BULLETIN DE LA S. M. F.

M.E. HERRERA

Integration on a semianalytic set

Bulletin de la S. M. F., tome 94 (1966), p. 141-180
<http://www.numdam.org/item?id=BSMF_1966__94__141_0>

Bull. Soc. math. France,
94, 1966, p. 141 à 180.

INTEGRATION ON A SEMIANALYTIC SET (*)

BY

MIGUEL E. HERRERA.

Let M be a complex analytic subset of dimension p of the complex analytic manifold X, and let M^* denote the submanifold of the regular points of dimension p of M. P. LELONG has proved in [8] that a differentiable form on X with compact support can be integrated on the naturally oriented manifold M^*, and that the current on X so obtained is closed. We extend this result to the case in which M is a locally closed semianalytic set of a real analytic manifold.

In chapter I, we summarize definitions and properties of semianalytic sets, currents and Borel-Moore homology which are used in this paper. Instead of relating the currents on a differentiable manifold to the Borel-Moore homology by means of the identification of the latter with singular differentiable homology, we have preferred to do it directly. Some proofs can then be shortened, although the definition of integration currents on a manifold appears to be somewhat different, formally, from the classical one [14].

In chapter II, we prove, following the method of [8], that if the locally closed semianalytic set M has dimension p, then each p-homology class c of M with real coefficients defines a current $I(M, c)$ on X, which can be called an integration current of M (Theorem II, A, 2.1). In particular, the current associated in [8] to a complex analytic set M is that defined by the fundamental class of M [2].

The second result is a Stokes' Theorem (Theorem II, B, 2.1). It is proved that the border of $I(M, c)$ is the integration current of the semianalytic set $bM = \overline{M} - M$ defined by the topological boundary ∂c of c. It follows immediately that the only closed integration currents

(*) Results in this paper are part of the author's thesis, presented at the University of Buenos Aires, 1965.

of M are those defined by projections onto M of homology classes of the closure $\overline{M}$. In particular, Lelong's current is closed, since a complex analytic set is closed.

Both theorems are proved locally, by projecting a semianalytic set in $\mathbf{R}^n$ into convenient subspaces. To do this, we use Łojasiewicz' normal decompositions, slightly modified.

Particular cases of these results have been announced in [6] and [7]. L. Bungart has announced a Stokes' theorem for real analytic sets in [3 *a*].

I am much indebted to Professors S. Łojasiewicz and R. Ricabarra for their valuable suggestions and help.

Chapter I.

Preliminaries.

A. Semianalytic sets.

In this chapter, a summary of the theory of semianalytic sets is presented. The references for definitions, properties and some of the proofs are [9], [10], [11], [12] and [13]. X is always a real analytic manifold of dimension n and $\mathbf{R}$ and $\mathbf{C}$ denote the field of the real and complex numbers, respectively.

1. Generalities.

1.1. Definition. — For each $x \in X$, let $S(x)$ be the smallest family of germs in x of subsets of X such that:

(1) $a, b \in S(x)$ implies $a \cup b \in S(x)$ and $a - b \in S(x)$;

(2) if f is a real analytic function on a neighborhood of x, the germ in x of the set $(f(y) > 0)$ belongs to $S(x)$. Then a subset M of X is semianalytic if for each $x \in X$ the germ of M in x belongs to $S(x)$.

1.2. — A locally finite union and a finite intersection of semianalytic sets is semianalytic. The complement of a semianalytic set is semianalytic. A subset of a closed analytic submanifold X' of X is semianalytic in X' if and only if it is so in X. The image of a semianalytic set under an analytic isomorphism is semianalytic. The closure, the interior and the boundary of a semianalytic set are semianalytic. A semianalytic set is locally connected. The family of the connected components of a semianalytic set is locally finite. A finite cartesian product of semianalytic sets is semianalytic.

1.3. — Let M be a semianalytic set in X and p a natural number > 0. A point $x \in M$ is *p-regular* if there is an open neighborhood U of x such that $M \cap U$ is an analytic submanifold of dimension p of U; x is o-*regular* if it is an isolated point of M. The sets of regular points of M (i. e., p-regular points for some p) is dense in M. The dimension $\dim M$ of M is $\leqq p$ if there are not q-regular points of M with $q > p$; $\dim M = p$ if $\dim M \leqq p$ but not $\dim M \leqq p - 1$.

Let $\dim M = p$; then $\dim \overline{M} = p$ and $\dim (\overline{M} - M) < p$ ($\overline{M}$ denotes the closure of M). We call the semianalytic set $bM = \overline{M} - M$ the *border* of M; bM is closed if and only if M is locally closed. The set $M^\star$ of the p-regular points of M is a p-dimensional analytic submanifold (not necessarily closed) of X, and $sM = M - M^\star$ is a semianalytic set of X with $\dim sM < p$. We call sM the *singular part* of M, and denote $\partial M = bM \cup sM$. $\dim \partial M < p$, and ∂M is closed if M is locally closed; in this case, $M^\star$ is a closed p-dimensional submanifold of $X - \partial M$. Observe that, again, we have the decomposition $sM = (sM)^\star + s(sM)$, and that by repeating the process it is possible to express M as a finite disjoint union of analytic submanifolds of X with strictly decreasing dimensions. If U is an open subset of X, then $M \cap U$ is a semianalytic subset of U and

$$M^\star \cap U = (M \cap U)^\star, \qquad (bM) \cap U = b(M \cap U) \qquad \text{and}$$
$$(\partial M) \cap U = \partial (M \cap U).$$

2. Normal decompositions.

2.1. — A function $H(z_1, \ldots, z_k; z_h)$ is called a distinguished polynomial in z_h centered at the origin of $\mathbf{C}^k \times \mathbf{C}$ if it is a polynomial in z_h with leading coefficient equal to 1 and whose other coefficients are holomorphic functions defined on a neighborhood of the origin of $\mathbf{C}^k$ and vanishing at the origin; it is called a real distinguished polynomial if its restriction $H(x_1, \ldots, x_k; x_h)$ to $\mathbf{R}^k \times \mathbf{R}$ is a real function.

A *normal system* in $\mathbf{R}^n$ centered at the origin is a family $(H_h^k; 0 \leqq k < h \leqq n)$ of real distinguished polynomials $H_h^k(x_1, \ldots, x_k; x_h)$ in the variable x_h, centered at the origin, with discriminants $D_h^k(x_1, \ldots, x_k) \not\equiv 0$, and such that in some neighborhood of the origin

(*a*) $H_k^{k-1}(z_1, \ldots, z_{k-1}; z_k) = H_h^k(z_1, \ldots, z_k; z_h) = 0$
implies $H_h^{k-1}(z_1, \ldots, z_{k-1}; z_h) = 0$;

(*b*) $D_h^k(z_1, \ldots, z_k) = 0$ implies $H_k^{k-1}(z_1, \ldots, z_{k-1}; z_k) = 0$

for all $1 \leqq k < h \leqq n$.

A neighborhood $Q = (x = (x_1, \ldots, x_n); |x_i| < d_i, i = 1, \ldots, n)$ of the origin of $\mathbf{R}^n$ is called *normal* (for the above normal system) if the functions $H_h^k(z_1, \ldots, z_k; z_h)$ are holomorphic on $((z_1, \ldots, z_n) \in \mathbf{C}^n; |z_i| \leqq d_i)$, satisfy (*a*) and (*b*) on $((z_1, \ldots, z_n) \in \mathbf{C}^n; |z_i| < d_i)$, and if

(*c*) $|z_i| < d_i$, $i = 1, \ldots, k$, and $H_h^k(z_1, \ldots, z_k; z_h) = 0$ implies $|z_h| < d_h$

for all k, h with $0 \leqq k < h \leqq n$.

There exists a fundamental system of normal neighborhoods of the origin.

Now let Q be a normal neighborhood for a normal system $H_h^k (0 \leqq k < h \leqq n)$. For each $k = 0, \ldots, n$, let us define

$$V^k = (x \in Q : H_n^{n-1} = \ldots = H_{k+1}^k = 0,\ H_k^{k-1} \neq 0) \quad (^1)$$

and let $\Gamma_\varkappa^k (\varkappa = \ldots)$ the family of the connected components of V^k. Then the decomposition

$$Q = \bigcup_{k=0}^{n} V^k = \bigcup_{k=0}^{n} \bigcup_{\varkappa} \Gamma_\varkappa^k$$

is called the *normal decomposition* of Q for the given normal system; the sets $\Gamma_\varkappa^k$ are the *members* of the decomposition.

Let c be a point of the real analytic manifold X. A *normal system* $\mathfrak{NS}$ *in* X *centered at* c is a pair $(\varphi, \mathfrak{NS}_0)$, where φ is a coordinate map $\varphi : U \to \varphi(U) \subset \mathbf{R}^n$ such that $c \in U \subset X$ and $\varphi(c) = 0 \in \mathbf{R}^n$ and where $\mathfrak{NS}_0$ is a normal system in $\mathbf{R}^n$ centered in $0 \in \mathbf{R}^n$. A *normal neighborhood for* $\mathfrak{NS}$ is a neighborhood $Q = \varphi^{-1}(Q_0)$, where $Q_0 \subset \varphi(U)$ is a normal neighborhood for $\mathfrak{NS}_0$. The *normal decomposition of* Q *for* $\mathfrak{NS}$ is the decomposition $Q = \bigcup_{k, \varkappa} \varphi^{-1}(\Gamma_\varkappa^k)$, the $\Gamma_\varkappa^k$ being the members of the normal decomposition of Q_0 for $\mathfrak{NS}_0$. In general, a *normal decomposition at* $c \in X$ is the normal decomposition of a normal neighborhood of c for a normal system in X centered at c.

2.2. — (*a*) Every normal decomposition in X is finite, and its members are semianalytic sets in X.

(*b*) Let $Q = \bigcup_{k, \varkappa} \Gamma_\varkappa^k$ be a normal decomposition. Each set $(\overline{\Gamma}_\varkappa^k - \Gamma_\varkappa^k) \cap Q$ is a union of some $\Gamma_\varkappa^j$ with $j < k$; consequently, $(\overline{V}^k - V^k) \cap Q \subset \bigcup_{i=0}^{k-1} V^i$ for each $k = 1, \ldots, n$, and $\bigcup_{i=0}^{k} V^i$ is closed in Q.

(1) $H_0^{-1} \equiv 1$.

(c) Let $Q = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition at $0 \in \mathbf{R}^n$ for the normal system $(H_h^k;\ 0 \leqq k < h \leqq n)$. Each $\Gamma_\varkappa^k$ with $0 < k < n$ has the expression

$\Gamma_\varkappa^k = (x \in Q;\ u = (x_1, \ldots, x_k) \in \Omega$ and $x_j = f_j(u), j = k+1, \ldots, n)$,
where Ω is an open set in $\mathbf{R}^k$, $0 \in \Omega^-$ and f_j are analytic functions on Ω such that $H_j^k(u; f_j(u)) = 0$ in Ω and $\lim_{u \to 0} f_j(u) = 0$ for each $j = k+1, \ldots, n$.

(d) Each member $\Gamma_\varkappa^k (0 < k \leqq n)$ of a normal decomposition at $c \in X$ is a k-dimensional analytic submanifold of X; the $\Gamma_\varkappa^n$ are open subsets of X and the only 0-dimensional member $\Gamma_\varkappa^0$ is equal to the set c; $c \in \Gamma_\varkappa^k$ for all $(k, \varkappa)$.

(e) Let $Q = (|x_i| < d_i;\ i = 1, \ldots, n) = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ be a normal decomposition for the normal system $\mathcal{NS} = (H_h^k;\ 0 \leqq k < h \leqq n)$ at $0 \in \mathbf{R}^n$, and let $0 < p < n$. Then $\mathcal{NS}_p = (H_h^k;\ 0 \leqq k < h \leqq p)$ is a normal system in $\mathbf{R}^p$ at $0 \in \mathbf{R}^p$ and $Q_p = (|x_i| < d_i,\ i = 1, \ldots, p)$ is a normal neighborhood for $\mathcal{NS}_p$. If

$$Q_p = \bigcup_{k=0}^{p} {}_pV^k = \bigcup_{k=0}^{p} \bigcup_{\nu} {}_p\Gamma_\nu^k$$

is the normal decomposition of Q_p for $\mathcal{NS}_p$ and $\pi : Q \to Q_p$ is the map $(x_1, \ldots, x_n) \to (x_1, \ldots, x_p)$, each $\Gamma_\varkappa^k$ with $k \leqq p$ verifies $\pi(\Gamma_\varkappa^k) = {}_p\Gamma_\nu^k$ for some ν.

2.3. Definition. — Let A be a subset of X. A normal decomposition $\bigcup_{k\varkappa} \Gamma_\varkappa^k$ in X is said compatible with A if $\Gamma_\varkappa^k \subset A$ or $\Gamma_\varkappa^k \subset X - A$ for each k, $\varkappa$. A normal system $\mathcal{NS}$ at $c \in X$ is said compatible with A if there is a neighborhood basis of c whose members are normal for $\mathcal{NS}$, and whose corresponding normal decompositions are compatible with A. Such neighborhoods are also said compatible with A.

(a) Let A_i $(i = 1, \ldots, s)$ be a finite family of semianalytic subsets of X and let $c \in X$. Then there is a normal system $\mathcal{NS}$ at c compatible with each set A_i and, in this case, a normal neighborhood basis for $\mathcal{NS}$ compatible with each A_i.

(b) If $Q = \bigcup_{i=0}^{n} V^i = \bigcup_{k,\varkappa} \Gamma_\varkappa^k$ is a normal decomposition compatible with the semianalytic set M and if $\dim M = p$, then

$$M = M \cap \left(\bigcap_{i=0}^{p} V^i \right) \quad \text{and} \quad \Gamma_\varkappa^p \subset M \ \text{ implies } \Gamma_\varkappa^p \subset M^* = M - sM.$$

(c) Let $O'(\mathbf{R}^n)$ be the family of all coordinate maps of $\mathbf{R}^n$ defined by the orthonormal bases of $\mathbf{R}^n$; $O'(\mathbf{R}^n)$ is identified with the space $O(n, \mathbf{R})$ of the orthogonal matrices of dimension n. Let A_i $(i = 1, \ldots, s)$ be semianalytic sets in a neighborhood of $0 \in \mathbf{R}^n$ and let $O'(A_1, \ldots, A_s)$ be the set of the maps in $O'(\mathbf{R}^n)$ for which there are normal systems at $0 \in \mathbf{R}^n$ compatible with each A_i. Then $O'(A_1, \ldots, A_s)$ is dense in $O'(\mathbf{R}^n)$.

B. Homology.

The homology theory used in this paper is that of Borel-Moore for locally compact spaces, as it is presented in [1] and [2] (or in the forthcoming book by G. Bredon [3]). We only state here for further reference, without proofs, some results not explicitly mentioned in the quoted papers, which are direct consequences of the theory. Unless specifically mentioned, notations and conventions of [2] are preserved. The cohomology is that defined in [4]. X is always a locally compact space, K a principal domain, and $\mathbf{Z}$ the integer ring.

1. — Let F be a closed subspace of X and $U = X - F$. There exists an exact sequence of homology

$$\ldots \to H_q(F; K) \xrightarrow{i_{FX}} H_q(X; K) \xrightarrow{j_{XU}} H_q(U; K) \xrightarrow{\partial_{UF}} H_{q-1}(F; K) \to \ldots;$$

∂_{UF} is called the boundary homomorphism for the pair $F \subset X$. The maps in this sequence will occasionally be abbreviated by i, j and ∂, respectively. Let Y be another closed subspace of X, and $V = X - Y$. Then in the diagram (1.1) (all homology groups with coefficients in K)

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"'] & \vdots \arrow[d, "i"'] & \vdots \arrow[d, "i"] & \\
\ldots \arrow[r] & H_q(F) \arrow[r, "i"] \arrow[d, "j"'] & H_q(X) \arrow[r, "j"] \arrow[d, "j"'] & H_q(U) \arrow[r, "\partial"] \arrow[d, "j"] & \ldots \\
\ldots \arrow[r] & H_q(F \cap V) \arrow[r, "i"] \arrow[d, "\partial"'] & H_q(V) \arrow[r, "j"] \arrow[d, "\partial"'] & H_q(U \cap V) \arrow[r, "\partial"] \arrow[d, "\partial"] & \ldots \\
\ldots \arrow[r] & H_{q-1}(F \cap Y) \arrow[r, "i"] \arrow[d] & H_{q-1}(Y) \arrow[r, "j"] \arrow[d] & H_{q-1}(U \cap Y) \arrow[r, "\partial"] \arrow[d] & \ldots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{1.1}$$

all squares commute, but the one with only boundaries, which anti-commutes. In the case $Y \subset F$, we have also the commutativity of

(1.2)
$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H_q(F) & \xrightarrow{i} & H_q(X) & \xrightarrow{j} & H_q(X-F) & \xrightarrow{\partial} & \ldots \\
& \uparrow i & & \updownarrow & & \uparrow j & & \\
\ldots \longrightarrow & H_q(Y) & \xrightarrow[i]{} & H_q(X) & \xrightarrow[j]{} & H_q(X-Y) & \xrightarrow[\partial]{} & \ldots
\end{array}$$

Let us suppose that the family $U_\alpha (\alpha \in J)$ of the connected components of $U = X - F$ is finite. Then

$$\sum_{\alpha \in J} j^{UU_\alpha} : \quad H_*(U; K) \to \sum_{\alpha \in J} H_*(U_\alpha; K)$$

is an isomorphism and $\sum_{\alpha \in J} i_{U_\alpha U}$ is the inverse isomorphism. For each $\alpha \in J$, let

$$\partial_{U_\alpha F} : \quad H_q(U_\alpha; K) \to H_{q-1}(F; K) \qquad (q \in Z)$$

be the boundary for $F \subset F \cup U_\alpha$. Then the following diagram commutes, as it is deduced from (1.1)

(1.3)
$$\begin{array}{ccc}
H_q(U; K) & \xrightarrow{\partial_{UF}} & H_{q-1}(F; K) \\
\Big\downarrow \sum_{\alpha \in J} j^{UU_\alpha} & \nearrow \sum_{\alpha \in J} \partial_{U_\alpha F} & \\
\sum_{\alpha \in J} H_q(U_\alpha; K) & &
\end{array} \qquad (q \in Z) \quad (2)$$

2. — For each integer q, there is a split exact sequence

$$0 \to \operatorname{Ext}(H_c^{q+1}(X; K), K) \to H_q(X; K) \xrightarrow{\alpha_X} \operatorname{Hom}(H_c^q(X; K), K) \to 0$$

which is compatible with the homomorphisms in the exact sequences of homology and compact cohomology ([4], II, 4.10) for the pair $F \subset X$, F being closed in X (3).

3. — The local homology sheaf $\mathcal{H}_*(X; K)$ on X is the sheaf generated by the presheaf $U \to H_*(U; K)$ (U open in X). For each family Φ of supports on X, there exists a natural homomorphism

$$\Delta : \quad H_*^\Phi(X; K) \to H_\Phi^0(X; \mathcal{H}_*(X; K)) = \Gamma_\Phi(\mathcal{H}_*(X; K)).$$

(2) *Cf.* Bredon [3], chap. V, § 5 and Ex. 30.
(3) *Id.*, Ex. 29.

The support of $c \in H_*^\Phi(X;\, K)$ is defined as the support of the section $\Delta\,(c)$. If the cohomologic dimension $\dim_K X$ is $\leqq n$,

$$\Delta:\quad H_n^\Phi(X;\, K) \to \Gamma_\Phi(\mathcal{H}_n(X;\, K))$$

is an isomorphism ([2], 1.10). Note that if M is a semianalytic set, then $\dim M = \dim_{\mathbf{Z}} M$.

4. — Let X be an n-dimensional manifold with a finite number of connected components. The sequence of 2 and the isomorphism

$$H_c^n(X;\, \mathbf{R}) \to H_c^n(X;\, \mathbf{Z}) \otimes \mathbf{R}$$

give natural isomorphisms

$$H_n(X;\, \mathbf{Z}) \otimes \mathbf{R} \to \mathrm{Hom}(H_c^n(X,\, \mathbf{Z}),\, \mathbf{Z}) \otimes \mathbf{R} \qquad \text{and}$$
$$H_n(X;\, \mathbf{R}) \to \mathrm{Hom}(H_c^n(X;\, \mathbf{Z}) \otimes \mathbf{R},\, \mathbf{R}).$$

As $H_c^n(X;\, \mathbf{Z})$ is a free finitely generated module, the natural homomorphism

$$\mathrm{Hom}(H_c^n(X;\, \mathbf{Z}),\, \mathbf{Z}) \otimes \mathbf{R} \to \mathrm{Hom}(H_c^n(X;\, \mathbf{Z}) \otimes \mathbf{R},\, \mathbf{R})$$

is an isomorphism, and by composition we obtain a natural isomorphism

$$H_n(X;\, \mathbf{R}) \to H_n(X;\, \mathbf{Z}) \otimes \mathbf{R}.$$

This isomorphism is compatible with the integer and real homology sequences for a pair $M \subset X$, where M is an $(n-1)$-dimensional closed submanifold of X such that M and $X - M$ have a finite number of connected components. Moreover, an isomorphism

$$\mathcal{H}_n(X;\, \mathbf{R}) \to \mathcal{H}_n(X;\, \mathbf{Z}) \otimes \mathbf{R}$$

is induced.

5. Cartesian product of homology classes.

Let X and Y be topological spaces with $\dim_K X = m$ and $\dim_K Y = n$. The sequence of 2 gives natural maps

$$\begin{array}{lccc}
 & H_m(X;K) \otimes H_n(Y;K) & \xrightarrow{\alpha_X \otimes \alpha_Y} & \mathrm{Hom}(H_c^m(X;K),\, K) \otimes \mathrm{Hom}(H_c^n(Y;K),\, K) \\
 & & & \downarrow \lambda \\
(5.1) & & & \mathrm{Hom}(H_c^m(X;\, K) \otimes H_c^n(Y;K),\, K) \\
 & & & \downarrow \eta \\
 & H_{m+n}(X \times Y;\, K)) & \xrightarrow[\alpha_{X \times Y}]{} & \mathrm{Hom}(H_c^{m+n}(X \times Y;\, K),\, K)
\end{array}$$

that, except λ, are isomorphisms, because of the dimensions of X, Y and $X\times Y$; η is deduced from the Künneth's isomorphism

$$H_c^{m+n}(X\times Y;\, K) \to H_c^m(X;\, K)\otimes H_c^n(Y;\, K).$$

The composition gives an homomorphism

$$(5.2)\qquad \odot:\ H_m(X;\, K)\otimes H_n(Y;\, K) \to H_{m+n}(X\times Y;\, K)$$

which can be proved to be associative. If $H_c^m(X;\, K)$ and $H_c^n(Y;\, K)$ are finitely generated free modules, in particular if X and Y are manifolds with a finite number of connected components, then $\odot$ is an isomorphism ([2], 2.11, *cf.* also [3], chap. V, 13.4). In general, we abbreviate $\odot\,(c\otimes c') = c\odot c'$, for each $c\in H_m(X, K)$ and $c'\in H_m(X, K)$.

Let F be a closed subset of X with $\dim_K F = m - 1$ and let $V = X - F$. Then the following diagram, in which all homology groups have coefficients in K, is commutative

$$(5.3)\qquad \begin{array}{ccccc} H_m(X)\otimes H_n(Y) & \xrightarrow{j^{X,V}\otimes i} & H_m(V)\otimes H_n(Y) & \xrightarrow{\partial_{VX}\otimes i} & H_{m-1}(F)\otimes H_n(Y) \\ \downarrow{\scriptstyle\odot} & & {\scriptstyle\odot}\downarrow & & {\scriptstyle\odot}\downarrow \\ H_{m+n}(X\times Y) & \xrightarrow[j]{} & H_{m+n}(V\times Y) & \xrightarrow[\partial]{} & H_{m+n-1}(F\times Y) \end{array}$$

6. The fundamental class of an open set in $\mathbf{R}^n$.

We denote $\mathbf{R}_- = (x\in\mathbf{R};\, x<0)$ and $\overline{\mathbf{R}}_- = (x\in\mathbf{R};\, x\leq 0)$. Let $e_1\in H_1(\mathbf{R};\mathbf{Z})$ be the fundamental class of $\mathbf{R}$ such that $\partial_{\mathbf{R}_-,0}\circ j^{\mathbf{R},\mathbf{R}_-}(e_1)$ is the canonical generator of the homology of the point 0, $\partial_{\mathbf{R}_-,0}$ being the boundary for $0\subset\overline{\mathbf{R}}_-$.

The *canonical fundamental class of* $\mathbf{R}^n$ is defined as the class

$$e_n = \overbrace{e_1\odot\ldots\odot e_1}^{n}\in H_n(\mathbf{R}^n;\, \mathbf{Z})\quad (4).$$

If $\mathbf{R}^n = \mathbf{R}^m\times\mathbf{R}^{n-m}$, then $e_n = e_m\odot e_{n-m}$, since $\odot$ is associative. Let W be an open set in $\mathbf{R}^n$; $e_W = j^{\mathbf{R}^n,W}(e_n)$ is called the *canonical fundamental class of* W. Then the commutativity of (5.3) and the definition of e_1 imply:

6.1. — Let U be an open set in $\mathbf{R}^{n-1}$ and $W = \mathbf{R}_-\times U\subset\mathbf{R}^n$. Then $\partial_{WU}(e_W) = e_U$, ∂_{WU} being the boundary for $0\times U\subset\overline{\mathbf{R}}_-\times U$.

6.2. — Let now U and V be open subsets in $\mathbf{R}^n$, $f: U\to V$ a differentiable homeomorphism of class $\mathcal{C}^s$ $(s\geq 1)$ and $f_*: H_n(U, \mathbf{Z})\to H_n(V;\mathbf{Z})$ the induced map. Then $f_*(e_U) = e_V$ if and only if the jacobian determinant of f is >0 on U. This follows from the similar fact for singular

(4) This definition is equivalent to that given in [2], 2.9.

homology and from the identification, on a manifold, between this homology and Borel-Moore homology ([2], 1.11).

6.3. — Let $f: X \to U$ be a homeomorphism between X and the open set U in $\mathbf{R}^n$. Then $e_f = f_*^{-1}(e_U) \in H_n(X; \mathbf{Z})$ is called the fundamental class of X defined by f. In particular, if E is a real vector space of dimension n, each base of E defines an isomorphism $E \to \mathbf{R}^n$ and consequently a fundamental class of E. Two bases of E define the same fundamental class if and only if the determinant of the matrix of change from one base to the other is positive. Thus an algebraic orientation of E defines a fundamental class of E.

Let E' and E'' be supplementary subspaces of E of dimensions p and $n-p$, respectively. Let $e' \in H_p(E'; \mathbf{Z})$ and $e'' \in H_{n-p}(E''; \mathbf{Z})$ be fundamental classes corresponding to algebraic orientations of E' and E''. Then $e' \odot e'' \in H_n(E; \mathbf{Z})$ is the fundamental class corresponding to the sum of the orientations of E' and E'' (*cf.* [2], 2.11).

C. Currents.

We assume familiarity with definitions and general properties of currents, as given in [8] and [14]. Unless otherwise stated, X is a paracompact differentiable manifold of class $\mathcal{C}^\infty$ and dimension n, not necessarily connected. $\mathcal{E}^p(X)$ denotes the space of differential forms on X of degree p and class $\mathcal{C}^\infty$, $\mathcal{D}^p(X)$ the subspace of forms with compact support and

$$\mathcal{E}(X) = \sum_{p=0}^{n} \mathcal{E}^p(X), \qquad \mathcal{D}(X) = \sum_{p=0}^{n} \mathcal{D}^p(X).$$

$\mathcal{E}(X)$ and $\mathcal{D}(X)$ are always considered with their usual structures of topological vector spaces. $\mathcal{D}'(X) = \sum \mathcal{D}'_p(X)$ denotes the space of (impair) currents on X, or topological dual of $\mathcal{D}(X)$, and $\mathcal{D}'_p(X)$ the subspace of currents of dimension p, or currents which are zero on $\sum_{q \neq p} \mathcal{D}'_q(X)$. $\mathcal{E}(X)$ and $\mathcal{D}'(X)$, together with the exterior differential d and the border b, respectively, are differential graded vector spaces, and $\mathcal{D}(X)$ is a differential subspace of $\mathcal{E}(X)$. If $T \in \mathcal{D}'(X)$, then $bT(a) = T(da)$ for each $a \in \mathcal{D}(X)$; T is said to be closed if $bT = 0$.

Let U be an open set of X and $T \in \mathcal{D}'(X)$. $T \mid U$ denotes the restriction of T on U; that is, $(T \mid U)(a) = T(a)$ for all $a \in \mathcal{D}(U)$. Let $S \in \mathcal{D}'(U)$; $T \in \mathcal{D}'(X)$ is called an extension of S on X if $T \mid U = S$.

Let X and Y be differentiable manifolds of class $\mathcal{C}^\infty$ and $f: X \to Y$ a differentiable map. f induces a homomorphism of differentiable vector spaces $f^*: \mathcal{E}(Y) \to \mathcal{E}(X)$ of degree 0. If f is proper, $f^*(\mathcal{D}(Y)) \subset \mathcal{D}(X)$ and a homomorphism $f: \mathcal{D}'(X) \to \mathcal{D}'(Y)$ is defined by $f(T)(b) = T(f^*(b))$ for all $b \in \mathcal{D}(Y)$; $f(T)$ is called the image of T under f.

1. Metric properties.

1.1. — Let A be an open set in $\mathbf{R}^n$. The norm $\|a\|$ of $a \in \mathcal{D}(A)$ is the supremum on A of the modules of the coefficients of a, under the natural coordinate system of $\mathbf{R}^n$. If T is a lineal form on $\mathcal{D}(A)$ and G is a relatively compact open set in $\mathbf{R}^n$, the norm $\|T\|_G$ of T on G is defined by

$$\|T\|_G = \sup \{ |T(a)|;\ a \in \mathcal{D}(G \cap A) \text{ and } \|a\| \leqq 1 \}.$$

T is said to be 0-continuous on A if $\|T\|_G$ is finite for each relatively compact open set G in $\mathbf{R}^n$ with $\overline{G} \subset A$. If this is so, $T \in \mathcal{D}'(A)$.

Let X be a manifold. $T \in \mathcal{D}'(X)$ is said to be 0-continuous if there exists a family $\varphi_U (U \in \mathfrak{U})$ of coordinate maps of X,

$$\varphi_U: \quad U \to \varphi_U(U) \subset \mathbf{R}^n, \qquad U \subset X,$$

whose domains are a covering of X, such that $\varphi_U(T \mid U) \in \mathcal{D}'(\varphi_U(U))$ is a 0-continuous current on $\varphi_U(U)$. The definition is independent of the particular family φ_U. $\mathcal{D}'^0(X)$ will denote the space of 0-continuous currents on X.

1.2. — Let X be a manifold and W an open set in X. $T \in \mathcal{D}'(W)$ is said to be bounded on X if there is a family $\varphi_U (U \in \mathfrak{U})$ of coordinate maps of X, $\varphi_U: U \to \varphi_U(U) \subset \mathbf{R}^n$, $U \subset X$, whose domains are a covering of X and such that

$$T_U = \varphi_U(T \mid U \cap W) \in \mathcal{D}'(\varphi_U(U \cap W))$$

satisfies the following condition, for each $U \in \mathfrak{U}$: $\|T_U\|_G$ is finite for every relatively compact open set G in $\mathbf{R}^n$ with $\overline{G} \subset \varphi(U)$.

The definition is independent of the particular family φ_U. If $T \in \mathcal{D}'(W)$ is bounded on X, T is 0-continuous on W.

1.3. Theorem ([5]).

(*a*) Let W be an open set in X. If $T \in \mathcal{D}'^0(W)$ has an extension $T' \in \mathcal{D}'^0(X)$, then T is bounded on X.

(*b*) Conversely, if $T \in \mathcal{D}'^0(W)$ is bounded on X, there exists a unique current, $T' \in \mathcal{D}'^0(X)$, called the simple extension of T on X, such that

---

([5]) *Cf.* [8], § 1 and 2.

for each coordinate map $\varphi : U \to \varphi(U) \subset \mathbf{R}^n$, $U \subset X$, of X, the following condition holds:

$$\| \varphi(T' \mid U) \|_G = \| \varphi(T \mid W \cap U) \|_G$$

for all relatively compact open sets G in $\mathbf{R}^n$ with $\overline{G} \subset \varphi(U)$. [It is observed that $\varphi(T' \mid U) \in \mathcal{D}'^0(\varphi(U))$ and $\varphi(T \mid U \cap W) \in \mathcal{D}'^0(\varphi(U \cap W))$.]

Under the conditions of 1.3 (*b*), let $a \in \mathcal{D}(X)$ and let K be the support of a. If K' is a relatively compact open neighborhood of K and u_j ($j = 1, 2, \ldots$) is a locally finite differentiable partition of the unity on $K' \cap W$, then

$$T'(a) = \lim_{m \to \infty} \sum_{j=1}^{m} T(u_j a) \quad (6).$$

Let W be an open set in X, and S and T be currents on W bounded on X. Then the simple extension of $S + T$ on X is the sum of the simple extensions of S and T.

1.4. — Let A be an open set in $\mathbf{R}^n$ and $T \in \mathcal{D}'^0(A)$. Let F be a subset of A. The norm of T on F is said to be zero, and denoted $\| T \|_F = 0$, if for each compact set K in A and each $\varepsilon > 0$, there is a relatively compact open set G such that $K \cap F \subset G \subset \overline{G} \subset A$ and $\| T \|_G < \varepsilon$.

Let $F \subset X$ and $T \in \mathcal{D}'^0(X)$. The norm of T on F is said to be zero ($\| T \|_F = 0$), if for each coordinate map $\varphi : U \to \varphi(U) \subset \mathbf{R}^n$, $U \subset X$, we have $\| \varphi(T \mid U) \|_{\varphi(U \cap F)} = 0$. It is proved that, if $T \in \mathcal{D}'^0(X)$ and F_i ($i = 1, 2, \ldots$) is a family of sets in X such that $\| T \|_{F_i} = 0$ for each i, then $\| T \|_{\cup F_i} = 0$.

(*a*) Let A be an open set in $\mathbf{R}^n$ and F a closed subset of A. Let $T' \in \mathcal{D}'^0(A)$ and $T = T' \mid A - F$. Then $\| T' \|_F = 0$ if and only if $\| T \|_G = \| T' \|_G$ for all relatively compact open sets G in $\mathbf{R}^n$.

Consequently :

(*b*) Let F be closed in the manifold X, $T \in \mathcal{D}'^0(X - F)$ and T' an extension of T on X. Then T' is the simple extension of T if and only if $\| T' \|_F = 0$ ([6], 2.2).

(*c*) Let $F' \subset F$ be closed sets in the manifold X and $T' \in \mathcal{D}'^0(X - F')$ such $\| T' \|_{F - F'} = 0$; let $T = T' \mid X - F$. Then, if T or T' have a simple extension on X, both simple extensions on X exist and are equal.

1.5. — Let $A \subset A'$ be open sets in $\mathbf{R}^n$ and λ a real number $\geqq 0$. $T \in \mathcal{D}'(A)$ is said to satisfy the condition C_λ *on* A' if, for each relatively

---

(6) *Cf.* [8], § 1 and 2.

compact open set G in $\mathbf{R}^n$ with $\overline{G} \subset A'$, there is a constant $k(G) > 0$ such that

$$\| T \|_{B_r} \leq k(G) . r^\lambda$$

for all open balls $B_r \subset G$ of radius r.

Let W be an open set in the manifold X and $T \in \mathcal{D}'(W)$. T satisfies the condition C_λ on X if there is a family $\varphi_U (U \in \mathfrak{U})$ of coordinate maps $\varphi_U : U \to \varphi_U(U) \subset \mathbf{R}^n$, $U \subset X$, such that, for each $U \in \mathfrak{U}$,

$$\varphi_U (T \mid W \cap U) \in \mathcal{D}(\varphi_U(U \cap W))$$

satisfies C_λ on $\varphi_U(U)$. If this is so, T is bounded on X. The definition is independent of the particular family φ_U.

(*a*) $T \in \mathcal{D}'(A)$ satisfies C_λ on A' ($A \subset A'$ open sets in $\mathbf{R}^n$) if and only if for each $x_0 \in A'$ there exists a ball $B_0 = B(x_0, r_0) \subset A'$ and a constant $k(B_0) > 0$ such that

$$\| T \|_{B_r} \leq k(B_0) \, r^\lambda$$

for all balls $B_r \subset B_0$ of radius r (*cf.* [8], theor. 5).

(*b*) Let $T \in \mathcal{D}'(X)$ satisfy condition C_λ on X, with $\lambda > 0$. Then $\| T \|_M = 0$ for any submanifold M of X of dimension $d < \lambda$ (*cf.* [6], 2.4 and [8], theor. 3).

(*c*) Let us suppose, under the conditions of (*b*), that X is a real analytic manifold. Then $\| T \|_M = 0$ for each semianalytic set M with $\dim M < \lambda$. This is a consequence of (*b*) and the decomposition of M in submanifolds of X of dimensions $< \lambda$ (*cf.* A, 1.3).

2. Integration on a manifold.

2.1. — Let a be a form in $\mathcal{D}^n(\mathbf{R}^n)$. If

$$a(x) = a_0(x) \, dx_1 \wedge \ldots \wedge dx_n,$$

where $(x_1, \ldots, x_n)$ are the natural coordinates of $\mathbf{R}^n$ and $a_0 \in \mathcal{D}^0(\mathbf{R}^n)$, we define $\int_{\mathbf{R}^n, e_n} a = \int_{\mathbf{R}^n} a_0 =$ integral of a_0 on the number space $\mathbf{R}^n$, e_n denoting the canonical fundamental class of $\mathbf{R}^n$ (B, 6).

Let now X be a connected manifold. If X is orientable, $H_n(X; \mathbf{Z}) \simeq \mathbf{Z}$; let $e \in H_n(X; \mathbf{Z})$ be a generator, or fundamental class, of X ([2], 1.11) It is possible to choose a family $\psi_\alpha (\alpha \in J)$ of coordinate maps

$$\psi_\alpha : \quad U_\alpha \to \psi_\alpha(U_\alpha) = V_\alpha \subset \mathbf{R}^n, \qquad U_\alpha \subset X,$$

such that:

(*a*) $U_\alpha (\alpha \in J)$ is a locally finite covering of X, and (*b*) $e_\alpha = j^{X, U_\alpha}(e) \in H_n(U_\alpha; \mathbf{Z})$ is the fundamental class of U_α defined by ψ_α (B, 6.3). Let $u_\alpha (\alpha \in J)$ a differentiable partition of the unity on X subordinated to $U_\alpha (\alpha \in J)$. We define, for each $a \in \mathcal{D}^n(X)$,

$$\int_{X, e} a = \sum_{\alpha \in J} \int_{\mathbf{R}^n, e_n} \psi_\alpha (u_\alpha a),$$

where $\psi_\alpha (u_\alpha a) \in \mathcal{D}^n(V_\alpha) \subset \mathcal{D}^n(\mathbf{R}^n)$ and only a finite number of terms of the summatory are non-zero. Because of B, 6.2, and known properties of Cauchy integral, $\int_{X,e} a$ is independent of the family ψ_α. Moreover, the family $U_\alpha (\alpha \in J)$ defines an orientation of X in the sense of [14], and $\int_{X,e}$ coincides with the current associated in [14] to such orientation.

It is immediate that $\int_{X,e}$ is a o-continuous current on X. It is called the integration current on X defined by $e \in H_n(X; \mathbf{Z})$. Clearly,

$$\int_{X, -e} = -\int_{X, e}.$$

If X is not orientable, $H_n(X; \mathbf{Z}) = 0$ and we agree $\int_{X, 0} = 0$.

Let X be now a manifold not necessarily connected, let $X_\alpha (\alpha \in J)$ be the family of its connected components and $X_\alpha (\alpha \in J' \subset J)$ be the subfamily of the orientable components. For each $c \in H_n(X; \mathbf{R})$ and $\alpha \in J$, let $c_\alpha = j^{X, X_\alpha}(c) \in H_n(X_\alpha; \mathbf{R})$. $c_\alpha = 0$ or $c_\alpha = \lambda_\alpha \otimes c_\alpha$, where $\lambda_\alpha \in \mathbf{R}$ and $e_\alpha \in H_n(X_\alpha; \mathbf{Z})$ is a fundamental class of X_α (B, 6), according to $\alpha \in J - J'$ or $\alpha \in J'$. We define

$$\int_{X, c} = \sum_{\alpha \in J'} \lambda_\alpha \wedge \int_{X_\alpha, e_\alpha}.$$

It is immediate that $\lambda_\alpha \wedge \int_{X, e_\alpha}$ is independent of the representation $c_\alpha = \lambda_\alpha \otimes e_\alpha (\alpha \in J')$.

2.2. Proposition. — *The map* $\int_X : H_n(X; \mathbf{R}) \to \mathcal{D}'_n(X)$, $c \to \int_{X, c}$ *has the following properties:*

(*a*) $\int_X$ *is a monomorphism;*

(*b*) $\int_{X, c}$ *satisfies condition* C_n *on* X *for each* $c \in H_n(X; \mathbf{R})$ (*cf.* 1.5);

(c) $\int_{X,c}$ *is compatible with restrictions to open subsets;*

(d) *The image of* $H_n(X; \mathbf{R})$ *under* $\int_X$ *is the subspace of the closed currents in* $\mathcal{D}'_n(X)$.

In fact, (a) and (c) are immediate; (b) follows because $\int_{\mathbf{R}^n, e_n}$ satisfies C_n on $\mathbf{R}^n$. (d) is contained in [14], § 5 and § 19.

It is observed that (c) implies that the supports of $\int_{X,c}$ and c are equal for each $c \in H_n(X; \mathbf{R})$; in particular, if Φ is a family of supports in X and $\mathcal{D}'_\Phi(X)$ is the space of currents on X with support in Φ, $\int_X$ induces a map $H_n^\Phi(X; \mathbf{R}) \to \mathcal{D}'_\Phi(X)$ by composing with the inclusion $H_n^\Phi(X; \mathbf{R}) \to H_n(X; \mathbf{R})$ ([2], 1.4).

2.3. — Let $c \in H_n(X; \mathbf{R})$. A form $a \in \mathcal{E}^n(X)$ is said to be c-integrable if $\sum_{\gamma \in J} \left| \int_{X,c} u_\gamma a \right| < \infty$ for a locally finite differentiable partition $u_\gamma (\gamma \in J)$ of the unity on X. In this case, it can be defined

$$\int_{X,c} a = \sum_{\gamma \in J} \int_{X,c} u_\gamma a,$$

and the number is independent of the partition.

If $f : X \to Y$ is a differentiable isomorphism between the manifolds X and Y, then $\int_{X,c} f^*(a) = \int_{Y, f_*(c)} a$ for each $c \in H_n(X; \mathbf{R})$ and for each $f_*(c)$-integrable form $a \in \mathcal{E}^n(Y)$; f_* denotes the map in homology induced by f. This follows from the definitions. In particular, $f\left(\int_{X,c}\right) = \int_{Y, f_*(c)}$.

Let M be a p-dimensional closed submanifold of X. The inclusion $i : M \to X$ is proper, hence the current $iT(a) = T(i^*(a)) = T(a \mid M)$, $a \in \mathcal{D}(X)$, is defined. If T satisfies condition C_λ on M, so does iT on X.

Let M be a p-dimensional submanifold of X, not necessarily closed, and $F = \overline{M} - M$. Then $i : M \to X - F$ is proper, and for each $c \in H_p(M; \mathbf{R})$ the current $i\left(\int_{M,c}\right) \in \mathcal{D}'_p(X - F)$ is defined. Let us suppose that the simple extension T of $i\left(\int_{M,c}\right)$ on X exists. Then by 1.3, the form $a|M \in \mathcal{E}^p(M)$ is c-integrable for each $a \in \mathcal{D}^p(X)$ and $T(a) = \int_{M,c} a|M$.

2.4. — Let X_i be differentiable manifolds of class $\mathcal{C}^\infty$ and dimension n_i $(i = 1, 2)$, $X = X_1 \times X_2$ the product manifold, $\pi_i : X \to X_i$ the projections and $\pi_i^\star : \mathcal{E}(X_i) \to \mathcal{E}(X)$ the induced maps. Let

$$\pi^\star = \pi_1^\star \wedge \pi_2^\star : \quad \mathcal{D}(X_1) \otimes \mathcal{D}(X_2) \to \mathcal{D}(X), \qquad a_1 \otimes a_2 \to \pi_1^\star(a_1) \wedge \pi_2^\star(a_2);$$

it is known that $\pi^\star(\mathcal{D}(X_1) \otimes \mathcal{D}(X_2))$ is dense in $\mathcal{D}(X)$ and that, if $\mathcal{D}(X_1) \otimes \mathcal{D}(X_2)$ is considered with its inductive tensor topology, then $\pi^\star$ can be extended to a topological vector space isomorphism

$$\pi : \quad \mathcal{D}(X_1) \overline{\otimes} \mathcal{D}(X_2) \to \mathcal{D}(X)$$

of the complexion $\mathcal{D}(X_1) \overline{\otimes} \mathcal{D}(X_2)$ of $\mathcal{D}(X_1) \otimes \mathcal{D}(X_2)$ onto $\mathcal{D}(X)$ (*see* [5], I, § 3 and [14], § 7). Hence, for each $p_i \leq n_i$ $(i = 1, 2)$ there exists a monomorphism

$$\overline{\otimes} : \quad \mathcal{D}'_{p_1}(X_1) \otimes \mathcal{D}'_{p_2}(X_2) \to \mathcal{D}'_{p_2+p_2}(X), \qquad S \otimes T \to S \overline{\otimes} T,$$

such that $S \overline{\otimes} T(\pi^\star(a_1 \otimes a_2)) = S(a_1).T(a_2)$ for each pair of forms $a_i \in \mathcal{D}(X_i)$ $(i = 1, 2)$. Moreover, the following diagram is commutative (*see* B, 5).

$$\begin{array}{ccc}
H_{n_1}(X; \mathbf{R}) \otimes H_{n_2}(X; \mathbf{R}) & \xrightarrow{\odot} & H_{n_1+n_2}(X; \mathbf{R}) \\
\int_{X_1} \otimes \int_{X_2} \Big\downarrow & & \Big\downarrow \int_X \\
\mathcal{D}'_{n_1}(X_1) \otimes \mathcal{D}'_{n_2}(X) & \xrightarrow[\overline{\otimes}]{} & \mathcal{D}'_{n_1+n_2}(X)
\end{array}$$

In fact, given $c_i \in H_{n_i}(X_i; \mathbf{R})$ $(i = 1, 2)$, it suffices to see that $\int_{X, c_1 \odot c_2}$ and $\int_{X_1, c_1} \overline{\otimes} \int_{X_2, c_2}$ are equal on the dense subspace $\pi^\star(\mathcal{D}(X_1) \otimes \mathcal{D}(X_2))$, which follows immediately from the definitions.

2.5. — We state here, for further reference, the following simple case of Stokes' Theorem.

LEMMA. — *Let $x = (x_1, \ldots, x_{p-1})$ be the natural coordinates of $\mathbf{R}^{p-1}$ and U a bounded open set in $\mathbf{R}^{p-1}$. Let $f : U \to \mathbf{R}$ be a $\mathcal{C}^\infty$-differentiable function such that $f > 0$ in U,*

$$\Gamma = ((x, t); x \in U, t = f(x)) \qquad \textit{and} \qquad L = ((x, t); x \in U, 0 < t < f(x)).$$

Let $a : \overline{L} \to \mathbf{R}$ be a continuous function such that $a \,|\, L \in \mathcal{E}^1(L)$, a is zero on a neighborhood of $U \times 0$ and $\dfrac{\partial g}{\partial t}$ is integrable on L. Then, for each $c \in H_p(L; \mathbf{R})$, we have

$$\int_{L,\,c} \frac{\partial g}{\partial t}\, dt \wedge dx_1 \wedge \ldots \wedge dx_{p-1} = \int_{\Gamma,\,\partial_{L,\Gamma}(c)} (g \,|\, \Gamma)\, dx_1 \wedge \ldots \wedge dx_{p-1},$$

where $\partial_{L,\Gamma}$ is the boundary : $H_p(L; \mathbf{R}) \to H_{p-1}(\Gamma; \mathbf{R})$ for the pair $\Gamma \subset L \cup \Gamma$ (*cf.* B, 1).

Let $c = \lambda \otimes e_0$, where $\lambda \in \mathbf{R}$ and $e_0 \in H_p(L; \mathbf{Z})$ is the fundamental class defined on L by the coordinates $(t, x_1, \ldots, x_{p-1})$ (B, 6.3). Since

$$\partial_{L,\Gamma}(c) = \lambda \otimes t_0 \qquad \text{and} \qquad t_0 = \partial_{L,\Gamma}(e_0) \in H_{p-1}(\Gamma; \mathbf{Z}) \quad (\text{B, } 4),$$

it suffices to prove the corresponding equality for $\displaystyle\int_{L,\,e_0}$ and $\displaystyle\int_{\Gamma,\,t_0}$. By definition,

$$\int_{L,\,e_0} \frac{\partial g}{\partial t}\, dt \wedge dx_1 \wedge \ldots \wedge dx_{p-1} = \int_L \frac{\partial g}{\partial t},$$

and partial integration on t gives $\displaystyle\int_L \frac{\partial g}{\partial t} = \int_U g(x, f(x))$. The last integral is equal to $\displaystyle\int_{\Gamma,\,t'_0} (g \,|\, \Gamma)\, dx_1 \wedge \ldots \wedge dx_{p-1}$, t'_0 being the fundamental class of Γ defined by the coordinate map $\Gamma \to U$, $(x, f(x)) \to x$, and we have to prove $t_0 = t'_0$. But this follows from B, 6.1, after a convenient homeomorphism.

3. de Rham's Theorem.

Let $\mathcal{E} = \sum_{p=0}^{n} \mathcal{E}^p$ the differential graded sheaf of germs of differentiable forms of class $\mathcal{C}^\infty$ on X; if U is an open set in X, $\mathcal{D}(U) = \Gamma_c(\mathcal{E} \,|\, U)$ is the space of differentiable forms on U with compact support. $\mathcal{E}$ is a soft resolution of $\mathbf{R}$ on X. Let $D = D(\mathcal{E})$ be the differential graded sheaf $U \to \operatorname{Hom}(\mathcal{D}(U), \mathbf{R})$, with the differential and grading deduced from those of $\mathcal{E}$. Then $\mathcal{D}'$, sheaf of germs of currents on X, is a differential graded subsheaf of D, and $j : \mathcal{D}' \to D$ will denote the inclusion.

If $\mathcal{H}(X; \mathbf{R}) = \mathcal{H}(X; \mathbf{Z}) \otimes \mathbf{R}$ is the local homology sheaf on X with coefficients in $\mathbf{R}$, $\displaystyle\int : \mathcal{H}(X; \mathbf{R}) \to \mathcal{D}'_n$ denotes the sheaf homomorphism induced by $\displaystyle\int_U : H_n(U; \mathbf{R}) \to \mathcal{D}'_n(U)$, for all open sets U in X (*cf.* 2.2).

Let $'\mathcal{D}$ and $'D$ be the sheafs $\mathcal{D}'$ and D with modified gradings $'\mathcal{D}_p = \mathcal{D}'_{n-p}$ and $'D_p = D_{n-p}$ $(p \in \mathbf{Z})$ and corresponding differentials of degree $+1$. The sheafs $'\mathcal{D}$ and $'D$, together with $\int$ and $j \circ \int$, are known to be soft resolutions of $\mathcal{H}(X; \mathbf{R})$ on X (*cf.* [14], § 19). Consequently, by [4], 4.7.1 and 4.7.2, we obtain:

$$H^p_\Phi(X; \mathcal{H}(X; \mathbf{R})) = H_{n-p}(\mathcal{D}'_\Phi(X)) = H_{n-p}(\Gamma_\Phi(D)) \qquad (p \in \mathbf{Z}),$$

for each family of supports Φ in X; here $\mathcal{D}'_\Phi(X) = \Gamma_\Phi(\mathcal{D}')$ denotes the space of currents with support in Φ. Since $\mathbf{R}$ is a field, $D(\mathcal{E})$ is also the dual sheaf of $\mathcal{E}$, according to [1], 2.6, and we have

$$H_q(\Gamma_\Phi(D)) = H^\Phi_q(X; \mathbf{R}) \quad \text{for each } q \in \mathbf{Z} \quad ([1], 3.4).$$

Then we obtain natural isomorphisms

$$(3.1) \qquad \nu(X): \ H^\Phi_q(X; \mathbf{R}) \to H_q(\mathcal{D}'_\Phi(X)) \qquad (q \in \mathbf{Z})$$

which, in dimension n, are induced by $\int_X$.

Finally, if M is a closed submanifold of X, then the following diagram commutes, as it can be seen from the definitions:

$$(3.2) \qquad \begin{array}{ccc} H^\Phi_*(X; \mathbf{R}) & \xrightarrow[\nu(X)]{} & H_*(\mathcal{D}'_\Phi(X)) \\ {\scriptstyle i_{MX}}\uparrow & & \uparrow{\scriptstyle i'_{MX}} \\ H^{\Phi \mid M}_*(M; \mathbf{R}) & \xrightarrow{\nu(M)} & H_*(\mathcal{D}'_{\Phi \mid M}(M)) \end{array}$$

i'_{MX} is the map induced by $i: \mathcal{D}'_{\Phi \mid M}(M) \to \mathcal{D}'_\Phi(X)$.

Chapter II.

Integration on a semianalytic set.

A. Integration currents.

1. A lemma on normal decompositions.

Let A_p be a p-dimensional vector subspace of $\mathbf{R}^n$. The linear map $t: \mathbf{R}^p \to A_p$ is said to be an orthogonal representation of A_p if it preserves the natural scalar products in $\mathbf{R}^p$ and $\mathbf{R}^n$, i. e., if $tu . tv = u . v$ for all $u, v \in \mathbf{R}^p$. Let $T = (t_{ij})$ the $n \times p$-matrix corresponding to t under the natural bases e_j $(j = 1, \ldots, p)$ and f_i $(i = 1, \ldots, n)$ of $\mathbf{R}^p$

and $\mathbf{R}^n$, respectively; $t(e_j) = \sum_{i=1}^{n} t_{ij} f_i$ for each $j = 1, \ldots, p$, and the functions $x_i = \sum_{j=1}^{p} t_{ij} y_j$ $(i = 1, \ldots, n)$ define the linear map $\mathbf{R}^p \to \mathbf{R}^n$, $y \to x$, induced by t. For each set $H = (i_1, \ldots, i_p)$ of p integers with $1 \leqq i_1 < \ldots < i_p \leqq n$, let T_H denote the $p \times p$-matrix defined by the rows $i_1, \ldots, i_p$ of T. Then, if t is orthogonal, the equality $\sum_H |T_H|^2 = 1$ holds, where $|T_H|$ is the determinant of T_H.

Let $\bigwedge^p \mathbf{R}^n$ be the p-exterior product of $\mathbf{R}^n$ $(0 < p < n)$; the p-vector $v \in \bigwedge^p \mathbf{R}^n$ is said to be associated to A_p if and only if $v \neq 0$ and $z \wedge v = 0$ for all $z \in A_p$. A family A_p^m $\left[m = 1, \ldots, N = \binom{n}{p}\right]$ of p-dimensional subspaces of $\mathbf{R}^n$ is called regular if each family of p-vectors v_m $(m = 1, \ldots, N)$ such that v_m is associated to A_p^m, is linearly independent, and consequently a base of $\bigwedge^p \mathbf{R}^n$.

Let now $t^m : \mathbf{R}^p \to A_p^m$ be orthogonal representations of p-dimensional subspaces A_p^m $\left[m = 1, \ldots, \binom{n}{p}\right]$ of $\mathbf{R}^n$. We denote $g_j^m = t^m(e_j)$ $(j = 1, \ldots, p;\ m = 1, \ldots, N)$; then, for each $m = 1, \ldots, N$, the p-vector $g^m = g_1^m \wedge \ldots \wedge g_p^m$ is associated to A_p^m, and can be expressed, under the natural base f_i $(i = 1, \ldots, n)$ of $\mathbf{R}^n$, as

$$g^m = \sum_H |T_H^m| f_H$$

$(H = (i_1, \ldots, i_p),\ 1 \leqq i_1 < \ldots < i_p \leqq n)$; for each $H = (i_1, \ldots, i_p)$, we denote here by T_H^m the $p \times p$-matrix defined by the rows $i_1, \ldots, i_p$ of the matrix T^m corresponding to t^m under the bases e_j and f_i $(m = 1, \ldots, N)$, and by f_H the p-vector $f_{i_1} \wedge \ldots \wedge f_{i_p}$. It follows that the family A_p^m is regular if and only if the determinant

$$(1.1) \quad \det(|T_H^m| : m = 1, \ldots, N;\ H = (i_1, \ldots, i_p),\ 1 \leqq i_1 < \ldots < i_p \leqq n)$$

is different from zero.

Under these conditions, let us define the following family of differential forms in $\mathcal{E}^p(\mathbf{R}^n)$:

$$\omega(A_p^m) = \sum (|T_H^m| \, dx_H;\ H = (i_1, \ldots, i_p);\ 1 \leqq i_1 < \ldots < i_p \leqq n)$$
$$(m = 1, \ldots, N),$$

where $dx_H = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ for each $H = (i_1, \ldots, i_p)$.

It is easily seen that:

(i) $\omega(A_p^m)$ is independent of the particular orthogonal representation t^m of A_p^m;

(ii) if $t^{m*}: \mathcal{E}^p(\mathbf{R}^n) \to \mathcal{E}^p(\mathbf{R}^p)$ is the map induced by t^m, then $t^{m*}(\omega(A_p^m)) = dy_1 \wedge \ldots \wedge dy_p$ $(m = 1, \ldots, N)$, where $y_1, \ldots, y_p$ are the coordinate functions of $\mathbf{R}^p$;

(iii) the family $\omega(A_p^m)$ $(m = 1, \ldots, N)$ is a base for the vector space of the forms in $\mathcal{E}^p(\mathbf{R}^n)$ with constant coefficients if and only if the determinant (1.1) is $\neq 0$ or, equivalently, if the family A_p^m $(m = 1, \ldots, N)$ is regular.

1.2. Lemma [7]. — *Let $M_1, \ldots, M_s$ be semianalytic sets in a neighborhood of the origin of $\mathbf{R}^n$, and let $0 < p < n$. There exist $N = \binom{n}{p}$ coordinate maps $x^m = (x_1^m, \ldots, x_n^m)$ $(m = 1, \ldots, N)$ in $O'(M_1, \ldots, M_s)$* [*cf.* A, 2.3 (c)] *such that the family of subspaces $A_p^m = (x_{p+1}^m = \ldots = x_n^m = 0)$ is regular.*

Let g_i $(i = 1, \ldots, n)$ be an orthonormal base in $\mathbf{R}^n$. For each $K = (i_1, \ldots, i_p)$, with $1 \leqq i_1 < \ldots < i_p \leqq n$, let $CK = (j_1, \ldots, j_{n-p})$, $1 \leqq j_1 < \ldots < j_{n-p} \leqq n$, be the complement of K in the set $(1, \ldots, n)$, and let $g_1^K = g_{i_1}, \ldots, g_p^K = g_{i_p}$, $g_{p+1}^K = g_{j_1}, \ldots, g_n^K = g_{j_{n-p}}$. The family of the $N = \binom{n}{p}$ p-vectors $g_K = g_1^K \wedge \ldots \wedge g_p^K$ $(K = \ldots)$ is linearly independent, so the family of the subspaces $A_p^K = (y_{p+1}^K = \ldots = y_n^K = 0)$ is regular, where $y^K = (y_1^K, \ldots, y_n^K)$ is the map in $O'(\mathbf{R}^n)$ defined by the orthonormal base $g^K = (g_1^K, \ldots, g_n^K)$.

Let T^K be the transition matrix between g^K and the natural base $(f_1, \ldots, f_n)$ of $\mathbf{R}^n$; for each K we have $g_K = \sum_H |T_H^K| f_H$, where

$$f_H = f_{i_1} \wedge \ldots \wedge f_{i_p}$$

if $H = (i_1, \ldots, i_p)$, and T_H^K is the minor of T^K defined by the rows $i_1, \ldots, i_p$ and the first p columns of T^K. As before, the regularity of A_p^K $(K = \ldots)$ implies $\det(|T_H^K|) \neq 0$. According to A, 2.3 (c), there exist N maps $x^K = (x_1^K, \ldots, x_n^K)$ in $O'(\mathrm{M}_1, \ldots, \mathrm{M}_s)$ so near the y^K as to imply $\det(|Q_H^K|) \neq 0$, where Q^K is the transition matrix between x^K and the natural coordinate map of $\mathbf{R}^n$. From this the regularity of the family $A_p^K = (x_{p+1}^K = \ldots = x_n^K = 0)$ $(K = \ldots)$ follows.

---

(7) For the complex case, *see* [8], § 5 and 6.

2. Construction of integration currents.

2.1. THEOREM. — *For each paracompact real analytic manifold X of dimension n and each locally closed semianalytic set M in X of dimension p ($0 \leq p \leq n$), there exists a unique monomorphism*

$$I(M): \begin{cases} H_p(M; \mathbf{R}) \to \mathcal{D}'_p(X) \\ \quad c \longrightarrow I(M, c) \end{cases}$$

of the real p-homology of M into the space of p-currents on X such that:

(*a*) *For each $c \in H_p(M; \mathbf{R})$, the support of $I(M, c)$ is the closure in X of the support of c. If $p = 0$, M is a point x_0 and $\lambda \in H_0(x_0; \mathbf{R}) = \mathbf{R}$, then $I(x_0, \lambda) = \lambda\delta(x_0)$; if $p > 0$, M^* is the p-regular part of M and $c^* = j^{M, M^*}(c) \in H_p(M^*; \mathbf{R})$, then $I(M, c)$ coincides on M^* with $\int_{M^*, c^*}$* (I, C, 2.2).

(*b*) *$I(M)$ is compatible with restrictions to open subsets, that is, the following diagram commutes for each open set W in X:*

$$\begin{array}{ccc} H_p(M; \mathbf{R}) & \xrightarrow{I(M)} & \mathcal{D}'_p(X) \\ \downarrow{\scriptstyle j^{M, M \cap W}} & & \downarrow{\scriptstyle \rho^{X, W}} \\ H_p(M \cap W; \mathbf{R}) & \xrightarrow[I(M \cap W)]{} & \mathcal{D}'_p(W) \end{array}$$

where $\rho^{X, W}$ denotes the restriction homomorphism of currents.

(*c*) *For all $c \in H_p(M; \mathbf{R})$, $I(M, c)$ satisfies condition C_p on X* (I, C, 1.5). *$I(M, c)$ is called the integration current of M defined by c.*

Proof. — As usual, $\delta(x_0)$ denotes the 0-current $f \to f(x_0)$ ($f \in \mathcal{D}^0(X)$). If $\dim M = 0$, M is discrete and, for each $c \in H_0(M; \mathbf{R})$, (*a*) and (*b*) determine $I(M, c) = \sum (\lambda_\nu \delta(x_\nu);\ x_\nu \in M$ and $\lambda_\nu = j^{M, x_\nu}(c))$; condition (*c*) is immediately verified.

Let us suppose in the following that $\dim M > 0$. It is recalled that $bM = \overline{M} - M$, $sM = M - M^*$ and $\partial M = bM \cup sM$ are semianalytic sets in X with dimensions $< p$ (I, A, 1.3), and that bM and ∂M are closed in X. In particular, $H_p(sM; \mathbf{R}) = 0$ in the homology sequence for the pair $sM \subset M$, hence $j^{MM^*}: H_p(M; \mathbf{R}) \to H_p(M^*; \mathbf{R})$ is injective. Since M^* is a closed analytic submanifold of dimension p of $X - \partial M$, the inclusion $i: \mathcal{D}'(M^*) \to \mathcal{D}'(X - \partial M)$ is defined.

We denote by $I'(M): H_p(M; \mathbf{R}) \to \mathcal{D}'_p(X - \partial M)$, $c \to I'(M, c)$, the composition $i \circ \int_{M^*} \circ j^{M, M^*}$; since i and $\int_{M^*}: H_p(M^*; \mathbf{R}) \to \mathcal{D}'_p(M^*)$ are injective (I, C, 2.2), $I'(M)$ is a monomorphism. If W is an open set in X, it is easily seen, according to I, B, 1.1 and I, C, 2.2, that the diagram

$$(2.2)\qquad \begin{array}{ccc} H_p(M;\mathbf{R}) & \xrightarrow{I'(M)} & \mathcal{D}'_p(X-\partial M) \\ {\scriptstyle j^{M, M\cup W}}\downarrow & & \downarrow{\scriptstyle \rho} \\ H_p(M\cap W;\mathbf{R}) & \xrightarrow[I'(M\cap W)]{} & \mathcal{D}'_p(W-\partial M) \end{array}$$

commutes, where ρ denotes the restriction of currents. Moreover, I, C, 2.2 (b) implies that $I'(M, c)$ satisfies condition C_p on $X-\partial M$, for each $c\in H_p(M;\mathbf{R})$.

We shall prove that $I'(M, c)$ satisfies C_p *on* X, for each $c\in H_p(M;\mathbf{R})$. This is trivial if $p=\dim X$, since in such case M^* is an open set in X. Let us suppose $0<p<n$.

Let $c\in H_p(M;\mathbf{R})$, $x\in X$ and $\varphi: U\to\varphi(U)=V\subset\mathbf{R}^n$ be an analytic coordinate map with open domain $U\subset X$ such that $x\in U$ and $\varphi(x)=0$. Because of (2.2), $I'(M, c)\mid U-\partial M=I'(M\cap U, c_U)$, where $c_U=j^{M, M\cap U}(c)$. Moreover,

$$\varphi(I'(M\cap U, c_U))=I'(M_1, c_1)\in\mathcal{D}'_p(V-\partial M_1),$$

where $M_1=\varphi(M\cap U)$ is a locally closed semianalytic set in V with $\dim M_1=p$, and $c_1=\varphi_*(c_U)\in H_p(M_1;\mathbf{R})$ (I, C, 2.3). It will suffice to prove that $I'(M_1, c_1)$ satisfies C_p on some neighborhood of the origin (I, C, 1.5).

2.3. Lemma. — *Let* $x=(x_1, \ldots, x_n)$ *be a coordinate map in* $O'(\overline{M}_1, \partial M_1)$ (I, A, 2.3 (c)), *let* $\mathcal{NS}$ *be a corresponding normal system and* $Q\subset V$ *a normal neighborhood for* $\mathcal{NS}$. *If* $\omega(A_p)$ *is the p-form associated to the subspace* $A_p=(x_{p+1}=\ldots=x_n=0)$ *of* $\mathbf{R}^n$ (*cf.* § 1), *there exists a constant* $k>0$ *such that the current* $\omega(A_p)\wedge I'(M_1, c_1)\in\mathcal{D}'_0(V-\partial M_1)$ *satisfies*

$$\|\omega(A_p)\wedge I'(M_1, c_1)\|_{B_r}\leq k.r^p$$

for each open ball $B_r\subset Q$ *of radius r.*

By definition, the normal decomposition $Q=\bigcup_{k,\chi}\Gamma^k_\chi$ for the system $\mathcal{NS}$ is compatible with $\overline{M}_1$ and ∂M_1. Then

$$\overline{M}_1=\bigcup(\Gamma^k_\chi;\ \Gamma^k_\chi\subset\overline{M}_1 \text{ and } k\leq p)$$

and, since $\dim\Gamma^p_\chi=p$ and $\dim\partial M_1<p$, we have

$$\bigcup(\Gamma^p_\chi;\ \Gamma^p_\chi\subset\overline{M}_1)\subset M_1-\partial\overline{M}_1=M_1^*,$$

and each $\Gamma^p_\varkappa \subset \overline{M}_1$ is an open subset of $M_1^\star$. Moreover,

$$S = \bigcup (\Gamma^k_\varkappa ;\ \Gamma^k_\varkappa \subset \overline{M}_1 \text{ and } k < p)$$

is a closed semianalytic set of Q, $\dim S < p$ and $\partial M_1 \subset S$. Then $S - \partial M_1$ is a semianalytic set in $Q - \partial M_1$ of dimension $< p$ and, since $I'(M_1, c_1)$ satisfies C_p on $V - \partial M_1$, it follows that

$$\| I'(M_1, c_1) \|_{S - \partial M_1} = 0 \quad [\text{I, C, } 1.5\,(c)];$$

in particular, $\| \omega(A_p) \wedge I'(M_1, c_1) \|_{S - \partial M_1} = 0$. Now, by I, C, 1.4 (a), both currents

$$\omega(A_p) \wedge I'(M_1, c_1) \mid Q - \partial M_1 \qquad \text{and} \qquad T' = \omega(A_p) \wedge I'(M_1, c_1) \mid Q - S$$

have the same norm on the relatively compact open sets G with $\overline{G} \subset Q$. Then it suffices to prove that T' satisfies C_p on Q.

Let $B_r \subset Q$ be an open ball of radius r and $a \in \mathcal{D}^0(B_r - S)$ a function with $\| a \| \leqq 1$. Let $J = (\varkappa : \Gamma^p_\varkappa \subset \overline{M}_1)$. We have

$$\begin{aligned} T'(a) &= I'(M_1, c_1)(a\,\omega(A_p)) \\ &= \int_{M_1^\star, c^\star} a\,\omega(A_p) = \int_{M_1^\star - S, c_\star} a\,\omega(A_p) = \sum_{\varkappa \in J} \int_{\Gamma^p_\varkappa, c_\varkappa} a\,\omega(A_p) \end{aligned}$$

where

$$c^\star = j^{M_1, M_1^\star}(c_1), \qquad c_\star = j^{M_1, M_1^\star - S}(c_1)$$

and

$$c_\varkappa = j^{M_1, \Gamma^p_\varkappa}(c_1) \quad \text{for each } \varkappa \in J.$$

According to I, A, 2.2, if $\pi : Q \to Q_p$ denotes the projection

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_p) = x^p,$$

each $\Gamma^p_\varkappa$ is the graphic of an analytic map $f_\varkappa : \pi(\Gamma^p_\varkappa) \to \mathbf{R}^{n-p}$, $\pi(\Gamma^p_\varkappa)$ being an open set in Q_p. Consequently,

$$\int_{\Gamma^p_\varkappa, c_\varkappa} a\,\omega(A_p) = \int_{\pi(\Gamma^p_\varkappa), c'_\varkappa} a(x^p, f_\varkappa(x^p))\, dx_1 \wedge \ldots \wedge dx_p$$

for each $\varkappa \in J$, where

$$c'_\varkappa = (\pi \mid \Gamma^p_\varkappa)_\star (c_\varkappa) \in H_p(\pi(\Gamma^p_\varkappa); \mathbf{R}).$$

Hence, if $c'_\varkappa = \lambda_\varkappa \otimes e_\varkappa$ where $\lambda_\varkappa \in \mathbf{R}$ and $e_\varkappa \in H_p(\pi(\Gamma^p_\varkappa); \mathbf{Z})$ is a generator ($\varkappa \in J$), we have

$$|T'(a)| \leqq \sum_{\varkappa \in J} \left| \lambda_\varkappa \int_{\pi(\Gamma^p_\varkappa), e_\varkappa} a(x^p, f_\varkappa(x^p))\, dx_1 \wedge \ldots \wedge dx_p \right|$$
$$= \sum_{\varkappa \in J} |\lambda_\varkappa| . \operatorname{area} \pi(\Gamma^p_\varkappa) \leqq \left(\sum_{\varkappa \in J} |\lambda_\varkappa| \right) \operatorname{area} \pi(B_r) \leqq k . r^p.$$

This implies the lemma.

Let now $x^m = (x^m_1, \ldots, x^m_n) \left[m = 1, \ldots, \binom{n}{p} \right]$ be maps in $O'(\overline{M}_1, \partial M_1)$ such that the family of p-dimensional subspaces $A^m_p = (x^m_{p+1} = \ldots = x^m_n = 0)$ is regular. The associated family $\omega(A^m_p)$ is then a base for the space of forms in $\mathcal{E}^p(\mathbf{R}^n)$ with constant coefficients, and each form

$$dx_H = dx_{i_1} \wedge \ldots \wedge dx_{i_p}, \qquad 1 \leqq i_1 < \ldots < i_p \leqq n,$$

where $(x_1, \ldots, x_n)$ are the natural coordinates of $\mathbf{R}^n$, can be expressed as $dx_H = \sum_m s^H_m \omega(A^m_p)$, with $s^H_m \in \mathbf{R}$ (*cf.* § 1). Then, for each form $a(x) = \sum_H a_H(x)\, dx_H$, we have

$$a = \sum_{H, m} a_H s^H_m \omega(A^m_p) = \sum_m P_m(\ldots, a_H, \ldots)\, \omega(A^m_p);$$

the polynomials P_m depend only on the maps x^m, and constants $k'_m \geqq 0$ exist such that $\| P_m(\ldots, a_H, \ldots) \| \leqq k'_m . \| a \|$ for each $a \in \mathcal{D}^p(\mathbf{R}^n)$ $\left[m = 1, \ldots, \binom{n}{p} \right]$.

Let Q_m be a normal neighborhood for the map x^m, and let $k_m > 0$ be the constant that 2.3 yields for $\omega(A^m_p) \wedge I'(M_1, c_1)$ and Q_m $\left[m = 1, \ldots, \binom{n}{p} \right]$. Then, if $B_r \subset Q = \bigcap_m Q_m$ is an open ball of radius r, and if $a \in \mathcal{D}^p(B_r - \partial M_1)$, 2.3 implies :

$$|I'(M_1, c_1)(a)| = \left| I'(M_1, c_1) \left(\sum_m P_m(\ldots, a_H, \ldots)\, \omega(A^m_p) \right) \right|$$
$$\leqq \sum_m |\omega(A^m_p) \wedge I'(M_1, c_1)(P_m)|$$
$$\leqq \sum_m \| \omega(A^m_p) \wedge I'(M_1, c_1) \|_{B_r} . \| P_m(\ldots, a_H, \ldots) \|$$
$$\leqq \left(\sum_m k_m k'_m \right) r^p \| a \|.$$

Hence $I'(M_1, c_1)$ satisfies C_p on Q and consequently so does $I'(M, c)$ on X. In particular, $I'(M, c)$ is bounded on X.

We now define $I(M, c)$ as the simple extension of $I'(M, c)$ on X, for each $c \in H_p(M; \mathbf{R})$; it exists, and satisfies C_p on X, because of I, C, 1.3. It is immediate that $I(M) : c \to I(M, c)$ is a monomorphism, since $I'(M)$ is.

Condition (a) of 2.1 is direct consequence of the definitions. As to (b), let W be an open set in X, $c \in H_p(M; \mathbf{R})$ and $c_W = j^{M, M\cap W}(c)$. $I(M, c) \mid W$ satisfies C_p on W, thus $\| I(M, c) \mid W \|_{\partial M \cap W} = 0$ and $I(M, c) \mid W$ is the simple extension on W of $I(M, c) \mid W - \partial M$ [I, C, 1.5 (c) and 1.4 (b)]; because of (2.2), the last current is equal to $I'(M \cap W, c_W)$, and this implies $I(M, c) \mid W = I(M \cap W, c_W)$.

Finally, to prove the uniqueness of $I(M)$, let $I^0(M) : H_p(M; \mathbf{R}) \to \mathcal{D}'_p(X)$, $c \to I^0(M, c)$, be another monomorphism which satisfies conditions (a), (b) and (c). (c) implies that $I^0(M, c)$ is the simple extension on X of $I^0(M, c) \mid X - \partial M$, and (a) and (b) imply that the last current is equal to $I(M, c) \mid X - \partial M$. The equality $I(M, c) = I^0(M, c)$ follows.

2.4. Remarks.

(1) In the conditions of 2.1, $I(M, c)(a) = \int_{M^*, c^*} a \mid M^*$ for all $c \in H_p(M; \mathbf{R})$ and $a \in \mathcal{D}^p(X)$, where $c^* = j^{M, M^*}(c)$, as it is deduced from I, C, 2.3.

(2) Let Φ be a family of supports in $\overline{M}$. The composition of $I(M)$ with the natural inclusion $H_p^{\Phi \cap M}(M; \mathbf{R}) \to H_p(M; \mathbf{R})$ ([2], 1.4 and 1.10) gives a map $I^\Phi(M) : H_p^{\Phi \cap M}(M; \mathbf{R}) \to \mathcal{D}'_p(X)$ whose images have all support in Φ. This is a consequence of 2.1 (a).

(3) Let X be a complex analytic manifold, M a complex analytic subset of X of (complex) dimension p and $e \in H_{2p}(M; \mathbf{Z})$ the fundamental class of M ([2], 3.4). Then $I(M, 1 \otimes e)$ coincides with the current associated by P. Lelong to M [8].

3. Properties of integration currents.

3.1. Proposition. — *Let $\varphi : X \to Y$ be an analytic isomorphism between the real analytic manifolds X and Y. Let M be a locally closed semianalytic set in X of dimension p. Then $\varphi(I(M, c)) = I(M', c')$ for each $c \in H_p(M; \mathbf{R})$, where $M' = \varphi(M)$ and $c' = (\varphi \mid M)_*(c) \in H_p(M'; \mathbf{R})$.*

This follows immediately from 2.4.(1) and I, C, 2.3.

3.2. Proposition. — *Let $N \subset M$ be locally closed semianalytic sets in X, with* $\dim N = \dim M = p$. *Then*

(*a*) *if N is a closed subset of M, $I(N, c) = I(M, i_{NM}(c))$ for all $c \in H_p(N; \mathbf{R})$;*

(*b*) *if S is a closed semianalytic set in X with* $\dim S < p$, *then* $I(M, c) = I(M - S, j^{M, M-S}(c)$ *for all* $c \in H_p(M; \mathbf{R})$ (I, B, 1).

On account of the condition C_p, it suffices to see that both currents in the case (*a*) coincide on $X - (\partial M \cup \partial N)$, and that the currents in the case (*b*) are equal on $X - S$, which is straightforward.

3.3. Proposition. — *Let $M_h (h \in J)$ be a locally finite family of locally closed semianalytic sets in X, with* $\dim M_h = p$ *for all $h \in J$. Let*

$$S = \bigcup_{h \in J} bM_h, \qquad M = \bigcup_{h \in J} (M_h - S),$$
$$j^h = j^{M_h, M_h - S}, \qquad i_h = i_{M_h - S, M} \qquad (h \in J)$$

and let

$$\mu_* : \quad \prod_{h \in J} H_p(M_h - S; \mathbf{R}) \to H_p(M; \mathbf{R})$$

be the homomorphism defined in [2], 1.7. *Then, for each element* $(c_h)_{h \in J}$ *in* $\prod_{h \in J} H_p(M_h; \mathbf{R})$, *we have*

$$\sum_{h \in J} I(M_h, c_h) = I(M, \mu_*(c)), \qquad \textit{where} \quad c = (j^h(c_h))_{h \in J}.$$

Observe that S is closed with $\dim S < p$ and M is locally closed with $\dim M = p$. It suffices to prove the equality on a relatively compact open set W in X. The restrictions of both currents to W are equal to

$$I_1 = \sum_{h \in J'} I(M_h \cap W, c_{hW}) \qquad \text{and} \qquad I_2 = I(M \cap W, c_W),$$

respectively; here $J' = (h \in J;\ M_h \cap W \neq \emptyset)$ is finite,

$$c_{hW} = j^{M_h, M_h \cap W}(c_h) \quad (h \in J') \qquad \text{and} \qquad c_W = j^{M, M \cap W}(\mu_*(c)).$$

By 3.2,

$$I_1 = \sum_{h \in J'} I((M_h - S) \cap W, j^{hW}(c_{hW})) = \sum_{h \in J'} I(M \cap W, i_{hW} \circ j^{hW}(c_{hW}))$$
$$= I(M \cap W, \sum_{h \in J'} i_{hW} \circ j^{hW}(c_{hW})),$$

where $j^{hW} = j^{M_h \cap W, (M_h - S) \cap W}$ and $i_{hW} = i_{(M_h - S) \cap W, M \cap W}$. By I, B, (1.1), the diagram

$$\begin{array}{ccccc} H_p(M_h; \mathbf{R}) & \xrightarrow{j^h} & H_p(M_h - S; \mathbf{R}) & \xrightarrow{i_h} & H_p(M; \mathbf{R}) \\ \downarrow j & & \downarrow j & & \downarrow j \\ H_p(M_h \cap W; \mathbf{R}) & \xrightarrow[j^{hW}]{} & H_p((M_h - S) \cap W; \mathbf{R}) & \xrightarrow[i_{hW}]{} & H_p(M \cap W; \mathbf{R}) \end{array}$$

commutes. Then, on account of [2], 1.7, we have

$$c_W = \sum_{h \in J'} j^{M, M \cap W} \circ i_h \circ (j^h(c)) = \sum_{h \in J'} i_{hW} \circ j^{hW}(c_{hW}),$$

and this implies $I_1 = I_2$, as wanted.

3.4. Proposition. — *Let M_h be a locally closed semianalytic set of dimension p_h in the real analytic manifold $X_h (h = 1, 2)$. Then the following diagram commutes* (I, C, 2.4) :

$$\begin{array}{ccc} H_{p_1}(M_1; \mathbf{R}) \otimes H_{p_2}(M_2; \mathbf{R}) & \xrightarrow{\odot} & H_{p_1+p_2}(M_1 \times M_2; \mathbf{R}) \\ \downarrow {\scriptstyle I(M_1) \otimes I(M_2)} & & \downarrow {\scriptstyle I(M_1 \times M_2)} \\ \mathcal{D}'_{p_1}(X_1) \otimes \mathcal{D}'_{p_2}(X_2) & \xrightarrow[\overline{\otimes}]{} & \mathcal{D}'_{p_1+p_2}(X_1 \times X_2) \end{array}$$

Let $c_h \in H_{p_h}(M_h; \mathbf{R})$ $(h = 1, 2)$ and $N = ((\partial M_1) \times M_2) \cup (M_1 \times (\partial M_2))$. $T = I(M_1 \times M_2, c_1 \odot c_2)$ satisfies condition $C_{p_1+p_2}$ on $X_1 \times X_2$ and, since $I(M_h, c_h)$ satisfies C_{p_h} on $X_h (h = 1, 2)$, it is immediate that

$$T' = I(M_1, c_1) \overline{\otimes} I(M_2, c_2)$$

satisfies $C_{p_1+p_2}$ on $X_1 \times X_2$. Clearly, $\dim N < p_1 + p_2$, and it suffices to prove that T and T' are equal on $W = X_1 \times X_2 - N$. But

$$T \mid W = i\left(\int_{M_1^\star \times M_2^\star, c_1^\star \odot c_2^\star}\right)$$

and

$$T' \mid W = i_1\left(\int_{M_1^\star, c_1^\star}\right) \overline{\otimes} i_2\left(\int_{M_2^\star, c_2^\star}\right),$$

where $c_h^\star = j^{M_h, M_h^\star}(c_h)$,

$$i_h : \ \mathcal{D}'(M_h^\star) \to \mathcal{D}'(X_h - \partial M_h) \quad (h = 1, 2)$$

and

$$i : \ \mathcal{D}'(M_1^\star \times M_2^\star) \to \mathcal{D}'(X_1 \times X_2 - N).$$

Then $T \mid W = T' \mid W$ because of I, C, 2.4.

B. Stokes' Theorem.

1. Projections of normal decompositions.

In this section a minor modification to normal decompositions is introduced, which will be useful for the proof of Theorem 2.1.

Let $\mathcal{NS} = (H_h^k(x_1, \ldots, x_k; x_h);\ 0 \leq k < h \leq n)$ be a normal system at $0 \in \mathbf{R}^n$, and let $Q = (|x_i| < d_i;\ i = 1, \ldots, n)$ be a normal neighborhood for $\mathcal{NS}$. For a fixed p, $0 < p \leq n$, we define

$$\begin{aligned} \Delta_{p-1} &= (x \in Q_{p-1};\ H_{p-1}^{p-2} \neq 0), \\ \Delta_p &= (x \in Q_p;\ H_{p-1}^{p-2} \neq 0), \\ \Delta &= (x \in Q;\ H_{p-1}^{p-2} \neq 0) \end{aligned}$$

and [I, A, 2.2 (e)] :

$$W^p = V^p \cap \Delta = (x \in Q;\ H_n^{n-1} = \ldots = H_{p+1}^p = 0,\ H_p^{p-1} \neq 0,\ H_{p-1}^{p-2} \neq 0),$$
$${}_pW^p = {}_pV^p \cap \Delta_p = (x \in Q_p;\ H_p^{p-1} \neq 0,\ H_{p-1}^{p-2} \neq 0).$$

If $p = n$, $W^n = {}_nW^n = (x \in Q;\ H_n^{n-1} \neq 0,\ H_{n-1}^{n-2} \neq 0)$.

Let $W^p = \bigcup_\tau L_\tau^p$ and ${}_pW^p = \bigcup_\sigma {}_pL_\sigma$ be the decompositions of W^p and ${}_pW^p$ in their connected components. The sets L_τ^p are called *modified members of* $\mathcal{NS}$. Each L_τ^p is an open subset of some connected component Γ_χ^p of V^p, hence it is a p-dimensional analytic submanifold, and also a semianalytic set of $\mathbf{R}^n$. Moreover, if $\mathcal{NS}$ is compatible with a set $A \subset \mathbf{R}^n$, so is the family $(L_\tau^p)_\tau$.

If $\mathcal{NS}$ is compatible with a p-dimensional semianalytic set M in $\mathbf{R}^n$, then

$$M = M \cap \left(\bigcup_{i=0}^{p} V^i \right) = M \cap W^p + M \cap \left(V^p \cap (H_{p-1}^{p-2} = 0) + \bigcup_{i=0}^{p-1} V^i \right).$$

The last term is a semianalytic set with dimension $< p$, because so is $\bigcup_{i=0}^{p-1} V^i$ and V^p is an analytic manifold of dimension p such that $H_{p-1}^{p-2} \neq 0$ on all its connected components. For each τ such that $L_\tau^p \subset M$,

$$bL_\tau^p = \bar{L}_\tau^p - L_\tau^p \subset \bar{M} - (M \cap W^p).$$

We denote

$$\pi_p :\quad Q \to Q_p, \quad (x_1, \ldots, x_n) \to (x_1, \ldots, x_p)$$

and

$$\pi_{p-1}^p :\quad Q_p \to Q_{p-1}, \quad (x_1, \ldots, x_p) \to (x_1, \ldots, x_{p-1}).$$

Each member ${}_p\Gamma_\varkappa^{p-1}$ of the normal system $\mathcal{NS}_p$ verifies

$$\pi_{p-1}^p({}_p\Gamma_\varkappa^{p-1}) = {}_{p-1}\Gamma_i^{p-1} \in \mathcal{NS}_{p-1} \quad [\text{I, A, 2.2 }(e)].$$

The sets ${}_{p-1}\Gamma_i^{p-1}$ are the connected components of ${}_{p-1}V^{p-1} = \Delta_{p-1}$.

For each ${}_{p-1}\Gamma_i^{p-1}$, let

$$f_{i,s}: \quad {}_{p-1}\Gamma_i^{p-1} \to \mathbf{R} \qquad (0 < s < l_i,\ 1 \leqq l_i)$$

be the analytic functions whose graphics in Q_p are the sets ${}_p\Gamma_\varkappa^{p-1}$ such that $\pi_{p-1}^p({}_p\Gamma_\varkappa^{p-1}) = {}_{p-1}\Gamma_i^{p-1}$; we can suppose $f_{i,s} < f_{i,s+1} (0 < s < l_i)$, and ${}_p\Gamma_s^{p-1}$ will denote the member ${}_p\Gamma_\varkappa^{p-1}$ corresponding in this way to $f_{i,s}$; $f_{i,0}$ and f_{i,l_i} will denote the constant functions equal to $-d_p$ and $+d_p$ on ${}_{p-1}\Gamma_i^{p-1}$, respectively.

1.1. Proposition.

(*a*) *Let* $0 < p \leqq n$. *Each* ${}_pL_\sigma$ *satisfies* $\pi_{p-1}^p({}_pL_\sigma) = {}_{p-1}\Gamma_i^{p-1}$ *for some* i, *and then, for some* s $(0 \leqq s < l_i)$,

$$(1.2) \qquad {}_pL_\sigma = ((x_1, \ldots, x_p) \in Q_p;\ f_{i,s}(x_1, \ldots, x_{p-1}) < x_p < f_{i,s+1}(x_1, \ldots, x_{p-1})).$$

In such case:

$$(1.3) \quad {}_p\bar{L}_\sigma \cap \Delta_p = \begin{cases} \text{(i)} & {}_p\Gamma_s^{p-1} \cup {}_pL_\sigma \cup {}_p\Gamma_{s+1}^{p-1} & \text{if } 0 < s,\ s+1 < l_i; \\ \text{(ii)} & {}_pL_\sigma \cup {}_p\Gamma_s^{p-1} & \text{if } 0 < s = l_i - 1 \\ & & \text{or } 0 = s,\ s+1 < l_i; \\ \text{(iii)} & {}_pL_\sigma & \text{if } 0 = s = l_i - 1. \end{cases}$$

(*b*) *Let* $0 < p < n$. *Then each* L_τ^p *verifies* $\pi_p(L_\tau^p) = {}_pL_\sigma$ *for some* σ; *in this case* $\pi_p \mid \bar{L}_\tau^p \cap \Delta$ *is a homeomorphism onto* ${}_p\bar{L}_\sigma \cap \Delta_p$ *and* $\pi_p \mid L_\tau^p$ *is an analytic isomorphism onto* ${}_pL_\sigma$. *Moreover,*

$$(1.4) \qquad \bar{L}_\tau^p \cap \Delta = \begin{cases} \text{(i)} & \Gamma_\alpha^{p-1} \cup L_\tau^p \cup \Gamma_\beta^{p-1}; \\ \text{(ii)} & \Gamma_\alpha^{p-1} \cup L_\tau^p; \\ \text{(iii)} & L_\tau^p \end{cases}$$

if ${}_pL_\sigma$ *verifies* (i), (ii) *or* (iii) *of* (*a*), *respectively. If* (i) *holds, then*

$$\pi_p(\Gamma_\alpha^{p-1}) = {}_p\Gamma_\alpha^{p-1} \qquad \textit{and} \qquad \pi_p(\Gamma_\beta^{p-1}) = {}_p\Gamma_{s+1}^{p-1};$$

if (ii) *holds, then*

$$\pi_p(\Gamma_s^{p-1}) = {}_p\Gamma_s^{p-1};$$

in all cases the maps $\pi_p \mid \Gamma^{p-1} \ldots$ *are analytic isomorphisms.*

Proof. — If $x' = (x'_1, \ldots, x'_p) \in {}_pW^p$, then $\pi^p_{p-1}(x') \in {}_{p-1}V^{p-1} = \Delta_{p-1}$; let us suppose that ${}_{p-1}\Gamma^{p-1}_i$ is the connected component of $\pi\,_{-1}(x')$ in ${}_{p-1}V^{p-1}$ Then

$$f_{i,s}(x'_1, \ldots, x'_{p-1}) < x'_p < f_{i,s+1}(x'_1, \ldots, x'_{p-1})$$

for some $s(0 \leq s < l_i)$, and it is immediate that the connected component ${}_pL_\sigma$ of x' in ${}_pW^p$ has the form (1.2) and $\pi^p_{p-1}({}_pL_\sigma) = {}_{p-1}\Gamma^{p-1}_\chi$. Hence ${}_p\bar{L}_\sigma \cap \Delta_p$ has the form (1.3).

To prove (*b*), let $M_p = (x \in Q;\ H^p_{p+1} = \ldots = H^p_n = 0)$; it follows from I, A, 2(*c*) that $\pi_p \mid M_p : M_p \to Q_p$ is proper [8], and then so is

$$\pi_p \mid M_p \cap (H^{p-1}_p \neq 0) \to Q_p \cap (H^{p-1}_p \neq 0);$$

consequently $\pi_p \mid V^p : V^p \to {}_pV^p$ is proper, since I, A, 2 (*a*) implies that V^p is a closed subspace of $M_p \cap (H^{p-1}_p \neq 0)$. $\pi_p \mid V^p$ is open too, since I, A, 2.2 (*c*) and (*e*) imply that V^p is a union of graphics of analytic functions defined on the connected components of ${}_pV^p$.

Since $\pi_p \mid V^p : V^p \to {}_pV^p$ is proper and open, so is $\pi_p \mid V^p \cap \Delta \to {}_pV^p \cap \Delta_p$, and it follows that for each connected component L^p_τ of W^p we have $\pi_p(L^p_\tau) = {}_pL_\sigma$ for some connected component ${}_pL_\sigma$ of ${}_pW^p$. In such case

$${}_pL_\sigma \subset \pi_p\left(\bar{L}^p_\tau \cap \Delta\right) \subset {}_p\bar{L}_\sigma \cap \Delta_p.$$

Moreover, $L^p_\tau \subset V^p \subset M_p$ implies that $\bar{L}^p_\tau \cap \Delta$ is a closed subset of $M_p \cap \Delta$ and, since $\pi_p \mid M_p \cap \Delta \to \Delta_p$ is proper, that $\pi_p\left(\bar{L}^p_\tau \cap \Delta\right)$ is closed in Δ_p. Consequently, $\pi_p\left(\bar{L}^p_\tau \cap \Delta\right) = {}_p\bar{L}_\sigma \cap \Delta_p$, as wanted. $\pi_p \mid L^p_\tau$ is an analytic isomorphism because L^p_τ is an open subset of some Γ^p_χ and, according to I, A, 2.2, $\pi \mid \Gamma^p_\chi$ is an analytic isomorphism.

To prove that $\pi_p \mid \bar{L}^p_\tau \cap \Delta$ is a homeomorphism onto ${}_p\bar{L}_\sigma \cap \Delta_p$, it suffices to see that it is injective, since it is continuous and proper. Clearly $\pi_p \mid L^p_\tau$ is injective. Let us suppose that a point $y' \in ({}_p\bar{L}_\sigma - {}_pL_\sigma) \cap \Delta_p$ exists such that $\pi^{-1}(y') \cap \bar{L}^p_\tau = (y^i;\ i = 1, \ldots, s)$ and $s > 1$; let $U^i = U_p \times U^i_{n-p}$ $(i = 1, \ldots, s)$ be a disjoint family of neighborhoods of the points y^i, where U_p is a neighborhood of y'. Then there is a neighborhood U'_p of y' in Q_p such that $(\pi_p \mid L^p_\tau)^{-1}(U'_p) \subset \bigcup_{i=1}^{s} U^i$. In fact, if this were not

---

[8] $f : X \to Y$ is proper if $f^{-1}(K)$ is compact for each compact set K in Y.

the case, a sequence $U_{p,n}(n \in \mathbf{Z})$ of neighborhoods of y' could be found, with diameters tending to zero, together with a sequence

$$y_n \in (\pi_p \mid L_\tau^p)^{-1} (U_{p,n}) - \bigcup_{i=1}^{s} U^i;$$

then $(y_n)_{n \in \mathbf{Z}}$ would have a cluster point $y \in \pi_p^{-1}(y') \cap \overline{L}_\tau^p$ different from y^i $(i = 1, \ldots, s)$, which is a contradiction. On account of (1.2), it is now easily seen that a neighborhood U_p'' of y' in Q_p exists such that $U_p'' \subset U_p'$ and $U_p'' \cap {}_pL_\sigma$ is connected. Then $(\pi_p \mid L_\tau^p)^{-1} (U_p'' \cap {}_pL_\sigma)$, as a connected subset of $\bigcup_{i=1}^{s} U^i$, is included in some U^{i_0}. This implies $y^i \notin \overline{L}_\tau^p$ if $i \neq i_0$, which is a contradiction. The first part of (*b*) is proved.

To prove (1.4), let us suppose that ${}_pL_\sigma = \pi_p(L_\tau^p)$ verifies

$${}_p\overline{L}_\sigma \cap \Delta_p = {}_p\Gamma_s^{p-1} \cup {}_pL_\sigma \cup {}_p\Gamma_{s+1}^{p-1}.$$

Then $L' = \left(\pi_p \,\middle|\, \overline{L}_\tau^p \cap \Delta\right)^{-1} ({}_p\Gamma_s^{p-1})$ and $L'' = \left(\pi_p \,\middle|\, \overline{L}_\tau^p \cap \Delta\right)^{-1} ({}_p\Gamma_{s+1}^{p-1})$ are connected subsets of $\overline{L}_\tau^p \cap \Delta$, $L' \cap L'' \neq \emptyset$ and $\overline{L}_\tau^p \cap \Delta - L_\tau^p = L' \cap L''$. Since L_τ^p is a connected component of $V^p \cap \Delta$,

$$\overline{L}_\tau^p \cap \Delta - L_\tau^p \subset \left(\overline{V}^p - V^p\right) \cap \Delta \subset V^{p-1} \quad \text{[I, A, 2.2 (}b\text{)].}$$

Then, if $x \in L'$, it follows that $x \in \Gamma_\alpha^{p-1}$ for some α and $x \notin \bigcup_{\beta \neq \alpha} \Gamma_\beta^{p-1}$; since the last set is closed in Δ [I, A, 2.2 (*b*)], a neighborhood W of x exists such that $L' \cap W \subset \Gamma_\alpha^{p-1}$, and the connection of L' implies $L' \subset \Gamma_\alpha^{p-1}$. By I, A, 2.2 (*e*), $\pi_p(\Gamma_\alpha^{p-1}) = {}_p\Gamma_i^{p-1}$ for some i; this, and $\pi_p(L') = {}_p\Gamma_s^{p-1}$, imply ${}_p\Gamma_s^{p-1} = {}_p\Gamma_i^{p-1}$. Then $L' = \Gamma_\alpha^{p-1}$, since $\pi_p \mid \Gamma_\alpha^{p-1}$ is injective, and also $L'' = \Gamma_\beta^{p-1}$. The other cases in (1.4) are proved similarly.

2. Stokes' Theorem.

It is recalled that if M is a locally closed semianalytic set with $\dim M = p$, then $bM = \overline{M} - M$ is a closed semianalytic set with $\dim bM < p$ (I, A, 1.3).

2.1. THEOREM. — *Let X be a paracompact real analytic manifold of dimension n and M a locally closed semianalytic set in X of dimension p* ($0 < p \leqq n$). *The following diagram commutes* (9) :

$$\begin{array}{ccc} H_p(M; \mathbf{R}) & \xrightarrow{I(M)} & \mathcal{D}'_p(X) \\ {\scriptstyle \partial_{M,bM}}\downarrow & & \downarrow{\scriptstyle b} \\ H_{p-1}(bM; \mathbf{R}) & \xrightarrow[I(bM)]{} & \mathcal{D}'_{p-1}(X) \end{array}$$

where b is the border in $\mathcal{D}'(X)$ *and* $\partial_{M,bM}$ *is the boundary in the exact sequence of homology*

$$0 \to H_p(\overline{M}; \mathbf{R}) \xrightarrow{j_{\overline{M},M}} H_p(M; \mathbf{R}) \xrightarrow{\partial_{M,bM}} H_{p-1}(bM; \mathbf{R}) \to \ldots$$

for the pair $bM \subset \overline{M}$.

Proof. — Let $c' \in H_p(M; \mathbf{R})$, $t' = \partial_{M,bM}(c')$ and $x \in X$. It suffices to prove that $bI(M, c')$ and $I(bM, t')$ are equal on some neighborhood of x. Let $\varphi : U \to \varphi(U) = V \subset \mathbf{R}^n$ be an analytic coordinate map such that $x \in U$ and $\varphi(x) = 0$. The set $M_1 = \varphi(M \cap U)$ is locally closed and semianalytic in V and, according to II, A, 1.2, there are maps $x^s = (x_1^s, \ldots, x_n^s)$ $\left[s = 1, \ldots, \binom{n}{p-1}\right]$ in $O'(\overline{M}_1, \partial M_1, sM_1)$ such that the family

$$A^s = (x_p^s = \ldots = x_n^s = 0)$$

of associated $(p-1)$-dimensional subspaces is regular.

For each $s = 1, \ldots, \binom{n}{p-1}$, let $\mathcal{NS}_s$ be a normal system for the map x^s compatible with $\overline{M}_1$, ∂M_1 and sM_1, and let Q^s be a corresponding compatible normal neighborhood. We denote $Q_0 = \cap\left(Q^s; s = 1, \ldots, \binom{n}{p-1}\right)$ ane $W = \varphi^{-1}(Q_0)$, and we are to prove $bI(M, c') \mid W = I(bM, t') \mid W$. Because of A, 2.1,

$$bI(M, c') \mid W = bI(M \cap W, c'_W) \qquad \text{and} \qquad I(bM, t') \mid W = I(bM \cap W, t'_W),$$

where

$$c'_W = j^{M, M\cap W}(c'), \qquad t'_W = j^{bM, bM\cap W}(t') = \partial_{M\cap W, bM\cap W}(c'_W) \quad \text{(I, B, 1.1)}.$$

Let $bI(N, c)$ and $I(bN, t) \in \mathcal{D}'_p(Q_0)$ denote the image currents of $bI(M \cap W, c'_W)$ and $I(bM \cap W, t'_W)$ under $\varphi \mid W$, where

$$N = M_1 \cap Q_0 = \varphi(M \cap W)$$

(9) $I(bM) = 0$ if $\dim bM < p - 1$.

is semianalytic in Q_0,

$$c = \varphi_*(c'_{W}) \in H_p(N; \mathbf{R}) \qquad \text{and} \qquad t = \varphi_*(t'_{W}) = \partial_{N\ bN}(c) \in H_{p-1}(bN; \mathbf{R}).$$

It is sufficient to prove

$$(2.2) \qquad bI(N, c) = I(bN, t).$$

Let us consider a fixed normal system $\mathcal{NS}_s$ and its neighborhood Q_s. In the following lemmas the other normal systems will not be used, so the subindex s will be dropped from all notations. $\overline{N}$ will always denote the closure of N in Q.

The decomposition $Q = \bigcup_{k,\varkappa} L^k_\varkappa$ corresponding to $\mathcal{NS}$ is compatible with $\overline{N}$, ∂N and sN, and consequently so is the family $(L^p_\tau)_\tau$ of modified members of $\mathcal{NS}$ (*see.* No. 1). It follows that $L^p_\tau \subset \overline{N}$ implies $L^p_\tau \subset N^* = N - \partial N$, since $\dim L^p_\tau = p$ and $\dim \partial N < p$ (I, A, 1.3); consequently, L^p_τ is an open submanifold of N^*. Moreover,

$$N \cap W^p = \bigcup (L^p_\tau : L^p_\tau \subset \overline{N}),$$

$S = \overline{N} - N \cap W^p$ is a closed semianalytic set in Q of dimension $< p$, and $(\overline{L}^p_\tau - L^p_\tau) \cap Q \subset S$ for each τ such that $L^p_\tau \subset \overline{N}$.

2.3. Lemma. — *Let* $J = (\tau : L^p_\tau \subset \overline{N})$. *Then*

$$I(N, c) = \sum_{\tau \in J} I(L^p_\tau, c_\tau),$$

where $c_\tau = j^{N, L^p_\tau}(c) \in H_p(L^p_\tau; \mathbf{R})$ $(\tau \in J)$.

Because of II, A, 3.2 (*b*), $I(N, c) = I(N - S, c_0)$, where $c_0 = j^{N, N-S}(c)$. But then II, A, 3.3 implies $I(N - S, c_0) = \sum_{\tau \in J} I(L^p_\tau, c_\tau)$, and the lemma follows.

2.4. Lemma. — *Let*

$$N_* = N \cap W^p, \qquad J_1 = (h; \Gamma^{p-1}_h \subset bN) \qquad \textit{and} \qquad J_0 = (h; \Gamma^{p-1}_h \subset \overline{N}).$$

Then :

$$I(bN, \partial_{N, bN}(c)) = \sum_{h \in J_0} I(\Gamma^{p-1}_h, t_h),$$

where $t_h = \partial_h \circ j^{N, N_*}(c)$ *and* ∂_h *is the boundary corresponding to the exact sequence of real homology*

$$(2.5) \qquad 0 \to H_p(N_* \cup \Gamma_h^{p-1}) \xrightarrow{j} H_p(N_*) \xrightarrow{\partial_h} H_{p-1}(\Gamma_h^{p-1}) \to \dots$$

of the pair $\Gamma_h^{p-1} \subset N_* \cup \Gamma_h^{p-1}$.

We observe that, for each $h \in J_0$, Γ_h^{p-1} is a closed subset of $N_* \cup \Gamma_h^{p-1}$, since $N_* \subset V^p$ and $\bar{\Gamma}_h^{p-1} - \Gamma_h^{p-1} \subset \bigcup_{i=0}^{p-2} V^i$ [I, A, 2.2 (*b*)]. Moreover, as $\mathfrak{N}\mathfrak{S}$ is compatible with bN, we have

$$bN = \bigcup_{h \in J_1} \Gamma_h^{p-1} + bN \cap \left(\bigcup_{i=0}^{p-2} V^i \right),$$

where $S' = bN \cap \left(\bigcup_{i=0}^{p-2} V^i \right)$ is a closed semianalytic set in Q of dimension $< p - 1$ and the Γ_h^{p-1} $(h \in J_1)$ are open subsets of bN. Reasoning as in the last lemma, we obtain

$$(2.6) \qquad I(bN, \partial_{N, bN}(c)) = \sum_{h \in J_1} I(\Gamma_h^{p-1}, t^h),$$

where

$$t^h = j^{bN, \Gamma_h^{p-1}}(\partial_{N, bN}(c)) \in H_{p-1}(\Gamma_h^{p-1}; \mathbf{R}) \qquad (h \in J_1).$$

Note that $J_1 \neq \emptyset$ if and only if $\dim bN = p - 1$; if $J_1 = \emptyset$, both members in (2.6) are zero. To prove the lemma it suffices to see that

(i) $h \in J_0 - J_1$ implies $t_h = 0$ and

(ii) $h \in J_1$ implies $t^h = t_h$.

In any case, $N_* \cup \Gamma_h^{p-1}$ is an open subset of $\bar{N}$. For

$$\bar{N} - (N_* \cup \Gamma_h^{p-1}) = (\bar{N} - \Gamma_h^{p-1}) \cap (\bar{N} - N_*)$$
$$= (\bar{N} - \Gamma_h^{p-1}) \cap \left(V^p \cap (H_{p-1}^{p-2} = 0) + \bigcup_{i=0}^{p-1} V^i \right) = A_1 \cup A_2$$

(*cf.* No. 1). Here

$$A_2 = (\bar{N} - \Gamma_h^{p-1}) \cap \left(\bigcup_{i=0}^{p-1} V^i \right) = \bigcup (\Gamma_\chi^k;\ \Gamma_\chi^k \subset \bar{N} - \Gamma_h^{p-1},\ k \leqq p - 1)$$

and, since $\overline{\Gamma}_{\varkappa}^{k}-\Gamma_{h}^{k}\subset\bigcup_{i=0}^{k-1}V^{i}$ for all $\varkappa$, it follows $\overline{A}_2=A_2$. Now let $x\in\overline{A}_1-A_1$, where $A_1=(\overline{N}-\Gamma_h^{p-1})\cap(V^p\cap(H_{p-1}^{p-2}=0))$; it follows that $x\in\overline{N}-\Gamma_h^{p-1}$, since $H_{p-1}^{p-2}\neq 0$ on Γ_h^{p-1}; hence

$$x\in\overline{V^p\cap(H_{p-1}^{p-2}=0)}-V^p\cap(H_{p-1}^{p-2}=0)\subset\bigcup_{i=0}^{p-2}V^i,$$

and we obtain $\overline{A}_1-A_1\subset A_2$. Then $A_1\cup A_2$ is closed, as wanted.

If $h\in J_0-J_1$, then $N_*\cup\Gamma_h^{p-1}\subset N$, and consequently

$$j^{N,N_*}(c)=j^{N_*\cup\Gamma_h^{p-1},N_*}\left(j^{N,N_*\cup\Gamma_h^{p-1}}(c)\right);$$

then $t_h=0$, since (2.5) is exact. If $h\in J_1$, $t_h=t^h$ follows from the commutativity of the diagram (I, B, 1.1):

$$\begin{array}{ccc}H_p(N;\mathbf{R}) & \xrightarrow{\partial_{N,bN}} & H_{p-1}(bN;\mathbf{R})\\ {\scriptstyle j^{N,N_*}}\big\downarrow & & \big\downarrow{\scriptstyle j^{bN,\Gamma_h^{p-1}}}\\ H_p(N_*;\mathbf{R}) & \xrightarrow[\partial_h]{} & H_{p-1}(\Gamma_h^{p-1};\mathbf{R})\end{array}$$

This prove the lemma.

2.7. Lemma. — *For each L_τ^p $(\tau\in J)$, each $c\in H_p(L_\tau^p;\mathbf{R})$ and each form $a\in\mathcal{D}^{p-1}(Q)$, $a(x)=a_0(x)\,dx_1\wedge\ldots\wedge dx_{p-1}$, where $x=(x_1,\ldots,x_n)$ is the coordinate map previously chosen, we have*

$$bI(L_\tau^p,c)(a)=\sum_{h\in J_0}I(\Gamma_h^{p-1},t_h)(a),$$

where $t_h=\partial_{\tau h}(c)$ and $\partial_{\tau h}:H_p(L_\tau^p;\mathbf{R})\to H_{p-1}(\Gamma_h^{p-1};\mathbf{R})$ is the boundary for the pair $\Gamma_h^{p-1}\subset L_\tau^p\cup\Gamma_h^{p-1}$.

Note that Γ_h^{p-1} is a closed subset of $L_\tau^p\cup\Gamma_h^{p-1}$, since $\overline{\Gamma}_h^{p-1}\subset\bigcup_{i=0}^{p-1}V^i$ and $L_\tau^p\subset V^p$. Let us first suppose $0<p<n$ and choose a member L_τ^p with $\tau\in J$.

$$\Gamma_h^{p-1}\cap\overline{L}_\tau^p=\Gamma_h^{p-1}\cap\overline{L}_\tau^p\cap\Delta\quad\text{for all } h,$$

since $H_{p-1}^{p-2}\neq 0$ on Γ_h^{p-1} (*see* No. 1), and $\overline{L}_\tau^p\cap\Delta$ has the form described in (1.4). Suppose, for example, that $\overline{L}_\tau^p\cap\Delta=\Gamma_\alpha^{p-1}\cup L_\tau^p$; then each Γ_h^{p-1} with $h\neq\alpha$ is a connected component of $L_\tau^p\cup\Gamma_h^{p-1}$, hence $t_h=0$ and we are reduced to prove

$$bI(L_\tau^p,c)(a)=I(\Gamma_\alpha^{p-1},t_\alpha)(a).$$

Let ${}_pL_\sigma \in {}_pW^p$ and ${}_p\Gamma_i^{p-1}$ be the members in Q_p such that $\pi_p \mid L_\tau^p \cup \Gamma_\alpha^{p-1}$ is a homeomorphism onto ${}_pL_\sigma \cup {}_p\Gamma_i^{p-1}$ and $\pi_p \mid L_\tau^p$ and $\pi_p \mid \Gamma_\alpha^{p-1}$ are analytic ismorphisms onto ${}_pL_\sigma$ and ${}_p\Gamma_i^{p-1}$, respectively. Let

$$g: \quad {}_pL_\sigma \cup {}_p\Gamma_i^{p-1} \to L_\tau^p \cup \Gamma_\alpha^{p-1}$$

be the inverse homeomorphism. We can suppose that

$${}_pL_\sigma = ((x_1, \ldots, x_p) \in Q_p ; (x_1, \ldots, x_{p-1}) \in {}_{p-1}\Gamma_i^{p-1}$$

and

$$-d_p < x_p < f(x_1, \ldots, x_{p-1})),$$

where $f: {}_{p-1}\Gamma_i^{p-1} \to \mathbf{R}$ is the analytic map whose graphic in Q_p is ${}_p\Gamma_i^{p-1}$.

Because of II, A, 2.4.(1),

$$I(L_{\tau, c}^p)(da) = \int_{L_\tau^p, c} da \qquad \text{and} \qquad I(\Gamma_\alpha^{p-1}, t_\alpha)(a) = \int_{\Gamma_\alpha^{p-1}, t_\alpha} a;$$

moreover,

$$\int_{L_\tau^p, c} da = \int_{{}_pL_\sigma, c_\sigma} (g \mid {}_pL_\sigma)^\star (da)$$

and

$$\int_{\Gamma_\alpha^{p-1}, t_\alpha} a = \int_{{}_p\Gamma_i^{p-1}, t_i} (g \mid {}_p\Gamma_i^{p-1})^\star (a),$$

where

$$c_\sigma = (\pi_p \mid L_\tau^p)_\star (c) \in H_p({}_pL_\sigma; \mathbf{R}),$$

$$t_i = (\pi_p \mid \Gamma_\alpha^{p-1})_\star (t_\alpha) \in H_{p-1}({}_p\Gamma_i^{p-1}; \mathbf{R}), \qquad (g \mid {}_pL_\sigma)^\star (da \mid L_\tau^p) \in \mathcal{E}({}_pL_\sigma)$$

and

$$(g \mid {}_p\Gamma_i^{p-1})^\star (a \mid \Gamma_\alpha^{p-1}) \in \mathcal{E}({}_p\Gamma_i^{p-1});$$

note that $t_i = \partial(c_\sigma)$, where ∂ is the boundary corresponding to the pair ${}_p\Gamma_i^{p-1} \subset {}_pL_\sigma \cup {}_p\Gamma_i^{p-1}$. Then it suffices to see that

$$\int_{{}_pL_\sigma, c_\sigma} (g \mid {}_pL_\sigma)^\star (da) = \int_{{}_p\Gamma_i^{p-1}, t_i} (g \mid {}_p\Gamma_i^{p-1})^\star (a),$$

which is a consequence of I, C, 2.5, since

$$(g \mid {}_p\Gamma_i^{p-1})^\star (a) = a_0(g)\, dx_1 \wedge \ldots \wedge dx_{p-1}$$

on ${}_p\Gamma_i^{p-1}$,

$$d(g \mid {}_pL_\sigma)^\star (a) = \frac{\partial a_0(g)}{\partial t}\, dt \wedge dx_1 \wedge \ldots \wedge dx_{p-1}$$

is integrable on ${}_pL_\sigma$ because so is da on L_τ^p, and $a_0(g)$ is continuous on ${}_pL_\sigma \cup {}_p\Gamma_i^{p-1}$. If $\overline{L}_\tau^p \cap \Delta$ has one of the other forms described in (1.4), the proof of the lemma is similar. The case $p = n$ is simpler, since then $\bar{L}_\tau^p \cap \Delta = {}_p\bar{L}_\tau \cap \Delta$ verifies (1.3) and I, C, 2.5 can be directly applied.

2.8. LEMMA. — *For each form* $a(x) = a_0(x)\, dx_1 \wedge \ldots \wedge dx_{p-1}$, *where* $a_0(x) \in \mathcal{D}^0(Q)$ *and* $x = (x_1, \ldots, x_n)$ *is the map previously chosen, we have*

$$bI(N, c)(a) = I(bN, \partial_{N, bN}(c)).$$

By 2.3 and 2.7,

$$bI(N, c)(a) = \sum_{\tau \in J} bI(L_\tau^p, c_\tau)(a) = \sum_{\tau, h \in J \times J_0} I(\Gamma_h^{p-1}, t_{\tau h}) = \sum_{h \in J_0} I(\Gamma_h^{p-1}, t^h)(a),$$

where $c_\tau = j^{N, L_\tau^p}(c)$ for each $\tau \in J$,

$$t_{\tau h} = \partial_{\tau h}(c_\tau), \qquad \partial_{\tau h} : \; H_p(L_\tau^p; \mathbf{R}) \to H_{p-1}(\Gamma_h^{p-1}; \mathbf{R})$$

being the boundary for the pair $\Gamma_h^{p-1} \subset L_\tau^p \cup \Gamma_h^{p-1}$ and $t^h = \sum_{\tau \in J} t_{\tau h}$. Because of 2.4, it suffices to prove $t_h = t^h$ for each $h \in J_0$, and this follows from the commutative diagram of real homology (I, B, 3):

$$\begin{tikzcd}
H_p(N) \arrow[r, "j^{N, N_*}"] \arrow[dr, "\sum_J j^{N, L_\tau^p}"'] & H_p(N_*) \arrow[r, "\partial_h"] \arrow[d, "\sum_J j^{N_*, L_\tau^p}"] & H_{p-1}(\Gamma_h^{p-1}) \\
& \sum_J H_p(L_\tau^p) \arrow[ur, "\sum_J \partial_{\tau h}"'] &
\end{tikzcd}$$

In fact,

$$t_h = \partial_h \circ j^{N, N_*}(c) = \sum_J \partial_{\tau h} \circ j^{N L_\tau^p}(c) = \sum_J \partial_{\tau h}(c_\tau) = t^h.$$

To finish the proof of the theorem, let us consider the chosen family of maps $x^s = (x_1^s, \ldots, x_n^s)$ $\left[s = 1, \ldots, \binom{n}{p-1}\right]$. Since the family of subspaces $A^s = (x_p^s = \ldots = x_n^s = 0)$ is regular, the family $\omega(A^s)$ $\left[s = 1, \ldots, \binom{n}{p-1}\right]$ of associated forms is a base for the space of forms in $\mathcal{E}^{p-1}(\mathbf{R}^n)$ with constant coefficients (A, 1). Consequently, each form $a \in \mathcal{D}^{p-1}(Q_0)$ can be expressed as $a = \sum_s a_s \omega(A^s)$, where $a_s \in \mathcal{D}^0(Q_0)$ and $\omega(A^s) = dx_1^s \wedge \ldots \wedge dx_{p-1}^s$ $\left[s = 1, \ldots, \binom{n}{p-1}\right]$. Then 2.8 can be applied to each term $a_s \omega(A^s)$, and we obtain

$$bI(N, c)(a) = I(bN, \partial_{N, bN}(c))(a),$$

as wanted.

2.9. Corollary. — *In the conditions of Theorem* 2.1, $I(M, c)$ *is closed if and only if* $c \in j^{\bar{M}, M}(H_p(\bar{M}; \mathbf{R}))$. *Hence* $I(M, c)$ *is closed if* $H_{p-1}(bM; \mathbf{R}) = 0$ *or, in particular, if* $\dim bM < p - 1$.

It suffices to consider the exact sequence of homology of $bM \subset \bar{M}$ and the fact that $I(M)$ is injective.

2.10. Corollary. — *The current associated by* P. Lelong *in* [8] *to a complex analytic set is closed* [A, 2.4.(3)].

2.11. Remarks.

(1) If Φ is a family of supports in $\bar{M}$, we have also a commutative diagram

$$\begin{array}{ccc} H_p^{\Phi \cap M}(M; \mathbf{R}) & \xrightarrow{I^{\Phi}(M)} & \mathcal{D}'_p(X) \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle b} \\ H_{p-1}^{\Phi \mid bM}(bM; \mathbf{R}) & \xrightarrow[I^{\Phi \mid bM}(bM)]{} & \mathcal{D}'_{p-1}(X) \end{array}$$

where ∂ has been defined in [2], 7.10 [*see* A, 2.4.(2)]. This follows because the natural homomorphisms

$$H_*^{\Phi \cap M}(M; \mathbf{R}) \to H_*(M; \mathbf{R}) \qquad \text{and} \qquad H_*^{\Phi \mid bM}(bM; \mathbf{R}) \to H\ (bM; \mathbf{R})$$

are compatible with ∂.

(2) If $\dim bM < p - 1$, it can be proved that $bI(M, c) = 0$ with the method of [8], without using Stokes' Theorem. This has been sketched in [6].

(3) If M is an oriented affine simplex of dimension p, the usual integration on M coincides with $I(M^*, 1 \otimes e)$, where $e \in H_p(M^*; \mathbf{Z})$ is the generator associated to the orientation of M ([1], p. 148). Then Theorem 2.1 reduces to the classical Stokes' Theorem, as presented for example in [14], § 6.

3. The homology class of an integration current.

According to 2.9, if M is a closed semianalytic set of dimension p of X, then $I(M, c)$ is closed for each $c \in H_p(M; \mathbf{R})$. Let us denote by $I_{M,X} : H_p(M; \mathbf{R}) \to H_p(\mathcal{D}'(X))$ the induced map into the p-homology of the currents on X.

3.1. PROPOSITION. — *The diagram*

$$\begin{array}{ccc} H_p(X;\mathbf{R}) & \xrightarrow{\nu(X)} & H_p(\mathcal{D}'(X)) \\ {\scriptstyle i_{M,X}}\uparrow & \nearrow{\scriptstyle I_{M,X}} & \\ H_p(M;\mathbf{R}) & & \end{array}$$

is commutative (I, C, 3.1) ([10]).

Since M is closed, $bM = \emptyset$ and $\partial M = sM = M - M^\star$; let $X^\star = X - \partial M$ and let

$$\rho^{X,X^\star}:\quad H_p(\mathcal{D}'(X)) \to H_p(\mathcal{D}'(X^\star))$$

be the map induced by the restriction of currents. Then

$$\rho^{X,X^\star}\circ\nu(X) = \nu(X^\star)\circ j^{X,X^\star} \quad \text{(I, C, 3.1)}$$

and $\rho^{X,X^\star}$ is injective, since $\dim\partial M < p$ implies that $j^{X,X^\star}$ is injective in dimension p. It will suffice to prove that

$$\rho^{X,X^\star}\circ I_{M,X} = \rho^{X,X^\star}\circ\nu(X)\circ i_{M,X}.$$

By A, 2.1 (*b*),

$$\rho^{X,X^\star}\circ I_{M,X} = I_{M^\star,X^\star}\circ j^{M,M^\star},$$

and the last map is equal to $\nu(X^\star)\circ i_{M^\star,X^\star}\circ j^{M,M^\star}$ [I, C, (3.2)], which coincides with $\nu(X^\star)\circ j^{X,X^\star}\circ i_{M,X} = \rho^{X,X^\star}\circ\nu(X)\circ i_{M,X}$, as wanted.

3.2. *Remarks.*

(1) The last proposition holds for any family of supports Φ on X, as it can be seen by composing $\nu(X)$ and $I_{M,X}$ with $H_p^\Phi(X;\mathbf{R}) \to H(X;\mathbf{R})$ and $H_p^{\Phi\mid M}(M;\mathbf{R}) \to H_p(M;\mathbf{R})$ [A, 2.4.(2)].

(2) Let X be oriented by a fundamental class $e \in H_n(X;\mathbf{Z})$. Let M_1 and M_2 be closed semianalytic sets in X such that $\dim M_1 = p$ and $\dim M_2 = q = n - p$. Let $c_1 \in H_p^c(M_1;\mathbf{R})$ and $c_2 \in H_q(M_2;\mathbf{R})$. Then the intersection product of $c_1' = i_{M_1X}(c_1)$ and $c_2' = i_{M_2X}(c_2)$ is an element $c_1'.c_2' \in H_0^c(X;\mathbf{R})$ ([2], 1.12). Let $\varepsilon : H_0^c(X;\mathbf{R}) \to \mathbf{R}$ be the map induced by mapping X into a point. Then $\varepsilon(c_1'.c_2') = I(M_1, c_1) \wedge I(M_2, c_2)$ (1), where the last symbol denotes the Kronecker index of the currents $I(M_1, c_1)$ and $I(M_2, c_2)$ ([14], § 20). This can be deduced from 3.1, the definition of the intersection product and the properties of the Kronecker index.

([10]) Particular cases of this property can be found in [2], 3.4 and [7], Theor. 3.

REFERENCES.

[1] Borel (A.) and Moore (J. C.). — Homology theory for locally compact spaces, *Mich. math. J.*, t. 7, 1960, p. 137-159.

[2] Borel (A.) and Haefliger (A.). — La classe d'homologie fondamentale d'un espace analytique, *Bull. Soc. math. France*, t. 89, 1961, p. 461-513.

[3] Bredon (Glen). — *Sheaf theory*. — New York, Mc Graw-Hill Comp. (to be published).

[3*a*] Bungart (Lutz). — Stokes' theorem on real analytic varieties, *Proc. Nat. Acad. Sci. U. S. A.*, 54, 1965, 343-344.

[4] Godement (Roger). — *Topologie algébrique et théorie des faisceaux*. — Paris, Hermann, 1958 (Act. scient. et ind., 1252; Publ. Inst. Math. Univ. Strasbourg, 13).

[5] Grothendieck (A.). — *Produits tensoriels topologiques et espaces nucléaires*. — Providence, American mathematical Society, 1955 (Memoirs of the American mathematical Society, 16).

[6] Herrera (Miguel E.). — On the extension of currents, *Rev. Union. mat. Argent.*, t. 22, 1965, p. 139-143.

[7] Herrera (Miguel E.). — Intégration sur un ensemble semi-analytique, *C. R. Acad. Sc.*, t. 260, 1965, p. 763-765.

[8] Lelong (P.). — Intégration sur un ensemble analytique complexe, *Bull. Soc. math. France*, t. 85, 1957, p. 239-262.

[9] Łojasiewicz (S.). — Une propriété topologique des sous-ensembles analytiques réels, *Colloques internationaux du C. N. R. S.* [117, 1962, Paris], p. 87-89.

[10] Łojasiewicz (S.). — Sur le problème de la division, *Studia Math.*, t. 18, 1959, p. 87-136.

[11] Łojasiewicz (S.). — Triangulation of semianalytic sets, *Ann. Scuola norm. sup. Pisa*, 3e série, t. 18, 1964, p. 449-474.

[12] Łojasiewicz (S.). — *Division de distribuciones y triangulacion de conjuntos semi-analiticos*, Universidad de Buenos Aires, Fac. C. exactas y naturales, Dep. Matematicas, 1965.

[13] Łojasiewicz (S.). — *Ensembles semi-analytiques*. — Paris, Presses universitaires de France, 1966 (Institut des Hautes Études Scientifiques).

[14] de Rham (G.). — *Variétés différentiables*, 2e éd. — Paris, Hermann, 1960 (Act. scient. et ind., 1222; Publ. Inst. Math. Univ. Nancago, 3).

(Manuscrit reçu le 8 janvier 1966.)

Miguel E. Herrera,
Facultad de Ciencias fisicomatematicas,
La Plata (Argentine),
Institute for advanced Study,
Princeton, N. J. 08540 (États-Unis).