose an open affine U as in the proof of 4.2, but such that additionally each $L_t \mid U$ and $F \mid Z \cap U$ is trivial. Now consider the restriction of (1) to U and the restriction of (2) to $Z \cap U$. The $\omega_t \mid U$ are independent of the resolution used in the restriction of (1). Hence the restriction of (1) may be replaced by a Koszul resolution as in page 122 of [0]. The result may now be read off from the construction used in [0] to obtain the isomorphism quoted just after (2).

A canonical element, λ_x, of $M(X^x) = \bigoplus_Z M(Z)$, the summation being taken over the set of fixed components of x, is now defined to have Z-component λ_Z.

5. **Formulation of the result.**

Recall the fundamental result of [0] :

QUOTE 5.1. — *Let $f : Y \to X$ be a morphism of irreducible varieties. Let $\eta \in K(Y)$.   Then $f_*(ch\,\eta \cdot \mathfrak{T}(Y)) = ch(f_!\eta) \cdot \mathfrak{T}(X) \in A(X) \otimes \mathbf{Q}$.*

COROLLARY 5.2. — *Let $f : Y \to X$ be a morphism of varieties.   Let $\eta \in M(Y)$ [or $M'(Y)$].   Then*

$$f_*(ct\,\eta \cdot \mathfrak{T}(Y)) = ct(f_!\eta) \cdot \mathfrak{T}(X) \in A(X) \otimes FW(k).$$

Proof. — As the above formula has to be interpreted componentwise, there is no loss of generality in assuming that Y and X are irreducible. It now follows from the definition of ct, proposition 2.8 and the above quote.

COROLLARY 5.3. — *Let the characteristic of k be $p \neq 0$.   Use the notation of section 4.   For $\eta \in M(x)$, and for $c : Z \to pt$,*

$$c_*(\mathfrak{T}(Z) \cdot (ct\,\lambda_Z)^{-1} \cdot ct(i^!\eta)) \in W(k) \subset FW(k).$$

Proof. — Apply 5.2 for the morphism c to $i^!(\eta') \cdot (\lambda_Z')^{-1} \in M'(Z)$, where η' is the image of η in $M'(x)$.

5.3 is an " integrality theorem " that is of importance in the non-periodic case as well as in 5.5.   It is convenient to make the following definition of the " *Todd class* ", $\mathfrak{T}(x)$, of an object, (X, x), of End : With the sum being taken over the set of fixed components, Z, of x, $\mathfrak{T}(x) = \bigoplus_Z \mathfrak{T}(Z) \cdot (ct\,\lambda_Z)^{-1} \in A(X^x) \otimes FW(k)$.   Then   the   " Lefschetz-Riemann-Roch " theorem is :

THEOREM 5.4. — *Let $f : (Y, y) \to (X, x)$ be a morphism of End.   Let $\eta \in M(y)$ [or $M'(y)$].   Then :*

$$f_*(ct\,\eta \cdot \mathfrak{T}(y)) = ct(f_!\eta) \cdot \mathfrak{T}(x) \in A(X^x) \otimes FW(k).$$

The proof will be given in the next three sections.   The important special case when X is a point is :

COROLLARY 5.5. — *Let $(Y, y) \in$ End.   If Z is a fixed component of y, write $i_Z : Z \to Y$ for the injection and $c_Z : Z \to pt$ for the unique morphism. Let $c : Y \to pt$.   Then the following equality holds in the ring $W(k)$ for $\eta \in M(y)$:*

$$\mathrm{B.tr.}(c_!\eta) = \sum_Z c_{Z*}(\mathfrak{T}(Z) \cdot (ct\,\lambda_Z)^{-1} \cdot ct(i_Z^!\eta)).$$

For an application, consider a finite group, G, acting on a variety, Y. Then a " *G*-sheaf " may be interpreted in terms of " covering homo-

morphisms " as follows : A *G-sheaf*, *F*, on *Y* is a coherent sheaf, also denoted by *F*, on *Y* together with, for each $g \in G$, a homomorphism $\Phi_g : g^* F \to F$ such that the Φ_g satisfy certain compatibility conditions which need not be written down explicitly here. These induce representations of *G* on the $H'[X; F]$, and corollary 5.5 gives a formula for the alternating sum of the Brauer characters of these representations.

The Chern number formulae of ILLUSIE may now be obtained in this context by arguments identical with those of [6], provided that the order of *G* is prime to the characteristic of the field. This assumption may be shown to be necessary by the example of the cyclic group of order *p*, where *p* is the characteristic of *k*, acting linearly on a projective space.

6. **The case of embeddings.**

This section gives a proof of theorem 5.4 for the vital special case when *f* is an embedding. 6.2 is used to reduce 6.1 to the already known 5.2.

PROPOSITION 6.1. — *Theorem 5.4 is valid under the additional hypothesis that f is an embedding.*

Proof. — Without loss of generality assume that both *X* and *Y* are irreducible. Let λ_x and λ_y be the canonical elements of $M(X^x)$ and $M(Y^y)$ of section 4. Let $i : Y^y \to Y$ and $j : X^x \to X$ be the embeddings. Let $f_0 : Y^y \to X^x$ be the restriction of *f*. Assume temporarily the following :

PROPOSITION 6.2. — *There is an element $\gamma \in M(Y^y)$ such that :*

(i) $$\lambda_y . \gamma = f_0^! \lambda_x,$$

and

(ii) $$j^! f_! \eta = f_{0!}(i^! \eta . \gamma) \quad \text{for all } \eta \in M(y).$$

Then

$$
\begin{aligned}
ct(f_! \eta) . \mathfrak{C}(x) &= ct(j^! f_! \eta) . \mathfrak{C}(X^x) . (ct \lambda_x)^{-1} && \text{by definition,} \\
&= ct(f_{0!}(i^! \eta . \gamma)) . \mathfrak{C}(X^x) . (ct \lambda_x)^{-1} && \text{by 6.2 (ii),} \\
&= f_{0*}(ct(i^! \eta . \gamma) . \mathfrak{C}(Y^y)) . (ct \lambda_x)^{-1} && \text{by 5.2 for } f_0, \\
&= f_{0*}(ct \eta . ct \gamma . \mathfrak{C}(Y^y) . f_{0*}(ct \lambda_x)^{-1}) && \text{by 3.1,} \\
&= f_*(ct \eta . \mathfrak{C}(Y^y) . (ct \lambda_y)^{-1}) && \text{by 6.2 (i),} \\
&= f_*(ct \eta . \mathfrak{C}(y)) && \text{by definition.}
\end{aligned}
$$

Proof of 6.2. — Let $X_1, X_2, \ldots$ be the fixed components of *x*. For convenience of notation, whenever a typical one has to be studied it will be taken to be X_1. Let $Y_1, Y_2, \ldots$ be the fixed components of *y*; each

is contained in some fixed component of x. For convenience of notation assume that only $Y_1, Y_2, \ldots, Y_r$ are contained in X_1, and let t be such that $1 \leq t \leq r$. Let $i_t : Y_t \to Y$ and $j_1 : X_1 \to X$ be the embeddings and let $f_t : Y_t \to X_1$ be the restriction of f. It is required to construct an element $\gamma_t \in M(Y_t)$ such that :

(i')
$$\lambda_{Y_t} \cdot \gamma_t = f_t^! \lambda_{X_1},$$

and

(ii')
$$j_1^! f_! \eta = \sum f_{t!}(i_t^! \eta \cdot \gamma_t) \in M(X_1) \quad \text{for all } \eta \in M(y).$$

The following construction will be given in outline only. The detailed checking with local co-ordinates is similar to that used to prove proposition 4.4, but is several times as long. First note that by theorem 2.5 it is sufficient to verify (i') and (ii') for η the image of a covering homomorphism $\psi : y^* F \to F$, where F is a locally free sheaf on Y.

There is a commutative diagram with exact rows on Y_t :

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_t^{\vee} & \longrightarrow & f_t^* N(X_1 \text{ in } X)^{\vee} & \overset{\beta}{\longrightarrow} & N(Y_t \text{ in } Y)^{\vee} & \longrightarrow & 0 \\
& & \downarrow{\omega_t} & & \downarrow{f_t^* \Phi_1} & & \downarrow{\Phi_t} & & \\
0 & \longrightarrow & M_t^{\vee} & \longrightarrow & f_t^* N(X_1 \text{ in } X)^{\vee} & \overset{\beta}{\longrightarrow} & N(Y_t \text{ in } Y)^{\vee} & \longrightarrow & 0
\end{array}
$$

in which $N(\ldots)^{\vee}$ denotes the dual of the appropriate normal bundle, or rather its associated locally free sheaf, β is the canonical homomorphism, and Φ_1 and Φ_t are the endomorphisms of 4.1. This defines $M_t^{\vee}$ and ω_t.

γ_t is defined to be $\sum (-)^u [\Lambda^u \omega_t]$. (i') is now an immediate consequence of the standard properties of the exterior power operation, *see* page 109 of [0]. Local co-ordinates may now be used to find isomorphisms $L^u j_1^* \cdot f_* F \approx \bigoplus_t f_{t*}(i_t^* F \underset{Y_t}{\otimes} \Lambda^u M_t^{\vee})$. The proof of (ii') may now be completed similarly to that of 4.4, i. e. the analogues of diagrams (1) and (2) have to be constructed for $R^0 f_! \psi$ and then the local co-ordinates may be used.

7. Another special case.

In this section, X is an irreducible variety of dimension d, $p : P \to P$ is a periodic (and therefore diagonalisable) linear endomorphism of the projective space P of dimension n, Y is the product $P \times X$, $y : Y \to Y$ is $p \times \text{id}_X$ and $f : Y \to X$ is the projection. Theorem 5.4 will be proved for this f.

There is a canonical " Hopf " invertible sheaf, $\mathcal{O}_P(1)$ in the notation of [EGA], II, 3.2.5, on P, and $\mathcal{O}_P(t)$ is written for its tth tensor power.

Say that a sheaf on Y is of type (H) if it is isomorphic to a finite direct sum of sheaves of the form $\mathcal{O}_P(t) \otimes L_t$, where L_t is a locally free sheaf on X and " $\otimes$ " denotes the Cartesian product. Recursively, say that a multiple extension, F, of sheaves of this form is of type (H') if it is of type (H) or if there exists an exact sequence :

$$0 \to \mathcal{O}_P(s) \otimes L \to F \to F' \to 0,$$

where L is a locally free sheaf on X and where F' has been constructed by such extensions using sheaves of the form $\mathcal{O}_P(t) \otimes L_t$ for $t < s$ only.

PROPOSITION 7.1. — Let $0 \to K \to H_r \to \ldots \to H_0 \to F \to 0$ be an exact sequence of sheaves on Y. If $r \geq n + d$ and if each H_t is of type (H), then K is of type (H').

Proof. — First note the corresponding fact about regular local rings, A, of dimension d. Let R be the graded ring $A[t_0, \ldots, t_n]$. Consider finitely generated $\mathbf{Z}$-graded R-modules. A is canonically such and tensor products of them will be taken ignoring the grading. Let $0 \to N \to M_r \to \ldots \to M_0 \to G \to 0$ be an exact sequence of them, with $r \geq n + d$ and each M_t free. Then, by [8], p. 210, th. 4.2, N is projective as an ungraded R-module. Hence $N \otimes A$ is a projective A-module, and so each of its graded components is free. Hence there is a free $\mathbf{Z}$-graded R-module M and a $\mu : M \to N$ such that $\mu \otimes A$ is an isomorphism. As Coker $(\mu) \otimes A = 0$, the argument of page 216 of [8] shows that Coker $(\mu) = 0$. As Tor$^1(N, A) = 0$, Ker $(\mu) \otimes A = 0$, and the same argument shows that Ker $(\mu) = 0$. So N is free.

Now let a be a point of X. [EGA], II, 2.7.5 and the above result show that there is an open neighbourhood, U, of a such that $K \mid P \times U$ is isomorphic to a finite direct sum of sheaves of the form $\mathcal{O}_P(t) \otimes L(U, t)$, where $L(U, t)$ is a locally free sheaf on U.

Write $K(t)$ for the tensor product of K with $\mathcal{O}_P(t) \otimes \mathcal{O}_X$. Then the above and [EGA], III, 2.1.15 plus the so-called projection formula show that there is a t_0 such that the adjunction morphism $f^*f_*(K(t)) \to K(t)$ has zero domain for $t < t_0$ and is injective for $t = t_0$. [EGA], III, 7.9.10 shows that $f_*(K(t_0))$ is locally free. Iteration of this argument shows that K is of type (H').

PROPOSITION 7.2. — *Let* $\Phi : y^*F \to F$ *be a covering homomorphism on* Y. *Then there exists an exact sequence, in which each* H_t *is of type* (H) *and* K *is of type* (H') :

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & y^*K & \longrightarrow & y^*H_{n+d} & \longrightarrow & \ldots & \longrightarrow & y^*H_0 & \longrightarrow & y^*F & \longrightarrow & 0 \\
& & \theta \downarrow & & \Psi_{n+d} \downarrow & & & & \Psi_0 \downarrow & & \Phi \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & H_{n+d} & \longrightarrow & \ldots & \longrightarrow & H_0 & \longrightarrow & F & \longrightarrow & 0
\end{array}
$$

Proof. — [EGA], III, 2.2.1 and [EGA], III, 7.9.10 may be used to construct Ψ_0 and H_0 as in the proof of 2.2. This process for constructing a resolution may be iterated $n + d$ times and then 7.1 is applicable.

LEMMA 7.3. — *Let L_s, L_t be locally free sheaves on X. Then, if $s > t$,*

$$\mathrm{Hom}[Y;\, y^*(\mathcal{O}_P(s) \otimes L_s),\, \mathcal{O}_P(t) \otimes L_t] = 0.$$

Proof. — As $K = \mathrm{Hom}_Y(y^*(\mathcal{O}_P(s) \otimes L_s),\, \mathcal{O}_P(t) \otimes L_t)$ is isomorphic to $\mathcal{O}_P(t - s) \otimes \mathrm{Hom}_X(L_s, L_t)$, [EGA], III, 2.1.15 plus the so-called projection formula shows that $f_* K = 0$. Whence the result.

PROPOSITION 7.4. — *$M(y)$ is additively generated by elements of the form $[\theta]$, where $\theta : y^*(\mathcal{O}_P(t) \otimes L) \to \mathcal{O}_P(t) \otimes L$ and where t is an integer and L is a locally free sheaf on X.*

Proof. — This is an immediate consequence of 7.2 and 7.3.

Note that another use of [EGA], III, 2.1.15 could be made to show that $M(y)$ is a free $M(X)$-module of rank $n + 1$. The corresponding result is standard in various K-theories.

PROPOSITION 7.5. — *Theorem 5.4 is valid under the additional hypothesis that f is as constructed in the first paragraph of this section.*

Proof. — In view of 7.4, it suffices to verify 5.4 for $\eta = [\theta]$. The fixed components of y are the products of certain projective subspaces of P with X; it is now seen to be sufficient to check 7.5 in the case when X is a point. This may be done in some two pages of explicit calculations; as these calculations are modifications of those on page 119 of [0] and on page 597 of [5], they will be omitted.

8. Completion of the proof.

Let $f : (Y, y) \to (X, x)$ be a morphism of End. Then, if $i : (Y, y) \to (P, p)$ is an embedding as in lemma 2.1, f may be factorised as $\pi . (i, f)$, where $(i, f) : (Y, y) \to (P \times X, p \times x)$ is also an embedding and $\pi : (P \times x, p \times x) \to (X, x)$ is the projection. By the usual formal argument (*cf.* page 113 of [0]), to prove theorem 5.4 for f it suffices to prove it for (i, f) and for π. Now embeddings were dealt with in section 6. In the special case when x is the identity, π was dealt with in section 7.

So 5.4 has already been proved for the special case when x is the identity, which is the case of greatest interest. To prove it in general, it remains to extend 7.5 to the above π.

The fixed components of the above x will be denoted by X_α, $\alpha = 1, 2, \ldots$ Write $j_\alpha : X_\alpha \to X$ for the embeddings. The fixed components of the above p will be denoted by P_t, $t = 1, 2, \ldots$ Write i_α for $\mathrm{id}_P \times j_\alpha : P \times X_\alpha \to P \times X$. Write $\pi_\alpha : P \times X_\alpha \to X_\alpha$ for the projection.

PROPOSITION 8.1. — *Theorem 5.4 is valid for the above* π.

Proof. — Assume temporarily that

$$(\bigstar) \qquad \pi_{\alpha !}.i_{\alpha}^{!}\eta = j_{\alpha}^{!}.\pi_{!}\eta \in M(X_{\alpha}), \quad \text{for all } \eta \in M(p \times x).$$

It is required to prove that, for all $\eta \in M(p \times x)$,

$$\pi_{*}(ct\pi.\mathfrak{C}(p \times x)) = ct(\pi_{!}\eta).\mathfrak{C}(x) \in \bigotimes_{\alpha} A(X_{\alpha}) \otimes FW(k).$$

Denote the α components of each side by $(\ldots)|_{\alpha}$: it is required to prove that, for each α, these α components are equal.

$$
\begin{aligned}
ct(\pi_{!}\eta).\mathfrak{C}(x)|_{\alpha} &= ct(j_{\alpha}^{!}.\pi_{!}\eta).\mathfrak{C}(X_{\alpha}).(ct\lambda_{X_{\alpha}})^{-1} &&\text{by definition,}\\
&= ct(\pi_{\alpha !}.i_{\alpha}^{!}\eta).\mathfrak{C}(X_{\alpha}).(ct\lambda_{X_{\alpha}})^{-1} &&\text{by } (\bigstar),\\
&= \pi_{\alpha *}(ct(i_{\alpha}^{!}\eta).\mathfrak{C}(p \times \mathrm{id}_{X_{\alpha}})).(ct\lambda_{X_{\alpha}})^{-1} &&\text{by 7.5 for } \pi_{\alpha},\\
&= \pi_{\alpha *}(ct(i_{\alpha}^{!}\eta).\mathfrak{C}(p \times \mathrm{id}_{X_{\alpha}}).\pi_{\alpha}^{*}(ct\lambda_{X_{\alpha}})^{-1}) &&\text{by 3.1,}\\
&= \pi_{*}(ct\eta.\mathfrak{C}(p \times x))|_{\alpha}.
\end{aligned}
$$

The last equality follows from expressing each side as a sum over t of contributions from the fixed components $P_{t} \times X_{\alpha}$ and then noting that the normal bundle of a product of embeddings is the product of the normal bundles.

My Oxford thesis contains an easy proof of $(\bigstar)$, but uses derived category theory. To establish $(\bigstar)$ by the methods of this paper, it would be necessary to use the spectral sequence of [EGA], III, 6.7.3 : for F a locally free sheaf on $P \times X$,

$$E_{2}^{p,q} = L^{-p}j_{\alpha}^{*}.R^{q}\pi_{*}F \implies R^{*}\pi_{\alpha *}.i_{\alpha}^{*}F.$$

The details may be omitted.

9. Remarks on the non–periodic case.

The above arguments depend very heavily on the hypothesis of the periodicity of the endomorphisms of varieties under consideration. If this hypothesis is removed, it is possible to use derived category theory to give a more general definition of the functor M. If lemmas 4.1 and 4.3 are made into hypotheses, a " Lefschetz-Riemann-Roch " theorem may be formulated for a far larger class of endomorphisms. In view of proposition 4.4, which does not depend on the hypothesis of periodicity, this is the only plausible fixed point formula.

I conjecture that such a theorem is valid if k has characteristic zero.

If k has characteristic $p \neq 0$, the Frobenius endomorphism, y, of the projective line, Y, (with $X = pt$) provides a counterexample to such a theorem. Indeed, for the canonical covering homomorphism $y^{*}\mathcal{O}_{Y} \to \mathcal{O}_{Y}$, it may be seen that the generalisation of 5.5 would yield $\mathrm{I} = p + \mathrm{I}$.

However my Oxford thesis gives a proof of corollary 5.5 with the equation replaced by a congruence mod p without the hypothesis of periodicity. (The proof there is complete only in the case when each fixed component is a point or a curve.)   This is therefore a formula for the classical, as opposed to the Brauer, trace.

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