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ON LIMITS TO THE ABSOLUTE VALUES OF THE ROOTS OF A POLYNOMIAL (1);

By Edward B. Van Vleck.

In a recent and very interesting article (2) *Montel* has shown that when in the equation

$$(1) \qquad 1 + a_1 x + a_2 x^2 + \ldots + a_p x^p + \ldots + a_n x^n = 0,$$

the values of the p consecutive coefficients $a_1, a_2, \ldots, a_p$ are given with $a_p \neq 0$, there exists an upper limit to the moduli of the p roots of smallest absolute value which is dependent only upon the values of the p given coefficients and upon the number of terms in the equation subsequent to $a_p x^p$ (i.e., the number of non-zero coefficients after a_p regardless of the degree). Denote

(1) Presented to the American Mathematical Society, 29 dec. 1923.

(2) *Annales Sc. de l'École Normale supérieure*, (3), vol. 40, 1923, p. 1. Only a part of Montel's results will be cited.

this number by k and the familiar number $\frac{n(n-1)\ldots(n-i+1)}{i!}$ by C_n^i. When a single coefficient a_p is given, the modulus of the root of smallest absolute value does not exceed $\sqrt[p]{\frac{C_{p+k}^p}{|a_p|}}$, and this upper limit can be attained by a properly constructed polynomial of degree $n = p + k$. When $p = 2$ and equation (1) has the form

$$(2) \qquad 1 + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n = 0,$$

the moduli of the two roots of smallest absolute value will not exceed $\sqrt{\frac{C_{2+k}^2}{|a_2|}}$, and this upper limit is realized only in a properly constructed polynomial of degree $n = 2 + k$. Montel conjectures that when a_p is given in the equation

$$(3) \qquad 1 + a_p x^p + a_{p+1} x^{p+1} + \ldots + a_n x^n = 0,$$

the corresponding upper limit to the moduli of the p roots of smallest absolute value is

$$(4) \qquad \sqrt[p]{\frac{C_{p+k}^p}{|a_p|}}.$$

To establish his theorems on the moduli of the roots Montel employs the method of mathematical induction in combination with a quasi-converse of a well-known theorem of *Lucas* concerning the roots of the derivative of a given polynomial ([1]). This combination is admirably adapted to demonstrate the existence of an upper limit for the moduli of the p roots dependent upon p and k, and the particular strength of his theorem and method is in taking account of the gaps in the equation subsequent to the last given coefficient a_p, thus making the upper limit dependent on k rather than upon the degree of the equation. On the other hand, the method is apparently not so well adapted to the actual determination of this upper limit except in the special cases treated by him.

In the following investigation the subject is approached by the consideration of symmetric functions of the roots. This method is well adapted to the specific determination of the upper limit to the

([1]) Use is also made of a theorem of **Walsh**.

moduli of the p smallest roots when the degree of the equation is given. It is shown that when the coefficient $a_p \neq 0$ is known in (3), the moduli of the p roots of smallest absolute value have $\sqrt[p]{\frac{C_n^p}{|a_p|}}$ as an upper limit, and this upper limit can be attained in a properly constructed polynomial, tho by only one of these roots. Thus the correctness of Montel's conjecture is established when $p + k = n$. The value of this upper limit is lowered when there are gaps in the equation subsequent to $a_p x^p$ so that $p + k < n$, and the amount by which it is lowered depends upon the position of the gaps. It is not easily shown by my method that the upper limit must be at least as small as (4), though I have no doubt of the correctness of Montel's conjecture.

Montel's attention was confined to the case in which the p coefficients given in (1) form the continuous suite $a_1, a_2, \ldots, a_p$. One may ask whether there are not other cases in which a finite upper limit exists for the moduli of the p roots of smallest absolute value when p coefficients a_i are given. This question is here considered and it is found, more generally, that such a limit exists when the suite $a_1, a_2, \ldots, a_{p-1}$ is given with *any* subsequent coefficient $a_{p+m} \neq 0$. Further, if $a_1, a_2, \ldots, a_{p-1}$ are all zero so that the equation has the form (3), an upper limit to the moduli of the p smallest roots is

$$\sqrt[p+m]{\frac{C_{m+p-1}^{p-1} \cdot C_n^{m+p}}{|a_{p+m}|}},$$

and this upper limit is realized in a properly constructed polynomial, tho by only one of the p roots. In part II it is shown that in no other case does a finite upper limit exist for the p smallest roots when p coefficients a_i are given.

I.

For the investigation below it is found convenient to replace x by $-\frac{1}{z}$. Then instead of seeking an upper limit U to the moduli of the p roots x_i of (1) which have the smallest absolute value, we must find a lower limit $L = \frac{1}{U}$ for the moduli of the corresponding roots $z_i = -\frac{1}{x_i}$ of

$$(5) \qquad z^n - a_1 z^{n-1} + a_2 z^{n-2} - \ldots + (-1)^p a_p z^{n-p} + \ldots \pm a_n = 0.$$

The results obtained below for the roots of this equation can be reformulated at once by the reader into corresponding results for the equation (1). We will suppose the subscripts to be so assigned that

$$|z_1| \geqq |z_2| \geqq \ldots \geqq |z_n|,$$

and for brevity we will call $z_1, \ldots, z_p$ the *p largest roots* of (5).

Suppose first a single coefficient $a_p \neq 0$ to be given in (5). Since $a_p = \Sigma z_1 z_2 \ldots z_p$, we have immediately

$$(6) \qquad |a_p| \equiv |\Sigma z_1 z_2 \ldots z_p| \leqq \Sigma |z_1 z_2 \ldots z_p| \leqq C_n^p . |z_1|^p,$$

and therefore a lower limit to the largest root of (5) is $\sqrt[p]{\frac{|a_p|}{C_n^p}}$. To attain this limit it is necessary and sufficient that the equality signs shall hold in (6). Hence all the terms of $\Sigma z_1 z_2 \ldots z_p$ must have the same argument and be equal in absolute value to z_1^p. Consequently we must take $z_1 = z_2 = \ldots = z_n$, unless $p = n$ when it suffices to have $|z_1| = |z_2| = \ldots = |z_n|$. When several coefficients $a_p \neq 0$ are given, the lower limit to the largest root of (5) is at least as great as the largest of the corresponding values $\sqrt[p]{\frac{|a_p|}{C_n^p}}$.

The lower limit just indicated for $|z_1|$ can be likewise raised when in addition to $a_p \neq 0$ we have given other coefficients $a_s = 0$. For then if we raise $a_p = \Sigma z_1 z_2 \ldots z_p$ to an appropriate i-th power, thereby obtaining a new symmetric function with $(C_n^p)^i$ terms, this number of terms can be reduced by the cancellation of one or more groups of terms owing to the given relations $a_s = \Sigma z_1 z_2 \ldots z_s = 0$. Let N be the number of remaining terms. Then $|z_1| \geqq \frac{\sqrt[p]{|a_p|}}{\sqrt[pi]{N}}$.

This method will be illustrated by considering the simple case

$$z^n - a_1 z^{n-1} + (-1)^{r+1} a_{r+1} z^{n-r-1} + \ldots \pm a_n = 0,$$

in which only a_1 is supposed to be given. We obtain

$$a_1^r \equiv (\Sigma z_1)^r = \Sigma z_1^r$$

since $a_2 = a_3 = \ldots = a_r = 0$. This result can be more rapidly derived from the familiar recurrent relation

$$s_k - a_1 s_{k-1} + a_2 s_{k-2} - \ldots + (-1)^k k a_k = 0 \qquad (k \leqq n),$$

where s_k denotes the sum of the k-th powers of the roots of (5). For the equation before us this gives

$$s_k - a_1 s_{k-1} = 0 \qquad (k \leqq r),$$

and consequently $s_r = a_1^r$. Hence we have

$$|a_1^r| \leqq \Sigma |z_1|^r \leqq n |z_1|^r,$$

and therefore $\frac{|a_1|}{\sqrt[r]{n}}$ is a lower limit for $|z_1|$. This is larger than the lower limit $\frac{|a_1|}{(n-r+1)}$ given by application of Montel's results, since for $n > r$

$$n \equiv (n-r) + r < [(n-r)+1]^r \qquad (r > 1).$$

In the case of the trinomial equation

$$z^n + a_r z^{n-r} + a_n = 0,$$

with given a_r it is extremely easy to specify a lower limit for the modulus of the largest root which is independent of the degree of the equation. Since $|a_n| \leqq |z_1^n|$, the equation

$$-a_r = z_1^r + \frac{a_n}{z_1^{n-r}}$$

gives immediately $|a_r| \leqq 2 |z_1|^r$. Consequently $\sqrt[r]{\frac{|a_r|}{2}}$ is a lower limit to the modulus of the largest root. Furthermore, this is the largest possible lower limit independent of the degree, inasmuch as this limit is attained in the case of the equation

$$z^{2r} + a_r z^r + \left(\frac{a_r}{2}\right)^2 = 0.$$

Pass next to the consideration of (5) when the p coefficients $a_1, a_2, \ldots, a_p$ are given with $a_p \neq 0$. Between the p equations $a_i = \Sigma z_1 z_2 \ldots z_i$ $(i = 1, 2, \ldots, p)$ we can eliminate $p-1$ roots of (5). Let the roots to be eliminated be called $\beta_1, \ldots, \beta_{p-1}$

and the remaining roots $\gamma_1, \ldots, \gamma_{n-p+1}$. Denote the sum of the products of the β_i taken i at a time by b_i and the corresponding sum for the γ_i by g_i. We have then

$$(7)\quad \left\{\begin{aligned} a_1 &= g_1 + b_1, \\ a_2 &= g_2 + b_1 g_1 + b_2, \\ a_3 &= g_3 + b_1 g_2 + b_2 g_1 + b_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ a_{p-1} &= g_{p-1} + b_1 g_{p-2} + \ldots + b_{p-1}, \\ a_p &= g_p \quad + b_1 g_{p-1} + \ldots + b_{p-1} g_1, \end{aligned}\right.$$

where $g_i \equiv 0$ when i exceeds $n - p + 1$.

Elimination of the b_i from these equations gives

$$(8)\quad \begin{vmatrix} -a_1 + g_1 & 1 & 0 & \ldots & 0 & 0 \\ -a_2 + g_2 & g_1 & 1 & \ldots & 0 & 0 \\ -a_3 = g_3 & g_2 & g_1 & \ldots & 0 & 0 \\ \ldots\ldots\ldots & \ldots & \ldots & \ldots & . & . \\ -a_{p-1} + g_{p-1} & g_{p-2} & g_{p-3} & \ldots & g_1 & 1 \\ -a_p + g_p & g_{p-1} & g_{p-2} & \ldots & g_2 & g_1 \end{vmatrix} = 0,$$

which forms the basis for our subsequent conclusions.

Put

$$(9)\quad \Delta_p = \begin{vmatrix} g_1 & 1 & 0 & \ldots & 0 & 0 \\ g_2 & g_1 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & . & \ldots & . & . \\ g_{p-1} & g_{p-2} & g_{p-3} & \ldots & g_1 & 1 \\ g_p & g_{p-1} & g_{p-2} & \ldots & g_2 & g_1 \end{vmatrix}.$$

Orthosymmetric determinants of this particular form occur occasionally in mathematical literature [1]. We have the obvious law of recurrence,

$$(10)\quad \Delta_p = g_1 \Delta_{p-1} - g_2 \Delta_{p-2} + g_3 \Delta_{p-3} - \ldots \pm g_{p-1} \Delta_1 \mp g_p.$$

In the particular case before us the g_i are the elementary symmetric functions formed from certain elements γ_i, and accordingly

$$\Delta_1 = g_1 = \Sigma\gamma_1, \qquad \Delta_2 = g_1^2 - g_2 = \Sigma\gamma_1^2 + \Sigma\gamma_1\gamma_2.$$

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(1) Cf. *Pascal's Determinants*, § 41.

Starting with these expressions, we will now establish by mathematical induction the following result :

LEMMA I. — *If the g_i in (9) are the elementary symmetric functions $\Sigma\gamma_1\gamma_2\ldots\gamma_i$ formed from any number r of elements taken i at a time (with $g_i \equiv 0$ for $i > r$), the determinant Δ_p is the sum of all possible products of the γ_i taken p at a time, repetition of γ_i being allowed in the formation of the products.*

Suppose that this is true of Δ_i up to the value $i = p - 1$ inclusive. In the first term $g_1 \Delta_{p-1}$ on the right-hand side of (10) there occur all possible products of the γ_i taken p at a time, repetition of the γ_i being permissible in the products. Consider any such product containing exactly m distinct elements γ_i. In the first term on the right-hand side of (10) the product occurs C_m^1 times, in the second term C_m^2 times, and so on until we reach the m-th term, after which it does not occur at all. The coefficient with which the product enters into Δ_p is therefore

$$C_m^1 - C_m^2 + C_m^3 - \ldots + (-1)^{m-1} C_m^m = 1.$$

It follows that Δ_p has the structure indicated in the *Lemma*.

Let the greatest of the absolue values $|\gamma_i|$ be denoted by $|\gamma|$. Since the number of combinations of r elements taken p at a time with repetition is C_{r+p-1}^p and since no term in Δ_p exceeds γ^p in absolute value, we obtain from the *Lemma* the useful inequality

$$(11) \qquad |\Delta_p| \leqq C_{r+p-1}^p \cdot |\gamma|^p.$$

Consider now the special case in which $a_1 = a_2 = \ldots = a_p = 0$. Suppose that the n roots of (5) have been divided in any way whatsoever into two classes, γ_i and β_i respectively, with the sole restriction that the number of the β_i shall be at least as great as p. The last equation of (7) must now be modified by adding b_p to its right-hand member. Then if $b_1, \ldots, b_{p-1}$ in (7) are eliminated as before, the resulting eliminant is the same as (8) except that a_p is there to be replaced by $a_p - b_p$. Since also $a_i = 0$ for $i \leqq p$, our equation (8) after this replacement may be written in the form

$$\Delta_p + (-1)^{p-1} b_p = 0.$$

Thus it appears that Δ_p, which explicitly contains only the g_i and hence the γ_i, can also be expressed in terms of the β_1 and is, except for the factor $(-1)^p$, identical with the elementary symmetric function $b_p = \Sigma\beta_1\beta_2\ldots\beta_p$. By combination of this result with *Lemma I* we reach immediately the following conclusion:

LEMMA II. — *When $a_i = 0$ $(i = 1, 2, \ldots, p)$, the sum b_i of the products of p or more roots β_i taken i at a time without repetition is for even values of $i \leqq p$ equal to Δ_i which is the sum of the products of the remaining roots taken i at a time with repetition, while for odd values of $i \leqq p$ it is equal to the negative of this sum.*

We are now ready to consider the special equation

$$(12)\qquad z^n + (-1)^p a_p z^{n-p} + (-1)^{p+1} a_{p+1} z^{n-p-1} + \ldots + (-1)^n a_n = 0,$$

which corresponds to Montel's equation (3). We suppose only a_p to be given. Let us choose the $p-1$ largest roots $z_1, z_2, \ldots, z_{p-1}$, as the β-roots to be eliminated through (7). Since

$$a_1 = a_2 = \ldots = a_{p-1} = 0,$$

our eliminantal equation (8) becomes

$$(13)\qquad (-1)^p a_p + \Delta_p = 0.$$

Now z_p is the largest of the roots remaining which enters into Δ_p. Putting $|\gamma| = |z_p|$, we find from (11) and (13) that

$$(14)\qquad |a_p| \leqq C_n^p |z_p|^p.$$

Thus it is established that the moduli of the p largest roots of (12) can not fall below $\sqrt[p]{\frac{|a_p|}{C_n^p}}$.

We proceed next to show that the lower limit just indicated for the modulus is the largest possible lower limit for the set of the p largest roots of (12). To prove this we must establish that our arbitrary coefficients $a_{p+1}, a_{p+2}, \ldots, a_n$ in (12) can be so chosen that the sign of equality will hold in (14). We first put aside the case $p = n$ as trivial, since equation (12) the becomes

$$z^n + (-1)^n a_n = 0$$

and all its roots have the modulus $\sqrt[n]{|a_n|}$, as demanded.

Suppose $p < n$. The sign of equality in (14) will hold when, and only when, the C_n^p terms of which Δ_p consists have all the same argument and a common modulus equal to $|z_p|^p$. Hence we must have $z_p = z_{p+1} = \ldots = z_n$, and by (13) their common value will be a p-th root of $\frac{(-1)^{p-1} a_p}{C_n^p}$. Except for the choice of this p-th root the determination of these $n - p + 1$ roots is unique, and correspondingly the determination of their elementary symmetric functions g_i in (7). Using the given value of a_p and setting

$$a_1 = a_2 = \ldots = a_{p-1} = 0,$$

we may now regard (7) as a system of equations to determine the $p - 1$ unknowns b_i. The first $p - 1$ equations of the system determine the b_i uniquely, while their consistency with the last equation of the system is guaranteed by (13). As the b_i are the elementary symmetric functions of the remaining $p - 1$ roots of (12) taken i at a time, these roots are accordingly uniquely determined.

It has thus been shown that when a_p is given, it is possible to take the roots of (12) — and, except for the choice of the above mentioned p-th root, in one way only — so that the sign of equality will hold in (14). Any set of p roots of (12) will include at least one of the $n - p + 1$ equal roots which have a modulus equal to $\sqrt[p]{\frac{|a_p|}{C_n^p}}$. Now it was proved earlier that the moduli of the p largest roots of (12) must be at least as great as this quantity. Consequently when only a_p is given, this is the greatest possible lower limit for the moduli of the set of the p largest roots of (12).

It remains to examine whether in the determination just made the values obtained for $z_1, z_2, \ldots, z_{p-1}$ through (7) are really as great in modulus as the $n - p + 1$ equal roots $z_i (i \geqq p)$. Denote by z' any one of the former set of roots, and suppose, if possible, that it has a modulus less than that of z_p. Let z' be exchanged with z_p in the preceding work so that z' enters into Δ_p in place of z_p. Thereby some of the terms of Δ_p will be lessened in absolute value. Since before the exchange all of its terms were equal to one another and their sum by (13) was equal to $\pm a_p$, it follows that after the exchange $|\Delta_p|$ will be less than $|a_p|$. This contradicts (13), and hence we conclude that $|z'|$ can not be less than $|z_p|$. The same contradiction arises if we suppose z' to be equal to z_p in absolute value but to differ from it in argument. For then on exchanging z' and z_p the terms of Δ_p, though equal in modulus, are no longer all equal in argument so that again we have $|\Delta_p| < |a_p|$.

We may, finally, remove the possibility that z' should be equal to z_p. For this purpose consider the equation

$$z^{p-1} - b_1 z^{p-2} + b_2 z^{p-3} - \ldots + (-1)^{p-1} b_{p-1} = 0,$$

which is satisfied by the $p-1$ largest roots of (12). Since

$$a_1 = a_2 = \ldots = a_{p-1} = 0,$$

we have $(-1)^i b_i = \Delta_i$ by *Lemma II*. But Δ_i is the sum of the products of the $n-p+1$ equal roots $z_i (i \geqq p)$ taken i at a time with repetition, and is therefore equal to $C'_{n-p+i} \cdot z_p^i$. Consequently the above equation becomes

$$(15) \quad z^{p-1} + C^1_{n-p+1} z_p z^{p-2} + C^2_{n-p+2} z_p^2 z^{p-3} + \ldots + C^{p-1}_{n-1} z_p^{p-1} = 0.$$

It is obviously impossible to satisfy this equation by taking $z = z_p$, as was to be shown.

The theorems reached in the last few paragraphs can be summed up as follows :

THEOREM I. — *When* $|a_p|$ *is given* in (12), *the quantity* $\sqrt[p]{\frac{|a_p|}{C_n^p}}$ *is a lower limit for the moduli of the p largest roots* $z_i (i \leqq p)$ *of* (12). *If* $p < n$, *this lower limit is reached by* z_p *when and only when* $z_p = z_{p+1} = \ldots = z_n$, *their common value being a p-th root of* $\frac{(-1)^{p-1} a_p}{C_n^p}$. *The remaining* $p-1$ *roots are of greater absolute value and satisfy equation* (15). *In the trivial case* $p = n$ *the* n *roots of* (12) *are the various* n-th *roots of* $(-1)^n a_n$.

Consider next the general equation (5) in which we will suppose $a_1, a_2, \ldots, a_p$ to be given with $a_p \neq 0$. The eliminant (8) may be written

$$(-1)^p a_p + (-1)^{p-1} a_{p-1} \Delta_1 + (-1)^{p-2} a_{p-2} \Delta_2 + \ldots - a_1 \Delta_{p-1} + \Delta_p = 0,$$

and accordingly, with the help of (11),

$$(16) \qquad \begin{aligned} |a_p| &\leqq |\Delta_p| + |a_1 \Delta_{p-1}| + \ldots + |a_{p-1} \Delta_1| \\ &\leqq C_n^p . |z_p|^p + C_{n-1}^{p-1} . |a_1| . |z_p|^{p-1} \\ &\quad + C_{n-2}^{p-2} |a_2| . |z_p|^{p-2} + \ldots + C_{n-p+1}^1 . |a_{p-1}| . |z_p| . \end{aligned}$$

This inequality is clearly impossible if $|z_p|$ is taken too small. We thus reach the conclusion :

THEOREM II. — *When $a_1, a_2, \ldots a_p$ are given with $|a_p| \neq 0$, there is a lower limit for the moduli of the p greatest roots of (5) which is at least as great as the smallest value of $|z_p|$ which satisfies the inequality* (16).

A similar treatment is possible when $a_1, a_2, \ldots, a_{p-1}$ are given with any subsequent coefficient a_{p+m} instead of a_p. Then in place of the last equation of (7) we must employ the equation

$$a_{p+m} = g_{p+m} + b_1 g_{p+m-1} + b_2 g_{p+m-2} + \ldots + b_{p-1} g_{m+1}.$$

Elimination of the $p - 1$ greatest roots now gives

$$(17) \qquad \begin{vmatrix} -a_1 + g_1 & 1 & 0 & \ldots & \ldots & 0 \\ -a_2 + g_2 & g_1 & 1 & \ldots & \ldots & 0 \\ \ldots\ldots\ldots & \ldots & \cdot & \ldots & \ldots & \cdot \\ -a_{p-1} + g_{p-1} & g_{p-2} & g_{p-3} & \ldots & \ldots & 1 \\ -a_{p+m} + g_{p+m} & g_{p+m-1} & g_{p+m-2} & \ldots & \ldots & g_{m+1} \end{vmatrix} = 0.$$

Expand in terms of the elements of the last row. The cofactor multiplying $-a_{p+m}$ is ± 1 while every other term contains one or more of the g_i as factors. It is therefore impossible to assign an arbitrarily small upper limit to the $|g_i|$. Now the g_i for $i \leqq n - p + 1$ are the elementary symmetric functions of the $n - p + 1$ roots

$$z_p, z_{p+1}, \ldots, z_n$$

taken i at a time. Consequently $|z_p|$, the greatest of the moduli of these roots, must have a lower limit greater than zero. Hence we conclude :

THEOREM III. — *When a_1, a_2, ..., a_{p-1}, a_{p+m} are given with $a_{p+m} \neq 0$, the moduli of the p largest roots of* (5) *have a lower limit greater than zero which depends only on the given coefficients and the degree n.*

A special case of interest is that in which all the p given coefficients are zero except a_{p+m}. Equation (5) has then the form

$$(18) \qquad \begin{aligned} &z^n + (-1)^p a_p z^{n-p} + (-1)^{p+1} a_{p+1} z^{n-p-1} \\ &+ \ldots + (-1)^{p+m} a_{p+m} z^{n-p-m} + \ldots \pm a_n = 0, \end{aligned}$$

where only a_{p+m} is given. Our eliminantal equation (17) now becomes

$$(19) \qquad (-1)^p a_{p+m} + \Delta_{p,m} = 0,$$

in which

$$(20) \qquad \Delta_{p,m} = \begin{vmatrix} g_1 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ g_2 & g_1 & 1 & \ldots & \ldots & 0 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot & \ldots & \ldots & \cdot & \cdot \\ g_{p-1} & g_{p-2} & g_{p-3} & \ldots & \ldots & g_1 & 1 \\ g_{p+m} & g_{p+m-1} & g_{p+m-2} & \ldots & \ldots & g_{m+2} & g_{m+1} \end{vmatrix}.$$

The expansion of (20) in terms of the elements of its last row gives

$$(21) \qquad \Delta_{p,m} = g_{m+1}\Delta_{p-1} - g_{m+2}\Delta_{p-2} + \ldots \pm g_{p+m},$$

where $g_i \equiv 0$ for $i > n - p + 1$. For convenience, designate by $\gamma_i (i = 1, 2, \ldots, n - p + 1)$, the roots $z_p, z_{p+1}, \ldots, z_n$ which enter into (20). Since g_i is the sum of their products taken i at a time without repetition while Δ is the sum of such products with repetition, $\Delta_{p,m}$ is a homogeneous function of the γ_i of degree $p + m$. By (21) each of its terms must contain at least $m + 1$ of the γ_i. Take any possible product $\gamma_1^{i_1}\gamma_2^{i_2}\ldots\gamma_k^{i_k}$, in which $k = m + j$ and $i_1 + i_2 + \ldots + i_k = p + m$. Here j is subject to the two conditions $j \leqq p$, $m + j \leqq n - p + 1$. The largest integral value of j satisfying both conditions will be denoted by q. Seek now the coefficient with which this product enters into the right-hand side of (21). The product occurs in $g_{m+i}\,\Delta_{p-i}$ only for $i \leqq j$, and then with the coefficient C_{m+j}^{m+i} since this is the number of terms in g_{m+i} which are factors of the product considerede. Hence the product enters into the right-hand side of (21) with the coefficient

$$C_{m+j}^{m+1} - C_{m+j}^{m+2} + \ldots + (-1)^{j-1} C_{m+j}^{m+j},$$

which can be condensed into the single term $C_{m+j-1}^{m} = C_{m+j-1}^{j-1}$ with the aid of the formula

$$C_{m+j}^{m+i} = C_{m+j-1}^{m+i-1} + C_{m+j-1}^{m+i} \qquad (i = 1, \ldots, j-1).$$

Our equation (21) may therefore be written,

$$(22) \qquad \Delta_{p,m} = \sum_{j=1}^{q} C_{m+j-1}^{j-1} \sum_{i_1+i_2+\ldots+i_k=m+p} [\Sigma \gamma_1^{i_1} \gamma_2^{i_2} \ldots \gamma_{k=m+j}^{i_k}],$$

where the triple summation is to be understood as follows. In the first summation we keep the exponents fixed but select the $m+j$ roots γ from $z_p, z_{p+1}, \ldots, z_n$ in all possible ways, and in the second summation we allow the exponents to take all possible sets of positive integral values consistent with the sum $p+m$.

We will next ask how many terms $\Delta_{p,m}$ contains. By the first summation we get a total of C_{n-p+1}^{m+j} terms. In consequence of the second summation this total is multiplied by C_{m+p-1}^{p-j}, for we then assign $p+m$ indistinguishable units as exponents to $k = m+j$ roots γ_i in all possible ways with at least one unit to each root. After the assignment of one unit to each root there are left $p-j$ units, for which we must select $p-j$ of $m+j$ roots in all possible ways with repetition allowed, and the number of ways in which this can be done is C_{m+p-1}^{p-j}. Finally, if each term in the triple summation is counted a number of times equal to its coefficient, we obtain as the total number of terms in (22)

$$(23) \qquad \sum_{j=1}^{q} C_{m+j-1}^{j-1} \, C_{m+p-1}^{p-j} \, C_{n-p+1}^{m+j}.$$

Since

$$C_{m+j-1}^{j-1} \, C_{m+p-1}^{p-j} = \frac{(p-1)(p-2)\ldots(p-j+1)}{(j-1)!} C_{m+p-1}^{p-1},$$

this may be written

$$C^{p-1}_{m+p-1}\left[C^{m+1}_{n-p+1}+(p-1)\,C^{m+2}_{n-p+1}+\frac{(p-1)(p-2)}{2!}\,C^{m+3}_{n-p+1}+\ldots\right.$$
$$\left.+\,C^{q-1}_{p-1}\;C^{m+q}_{n-p+1}\right]=C^{p-1}_{m+p-1}\;C^{m+p}_{n}.$$

Return now to equation (19). We have just shown that $\Delta_{m,p}$ may be regarded as consisting of $C^{p-1}_{m+p-1}\;C^{m+p}_{n}$ terms with coefficient $+1$, each term being of degree $m+p$ in terms of the $n-p+1$ roots $z_i(i\geqq p)$. Since none of these roots exceeds z_p in absolute value, equation (19) furnishes the inequality

$$|a_{p+m}|=|\Delta_{m,p}|\leqq C^{p-1}_{m+p-1}\;C^{m+p}_{n}\;|z_p|^{m+p}.$$

Thus we arrive at the following result :

THEOREM IV. — *When a_{p+m} is given in* (18), *the p roots of greatest absolute value have the lower limit*

$$\sqrt[p+m]{\frac{|a_{p+m}|}{C^{p-1}_{m+p-1}\;C^{m+p}_{n}}}$$

for their moduli.

By reasoning like that used for equation (12) when a_p was given, it is clear that the lower limit can be attained only by taking $z_p=z_{p+1}=\ldots=z_n$. The first $p-1$ equations of (7) may again be used to determine the remaining roots $z_1, \ldots, z_{p-1}$ which again satisfy (15). The same considerations as before apply to prove that the moduli of the latter set of roots are then actually greater than that of z_p.

At the end of part II it is shown that not more than p roots are conditioned to have a lower limit greater than zero for their moduli in the case before us.

II.

In conclusion we will show that *there are no cases other than those included under Theorem III in which a lower limit greater than zero for the moduli is imposed upon p roots by giving p coefficients a_i.* In any other case there will be given a suite of only $p - m$ consecutive coefficients $a_1, a_2, \ldots, a_{p-m} (2 \leqq m \leqq p)$ with m subsequent coefficients. The two given coefficients of greatest subscript will be denoted by a_{p-1+k}, $a_{p-1+l} (0 < k < l)$. The desired conclusion will be established by proving that $n - p + 1$ roots of (5) can be taken as small as we please in absolute value.

As before, we will divide the roots of (5) into two classes, the one class containing the $p - 1$ largest roots $z_i (i < p)$ which have the b_i for their elementary symmetric functions, and the other class containing the remaining $n - p + 1$ roots with the g_i for their elementary symmetric functions. We will again eliminate the former set of roots. The first $p - m$ equations of (7) hold, but in place of the last m equations of system (7) we now have m equations of the form

$$(24) \qquad a_{p-1+i} = g_{p-1+i} + b_1 g_{p-2+i} + \ldots + b_{p-1} g_i.$$

A necessary condition for the consistency of the system is that the eliminant Δ resulting from the elimination of the b_i shall be zero. For convenience of reference we shall write down the eliminantal equation for $m = 3$, which is

$$(25) \qquad \Delta \equiv \begin{vmatrix} -a_1 + g_1 & 1 & \ldots & \ldots & 0 \\ -a_2 + g_2 & g_1 & \ldots & \ldots & 0 \\ \ldots\ldots\ldots & \ldots & \ldots & \ldots & 0 \\ -a_{p-3} + g_{p-3} & g_{p-4} & \ldots & \ldots & 0 \\ -a_{p-1+j} + g_{p-1+j} & g_{p-2+j} & \ldots & \ldots & g_j \\ -a_{p-1+k} + g_{p-1+k} & g_{p-2+k} & \ldots & \ldots & g_k \\ -a_{p-1+l} + g_{p-1+l} & g_{p-2+l} & \ldots & \ldots & g_l \end{vmatrix} = 0.$$

It is to be noted, however, that the subsequent argument will hold for every value of $m > 1$. In any case the system will be consistent and admit a unique solution if the first minor M_1 of Δ obtained by omitting its first column and last row is not zero, or just as well if any other first minor is not zero which is taken from the matrix $\| M \|$ remaining after the omission of the first colum of Δ.

A simplification of the problem may be made by equating to zero all the coefficients in (5) after the last given coefficient a_{p-1+l}, or, in other words, by taking $n - (p - 1 + l)$ roots equal to zero. Then after the removal of the factor $x^{n-(p-1+l)}$ there is left an equation of degree $p - 1 + l$ with p given coefficients, for which we must prove that l roots can be taken as small as we please in absolute value. Since only l roots now enter into the g_i, every g_i is identically zero for $i > l$. In consequence, (25) takes the form

$$(26) \qquad \Delta \equiv N g_l + (-1)^p a_{p-1+l} M_1 = 0,$$

in which N denotes that minor of Δ which is obtained by deleting its last row and column.

It will suffice to show that the $g_i (i \leqq l)$ can be made as small as we choose in absolute value, for then the same is true of the l roots which enter therein. The method of proof will be based on the form of Δ. As we pass from left to right along any row, the subscript steadily diminishes by a unit, all elements to the right of $g_0 \equiv 1$ being zero. In passing down the principal diagonal or any parallel file the subscript never diminishes, and the same holds for any minor taken from r consecutive columns. Whatever be the value of $m > 1$, no element in the principal diagonal of Δ is identically zero nor in the parallel file just above, and the last two rows are the same as the last two of (25) with now $g_i \equiv 0$ for $i > l$. On these simple facts the proof is built.

It will be shown first that *any minor of Δ taken from r consecutive columns (inclusive of Δ itself) will not vanish identically if the product of the elements in its principal diagonal is not zero.* To see this, begin with the top row of the minor. Its first element has a subscript greater than that of any other element of the same row. In case it is an element $a_i - g_i$ from the first column of Δ, we will use only the g_i. In the next row of the minor the element with greatest subscript which can be used as its multiplier is the element in its principal diagonal. In the third row the element with greatest subscript which can be used to multiply the product of the two elements already selected lies also in the principal diagonal; and so on. Consequently, if there is no zero element in the principal diagonal, the product of all these elements will be unique among the products which make up the minor, and hence the minor cannot then vanish identically. It may be added, incidentally, that if the first element is an element $a_i - g_i$ with $a_i \neq 0$, we will obtain two unique terms.

By direct application of the result just established it follows that neither M_1 nor N vanishes identically. The same is true of the second minor M_2 of Δ obtained by suppressing the last row and column of M_1, or any r-th minor M_r obtained by suppressing the last $r-1$ rows and columns of M_1.

We are now ready for the consideration of our equation (26). If $a_{p-1+l}=0$, it may be satisfied by merely taking $g_l=0$. This does not cause M_1 to vanish since g_l is not contained among the elements of its principal diagonal. Then the other g_i with $i<l$ can be chosen as small as we please in absolute value but so as not to make $M_1=0$. All conditions desired are then fulfilled, and hence $n-p+1$ roots of (5) can be taken as small as we please in absolute value.

We may suppose henceforth $a_{p-1+l}\neq 0$. Let M_1 be then expanded in terms of the elements of its last row and their cofactors. Equation (26) thereby takes the form

$$(27)\qquad N g_l+(-1)^p a_{p-1+l}(g_k M_2+g_{k+1}M'_2+\ldots)=0.$$

In appearance the form is homogeneous in $g_k, g_{k+1}, \ldots, g_l$, but it is to be born in mind that these quantities are contained in N, M_2, M'_2, ... We will now regard (27) as an equation to determine g_k when the remaining $g_i (i\leqq l)$ are given. It has already been pointed out that M_2 does not vanish identically, and this still holds true if all elements g_i with subscript greater than k are equated to zero, inasmuch as all elements in the principal diagonal of M_2 have a subscript k or less. Then M_2 becomes a polynomial in some or all of the quantities $g_1, g_2, \ldots, g_k$. We will choose for $g_1, g_2, \ldots, g_{k-1}$ a set of values which does not cause this polynomial to vanish identically. This will make M_2 either a constant or a polynomial in g_k. We will also suppose that the values just selected are less in absolute magnitude than an arbitrarily prescribed positive ε. These values of $g_1, \ldots, g_{k-1}$ we will now employ in (27) and holding them fixed, we will let $g_{k+1}, g_{k+2}, \ldots, g_l$ approach zero in any manner. The left hand side which is a polynomial in g_k with varying coefficients will approach as its limit a polynomial with fixed coefficients; namely, the limit of $(-1)^p a_{p-1+l} g_k M_2$. Since the roots of a polynomial are continuous functions of its coefficients, there must be a root of the polynomial which is either zero or approaches zero as its limit, and this root we will take as the value of g_k. Thus all our $|g_i|$ may be made simultaneously as small as we please.

It only remains to make sure that we can thus take our $|g_i|$ arbitrarily small without causing to vanish all the first minors of Δ which can be formed from the matrix $\|M\|$. In showing this we will treat successively the two possibilities $l = k + 1$ and $l > k + 1$.

When $l = k + 1$, every g_i with $i > k + 1$ is identically zero. Consider then the first minor $g_l M_2$ of Δ which results from the omission of the next to the last row of $\|M\|$. To keep this and other first minors later under consideration different from zero, we will henceforth impose the condition that g_l shall be different from zero in its approach to zero. Since the last element in the principal diagonal of M_1 is g_k, the subscript of the last element in the principal diagonal of M_2 or of any other principal minor of M_1 must be k or less. Suppose first that the last element in the principal diagonal of M_2 has a subscript less than k. This renders it impossible for it to vanish identically for $g_l = g_k = 0$, and clearly we can impose the requirement that the fixed values given above to $g_1, g_2, \ldots, g_{k-1}$ are such that M_2 is then different from zero. Accordingly when in (27) we make $|g_l|$ sufficiently small and with it $|g_k|$ also, we obtain a first minor $g_l M_2 \neq 0$ as demanded, and all conditions desired are therefore met.

Suppose, on the other hand, that the last element in the principal diagonal of M_2 is g_k. Then the adjacent element to its left in the principal diagonal of Δ is g_{k+1}, so that the three last elements of the diagonal are $g_l = g_{k+1}$. In this case consider the first minor $g_{k+1}^2 M_3$ of Δ which is obtained by omitting the second row preceding the last in $\|M\|$. If the last element in the principal diagonal of M_3 has a subscript less than k, then M_3 does not vanish identically for $g_{k+1} = g_k = 0$, and we may suppose the fixed values already given to $g_1, g_2, \ldots, g_{k-1}$ to be subject to the restriction that the value of M_3 is not then zero. Accordingly, when $|g_{k+1}| \equiv |g_l|$ becomes sufficiently small and with it $|g_k|$ also, we get a first minor $g_{k+1}^2 M_3 \neq 0$, as desired. On the other hand, if the last element in the principal diagonal of M_3 is g_k, the adjacent element to its left is g_{k+1} and consequently the last four elements in the principal diagonal of Δ are g_{k+1}. In this case consider in similar fashion the first minor $g_{k+1}^3 M_4$ resulting from the omission of the third row preceding the last in $\|M\|$. Continuing thus, we come finally to the case in which all the elements after the first in the principal diagonal of Δ are g_{k+1}, and then the first minor g_{k+1}^{p-1} obtained by omitting the first row of $\|M\|$ is different from zero.

The case $l > k + 1$ can be handled in much the same manner. For simplification we will equate to zero all g_i with subscripts between l and $k + 1$. Then when g_{k+1} and g_l in (27) approach zero in any manner whatsoever, g_k will also approach zero. We will hereafter keep g_{k+1} as well as g_l different from zero in this approach.

Consider again the first minor $g_l M_2$. Suppose first that the last element in the principal diagonal of M_2 has a subscript less than k. It will not vanish identically for $g_l = g_{k+1} = g_k = 0$, and, as before, we will suppose the fixed $g_1, g_2, \ldots, g_{k-l}$ so chosen that its value is not then zero. If this has been done, the first minor $g_l M_2$ will be different from zero for sufficiently small $|g_{k+1}|$ and $|g_l|$.

Suppose next the last element in the principal diagonal of M_2 to be g_k, the adjacent element to its left being g_{k+1}. Then we again consider the first minor resulting from the omission of the second row preceding the last in $\|M\|$. Besides the term $g_l g_{k+1} M_3$ this may also contain another term $g_l^2 M_3'$ if the element g_l occurs in the next to last row of $\|M\|$. If the last element in the principal diagonal of M_3 has a subscript less than k, clearly M_3 will not vanish identically for $g_l = g_{k+1} = g_k = 0$, and we can impose upon the fixed values of $g_1, g_2, \ldots, g_{k-1}$ the further restriction that the value of M_3 shall not then be zero. Now let g_{k+1} and g_l approach zero, making g_l infinitesimal in comparison with g_{k+1}. The term $g_l^2 M_3'$ in the minor ultimately becomes negligible in comparison with $g_l g_{k+1} M_3 \neq 0$. Consequently we get a first minor of Δ which is different from zero.

On the other hand, suppose that the last element in the principal diagonal of M_3 is g_k. Then the last four elements in the principal diagonal of Δ are three g_{k+1} followed by g_l. In this case we consider again the first minor obtained by omitting in $\|M\|$ the third row preceding the last. In addition to $g_l g_{k+1}^2 M_4$ this minor may now contain other terms such as $g_l^3 M_4'$, $g_l^2 g_k M_4''$ due to the occurrence of g_l to the left of the principal diagonal in the second or second and third rows preceding the last in $\|M\|$, but each of these additional terms will certainly contain g_l^2 as a factor. If the last element of M_4 is not g_k, we may proceed in the same manner as before and by taking finally g_l infinitesimal in comparison with g_{k+1}^2 make $g_l g_{k+1}^2 M_4$ the dominant term of our first minor. Since this is different from zero for sufficiently small $|g_l|$, $|g_{k+1}|$, $|g_k|$, we get thus a first minor different from zero, as desired. When the last element in the principal diagonal of M_4 is g_k, the last five elements in the principal diagonal of Δ are four g_{k+1} followed by g_l. We then consider the minor obtained from $\|M\|$ by suppressing the fourth row from the last; and so on. With each succeeding stage we have a minor containing a term $g_l g_{k+1}^{i-2} M_i$ while every other term contains g_l^2 as a factor. Hence by taking g_l infinitesimal in comparison with g_{k+1}^{i-2} we can make the first mentioned term become the dominant one. We close finally with a first minor of Δ whose principal diagonal consists of $p-2$ elements g_{k+1} followed by the final element g_l. Since in each successive case the dominant term does not vanish for sufficiently small $|g_l|$, $|g_{k+1}|$, $|g_k|$ provided that $|g_l|$ is conditioned in the manner indicated, we conclude that we can always get a first minor different from zero. This completes the proof that not more than $p-1$ roots are conditioned to have a lower limit greater than zero for their moduli when the p given coefficients do not accord with Theorem III.

We may now supplement Theorem III. This stated that when $a_1, \ldots, a_{p-1}, a_{p+m}$ were given with $a_{p+m} \neq 0$, the set of the p largest roots were thereby conditioned to have a lower limit greater than zero for their moduli. It may now be shown that no more roots are thus conditioned. For if $m > 1$ and we arbitrarily assign a_{p+m-1}, we thereby bring the case under the investigation just made, from which it appears that no more than p roots will be conditioned to have a lower limit greater than zero for their moduli. The case $m = 1$ demands separate treatment. We may then equate to zero all coefficients of (5) subsequent to a_{p+1}, thus making all but $p+1$ roots zero. Let z' denote that one of the remaining roots which has the smallest absolute value. We have the relations

$$a_i = z' s_{i-1} + s_i \qquad (i = 1, 2, \ldots, p-1),$$
$$a_{p+1} = z' s_p,$$

where s_i denotes the sums of the products of the other p roots taken i at a time. The set of equations for the s_i are obviously consistent for any value of z' not zero, no matter how small its absolute value. Lastly, if $m = 0$, it is obvious that we may make all but p roots equal to zero by equating to zero all coefficients after a_p. Consequently, no more than p roots of our equation are conditioned in the manner above stated in the Theorem.