BULLETIN DE LA S. M. F.

N.T. VAROPOULOS

The functions that operate on $B_0(\Gamma)$ of a discrete group Γ

Bulletin de la S. M. F., tome 93 (1965), p. 301-321

<http://www.numdam.org/item?id=BSMF_1965__93__301_0>

Bull. Soc. math. France,
93, 1965, p. 301 à 321.

THE FUNCTIONS THAT OPERATE ON $B_0(\Gamma)$ OF A DISCRETE GROUP Γ;

BY

NICHOLAS TH. VAROPOULOS.

Introduction and notations. — Let G be a locally compact abelian group, and let $\hat{G}$ be the dual group. We shall, throughout in this paper, follow well established and standardised notations.

We shall denote by $L_1(G)$ the Banach algebra of bounded Radon measures on G which are absolutely continuous with respect to the Haar measure of G; $\mathcal{K}(G) \subset L_1(G)$ will denote the space of continuous functions on G with compact support, and, when G is compact, h_G will denote the normalised Haar measure of G.

We shall also denote by $M(G) \supset M_0(G)$ the Banach algebra of bounded Radon measures on G, and the closed ideal of those measures whose Fourier transform vanishes at the infinity of $\hat{G}$. $M(G)$ has a natural involution $\mu \to \tilde{\mu} = \overline{\mu(-x)}$. Finally we shall denote by $B(\hat{G})$ the function algebra on $\hat{G}$ of all Fourier transforms of elements of $M(G)$.

Let now G be a compact abelian group and let $L_1(G) \subset A \subset M(G)$ be any, not necessarily closed, subalgebra of $M(G)$ containing $L_1(G)$, we then introduce the :

DEFINITION. — We shall say that the complex function Φ operates on A in $[-a, a]$, for some $a > 0$, if Φ is defined in $[-b, b]$ and $b \geq a$, and if for all $\alpha \in A$ such that $-a \leq \hat{\alpha}(\chi) \leq a$ we have $\Phi[\hat{\alpha}(\chi)] \in B(\hat{G})$, i. e. if there exists a measure in $M(G)$, which we shall denote by $\Phi[\alpha] \in M(G)$, such that $(\Phi[\alpha])\hat{}\,(\chi) \equiv \Phi[\hat{\alpha}(\chi)]$.

If now for some $\mu \in M(G)$ we denote by $\{ L_1(G); \mu \}$ the subalgebra of $M(G)$ generated by $L_1(G)$ and μ, we can state the main result of this paper as follows :

THEOREM (F). — *In every infinite compact abelian group G, there exists $\mu \in M_0(G)$ such that the only complex functions that operate on $\{ L_1(G); \mu \}$ in* $[-1, 1]$ *are those that coincide with an entire function in some neighbourhood of* 0.

The material of this paper is divided as follows :

§ 1. We make some general remarks and give an equivalent form to the theorem (F).

§ 2. We prove the theorem for the particular case when $G = T$ the one dimensional torus.

§ 3. We prove the theorem for the particular case when

$$G = \prod_{n=1}^{\infty} \mathbf{Z}(p_n),$$

for prime numbers p_n ($n \geq 1$).

§ 4. We prove the theorem for the particular case when

$$G = U(p) = [\mathbf{Z}(p^{\infty})]^{\wedge}$$

the group of p-adic integers, for some prime number p.

§ 5. We deduce the proof of the general theorem.

1. General remarks.

G denotes an infinite compact abelian group in this paragraph.

Then it is a well known theorem of Kahane-Katznelson (*cf.* [6], 6.5.4) that if Φ operates on $L_1(G)$ in $[-a, a]$ for some $a > 0$, then there exists $a \geq \delta > 0$ such that

$$\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j \qquad \text{for} \quad -\delta < \zeta < \delta.$$

It is also immediate to verify that if $L_1(G) \subset A \subset M_0(G)$ and $a > 0$, and if Φ operates on A in $[-a, a]$ then the function $\Phi_R(\zeta) \equiv \Phi(R\zeta)$ for any $R > 0$ operates on A in $\left[-\frac{a}{R}, \frac{a}{R}\right]$ and if

$$\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j \qquad \text{for} \quad -\delta < \zeta < \delta;$$

$$\Phi_R(\zeta) = \sum_{j=0}^{\infty} \alpha_j R^j \zeta^j \qquad \text{for} \quad -\frac{\delta}{R} < \zeta < \frac{\delta}{R};$$

using these observations it is easy to see that our theorem (F) is equivalent to the following :

THEOREM (Φ). — *In every infinite compact abelian group G, there exists $\mu \in M_0(G)$ such that for every $\delta > 0$ and every $\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j$ (convergent for $\zeta \in [-\delta, \delta]$) which operates on $\{ L_1(G); \mu \}$ in $[-\delta, \delta]$ we must have $\alpha_j = O(1)$ as $j \to \infty$.*

The above theorem motivates the following :

DEFINITION. — We shall say that, for a compact abelian group G, $\mu \in M_0(G)$ is a Φ-measure if for every $\delta > 0$, and every complex function Φ, such that

$$\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j \qquad \text{for} \quad \zeta \in [-\delta, \delta]$$

and which operates on $\{ L_1(G); \mu \}$ in $[-\delta, \delta]$, we can deduce that $\alpha_j = O(1)$ as $j \to \infty$. We prove the obvious :

LEMMA. — *Let G be a compact abelian group and let H be a closed subgroup, and suppose that G/H has Φ-measures, then G has Φ-measures also.*

Proof. — Indeed it suffices to observe that there exists a natural identification of $M(G/H)$ with the subalgebra of $M(G)$ consisting of all those elements of $M(G)$ whose Fourier transform is identically zero outside $(G/H)^\wedge [(G/H)^\wedge \subset \hat{G}]$; and that in that identification $M_0(G/H) \subset M_0(G)$ and $L_1(G/H) \subset L_1(G)$ ([2], chap. 7, § 2).

2. The one dimensional torus.

In this paragraph, we shall prove :

THEOREM (T). — *T the one dimensional torus has Φ-measures.*

The proof of this theorem will not be given before the end of the paragraph; before that, we shall introduce some notations and definitions and also prove some lemmas which are interesting for their own sake.

Let us denote by

$$\Omega = \Omega^{(2)} = \prod_{n=1}^{\infty} \Omega_n^{(2)} = \prod_{n=1}^{\infty} \Omega_n, \qquad \text{where} \quad \Omega_n^{(2)} = \Omega_n = \{ 0; 1 \} \qquad (n \geq 1)$$

(the space of two points). Then using the binary expansion of the real numbers in $[0, 1]$ we can find an onto mapp :

$$s : \quad \Omega \quad \to \quad T = \mathbf{R}/\mathbf{Z}$$

which identifies the two spaces modulo a denumerable set, using s the continuous (diffused) measures on T and Ω can be identified, i. e. $M_c(T) = M_c(\Omega)$, where $M_c(X)$ for a general locally compact space X denotes the space of continuous bounded Radon measures on X. We shall also denote by

$$\overset{N}{\underset{M}{\varpi}} : \Omega \quad \to \quad \prod_{n=M}^{N} \Omega_n$$

the natural projections of the Cartesian product $\Omega (= \Omega^{(2)})$.

Let now

$$Q = \{ q_n \}_{n=1}^{\infty}, \qquad R = \{ r_n \}_{n=1}^{\infty}$$

be two sequences of positive integers such that

(1) $\quad r_n - q_n \geqq 5n; \qquad q_{n+1} - r_n \geqq 5n \quad (n \geqq 1) \ [q_n < r_n < q_{n+1}].$

Let also :

$$E = \{ \varepsilon_n \}_{n=1}^{\infty}$$

be a sequence of real numbers such that

$$(2) \qquad \left\{ 0 < \varepsilon_n < \frac{1}{8}; \qquad \varepsilon_n \underset{n \to \infty}{\longrightarrow} 0; \qquad \sum_{n=1}^{\infty} \varepsilon_n^{\sigma} = +\infty \quad \text{all real } \sigma \geqq 0 \quad [\varepsilon_n^{\sigma} \geqq 0]; \right.$$

Q, R and E being fixed once and for all in this paragraph we can define for all $\sigma \in [1, +\infty)$

$$\mu_{\sigma, N} = \left[\prod_{n=1}^{N} [1 + 2\varepsilon_n^{\sigma} \cos(2^{q_n} t)] \right] h_T.$$

Then $\mu_{\sigma, N} \in M_c^+(T) = M_c^+(\Omega)$, $t \in T$ denotes an integration variable, and where in general $M_c^+(X) = M_c(X) \cap M^+(X)$ for some locally compact space X denotes the set of positive bounded continuous Radon measures on X.

Let us also define

$$\mu_\sigma = \lim_N \mu_{\sigma, N} \qquad \text{for} \quad \sigma \in [1, +\infty)$$

the limit being taken in the vague topology of measures. [It exists because $\| \mu_{\sigma, N} \| = 1 \ (N \geqq 1)$ and, for every $\chi \in \hat{T}$, $\hat{\mu}_{\sigma, N}(\chi)$ converges as $N \to \infty$, as we see in what follows.]

Let us now denote the subset of $\mathbf{Z}$ (= The integers)

$$\mathfrak{S}_M = \{ \eta_1 2^{q_1} + \eta_2 2^{q_2} + \ldots + \eta_M 2^{q_M}; \ \eta_r = 0, +1, -1 \}.$$

We have then for all $\sigma \in [1, +\infty)$,

$$(3) \quad \{0\} \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \ldots \subset \mathfrak{S}_M \subset \ldots \subset \bigcup_M \mathfrak{S}_M = \Sigma = \operatorname{supp} \hat{\mu}_\sigma \subset \mathbf{Z} = \hat{T}$$

and also

(4) $\mathfrak{S}_M \subset [-M\,2^{q_M}, M\,2^{q_M}] = I_M$ and $\mathfrak{S}_M = I_M \cap \Sigma$

and

(5) $m, n \in \mathbf{Z},\quad m \neq n \;\Rightarrow\; 2^{q_{M+1}}m + I_M \cap 2^{q_{M+1}}n + I_M = \emptyset.$

Also if we denote in general by $K\mathbf{Z} = \{Km; m \in \mathbf{Z}\}$ for any $K \in \mathbf{Z}$, then it is immediate that for each $M \geq 1$

(6) $$\Sigma = (\Sigma \cap 2^{q_{M+1}}\mathbf{Z}) + \mathfrak{S}_M.$$

So from (3), (4), (5) and (6), it follows that

(7) $$\hat{\mu}_\sigma\left[\sum_{r=1}^{N} \eta_r\, 2^{q_r}\right] = \prod_{r=1}^{N} \varepsilon_r^{\sigma|\eta_r|} \qquad (\eta_r = 0, +1, -1)$$

and using this it follows at once that for all $\sigma \in [1, +\infty)$

(8) $$\hat{\mu}_\sigma(n) \xrightarrow[|n| \to \infty]{} 0 \qquad \text{and} \qquad \|\hat{\mu}_\sigma - \hat{h}_T\|_\infty \leq 8^{-\sigma}.$$

From (8), it follows that $\mu_\sigma \in M_c^+(T)$, and from (7), it follows that

(9) $$\mu_\sigma \star \mu_\rho = \mu_{\sigma+\rho}.$$

We now prove :

LEMMA 1. — *If* φ, $f \in \mathcal{K}(\mathbf{Z})$ *and if for some* $N \geq 1$,

$$\operatorname{supp} f \subset [-2^{q_N}, 2^{q_N}] \quad \textit{and} \quad \operatorname{supp}\varphi \subset 2^{q_{N+1}}\mathbf{Z}$$

[*this means that* φ *can be considered as* $\in \mathcal{K}(2^{q_{N+1}}\mathbf{Z})$]. *Then for every* $\sigma \in [1, +\infty)$, *we have*

$$\langle \mu_\sigma, \hat{f} \rangle \langle \mu_\sigma, \hat{\varphi} \rangle = \langle \mu_\sigma, \hat{f}.\hat{\varphi} \rangle = \langle \mu_\sigma, (f \star \varphi)\hat{}\, \rangle.$$

Proof. — Observe first that in general for $\alpha \in M(G)$ and $\psi \in L_1(\hat{G})$, and any locally compact group G, we have

(10) $$\langle \alpha, \hat{\psi} \rangle = \langle \hat{\alpha}, \psi \rangle = \int_{s \in G} \int_{\chi \in \hat{G}} \chi(s)\, d\psi(\chi)\, d\alpha(s).$$

Observe next that to prove the lemma, we may assume that

$\varphi = \lambda\delta_\nu$ and $f = l\,\delta_n$, $(n, \nu \in \mathbf{Z})$ ($\lambda \neq 0$, $l \neq 0$ complex numbers),

i. e. that the support of φ and f consist of single points, since then the lemma would follow by bilinearity.

Then it follows from (10) and (4) that

(11) $$\langle \mu_\sigma, \hat{f} \rangle \langle \mu_\sigma, \hat{\varphi} \rangle \neq 0 \iff \nu \in \Sigma \cap 2^{q_{N+1}}\mathbf{Z}, \qquad n \in \mathfrak{S}_N$$

which implies from (4) and (5) that

$$n = \sum_{p=1}^{N} \eta_p \, 2^{q_p}, \qquad \nu = \sum_{p=N+1}^{R} \eta_p \, 2^{q_p} \qquad \text{some} \quad R > N,$$

and (7) and (10) implie then that

$$\langle \mu_\sigma, \hat{f} \rangle \langle \mu_\sigma, \hat{\varphi} \rangle = \lambda l \prod_{p=1}^{R} \varepsilon_p^{\sigma \, | \, \eta_p |} = \langle \mu_\sigma, (f \star \varphi)^\wedge \rangle.$$

Also (11), taking into account (4), (5) and (6) implies that

$$\langle \mu_\sigma, \hat{f} \rangle \langle \mu_\sigma, \hat{\varphi} \rangle = 0 \quad \Rightarrow \quad n + \nu \notin \Sigma \quad \Rightarrow \quad \langle \mu_\sigma, (f \star \varphi)^\wedge \rangle = 0$$

and this completes the proof of the lemma.

We now introduce some more notations and definitions :

For $t \in \Omega \, (= \Omega^{(2)})$, we define $t^{(N)} \in \Omega$ by

$$\overset{q_{N+1}}{\underset{1}{\varpi}} (t^{(N)}) = \overset{q_{N+1}}{\underset{1}{\varpi}} (t) \qquad \text{and} \qquad \overset{\infty}{\underset{q_{N+1}+1}{\varpi}} (t^{(N)}) = \overset{\infty}{\underset{q_{N+1}+1}{\varpi}} (0),$$

where $0 = (0, 0, 0, 0, \ldots) \in \Omega$.

We also define, for all integer $N \geq 1$,

$$\theta_N(t) = 2 \cos\left(2^{q_N} t^{(N)}\right) - 2 \int_\Omega \cos\left(2^{q_N} t^{(N)}\right) dh_T(t).$$

It is immediate then that

$$(12) \qquad \overset{q_{N+1}}{\underset{q_N+1}{\varpi}} (t) = \overset{q_{N+1}}{\underset{q_N+1}{\varpi}} (t') \quad \Rightarrow \quad \theta_N(t) = \theta_N(t')$$

and

$$(13) \qquad \int_\Omega \theta_N(t) \, dh_T(t) = 0 \qquad \text{and} \qquad \| \theta_N \|_\infty \leq 4 \qquad (N \geq 1)$$

and also since

$$\| t - t^{(N)} \|_{t, \infty} = O\left[2^{-q_{N+1}}\right] \qquad \text{as} \quad N \to \infty$$

[where in general for $f(t) \in \mathbf{C}(X)$, X a topological space, we denote $\| f(t) \|_{t, \infty} = \sup_{t \in X} | f(t) |$].

We see using (1) that

$$(14) \quad \| \theta_N(t) - 2 \cos\left(2^{q_N} t\right) \|_{t, \infty} = O\left[2^{q_N - q_{N+1}}\right] = O\left[2^{-10 N}\right] \qquad \text{as} \quad N \to \infty.$$

Let us further introduce the nets on Ω

$$(15)\qquad \begin{cases} R_N(t) = \left\{ \omega \in \Omega;\ \overset{r_N}{\underset{1}{\varpi}}(t) = \overset{r_N}{\underset{1}{\varpi}}(\omega) \right\} \subset \Omega, \\ S_N(t) = \left\{ \omega \in \Omega;\ \overset{q_{N+1}}{\underset{1}{\varpi}}(t) = \overset{q_{N+1}}{\underset{1}{\varpi}}(\omega) \right\} \subset \Omega, \end{cases}$$

and let us finally define using (2), (12) and (13)

$$\nu_{\sigma,N} = \left[\prod_{n=1}^{N} [1 + \varepsilon_n^\sigma \theta_n(t)] \right] h_T \in M^+(\Omega),$$

$$\nu_\sigma = \lim_N \nu_{\sigma,N}$$

for all $\sigma \in [1, +\infty)$.

Our next task is to prove the

LEMMA 2. — *ν_σ and μ_σ are equivalent measures for all $\sigma \in [1, +\infty)$.* This lemma is the analogue of equation (3) of [8].

Proof. — We first compute a certain number of estimates :

ESTIMATE (A) :

$$\left\| 2^{r_N} \mu_{\sigma,N}[R_N(t)] - \prod_{n=1}^{N} [1 + 2\varepsilon_n^\sigma \cos(2^{q_n} t)] \right\|_{t,\infty}$$

$$\leq \left\| \underset{\omega \in R_N(t)}{OSC} \left\{ \prod_{n=1}^{N} [1 + 2\varepsilon_n^\sigma \cos(2^{q_n}\omega)] \right\} \right\|_{t,\infty} = O[2^{N+q_N-r_N}] = O[2^{-4N}]$$

as $N \to \infty$ by (1).

ESTIMATE (B). — Using (3), we see that

$$\| \mu_\sigma[R_N(t)] - \mu_{\sigma,N}[R_N(t)] \|_{t,\infty}$$

$$\leq \left\| \sum_{n \geq N+1} \sum_{s \in \mathfrak{H}_n \setminus \mathfrak{H}_{n-1}} \left| \int_{R_N(t)} \cos(s\omega)\, dh_T(\omega) \right| \right\|_{t,\infty}$$

$$= O\left[\sum_{n=N+1}^{\infty} 2^{n-q_n} \right] = O[2^{N-q_{N+1}}] \qquad \text{as} \quad N \to \infty.$$

ESTIMATE (C). — Putting together (A) and (B), and using (1), we see that

$$\left\| 2^{r_N} \mu_\sigma[R_N(t)] - \prod_{n=1}^{N} [1 + 2\varepsilon_n^\sigma \cos(2^{q_n} t)] \right\|_{t,\infty} = O[2^{-4N}] \qquad \text{as } N \to \infty.$$

ESTIMATE (D). — Using (12) and (14), we see that

$$\nu_\sigma[R_N(t)] = \nu_{\sigma, N}[R_N(t)]$$

and

$$\left\| 2^{r_N} \nu_\sigma[R_N(t)] - \prod_{n=1}^{N} [1 + \varepsilon_n^\sigma \theta_n(t)] \right\|_{t, \infty}$$

$$\leq \left\| \underset{\omega \in R_N(t)}{OSC} \left\{ \prod_{n=1}^{N} [1 + \varepsilon_n^\sigma \theta_n(\omega)] \right\} \right\|_{t, \infty} = O[2^{N+q_N-r_N}] = O[2^{-4N}]$$

as $N \to \infty$ from (1).

ESTIMATE (E). — Using (C) and (D) above and the fact that [(2), (13)]

$$2^{-N} \leq \prod_{n=1}^{N} [1 + \varepsilon_n^\sigma \theta_n(t)] \leq 2^N$$

for all $\sigma \in [1, +\infty)$, we deduce that

$$\left\| \frac{\mu_\sigma[R_N(t)]}{\nu_\sigma[R_N(t)]} - \prod_{n=1}^{N} \frac{1 + 2\varepsilon_n^\sigma \cos(2^{q_n} t)}{1 + \varepsilon_n^\sigma \theta_n(t)} \right\|_{t, \infty} = O[2^{-N}] \quad \text{as} \quad N \to \infty.$$

To complete now the proof of the lemma observe that because of (14)

$$(16) \qquad \prod_{n=1}^{N} \frac{1 + 2\varepsilon_n^\sigma \cos(2^{q_n} t)}{1 + \varepsilon_n^\sigma \theta_n(t)} \xrightarrow[N \to \infty]{} \Delta^{(\sigma)}(t) \quad \text{uniformly for } t$$

for some function of t, $\Delta^{(\sigma)}(t)$; and that there exists $\beta \geq 1$ such that

$$(17) \qquad \beta^{-1} \leq \Delta^{(\sigma)}(t) \leq \beta.$$

Then the estimate (E) and (16) imply that

$$\frac{\mu_\sigma[R_N(t)]}{\nu_\sigma[R_N(t)]} \xrightarrow[N \to \infty]{} \Delta^{(\sigma)}(t) \quad \text{uniformly as } t \in \Omega$$

and this together with (17) imply the required result that μ_σ and ν_σ are equivalent measures. We prove next the

LEMMA 3. — $\rho, \sigma \in [1, +\infty), \rho \neq \sigma \Rightarrow \nu_\rho \perp \nu_\sigma$.

Proof. — For the proof of the lemma the technique developed in [7] and [8] is very closely followed.

We introduce the following functions of $t \in \Omega$,

$$Z_n^{[\rho/\sigma]} = Z_n^{[\rho/\sigma]}(t) = \log \frac{1 + \varepsilon_n^\rho \theta_n(t)}{1 + \varepsilon_n^\sigma \theta_n(t)} \qquad \text{for} \quad \rho, \sigma \in [1, +\infty)$$

and consider $\{Z_n^{[\rho/\sigma]}\}_{n=1}^{\infty}$ as a sequence of random variables with respect to the probability distribution ν_σ.

Then (2) and (13) imply that $\{Z_n^{[\rho/\sigma]}\}_{n=1}^{\infty}$ are uniformly bounded; and (12) that they are independent.

Assume now that $\sigma > \rho \geqq 1$. We have

$$Z_n^{[\rho/\sigma]} = \varepsilon_n^\rho [\theta_n(t) + o(1)] \quad \text{as } n \to \infty \text{ uniformly in } t \in \Omega.$$

And therefore applying (8), (14) and (17) and lemma 2, we see that ([5], chap. 7, sect. 43)

$$(18) \quad \mathbf{E}(Z_n^{[\rho/\sigma]})^2 = \varepsilon_n^{2\rho}\left(\int_\Omega \theta_n^2(t)\, d\nu_\sigma + o(1)\right) \geqq \beta^{-1}\varepsilon_n^{2\rho}\left(\int_\Omega \theta_n^2(t)\, d\mu_\sigma + o(1)\right)$$
$$= \beta^{-1}\varepsilon_n^{2\rho}\left(4\int_\Omega \cos^2(2^{q_n}t)\, d\mu_\sigma + o(1)\right)$$
$$= \beta^{-1}\varepsilon_n^{2\rho}(2 + o(1)) \qquad \text{as} \quad n \to \infty,$$

also we have from (2) and (13) (the complete analogue of equation (4) of [8]) that

$$\mathbf{E}Z_n^{[\rho/\sigma]} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\varepsilon_n^{\rho k} - \varepsilon_n^{\sigma k}}{k} \int_\Omega \theta_n^k(t)\, d\nu_\sigma = O[\varepsilon_n^{2\rho}]$$

and that together with (18) implies that

$$\sigma^2(Z_n^{[\rho/\sigma]}) \geqq \varepsilon_n^{2\rho}[2\beta^{-1} + o(1)] \qquad \text{as} \quad n \to \infty$$

and this in turn implies that

$$(19) \qquad \sum_{n=1}^{\infty} \sigma^2(Z_n^{[\rho/\sigma]}) = +\infty.$$

Now just as in [7] and [8], we use the following proposition of probability theory :

If $\{U_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of independent random variables such that $\sum_{n=1}^{\infty} \sigma^2 U_n = +\infty$; then we have $\overline{\lim_N} \left|\sum_{n=1}^{N} U_n\right| = +\infty$ almost surely.

From that proposition and (19), we deduce that for $\sigma > \rho \geqq 1$

$$(20) \qquad \overline{\lim_N} \left| \sum_{n=1}^{N} Z_n^{[\rho/\sigma]} \right| = +\infty \quad \text{p. p. } \nu_\sigma.$$

For the sake of completeness, we give here a proof of the above proposition :

We consider another sequence $\{U'_n\}_{n=1}^{\infty}$ of random variables identically distributed with the sequence $\{U_n\}_{n=1}^{\infty}$ and such that the family of random variables $\{U_m; U'_n\}_{m,\,n=1}^{\infty}$ is independent. We then consider the uniformly bounded sequence of independent random variables $\{V_n = U_n - U'_n\}_{n=1}^{\infty}$, and observe that $\sigma^2 V_n = 2\sigma^2 U_n$ and $\mathbf{E}V_n = 0$ all $n \geq 1$. Now an application of Kolmogorov's inequality ([4], 16.2, A) gives $\overline{\lim_N} \left| \sum_{n=1}^{N} V_n \right| = +\infty$ almost surely; to deduce from that the proposition it suffices to observe that if

$$P\left[\overline{\lim_N} \left| \sum_{n=1}^{N} U_n \right| = +\infty \right] < 1,$$

then the zero-one law of probability theory would give

$$P\left[\overline{\lim_N} \left| \sum_{n=1}^{N} U_n \right| < +\infty \right] = 1$$

and therefore also

$$P\left[\overline{\lim_N} \left| \sum_{n=1}^{N} V_n \right| < +\infty \right] = 1$$

which is a contradiction, and proves the proposition.

We are now in a position to complete the proof of the lemma just as in [7] and [8].

For ρ and $\sigma \in [1, +\infty)$ arbitrary, the lower derivative of ν_ρ with respect ν_σ along a natural net (15) of Ω is given by

$$D^{[\rho/\sigma]}(t) = \underline{\lim_N} \frac{\nu_\rho[S_N(t)]}{\nu_\sigma[S_N(t)]} = \underline{\lim_N} \prod_{n=1}^{N} \frac{1 + \varepsilon_n^\rho\,\theta_n(t)}{1 + \varepsilon_n^\sigma\,\theta_n(t)}.$$

Therefore it follows ([5], chap. 7, sect. 43; [1], § 5, n° 7) that the following condition :

$$\text{(i)} \qquad \underline{\lim_N} \sum_{n=1}^{N} Z_n^{[\rho/\sigma]} = -\infty \text{ p. p. } \nu_\sigma \iff D^{[\rho/\sigma]}(t) = 0 \text{ p. p. } \nu_\sigma$$

implies that $\nu_\rho \perp \nu_\sigma$.

But it is also true that the following condition

$$\text{(ii)} \qquad \lim_N \sum_{n=1}^{N} Z_n^{[\rho/\sigma]} = +\infty \text{ p. p. } \nu_\sigma \quad \Leftrightarrow \quad D^{[\sigma/\rho]}(t) = 0 \text{ p. p. } \nu_\sigma$$

implies that $\nu_\rho \perp \nu_\sigma$.

To see that we assume that (ii) holds and yet ν_ρ and ν_σ are not orthogonal. Then there exists $E \subset \Omega$ a Borel subset such that $\nu_\sigma(E) = 1$ and such that $D^{[\sigma/\rho]}(t) = 0$ for all $t \in E$. But then since $\nu_\rho(E) > 0$ (ν_ρ and ν_σ not being orthogonal) the zero-one law of probability applied to the sequence $\{\theta_n\}_{n=1}^{\infty}$, considered as a sequence of random variables for the probability distribution ν_ρ, implies that $\nu_\rho[D^{[\sigma/\rho]}(t) = 0] = 1$, which by condition (i) implies that $\nu_\rho \perp \nu_\sigma$. And that is a contradiction.

Now using (20) and the zero-one law of probability, we see that for $\sigma > \rho$ either (i) or (ii) must hold; so in either case $\nu_\rho \perp \nu_\sigma$, which proves the lemma.

LEMMA 4. — $\rho \neq \sigma$, ρ, $\sigma \in [1, +\infty) \Rightarrow \mu_\rho \perp \mu_\sigma$.

This is a consequence of lemma 2 and lemma 3.

From lemma 4 and (9), it follows in particular that μ_σ is a singular measure for all $\sigma \in [1, +\infty)$.

We make a final observation before proving theorem (T).

If we denote by

$$\mu_\sigma^{(R)} = \lim_N \left\{ \left[\prod_{n=R}^{N} [1 + \varepsilon_n^\sigma \cos(2^{q_n} t)] \right] h_T \right\} \in M_0(T)$$

for $R \geqq 1$, then using the identification of the proof of the lemma of paragraph 1, we see that

$$\mu_\sigma^{(R)} \in M_0 \left[T/\mathbf{Z}\left(2^{q_R}\right) \right] \subset M_0(T).$$

And if we apply lemma 4 to the measures $\mu_\sigma^{(R)}$ which are all of the same type as $\mu_\sigma = \mu_\sigma^{(1)}$ [for different but admissible choises of Q, R, and E], we deduce that $\mu_\sigma^{(R)}$ is singular for $R \geqq 1$ and $\sigma \in [1, +\infty)$ and

$$\text{(21)} \qquad R \geqq 1 \text{ and } \sigma \neq \rho; \quad \sigma, \rho \in [1, +\infty) \quad \Rightarrow \quad \mu_\sigma^{(R)} \perp \mu_\rho^{(R)}.$$

Proof of theorem (T). — We prove that μ_1, as defined above, is a Φ-measure of T.

Let $\delta > 0$ arbitrary and let $\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j$ for $\zeta \in [-\delta, \delta]$ operate in $A = \{L_1(T); \mu_1\}$, we prove that $\alpha_j = O(1)$ as $j \to \infty$.

Let s be a positive integer such that $8^{-s} \leqq \frac{\delta}{2}$ and let us denote by

$$\mu = \mu_s = \mu_1^s \in A; \qquad \lambda = \mu - h_T = (\mu_1 - h_T)^s \in A,$$

because of (9). We observe at once that, for q positive integer, we have

$$\lambda^q = \mu^q - h_T = \mu_1^{sq} - h_T = (\mu_1 - h_T)^{sq} \in A, \tag{22}$$

and, also because of (8), that

$$\| \hat{\lambda} \|_\infty \leqq \frac{\delta}{2}. \tag{23}$$

Now fix j a positive integer; and using the fact that $\mu^j = \mu_{sj}$ is a singular measure, and also the fact that the real trigonometric polynomials of T are dense in $\mathbf{C}_\mathbf{R}(T)$, we see that there exists N_j a positive integer and $f_j \in \mathcal{K}(\mathbf{Z})$ such that

$$\operatorname{supp} f_j \subset \left[-2^{q_{N_j}}, 2^{q_{N_j}}\right] \quad \text{and} \quad \langle h_T, \hat{f}_j \rangle = 0 \tag{24}$$

and

$$f_j = \tilde{f}_j; \qquad \| \hat{f}_j \|_\infty \leqq 1; \qquad | \alpha_j \langle \lambda^j, \hat{f}_j \rangle - \alpha_j | \leqq 1. \tag{25}$$

Now using (10) and (23), we see that there exists $M_j > j$ a positive integer such that

$$\sum_{r > M_j} | \alpha_r \langle \lambda^r, \hat{f}_j \rangle | \leqq 1. \tag{26}$$

Since now by (21) the measures

$$\mu_{sp}^{(N_j+1)} \in M_0\left[T / \mathbf{Z}\left(2^{q_{N_j+1}}\right)\right] \qquad (p = 1, 2, \ldots, M_j)$$

are singular and mutually orthogonal, and since the real trigonometric polynomials of $T / \mathbf{Z}\left(2^{q_{N_j+1}}\right)$ are uniformly dence in $\mathbf{C}_\mathbf{R}\left[T / \mathbf{Z}\left(2^{q_{N_j+1}}\right)\right]$, we see that we can find $\varphi_j \in \mathcal{K}(\mathbf{Z})$ such that

$$\left\{ \begin{array}{l} \operatorname{supp} \varphi_j \subset 2^{q_{N_j+1}} \mathbf{Z} \left(\Leftrightarrow \varphi_j \in \mathcal{K}\left[\left(T / \mathbf{Z}\left(2^{q_{N_j+1}}\right)\right)^\wedge\right] \right. \\ \qquad \text{and} \quad \langle h_T, \hat{\varphi}_j \rangle = 0 \end{array} \right. \tag{27}$$

and

$$\varphi_j = \tilde{\varphi}_j \quad \text{and} \quad \| \hat{\varphi}_j \|_\infty \leqq 1 \tag{28}$$

and also using (25)

$$\left| \sum_{p=0}^{M_j} \alpha_p \langle \mu_{sp}^{(N_j+1)}, \hat{\varphi}_j \rangle \langle \lambda^p \hat{f}_j \rangle - \alpha_j \right| \leqq 2$$

and from this and (27), we deduce that

$$(29) \qquad \left| \sum_{p=0}^{M_j} \alpha_p \langle \lambda^p, \hat{\varphi}_j \rangle \langle \lambda^p, \hat{f}_j \rangle - \alpha_j \right| \leq 2.$$

Now from (22), (27) and (28), we deduce that for all positive integer q, we have

$$|\langle \lambda^q, \hat{\varphi}_j \rangle| \leq 1$$

and from that and (26) and (29), we deduce that

$$\left| \alpha_j - \sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, \hat{\varphi}_j \rangle \langle \lambda^r, \hat{f}_j \rangle \right| \leq 3$$

and this together with the lemma 1 and (24) and (27) implies that

$$(30) \qquad \left| \alpha_j - \sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, \hat{\varphi}_j \hat{f}_j \rangle \right| = O(1) \qquad \text{as} \quad j \to \infty.$$

Finally using (10), (23), (25) and (28), we see that

$$\sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, \hat{\varphi}_j \hat{f}_j \rangle = \sum_{r=0}^{\infty} \alpha_r \langle \hat{\lambda}_r, \varphi_j \star f_j \rangle = \langle (\Phi[\lambda])^\wedge, \varphi_j \star f_j \rangle$$
$$= \langle \Phi[\lambda], \hat{\varphi}_j \hat{f}_j \rangle = O(1) \qquad \text{as} \quad j \to \infty$$

and that together with (30) gives

$$\alpha_j = O(1) \qquad \text{as} \quad j \to \infty$$

which completes the proof of theorem (T).

3. The Cartesian product.

THEOREM (II). — If $G = \prod_{n=1}^{\infty} G_n$ where $G_n \cong \mathbf{Z}(p_n)$ with p_n prime numbers $(n \geq 1)$, then G has a Φ-measure.

Proof. — We shall prove the theorem in the following two particular cases :

Case A : $p_n = p$ $(n \geq 1)$ a fixed prime;

Case B : $p_n \geq n$ all $n \geq 1$.

The general result follows from those cases and the lemma of paragraph 1, since every group of the type considered in theorem (II) can be written as $H \times K$, where H is either as in *case* A or *case* B.

So for the rest of the proof, we shall assume that we are in either *case* A or *case* B. Then using the material of [7], we see that there exist for each $n \geq 1$:

$$\mu_n \in M^+(G_n); \qquad \|\mu_n\| = 1; \qquad \mu_n = \tilde{\mu}_n$$

such that for every $N \geq 1$,

$$(31) \qquad \bigotimes_{n \geq N} \mu_n \text{ is } \Theta\text{-measure of} \prod_{n \geq N} G_n \quad [7]$$

and such that if we denote by $\mu = \bigotimes_{n=1}^{\infty} \mu_n$, then

$$(32) \qquad \|\hat{\mu} - \hat{h}_G\|_\infty < 1.$$

We proceed to prove that this μ above is the required Φ-measure of G. To that effect let $\delta > 0$ be arbitrary and let $\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j$ for $\zeta \in [-\delta, \delta]$ operate on $\{L_1(G); \mu\}$ in $[-\delta, \delta]$, we shall prove that $\alpha_j = O(1)$ as $j \to \infty$.

By (32), we may chose s positive integer s. t., if we define for all $n \geq 1$:

$$(33) \quad \nu_n = \mu_n^s; \qquad \nu = \bigotimes_{n=1}^{\infty} \nu_n = \mu^s \in A; \qquad \lambda = \mu^s - h_G = \nu - h_G = (\mu - h_G)^s,$$

then

$$(34) \qquad \|\hat{\lambda}\|_\infty \leq \frac{\delta}{2} \qquad \text{and} \qquad \lambda^p = \nu^p - h_G$$

for every positive integer p.

Fix now j a positive integer. Then using the fact that ν^j is a singular measure and that the real trigonometric polynomials on G are dense in $\mathbf{C_R}(G)$, we see that there exists $N_j \geq 1$ and $f_j \in \mathcal{K}(\hat{G})$ such that

$$(35) \qquad \operatorname{supp} f_j \subset \sum_{n=1}^{N_j} \hat{G}_n \qquad \text{and} \qquad \langle h_G, \hat{f}_j \rangle = 0$$

and

$$(36) \qquad f_j = \tilde{f}_j; \qquad \|\hat{f}_j\|_\infty \leq 1; \qquad |\alpha_j \langle \lambda^j, \hat{f}_j \rangle - \alpha_j| \leq 1.$$

Using then (34) and (10), we see that there exists $M_j > j$ such that

$$(37) \qquad \sum_{r > M_j} |\alpha_r \langle \lambda^r, \hat{f}_j \rangle| \leq 1.$$

Finally using (36), the fact that $\bigotimes_{n>N_j} \nu_n$ is a Θ-measure of $\prod_{n>N_j} G_n = G^{(N_j)}$, and the fact that the real trigonometric polynomials of $G^{(N_j)}$ are uniformly dense in $\mathbf{C_R}[G^{(N_j)}]$, we see that there exists $\varphi_j \in \mathcal{K}(\hat{G})$ such that

$$(38) \qquad \operatorname{supp} \varphi_j \subset \mathcal{K}\left(\sum_{n>N_j} \hat{G}_n\right) \qquad \text{and} \qquad \langle h_G, \hat{\varphi}_j \rangle = 0$$

and

$$(39) \quad \varphi_j = \tilde{\varphi}_j; \qquad \|\hat{\varphi}_j\|_\infty \leqq 1; \qquad \left| \sum_{r=0}^{M_j} \alpha_r \langle \lambda^r, \hat{\varphi}_j \rangle \langle \lambda^r, \hat{f}_j \rangle - \alpha_j \right| \leqq 2,$$

but since from (34), (38) and (39), we have for every positive integer q

$$|\langle \lambda^q, \hat{\varphi}_j \rangle| \leqq 1.$$

We see that (37) and (39) give

$$(40) \qquad \left| \alpha_j - \sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, \hat{\varphi}_j \rangle \langle \lambda^r, \hat{f}_j \rangle \right| \leqq 3,$$

but by (35) and (38), we have, for all r,

$$\langle \lambda^r, \hat{\varphi}_j \rangle \langle \lambda^r, \hat{f}_j \rangle = \langle \lambda^r, \hat{f}_j \otimes \hat{\varphi}_j \rangle = \langle \lambda^r, (f_j \otimes \varphi_j)^\wedge \rangle$$

and that together with (10), (34), (36) and (39) gives

$$\sum_{r=0}^{\infty} \alpha^r \langle \lambda^r, \hat{\varphi}_j \rangle \langle \lambda^r, \hat{f}_j \rangle = \sum_{r=0}^{\infty} \alpha_r \langle \hat{\lambda}^r, f_j \otimes \varphi_j \rangle = \langle (\Phi[\lambda])^\wedge, f_j \otimes \varphi_j \rangle$$
$$= \langle \Phi[\lambda], \hat{f}_j \otimes \hat{\varphi}_j \rangle = O(1) \qquad \text{as} \quad j \to \infty$$

which together with (40) gives the required result that

$$\alpha_j = O(1) \qquad \text{as} \quad j \to \infty.$$

And that completes the proof of theorem (II).

4. The p-adic integers.

We shall prove in this paragraph the

THEOREM (U). — If $G = U(p)$ is the additive group of p-adic integers for some prime p, then G has a Φ-measure.

Observe at once, that

$$U(p) = \lim_{\overleftarrow{N}} \mathbf{Z}(p^N) = [\mathbf{Z}(p^\infty)]^\wedge.$$

The key reference for the proof of theorem (U) is [8], and to avoid unnecessary repetitions we follow it very closely indeed; we start by giving a:

Summary of results and notations of [8]. — There exist canonical identifications of $\mathbf{Z}(p^N)$ with

$$\prod_{n=1}^{N} \Omega_n = \prod_{n=1}^{N} \Omega_n^{(p)}$$

and of $G = U(p)$ with

$$\Omega = \Omega^{(p)} = \prod_{n=1}^{\infty} \Omega_n = \prod_{n=1}^{\infty} \Omega_n^{(p)}, \qquad \text{where} \quad \Omega_n = \Omega_n^{(p)} = \{ 0; 1; \ldots; p-1 \}$$

(the space of p elements); and with those identifications the canonical projections $\pi_N : U(p) \to \mathbf{Z}(p^N)$ are identified with $\overset{N}{\underset{1}{\varpi}}$, where $\overset{N}{\underset{M}{\varpi}} : \Omega \to \prod_{n=M}^{N} \Omega_n$ are the canonical projections of the cartesian product.

Now let $\{ K_n \}_{n=0}^{\infty}$ be a sequence of integers and $\{ \varepsilon_n \}_{n=1}^{\infty}$ a sequence of real numbers such that $K_0 \geqq 2$ and $K_{n+1} - K_n \geqq n+1$ and

$$0 \leqq \varepsilon_n \leqq \frac{1}{8}; \qquad \varepsilon_n \xrightarrow[n \to \infty]{} 0; \qquad \sum_{n=1}^{\infty} \varepsilon_n^\sigma = +\infty \quad \text{all real } \sigma \geqq 0 \qquad [\varepsilon_n^\sigma \geqq 0].$$

For those sequences, we can construct two families of singular continuous measures on $G = U(p)$

$$(\mu_\sigma)_{1 \leqq \sigma < +\infty} \quad \text{and} \quad (\nu_\sigma)_{1 \leqq \sigma < +\infty}$$

such that for all $\sigma \in [1, +\infty)$ [equation (3) of [8]]

(41) μ_σ and ν_σ are equivalent

and such that

(42) $\mu_\sigma \geqq 0, \qquad \nu_\sigma \geqq 0; \qquad \| \mu_\sigma \| = \| \nu_\sigma \| = 1$

and which also satisfy the following list of properties

(43) $\mu_\sigma \bigstar \mu_\tau = \mu_{\sigma+\tau}; \qquad \mu_\sigma = \tilde{\mu}_\sigma; \qquad \mu_\sigma \in M_0(G)$

for all $\sigma, \tau \in [1, +\infty)$ [equation (1) of [8]]

(44) $\| \hat{\mu}_\sigma - \hat{h}_G \|_\infty \leqq 8^{-\sigma}.$

This follows from straight computation of the Fourier transform of μ_σ, as defined in [8], and the fact that $0 \leqq \varepsilon_n \leqq 1/8$.

(45) $$\nu_\sigma = \bigotimes_{n=1}^{\infty} \nu_n^{(\sigma)} \qquad \text{for some} \quad \nu_n^{(\sigma)} \in M^+\left(\prod_{K_{n-1}+1}^{K_n} \Omega_n\right),$$

where we use the identification of G and Ω

(46) $$\bigotimes_{n \geqq N} \nu_n^{(\sigma)} \perp \bigotimes_{n \geqq N} \nu_n^{(\tau)} \qquad \text{for} \quad \sigma, \tau \in [1, +\infty) \text{ and } \sigma \neq \tau,$$

where of course

$$\bigotimes_{n \geqq M} \nu_n^{(\sigma)} \in M^+\left(\prod_{n \geqq K_{M-1}+1} \Omega_n\right), \qquad \sigma \in [1, +\infty)$$

this is seen the same way as the equation (7) of [8].

(47) $$\|\mu_\sigma - \nu_\sigma\| = O[8^{-\sigma}] \qquad \text{as} \quad \sigma \to \infty$$

this is a consequence of (41) and the relation (2) of [8] together with ([5], chap. 7, sect. 43). [It is in fact a refinement of equation (3) of [8].]

Proof of theorem (U). — We prove that the μ_1 defined in [8], some of whose properties we have summarised above is a Φ-measure of G. To that effect let $\delta > 0$ be arbitrary and let

$$\Phi(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j \qquad \text{for} \quad \zeta \in [-\delta, \delta]$$

operate on $A = \{L_1(G); \mu_1\}$ in $[-\delta, \delta]$, we shall prove that $\alpha_j = O(1)$ as $j \to \infty$.

Let us chose s a positive integer such that $8^{-s} \leqq \delta/2$ and let us denote using (43) by

$$\mu = \mu_1^s = \mu_s \in A; \qquad \lambda = \mu - h_G; \qquad \eta_k = \nu_{sk} - h_G$$

for k all positive integers.

We then observe at once using (43) that, for all q positive integers, we have

(48) $$\lambda^q = \mu_{sq} - h_G = \mu^q - h_G = \mu_1^{sq} - h_G = (\mu_1 - h_G)^{sq}.$$

We have also by (44)

(49) $$\|\hat{\lambda}\|_\infty = \|\hat{\mu}_1^s - \hat{h}_G\|_\infty = \|[(\mu_1 - h_G)^s]^\wedge\|_\infty \leqq 8^{-s} \leqq \frac{\delta}{2},$$

using (47) and (48), we see also that

$$(50) \qquad \| \lambda^q - \eta_q \| = O\left[\left(\frac{\delta}{2}\right)^q\right] \qquad \text{as} \quad q \to \infty.$$

Fix now j an arbitrary positive integer. Since ν_{sj} is a singular measure on G, and since the set of functions $\left\{ \check{P} \circ \overset{N}{\underset{1}{\varpi}};\ N \geq 1,\ \check{P} \in \mathbf{C_R}\left(\prod_{n=1}^{N} \Omega_n\right)\right\}$ are uniformly dense in $\mathbf{C_R}(G)$ [the identification of G and Ω being tacitly used here and subsequently], we see that there exists $N_j \geq 1$ and

$$P_j = \check{P}_j \circ \overset{K_{N_j}}{\underset{1}{\varpi}} \quad \text{for some } \check{P}_j \in \mathbf{C_R}\left(\prod_{n=1}^{K_{N_j}} \Omega_n\right)$$

such that

$$(51) \qquad \| P_j \|_\infty \leq 1; \qquad \langle h_G, P_j \rangle = 0 \qquad | \alpha_j \langle \eta_j, P_j \rangle - \alpha_j | \leq 1.$$

Now since P_j can be factored through $\pi_{K_{N_j}}$ there exists $f_j \in \mathcal{K}(\hat{G})$ such that $\hat{f}_j \equiv P_j$. Therefore by (10) and (49) we see that

$$\sum_{r=0}^{\infty} | \alpha_r \langle \lambda^r, P_j \rangle | < +\infty$$

which implies by (50) that

$$\sum_{r=0}^{\infty} | \alpha_r \langle \eta_r, P_j \rangle | < +\infty.$$

Thus we can find $M_j > j$ such that

$$(52) \qquad \sum_{r > M_j} | \alpha_r \langle \eta_r, P_j \rangle | \leq 1.$$

Now using (45), (46), (51) and the fact that $(\nu_\sigma)_{1 \leq \sigma < +\infty}$ are all singular and continuous, we see that there exists $\check{f} \in \mathbf{C_R}\left(\prod_{K_{N_j}+1}^{\infty} \Omega_n\right)$ such that if $f = \check{f} \circ \underset{K_{N_j}+1}{\overset{\infty}{\varpi}} \in \mathbf{C_R}(\Omega)$, then

$$\| f \|_\infty \leq 1; \qquad \langle h_G, f \rangle = 0; \qquad \left| \sum_{r=0}^{M_j} \alpha_r \langle \eta_r, f \rangle \langle \eta_r, P_j \rangle - \alpha_j \right| < 2.$$

We can then approximate uniformly f by a function of the form $Q_j = \check{Q}_j \circ \overset{K_{R_j}}{\underset{K_{N_j}+1}{\varpi'}}$ for some $R_j > N_j$ and some $\check{Q}_j \in C_{\mathbf{R}}\left[\prod_{K_{N_j}+1}^{K_{R_j}} \Omega_n\right]$ and which satisfies

$$(53)\quad \|Q_j\|_\infty \leq 1; \qquad \langle h_G, Q_j \rangle = 0; \qquad \left|\sum_{r=0}^{M_j} \alpha_r \langle \eta_r, Q_j \rangle \langle \eta_r, P_j \rangle - \alpha_j\right| \leq 2$$

let then

$$(54)\quad \check{S}_j = \check{P}_j \otimes \check{Q}_j \in C_{\mathbf{R}}\left[\prod_{n=1}^{K_{R_j}} \Omega_n\right] \qquad \text{and} \qquad S_j = \check{S}_j \circ \overset{K_{R_j}}{\underset{1}{\varpi'}} = \hat{\varphi}_j$$

for some $\varphi_j \in \mathcal{K}(\hat{G})$.

Then we see from (51) and (53) that for all positive integer q we have

$$|\langle \eta_q, Q_j \rangle| \leq 1 \qquad \text{and} \qquad \langle \eta_q, P_j \rangle \langle \eta_q, Q_j \rangle = \langle \eta_q, S_j \rangle$$

and that together with (52) and (53) implies that

$$\left|\alpha_j - \sum_{r=0}^{\infty} \alpha_r \langle \eta_r, S_j \rangle\right| \leq 3.$$

From this using (50) and the fact that $\|S_j\|_\infty \leq 1$, we deduce that

$$(55)\qquad \left|\alpha_j - \sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, S_j \rangle\right| = O(1) \qquad \text{as} \quad j \to \infty.$$

On the other hands from (10), (49) and (54) we have

$$\sum_{r=0}^{\infty} \alpha_r \langle \lambda^r, S_j \rangle = \sum_{r=0}^{\infty} \alpha_r \langle \hat{\lambda}_r, \varphi_j \rangle = \langle \Phi[\lambda], \hat{\varphi}_j \rangle = O(1)$$

as $j \to \infty$ since $\|\hat{\varphi}_j\|_\infty \leq 1$, and this together with (55) gives the required result that

$$\alpha_j = O(1) \qquad \text{as} \quad j \to \infty$$

which completes the proof of theorem (U).

5. **Proof of the theorem Φ.**

To prove the theorem for a general infinite compact group G, we consider $\hat{G}$ the discrete dual group of G, and observe that we can distinguish three mutually exclusive and exhaustive possibilities for $\hat{G}$.

(A) $\hat{G}$ is not a torsion group.

(B) $\hat{G}$ is a torsion group and has a subgroup $K \subset \hat{G}$ which for some prime p is $K \cong \mathbf{Z}(p^\infty)$.

(C) $\hat{G}$ is a torsion group, and no subgroup of $\hat{G}$ is isomorphic to any $\mathbf{Z}(p^\infty)$ for any prime p.

We prove that in case (C), $\hat{G}$ has a subgroup $K \subset \hat{G}$ which is $K \cong \sum_{n=1}^{\infty} \mathbf{Z}(p_n)$ for prime numbers p_n.

To that effect, we shall need the following lemma due to Prüfer ([3], § 25, vol. 1, p. 181) stating that :

The only infinite indecomposable, torsion, abelian groups are the $\mathbf{Z}(p^\infty)$, *where p runs through all primes.*

Using that lemma, we see that we can construct inductively two sequences of subgroups of $\hat{G}$

$$\{A_n\}_{n=1}^{\infty}; \quad \{B_n\}_{n=0}^{\infty}$$

such that $B_0 = \hat{G}$; $A_n \neq \{0\}$ $(n \geq 1)$; B_n is infinite $(n \geq 0)$. And also such that

$$B_n = A_{n+1} \oplus B_{n+1} \quad \text{for all } n \geq 0.$$

Indeed it suffices to observe that if B_n has been constructed, since it is infinite and $\not\cong \mathbf{Z}(p^\infty)$ for any prime p, it is non indecomposable and thus can be written as the product of two non trivial subgroups $B_n = A_{n+1} \oplus B_{n+1}$ were we may assume B_{n+1} to be infinite and of course $A_{n+1} \neq \{0\}$.

We see then that

$$\sum_{n=1}^{\infty} \mathbf{Z}(p_n) \cong K \subseteq \sum_{n=1}^{\infty} A_n = A$$

and A can be identified with a subgroup of $\hat{G}$ which completes the proof of our assertion.

Now by duality it follows that in

Case A : $G \supset H$ a closed subgroup such that $G/H \cong T$.

Case B : $G \supset H$ a closed subgroup such that $G/H \cong U(p) = [\mathbf{Z}(p^\infty)]^\wedge$ some prime p.

Case C : $G \supset H$ a closed subgroup such that $G/H \cong \prod_{n=1}^{\infty} \mathbf{Z}(p_n)$ for prime numbers p_n.

From that our theorem (Φ) is seen to be correct in each *case* (A), (B) and (C) by the lemma of paragraph 1 and theorems (T), (U) and (Π) respectively. The proof of our result is completed.

It is my great pleasure to finish up by expressing my gratitude to Paul MALLIAVIN, who has read throughout several versions and drafts of those methods and results and has made many valuable suggestions and criticisms.

BIBLIOGRAPHY.

[1] BOURBAKI (Nicolas). — *Intégration*, chap. 5. — Paris, Hermann, 1956 (*Act. scient. et ind.*, 1244; Bourbaki, 21).

[2] BOURBAKI (Nicolas). — *Intégration*, chap. 7-8. — Paris, Hermann, 1964 (*Act. scient. et ind.*, 1306; Bourbaki, 29).

[3] KUROŠ (A. G.). — *The theory of groups*, vol. 1. — New York, Chelsea publishing Comp., 1955.

[4] LOÈVE (Michel). — *Probability theory*, 3rd edition. — Princeton, D. Van Nostrand Comp., 1963 (The University Series in higher Mathematics).

[5] MUNROE (M. E.). — *Introduction to measure and integration*. — Cambridge, Addison-Wesley publishing Comp., 1953 (Addison-Wesley Mathematics Series).

[6] RUDIN (Walter). — *Fourier analysis on groups*. — New York, Interscience Publishers, 1962 (*Interscience Tracts in pure and applied Mathematics*, 12).

[7] VAROPOULOS (Nicholas T.). — Sur les mesures de Radon d'un groupe localement compact abélien, *C. R. Acad. Sc.*, t. 258, 1964, p. 3805-3808.

[8] VAROPOULOS (Nicholas T.). — Sur les mesures de Radon d'un groupe localement compact, *C. R. Acad. Sc.*, t. 258, 1964, p. 4896-4899.

(Manuscrit reçu le 14 octobre 1964.)

Nicholas Th. VAROPOULOS,
Trinity College, Cambridge, G.-B.
et Dép. des Math. de la Faculté des Sciences
d'Orsay, Seine-et-Oise.