

quatrième série - tome 43 fascicule 2 mars-avril 2010

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

Kenichi BANNAI & Shinichi KOBAYASHI & Takeshi TSUJI

*On the de Rham and p -adic realizations of the
Elliptic Polylogarithm for CM elliptic curves*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON THE DE RHAM AND p -ADIC REALIZATIONS OF THE ELLIPTIC POLYLOGARITHM FOR CM ELLIPTIC CURVES

BY KENICHI BANNAI, SHINICHI KOBAYASHI AND TAKESHI TSUJI

ABSTRACT. – In this paper, we give an explicit description of the de Rham and p -adic polylogarithms for elliptic curves using the Kronecker theta function. In particular, consider an elliptic curve E defined over an imaginary quadratic field \mathbb{K} with complex multiplication by the full ring of integers $\mathcal{O}_{\mathbb{K}}$ of \mathbb{K} . Note that our condition implies that \mathbb{K} has class number one. Assume in addition that E has good reduction above a prime $p \geq 5$ unramified in $\mathcal{O}_{\mathbb{K}}$. In this case, we prove that the specializations of the p -adic elliptic polylogarithm to torsion points of E of order prime to p are related to p -adic Eisenstein-Kronecker numbers. Our result is valid even if E has supersingular reduction at p . This is a p -adic analogue in a special case of the result of Beilinson and Levin, expressing the Hodge realization of the elliptic polylogarithm in terms of Eisenstein-Kronecker-Lerch series. When p is ordinary, then we relate the p -adic Eisenstein-Kronecker numbers to special values of p -adic L -functions associated to certain Hecke characters of \mathbb{K} .

RÉSUMÉ. – Dans cet article, nous donnons une description explicite des réalisations de de Rham et p -adiques des polylogarithmes elliptiques en utilisant la fonction thêta de Kronecker. Considérons en particulier une courbe elliptique E définie sur un corps quadratique imaginaire \mathbb{K} , à multiplication complexe par l'anneau des entiers $\mathcal{O}_{\mathbb{K}}$ de \mathbb{K} , et ayant bonne réduction en chaque place au-dessus d'un nombre premier $p \geq 5$ non ramifié dans \mathbb{K} . On notera que le nombre de classe de \mathbb{K} est nécessairement égal à un. Nous montrons alors que les spécialisations des polylogarithmes p -adiques aux points de torsion de E d'ordre premier à p sont reliées aux nombres d'Eisenstein-Kronecker p -adiques. Ce résultat est valable même si E a une réduction supersingulière en p . C'est un analogue p -adique d'un cas spécial du résultat de Beilinson et Levin exprimant la réalisation de Hodge du polylogarithme elliptique en utilisant les séries d'Eisenstein-Kronecker-Lerch. Si p est quelconque, nous établissons un lien entre les nombres d'Eisenstein-Kronecker p -adiques et les valeurs spéciales des fonctions L associées aux caractères de Hecke de \mathbb{K} .

The first and second authors were supported by the JSPS Postdoctoral Fellowships for Research Abroad 2005-2007/2004-2006. This work was also supported in part by KAKENHI (18540015, 18740006, 19740010, 21674001).

0. Introduction

0.1. Introduction

In the paper [7], Beilinson and Levin constructed the elliptic polylogarithm, which is an element in absolute Hodge or ℓ -adic cohomology of an elliptic curve minus the identity. This construction is a generalization to the case of elliptic curves of the construction by Beilinson and Deligne of the polylogarithm sheaf on the projective line minus three points. The purpose of this paper is to study the p -adic realization of the elliptic polylogarithm for an elliptic curve with complex multiplication, even when the elliptic curve has supersingular reduction at the prime p .

To achieve our goal, we first describe the de Rham realization of the elliptic polylogarithm for a general elliptic curve defined over a subfield of \mathbb{C} . In particular, we explicitly describe the connection of the elliptic polylogarithm using rational functions. Similar results were obtained by Levin and Racinet [21] Section 5.1.3, and Besser and Solomon [28].

The de Rham realization of the elliptic polylogarithm gives the coherent module with connection underlying the polylogarithm sheaf in the Hodge and p -adic cases. We construct the p -adic realization of the elliptic polylogarithm as a filtered overconvergent F -isocrystal on the elliptic curve minus the identity, when the elliptic curve is defined over an imaginary quadratic field, has complex multiplication by its ring of full integers, and has good reduction over a fixed prime $p \geq 5$ unramified in the ring of complex multiplication. The Frobenius structure on the p -adic elliptic polylogarithm by definition is compatible with the connection of the underlying de Rham realization. Our main result, Theorem 4.19, is an explicit description of the Frobenius structure on the p -adic elliptic polylogarithm sheaf in terms of overconvergent functions characterized as the solutions of the p -adic differential equations arising from the compatibility of the Frobenius with the connection.

Using this description, we calculate the specializations of the p -adic elliptic polylogarithm to torsion points of order prime to p (more precisely, torsion points of order prime to \mathfrak{p}) and prove that the specializations give the p -adic Eisenstein-Kronecker numbers, which are special values of the p -adic distribution interpolating Eisenstein-Kronecker numbers (see Theorem 5.7). This result is a generalization of the result of [4], where we have dealt only with the one variable case for an ordinary prime. A similar result concerning the specialization in two-variables was obtained in [5], again for ordinary primes, using a very different method. The result of the current paper is valid even when p is supersingular.

When the elliptic curve has good ordinary reduction at the primes above p , the p -adic Eisenstein-Kronecker numbers are related to special values of p -adic L -functions which p -adically interpolate special values of certain Hecke L -functions associated to imaginary quadratic fields (see Proposition 2.27). Hence our main result in the ordinary case implies that the specialization of the p -adic elliptic polylogarithm to torsion points as above are related to special values of certain p -adic L -functions (Corollary 5.10). Recently, Solomon [28] has announced that the p -adic elliptic polylogarithm as constructed in this paper is the image by the syntomic regulator of the motivic elliptic polylogarithm. Assuming this fact, our result may be interpreted as a p -adic analogue of Beilinson's conjecture.

In the appendix, modeling on our approach of the p -adic case, we calculate the real Hodge realization of the elliptic polylogarithm by solving certain iterated differential equations as

in the p -adic case. The Hodge realization of the elliptic polylogarithm was first described by Beilinson-Levin [7] and Wildeshaus [32]. We give an alternative description of the real Hodge realization in terms of multi-valued meromorphic functions given as the solutions of these differential equations. Our method highlights the striking similarity between the classical and the p -adic cases.

0.2. Overview

The detailed content of this paper is as follows. In §1, we introduce the Kronecker theta function $\Theta(z, w)$, which is our main tool in describing the elliptic polylogarithm. A slightly modified version of this function was previously used by Levin [20] to describe the analytic aspect of the elliptic polylogarithm. We use this function to construct rational functions L_n on the elliptic curve, which we call the connection functions. The main result of the first section is the explicit description of the de Rham realization of the elliptic polylogarithm in terms of L_n (Corollary 1.42).

The main result of this paper is an explicit description of the p -adic elliptic polylogarithm for CM elliptic curves. Let \mathbb{K} be an imaginary quadratic field, and let E be an elliptic curve defined over \mathbb{K} with complex multiplication by the full ring of integers $\mathcal{O}_{\mathbb{K}}$ of \mathbb{K} . Note that by the theory of complex multiplication, the existence of E implies that the class number of \mathbb{K} is one. We assume in addition that E has good reduction above a prime $p \geq 5$ unramified in $\mathcal{O}_{\mathbb{K}}$. We denote by $\psi_{E/\mathbb{K}}$ the Grössencharacter of E over \mathbb{K} . We fix a prime \mathfrak{p} of $\mathcal{O}_{\mathbb{K}}$ over p , and we let $\pi := \psi_{E/\mathbb{K}}(\mathfrak{p})$. Let Γ be the period lattice of E for some invariant differential ω defined over $\mathcal{O}_{\mathbb{K}}$.

In §2, we introduce the Eisenstein-Kronecker-Lerch series and Eisenstein-Kronecker numbers. We fix a lattice Γ in \mathbb{C} . Let $z_0 \in \mathbb{C} \setminus \Gamma$. We define the Eisenstein-Kronecker numbers $e_{a,b}^*(z_0)$ for integers a and b by the formula

$$e_{a,b}^*(z_0) = \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{\bar{\gamma}^a}{\gamma^b} \langle \gamma, z_0 \rangle,$$

where $\langle \gamma, z_0 \rangle := \exp((\gamma \bar{z}_0 - z_0 \bar{\gamma})/A)$ and A is the fundamental area of Γ divided by $\varpi = 3.14159 \dots$. The above sum converges only for $b > a + 2$, but one may give it meaning for all a and b by analytic continuation. Let z_0 be a point in $\mathbb{C} \setminus \Gamma$ which defines a non-zero torsion point in $E(\mathbb{Q}) \cong \mathbb{C}/\Gamma$, which we again denoted by z_0 . By Damerell’s theorem, the numbers $e_{a,b}^*(z_0)/A^a$ are algebraic over \mathbb{K} when $a, b \geq 0$. We fix once and for all an embedding $i_{\mathfrak{p}} : \overline{\mathbb{K}} \hookrightarrow \mathbb{C}_p$ continuous for the \mathfrak{p} -adic topology on \mathbb{K} , and we regard $e_{a,b}^*(z_0)/A^a$ for $a, b \geq 0$ as p -adic numbers through this embedding. The Hodge realization of the elliptic polylogarithm is related to Eisenstein-Kronecker numbers $e_{a,b}^*(z_0)$ for $a < 0$ (see Theorem A.29), which are complex numbers expected to be transcendental. We use p -adic interpolation to define p -adic versions of $e_{a,b}^*(z_0)$ for $a < 0$.

Assume now that p is ordinary of the form $(p) = \mathfrak{p}\mathfrak{p}^*$ in $\mathcal{O}_{\mathbb{K}}$, and suppose that z_0 is a non-zero torsion point of order prime to p . For their construction of the two-variable p -adic L -function for CM elliptic curves (see also [6]), Manin-Vishik [22] and Katz [19] constructed

a p -adic measure $\mu_{z_0,0}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ satisfying

$$\frac{1}{\Omega_p^{a+b}} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x, y) = (-1)^{a+b} \left(\frac{e_{a,b+1}^*(z_0)}{A^a} - \frac{\pi^a e_{a,b+1}^*(\pi z_0)}{\pi^{b+1} A^a} \right)$$

for any $a, b \geq 0$, where Ω_p is a certain p -adic period in W^\times for the ring of integers W of the maximal unramified extension of \mathbb{Q}_p in \mathbb{C}_p . Using this measure, we define the p -adic Eisenstein-Kronecker numbers as follows.

DEFINITION 0.1. – Suppose z_0 is a non-zero torsion point of $E(\overline{\mathbb{Q}})$ of order prime to p . For any integer a, b such that $b \geq 0$, we define the p -adic Eisenstein-Kronecker number $e_{a,b+1}^{(p)}(z_0)$ by the formula

$$e_{a,b+1}^{(p)}(z_0) := \frac{1}{\Omega_p^{a+b} b!} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x, y).$$

Note that this definition is valid even for $a < 0$.

If p is supersingular, which due to our assumption that p be unramified in \mathcal{O}_K is equivalent to the condition that p remains prime in \mathcal{O}_K , then a two-variable measure as above interpolating Eisenstein-Kronecker numbers does not exist. We define $e_{a,b+1}^{(p)}(z_0)$ using p -adic distributions, constructed originally by Boxall [12][11], Schneider-Teitelbaum [25], Fourquaux and Yamamoto [33], which interpolate in one-variable Eisenstein-Kronecker numbers for fixed $b \geq 0$. The latter construction is valid even when p is ordinary, and the definition in this case is equivalent to the one given above. In both constructions, the fact that the generating function for Eisenstein-Kronecker numbers is given by the Kronecker theta function $\Theta_{z_0, w_0}(z, w)$ ([6] Theorem 1.17) is crucial.

In §3, we give the definition of the p -adic elliptic polylogarithm functions, which are overconvergent functions on the elliptic curve minus the residue disc around the identity characterized as the solutions of a certain differential equation. We then give the relation of these functions to the p -adic Eisenstein-Kronecker numbers.

In §4, we construct and explicitly calculate the p -adic elliptic polylogarithm. Let K be a finite unramified extension of \mathbb{K}_p . We denote by \mathcal{O}_K the ring of integers of K and by k its residue field. We denote again by E a model of our elliptic curve over \mathcal{O}_K . The rigid cohomology $H_{\text{rig}}^1(E_k/K)$ of $E_k := E \otimes k$ is a Frobenius K -module with Hodge filtration coming from the Hodge filtration of de Rham cohomology of $E_K := E \otimes K$ through the canonical isomorphism

$$H_{\text{dR}}^1(E_K/K) \cong H_{\text{rig}}^1(E_k/K).$$

This cohomology group is a K -vector space with certain basis $\underline{\omega}$ and $\underline{\omega}^*$. We let \mathcal{H} be the filtered Frobenius module dual to $H_{\text{rig}}^1(E_k/K)$, and we denote by $\underline{\omega}^\vee$ and $\underline{\omega}^{*\vee}$ the dual basis.

Let $S(E)$ be the category of filtered overconvergent F -isocrystals on E , referred to as the category of syntomic coefficients in our previous papers, which plays a rough p -adic analogue of the category of variations of mixed Hodge structures on E . We denote by $S(\mathcal{V})$ the same category on $\mathcal{V} := \text{Spec } \mathcal{O}_K$, which is simply the category of filtered Frobenius modules. The elliptic logarithm sheaf $\mathcal{L}og$ is a pro-object $\mathcal{L}og = \varprojlim \mathcal{L}og^N$ in $S(E)$. One of its main features is the splitting principle, given as follows.

LEMMA 0.2 (= Lemma 5.3). – Let $z_0 \in E(K)$ be a torsion point of order prime to p . Then we have a canonical isomorphism

$$i_{z_0}^* \text{Log} \cong \prod_{j \geq 0} \text{Sym}^j \mathcal{H}$$

as filtered Frobenius modules in $S(\mathcal{V})$.

We let \mathcal{H}^\vee be the dual of \mathcal{H} , and we denote by \mathcal{H}_E and \mathcal{H}_E^\vee the pull-backs of \mathcal{H} and \mathcal{H}^\vee to E by the structure morphism. We let $U = E \setminus [0]$, where $[0]$ is the identity element of E . The p -adic elliptic polylogarithm class is an element pol_{syn} in the rigid syntomic cohomology group

$$\text{pol}_{\text{syn}} \in H_{\text{syn}}^1(U, \mathcal{H}_E^\vee \otimes \text{Log}(1))$$

characterized by a certain residue condition. The importance of this element is that it is the image by the syntomic regulator of the motivic elliptic polylogarithm [28]. Our main theorem, Theorem 4.19, is an explicit description of the p -adic elliptic polylogarithm sheaf, which is an extension of $\text{Log}(1)$ by \mathcal{H}_E in the category $S(U)$ of filtered overconvergent F -isocrystals on U whose extension class corresponds to pol_{syn} .

Finally, in §5, we calculate the specialization of the p -adic elliptic polylogarithm to non-zero torsion points of order prime to p . We now let K be the maximal unramified extension of \mathbb{K}_p . Let $z_0 \in E(K)$ be any non-zero torsion point of order prime to p , and let $i_{z_0} : \text{Spec } \mathcal{O}_K \rightarrow U$ be the inclusion induced by z_0 . By the splitting principle, the pull-back of pol_{syn} to z_0 gives an element

$$i_{z_0}^* \text{pol}_{\text{syn}} \in \prod_{j \geq 0} H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes \text{Sym}^j \mathcal{H}(1)).$$

The calculation of syntomic cohomology gives an isomorphism

$$(0.3) \quad H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes \text{Sym}^j \mathcal{H}(1)) \cong \mathcal{H}^\vee \otimes \text{Sym}^j \mathcal{H} / K\underline{\omega} \otimes \underline{\omega}^{*\vee j}.$$

Our main result is the following.

THEOREM 0.4 (= Theorem 5.7). – Suppose $p \geq 5$ is unramified in $\mathcal{O}_{\mathbb{K}}$. If we let $\underline{\omega}^{m,k} := \underline{\omega}^{\vee m} \underline{\omega}^{*\vee k}$, then the image of $i_{z_0}^* \text{pol}_{\text{syn}}$ in $H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes \text{Sym}^j \mathcal{H}(1))$ through the isomorphism (0.3) is given as

$$- \sum_{\substack{m+k=j \\ m \geq 1, k \geq 0}} e_{-m, k+1}^{(p)}(z_0) \underline{\omega} \otimes \underline{\omega}^{m,k} - \sum_{\substack{m+k=j \\ m \geq 0, k \geq 1}} e_{-m-1, k}^{(p)}(z_0) \underline{\omega}^* \otimes \underline{\omega}^{m,k}.$$

The above result shows that the p -adic elliptic polylogarithm specializes to give the p -adic Eisenstein-Kronecker numbers. This is a p -adic analogue of the result of Beilinson-Levin and Wildeshaus in the Hodge case. When p is ordinary, we may interpret the above result in terms of special values of p -adic L -functions of imaginary quadratic fields (Corollary 5.10). Since the elliptic polylogarithm is motivic in origin, our result is in the direction of the p -adic Beilinson conjecture relating motivic elements to special values of p -adic L -functions. The implication of our result with regards to the precise formulation of the p -adic Beilinson conjecture of Perrin-Riou ([24], see also [13] Conjecture 2.7) as well as the precise relation in the supersingular case between p -adic Eisenstein-Kronecker numbers and special values of p -adic L -functions will be investigated in subsequent research.

0.3. Acknowledgment

Part of this research was conducted while the first author was visiting the École Normale Supérieure in Paris, and the second author the Institut de Mathématiques de Jussieu. We would like to thank our hosts Yves André and Pierre Colmez for hospitality. The authors would also like to thank John Coates and Hidekazu Furusho for pointing out the importance of the distribution relation, and Daniel Bertrand for comments concerning the definition of the connection function. The authors would like to thank the referees for valuable comments which helped to greatly improve the overall quality of this paper.

1. de Rham realization of the elliptic polylogarithm

1.1. Kronecker theta function

Here, we first review the definition of the Kronecker theta function $\Theta(z, w)$. Then we define the connection function $L_n(z)$, which will later be used to describe the connection of the elliptic polylogarithm. We fix a lattice $\Gamma \subset \mathbb{C}$, and we define A to be the fundamental area of Γ divided by $\varpi = 3.14159 \dots$. In other words, if $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ such that $\text{Im}(\gamma_1/\gamma_2) > 0$, then $A = (\gamma_1\bar{\gamma}_2 - \gamma_2\bar{\gamma}_1)/2\varpi i$.

DEFINITION 1.1. – We define $\theta(z)$ to be the reduced theta function (in the sense of [6] §1.2) for $E := \mathbb{C}/\Gamma$ corresponding to the divisor $[0]$, normalized so that $\theta'(0) = 1$.

Let $e_2^* := \lim_{u \rightarrow 0^+} \sum_{\gamma \in \Gamma \setminus \{0\}} \gamma^{-2} |\gamma|^{-2u}$, and let $\sigma(z)$ be the Weierstrass σ -function. Then $\theta(z)$ may be given explicitly as

$$(1.2) \quad \theta(z) = \exp \left[-\frac{1}{2} e_2^* z^2 \right] \sigma(z).$$

The theta function $\theta(z)$ is a holomorphic function on \mathbb{C} whose only zeroes are simple zeroes at $z \in \Gamma$, and satisfies the transformation formula

$$(1.3) \quad \theta(z + \gamma) = \varepsilon(\gamma) \exp \left[\frac{\bar{\gamma}}{A} \left(z + \frac{\gamma}{2} \right) \right] \theta(z)$$

for any $\gamma \in \Gamma$, where $\varepsilon(\gamma) = -1$ if $\gamma \notin 2\Gamma$ and $\varepsilon(\gamma) = 1$ otherwise.

DEFINITION 1.4 (Kronecker theta function). – We define the Kronecker theta function $\Theta(z, w)$ to be the function

$$\Theta(z, w) := \frac{\theta(z+w)}{\theta(z)\theta(w)}.$$

Note that since $\sigma(z)$ hence $\theta(z)$ is an odd function, we have $\theta''(0) = 0$. We let $F_1(z)$ be the meromorphic function

$$F_1(z) := \lim_{w \rightarrow 0} (\Theta(z, w) - w^{-1}) = \theta'(z)/\theta(z).$$

Then $F_1(z) = \zeta(z) - e_2^* z$, where $\zeta(z) := \sigma'(z)/\sigma(z)$ is the Weierstrass zeta function. $F_1(z)$ satisfies the transformation formula $F_1(z + \gamma) = F_1(z) + \bar{\gamma}/A$ for any $\gamma \in \Gamma$.

The existence of $F_1(z) := \lim_{w \rightarrow 0} (\Theta(z, w) - w^{-1})$ shows that the function $\Theta(z, w) - w^{-1}$ is holomorphic in a neighborhood of $w = 0$ for a fixed $z \notin \Gamma$. By exchanging z and w and combining the results, we see that the function $\Theta(z, w) - z^{-1} - w^{-1}$

is holomorphic in a neighborhood of $z = w = 0$. Hence we may consider the two-variable Taylor expansion of this function at $(z, w) = (0, 0)$. As a result, we have an expansion

$$\Theta(z, w) = \sum_{b \geq 0} F_b(z)w^{b-1},$$

where $F_0(z) \equiv 1$, $F_1(z)$ is as before, and $F_b(z)$ is holomorphic in z for $b > 1$.

DEFINITION 1.5. – We define the function $\Xi(z, w)$ by

$$\Xi(z, w) = \exp(-F_1(z)w)\Theta(z, w).$$

From the transformation Formula (1.3), we see that $\Xi(z + \gamma, w) = \Xi(z, w)$ for any $\gamma \in \Gamma$. Since $\theta'(0) = 1$, for any $z \notin \Gamma$, the function $\Xi(z, w) - w^{-1}$ is holomorphic with respect to w in a neighborhood of $w = 0$.

DEFINITION 1.6. – We define the *connection function* $L_n(z)$ to be the function in z whose value at a fixed $z \in \mathbb{C} \setminus \Gamma$ is defined as the coefficients of the Laurent expansion of $\Xi(z, w)$ with respect to w at $w = 0$:

$$(1.7) \quad \Xi(z, w) = \sum_{n \geq 0} L_n(z)w^{n-1}.$$

The connection function is given explicitly as $L_0(z) \equiv 1$ when $n = 0$. The function $L_n(z)$ for general $n \geq 0$ is explicitly given by the formula

$$L_n(z) = \sum_{b=0}^n \frac{(-F_1(z))^{n-b}}{(n-b)!} F_b(z).$$

Since $L_n(z + \gamma) = L_n(z)$ for any $\gamma \in \Gamma$, the function $L_n(z)$ is an elliptic function, holomorphic except for poles at $z \in \Gamma$.

We next prove the algebraicity of the connection function when the complex torus \mathbb{C}/Γ has a model over a subfield F of \mathbb{C} . Let E be an elliptic curve defined over F , given by the Weierstrass equation

$$(1.8) \quad E : y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in F.$$

Let Γ be the period lattice of E with respect to the invariant differential $\omega = dx/y$, and denote by ξ the complex uniformization

$$(1.9) \quad \xi : \mathbb{C}/\Gamma \xrightarrow{\cong} E(\mathbb{C}), \quad z \mapsto (\wp(z), \wp'(z)).$$

Then the following results from the recurrence relation for the Taylor coefficients of $\sigma(z)$ at the origin (See [1] pp. 635-636 or [31]).

LEMMA 1.10. – *The Taylor expansion of $\sigma(z)$ at $z = 0$ has coefficients in F .*

Lemma 1.10 implies that the Laurent coefficients of $\zeta(z) := \sigma'(z)/\sigma(z)$, $\wp(z) = -\zeta'(z)$ and $\wp'(z)$ at $z = 0$ are also in F . By the definition of $\Xi(z, w)$ and (1.2), we have

$$(1.11) \quad \Xi(z, w) = \exp[-\zeta(z)w] \frac{\sigma(z+w)}{\sigma(z)\sigma(w)}.$$

Hence the coefficients of the two-variable expansion of $\Xi(z, w)$ at $(z, w) = (0, 0)$ with respect to variables z and w are also in F . For any rational function f on E defined over F , we denote by $f(z)$ the pullback of f by ξ .

PROPOSITION 1.12 (Algebraicity). – *For any integer $n \geq 0$, the connection function $L_n(z)$ is obtained as a pullback by ξ of a rational function L_n on E defined over F .*

Proof. – The function $L_n(z)$ is holomorphic outside $[0] \in E(\mathbb{C})$. By the previous lemma, the Laurent coefficients of $L_n(z)$ at $z = 0$ are in F . The Laurent coefficients of $\wp(z)$ and $\wp'(z)$ at the origin are also in F . We may remove the non-positive degree of the Laurent expansion of $L_n(z)$ at 0 by subtracting a suitable function $h(z) \in F[\wp(z), \wp'(z)]$. Then $L_n(z) = h(z) \in F[\wp(z), \wp'(z)]$, proving our assertion. \square

1.2. Review of de Rham cohomology

In order to fix notations, we review facts about de Rham cohomology of smooth algebraic varieties defined over a field F with characteristic 0. Let X be a smooth algebraic variety defined over F , and let Ω_X^\bullet be the de Rham complex on X .

DEFINITION 1.13. – We denote by $M(X)$ the abelian category consisting of pairs (\mathcal{F}, ∇) , where \mathcal{F} is a locally free module on X and $\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_X^1$ is an integrable connection on \mathcal{F} .

DEFINITION 1.14. – For any object \mathcal{F} in $M(X)$, we define the de Rham cohomology $H_{\text{dR}}^i(X, \mathcal{F})$ of X with coefficients in \mathcal{F} by

$$H_{\text{dR}}^i(X, \mathcal{F}) := R^i\Gamma(X, \Omega^\bullet(\mathcal{F})),$$

where $\Omega^\bullet(\mathcal{F})$ denotes the de Rham complex $\mathcal{F} \otimes \Omega_X^\bullet$.

PROPOSITION 1.15. – *We have a canonical isomorphism*

$$\text{Ext}_{M(X)}^1(\mathcal{O}_X, \mathcal{F}) \xrightarrow{\cong} H_{\text{dR}}^1(X, \mathcal{F}).$$

Proof. – The canonical homomorphism is given as follows. For an extension

$$(1.16) \quad 0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0$$

in $M(X)$, we define the class $[\mathcal{E}]$ in $H_{\text{dR}}^1(X, \mathcal{F})$ to be the image of $1 \in F \subset H_{\text{dR}}^0(X, \mathcal{O}_X)$ by the boundary map $H_{\text{dR}}^0(X, \mathcal{O}_X) \rightarrow H_{\text{dR}}^1(X, \mathcal{F})$ associated to (1.16). The inverse homomorphism is defined as follows. Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is an affine open covering of X . Then any cohomology class $[\xi]$ in $H_{\text{dR}}^1(X, \mathcal{F})$ may be represented by a Čech cocycle

$$(\xi_i, u_{ij}) \in \prod_{i \in I} \Gamma(U_i, \mathcal{F} \otimes \Omega_X^1) \oplus \prod_{i, j \in I} \Gamma(U_i \cap U_j, \mathcal{F}),$$

for this covering satisfying $d\xi_i = 0$, $du_{ij} = \xi_j - \xi_i$ and $u_{ij} + u_{jk} = u_{ik}$ for any $i, j, k \in I$. The extension \mathcal{E} whose class in $\text{Ext}_{M(X)}^1(\mathcal{O}_X, \mathcal{F})$ corresponds to $[\xi]$ is constructed as follows: We define \mathcal{E}_i to be the coherent \mathcal{O}_{U_i} -module $\mathcal{E}_i := \mathcal{O}_{U_i} \underline{e}_i \oplus \mathcal{F}|_{U_i}$, with connection $\nabla(\underline{e}_i) := \xi_i$. We define \mathcal{E} to be the extension obtained by pasting together \mathcal{E}_i on $U_i \cap U_j$ through the isomorphism $\mathcal{E}_i|_{U_i \cap U_j} \cong \mathcal{E}_j|_{U_i \cap U_j}$, $\underline{e}_i = \underline{e}_j - u_{ij}$, which is compatible with the connection. \square

Let X be a smooth algebraic variety over F and $D \hookrightarrow X$ be a normal crossing divisor of X over F . We denote by $\Omega_X^1(\log D)$ the sheaf of differentials on X with logarithmic poles along D . For any $\mathcal{F} \in M(X)$, we define the logarithmic de Rham cohomology of X with logarithmic poles along D and coefficients in \mathcal{F} by

$$H_{\log \text{dR}}^i(X, \mathcal{F}) := R^i\Gamma(X, \Omega_{\log}^\bullet(\mathcal{F})),$$

where $\Omega_{\log}^\bullet(\mathcal{F})$ denotes the de Rham complex $\mathcal{F} \otimes \Omega_X^\bullet(\log D)$ of \mathcal{F} with log poles along D . Let $U := X \setminus D$ and let $j : U \hookrightarrow X$ be the natural inclusion. The morphism $\Omega_{\log}^\bullet(\mathcal{F}) \rightarrow j_*\Omega^\bullet(\mathcal{F})|_U$ induces a canonical isomorphism

$$(1.17) \quad H_{\log \text{dR}}^i(X, \mathcal{F}) \xrightarrow{\cong} H_{\text{dR}}^i(U, \mathcal{F}).$$

Suppose X is a smooth curve defined over F , and let $i : D \hookrightarrow X$ be a smooth divisor of X defined over F . Then for \mathcal{F} in $M(X)$, the localization sequence in this case is the isomorphism $H_{\text{dR}}^0(X, \mathcal{F}) \cong H_{\text{dR}}^0(U, \mathcal{F})$ and the exact sequence

$$\begin{aligned} 0 \rightarrow H_{\text{dR}}^1(X, \mathcal{F}) \rightarrow H_{\text{dR}}^1(U, \mathcal{F}) \xrightarrow{\text{res}} H_{\text{dR}}^0(D, i^*\mathcal{F}) \\ \rightarrow H_{\text{dR}}^2(X, \mathcal{F}) \rightarrow H_{\text{dR}}^2(U, \mathcal{F}) \rightarrow 0, \end{aligned}$$

obtained from the long exact sequence associated to the exact sequence

$$(1.18) \quad 0 \rightarrow \Omega_X^\bullet(\mathcal{F}) \rightarrow \Omega_{\log}^\bullet(\mathcal{F}) \rightarrow i_*i^*\mathcal{F}[-1] \rightarrow 0$$

and the isomorphism (1.17). The boundary map

$$(1.19) \quad \text{res} : H_{\log \text{dR}}^1(X, \mathcal{F}) \rightarrow H_{\text{dR}}^0(D, i^*\mathcal{F})$$

of the long exact sequence associated to (1.18) is given as follows. Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an affine open covering of X . Then any cohomology class ξ in $H_{\log \text{dR}}^1(X, \mathcal{F})$ may be represented by a cocycle

$$(\xi_i, u_{ij}) \in \prod_{i \in I} \Gamma(U_i, \mathcal{F} \otimes \Omega_X^1(\log D)) \oplus \prod_{i, j \in I} \Gamma(U_i \cap U_j, \mathcal{F}),$$

satisfying $du_{ij} = \xi_j - \xi_i$ and $u_{ij} + u_{jk} = u_{ik}$ for any $i, j, k \in I$. For each point $P \in D$, the image of $\text{res}(\xi)$ in $\Gamma(P, i^*\mathcal{F})$ is precisely $\text{res}_P(\xi_i)$ for $i \in I$ such that $P \in U_i$.

1.3. The logarithm sheaf

We return to the case of an elliptic curve E defined over a field $F \subset \mathbb{C}$ as in (1.8). Let $H_{\text{dR}}^1(E)$ be the first de Rham cohomology of E , which may be calculated as

$$(1.20) \quad H_{\text{dR}}^1(E) \xrightarrow{\cong} R^1\Gamma\left(E, \mathcal{O}_E([0]) \xrightarrow{d} \Omega_E^1(2[0])\right) \xleftarrow{\cong} \Gamma(E, \Omega_E^1(2[0])).$$

Let ω^* be the algebraic differential defined over \mathbb{C} corresponding to $dF_1 = d \log \theta(z)$ through the complex uniformization (1.9). Then ω^* is a differential of the second kind. As it is the case for applications in this paper, we assume that ω^* is defined over F . We denote by $\underline{\omega}$ and $\underline{\omega}^*$ the classes in $H_{\text{dR}}^1(E)$ corresponding to the differentials $\omega := dx/y$ and ω^* in $\Gamma(E, \Omega_E^1(2[0]))$, which form a basis $\{\underline{\omega}, \underline{\omega}^*\}$ of $H_{\text{dR}}^1(E)$. We denote by $\mathcal{H} := H_{\text{dR}}^1(E)^\vee$ the dual of $H_{\text{dR}}^1(E)$, and we let $\{\underline{\omega}^\vee, \underline{\omega}^{*\vee}\}$ be the dual basis of $\{\underline{\omega}, \underline{\omega}^*\}$. For any smooth scheme X over F , we denote by \mathcal{H}_X the coherent module $\mathcal{H} \otimes \mathcal{O}_X$ on X with connection such that

$\nabla(\underline{\omega}^\vee) = \nabla(\underline{\omega}^{*\vee}) = 0$. Since the connection on $\mathcal{H}_E := \mathcal{H} \otimes \mathcal{O}_E$ is simply the differentiation on \mathcal{O}_E , we have a canonical isomorphism

$$(1.21) \quad H_{\text{dR}}^1(E, \mathcal{H}_E) = H_{\text{dR}}^1(E) \otimes \mathcal{H} = \mathcal{H}^\vee \otimes \mathcal{H}.$$

DEFINITION 1.22. – The first logarithm sheaf $\mathcal{L}og^{(1)}$ is defined to be any extension of \mathcal{H}_E by \mathcal{O}_E in $M(E)$, whose extension class in

$$\text{Ext}_{M(E)}^1(\mathcal{O}_E, \mathcal{H}_E) \cong H_{\text{dR}}^1(E, \mathcal{H}_E)$$

is mapped by (1.21) to the identity $\underline{\omega} \otimes \underline{\omega}^\vee + \underline{\omega}^* \otimes \underline{\omega}^{*\vee}$. We define the N -th logarithm sheaf $\mathcal{L}og^N$ to be the N -th symmetric tensor product of $\mathcal{L}og^{(1)}$.

The homomorphism $\mathcal{L}og^{(1)} \rightarrow \mathcal{O}_E$ induces a natural projection $\mathcal{L}og^{N+1} \rightarrow \mathcal{L}og^N$. Let $i_{[0]}: [0] \hookrightarrow E$ be the natural inclusion of the identity $[0]$ in E . Since a connection on a point is zero, we have a splitting

$$(1.23) \quad \epsilon: i_{[0]}^* \mathcal{L}og^{(1)} \cong F \bigoplus \mathcal{H}$$

on $M(\text{Spec } F)$. A choice of ϵ as above induces a splitting $\epsilon: i_{[0]}^* \mathcal{L}og^N \cong \prod_{j=0}^N \text{Sym}^j \mathcal{H}$ on the N -th symmetric tensor product, which is compatible with the projection $\mathcal{L}og^{N+1} \rightarrow \mathcal{L}og^N$.

REMARK 1.24. – The first logarithm sheaf $\mathcal{L}og^{(1)}$ has non-trivial automorphisms as extensions of \mathcal{H}_E by \mathcal{O}_E . However, if we choose a pair $(\mathcal{L}og^{(1)}, \epsilon)$ consisting of a first logarithm sheaf and a splitting ϵ as in (1.23), then the pair $(\mathcal{L}og^{(1)}, \epsilon)$ is unique up to unique isomorphism.

We now give an explicit construction of $\mathcal{L}og^{(1)}$. We take an affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of E . Then there exists an element

$$(1.25) \quad (\omega_i^*, \mathbf{u}_{ij}) \in \prod_{i \in I} \Gamma(U_i, \Omega_E^1) \oplus \prod_{i, j \in I} \Gamma(U_i \cap U_j, \mathcal{O}_E)$$

satisfying the cocycle conditions $d u_{ij} = \omega_j^* - \omega_i^*$, $u_{ij} + u_{jk} = u_{ik}$ for any $i, j, k \in I$, which represents the class of $\underline{\omega}^*$ in $H_{\text{dR}}^1(E)$. Then, if we set $\nu_i = \underline{\omega}^\vee \otimes \omega + \underline{\omega}^{*\vee} \otimes \omega_i^*$ and $\mathbf{u}_{ij} = u_{ij} \underline{\omega}^{*\vee}$, the pair

$$(\nu_i, \mathbf{u}_{ij}) \in \prod_{i \in I} \Gamma(U_i, \mathcal{H}_E \otimes \Omega_E^1) \oplus \prod_{i, j \in I} \Gamma(U_i \cap U_j, \mathcal{H}_E)$$

is a cocycle which represents the cohomology class in $H_{\text{dR}}^1(E, \mathcal{H}_E)$ that maps to the identity by (1.21). Hence Proposition 1.15 gives the following.

PROPOSITION 1.26. – On each U_i , we let $\mathcal{L}og_i^{(1)}$ be $\mathcal{O}_{U_i} e_i \bigoplus \mathcal{H}_{U_i}$ with connection horizontal on \mathcal{H} and $\nabla(e_i) = \nu_i \in \Gamma(U_i, \mathcal{H}_E \otimes \Omega_E^1)$. Then $\mathcal{L}og^{(1)}$ is obtained by pasting together $\mathcal{L}og_i^{(1)}$ through the isomorphism on $U_i \cap U_j$:

$$(1.27) \quad \mathcal{L}og_i^{(1)}|_{U_i \cap U_j} \cong \mathcal{L}og_j^{(1)}|_{U_i \cap U_j}, \quad e_i = e_j - \mathbf{u}_{ij} = e_j - u_{ij} \underline{\omega}^{*\vee}.$$

Now, let $\underline{\omega}_i^{m,n} := e_i^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a!$ for $a = N - m - n$.

COROLLARY 1.28. – On each U_i , we let

$$\mathcal{L}og_i^N := \bigoplus_{0 \leq m+n \leq N} \mathcal{O}_{U_i} \underline{\omega}_i^{m,n},$$

with connection $\nabla_{\mathcal{L}} = d + \nu_i$. Then the N -th logarithm sheaf $\mathcal{L}og^N$ is given by pasting together $\mathcal{L}og_i^N$ on $U_i \cap U_j$ through the isomorphism

$$(1.29) \quad \mathcal{L}og_i^N|_{U_i \cap U_j} \cong \mathcal{L}og_j^N|_{U_i \cap U_j}, \quad \underline{\omega}_i^{m,n} = \sum_{k=n}^{N-m} \frac{(-u_{ij})^{k-n}}{(k-n)!} \underline{\omega}_j^{m,k}.$$

Proof. – This follows by calculating $\underline{\omega}_i^{m,n}$ in terms of $\underline{\omega}_j^{m,k}$. □

The projection $\mathcal{L}og^{N+1} \rightarrow \mathcal{L}og^N$ on each U_i is defined by mapping $\underline{\omega}_i^{m,n}$ to $\underline{\omega}_i^{m,n}$ if $m+n \leq N$, and to zero if $m+n = N+1$.

One may describe the restriction of $\mathcal{L}og^{(1)}$ to $U = E \setminus [0]$ by using differentials of the second kind. Since (ω_i^*, u_{ij}) and ω^* represent the same class in $R^1\Gamma(E, \mathcal{O}_E([0]) \xrightarrow{d} \Omega_E^1(2[0]))$, this implies that there exists an element

$$(1.30) \quad (u_i) \in \prod_{i \in I} \Gamma(U_i, \mathcal{O}_E([0]))$$

satisfying $\omega^* = \omega_i^* - du_i$ and $u_{ij} = u_j - u_i$ for any $i, j \in I$.

PROPOSITION 1.31. – The restriction of $\mathcal{L}og^{(1)}$ to $U = E \setminus [0]$ is given by the free \mathcal{O}_U -module $\mathcal{L} := \mathcal{O}_U \underline{e} \oplus \mathcal{H}_U$, with connection horizontal on \mathcal{H} and satisfying $\nabla(\underline{e}) = \nu$, where $\nu = \underline{\omega}^\vee \otimes \omega + \underline{\omega}^{*\vee} \otimes \omega^*$.

Proof. – For each $i \in I$, we have an isomorphism $\mathcal{L}|_{U \cap U_i} \cong \mathcal{L}og_i^{(1)}|_{U \cap U_i}$ on $U \cap U_i$ which is the identity on \mathcal{H} and $\underline{e} = \underline{e}_i - u_i \underline{\omega}^{*\vee}$. The coboundary condition for u_i implies that it is compatible with the connection. □

Let $\underline{\omega}^{m,n} := \underline{e}^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a!$ for $a = N - m - n$. Then we have the following.

COROLLARY 1.32. – The restriction of $\mathcal{L}og^N$ to U is given by the free \mathcal{O}_U -module

$$\mathcal{L}^N := \bigoplus_{0 \leq m+n \leq N} \mathcal{O}_U \underline{\omega}^{m,n}$$

with connection $\nabla_{\mathcal{L}} = d + \nu$. For any $i \in I$, the isomorphism $\mathcal{L}^N|_{U \cap U_i} \cong \mathcal{L}og_i^N|_{U \cap U_i}$ is given on $U \cap U_i$ by

$$(1.33) \quad \underline{\omega}^{m,n} = \sum_{k=n}^{N-m} \frac{(-u_i)^{k-n}}{(k-n)!} \underline{\omega}_i^{m,k}.$$

To finish this section, we give a choice of a splitting $\epsilon: i_{[0]}^* \mathcal{L}og^{(1)} \cong F \oplus \mathcal{H}$ as in (1.23). We fix an affine open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of E , and we choose (ω_i^*, u_{ij}) and (u_i) as in (1.25) and (1.30). Consider any $i \in I$ such that $[0] \in U_i$. Since $\omega_i^* = \omega^* + du_i$ is a meromorphic differential form without poles on U_i , the function $\eta_i(z) := F_1(z) + u_i(z)$ is a meromorphic function on \mathbb{C} , holomorphic on the inverse image of U_i by the uniformization $\mathbb{C} \twoheadrightarrow \mathbb{C}/\Gamma \cong E(\mathbb{C})$, whose value at 0 is an element in F . Let $\tilde{u}_i := u_i - \eta_i(0)$ if $[0] \in U_i$. By replacing u_i by \tilde{u}_i and u_{ij} by $\tilde{u}_j - \tilde{u}_i$, we may assume that $\eta_i(0) = (F_1 + u_i)(0) = 0$ if

$[0] \in U_i$. Then for any $U_i, U_j \in \mathfrak{U}$ such that $[0] \in U_i \cap U_j$, we have $u_{ij}(0) = (u_j - u_i)(0) = (\eta_j - \eta_i)(0) = 0$. Hence we may define the splitting

$$(1.34) \quad \epsilon : i_{[0]}^* \mathcal{L}og^{(1)} \cong F \oplus \mathcal{H}$$

by mapping e_i to the identity element $1 \in F$ for $i \in I$ such that $[0] \in U_i$.

DEFINITION 1.35. – We fix (ω_i^*, u_{ij}) and (u_i) as above. We let $\{\omega_{[0]}^{m,n}\}$ be the basis of $i_{[0]}^* \mathcal{L}og^N$ obtained as the restriction of the basis $\{\omega_i^{m,n}\}$ of $\mathcal{L}og_i^N$ for some $i \in I$ such that $[0] \in U_i$. This basis is independent of the choice of i .

The splitting

$$(1.36) \quad \epsilon : i_{[0]}^* \mathcal{L}og^N \cong \prod_{k=0}^N \text{Sym}^k \mathcal{H}$$

induced from the splitting (1.34) is given by mapping $\omega_{[0]}^{m,n}$ to $\omega^{\vee m} \omega^{*\vee n}$.

1.4. The polylogarithm sheaf

Here, we define and explicitly describe the elliptic polylogarithm sheaf, originally constructed by Beilinson and Levin [7]. Let the notations be as before. In particular, let $\mathcal{L}og^N$ be the logarithm sheaf explicitly described in Corollary 1.28 and let ϵ be the splitting given in (1.36).

LEMMA 1.37. – We have $\varprojlim_N H_{\text{dR}}^i(E, \mathcal{L}og^N) = 0$ for $i = 0, 1$, and the projection gives an isomorphism $\varprojlim_N H_{\text{dR}}^2(E, \mathcal{L}og^N) \xrightarrow{\cong} H_{\text{dR}}^2(E) \cong F$.

Proof. – This statement is proved in [7] 1.1.2. See also [17] Lemma A.1.4 or [4] Lemma 3.4. □

Let $D = [0]$ and $U = E \setminus [0]$. By Lemma 1.37, the residue map induces an isomorphism

$$\varprojlim_N H_{\text{dR}}^1(U, \mathcal{L}og^N) \xrightarrow{\cong} \varprojlim_N \text{Ker} \left(H_{\text{dR}}^0(D, i_{[0]}^* \mathcal{L}og^N) \rightarrow H_{\text{dR}}^2(E, \mathcal{L}og^N) \right).$$

By (1.36) and Lemma 1.37 for H_{dR}^2 , we have an isomorphism

$$(1.38) \quad \text{res} : \varprojlim_N H_{\text{dR}}^1(U, \mathcal{H}_U^\vee \otimes \mathcal{L}og^N) \cong \mathcal{H}^\vee \otimes \prod_{k \geq 1} \text{Sym}^k \mathcal{H}.$$

Note that this isomorphism depends on the choice of (1.36).

DEFINITION 1.39. – We define the *polylogarithm class* (for the splitting (1.36)) to be a system of classes $\text{pol}_{\text{dR}}^N \in H_{\text{dR}}^1(U, \mathcal{H}_U^\vee \otimes \mathcal{L}og^N)$ which maps through (1.38) to the identity in $\mathcal{H}^\vee \otimes \mathcal{H} \subset \mathcal{H}^\vee \otimes \prod_{k \geq 1} \text{Sym}^k \mathcal{H}$. Furthermore, we define the *elliptic polylogarithm sheaf* on U to be a system of sheaves \mathcal{P}^N in $M(U)$ given as an extension

$$0 \rightarrow \mathcal{L}og^N \rightarrow \mathcal{P}^N \rightarrow \mathcal{H}_U \rightarrow 0$$

whose extension class in $\text{Ext}_{M(U)}^1(\mathcal{H}_U, \mathcal{L}og^N) \cong H_{\text{dR}}^1(U, \mathcal{H}_U^\vee \otimes \mathcal{L}og^N)$ is pol_{dR}^N .

The rest of this section is devoted to explicitly describing the extension class pol_{dR}^N and the polylogarithm sheaf \mathcal{P}^N . We first construct certain classes in $H_{\text{dR}}^1(U, \mathcal{L}og^N)$.

DEFINITION 1.40. – We define $\omega^\vee, \omega^{*\vee}$ in $\Gamma(U, \mathcal{L}og^N \otimes \Omega_U^1)$ to be the sections

$$\omega^\vee = -\underline{\omega}^{0,0} \otimes \omega^* + \sum_{n=1}^N L_n \underline{\omega}^{1,n-1} \otimes \omega, \quad \omega^{*\vee} = \sum_{n=0}^N L_n \underline{\omega}^{0,n} \otimes \omega.$$

The sections ω^\vee and $\omega^{*\vee}$ define classes $[\omega^\vee]$ and $[\omega^{*\vee}]$ in $H_{\text{dR}}^1(U, \mathcal{L}og^N)$.

THEOREM 1.41. – We let \mathbf{p}^N be the cohomology class

$$\mathbf{p}^N := \underline{\omega} \otimes [\omega^\vee] + \underline{\omega}^* \otimes [\omega^{*\vee}] \in \mathcal{H}^\vee \otimes H_{\text{dR}}^1(U, \mathcal{L}og^N).$$

Then the image of \mathbf{p}^N by (1.38) is the identity $\underline{\omega} \otimes \underline{\omega}^\vee + \underline{\omega}^* \otimes \underline{\omega}^{*\vee}$ in $\mathcal{H}^\vee \otimes \mathcal{H} \subset \mathcal{H}^\vee \otimes \prod_{k \geq 1} \text{Sym}^k \mathcal{H}$. In particular, we have

$$\text{pol}_{\text{dR}} := \lim_{\leftarrow N} \text{pol}_{\text{dR}}^N = \lim_{\leftarrow N} \mathbf{p}^N.$$

The proof of this theorem will be given at the end of this section.

COROLLARY 1.42. – The polylogarithm sheaf \mathcal{P}^N may be constructed as follows. As a coherent \mathcal{O}_U -module, it is given as the sum

$$\mathcal{P}^N := \mathcal{H}_U \bigoplus \mathcal{L}og_U^N,$$

with connection ∇_φ given by ∇_φ on $\mathcal{L}og^N$ and $\nabla_\varphi(\underline{\omega}^\vee) = \omega^\vee, \nabla_\varphi(\underline{\omega}^{*\vee}) = \omega^{*\vee}$.

We now prepare some results necessary for the proof of the theorem. In order to calculate the residue (1.38), we must express the classes $[\omega^\vee]$ and $[\omega^{*\vee}]$ in terms of cocycles in logarithmic de Rham cohomology

$$H_{\log \text{dR}}^1(E, \mathcal{L}og^N) := R^1 \Gamma(E, \Omega_{\log}^\bullet(\mathcal{L}og^N)).$$

We take an open affine covering $\mathfrak{U} = \{U_i\}$ of E , and we fix a cocycle (ω_i^*, u_{ij}) and a coboundary (u_i) as in Definition 1.35. We let

$$\Xi_i(z, w) := \exp(-u_i(z)w) \Xi(z, w) = \exp(-\eta_i(z)w) \Theta(z, w)$$

for $\eta_i(z) = F_1(z) + u_i(z)$.

DEFINITION 1.43. – For any integer $k \geq 0$ and $i \in I$, we define $L_{k,i}(z)$ to be the function in z given by

$$\Xi_i(z, w) = \sum_{k \geq 0} L_{k,i}(z) w^{k-1}.$$

By definition, $L_{k,i}(z)$ may be expressed in terms of $L_n(z)$ and $u_i(z)$, hence it comes from a rational function $L_{k,i}$ on E defined over F . The importance of this function is the following property.

PROPOSITION 1.44. – The functions $L_{k,i}$ are holomorphic on $U_i \setminus [0]$. It has a simple pole of residue one at $[0]$ if $k = 1$ and is holomorphic on U_i if $k \neq 1$.

Proof. – By definition, we have $L_{0,i} = L_0 \equiv 1$, so the statement is true for $k = 0$. Similarly, we have $L_{1,i} = -u_i + L_1 = -u_i$. By definition, u_i is holomorphic on $U_i \setminus [0]$. When $[0] \in U_i$, the function $\eta_i(z) = F_1(z) + u_i(z)$ is holomorphic at $z = 0$. Since the residue of $F_1(z)$ at $z = 0$ is one, the residue of $u_i(z)$ at $z = 0$ is -1 . This gives the assertion for $k = 1$. For the general case, note that for a fixed $w \notin \Gamma$, the function $\Theta(z, w) - z^{-1}$ is holomorphic in a neighborhood of $z = 0$. Since we have chosen $\eta_i(z)$ so that $\eta_i(0) = 0$, this implies that $L_{k,i}$ is holomorphic on U_i when $k > 1$. \square

COROLLARY 1.45. – *Let*

$$\omega_i^\vee := -\omega_i^{0,0} \otimes \omega_i^* + \sum_{k=1}^N L_{k,i} \omega_i^{1,k-1} \otimes \omega, \quad \omega_i^{*\vee} := \sum_{k=0}^N L_{k,i} \omega_i^{0,k} \otimes \omega.$$

Then we have $\omega_i^\vee, \omega_i^{*\vee} \in \prod_{i \in I} \Gamma(U_i, \mathcal{L}og_i^N \otimes \Omega_E^1(\log[0]))$.

Let the notations be as above.

PROPOSITION 1.46. – *For any $i, j \in I$, we define $\alpha_{ij} \in \Gamma(U_i \cap U_j, \mathcal{L}og^N)$ to be the element*

$$\alpha_{ij} := \sum_{k=0}^N \frac{u_{ij}^{k+1}}{(k+1)!} \omega_i^{0,k}.$$

Then $(\omega_i^\vee, \alpha_{ij})$ and $(\omega_i^{*\vee}, 0)$ satisfy the cocycle conditions

$$\nabla_{\mathcal{L}}(\alpha_{ij}) = \omega_j^\vee - \omega_i^\vee, \quad \alpha_{ij} + \alpha_{jk} = \alpha_{ik}, \quad \omega_j^{*\vee} - \omega_i^{*\vee} = 0,$$

hence define cohomology classes in $H_{\log \text{dR}}^1(E, \mathcal{L}og^N)$. These classes coincide with the classes $[\omega^\vee]$ and $[\omega^{*\vee}]$ in $H_{\text{dR}}^1(U, \mathcal{L}og^N)$ through the isomorphism $H_{\log \text{dR}}^1(E, \mathcal{L}og^N) \cong H_{\text{dR}}^1(U, \mathcal{L}og^N)$.

Proof. – By (1.33) and the definition of L_n and $L_{k,i}$, we have

$$(1.47) \quad \sum_{n=0}^N L_n \omega^{0,n} = \sum_{k=0}^N \sum_{n=0}^k L_n \frac{(-u_i)^{k-n}}{(k-n)!} \omega_i^{0,k} = \sum_{k=0}^N L_{k,i} \omega_i^{0,k}.$$

Hence $\omega^{*\vee} = \omega_i^{*\vee} = \omega_j^{*\vee}$, which proves that $\omega_j^{*\vee} - \omega_i^{*\vee} = 0$. For any $i \in I$, we define $\alpha_i \in \Gamma(U_i \setminus [0], \mathcal{L}og^N)$ to be the element

$$\alpha_i := \sum_{k=0}^N \frac{(-u_i)^{k+1}}{(k+1)!} \omega_i^{0,k}.$$

Then using (1.29), we see that $\alpha_{ij} = \alpha_j - \alpha_i$ if $U_i \cap U_j \neq \emptyset$. The equality $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ follows immediately from this fact. By definition of the connection, we have

$$\nabla_{\mathcal{L}}(\alpha_i) = -\omega_i^{0,0} \otimes du_i + \sum_{k=1}^N \frac{(-u_i)^k}{k!} \omega_i^{0,k} \otimes \eta + \sum_{k=1}^N \frac{(-u_i)^k}{k!} \omega_i^{1,k-1} \otimes \omega.$$

Here, we have used the fact that $du_i = \omega^* - \omega_i^*$. Again by (1.33), we have

$$\sum_{n=1}^N L_n \omega^{1,n-1} = \sum_{k=1}^N \sum_{n=1}^k L_n \frac{(-u_i)^{k-n}}{(k-n)!} \omega_i^{1,k-1} = \sum_{k=1}^N \left(L_{k,i} - \frac{(-u_i)^k}{k!} \right) \omega_i^{1,k-1}.$$

Applying this equality and (1.33) of Corollary 1.32, we see that

$$(1.48) \quad \nabla_{\mathcal{L}}(\alpha_i) = \omega_i^{\vee} - \omega^{\vee}.$$

This shows that $\nabla_{\mathcal{L}}(\alpha_{ij}) = \nabla_{\mathcal{L}}(\alpha_j) - \nabla_{\mathcal{L}}(\alpha_i) = \omega_j^{\vee} - \omega_i^{\vee}$. The classes $(\omega_i^{\vee}, \alpha_{ij})$ and $(\omega_i^{*\vee}, 0)$ coincide with $[\omega^{\vee}]$ and $[\omega^{*\vee}]$, since by (1.48), the element $(\alpha_i)_{i \in I} \in \prod_i \Gamma(U_i \setminus [0], \mathcal{L}og^N)$ gives the coboundary of the difference between ω^{\vee} and ω_i^{\vee} and $\omega^{*\vee} = \omega_i^{*\vee}$ on each U_i . \square

Proof of Theorem 1.41. – By Proposition 1.46, we may calculate the residues of $[\omega^{\vee}]$ and $[\omega^{*\vee}]$ using the cocycles $(\omega_i^{\vee}, \alpha_{ij})$ and $(\omega_i^{*\vee}, 0)$. By Proposition 1.44 and the description of the residue map given at the end of §1.2, we have $\text{res}([\omega^{\vee}]) = \omega_{[0]}^{1,0}$ and $\text{res}([\omega^{*\vee}]) = \omega_{[0]}^{0,1}$. Our theorem now follows from the definition of p^N . \square

REMARK 1.49. – As it is useful for applications, we assumed that ω^* is defined over F and used this differential to describe the de Rham realization of the elliptic polylogarithm. It is possible to give an algebraic construction without this assumption, by taking the basis $\{\omega, \eta\}$ of $H_{\text{dR}}^1(E)$, where η is the class corresponding to the algebraic differential of the second kind $\eta = xdx/y$ defined over F . The relation is given by $\eta = -\omega^* - e_2^*\omega$.

2. Classical and p -adic Eisenstein-Kronecker numbers

In this section, we will first review the definition of Eisenstein-Kronecker-Lerch series and Eisenstein-Kronecker numbers. We will then review the construction of the p -adic distribution interpolating Eisenstein-Kronecker numbers, when the corresponding elliptic curve has complex multiplication. This distribution will be used to define the p -adic Eisenstein-Kronecker numbers. In what follows, we fix a lattice $\Gamma \subset \mathbb{C}$.

2.1. Eisenstein-Kronecker numbers

Let $\langle z, w \rangle := \exp[(z\bar{w} - \bar{z}w)/A]$, where A is again the fundamental area of Γ divided by ϖ .

DEFINITION 2.1 ([30] VIII §12). – Let $z_0, w_0 \in \mathbb{C}$. For any integer $a \geq 0$, we define the Eisenstein-Kronecker-Lerch series $K_a^*(z_0, w_0, s)$ by

$$(2.2) \quad K_a^*(z_0, w_0, s) := \sum_{\gamma \in \Gamma}^* \frac{(\bar{z}_0 + \bar{\gamma})^a}{|z_0 + \gamma|^{2s}} \langle \gamma, w_0 \rangle,$$

where $*$ denotes the sum over all $\gamma \in \Gamma$ such that $\gamma \neq -z_0$. The series converges for $\text{Re}(s) > \frac{a}{2} + 1$, and extends to a meromorphic function on the complex plane by analytic continuation.

We extend the definition of the above series to integers $a < 0$ by

$$(2.3) \quad K_a^*(z_0, w_0, s) = (-1)^a \overline{K_{-a}^*(-z_0, w_0, \bar{s} - a)}.$$

This function also satisfies (2.2) for $\text{Re}(s) > a/2 + 1$.

PROPOSITION 2.4. – *Let a be an integer.*

- (i) *The function $K_a^*(z_0, w_0, s)$ for s continues meromorphically to a function on the whole s -plane, with a simple pole only at $s = 1$ if $a = 0$ and $w_0 \in \Gamma$.*

(ii) *The functions $K_a^*(z_0, w_0, s)$ satisfy the functional equation*

$$(2.5) \quad \Gamma(s)K_a^*(z_0, w_0, s) = A^{a+1-2s}\Gamma(a+1-s)K_a^*(w_0, z_0, a+1-s)\langle w_0, z_0 \rangle.$$

Proof. – The assertions (i) for $a > 0$ and (ii) for $a \geq 0$ are given in [30] VIII §13. We next prove (i) for $a = 0$. By [30] VIII §13 (13), the only poles of $\Gamma(s)K_0^*(z_0, w_0, s)$ are a simple pole at $s = 0$ if $z_0 \in \Gamma$ and at $s = 1$ if $w_0 \in \Gamma$. Since $\Gamma(s)$ has a simple pole at $s = 0$ and $\Gamma(1) = 1$, we see that the unique pole of $K_0^*(z_0, w_0, s)$ is a simple pole at $s = 1$ if $w_0 \in \Gamma$. Finally, for integers $a < 0$, the fact that $K_a^*(z_0, w_0, s)$ is holomorphic in s follows from (2.3) and the statement for $a > 0$. The functional equation applied to $\Gamma(\bar{s}-a)K_{-a}^*(-z_0, w_0, \bar{s}-a)$ gives the functional equation

$$(2.6) \quad \Gamma(s-a)K_a^*(z_0, w_0, s) = (-1)^a A^{a+1-2s}\Gamma(1-s)K_a^*(w_0, z_0, a+1-s)\langle w_0, z_0 \rangle.$$

Since $\Gamma(s)\Gamma(1-s) = \varpi / \sin \varpi s$, we have $\Gamma(s)/\Gamma(s-a) = (-1)^a \Gamma(a+1-s)/\Gamma(1-s)$. The functional equation for $a < 0$ follows by multiplying this quotient to both sides of (2.6). \square

Suppose $z_0, w_0 \in \mathbb{C}$. In [6] Definition 1.5, we defined the Eisenstein-Kronecker numbers $e_{a,b+1}^*(z_0, w_0)$ for integers $a, b \geq 0$. We extend the definition to general integers a, b as follows.

DEFINITION 2.7. – Let a, b be integers such that $(a, b) \neq (-1, 1)$ if $w_0 \in \Gamma$. We define the Eisenstein-Kronecker numbers $e_{a,b}^*(z_0, w_0)$ by

$$e_{a,b}^*(z_0, w_0) := K_{a+b}^*(z_0, w_0, b).$$

For $(a, b) = (0, 0)$, we have $e_{0,0}^*(z_0, w_0) := K_0^*(z_0, w_0, 0) = -\langle w_0, z_0 \rangle$.

Part of the importance of Eisenstein-Kronecker numbers stems from its relation to special values of Hecke L -functions associated to imaginary quadratic fields (see for example [6] Proposition 1.6). We will use the following version of Eisenstein-Kronecker numbers.

DEFINITION 2.8. – Suppose a, b are integers, and let $z_0 \in \mathbb{C}$ such that $z_0 \notin \Gamma$ if $(a, b) = (-1, 1)$. We let

$$e_{a,b}^*(z_0) := e_{a,b}^*(0, z_0) = K_{a+b}^*(0, z_0, b).$$

We next consider the generating function of Eisenstein-Kronecker numbers. For any $z_0, w_0 \in \mathbb{C}$, let

$$\Theta_{z_0, w_0}(z, w) := \exp\left[-\frac{z_0 \bar{w}_0}{A}\right] \exp\left[-\frac{z \bar{w}_0 + w \bar{z}_0}{A}\right] \Theta(z + z_0, w + w_0).$$

Then we have the following.

THEOREM 2.9. – *The expansion of $\Theta_{z_0, w_0}(z, w)$ at $(z, w) = (0, 0)$ is given by*

$$\Theta_{z_0, w_0}(z, w) = \langle w_0, z_0 \rangle \frac{\delta_{z_0}}{z} + \frac{\delta_{w_0}}{w} + \sum_{a, b \geq 0} (-1)^{a+b} \frac{e_{a,b+1}^*(z_0, w_0)}{a! A^a} z^b w^a,$$

where $\delta_x = 1$ if $x \in \Gamma$ and $\delta_x = 0$ otherwise.

Proof. – This is obtained from §1.4 Theorem 1.17 of [6], by replacing the index a, b by $a, b + 1$. \square

By definition of $\Theta_{z_0, w_0}(z, w)$, we have $\Theta_{z_0, w_0}(z, w) = \langle w_0, z_0 \rangle \Theta_{w_0, z_0}(w, z)$. Hence by Theorem 2.9, we have

$$\Theta_{z_0, 0}(z, w) = \Theta_{0, z_0}(w, z) = \frac{\delta_{z_0}}{z} + \frac{1}{w} + \sum_{a, b \geq 0} (-1)^{a+b} \frac{e_{a, b+1}^*(z_0)}{a! A^a} z^a w^b.$$

Using this function, we next define the function $F_{z_0, b}(z)$ as follows.

DEFINITION 2.10. – For any $z_0 \in \mathbb{C}$, we let $F_{z_0, b}(z)$ be the function such that

$$\Theta_{z_0, 0}(z, w) = \sum_{b \geq 0} F_{z_0, b}(z) w^{b-1}.$$

In particular, $F_{z_0, 1}(z) = F_1(z + z_0) - \bar{z}_0/A$. For any $\gamma \in \Gamma$, we have

$$\Theta_{z_0 + \gamma, 0}(z, w) = \exp \left[-\frac{w(\bar{z}_0 + \bar{\gamma})}{A} \right] \Theta(z + z_0 + \gamma, w) = \Theta_{z_0, 0}(z, w).$$

Hence $F_{z_0, b}(z)$ depends only on the choice of z_0 modulo Γ . Henceforth, the z_0 of $F_{z_0, b}$ will either denote an element in \mathbb{C} or a class in \mathbb{C}/Γ . We have from Theorem 2.9 the following corollary.

COROLLARY 2.11. – For any $b \geq 0$, the Laurent expansion of $F_{z_0, b}(z)$ at 0 is given by

$$F_{z_0, b}(z) = \frac{\delta_{b-1, z_0}}{z} + \sum_{a \geq 0} (-1)^{a+b-1} \frac{e_{a, b}^*(z_0)}{a! A^a} z^a,$$

where $\delta_{b, x} = 1$ if $b = 0$ and $x \in \Gamma$, and is zero otherwise.

Proof. – The statement for $b > 0$ is Theorem 2.9. The statement for $b = 0$ follows from the fact that $e_{0, 0}^*(z_0) = -1$ and $e_{a, 0}^*(z_0) = 0$ for any $a > 0$ (see Remark A.5 for a proof of this fact). \square

2.2. p -adic Eisenstein-Kronecker numbers

We next give the definition of p -adic Eisenstein-Kronecker numbers, when the lattice Γ is the period lattice of an elliptic curve with complex multiplication given as follows. Let \mathbb{K} be an imaginary quadratic field, and let E be an elliptic curve defined over \mathbb{K} with complex multiplication by the full ring of integers $\mathcal{O}_{\mathbb{K}}$ of \mathbb{K} . We note that the existence of such E implies that \mathbb{K} is of class number one. We fix a Weierstrass model

$$(2.12) \quad E : y^2 = 4x^3 - g_2x - g_3, \quad \omega = dx/y$$

of E over $\mathcal{O}_{\mathbb{K}}$ with good reduction at the primes above $p \geq 5$. We assume in addition that p is unramified in $\mathcal{O}_{\mathbb{K}}$. We let Γ be the period lattice of E with respect to the invariant differential $\omega = dx/y$. In this case, the following theorem was proved by Damerell ([14], [15]).

THEOREM 2.13 (Damerell). – Let a and b be integers ≥ 0 , and let $z_0 \in \Gamma \otimes \mathbb{Q}$, which corresponds to a torsion point of E . Then we have

$$e_{a, b}^*(z_0)/A^a \in \overline{\mathbb{K}}.$$

Moreover, we have $e_2^* := e_{0, 2}^*(0) \in \mathbb{K}$ for the constant e_2^* defined in §1.1.

See [6] Corollary 2.10 for a proof using $\Theta(z, w)$. The above theorem shows that the Laurent coefficients of $\Theta_{z_0,0}(z, w)$ at the origin are in $\overline{\mathbb{K}}$. In what follows, we assume that $z_0 \in \Gamma \otimes \mathbb{Q}$.

We denote by $\psi_{E/\mathbb{K}}$ the Grössencharacter of \mathbb{K} associated to E . We fix a prime \mathfrak{p} of $\mathcal{O}_{\mathbb{K}}$ over p , and we let $\pi := \psi_{E/\mathbb{K}}(\mathfrak{p}) \in \mathcal{O}_{\mathbb{K}}$. Then π is a generator of \mathfrak{p} . Henceforth, we fix an embedding $i_p : \overline{\mathbb{K}} \hookrightarrow \mathbb{C}_p$ such that the completion of \mathbb{K} in \mathbb{C}_p is $\mathbb{K}_{\mathfrak{p}}$. We let K be a finite unramified extension of $\mathbb{K}_{\mathfrak{p}}$ in \mathbb{C}_p . Let \widehat{E} be the formal group associated to $E \otimes_{\mathcal{O}_{\mathbb{K}}} \mathcal{O}_{\mathbb{K}_{\mathfrak{p}}}$ with respect to the formal parameter $s = -2x/y$. Then \widehat{E} is a Lubin-Tate group over $\mathcal{O}_{\mathbb{K}_{\mathfrak{p}}}$, of height one or two depending on whether E has ordinary or supersingular reduction at \mathfrak{p} . Denote by $\lambda(t)$ the formal logarithm of \widehat{E} normalized so that $\lambda'(0) = 1$, and denote by $\widehat{\Theta}_{z_0,0}(s, t)$ the formal composition of the two-variable Laurent expansion of $\Theta_{z_0,0}(z, w)$ at $z = 0$ and $w = 0$ with the formal power series $z = \lambda(s)$, $w = \lambda(t)$. Let $\partial_{s,\log} := \lambda'(s)^{-1} \partial_s = \partial_z$.

We denote by $\widehat{F}_{z_0,b}(s)$ the formal composition of the Taylor expansion of $F_{z_0,b}(z)$ at $z = 0$ with the power series $z = \lambda(s)$. By definition, we have

$$\Theta_{z_0,0}(z, w) = \exp(F_{z_0,1}(z)w) \Xi(z + z_0, w).$$

If we let $\widehat{L}_{z_0,n}(s) := L_n(z + z_0)|_{z=\lambda(s)}$, then the above gives the equality

$$(2.14) \quad \widehat{F}_{z_0,b}(s) = \sum_{n=0}^b \frac{\widehat{F}_{z_0,1}(s)^{b-n}}{(b-n)!} \widehat{L}_{z_0,n}(s).$$

If $f(z)$ is a meromorphic function on \mathbb{C}/Γ corresponding to a rational function f on E , then the power series $\widehat{f}(s) := f(z)|_{z=\lambda(s)}$ is the expansion of the rational function f with respect to the formal parameter $s = -2x/y$ of the elliptic curve.

LEMMA 2.15. – *Let K be a finite extension of $\mathbb{K}_{\mathfrak{p}}$, and suppose that the meromorphic function $f(z)$ on \mathbb{C}/Γ corresponds to a rational function f on E defined over K , without any pole on $\widehat{E}(\mathfrak{m}_{\mathbb{C}_p}) \setminus [0]$. Then $\widehat{f}(s) := f(z)|_{z=\lambda(s)}$ has bounded coefficients.*

Proof. – Consider the embedding $K(E) \hookrightarrow \text{Frac}(\mathcal{O}_K[[s]])$ of the functional field of E to the fractional field of the ring of formal power series with respect to s . Note that the image $\widehat{f}(s)$ of f is of the form $\widehat{f}(s) = \widehat{P}(s)/\widehat{Q}(s)$, where $\widehat{P}(s), \widehat{Q}(s) \in \mathcal{O}_K[[s]]$ are relatively prime. By the p -adic Weierstrass preparation theorem, $\widehat{Q}(s)$ is of the form $\widehat{Q}(s) = p^m \widehat{U}(s)R(s)$, where $\widehat{U}(s) \in \mathcal{O}_K[[s]]^\times$ and $R(s)$ is a distinguished polynomial, in other words $R(s) \equiv s^N$ modulo \mathfrak{p} for $N = \deg R(s)$. If $R(s) \not\equiv s^N$, then the non-zero roots of this polynomial would correspond to poles of f on $\widehat{E}(\mathfrak{m}_{\mathbb{C}_p})$ other than zero. Hence by our assumption, we must have $R(s) = s^N$. This shows that $p^m \widehat{f}(s) \in \mathcal{O}_K[[s]][s^{-1}]$, proving the lemma. \square

For any torsion point $z_0 \in E(\overline{\mathbb{Q}})$, we define the order of z_0 to be the annihilator of z_0 as an element in the $\mathcal{O}_{\mathbb{K}}$ -module $E(\overline{\mathbb{Q}})$.

PROPOSITION 2.16. – *Suppose $z_0 \in E(\overline{\mathbb{Q}})$ is a non-zero torsion point of order \mathfrak{n} prime to \mathfrak{p} . Then the power series*

$$\widehat{F}_{z_0,b}(s) \in \overline{\mathbb{K}}[[s]]$$

converges on the open unit disc $B^-(0, 1) := \{s \in \mathbb{C}_p \mid |s|_p < 1\}$. In particular, this series defines a rigid analytic function on $B^-(0, 1)$.

Proof. – When $b = 0$ then there is nothing to prove. Since $F_1(z) = \zeta(z) - e_2^*z$, we have $\partial_z F_1(z) = -\wp(z) - e_2^*$. Hence

$$\partial_z F_{z_0,1}(z) = \partial_z F_1(z + z_0) = -\wp(z + z_0) - e_2^*.$$

By Lemma 2.15, $\widehat{\wp}_{z_0}(s) := \wp(z + z_0)|_{z=\lambda(s)}$ is known to have bounded coefficients (in fact, one may prove that the coefficients are \mathfrak{p} -integral). Hence this power series converges on $B^-(0, 1)$, which implies that $\widehat{F}_{z_0,1}(s)$ also converges on $B^-(0, 1)$. The assertion for general b follows from (2.14), noting that by Lemma 2.15, $\widehat{L}_{z_0,n}(s)$ also has bounded coefficients. \square

REMARK 2.17. – If p is a prime of ordinary reduction, then we may prove that the coefficients of $\widehat{F}_{z_0,1}(s)$, hence that of $\widehat{F}_{z_0,b}(s)$, is in fact bounded.

We will use $\widehat{F}_{z_0,b}(s)$ to construct our p -adic distribution. Since \widehat{E} is a Lubin-Tate group, it has an action of $\mathcal{O}_{\mathbb{K}_p}$. We have $\mathcal{O}_{\mathbb{K}_p}$ -linear isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\widehat{E}, \widehat{\mathbb{G}}_m) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p \widehat{E}, T_p \widehat{\mathbb{G}}_m) \xleftarrow{\cong} \mathcal{O}_{\mathbb{K}_p}.$$

The last isomorphism depends on the choice of a p -adic period as follows. There exists $\Omega_p \in \mathbb{C}_p^\times$ such that the formal power series $\exp(\lambda(s)/\Omega_p)$, defined as the formal composition of the power series $\exp(S/\Omega_p)$ in S with the power series $\lambda(s)$ in s , is an element in $\mathcal{O}_{\mathbb{C}_p}[[s]]$. The second isomorphism is given by associating to any $x \in \mathcal{O}_{\mathbb{K}_p}$ the homomorphism of formal groups defined by $\exp(x\lambda(s)/\Omega_p)$, and depends on the choice of Ω_p . The notation $\exp(x\lambda(s)/\Omega_p)$ needs some care, since if s_m is a primitive p^m -torsion point in $\widehat{E}(\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}})$, then $\lambda(s_m) = 0$ but $\exp(x\lambda(s)/\Omega_p)|_{s=s_m}$ is a primitive p^m -th root of unity.

In what follows, we fix once and for all a choice of a p -adic period Ω_p . Let $C^{\mathrm{an}}(\mathcal{O}_{\mathbb{K}_p}, \mathbb{C}_p)$ be the set consisting of locally \mathbb{K}_p -analytic functions on $\mathcal{O}_{\mathbb{K}_p}$. We define our p -adic distribution $\mu_{z_0,b}$ as follows.

DEFINITION 2.18. – Let z_0 be a non-zero torsion point in $E(\overline{\mathbb{Q}})$ of order prime to \mathfrak{p} . For any integer $b \geq 0$, we define $\mu_{z_0,b}$ to be the p -adic distribution on $C^{\mathrm{an}}(\mathcal{O}_{\mathbb{K}_p}, \mathbb{C}_p)$ associated to $\widehat{F}_{z_0,b}(s)$. Such distribution satisfies the relation

$$\int_{\mathcal{O}_{\mathbb{K}_p}} \exp(x\lambda(s)/\Omega_p) d\mu_{z_0,b}(x) = \widehat{F}_{z_0,b}(s).$$

When p is ordinary, then this is the p -adic measure associated to bounded power series. When p is supersingular, then this is the p -adic distribution associated to rigid analytic functions on the open unit disc constructed in [25] Theorem 2.3 and Theorem 3.6.

When p is ordinary, the above distribution is related to the two-variable measure used by Manin-Vishik and Katz in defining the two-variable p -adic L -function interpolating special values of Hecke L -function of imaginary quadratic fields (See Proposition 2.25). When p is supersingular, the above distribution was considered by Boxall [12] [11] and Schneider-Teitelbaum [25] for the case $b = 0$, and by Fourquaux and Yamamoto [33] for any $b \geq 0$. We define the p -adic Eisenstein-Kronecker numbers using this distribution.

DEFINITION 2.19. – Let z_0 be a non-zero torsion point in $E(\overline{\mathbb{Q}})$ of order prime to \mathfrak{p} . For any integers a and b such that $b \geq 0$, we define the p -adic Eisenstein-Kronecker number $e_{a,b}^{(p)}(z_0)$ by

$$e_{a,b}^{(p)}(z_0) := \Omega_{\mathfrak{p}}^{-a} \int_{\mathcal{O}_{\mathbb{K}_{\mathfrak{p}}}^{\times}} x^a d\mu_{z_0,b}(x),$$

where we denote again by $\mu_{z_0,b}$ the restriction of $\mu_{z_0,b}$ to $\mathcal{O}_{\mathbb{K}_{\mathfrak{p}}}^{\times}$.

The above definition is justified since $e_{a,b}^{(p)}$ is related to the Eisenstein-Kronecker numbers $e_{a,b}^*(z_0)$ when $a, b \geq 0$ (see §2.3 Corollary 2.24).

2.3. Interpolation property of the p -adic distribution

Here we will relate the p -adic Eisenstein-Kronecker numbers to the Eisenstein-Kronecker numbers when $a, b \geq 0$. We keep the notations of §2.2. We first begin with the distribution property of $\Theta_{z_0,0}(z, w)$.

PROPOSITION 2.20 (Distribution relation). – For any $z_0 \in \mathbb{C}$, we have

$$\Theta_{\pi^m z_0, 0}(\pi^m z, \overline{\pi}^{-m} w) = \frac{1}{\pi^m} \sum_{z_m \in \frac{1}{\pi^m} \Gamma / \Gamma} \Theta_{z_0 + z_m, 0}(z, w).$$

Proof. – Note that $\Theta_{\pi^m z_0, 0}(\pi^m z, \overline{\pi}^{-m} w; \Gamma) = \Theta_{N\mathfrak{p}^m z_0, 0}(N\mathfrak{p}^m z, w; \overline{\pi}^m \Gamma)$. Our assertion is a special case of [6] Proposition 1.16. \square

COROLLARY 2.21. – The function $F_{z_0,b}(z)$ satisfies the relation

$$F_{\pi^m z_0, b}(\pi^m z) = \frac{\overline{\pi}^{mb}}{N(\mathfrak{p})^m} \sum_{z_m \in \frac{1}{\pi^m} \Gamma / \Gamma} F_{z_0 + z_m, b}(z).$$

Proof. – The statement is trivial when $b = 0$. The case for $b \geq 1$ follows from the distribution relation Proposition 2.20 for the Kronecker theta function and the definition of $F_{z_0,b}(z)$. \square

In what follows, we again let z_0 be a non-zero torsion point of E of order prime to \mathfrak{p} . The power series $\widehat{F}_{z_0,b}(s)$ satisfies the following translation formula with respect to \mathfrak{p}^m -torsion points.

LEMMA 2.22 (Translation). – Recall that $\widehat{F}_{z_0,b}(s)$ is the formal power series composition of the Taylor expansion of $F_{z_0,b}(z)$ at the origin with $z = \lambda(s)$. If we denote by \oplus the formal group law of \widehat{E} , then we have

$$\widehat{F}_{z_0,b}(s \oplus s_m) = \widehat{F}_{z_0 + z_m, b}(s),$$

where s_m is a torsion point in $\widehat{E}[\mathfrak{p}^m]$, and z_m is the image of s_m through the inclusion $\widehat{E}(\mathfrak{m}_{\mathfrak{p}})_{\text{tor}} \subset E(\overline{\mathbb{Q}})_{\text{tor}} \subset \mathbb{C}/\Gamma$.

Proof. – The statement is trivial when $b = 0$. When $b = 1$, the function

$$F(z) := F_{z_0,1}(z) - \bar{\pi}^{-m} F_{z_0,1}(\pi^m z)$$

is elliptic, hence satisfies $\widehat{F}(s \oplus s_m) = F(z + z_m)|_{z=\lambda(s)}$. By the equalities $F_{z_0,1}(\pi^m z)|_{z=\lambda(s)} = \widehat{F}_{z_0,1}([\pi^m]s)$ and $\widehat{F}_{z_0,1}([\pi^m](s \oplus s_m)) = \widehat{F}_{z_0,1}([\pi^m]s)$, we have

$$\widehat{F}(s \oplus s_m) = \widehat{F}_{z_0,1}(s \oplus s_m) - \bar{\pi}^{-m} \widehat{F}_{z_0,1}([\pi^m]s).$$

In addition, we have by definition $F_{z_0,1}(z + z_m) = F_{z_0+z_m,1}(z) + \bar{z}_m/A$. Hence applying the equality $F_{z_0+\pi^m z_m,1}(\pi^m z) = F_{z_0,1}(\pi^m z)$ for $\pi^m z_m \in \Gamma$, we have

$$F(z + z_m) := F_{z_0+z_m,1}(z) - \bar{\pi}^{-m} F_{z_0,1}(\pi^m z).$$

Our assertion follows by combining the above results. The case for $b > 1$ follows from (2.14), applying our lemma for $b = 1$ and noting that $\widehat{L}_{z_0,n}(s \oplus s_m) = \widehat{L}_{z_0+z_m,n}(s)$ since $L_n(z)$ corresponds to a rational function. \square

Using the above lemma, we have the following.

PROPOSITION 2.23. – *The distribution $\mu_{z_0,b}$ restricted to $\mathcal{O}_{\mathbb{K}_p}^\times$ satisfies*

$$\int_{\mathcal{O}_{\mathbb{K}_p}^\times} \exp(x\lambda(s)/\Omega_p) d\mu_{z_0,b}(x) = \widehat{F}_{z_0,b}(s) - \frac{1}{\bar{\pi}^b} \widehat{F}_{\pi z_0,b}([\pi]s).$$

Proof. – Note that for any primitive \mathfrak{p} -torsion point s_1 in $E[\mathfrak{p}]$, the value $\exp(\lambda(s)/\Omega_p)|_{s=s_1}$ is a primitive p -th root of unity. Hence standard argument shows that the restriction of distributions from $\mathcal{O}_{\mathbb{K}_p}$ to $\mathcal{O}_{\mathbb{K}_p}^\times$ is given by

$$\int_{\mathcal{O}_{\mathbb{K}_p}^\times} \exp(x\lambda(s)/\Omega_p) d\mu_{z_0,b}(x) = \widehat{F}_{z_0,b}(s) - \frac{1}{N(\mathfrak{p})} \sum_{s_1 \in E[\mathfrak{p}]} \widehat{F}_{z_0,b}(s \oplus s_1).$$

Our result now follows from Lemma 2.22 and the distribution relation (Corollary 2.21) of $F_{z_0,b}(z)$. \square

The expansion of $F_{z_0,b}(z)$ given in Corollary 2.11 gives the following.

COROLLARY 2.24. – *The distribution $\mu_{z_0,b}$ satisfies*

$$\Omega_p^{-a} \int_{\mathcal{O}_{\mathbb{K}_p}^\times} x^a d\mu_{z_0,b}(x) = (-1)^{a+b-1} \left(\frac{e_{a,b}^*(z_0)}{A^a} - \frac{\pi^a e_{a,b}^*(\pi z_0)}{\bar{\pi}^b A^a} \right)$$

for any integer $a \geq 0$. In particular, we have

$$e_{a,b}^{(p)}(z_0) = (-1)^{a+b-1} \left(\frac{e_{a,b}^*(z_0)}{A^a} - \frac{\pi^a e_{a,b}^*(\pi z_0)}{\bar{\pi}^b A^a} \right).$$

The above result shows that the p -adic Eisenstein-Kronecker numbers are related to the usual Eisenstein-Kronecker numbers when $a, b \geq 0$.

2.4. The relation to the p -adic L -function

Let the notations be as in §2.2. In this section, we suppose (2.12) has good *ordinary* reduction at the primes above p . In this case, we relate the p -adic Eisenstein-Kronecker numbers to special values of p -adic L -functions associated to Hecke characters of imaginary quadratic fields. The condition on p implies that we have canonical isomorphisms $\mathcal{O}_{\mathbb{K}_p} \cong \mathbb{Z}_p$ and $\mathcal{O}_{\mathbb{K}_p^*} \cong \mathbb{Z}_p$, hence we have a canonical isomorphism $(\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. We denote by κ_1, κ_2 the projections to the first and second factors of the above isomorphism.

Let z_0 be a non-zero torsion point of $E(\overline{\mathbb{K}})$ order prime to \mathfrak{p} . In [6], we defined a two-variable p -adic measure $\mu_{z_0,0}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$. By substituting $\eta_p(t) = \exp(\lambda(t)/\Omega_p) - 1$ into Definition 3.2 of [6], we see that the p -adic measure $\mu_{z_0,0}$ is defined to satisfy

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp(x\lambda(s)/\Omega_p) \exp(y\lambda(t)/\Omega_p) d\mu_{z_0,0}(x, y) = \widehat{\Theta}_{z_0,0}^*(s, t),$$

where $\widehat{\Theta}_{z_0,0}^*(s, t) := \widehat{\Theta}_{z_0,0}(s, t) - t^{-1}$. By taking $\partial_{t, \log}^b$ of both sides and substituting $t = 0$, we obtain the equality

$$\frac{1}{\Omega_p^b} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp(x\lambda(s)/\Omega_p) y^b d\mu_{z_0,0}(x, y) = b! \widehat{F}_{z_0, b+1}^*(s),$$

where $\widehat{F}_{z_0, b+1}^*(s) = \frac{1}{b!} \lim_{t \rightarrow 0} \partial_{t, \log}^b \widehat{\Theta}_{z_0,0}^*(s, t)$. Then $\widehat{F}_{z_0, b+1}^*(s) = \widehat{F}_{z_0, b+1}(s) + c_{b+1}$ for some constant c_{b+1} in K . The usual formula for the restriction of the distribution to $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ gives the equality

$$\begin{aligned} \frac{1}{b! \Omega_p^b} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \exp(x\lambda(s)/\Omega_p) y^b d\mu_{z_0,0}(x, y) \\ = \widehat{F}_{z_0, b}(s) - \frac{1}{N(\mathfrak{p})} \sum_{s_1 \in E[\mathfrak{p}]} \widehat{F}_{z_0, b}(s \oplus s_1) = \widehat{F}_{z_0, b}(s) - \frac{1}{\pi^b} \widehat{F}_{\pi z_0, b}([\pi]s). \end{aligned}$$

Note that the $*$ is not required in the middle term since the constant c_{b+1} cancels in the sum. This shows that we have

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x, y) = b! \Omega_p^b \int_{\mathbb{Z}_p^\times} x^a d\mu_{z_0, b+1}(x).$$

Then the definition of $e_{a,b}^{(p)}(z_0)$ gives the following.

PROPOSITION 2.25. – *Suppose $p \geq 5$ is an ordinary prime. Let $e_{a,b}^{(p)}(z_0)$ be the p -adic Eisenstein-Kronecker number of Definition 2.19. Then for any integer a, b such that $b \geq 0$, we have*

$$e_{a, b+1}^{(p)}(z_0) = \frac{1}{\Omega_p^{a+b} b!} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x, y).$$

The above result shows that our definition of $e_{a, b+1}^{(p)}(z_0)$ coincides with the definition given in the introduction. We use the above formula to relate the p -adic Eisenstein-Kronecker number to special values of the p -adic L -function associated to Hecke characters of \mathbb{K} . Suppose \mathfrak{g} is an ideal of $\mathcal{O}_{\mathbb{K}}$ prime to p and divisible by the conductor \mathfrak{f} of $\psi_{E/\mathbb{K}}$. We fix a

complex period Ω , which is any complex number satisfying $\Gamma = \mathcal{O}_{\mathbb{K}}\Omega$. Let $g \in \mathcal{O}_{\mathbb{K}}$ be a generator of \mathfrak{g} . For z_0 as above, we define a variant $\mu_{z_0,0}^{(g)}$ of the measure $\mu_{z_0,0}$ by the formula

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp(x\lambda(s)/\Omega_p) \exp(y\lambda(t)/\Omega_p) d\mu_{z_0,0}^{(g)}(x, y) = \widehat{\Theta}_{z_0,0}^*([g^{-1}]s, [\bar{g}]t).$$

For $\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times$, the value $\alpha_0\Omega/g$ defines a primitive \mathfrak{g} -torsion point of \mathbb{C}/Γ , and $\mu_{\alpha_0\Omega/g,0}^{(g)}$ induces a measure on $(\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times$ through the isomorphism

$$(\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times \cong \mathcal{O}_{\mathbb{K}_p}^\times \times \mathcal{O}_{\mathbb{K}_p^*}^\times \xrightarrow{\cong} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times,$$

where the right arrow is defined by $(\alpha_1, \alpha_2) \mapsto (\alpha_1^{-1}, \alpha_2)$. We define the measure μ_g on $\mathbb{X} := \varprojlim_n (\mathcal{O}_{\mathbb{K}}/\mathfrak{g}p^n)^\times = (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times \times (\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times$ by

$$\int_{\mathbb{X}} f(\alpha) d\mu_g(\alpha) = g^{-1}\Omega_p \sum_{\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times} \int_{(\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times} f(\alpha_0\alpha) \kappa_1(\alpha) d\mu_{\alpha_0\Omega/g,0}^{(g)}(\alpha).$$

The measure μ_g corresponds to the measure denoted by μ in [26] Theorem 4.14 through the canonical isomorphism $\mathcal{G} := \text{Gal}(\mathbb{K}(\mathfrak{g}p^\infty)/\mathbb{K}) \cong \mathbb{X}$. Let $\varphi_p : \mathbb{X} \rightarrow \mathbb{C}_p^\times$ be a p -adic character. Similarly to [26] (49), we define the value of the p -adic L -function at the character φ_p by

$$L_p(\varphi_p) := \int_{\mathbb{X}} \varphi_p(\alpha) d\mu_g(\alpha).$$

The p -adic L -function satisfies the following interpolation property. Let $I(\mathfrak{g})$ be the group of fractional ideals of $\mathcal{O}_{\mathbb{K}}$ prime to \mathfrak{g} , and let $\varphi : I(\mathfrak{g}) \rightarrow \overline{\mathbb{K}}^\times$ be an algebraic Hecke character whose conductor divides \mathfrak{g} . Suppose the infinity type of φ is (m, n) . The Hecke L -function of φ is defined for $\text{Re}(s) > \frac{m+n}{2} + 1$ as the series

$$L_{\mathfrak{g}}(\varphi, s) = \sum_{(\mathfrak{a}, \mathfrak{g})=1} \frac{\varphi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where the sum is over all integral ideals \mathfrak{a} of $\mathcal{O}_{\mathbb{K}}$ prime to \mathfrak{g} . By analytic continuation, $L_{\mathfrak{g}}(\varphi, s)$ extends to a meromorphic function on \mathbb{C} . Then there exists a finite character $\chi : (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times \rightarrow \overline{\mathbb{K}}^\times$ such that φ is of the form $\varphi((\alpha)) = \chi(\alpha)\alpha^m\bar{\alpha}^n$ for any $\alpha \in \mathcal{O}_{\mathbb{K}}$ prime to \mathfrak{g} . Then we have a p -adic character $\varphi_p : \mathbb{X} \rightarrow \mathbb{C}_p^\times$ defined by $\varphi_p(\alpha) := \varphi((\alpha))$ for any $\alpha \in \mathcal{O}_{\mathbb{K}}$ prime to $\mathfrak{g}p$. We have $\varphi_p = \chi_p \kappa_1^m \kappa_2^n$ as p -adic characters on \mathbb{X} , where χ_p is obtained as the composition of χ with the fixed embedding $\overline{\mathbb{K}} \hookrightarrow \mathbb{C}_p$.

PROPOSITION 2.26. – *Let φ be an algebraic Hecke character of \mathbb{K} of conductor dividing \mathfrak{g} and infinity type (m, n) , and let $d_{\mathbb{K}}$ be the discriminant of \mathbb{K} . If $m < 0$ and $n \geq 0$, then we have*

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = (-m-1)! \left(\frac{2\omega}{\sqrt{d_{\mathbb{K}}}} \right)^n \left(1 - \frac{\varphi^{-1}(\mathfrak{p})}{p} \right) (1 - \varphi(\mathfrak{p}^*)) \frac{L_{\mathfrak{g}}(\varphi, 0)}{\Omega^{n-m}}.$$

Proof. – Note that we have $A = \Omega\bar{\Omega}\sqrt{d_{\mathbb{K}}}/2\omega$. Our assertion follows from the interpolation property of $\mu_{z_0,0}$ and $\mu_{z_0,0}^{(g)}$ ([6] Proposition 3.3), the relation between Eisenstein-Kronecker numbers and special values of Hecke L -functions ([6] Proposition 1.6), and the formula for the restriction to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ of the measure $\mu_{z_0,0}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ ([6] Proposition 3.5). \square

The relation between the p -adic L -function and the p -adic Eisenstein-Kronecker numbers is given as follows.

PROPOSITION 2.27. – *Let the notations be as above. For a p -adic character $\varphi_p : \mathbb{X} \rightarrow \mathbb{C}_p^\times$, we have*

$$\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = (1 - \varphi_p(\bar{\pi})) g^m \bar{g}^n \sum_{\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times} \chi(\alpha_0) e_{-m-1, n+1}^{(p)}(\alpha_0 \Omega/g).$$

Proof. – By definition of $e_{a,b}^{(p)}(z_0)$ and the measure $\mu_{z_0,0}^{(g)}$, we have

$$\int_{(\mathcal{O}_{\mathbb{K}} \otimes \mathbb{Z}_p)^\times} \kappa_1^{m+1}(\alpha) \kappa_2^n(\alpha) d\mu_{\alpha_0 \Omega/g, 0}^{(g)}(\alpha) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{-m-1} y^n d\mu_{\alpha_0 \Omega/g, 0}^{(g)}(x, y).$$

By [6] Proposition 3.5, the above integral is equal to

$$g^{m+1} \bar{g}^n \left(e_{-m-1, n+1}^{(p)}(\alpha_0 \Omega/g) - \pi^n \bar{\pi}^m e_{-m-1, n+1}^{(p)}(\alpha_0 \Omega/g \bar{\pi}) \right).$$

Since $\varphi_p(\bar{\pi}) = \chi(\bar{\pi}) \bar{\pi}^m \pi^n$, our assertion follows by taking $g^{-1} \Omega_p \sum \chi(\alpha_0)$ on both sides of the above equality, where the sum is over $\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times$. \square

3. p -adic elliptic polylogarithm functions

In this section, we give the definition of the p -adic elliptic polylogarithm function. We then prove the relation of this function to the p -adic distribution defined in the previous section. We keep the notations of §2.2. We again assume that E has good reduction at the primes above $p \geq 5$ and that p is unramified in $\mathcal{O}_{\mathbb{K}}$.

3.1. p -adic elliptic polylogarithm functions

In this section, we define the p -adic elliptic polylogarithm function. We first begin by defining a p -modified variant of the connection function. For any $z_0 \in \mathbb{C}$, let

$$\Theta_{z_0,0}^{(p)}(z, w) := \Theta_{z_0,0}(z, w) - \frac{1}{\bar{\pi}} \Theta_{\pi z_0,0}(\pi z, \bar{\pi}^{-1} w),$$

and $\Theta^{(p)}(z, w) := \Theta_{0,0}^{(p)}(z, w)$.

DEFINITION 3.1. – Let $\Xi^{(p)}(z, w) := \exp(-F_1(z)w) \Theta^{(p)}(z, w)$. For any integer $n \geq 0$, we define $L_n^{(p)}(z)$ to be the function given by

$$\Xi^{(p)}(z, w) = \sum_{n \geq 0} L_n^{(p)}(z) w^{n-1}.$$

We let $F_1^{(p)}(z)$ be the meromorphic function

$$(3.2) \quad F_1^{(p)}(z) := F_1(z) - \frac{1}{\bar{\pi}} F_1(\pi z) = \frac{1}{N(\mathfrak{p})} \left(\log \left(\theta(z)^{N(\mathfrak{p})} / \theta(\pi z) \right) \right)'$$

Then the transformation formula $F_1(z + \gamma) = F_1(z) + \bar{\gamma}/A$ for $\gamma \in \Gamma$ shows that $F_1^{(p)}(z)$ is a rational function defined over \mathbb{K} . By definition, $F_{z_0,1}(z) = F_1(z + z_0) - \bar{z}_0/A$. Hence for any $z_0 \in \mathbb{C}$, we have

$$(3.3) \quad F_1^{(p)}(z + z_0) = F_{z_0,1}(z) - \bar{\pi}^{-1} F_{\pi z_0,1}(\pi z).$$

LEMMA 3.4. – Let n be an integer ≥ 0 . The function $L_n^{(p)}(z)$ is periodic with respect to Γ and is holomorphic outside $\pi^{-1}\Gamma$. The meromorphic function $L_n^{(p)}(z)$ corresponds to a rational function on E defined over \mathbb{K} . In addition, for $\Xi_{z_0}^{(p)}(z, w) := \exp(-F_{z_0,1}(z)w)\Theta_{z_0,0}^{(p)}(z, w)$, we have

$$(3.5) \quad \Xi_{z_0}^{(p)}(z, w) = \sum_{n \geq 0} L_n^{(p)}(z + z_0)w^{n-1}.$$

Proof. – From the equality $\Theta(z, w) = \exp(F_1(z)w)\Xi(z, w)$, we have

$$\Theta^{(p)}(z, w) = \exp(F_1(z)w)\Xi(z, w) - \bar{\pi}^{-1} \exp(F_1(\pi z)w/\bar{\pi})\Xi(\pi z, \bar{\pi}^{-1}w).$$

Since $F_1^{(p)}(z) = F_1(z) - \bar{\pi}^{-1}F_1(\pi z)$, the above equality and the definition of $\Xi^{(p)}(z, w)$ show that we have

$$\Xi^{(p)}(z, w) = \Xi(z, w) - \bar{\pi}^{-1} \exp(-F_1^{(p)}(z)w)\Xi(\pi z, \bar{\pi}^{-1}w).$$

This implies that $L_n^{(p)}(z)$ may be expressed as the difference between $L_n(z)$ and a sum over products of $\bar{\pi}$, $F_1^{(p)}(z)$ and $[\pi]^*L_k(z)$. This proves that $L_n^{(p)}(z)$ is defined over \mathbb{K} and holomorphic outside $\pi^{-1}\Gamma$, since the same holds true for $L_n(z)$, $F_1^{(p)}(z)$ and $[\pi]^*L_k(z)$. Equality (3.5) follows from the fact that we have $\Xi^{(p)}(z, w) = \Xi_{z_0}^{(p)}(z + z_0, w)$ for $z_0 \in \mathbb{C}$. \square

Let \mathcal{U} be the formal completion of U with respect to the special fiber, and we denote by \mathcal{U}_K the rigid analytic space associated to \mathcal{U} . Let $R = \Gamma(U, \mathcal{O}_U)$, and we denote by R^\dagger the weak completion of R ([23] Definition 1.1). Then $R_K^\dagger := R^\dagger \otimes K$ is the ring of overconvergent functions on \mathcal{U}_K . The following are the p -adic elliptic polylogarithm functions.

THEOREM 3.6. – Let $D_{0,n}^{(p)} = L_n^{(p)}$ for any integer $n > 0$, and $D_{m,n}^{(p)} = 0$ if $n \leq 0$. Then for integers $m, n > 0$, there exists a unique system of overconvergent functions $D_{m,n}^{(p)}$ on \mathcal{U}_K iteratedly satisfying

$$(3.7) \quad dD_{m,n}^{(p)} = -D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^*.$$

We call the functions $D_{m,n}^{(p)}$ the p -adic elliptic polylogarithm functions.

See Lemma A.11 for the differential equations satisfied by the elliptic polylogarithm functions in the Hodge case. The theorem is proved using the following calculation of rigid cohomology. Berthelot defined rigid cohomology for any scheme of finite type over a field of characteristics $p > 0$ ([8], [10]). When the scheme is affine and smooth over the base, then rigid cohomology is canonically isomorphic to Monsky-Washnitzer cohomology ([10] Proposition 1.10), which by definition may be calculated using overconvergent functions and differentials. Rigid cohomology $H_{\text{rig}}^i(U_k/K)$ of U_k may be calculated as

$$H_{\text{rig}}^i(U_k/K) \cong H^i \left[R_K^\dagger \xrightarrow{d} R_K^\dagger \otimes \Omega_R^1 \right].$$

In order to determine if a differential form in $R^\dagger \otimes \Omega_R^1$ is integrable by overconvergent functions, it is sufficient to determine the vanishing of the corresponding cohomology class in $H_{\text{rig}}^1(U_k/K)$. We have isomorphisms

$$H_{\text{rig}}^1(U_k/K) = H_{\text{dR}}^1(U_K) \cong H_{\text{dR}}^1(E_K) = K\omega \oplus K\omega^*$$

for this cohomology group.

Proof of Theorem 3.6. – We first prove the uniqueness of the solution. Let $D_{m,n}^{(p)}$ and $\widetilde{D}_{m,n}^{(p)}$ be two systems of solutions satisfying (3.7). We prove by induction that $D_{m,n}^{(p)} = \widetilde{D}_{m,n}^{(p)}$. Suppose $D_{a,b}^{(p)} = \widetilde{D}_{a,b}^{(p)}$ for $a + b < N$. We have by assumption $dD_{m,n}^{(p)} = d\widetilde{D}_{m,n}^{(p)}$ for integers $m, n > 0$ satisfying $m + n = N$. Hence there exist constants $c_{m,n}$ such that $D_{m,n}^{(p)} = \widetilde{D}_{m,n}^{(p)} + c_{m,n}$. We let $c_{N,0} = 0$. By (3.7) and the induction hypotheses, we have

$$d\left(D_{m+1,n}^{(p)} - \widetilde{D}_{m+1,n}^{(p)}\right) = c_{m,n}\omega + c_{m+1,n-1}\omega^*.$$

This shows that the class of $c_{m,n}\omega + c_{m+1,n-1}\omega^*$ in $H_{\text{rig}}^1(U_k/K)$ must be zero, hence $c_{m,n} = 0$. Our assertion now follows by induction. The existence of the solution for (3.7) follows from Proposition 3.8 below. \square

PROPOSITION 3.8. – *Let $D_{m,n}^{(p)}$ for $m \leq 0$ or $n \leq 0$ as in Theorem 3.6. Then for integers $m, n > 0$, there exist overconvergent functions $D_{m,n}^{(p)}$ on \mathcal{U}_K iteratedly satisfying the differential Equations (3.7) and the distribution relation $\sum_{z_1 \in E[\mathfrak{p}](K)} G_{m,n}^{(p)}(z + z_1) = 0$, where*

$$(3.9) \quad G_{m,n}^{(p)} = \sum_{k=0}^n \frac{(F_1^{(p)})^{n-k}}{(n-k)!} D_{m,k}^{(p)}.$$

Before proving Proposition 3.8, we first give a lemma.

LEMMA 3.10. – *The distribution relation above is true if $m = 0$ or $n = 0$.*

Proof. – The statement for $n = 0$ follows from the fact that $G_{m,0}^{(p)} = D_{m,0}^{(p)} = 0$. By the definition of $\Theta_{z_0,w_0}^{(p)}(z, w)$, the distribution relation in Proposition 2.20 for $\Theta_{z_0,w_0}^{(p)}(z, w)$ gives the relation

$$(3.11) \quad \sum_{z_1 \in E[\mathfrak{p}]} \Theta_{z_1,0}^{(p)}(z, w) = 0.$$

If $z_1 \in E[\mathfrak{p}]$, then (3.3) gives the equality $F_1^{(p)}(z + z_1) = F_{z_1,1}(z) - \bar{\pi}^{-1}F_1(\pi z)$. This equality and (3.5) show that

$$\Theta_{z_1,0}^{(p)}(z, w) = \exp\left(\frac{1}{\bar{\pi}}F_1(\pi z)w\right) \exp\left(F_1^{(p)}(z + z_1)w\right) \sum_{n \geq 0} L_n^{(p)}(z + z_1)w^{n-1}$$

for any $z_1 \in E[\mathfrak{p}]$. Then (3.11) translates to the equality

$$\sum_{z_1 \in E[\mathfrak{p}]} \exp\left(F_1^{(p)}(z + z_1)w\right) \sum_{n \geq 0} L_n^{(p)}(z + z_1)w^{n-1} = 0.$$

By noting that $D_{0,n}^{(p)} = L_n^{(p)}$ and writing out the coefficient of w^{n-1} in the above equality, we obtain the distribution relation for $m = 0$ and $n > 0$. \square

Proof of Proposition 3.8. – The statement for $m = 0$ or $n = 0$ is given by the previous lemma. We define $D_{m,n}^{(p)}$ by induction on $N = m + n$. Suppose $N > 1$ and $D_{a,b}^{(p)}$ exists for integers a, b such that $a + b < N$. Let m and n be integers > 0 such that $m + n = N$. Then $-D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^*$ defines a class in $H_{\text{rig}}^1(U_k/K)$. Since $H_{\text{rig}}^1(U_k/K) = K\omega \oplus K\omega^*$, there exist unique constants $c_{m,n}, c_{m,n}^* \in K$ such that the cohomology class of

$$-D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^* + c_{m,n}\omega + c_{m,n}^*\omega^*$$

vanishes in $H_{\text{rig}}^1(U_k/K)$. This implies that for any m and n such that $m + n = N$, there exist overconvergent functions $\widetilde{D}_{m,n}$ on \mathcal{U}_K such that

$$d\widetilde{D}_{m,n} = -D_{m-1,n}^{(p)}\omega - D_{m,n-1}^{(p)}\omega^* + c_{m,n}\omega + c_{m,n}^*\omega^*.$$

We define $\widetilde{G}_{m,n}$ as in (3.9), with the highest term $D_{m,n}^{(p)}$ replaced by $\widetilde{D}_{m,n}$. This function is again overconvergent, and satisfies

$$d\widetilde{G}_{m,n} = -\bar{\pi}^{-1}G_{m,n-1}^{(p)}[\pi]^*\omega^* - G_{m-1,n}^{(p)}\omega + c_{m,n}\omega + c_{m,n}^*\omega^*.$$

Let $C_{m,n} := \sum_{z_1 \in E[\mathfrak{p}]} \tau_{z_1}^* (\widetilde{G}_{m,n})$. The differential forms ω and $[\pi]^*\omega^*$ are invariant under translations $\tau_{z_1} : z \mapsto z + z_1$ for $z_1 \in E[\mathfrak{p}]$. Hence if we sum the above equation with respect to translations τ_{z_1} , then the distribution relations for $G_{m,n-1}^{(p)}$ and $G_{m-1,n}^{(p)}$ give the relation

$$dC_{m,n} = \sum_{z_1 \in E[\mathfrak{p}]} \tau_{z_1}^* (d\widetilde{G}_{m,n}) = N(\mathfrak{p})c_{m,n}\omega + c_{m,n}^* \sum_{z_1 \in E[\mathfrak{p}]} \tau_{z_1}^* (\omega^*).$$

Since the cohomology class of $\tau_{z_1}^* (\omega^*)$ is $\underline{\omega}^*$ in $H_{\text{rig}}^1(U_k/K)$, and since $C_{m,n}$ is an overconvergent function, the above formula implies that the cohomology class $N(\mathfrak{p})(c_{m,n}\omega + c_{m,n}^*\underline{\omega}^*)$ is zero in $H_{\text{rig}}^1(U_k/K)$. This implies that $c_{m,n} = c_{m,n}^* = 0$, hence the function $C_{m,n}$ is constant. We let

$$D_{m,n}^{(p)} := \widetilde{D}_{m,n} - (C_{m,n}/N(\mathfrak{p})).$$

Then $D_{m,n}^{(p)}$ satisfies (3.7). If we let $G_{m,n}^{(p)}$ as in (3.9), then we have $G_{m,n}^{(p)} = \widetilde{G}_{m,n} - (C_{m,n}/N(\mathfrak{p}))$. By the construction of $C_{m,n}$, the function $G_{m,n}^{(p)}$ satisfies the distribution relation. \square

3.2. p -adic elliptic polylogarithm functions and p -adic Eisenstein-Kronecker numbers

We next compare the p -adic elliptic polylogarithm function $D_{m,n}^{(p)}$ constructed in the previous section with the p -adic distribution used in defining the p -adic Eisenstein-Kronecker numbers. We first begin by describing the residue discs of E . Let $E_{\mathbb{C}_p}^{\text{an}}$ be the extension to \mathbb{C}_p of the rigid analytic space E_K^{an} . There is a morphism

$$\text{red} : E_{\mathbb{C}_p}^{\text{an}} \rightarrow E_{\overline{\mathbb{F}}_p}$$

called the reduction map. The inverse image of a point in $E_{\overline{\mathbb{F}}_p}$ is called a residue disc, which is an admissible open subset in $E_{\mathbb{C}_p}^{\text{an}}$ (See [BGR, Sec. 9.1.4, Prop. 5]). The formal parameter $s = -2x/y$ at the identity of the elliptic curve parameterizes the residue disc around $[0]$, and we have a natural inclusion

$$\iota : B^-(0, 1) \hookrightarrow E_{\mathbb{C}_p}^{\text{an}},$$

where $B^-(0, 1)$ is the rigid analytic open disc $B^-(0, 1) := \{s \in \mathbb{C}_p \mid |s| < 1\}$.

DEFINITION 3.12. – Suppose we are given a rigid analytic function $f(z)$ on $E_{\mathbb{C}_p}^{\text{an}}$. Then following the convention of the complex case (see Lemma 2.15), we denote by $f(z)|_{z=\lambda(s)}$ the rigid analytic function on $B^-(0, 1)$ obtained as the pull-back of $f(z)$ by ι .

We denote by D the translation invariant derivation on E defined by $df = D(f)\omega$ for any overconvergent function f . Then D restricts on each residue disc to the derivation $\partial_{s,\log} := \lambda'(s)^{-1}\partial_s$. We define a p -modified variant of $F_{z_0,b}(z)$ using the right hand side of Proposition 2.23.

DEFINITION 3.13. – For any integer $b \geq 0$, we define $F_{z_0,b}^{(p)}(z)$ to be the function

$$F_{z_0,b}^{(p)}(z) = F_{z_0,b}(z) - \bar{\pi}^{-b} F_{\pi z_0,b}(\pi z).$$

Note that we have

$$\Theta_{z_0,0}^{(p)}(z, w) = \sum_{b \geq 0} F_{z_0,b}^{(p)}(z) w^{b-1}.$$

The relation $\Theta_{z_0,0}^{(p)}(z, w) = \exp(F_{z_0,1}(z)w) \Xi_{z_0}^{(p)}(z, w)$ of Lemma 3.4 gives the relation

$$(3.14) \quad F_{z_0,b}^{(p)}(z) = \sum_{n=0}^b \frac{F_{z_0,1}^{(p)}(z)^{b-n}}{(b-n)!} L_n^{(p)}(z+z_0).$$

When $b = 1$, then $F_{z_0,1}^{(p)}(z) = F_1^{(p)}(z+z_0)$, which corresponds to a rational function defined over $\mathbb{K}(z_0)$. Assume now that z_0 is a torsion point in $E(\mathbb{C}_p)$ of order prime to \mathfrak{p} . Then Proposition 2.23 implies that we have

$$\widehat{F}_{z_0,b}^{(p)}(s) = \int_{\mathcal{O}_{\mathbb{K}_p}^\times} \exp(x\lambda(s)/\Omega_{\mathfrak{p}}) d\mu_{z_0,b}(x),$$

where $\widehat{F}_{z_0,b}^{(p)}(s) := F_{z_0,b}^{(p)}(z)|_{z=\lambda(s)}$.

DEFINITION 3.15. – Suppose z_0 is a torsion point of $E(\mathbb{C}_p)$ of order prime to \mathfrak{p} . For any integer m and b such that $b \geq 0$, we define $\widehat{E}_{z_0,m,b}^{(p)}(s)$ to be the function on $B^-(0, 1)$ given by the power series

$$\widehat{E}_{z_0,m,b}^{(p)}(s) := (-\Omega_{\mathfrak{p}})^m \int_{\mathcal{O}_{\mathbb{K}_p}^\times} x^{-m} \exp(x\lambda(s)/\Omega_{\mathfrak{p}}) d\mu_{z_0,b}(x).$$

By the definition of p -adic Eisenstein-Kronecker numbers, we have

$$(3.16) \quad \widehat{E}_{z_0,m,b}^{(p)}(0) = e_{-m,b}^{(p)}(z_0)$$

for any integer m and b such that $b \geq 0$. By construction, the function $\widehat{E}_{z_0,m,b}^{(p)}(s)$ satisfies the differential equation

$$\partial_{s, \log} \widehat{E}_{z_0,m,b}^{(p)}(s) = -\widehat{E}_{z_0,m-1,b}^{(p)}(s).$$

Since the p -adic distribution corresponding to the power series $\widehat{E}_{z_0,m,b}^{(p)}(s)$ has support on $\mathcal{O}_{\mathbb{K}_p}^\times$, integration by this distribution on $\mathfrak{p}\mathcal{O}_{\mathbb{K}_p}$ must be zero. Denote by \oplus the formal group law of \widehat{E} . By calculating the restriction to $\mathfrak{p}\mathcal{O}_{\mathbb{K}_p}$ of the distribution on $\mathcal{O}_{\mathbb{K}_p}$, we obtain the distribution relation

$$\sum_{s_1 \in E[\mathfrak{p}]} \widehat{E}_{z_0,m,b}^{(p)}(s \oplus s_1) = 0.$$

We will use these properties to give the relation between $\widehat{E}_{z_0,m,b}^{(p)}$ and $D_{m,n}^{(p)}$.

PROPOSITION 3.17. – For any integer $m, b \geq 0$, we have the equality

$$(3.18) \quad \widehat{E}_{z_0,m,b}^{(p)}(s) = \sum_{n=0}^b \frac{\widehat{F}_{z_0,1}^{(p)}(s)^{b-n}}{(b-n)!} D_{m,n}^{(p)}(z+z_0)|_{z=\lambda(s)}.$$

Proof. – We prove the statement by induction on $m \geq 0$. Denote by $\widetilde{E}_{z_0,m,b}(s)$ the right hand side of (3.18). When $m = 0$, then $D_{0,n}^{(p)} = L_n^{(p)}$, and the relation between $L_n^{(p)}(z)$ and $F_{z_0,b}^{(p)}(z)$ of (3.14) gives the relation

$$\widetilde{E}_{z_0,0,b}(s) = \sum_{n=0}^b \frac{\widehat{F}_{z_0,1}(z)^{b-n}}{(b-n)!} L_n^{(p)}(z+z_0) \Big|_{z=\lambda(s)} = \widehat{F}_{z_0,b}^{(p)}(s),$$

which is equal to $\widehat{E}_{z_0,0,b}^{(p)}(s)$ by definition. Suppose now that the statement is true for $m \geq 0$. Since $F_{z_0,1}(z) = F_1(z+z_0) - \bar{z}_0/A$, we have $dF_{z_0,1} = d(\tau_{z_0}^* F_1) = \tau_{z_0}^*(\omega^*)$. From this fact and the differential equation satisfied by $D_{m+1,n}^{(p)}$, we see that $d\widetilde{E}_{z_0,m+1,b}(s) = -\widetilde{E}_{z_0,m,b}(s)\omega$. Hence by induction,

$$(3.19) \quad \partial_{s, \log} \widetilde{E}_{z_0,m+1,b}(s) = -\widehat{E}_{z_0,m,b}^{(p)}(s).$$

Furthermore, we have from the definition of $G_{m+1,k}^{(p)}$ and (3.3) that

$$\widetilde{E}_{z_0,m+1,b}(s) = \sum_{k=0}^b \frac{\widehat{F}_{\pi z_0,1}([\pi]s)^{b-k}}{\pi^{b-k}(b-k)!} G_{m+1,k}^{(p)}(z+z_0) \Big|_{z=\lambda(s)}.$$

Since $\widehat{F}_{\pi z_0,1}([\pi](s \oplus s_1)) = \widehat{F}_{\pi z_0,1}([\pi]s)$, the distribution relation for $G_{m+1,k}^{(p)}$ gives the distribution relation

$$(3.20) \quad \sum_{s_1 \in E[p]} \widetilde{E}_{z_0,m+1,b}(s \oplus s_1) = 0.$$

The power series $\widehat{E}_{z_0,m+1,b}(s)$ and $\widetilde{E}_{z_0,m+1,b}(s)$ satisfy the same differential Equation (3.19), hence differ only by a constant. Since both power series satisfy the same distribution relation (3.20), we see that the constant is in fact zero, proving our assertion. \square

4. p -adic realization of the elliptic polylogarithm

We keep the notation of §2.2. In this section, we will explicitly determine the p -adic elliptic polylogarithm sheaf, by showing that the functions $D_{m,n}^{(p)}$ describe the Frobenius isomorphism of the p -adic elliptic polylogarithm.

4.1. Rigid syntomic cohomology

We first briefly recall the theory of filtered overconvergent F -isocrystals (or syntomic coefficients) and rigid syntomic cohomology developed in [3]. Let K be a finite unramified extension of \mathbb{Q}_p with ring of integers \mathcal{O}_K and residue field k . We fix an integer $q = p^m$ for some $m \geq 1$, and we denote by Frob_q the Frobenius $x \mapsto x^q$ on k . We let σ be the extension to \mathcal{O}_K and K of the Frobenius Frob_q on k .

Let X be a smooth scheme of finite type over \mathcal{O}_K , with smooth compactification $j : X \hookrightarrow \overline{X}$ over \mathcal{O}_K such that the complement $D := \overline{X} \setminus X$ is a relative strict normal crossing divisor over \mathcal{O}_K . Denote by \mathcal{X} and $\overline{\mathcal{X}}$ the formal completion of X and \overline{X} with respect to the special fiber. We assume in addition that there exists a Frobenius $\phi : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$ lifting the Frobenius Frob_q on $\overline{X}_k := \overline{X} \otimes k$, such that $\phi(\mathcal{X}) \subset \mathcal{X}$. Then the triple

$$\mathcal{X} := (X, \overline{X}, \phi)$$

is a syntomic datum in the sense of [3] Definition 1.1.

Suppose \mathcal{M} is an $\mathcal{O}_{\overline{X}_K}$ -module with integrable connection $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_{\overline{X}_K}^1(\log D)$ with logarithmic poles along D . Then as in [3], paragraph before Definition 1.8, we let $\mathcal{M}_{\text{rig}} := j^\dagger \mathcal{M}^{\text{an}}$ with connection $\nabla_{\text{rig}} : \mathcal{M}_{\text{rig}} \rightarrow \mathcal{M}_{\text{rig}} \otimes \Omega_{\overline{X}_K}^1$. A Frobenius structure on \mathcal{M}_{rig} is an isomorphism

$$\Phi : \phi^*(\mathcal{M}_{\text{rig}}) \xrightarrow{\cong} \mathcal{M}_{\text{rig}}$$

of $j^\dagger \mathcal{O}_{\overline{X}_K}$ -modules compatible with the connection. If such a structure exists on \mathcal{M}_{rig} , then by [9] Theorem 2.5.7, the connection ∇_{rig} on \mathcal{M}_{rig} becomes overconvergent (in the sense of [9] Definition 2.2.5), and the pair $(\mathcal{M}_{\text{rig}}, \Phi)$ gives a realization (in the sense of [9] p. 68) of an overconvergent F -isocrystal in $F\text{-Isoc}^\dagger(X_k/K)$.

DEFINITION 4.1. – We define the category of filtered overconvergent F -isocrystals on \mathcal{X} to be the category $S(\mathcal{X})$ whose objects are the 4-uples $\mathcal{M} := (\mathcal{M}, \nabla, F^\bullet, \Phi)$ consisting of:

1. \mathcal{M} is a coherent $\mathcal{O}_{\overline{X}_K}$ -module.
2. $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_{\overline{X}_K}^1(\log D)$ is an integrable connection on \mathcal{M} with logarithmic singularities along D .
3. F^\bullet is a descending exhaustive separated filtration by coherent $\mathcal{O}_{\overline{X}_K}$ -submodules on \mathcal{M} , satisfying the Griffiths transversality

$$\nabla(F^m \mathcal{M}) \subset F^{m-1} \mathcal{M} \otimes \Omega_{\overline{X}_K}^1(\log D).$$

4. $\Phi : \phi^*(\mathcal{M}_{\text{rig}}) \xrightarrow{\cong} \mathcal{M}_{\text{rig}}$ is a Frobenius structure on \mathcal{M}_{rig} .

REMARK 4.2. – We will denote any object in $M(X)$ by fonts such as $\mathcal{F}, \mathcal{G}, \mathcal{M}$. Any filtered overconvergent F -isocrystal will be denoted by fonts such as $\mathcal{F}, \mathcal{G}, \mathcal{M}$, and the same convention will be used for the logarithm sheaf $\mathcal{L}og, \mathcal{L}og$ and the polylogarithm sheaf \mathcal{P}, \mathcal{P} .

The Tate objects in this category is given as follows.

DEFINITION 4.3. – We define the *Tate object* in $S(\mathcal{X})$ to be the filtered overconvergent F -isocrystal $K(j) = (\mathcal{M}, \nabla, F^\bullet, \Phi)$, such that $\mathcal{M} = \mathcal{O}_{\overline{X}_K}$ with the trivial connection, the Hodge filtration is such that $F^{-j} \mathcal{M} = \mathcal{M}$ and $F^{-j+1} \mathcal{M} = 0$, and the Frobenius is multiplication by p^{-j} .

For any filtered overconvergent F -isocrystal \mathcal{M} in $S(\mathcal{X})$, we may define the de Rham and rigid cohomology $H_{\text{dR}}^i(\mathcal{X}, \mathcal{M})$ and $H_{\text{rig}}^i(\mathcal{X}, \mathcal{M}_{\text{rig}})$ of \mathcal{X} with coefficients in \mathcal{M} and \mathcal{M}_{rig} . The de Rham cohomology has a Hodge filtration induced from the Hodge filtration on \mathcal{M} , and rigid cohomology has a Frobenius $\phi : H_{\text{rig}}^i(\mathcal{X}, \mathcal{M}_{\text{rig}}) \otimes_{\sigma} K \xrightarrow{\cong} H_{\text{rig}}^i(\mathcal{X}, \mathcal{M}_{\text{rig}})$ induced from the Frobenius on \mathcal{M}_{rig} . There exists a natural homomorphism

$$(4.4) \quad \theta : H_{\text{dR}}^i(\mathcal{X}, \mathcal{M}) \rightarrow H_{\text{rig}}^i(\mathcal{X}, \mathcal{M}_{\text{rig}})$$

(see for example [2] §2 or [27]).

REMARK 4.5. – In order to define rigid syntomic cohomology, we consider filtered overconvergent F -isocrystals \mathcal{M} on \mathcal{X} satisfying the following additional conditions.

1. The Hodge to de Rham spectral sequence

$$E_1^{i,j} = R\Gamma^{i+j}(\overline{X}_K, \text{Gr}_F^i(\mathcal{M} \otimes \Omega_{\overline{X}_K}^\bullet(\log D))) \Rightarrow H_{\text{dR}}^{i+j}(\mathcal{X}, \mathcal{M})$$

degenerates at E_1 .

2. The θ of (4.4) is an isomorphism.

In what follows, suppose \mathcal{M} satisfies the condition of Remark 4.5. Any filtered overconvergent F -isocrystal which appears in our application will satisfy the above conditions.

DEFINITION 4.6. – We define the filtered Frobenius module $H^i(\mathcal{X}, \mathcal{M})$ of \mathcal{X} with coefficients in \mathcal{M} to be the cohomology group $H_{\text{rig}}^i(\mathcal{X}, \mathcal{M}_{\text{rig}})$ with its natural Frobenius and Hodge filtration induced from the Hodge filtration on de Rham cohomology $H_{\text{dR}}^i(\mathcal{X}, \mathcal{M})$ through the isomorphism θ .

REMARK 4.7. – In our previous paper, we imposed an additional condition, that the filtered Frobenius module $H^i(\mathcal{X}, \mathcal{M})$ defined above is a weakly admissible filtered Frobenius module in the sense of Fontaine ([16] 4.1.4). However, in the current paper for the supersingular case, we are considering Frobenius σ which is not absolute, hence this notion is not useful. The morphisms between $H^i(\mathcal{X}, \mathcal{M})$ may not a priori be strictly compatible with the Hodge filtration. For our application, any such morphism we use will be strictly compatible with the Hodge filtration, since it underlies a morphism of mixed Hodge structures.

We denote by $H_{\text{syn}}^i(\mathcal{X}, \mathcal{M})$ the rigid syntomic cohomology (or simply syntomic cohomology) of \mathcal{X} with coefficients in \mathcal{M} , defined in [3] Definition 2.4. Syntomic cohomology is related to extension classes as follows.

PROPOSITION 4.8 ([3] Theorem 1). – We have a canonical isomorphism

$$\text{Ext}_{S(\mathcal{X})}^i(K(0), \mathcal{M}) \xrightarrow{\cong} H_{\text{syn}}^i(\mathcal{X}, \mathcal{M})$$

when $i = 0, 1$, and $K(0)$ is the Tate object in $S(\mathcal{X})$.

Denote by $\mathcal{V} = (\text{Spec } \mathcal{O}_K, \text{Spec } \mathcal{O}_K, \sigma)$ the trivial syntomic datum. In this case, any object \mathcal{M} in $S(\mathcal{V})$ is simply a filtered Frobenius module, and

$$(4.9) \quad H_{\text{syn}}^i(\mathcal{V}, \mathcal{M}) = H^i \left[F^0 \mathcal{M} \xrightarrow{1-\phi^*} \mathcal{M} \right],$$

where the $F^0 \mathcal{M}$ in the complex on the right is in degree zero. The relation between rigid cohomology and syntomic cohomology is given as follows ([3] Proposition 2.7).

LEMMA 4.10. – We have the short exact sequence

$$0 \rightarrow H_{\text{syn}}^1(\mathcal{V}, H^i(\mathcal{X}, \mathcal{M})) \rightarrow H_{\text{syn}}^{i+1}(\mathcal{X}, \mathcal{M}) \rightarrow H_{\text{syn}}^0(\mathcal{V}, H^{i+1}(\mathcal{X}, \mathcal{M})) \rightarrow 0.$$

We have an inclusion $H_{\text{syn}}^0(\mathcal{V}, H^1(\mathcal{X}, \mathcal{M})) \hookrightarrow H_{\text{rig}}^1(\mathcal{X}, \mathcal{M}_{\text{rig}})$. Let

$$H_{\text{syn}}^1(\mathcal{X}, \mathcal{M}) \rightarrow H_{\text{rig}}^1(\mathcal{X}, \mathcal{M}_{\text{rig}}) \xrightarrow{\theta^{-1}} H_{\text{dR}}^1(\mathcal{X}, \mathcal{M}) \rightarrow H_{\text{dR}}^1(X_K, \mathcal{M})$$

be the morphism induced from the surjection of the short exact sequence in Lemma 4.10. The relation between this map and the isomorphism of Proposition 4.8 is given by the following.

LEMMA 4.11 ([3] Proposition 4.4). – *The following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ext}_{S(\mathcal{X})}^1(K(0), \mathcal{M}) & \xrightarrow{\mathrm{For}} & \mathrm{Ext}_{M(X_K)}^1(K(0), \mathcal{M}) \\ \cong \downarrow & & \cong \downarrow \\ H_{\mathrm{syn}}^1(\mathcal{X}, \mathcal{M}) & \longrightarrow & H_{\mathrm{dR}}^1(X_K, \mathcal{M}), \end{array}$$

where $M(X_K)$ is the category of coherent modules with integrable connection on X_K , and $\mathrm{For} : S(\mathcal{X}) \rightarrow M(X_K)$ is the functor obtained by forgetting the Hodge filtration and the Frobenius structure.

4.2. The logarithm sheaf

We now describe the logarithm sheaf. We fix a model E over $\mathcal{O}_{\mathbb{K}}$ of our elliptic curve as in (2.12). We let K be a finite unramified extension of $\mathbb{K}_{\mathfrak{p}}$ in \mathbb{C}_p . Let $\pi := \psi_{E/\mathbb{K}}(\mathfrak{p})$ as before, and let $[\pi] : E \rightarrow E$ be the multiplication induced on E . We define $\phi := [\pi] \otimes \sigma$ to be the Frobenius on $E_{\mathcal{O}_K} := E \otimes_{\mathcal{O}_{\mathbb{K}}} \mathcal{O}_K$ induced from $[\pi]$ and the Frobenius on \mathcal{O}_K . Then ϕ is a Frobenius morphism of degree $[K_{\mathfrak{p}} : \mathbb{Q}_p]$, and the triple $\mathcal{E} = (E_{\mathcal{O}_K}, E_{\mathcal{O}_K}, \phi)$ is a syntomic datum.

By Damerell's theorem, we have $e_2^* \in \mathbb{K} \subset K$. We let the notations be as in §1.4. We let $H^1(\mathcal{E})$ be the filtered Frobenius module defined in Definition 4.6 associated to the trivial object in $S(\mathcal{E})$. Then the underlying K -vector space of $H^1(\mathcal{E})$ is given by $H_{\mathrm{dR}}^1(E_K/K)$, hence we have $H^1(\mathcal{E}) = K\omega \oplus K\omega^*$, with Hodge filtration such that $F^0 H^1(\mathcal{E}) = H^1(\mathcal{E})$, $F^1 H^1(\mathcal{E}) = K\omega$, and $F^2 H^1(\mathcal{E}) = 0$. By the theory of complex multiplication, the action of the Frobenius $\phi^* : H^1(\mathcal{E}) \rightarrow H^1(\mathcal{E})$ is given by $\phi^*(\omega) = \pi\omega$ and $\phi^*(\omega^*) = \bar{\pi}\omega^*$, since $\phi = [\pi] \otimes \sigma$ and the class of ω^* is that of $d\bar{z}/A$. Hence the action of the Frobenius on $\mathcal{H} := H^1(\mathcal{E})^\vee$ is given by

$$\phi^*(\omega^\vee) = \pi^{-1}\omega^\vee, \quad \phi^*(\omega^{*\vee}) = \bar{\pi}^{-1}\omega^{*\vee}.$$

For any syntomic datum \mathcal{X} , we denote by $\mathcal{H}_{\mathcal{X}}$ the constant filtered overconvergent F -isocrystal on \mathcal{X} obtained as the pull-back of \mathcal{H} to \mathcal{X} . We have the natural isomorphism

$$H_{\mathrm{syn}}^0(\mathcal{V}, H^1(\mathcal{E}, \mathcal{H}_{\mathcal{E}})) = H_{\mathrm{syn}}^0(\mathcal{V}, \mathcal{H}^\vee \otimes \mathcal{H}) = \mathrm{Hom}_{S(\mathcal{V})}(\mathcal{H}, \mathcal{H}).$$

The short exact sequence of Lemma 4.10 in this case gives

$$(4.12) \quad 0 \rightarrow H_{\mathrm{syn}}^1(\mathcal{V}, \mathcal{H}) \rightarrow H_{\mathrm{syn}}^1(\mathcal{E}, \mathcal{H}_{\mathcal{E}}) \rightarrow \mathrm{Hom}_{S(\mathcal{V})}(\mathcal{H}, \mathcal{H}) \rightarrow 0.$$

The pullback $i_{[0]}^*$ by the identity $[0] : \mathcal{V} \rightarrow \mathcal{E}$ of the elliptic curve gives a splitting $i_{[0]}^* : H_{\mathrm{syn}}^1(\mathcal{E}, \mathcal{H}_{\mathcal{E}}) \rightarrow H_{\mathrm{syn}}^1(\mathcal{V}, \mathcal{H})$ of the above exact sequence.

DEFINITION 4.13. – We define the first logarithm sheaf $\mathrm{Log}^{(1)}$ to be any extension of $\mathcal{H}_{\mathcal{E}}$ by $K(0)$ in $S(\mathcal{E})$, whose extension class in

$$\mathrm{Ext}_{S(\mathcal{E})}^1(K(0), \mathcal{H}_{\mathcal{E}}) \cong H_{\mathrm{syn}}^1(\mathcal{E}, \mathcal{H}_{\mathcal{E}})$$

is mapped by the surjection of (4.12) to the identity and to zero by the splitting $i_{[0]}^*$. We define the N -th logarithm sheaf Log^N to be the N -th symmetric product of $\mathrm{Log}^{(1)}$.

Next, we explicitly construct $\mathcal{L}og^{(1)}$ in the p -adic case. We take an affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $E \otimes_{\mathcal{O}_K} \mathbb{K}$, and we fix (η_i, u_{ij}) and (u_i) as in Definition 1.35. By abuse of notation, we again denote by U_i the base extension $U_i \otimes_{\mathbb{K}} K$ of U_i to K . Then $\{U_i\}_{i \in I}$ is an affine covering of E_K . We denote by $\mathcal{L}og^{(1)}$ the module defined in Proposition 1.26, whose restriction to U_i is given by the module $\mathcal{L}og_i^{(1)}$ defined as

$$\mathcal{L}og_i^{(1)} := \mathcal{O}_{U_i} \underline{e}_i \bigoplus \mathcal{H}_{U_i},$$

with Hodge filtration as the direct sum and connection $\nabla(\underline{e}_i) = \underline{\omega}^\vee \otimes \omega + \underline{\omega}^{*\vee} \otimes \omega_i^*$. We define the Frobenius $\Phi : \phi^*(\mathcal{L}og^{(1)}) \rightarrow \mathcal{L}og^{(1)}$ as follows. The connection of $\phi^*\mathcal{L}og_i^{(1)}$ on $\phi^{-1}(U_i)$ is given by

$$\nabla_\phi(\underline{e}_i^\phi) = \underline{\omega}^\vee \otimes [\pi]^*\omega + \underline{\omega}^{*\vee} \otimes [\pi]^*\omega_i^*.$$

We let $F_1^{(p)}$ as in (3.2). We define $\xi_{ij}^{(p)}$ to be the rational function $\xi_{ij}^{(p)} := F_1^{(p)} - u_j + ([\pi]^*u_i/\bar{\pi})$ on $\phi^{-1}(U_i) \cap U_j$. Then the differential $d\xi_{ij}^{(p)} = \omega_j^* - [\pi]^*\omega_i^*/\bar{\pi}$ is holomorphic on $\phi^{-1}(U_i) \cap U_j$, hence $\xi_{ij}^{(p)}$ is also holomorphic on $\phi^{-1}(U_i) \cap U_j$. We define the Frobenius $\Phi : \phi^*(\mathcal{L}og^{(1)}) \rightarrow \mathcal{L}og^{(1)}$ to be the morphism given locally on $\phi^{-1}(U_i) \cap U_j$ by the morphisms

$$\Phi_{ij} : \phi^*(\mathcal{L}og_i^{(1)}) \xrightarrow{\cong} \mathcal{L}og_j^{(1)}$$

given by $\Phi_{ij}(\underline{e}_i^\phi) = \underline{e}_j - \xi_{ij}^{(p)}\omega^{*\vee}$, $\Phi_{ij}(\underline{\omega}^\vee) = \pi^{-1}\underline{\omega}^\vee$, $\Phi_{ij}(\omega^{*\vee}) = \bar{\pi}^{-1}\omega^{*\vee}$. Then Φ is an isomorphism compatible with the connection.

PROPOSITION 4.14. – *The first logarithm sheaf $\mathcal{L}og^{(1)}$ is given explicitly by the filtered overconvergent F -isocrystal whose underlying coherent module with connection is $\mathcal{L}og^{(1)}$, whose Hodge filtration is the direct sum of the Hodge filtration on each U_i , and the Frobenius $\Phi : \phi^*(\mathcal{L}og^{(1)}) \rightarrow \mathcal{L}og^{(1)}$ is given as above.*

Proof. – We check that the module described above satisfies the necessary properties of the first logarithm sheaf. By construction, the underlying coherent module with connection of $\mathcal{L}og^{(1)}$ maps to that of de Rham cohomology, which by definition corresponds to the identity. Hence by Lemma 4.11, the surjection of (4.12) maps $\mathcal{L}og^{(1)}$ to the identity. From our choice of u_i , we have $\zeta_i(0) = 0$ for $\zeta_i(z) = F_1(z) - u_i(z)$. Hence $\xi_{ij}^{(p)}(0) = 0$, since we have from the definition $\xi_{ij}^{(p)}(z) = \zeta_j(z) - [\pi]^*\zeta_i(z)/\bar{\pi}$. This implies that the splitting $\epsilon : i_{[0]}^*\mathcal{L}og^{(1)} \cong K \bigoplus \mathcal{H}$ given in (1.34) of underlying K -vector spaces gives a splitting compatible with the Hodge filtration and the Frobenius, proving our assertion. \square

Let $U_{\mathcal{O}_K} = E_{\mathcal{O}_K} \setminus [0]$ and $j : U_{\mathcal{O}_K} \hookrightarrow E_{\mathcal{O}_K}$ be the natural inclusion. The Frobenius Φ_{ij} on each $\phi^{-1}(U_i) \cap U_j$ paste together to give the Frobenius

$$\Phi_{\mathcal{L}}(\underline{e}^\phi) = \underline{e} - F_1^{(p)}\omega^{*\vee}$$

for $\mathcal{L} := j^\dagger \mathcal{O}_{\mathcal{E}_K} \underline{e} \bigoplus j^\dagger \mathcal{O}_{\mathcal{E}_K} \underline{\omega}^\vee \bigoplus j^\dagger \mathcal{O}_{\mathcal{E}_K} \omega^{*\vee}$ on \mathcal{E}_K . This gives the following.

COROLLARY 4.15. – *The base extension of the overconvergent F -isocrystal underlying the logarithm sheaf $\mathcal{L}og^N$ to $j^\dagger \mathcal{O}_{\mathcal{E}_K}$ is given as follows. The underlying module is*

$$\mathcal{L}^N = \bigoplus_{0 \leq m+n \leq N} j^\dagger \mathcal{O}_{\mathcal{E}_K} \omega^{m,n},$$

with connection given by $\nabla_{\mathcal{L}} = d + \nu$ and the Frobenius given by

$$\Phi_{\mathcal{L}}(\omega_{\phi}^{m,n}) = \frac{1}{\pi^m \bar{\pi}^n} \sum_{k=n}^{N-m} \frac{(-F_1^{(p)})^{k-n}}{(k-n)!} \omega^{m,k}.$$

The above overconvergent F -isocrystal underlies the filtered overconvergent F -isocrystal obtained as the restriction of $\mathcal{L}og^N$ to $\mathcal{U} = (U_{\theta_K}, E_{\theta_K}, \phi)$.

4.3. The polylogarithm sheaf

In this section, we define and calculate the p -adic polylogarithm sheaf. We again let $U_{\theta_K} = E_{\theta_K} \setminus [0]$ and we define \mathcal{U} to be the syntomic datum $\mathcal{U} = (U_{\theta_K}, E_{\theta_K}, \phi)$. We also let $\mathcal{D} = ([0], [0], \sigma)$. Similarly to the Hodge case, we have the following.

LEMMA 4.16. – *We have isomorphisms of filtered Frobenius modules*

$$\begin{aligned} \varprojlim_N H^0(\mathcal{U}, \mathcal{L}og^N(1)) &\xrightarrow{\cong} \varprojlim_N H^0(\mathcal{E}, \mathcal{L}og^N(1)) = 0 \\ \text{res} : \varprojlim_N H^1(\mathcal{U}, \mathcal{L}og^N(1)) &\xrightarrow{\cong} \varprojlim_N H^0(\mathcal{D}, i_{[0]}^* \mathcal{L}og^N) = \varprojlim_N i_{[0]}^* \mathcal{L}og^N. \end{aligned}$$

Proof. – By Tsuzuki [29], the localization maps in rigid cohomology are compatible with the Frobenius. The isomorphisms follow from the calculation for de Rham cohomology given in Lemma 1.37. The Hodge filtration is strictly compatible, since the isomorphism for de Rham cohomology underlies an isomorphism of mixed Hodge structures. \square

The exact sequence of Lemma 4.10 gives the short exact sequence

$$\begin{aligned} 0 \rightarrow H_{\text{syn}}^1(\mathcal{V}, H^0(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1))) &\rightarrow H_{\text{syn}}^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1)) \\ &\rightarrow H_{\text{syn}}^0(\mathcal{V}, H^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1))) \rightarrow 0. \end{aligned}$$

By Lemma 4.16, the projective limit of $H^0(\mathcal{U}, \mathcal{L}og^N)$ is zero. Hence the above sequence gives a natural isomorphism

$$\varprojlim_N H_{\text{syn}}^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1)) \xrightarrow{\cong} \varprojlim_N H_{\text{syn}}^0(\mathcal{V}, H^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1))).$$

We have a natural isomorphism

$$H_{\text{syn}}^0(\mathcal{V}, H^0(\mathcal{D}, \mathcal{H}_{\mathcal{E}}^{\vee} \otimes i_{[0]}^* \mathcal{L}og^N)) \xrightarrow{\cong} H_{\text{syn}}^0(\mathcal{V}, \mathcal{H}^{\vee} \otimes i_{[0]}^* \mathcal{L}og^N) \cong \mathbb{Q}_p,$$

where the last map is obtained by mapping the identity element in $\mathcal{H}^{\vee} \otimes \mathcal{H} = \text{Hom}(\mathcal{H}, \mathcal{H})$ to $1 \in \mathbb{Q}_p$, and is an isomorphism by reasons of weights. Combining the above isomorphisms, we obtain an isomorphism

$$(4.17) \quad \varprojlim_N H_{\text{syn}}^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \mathcal{L}og^N(1)) \xrightarrow{\cong} \mathbb{Q}_p.$$

DEFINITION 4.18. – We define the *polylogarithm class* to be a system of classes $\text{pol}_{\text{syn}}^N \in H_{\text{syn}}^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \text{Log}^N(1))$ which maps to 1 through (4.17). We define the *elliptic polylogarithm sheaf* on \mathcal{U} to be a system of filtered overconvergent F -isocrystals \mathcal{P}^N on \mathcal{U} given as an extension

$$0 \rightarrow \text{Log}^N(1) \rightarrow \mathcal{P}^N \rightarrow \mathcal{H}_{\mathcal{U}} \rightarrow 0$$

whose extension class corresponds to $\text{pol}_{\text{syn}}^N$ in

$$\text{Ext}_{S(\mathcal{U})}^1(\mathcal{H}_{\mathcal{U}}, \text{Log}^N(1)) \cong H_{\text{syn}}^1(\mathcal{U}, \mathcal{H}_{\mathcal{U}}^{\vee} \otimes \text{Log}^N(1)).$$

We now prove the main result of this paper, explicitly describing the p -adic elliptic polylogarithm sheaf \mathcal{P}^N on \mathcal{U} .

THEOREM 4.19. – *The elliptic polylogarithm sheaf \mathcal{P}^N on \mathcal{U} is given by the filtered overconvergent F -isocrystal $\mathcal{P}^N := (\mathcal{P}^N, \nabla, F^{\bullet}, \Phi)$ defined as follows.*

1. \mathcal{P}^N is the coherent module $\mathcal{P}^N = \mathcal{H}_E \oplus \text{Log}^N$, with integrable connection $\nabla(\underline{\omega}^{\vee}) = \underline{\omega}^{\vee}$ and $\nabla(\underline{\omega}^{*\vee}) = \underline{\omega}^{*\vee}$.
2. F^{\bullet} is the filtration on \mathcal{P}^N given by $F^m \mathcal{P}^N = F^m \mathcal{H}_E \oplus F^{m+1} \text{Log}^N$.
3. We denote by $\Phi_{\mathcal{P}}$ the morphism $\Phi_{\mathcal{P}} : \phi^* \mathcal{P}_{\text{rig}}^N \rightarrow \mathcal{P}_{\text{rig}}^N$ which extends the Frobenius $\Phi_{\mathcal{L}(1)} := N(\mathfrak{p})^{-1} \Phi_{\mathcal{L}}$ on $\text{Log}_{\text{rig}}^N(1)$ and is given by

$$\begin{aligned} \Phi_{\mathcal{P}}(\pi \underline{\omega}^{\vee}) &= \underline{\omega}^{\vee} - \sum_{n=0}^N \frac{(-F_1^{(p)})^{n+1}}{(n+1)!} \underline{\omega}^{0,n} + \sum_{m=1}^N \sum_{n=0}^{N-m} D_{m,n+1}^{(p)} \underline{\omega}^{m,n} \\ \Phi_{\mathcal{P}}(\pi \underline{\omega}^{*\vee}) &= \underline{\omega}^{*\vee} + \sum_{m=0}^N \sum_{n=0}^{N-m} D_{m+1,n}^{(p)} \underline{\omega}^{m,n}. \end{aligned}$$

By definition, $\Phi_{\mathcal{P}}$ is compatible with the projection $\mathcal{P}^{N+1} \rightarrow \mathcal{P}^N$.

Proof. – Since the polylogarithm sheaf as an extension class is uniquely characterized by the property of Definition 4.18, it is sufficient to prove that \mathcal{P}^N is a filtered overconvergent F -isocrystal satisfying the required property. By Lemma 4.11, we may take the underlying module with connection of the polylogarithm sheaf to be that of the de Rham realization of §1, and we may take the Hodge filtration as in §3, as the direct sum. Hence it is sufficient to prove that $\Phi_{\mathcal{P}}$ is compatible with the connection and indeed gives a Frobenius structure on the overconvergent isocrystal underlying \mathcal{P}^N . This fact is Proposition 4.20 below. \square

The rest of this subsection is devoted to proving the following proposition.

PROPOSITION 4.20. – *Let the notations be as in Theorem 4.19. Then the morphism $\Phi_{\mathcal{P}} : \phi^* \mathcal{P}_{\text{rig}}^N \rightarrow \mathcal{P}_{\text{rig}}^N$ is compatible with the connection on $\mathcal{P}_{\text{rig}}^N$.*

The proof of Proposition 4.20 will be given at the end of this section. We first start with a lemma concerning the action of $\Phi_{\mathcal{P}}$ on $\underline{\omega}^{\vee}$ and $\underline{\omega}^{*\vee}$. Recall that the Frobenius ϕ on \mathcal{E} is defined as $\phi := [\pi] \otimes \sigma$.

LEMMA 4.21. – *We have*

$$\begin{aligned} (1 - \pi\Phi_{\mathcal{L}(1)}) \omega^\vee &= -\underline{\omega}^{0,0} \otimes \left(1 - \frac{[\pi]^*}{\bar{\pi}}\right) \omega^* + \sum_{k=1}^N \frac{(-F_1^{(p)})^k}{k!} \underline{\omega}^{0,k} \otimes \frac{[\pi]^*}{\bar{\pi}} \omega^* \\ &\quad + \sum_{k=0}^{N-1} \left(L_{k+1}^{(p)} + \frac{(-F_1^{(p)})^{k+1}}{(k+1)!} \right) \underline{\omega}^{1,k} \otimes \omega, \\ (1 - \bar{\pi}\Phi_{\mathcal{L}(1)}) \omega^{*\vee} &= \sum_{k=1}^N L_k^{(p)} \underline{\omega}^{0,k} \otimes \omega. \end{aligned}$$

Proof. – The action of the Frobenius on ω^\vee is given by

$$\begin{aligned} \Phi_{\mathcal{L}(1)}(\omega^\vee) &= -\frac{1}{N(\mathfrak{p})} \sum_{k=0}^N \frac{(-F_1^{(p)})^k}{k!} \underline{\omega}^{0,k} \otimes [\pi]^* \omega^* \\ &\quad + \frac{1}{\pi} \sum_{k=0}^{N-1} \sum_{n=0}^k \frac{([\pi]^* L_{n+1})(-F_1^{(p)})^{k-n}}{\bar{\pi}^{n+1}(k-n)!} \underline{\omega}^{1,k} \otimes \omega. \end{aligned}$$

The last sum may be expressed as

$$\sum_{n=0}^k \frac{([\pi]^* L_{n+1})(-F_1^{(p)})^{k-n}}{\bar{\pi}^{n+1}(k-n)!} = \left(\sum_{n=0}^{k+1} \frac{([\pi]^* L_n)(-F_1^{(p)})^{k+1-n}}{\bar{\pi}^n(k+1-n)!} \right) - \frac{(-F_1^{(p)})^{k+1}}{(k+1)!}.$$

If we expand the L_n in the first sum of the right hand side, we have

$$\sum_{b=0}^{k+1} \sum_{n=b}^{k+1} \frac{(-[\pi]^* F_1)^{n-b} (-F_1^{(p)})^{k+1-n} [\pi]^* F_b}{\bar{\pi}^{n-b}(n-b)!(k+1-n)! \bar{\pi}^b} = \sum_{b=0}^{k+1} \frac{(-F_1)^{k+1-b} [\pi]^* F_b}{(k+1-b)! \bar{\pi}^b}.$$

Hence we have

$$L_{k+1} - \sum_{n=0}^k \frac{([\pi]^* L_{n+1})(-F_1^{(p)})^{k-n}}{\bar{\pi}^{n+1}(k-n)!} = L_{k+1}^{(p)} + \frac{(-F_1^{(p)})^{k+1}}{(k+1)!}$$

as desired. The second equality may be proved in a similar fashion, again by direct calculation. \square

Proof of Proposition 4.20. – For the basis $\pi\omega^\vee$, if we calculate the composition of the connection with the Frobenius, then we have $\Phi_{\mathcal{F}} \circ \nabla_{\mathcal{F}}(\pi\omega^\vee) = \pi\Phi_{\mathcal{L}(1)}(\omega^\vee)$ and

$$\nabla_{\mathcal{F}} \circ \Phi_{\mathcal{F}}(\pi\omega^\vee) = \nabla_{\mathcal{F}} \left(\omega^\vee - \sum_{n=0}^N \frac{(-F_1^{(p)})^{n+1}}{(n+1)!} \underline{\omega}^{0,n} + \sum_{m=1}^N \sum_{n=0}^{N-m} D_{m,n+1}^{(p)} \underline{\omega}^{m,n} \right).$$

By the differential equation satisfied by $D_{m,n}^{(p)}$ and the fact that $dF_1 = \omega^*$, we see that the above is equal to

$$\begin{aligned} \omega^\vee + \sum_{n=0}^N \frac{(-F_1^{(p)})^n}{n!} \underline{\omega}^{0,n} \otimes dF_1^{(p)} - \sum_{n=0}^{N-1} \frac{(-F_1^{(p)})^{n+1}}{(n+1)!} \underline{\omega}^{0,n+1} \otimes \omega^* \\ - \sum_{k=0}^{N-1} \left(L_{k+1}^{(p)} + \frac{(-F_1^{(p)})^{k+1}}{(k+1)!} \right) \underline{\omega}^{1,k}. \end{aligned}$$

Hence Lemma 4.21 gives the compatibility $\Phi_{\mathcal{D}} \circ \nabla_{\mathcal{D}}(\pi\omega^{\vee}) = \nabla_{\mathcal{D}} \circ \Phi_{\mathcal{D}}(\pi\omega^{\vee})$ of the Frobenius with the connection on $\pi\omega^*$. A similar calculation gives the compatibility $\Phi_{\mathcal{D}} \circ \nabla_{\mathcal{D}}(\bar{\pi}\omega^{*\vee}) = \nabla_{\mathcal{D}} \circ \Phi_{\mathcal{D}}(\bar{\pi}\omega^{*\vee})$ of the Frobenius on $\bar{\pi}\omega^{*\vee}$. This proves our assertion. \square

5. Specialization of the p -adic elliptic polylogarithm sheaf

5.1. Calculation of the specialization

We next calculate the specialization of the p -adic elliptic polylogarithm to non-zero torsion point z_0 of $E(K)$ of order prime to p . Note that by the theory of complex multiplication, we have a commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathcal{O}_K & \xrightarrow{i_{z_0}} & E \otimes_{\mathcal{O}_K} \mathcal{O}_K \\ \sigma \uparrow & & \phi \uparrow \\ \text{Spec } \mathcal{O}_K & \xrightarrow{i_{z_0}} & E \otimes_{\mathcal{O}_K} \mathcal{O}_K. \end{array}$$

Hence the induced map $i_{z_0} : \mathcal{V} \rightarrow \mathcal{E}$ is a morphism of syntomic data. Denote by $i_{z_0}^* \mathcal{L}og^N$ the filtered Frobenius module defined as the pullback of the logarithm sheaf $\mathcal{L}og^N$ to z_0 . Then

$$i_{z_0}^* \mathcal{L}og^N = \bigoplus_{0 \leq m+n \leq N} K \omega^{m,n},$$

with the Hodge filtration given by the direct sum and the Frobenius $\Phi : K \otimes_{\sigma} i_{z_0}^* \mathcal{L}og^N \xrightarrow{\cong} i_{z_0}^* \mathcal{L}og^N$ given by

$$\Phi(\omega^{m,n}) := \frac{1}{\pi^m \bar{\pi}^n} \sum_{k=n}^{N-m} \frac{(-F_1^{(p)}(z_0))^{k-n}}{(k-n)!} \omega^{m,k}.$$

In order to calculate the specialization of the polylogarithm sheaf, we will first describe the splitting of the sheaf $i_{z_0}^* \mathcal{L}og^N$.

LEMMA 5.1. – We have $F_{z_0,1}(0) \in K$ and

$$\left(1 - \frac{\sigma}{\bar{\pi}}\right) F_{z_0,1}(0) = F_1^{(p)}(z_0).$$

Proof. – Let n be the smallest integer > 1 such that $\pi^n z_0 = z_0$ in $E(K)$. Then repeatedly using the equality $F_{z_0,1}(z) = F_1^{(p)}(z + z_0) + \bar{\pi}^{-1} F_{\pi z_0,1}(\pi z)$ of (3.3), we have

$$F_{z_0,1}(0) = (1 - \bar{\pi}^{-n})^{-1} \sum_{k=0}^{n-1} \bar{\pi}^{-k} F_1^{(p)}(\pi^k z_0).$$

Since $F_1^{(p)}(z)$ is a rational function defined over \mathbb{K} , the above equality shows that $F_{z_0,1}(0) \in K$. Furthermore, we have by the theory of complex multiplication $(F_1^{(p)}(z_0))^{\sigma} = F_1^{(p)}(\pi z_0)$. Then the above formula shows that $\sigma(F_{z_0,1}(0)) = F_{\pi z_0,1}(0)$. Hence our assertion now follows from the fact that $F_1^{(p)}(z_0) = F_{z_0,1}(0) - \bar{\pi}^{-1} F_{\pi z_0,1}(0)$. \square

We first describe the unique splitting of $i_{z_0}^* \mathcal{L}og^{(1)}$ as a filtered Frobenius module. We define an element \underline{e}' which corresponds to the basis denoted by the same character in the Hodge case.

LEMMA 5.2. – Let $\underline{e}' = \underline{e} - F_{z_0,1}(0)\underline{\omega}^{*\vee} \in F^0(i_{z_0}^* \mathcal{L}og^{(1)})$. Then the Frobenius acts on this element as $\Phi(\underline{e}') = \underline{e}'$. In particular, by mapping the basis of $K(0)$ to \underline{e}' , we have a splitting of filtered Frobenius modules of the sequence

$$0 \rightarrow \mathcal{H} \rightarrow i_{z_0}^* \mathcal{L}og^{(1)} \rightarrow K(0) \rightarrow 0.$$

Proof. – Since we have $\Phi(\underline{\omega}^{*\vee}) = \underline{\omega}^{*\vee}/\bar{\pi}$, the previous lemma gives the equality $\sigma(F_{z_0,1}(0))\Phi(\underline{\omega}^{*\vee}) = (F_{z_0,1}(0) - F_1^{(p)}(z_0))\underline{\omega}^{*\vee}$. Since $\Phi(\underline{e}) = \underline{e} - F_1^{(p)}(z_0)\underline{\omega}^{*\vee}$, we have $\Phi(\underline{e}') = \Phi(\underline{e}) - \sigma(F_{z_0,1}(0))\Phi(\underline{\omega}^{*\vee}) = \underline{e}'$ as desired. \square

We let $\underline{e}^{m,n} := \underline{e}'^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a!$, where $a = N - m - n$. This basis gives a splitting of $i_{z_0}^* \mathcal{L}og^N$ as filtered Frobenius modules as follows. See Lemma A.28 for the splitting principle in the Hodge case.

LEMMA 5.3 (Splitting Principle). – We have a splitting

$$i_{z_0}^* \mathcal{L}og^N \cong \prod_{j=0}^N \text{Sym}^j \mathcal{H}$$

of filtered Frobenius modules, given by mapping $\underline{e}^{m,n}$ to $\underline{\omega}^{\vee m} \underline{\omega}^{*\vee n}$.

Let z_0 be a torsion point in $E(K)$ of order prime to \mathfrak{p} as above. We denote by $i_{z_0}^* \mathcal{P}^N$ the pullback of the polylogarithm sheaf \mathcal{P}^N to z_0 . We now describe $i_{z_0}^* \mathcal{P}^N$ using the basis $\underline{e}^{m,k}$. By construction of the polylogarithm sheaf, $i_{z_0}^* \mathcal{P}^N$ is a K -vector space

$$i_{z_0}^* \mathcal{P}^N = \mathcal{H} \bigoplus i_{z_0}^* \mathcal{L}og^N,$$

endowed with a Frobenius $\Phi : K \otimes_{\sigma} i_{z_0}^* \mathcal{P}^N \xrightarrow{\cong} i_{z_0}^* \mathcal{P}^N$ induced from that of \mathcal{P}^N . Using the basis $\underline{e}^{m,k}$ of $i_{z_0}^* \mathcal{L}og^N$, we have the following.

PROPOSITION 5.4. – Let $e_{a,b}^{(p)}(z_0)$ be the p -adic Eisenstein-Kronecker numbers defined in Definition 2.19. The Frobenius on $i_{z_0}^* \mathcal{P}^N$ is expressed as

$$\begin{aligned} \Phi(\pi \underline{\omega}^{\vee}) &= \underline{\omega}^{\vee} + \sum_{k=0}^N \left(1 - \frac{\sigma}{\bar{\pi}^{k+1}}\right) \frac{F_{z_0,1}(0)^{k+1}}{(k+1)!} \underline{e}^{0,k} \\ &\quad + \sum_{m=1}^N \sum_{k=0}^{N-m} e_{-m,k+1}^{(p)}(z_0) \underline{e}^{m,k}, \\ \Phi(\bar{\pi} \underline{\omega}^{*\vee}) &= \underline{\omega}^{*\vee} + \sum_{m=0}^N \sum_{k=0}^{N-m} e_{-m-1,k}^{(p)}(z_0) \underline{e}^{m,k}. \end{aligned}$$

Proof. – By definition of $\underline{\omega}^{m,n}$ and \underline{e}' , we have $\underline{\omega}^{m,n} := \underline{e}'^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a! = (\underline{e}' + F_{z_0,1}(0)\underline{\omega}^{*\vee})^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a!$ for $a = N - m - n$. This implies that

$$\underline{\omega}^{m,n} = \sum_{k=n}^{N-m} \frac{F_{z_0,1}(0)^{k-n}}{(k-n)!} \underline{e}^{m,k}.$$

Using this formula, we have

$$\sum_{n=0}^N \frac{(-F_1^{(p)}(z_0))^{n+1}}{(n+1)!} \underline{\omega}^{0,n} = \sum_{k=0}^N \sum_{n=0}^k \frac{(-F_1^{(p)}(z_0))^{n+1} F_{z_0,1}(0)^{k-n}}{(n+1)!(k-n)!} \underline{e}^{0,k}.$$

By Lemma 5.1, the sum in the coefficient of $\underline{e}^{0,k}$ is thus

$$\frac{F_{z_0,1}(0)^{k+1}}{(k+1)!} - \frac{\sigma(F_{z_0,1}(0))^{k+1}}{\bar{\pi}^{k+1}(k+1)!} = \left(1 - \frac{\sigma}{\bar{\pi}^{k+1}}\right) \frac{F_{z_0,1}(0)^{k+1}}{(k+1)!}.$$

For the coefficients for the other basis, we have

$$\sum_{n=0}^{N-m} D_{m,n+1}^{(p)}(z_0) \underline{\omega}^{m,n} = \sum_{k=0}^{N-m} \sum_{n=0}^k D_{m,n+1}^{(p)}(z_0) \frac{F_{z_0,1}(0)^{k-n}}{(k-n)!} \underline{e}^{m,k}.$$

By (3.18), the coefficient of $\underline{e}^{m,k}$ is given by

$$\sum_{n=1}^{k+1} D_{m,n}^{(p)}(z_0) \frac{F_{z_0,1}(0)^{k+1-n}}{(k+1-n)!} = \widehat{E}_{z_0,m,k+1}^{(p)}(0),$$

where the equality holds since $D_{m,0}^{(p)} \equiv 0$. This and (3.16) give the first equality. The second equality may be proved by direct calculation in a similar fashion. \square

5.2. Specialization in syntomic cohomology

We now calculate the extension class of the specialization of the elliptic polylogarithm in syntomic cohomology. We let K be the maximal unramified extension of \mathbb{Q}_p , and let $\mathcal{V} = \text{Spec } \mathcal{O}_K$. As in §5.1, let $z_0 \in E(K)$ be a non-zero torsion point of order prime to p . The specialization of the elliptic polylogarithm $i_{z_0}^* \mathcal{P}^N$ gives a cohomology class

$$\text{pol}_{z_0}^N \in \text{Ext}_{S(\mathcal{V})}^1(\mathcal{H}, \text{Log}(1)) = H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N(1)).$$

We use the results of the previous section to calculate this element explicitly.

LEMMA 5.5. – *We have an isomorphism of K -vector spaces*

$$(5.6) \quad H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N(1)) \xrightarrow{\cong} \mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N / \bigoplus_{n=0}^N K \underline{\omega} \otimes \underline{e}^{0,n}.$$

Proof. – By (4.9), we have $H_{\text{syn}}^1(\mathcal{V}, M) \xrightarrow{\cong} M/(1-\Phi)F^0M$ for any filtered Frobenius module M in $S(\mathcal{V})$, where the map is given by mapping the extension class $[M'] \in \text{Ext}_{S(\mathcal{V})}^1(K(0), M) = H_{\text{syn}}^1(\mathcal{V}, M)$ of an extension

$$0 \rightarrow M \rightarrow M' \rightarrow K(0) \rightarrow 0$$

to $(1-\Phi)e \in M$, where e is any lifting of the fixed basis of $K(0)$ to F^0M' . The lemma follows from the fact that $F^0(\mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N(1)) = \bigoplus_{n=0}^N K \underline{\omega} \otimes \underline{e}^{0,n}$ and that $(1-\Phi)$ gives a surjection on this space. \square

THEOREM 5.7. – *Let z_0 be a non-zero torsion point in $E(K)$ of order prime to \mathfrak{p} . Then the image of $\text{pol}_{z_0}^N$ through the isomorphism*

$$\text{Ext}_{S(\gamma)}^1(\mathcal{H}, i_{z_0}^* \text{Log}^N(1)) \xrightarrow{\cong} \mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N / \bigoplus_{n=0}^N K \underline{\omega} \otimes \underline{e}^{0,n}$$

of (5.6) is

$$- \sum_{m=1}^N \sum_{k=0}^{N-m} e_{-m, k+1}^{(p)}(z_0) \underline{\omega} \otimes \underline{e}^{m,k} - \sum_{m=0}^N \sum_{k=1}^{N-m} e_{-m-1, k}^{(p)}(z_0) \underline{\omega}^* \otimes \underline{e}^{m,k},$$

where the $e_{a,b}^{(p)}$ are the p -adic Eisenstein-Kronecker numbers defined in Definition 2.19.

Proof. – The theorem follows from Proposition 5.4, (5.6) and the definition of the p -adic Eisenstein-Kronecker numbers. The terms containing $\underline{\omega} \otimes \underline{e}^{0,n}$ maps to zero, since $\underline{\omega} \otimes \underline{e}^{0,n} \in F^0(\mathcal{H}^\vee \otimes i_{z_0}^* \text{Log}^N(1))$. □

The real Hodge analogue of the above result is given in Theorem A.29.

5.3. Relation to the p -adic L -function

Suppose E has good ordinary reduction over $p \geq 5$. Using the calculations in §2.4, we may interpret the result of Theorem 5.7 in terms of special values of p -adic L -functions. Let the notations be as in §2.4. In particular, let Ω be such that $\Gamma = \Omega \mathcal{O}_{\mathbb{K}}$. We let \mathfrak{g} be an integral ideal of $\mathcal{O}_{\mathbb{K}}$ prime to p and divisible by the conductor \mathfrak{f} of $\psi_{E/\mathbb{K}}$, and we fix a generator g of \mathfrak{g} . For any $\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times$, the element $\alpha_0 \Omega/g$ defines a primitive \mathfrak{g} -torsion point in $E(\mathbb{C}_p)$. We let K be the maximal unramified extension of \mathbb{Q}_p , so that $\alpha_0 \Omega/g$ defines a point in $E(K)$ for any $\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times$.

For any torsion point $z_0 \in E(K)$, we identify $i_{z_0}^* \text{Log}$ with $\prod_{k \geq 0} \text{Sym}^k \mathcal{H}$ through the splitting principle. We define the projection map

$$(5.8) \quad \varrho : H_{\text{syn}}^1(\mathcal{V}, \mathcal{H}^\vee \otimes \prod_{k \geq 0} \text{Sym}^k \mathcal{H}(1)) \rightarrow H_{\text{syn}}^1(\mathcal{V}, \prod_{k \geq 0} \text{Sym}^k \mathcal{H}(1))$$

to be the morphism induced from $\mathcal{H}^\vee \otimes \prod_{k \geq 0} \text{Sym}^k \mathcal{H} \rightarrow \prod_{k \geq 0} \text{Sym}^k \mathcal{H}$ given on the basis by $\underline{\omega} \otimes \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} \mapsto \underline{\omega}^{\vee m-1} \underline{\omega}^{*\vee n}$ and $\underline{\omega}^* \otimes \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} \mapsto \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n-1}$, with the convention that $\underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} = 0$ if m or $n < 0$. (Compare [17] Lemma 2.2.3). As in (5.6), we have an isomorphism

$$(5.9) \quad H_{\text{syn}}^1(\mathcal{V}, \prod_{j \geq 0} \text{Sym}^j \mathcal{H}(1)) \cong \prod_{j \geq 0} \text{Sym}^j \mathcal{H}.$$

Then we have the following.

COROLLARY 5.10. – *Let $\chi_p : (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times \rightarrow \mathbb{C}_p^\times$ be a finite p -adic character. The image by the projection (5.8) of*

$$\sum_{\alpha_0 \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{g})^\times} \chi_p(\alpha_0) \text{pol}_{\alpha_0 \Omega/g}$$

in $H_{\text{syn}}^1(\mathcal{V}, \prod_{j \geq 0} \text{Sym}^j \mathcal{H}(1))$ through the isomorphism of (5.9) is given in terms of the p -adic L -function $L_p(\varphi_{m,k})$ by

$$-2 \sum_{m,k \geq 0} \left((1 - \varphi_{m,k}(\bar{\pi}))^{-1} \frac{L_p(\varphi_{m,k})}{\Omega_{\mathfrak{p}}^{k-m}} \right) \frac{\omega^{\vee m} \omega^{* \vee k}}{g^m \bar{g}^k},$$

where $\varphi_{m,k}$ is the p -adic character $\varphi_{m,k} := \chi_p \kappa_1^m \kappa_2^k$ on $\mathbb{X} = \varprojlim_n (\mathcal{O}_{\mathbb{K}}/\mathfrak{g}p^n)^\times$.

Proof. – Our assertion follows from Theorem 5.7 and Proposition 2.27. □

Appendix

The real Hodge realization

The Hodge realization of the elliptic polylogarithm was calculated in the original paper by Beilinson and Levin [7], as well as [32]. In the above papers, the Hodge realization was expressed in terms of the q -averaged polylogarithm function, obtained from the classical polylogarithm function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In the appendix, closely following our method of the p -adic case, we will describe how to explicitly describe the \mathbb{R} -Hodge realization of the elliptic polylogarithm using functions given by certain iterated integrals starting from the connection functions $L_n(z)$.

A.1. Real analytic elliptic polylogarithm functions

In this subsection, we will define the Eisenstein-Kronecker functions $E_{m,b}(z)$ and the real analytic elliptic polylogarithm function $G_{m,b}(z)$. We first investigate the properties of Eisenstein-Kronecker-Lerch series viewed as a function in z and w . For $z, w \in (\mathbb{C} \setminus \Gamma)$, we let

$$K_a(z, w, s) := K_a^*(z, w, s).$$

The right hand side is defined even for $z, w \in \Gamma$, whereas the left hand side is not. Then $K_a(z, w, s)$ is a \mathcal{C}^∞ -function for (z, w) for any integer a . Moreover, the derivatives of $K_a(z, w, s)$ with respect to $\partial_z, \partial_{\bar{z}}, \partial_w, \partial_{\bar{w}}$ are all analytic in s . The Eisenstein-Kronecker-Lerch series for various integers a and s are related by the following differential equations.

LEMMA A.1. – *Let a be an integer, and consider $K_a(z, w, s)$ to be a function for $z, w \in (\mathbb{C} \setminus \Gamma)$. Then $K_a(z, w, s)$ satisfies the differential equations*

$$\begin{aligned} \partial_z K_a(z, w, s) &= -s K_{a+1}(z, w, s + 1) \\ \partial_{\bar{z}} K_a(z, w, s) &= (a - s) K_{a-1}(z, w, s) \\ \partial_w K_a(z, w, s) &= -(K_{a+1}(z, w, s) - \bar{z} K_a(z, w, s))/A \\ \partial_{\bar{w}} K_a(z, w, s) &= (K_{a-1}(z, w, s - 1) - z K_a(z, w, s))/A. \end{aligned}$$

The Eisenstein functions are defined as follows.

DEFINITION A.2. – For any integer m and b , we define the Eisenstein function $E_{m,b}(z)$ to be the \mathcal{C}^∞ -function on $\mathbb{C} \setminus \Gamma$ given by

$$E_{m,b}(z) := K_{b-m}^*(0, z, b).$$

Compare [30]VIII §14, where a slightly different definition is used. By definition, we have

$$(A.3) \quad E_{m,b}(z_0) = e_{-m,b}^*(z_0)$$

for $z_0 \in \mathbb{C} \setminus \Gamma$. Note that $E_{m,b}(z)$ satisfies $\overline{E_{m,b}(z)} = (-1)^{b-m} E_{b,m}(z)$ under complex conjugation. By definition, we have $E_{m,b}(z + \gamma) = E_{m,b}(z)$ for any $\gamma \in \Gamma$. Calculations similar to Lemma A.1 give the equality

$$(A.4) \quad \partial_z E_{m+1,b}(z) = -E_{m,b}(z)/A, \quad \partial_{\bar{z}} E_{m,b+1}(z) = E_{m,b}(z)/A.$$

REMARK A.5. – Note that by Definition 2.7, we have $E_{0,0}(z) \equiv -1$, and (A.4) shows that we have $E_{-a,0}(z) = 0$ for any $a > 0$. This implies that for any $z_0 \in \mathbb{C}$, we have $e_{0,0}^*(z_0) = -1$ and $e_{a,0}^*(z_0) = 0$ if $a > 0$.

Using Lemma A.1, we may prove that

$$K_1(z, w, 1) = \sum_{b \geq 0} (-1)^{b-1} E_{0,b}(w) z^{b-1}.$$

By Kronecker's equality $\Theta(z, w) = \exp(z\bar{w}/A) K_1(z, w, 1)$ (see [30] VIII §4, p. 71 (7) or [6] Theorem 1.13 for a proof), Definition 1.3, Definition 1.4 and the fact that $E_{0,1}(z) = F_1(z) - \bar{z}/A$, we have

$$(A.6) \quad L_n(z) = (-1)^{n-1} \sum_{b=0}^n \frac{E_{0,1}^{n-b}(z)}{(n-b)!} E_{0,b}(z).$$

We next define real analytic functions which correspond to the periods of the \mathcal{C}^∞ -sheaf associated to the elliptic polylogarithm sheaf.

DEFINITION A.7 (Real analytic elliptic polylogarithm function). – We define the multi-valued functions $G_{m,b}(z)$ on $\mathbb{C} \setminus \Gamma$ by first letting $G_{m,-1}(z) = 0$ and $G_{0,b}(z) = E_{0,b}(z)$ for integers $m, b \geq 0$, and we then iteratedly define $G_{m,b}(z)$ for $m, b \geq 0$, to be any function satisfying

$$\partial_z G_{m+1,b}(z) = -G_{m,b}(z), \quad \partial_{\bar{z}} G_{m,b+1}(z) = G_{m,b}(z)/A.$$

If we assume the existence of $G_{m+1,b}(z)$ and $G_{m,b+1}(z)$, then we have

$$\partial_z (G_{m+1,b}(z)/A) = -G_{m,b}(z)/A = -\partial_{\bar{z}} G_{m,b+1}(z),$$

which implies that $G_{m,b+1}(z)dz + G_{m+1,b}(z)d\bar{z}/A$ is a closed form. Hence the function $G_{m+1,b+1}(z)$ exists in this case. We fix a choice of $G_{m,b}(z)$ satisfying the condition in Proposition A.8 below.

PROPOSITION A.8. – We may iteratedly choose $G_{m,b}(z)$ so that

$$A^b G_{m,b}(z) + (-1)^{m+b} A^m \overline{G_{b,m}(z)} = A^{m+b} E_{m,b}(z) - \frac{(-z)^m \bar{z}^b}{m!b!}$$

for any integers $m, b \geq 0$.

A.2. Elliptic polylogarithm functions

We next construct the elliptic polylogarithm functions $D_{m,n}(z)$ and $D_{m,n}^*(z)$, which are holomorphic multi-valued functions on $\mathbb{C} \setminus \Gamma$.

DEFINITION A.9. – For integers $m, n \geq 0$, we define the elliptic polylogarithm functions $D_{m,n}(z)$ and $D_{m,n}^*(z)$ by

$$D_{m,n}(z) = (-1)^{n-1} \sum_{k=0}^n \frac{E_{0,1}(z)^{n-k}}{(n-k)!} G_{m,k}(z)$$

$$D_{m,n}^*(z) = D_{m,n}(z) - \frac{(-1)^{m+n}}{m!n!} z^m F_1(z)^n.$$

LEMMA A.10. – Both $D_{m,n}(z)$ and $D_{m,n}^*(z)$ are holomorphic functions on the universal covering of $\mathbb{C} \setminus \Gamma$.

Proof. – The statement for $D_{m,n}(z)$ follows from the fact that the functions in the sum are defined on $\mathbb{C} \setminus \Gamma$ and that $\partial_{\bar{z}} D_{m,n}(z) = 0$, which follows from the fact that $\partial_{\bar{z}} E_{0,1}(z) = -1$, $\partial_{\bar{z}} G_{m,0}(z) = 0$, and $\partial_{\bar{z}} G_{m,k}(z) = G_{m,k-1}(z)/A$ for $k \geq 1$. The statement for $D_{m,n}^*(z)$ follows from the fact that $F_1(z)$ is holomorphic on $\mathbb{C} \setminus \Gamma$. \square

LEMMA A.11. – The functions $D_{m,n}(z)$ and $D_{m,n}^*(z)$ for $m, n > 0$ satisfy

$$dD_{m,n}(z) = -D_{m-1,n}(z)dz - D_{m,n-1}(z)dF_1$$

$$dD_{m,n}^*(z) = -D_{m-1,n}^*(z)dz - D_{m,n-1}^*(z)dF_1.$$

The p -adic analogues of $D_{m,n}(z)$ and $D_{m,n}^*(z)$ are the overconvergent functions $D_{m,n}^{(p)}$ given in Theorem 3.6. By definition, we have

$$D_{0,n}(z) = (-1)^{n-1} \sum_{k=0}^n \frac{E_{0,1}(z)^{n-k}}{(n-k)!} E_{0,k}(z) = L_n(z),$$

$$D_{0,n}^*(z) = D_{0,n}(z) - \frac{(-1)^n}{n!} F_1(z)^n = L_n(z) - \frac{(-1)^n}{n!} F_1(z)^n.$$

Hence the elliptic polylogarithm functions $D_{m,n}(z)$ are obtained from $L_n(z)$ by iterated integration.

A.3. Absolute Hodge cohomology

In this section, we will freely use terminology concerning variation of mixed Hodge structures and absolute Hodge cohomology. See for example [18], Appendix A, for details. For any variety X smooth and separated of finite type over \mathbb{C} , we denote by $\text{VMHS}_{\mathbb{R}}(X)$ the category of polarizable admissible variation of mixed \mathbb{R} -Hodge structures on X . For any \mathcal{F} in $\text{VMHS}_{\mathbb{R}}(X)$, we denote by \mathcal{F} the underlying locally free \mathcal{O}_X -module with integrable connection, and by $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{C}}$ the underlying \mathbb{R} - and \mathbb{C} -local systems. If we let $X^{\text{an}} := X(\mathbb{C})$, then the theorem of de Rham gives a canonical isomorphism

$$H_{\text{dR}}^i(X, \mathcal{F}) \cong H_B^i(X^{\text{an}}, \mathcal{F}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$$

between de Rham and Betti cohomologies. We denote by $H^i(X^{\text{an}}, \mathcal{F})$ the mixed \mathbb{R} -Hodge structure induced from the above isomorphism, and by $H_{\mathcal{A}}^i(X, \mathcal{F})$ the m -th absolute Hodge

cohomology of X with coefficients in \mathcal{F} . Note that for $i = 0, 1$, there exists a canonical isomorphism

$$(A.12) \quad \mathrm{Ext}_{\mathrm{VMHS}_{\mathbb{R}}(X)}^i(\mathbb{R}(0), \mathcal{F}) \xrightarrow{\cong} H_{\mathcal{A}}^i(X, \mathcal{F}).$$

Let $S = \mathrm{Spec} \mathbb{C}$. Then $\mathrm{VMHS}_{\mathbb{R}}(S)$ is the category of polarizable mixed \mathbb{R} -Hodge structures $\mathrm{MHS}_{\mathbb{R}}$. The Leray spectral sequence

$$E_2^{p,q} = H_{\mathcal{A}}^p(S, H^q(X^{\mathrm{an}}, \mathcal{F})) \Rightarrow H_{\mathcal{A}}^{p+q}(X, \mathcal{F})$$

degenerates to give the short exact sequence

$$(A.13) \quad 0 \rightarrow H_{\mathcal{A}}^1(S, H^0(X^{\mathrm{an}}, \mathcal{F})) \rightarrow H_{\mathcal{A}}^1(X, \mathcal{F}) \rightarrow H_{\mathcal{A}}^0(S, H^1(X^{\mathrm{an}}, \mathcal{F})) \rightarrow 0.$$

Consider the map $H_{\mathcal{A}}^0(S, H^1(X^{\mathrm{an}}, \mathcal{F})) \hookrightarrow H_B^1(X^{\mathrm{an}}, \mathcal{F}_{\mathbb{C}}) \cong H_{\mathrm{dR}}^1(X, \mathcal{F})$, where the last isomorphism is de Rham's theorem. Then we have a commutative diagram

$$(A.14) \quad \begin{array}{ccc} \mathrm{Ext}_{\mathrm{VMHS}_{\mathbb{R}}(X)}^1(\mathbb{R}(0), \mathcal{F}) & \xrightarrow{\mathrm{For}} & \mathrm{Ext}_{M(X)}^1(\mathcal{O}_X, \mathcal{F}) \\ \cong \downarrow & & \cong \downarrow \\ H_{\mathcal{A}}^1(X, \mathcal{F}) & \longrightarrow & H_{\mathrm{dR}}^1(X, \mathcal{F}), \end{array}$$

where $M(X)$ is the category of locally free \mathcal{O}_X -modules with integrable connection defined in Definition 1.13, and $\mathrm{For} : \mathrm{VMHS}_{\mathbb{R}}(X) \rightarrow M(X)$ is the functor associating to any \mathcal{F} the underlying coherent module with connection \mathcal{F} on X , forgetting the Hodge filtration and the \mathbb{R} -structure.

A.4. The logarithm sheaf

We now define the logarithm sheaf. Let E be an elliptic curve defined over $S = \mathrm{Spec} \mathbb{C}$. We let $H^1(E, \mathbb{R})$ be the pure \mathbb{R} -Hodge structure of weight 1 given by $H_{\mathrm{dR}}^1(E) \cong H_B^1(E^{\mathrm{an}}, \mathbb{R}) \otimes \mathbb{C}$, and we let $\mathcal{H} := H^1(E, \mathbb{R})^{\vee}$ be the dual Hodge structure.

We denote by ϕ_{∞} the complex conjugation on $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ defined by the action of complex conjugation on \mathbb{C} . The class $\underline{\omega}$ is represented by dz , and since $dE_{0,1}(z) = dF_1 - d\bar{z}/A = \omega^* - d\bar{z}/A$ where $E_{0,1}(z)$ is a single valued real analytic function on U (see Remark A.5), the class of $\underline{\omega}^*$ is represented by $d\bar{z}/A$. Hence the complex conjugation ϕ_{∞} acts on these classes as

$$(A.15) \quad \phi_{\infty}(\underline{\omega}^{\vee}) = \underline{\omega}^{*\vee}/A, \quad \phi_{\infty}(\underline{\omega}^{*\vee}) = A\underline{\omega}^{\vee}.$$

If we let $\gamma_1 := (\underline{\omega}^{\vee} + (\underline{\omega}^{*\vee}/A))$ and $\gamma_2 := i(\underline{\omega}^{\vee} - (\underline{\omega}^{*\vee}/A))$, then γ_1 and γ_2 form a basis of $\mathcal{H}_{\mathbb{R}}$.

For any smooth scheme X over S , we denote by \mathcal{H}_X the constant variation of \mathbb{R} -Hodge structures on X . The underlying coherent module with connection of \mathcal{H} is \mathcal{H}_X . We have a natural isomorphism

$$H_{\mathcal{A}}^0(S, H^1(E^{\mathrm{an}}, \mathcal{H}_E)) = H_{\mathcal{A}}^0(S, \mathcal{H}^{\vee} \otimes \mathcal{H}) = \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}).$$

Hence the short exact sequence (A.13) defined from the Leray spectral sequence gives the short exact sequence

$$0 \rightarrow H_{\mathcal{A}}^1(S, \mathcal{H}) \rightarrow H_{\mathcal{A}}^1(E, \mathcal{H}_E) \rightarrow \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}) \rightarrow 0.$$

In addition, we have in this case

$$H^1_{\mathcal{A}}(S, \mathcal{H}) = \text{Ext}^1_{\text{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathcal{H}) = \mathcal{H}_{\mathbb{C}} / (\mathcal{H}_{\mathbb{R}} + F^0 \mathcal{H}_{\mathbb{C}}) = 0.$$

Hence the above exact sequence gives an isomorphism

$$(A.16) \quad H^1_{\mathcal{A}}(E, \mathcal{H}_E) \xrightarrow{\cong} \text{Hom}_{\text{MHS}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}).$$

DEFINITION A.17. – The sheaf $\mathcal{L}og^{(1)}$ is defined to be any extension of \mathcal{H}_E by $\mathbb{R}(0)$ in $\text{VMHS}_{\mathbb{R}}(E)$, whose extension class in

$$\text{Ext}^1_{\text{VMHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathcal{H}_E) \cong H^1_{\mathcal{A}}(E, \mathcal{H}_E)$$

is mapped to the identity through (A.16). We define the N -th logarithm sheaf $\mathcal{L}og^N$ to be the N -th symmetric tensor product of $\mathcal{L}og^{(1)}$.

The logarithm sheaf for the Hodge realization is determined uniquely up to unique isomorphism. Denote by $i_{[0]}^*$ the pull-back by the identity $i_{[0]} : S \rightarrow E$ of the elliptic curve. Since the extension of \mathcal{H} by $\mathbb{R}(0)$ is split on S , there exists a splitting

$$\epsilon : i_{[0]}^* \mathcal{L}og^{(1)} \cong \mathbb{R}(0) \oplus \mathcal{H}$$

as mixed \mathbb{R} -Hodge structures. This splitting is unique due to weight reasons.

We will next describe $\mathcal{L}og^{(1)}$ as a variation of mixed \mathbb{R} -Hodge structures on E . As in the p -adic case, by (A.14), the underlying coherent module with connection on E of $\mathcal{L}og^{(1)}$ is the locally free \mathcal{O}_E -module with connection $\mathcal{L}og^{(1)}$ of Proposition 1.26. Take an affine open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of E , and cohomology classes (η_i, u_{ij}) and (u_i) as in Definition 1.35. We denote by $\mathcal{L}og_i^{(1)}$ the module

$$\mathcal{L}og_i^{(1)} := \mathcal{O}_{U_i, \underline{e}_i} \oplus \mathcal{H}_{U_i}$$

defined in Proposition 1.26, with connection $\nabla(\underline{e}_i) = \underline{\omega}^{\vee} \otimes \omega + \underline{\omega}^{*\vee} \otimes \omega_i^*$. The Hodge filtration is defined to be the direct sum, which extends to a filtration of $\mathcal{L}og^{(1)}$ on E . Let $v_i := \underline{e}_i - z\underline{\omega}^{\vee} - \xi_i(z)\underline{\omega}^{*\vee}$, where $\xi_i(z) = F_1(z) - u_i(z)$. Then the v_i are horizontal and paste together to define a multi-valued section v_0 of $\mathcal{L}og^{(1)}$ on E^{an} . Since the \mathbb{R} -structure of $\mathcal{L}og^{(1)}$ on the universal covering space of E^{an} given by v_0 and the \mathbb{R} -structure $\mathcal{H}_{\mathbb{R}}$ of $\mathcal{H}_{E^{\text{an}}}$ is invariant under monodromy, this structure descends to give an \mathbb{R} -structure $\mathcal{L}og_{\mathbb{R}}^{(1)}$ on E^{an} . We define the weight filtration W_{\bullet} on $\mathcal{L}og_{\mathbb{R}}^{(1)}$ by $W_{-2}\mathcal{L}og_{\mathbb{R}}^{(1)} = 0, W_{-1}\mathcal{L}og_{\mathbb{R}}^{(1)} = \mathcal{H}_{\mathbb{R}}$ and $W_0\mathcal{L}og_{\mathbb{R}}^{(1)} = \mathcal{L}og_{\mathbb{R}}^{(1)}$. The above give the structure as a variation of mixed \mathbb{R} -Hodge structures of $\mathcal{L}og^{(1)}$.

PROPOSITION A.18. – *The variation of mixed \mathbb{R} -Hodge structures on E given above satisfies the property of the first logarithm sheaf of Definition A.17.*

Proof. – By construction, the class of the underlying coherent module with connection $\mathcal{L}og^{(1)}$ of $\mathcal{L}og^{(1)}$ maps to the element in the first de Rham cohomology which corresponds to the identity. Hence by (A.14), the isomorphism of (A.16) maps the class of $\mathcal{L}og^{(1)}$ to the identity. This gives our assertion. \square

REMARK A.19. – By construction, the splitting $\epsilon : i_{[0]}^* \mathcal{L}og^{(1)} \cong \mathbb{C} \oplus \mathcal{H}$ given in (1.34) of the underlying \mathbb{C} -vector spaces gives the unique splitting of $i_{[0]}^* \mathcal{L}og^{(1)}$ compatible with the Hodge filtration and the \mathbb{R} -structure.

The N -th logarithm sheaf is the N -th symmetric tensor product of $\mathcal{L}og^{(1)}$. The Hodge filtration, the \mathbb{R} -structure and the weight filtration are defined naturally by taking the symmetric tensor product of each structure.

A.5. The polylogarithm sheaf

Next, we use the logarithm sheaf of the previous subsection to define the polylogarithm class in absolute Hodge cohomology. Then we will explicitly describe the polylogarithm sheaf, which is defined to be the pro-variation of mixed Hodge structures corresponding to the polylogarithm class.

Let $D = [0]$ and $U = E \setminus [0]$. Similarly to the p -adic case, the residue map gives an isomorphism

$$(A.20) \quad \varprojlim_N H_{\mathcal{A}}^1(U, \mathcal{H}_U^{\vee} \otimes \mathcal{L}og^N(1)) \xrightarrow{\cong} \mathbb{R}.$$

DEFINITION A.21. – By [7] (see also [17] Appendix A), the *polylogarithm class* is defined to be a system of classes $\text{pol}_{\mathcal{A}}^N \in H_{\mathcal{A}}^1(U, \mathcal{H}_U^{\vee} \otimes \mathcal{L}og^N(1))$ which maps to 1 through (A.20). The *elliptic polylogarithm sheaf* on U is defined to be a system of variation of mixed \mathbb{R} -Hodge structures \mathcal{P}^N on U given as an extension

$$0 \rightarrow \mathcal{L}og^N(1) \rightarrow \mathcal{P}^N \rightarrow \mathcal{H}_U \rightarrow 0$$

whose extension class corresponds to $\text{pol}_{\mathcal{A}}^N$ in

$$\text{Ext}_{\text{VMHS}_{\mathbb{R}}(U)}^1(\mathcal{H}_U, \mathcal{L}og^N(1)) \cong H_{\mathcal{A}}^1(U, \mathcal{H}_U^{\vee} \otimes \mathcal{L}og^N(1)).$$

We next explicitly describe the variations of mixed \mathbb{R} -Hodge structures \mathcal{P}^N . By definition, \mathcal{P}^N is given as an extension

$$0 \rightarrow \mathcal{L}og^N(1) \rightarrow \mathcal{P}^N \rightarrow \mathcal{H}_U \rightarrow 0.$$

As in the p -adic case, the polylogarithm sheaf is characterized by the image of its cohomology class mapped to de Rham cohomology. By (A.14), this implies that the underlying coherent module with connection of \mathcal{P}^N is the de Rham realization \mathcal{P}^N given in Corollary 1.42, and the injectivity of (A.14) implies that \mathcal{P}^N is the unique variation of mixed \mathbb{R} -Hodge structures up to canonical isomorphism which one may equip on \mathcal{P}^N . We define the Hodge and weight filtrations of \mathcal{P}^N as the direct sum of the Hodge and weight filtrations on $\mathcal{L}og^N(1)$ and \mathcal{H} . In order to define the \mathbb{R} -structure $\mathcal{P}_{\mathbb{R}}^N$ on \mathcal{P}^N , we first introduce certain horizontal sections v and v^* of \mathcal{P}^N on the universal covering space of U^{an} given as follows.

LEMMA A.22. – *Let*

$$(A.23) \quad \begin{aligned} v &= \underline{\omega}^{\vee} - \sum_{n=0}^N \frac{(-F_1(z))^{n+1}}{(n+1)!} \underline{\omega}^{0,n} + \sum_{m=1}^N \sum_{n=0}^{N-m} D_{m,n+1}^*(z) \underline{\omega}^{m,n} \\ v^* &= \underline{\omega}^{*\vee} + \sum_{m=0}^N \sum_{n=0}^{N-m} D_{m+1,n}(z) \underline{\omega}^{m,n}. \end{aligned}$$

Then v and v^* are horizontal sections of \mathcal{P}^N .

Proof. – The statement follows from the fact that $dF_1 = \omega^*$ and the differential equations Lemma A.11 satisfied by $D_{m,n}(z)$ and $D_{m,n}^*(z)$. \square

Our choice of the real analytic elliptic polylogarithm function given in Proposition A.8 gives the following.

PROPOSITION A.24. – *Let $v_1 := v + (v^*/A)$ and $v_2 = i(v - (v^*/A))$. Then the \mathbb{R} -structure defined by v_1 , v_2 and $\mathcal{L}og_{\mathbb{R}}^N(1)$ on the universal covering space of U^{an} descends to give an \mathbb{R} -structure $\mathcal{P}_{\mathbb{R}}^N$ of \mathcal{P}^N on U^{an} , which fits into the exact sequence*

$$0 \rightarrow \mathcal{L}og_{\mathbb{R}}^N(1) \rightarrow \mathcal{P}_{\mathbb{R}}^N \rightarrow \mathcal{H}_{\mathbb{R}} \rightarrow 0.$$

The above proposition may be proved by explicitly calculating the action of the complex conjugation ϕ_{∞} on v and v^* . By the construction of the elliptic polylogarithm, the choice of an \mathbb{R} -structure on \mathcal{P}^N is unique up to isomorphism. Hence the \mathbb{R} -structure defined above is the real structure of the elliptic polylogarithm. This gives the main result of our appendix.

THEOREM A.25. – *The elliptic polylogarithm sheaf \mathcal{P}^N is the variation of mixed \mathbb{R} -Hodge structures on U given as an extension*

$$0 \rightarrow \mathcal{L}og^N(1) \rightarrow \mathcal{P}^N \rightarrow \mathcal{H}_U \rightarrow 0,$$

whose underlying coherent module with connection is the object \mathcal{P}^N given in Definition 1.39, whose Hodge filtration is given as the direct sum of the Hodge filtrations on \mathcal{H}_U and $\mathcal{L}og^N(1)$, whose real structure is the structure $\mathcal{P}_{\mathbb{R}}^N$ given in Proposition A.24, and whose weight filtration is the direct sum of the weight filtrations on \mathcal{H} and $\mathcal{L}og^N(1)$.

This shows that the holomorphic functions $D_{m,n}(z)$, $D_{m,n}^*(z)$ of Definition A.9 are in fact periods of the elliptic polylogarithm sheaf.

A.6. Specialization to points

We now calculate the specialization of the elliptic polylogarithm sheaf to points of the elliptic curve. Suppose $z_0 \in \mathbb{C}$, and we denote by $i_{z_0}^* \mathcal{L}og^N$ the restriction of the variation of mixed Hodge structures $\mathcal{L}og^N$ to the point z_0 . We let $S = \text{Spec } \mathbb{C}$.

Let $\mathcal{M} := \mathcal{H}^{\vee} \otimes i_{z_0}^* \mathcal{L}og^N$. We have an isomorphism

$$(A.26) \quad H_{\mathcal{A}}^1(S, \mathcal{H}^{\vee} \otimes i_{z_0}^* \mathcal{L}og^N(1)) = M_{\mathbb{C}} / (M_{\mathbb{R}}(1) + F^1 M_{\mathbb{C}}).$$

Denote by $i_{z_0}^* \mathcal{P}^N$ the pull-back of the polylogarithm sheaf to the point z_0 . Note that $\underline{\omega} \otimes \omega^{0,n} \in F^1 M_{\mathbb{C}}$. Hence the explicit calculation of the period of $i_{z_0}^* \mathcal{P}^N$ shows that the extension class

$$[i_{z_0}^* \mathcal{P}^N] \in \text{Ext}_{\text{MHS}_{\mathbb{R}}}^1(\mathcal{H}, i_{z_0}^* \mathcal{L}og^N(1)) = H_{\mathcal{A}}^1(S, \mathcal{H}^{\vee} \otimes i_{z_0}^* \mathcal{L}og^N(1))$$

corresponds through (A.26) to the element

$$\text{pol}_{z_0}^N := \gamma_1^{\vee} \otimes (v_1 - \gamma_1) + \gamma_2^{\vee} \otimes (v_2 - \gamma_2)$$

in $M_{\mathbb{C}}/(M_{\mathbb{R}}(1) + F^1 M_{\mathbb{C}})$. Here $\gamma_1^{\vee}, \gamma_2^{\vee}$ is a basis of $\mathcal{H}_{\mathbb{R}}^{\vee}$ dual to γ_1, γ_2 . Our main result is the explicit calculation of the image of the above element through the isomorphism

$$(A.27) \quad M_{\mathbb{C}}/(M_{\mathbb{R}}(1) + F^1 M_{\mathbb{C}}) \xrightarrow{\cong} M_{\mathbb{R}}/M_{\mathbb{R}} \cap (M_{\mathbb{R}}(1) + F^1 M_{\mathbb{C}})$$

defined by $u \mapsto u + \phi_{\infty}(u)$, where ϕ_{∞} denotes the complex conjugation on $M_{\mathbb{C}}$. In order to state our result, we will use a basis of $i_{z_0}^* \text{Log}^N$ which gives the isomorphism of mixed \mathbb{R} -Hodge structures between $i_{z_0}^* \text{Log}^N$ and $\prod_{k=0}^N \text{Sym}^k \mathcal{H}$. Let $\underline{e}' := \underline{e} - E_{0,1}(z_0) \underline{\omega}^{*\vee}$ be an element of $i_{z_0}^* \text{Log}^{(1)}$. We let $\underline{e}^{m,n} := \underline{e}'^a \underline{\omega}^{\vee m} \underline{\omega}^{*\vee n} / a!$ for $a = N - m - n$, which form a basis of $i_{z_0}^* \text{Log}^N$. The complex conjugation acts on this base by $\phi_{\infty}(\underline{e}^{m,n}) = A^{n-m} \underline{e}^{n,m}$.

As in the p -adic case given in Lemma 5.3, we have the following.

LEMMA A.28 (Splitting Principle). – *By mapping $\underline{e}^{m,k}$ to $\underline{\omega}^{\vee m} \underline{\omega}^{*\vee k}$, we have an isomorphism of mixed \mathbb{R} -structures*

$$i_{z_0}^* \text{Log}^N \cong \prod_{j=0}^N \text{Sym}^j \mathcal{H}.$$

Applying our choice of $G_{m,k}(z_0)$ given in Proposition A.8 to the explicit description of $\text{pol}_{z_0}^N + \phi_{\infty}(\text{pol}_{z_0}^N)$, we now have the following result, originally due to Beilinson and Levin [7] (see also [32] III Theorem 4.8).

THEOREM A.29. – *Let z_0 be a non-zero point in $E(\mathbb{C})$, and let*

$$\text{pol}_{z_0}^N := [\mathcal{P}_{z_0}^N] \in H_{\mathcal{A}}^1(S, \mathcal{H}^{\vee} \otimes i_{z_0}^* \text{Log}^N(1))$$

be the pullback by z_0 of the cohomology class of the elliptic polylogarithm sheaf \mathcal{P}^N . Then the image of this class through the isomorphism

$$H_{\mathcal{A}}^1(S, \mathcal{H}^{\vee} \otimes i_{z_0}^* \text{Log}^N(1)) \xrightarrow{\cong} M_{\mathbb{R}}/M_{\mathbb{R}} \cap (M_{\mathbb{R}}(1) + F^1 M_{\mathbb{C}})$$

for $\mathcal{M} := \mathcal{H}^{\vee} \otimes i_{z_0}^ \text{Log}^N$ is given by*

$$\sum_{m=1}^N \sum_{k=0}^{N-m} \frac{(-1)^k}{A^{-m}} e_{-m,k+1}^*(z_0) \underline{\omega} \otimes \underline{e}^{m,k} + \sum_{m=0}^N \sum_{k=1}^{N-m} \frac{(-1)^{k-1}}{A^{-m-1}} e_{-m-1,k}^*(z_0) \underline{\omega}^* \otimes \underline{e}^{m,k},$$

where $e_{a,b}^(z_0)$ are the Eisenstein-Kronecker numbers of Definition 2.8.*

The p -adic analogue of the above result is given in Theorem 5.7.

REFERENCES

- [1] M. ABRAMOWITZ, I. A. STEGUN (eds.), Weierstrass elliptic and related functions, Ch. 18, in *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., 1992, p. 627–671.
- [2] F. BALDASSARRI, B. CHIARELLOTTA, Algebraic versus rigid cohomology with logarithmic coefficients, in *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, Perspect. Math. **15**, Academic Press, 1994, 11–50.
- [3] K. BANNAI, Rigid syntomic cohomology and p -adic polylogarithms, *J. reine angew. Math.* **529** (2000), 205–237.

- [4] K. BANNAI, On the p -adic realization of elliptic polylogarithms for CM-elliptic curves, *Duke Math. J.* **113** (2002), 193–236.
- [5] K. BANNAI, G. KINGS, p -adic elliptic polylogarithm, p -adic Eisenstein series and Katz measure, preprint [arXiv:0707.3747](https://arxiv.org/abs/0707.3747), to appear in *Amer. J. Math.*
- [6] K. BANNAI, S. KOBAYASHI, Algebraic theta functions and p -adic interpolation of Eisenstein-Kronecker numbers, preprint [arXiv:math.NT/0610163](https://arxiv.org/abs/math/0610163), to appear in *Duke Math. J.*
- [7] A. BEĪLINSON, A. LEVIN, The elliptic polylogarithm, in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., 1994, 123–190.
- [8] P. BERTHELOT, Géométrie rigide et cohomologie des variétés algébriques de caractéristique p , *Mém. Soc. Math. France (N.S.)* **23** (1986), 7–32.
- [9] P. BERTHELOT, Cohomologie rigide et cohomologie rigide à support propre, première partie, preprint IRMAR 96-03, 1996.
- [10] P. BERTHELOT, Finitude et pureté cohomologique en cohomologie rigide, *Invent. Math.* **128** (1997), 329–377.
- [11] J. L. BOXALL, A new construction of p -adic L -functions attached to certain elliptic curves with complex multiplication, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 31–68.
- [12] J. L. BOXALL, p -adic interpolation of logarithmic derivatives associated to certain Lubin-Tate formal groups, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 1–27.
- [13] P. COLMEZ, Fonctions L p -adiques, Séminaire Bourbaki, vol. 1998/99, exposé n° 851, *Astérisque* **266** (2000), 21–58.
- [14] R. M. DAMERELL, L -functions of elliptic curves with complex multiplication. I, *Acta Arith.* **17** (1970), 287–301.
- [15] R. M. DAMERELL, L -functions of elliptic curves with complex multiplication. II, *Acta Arith.* **19** (1971), 311–317.
- [16] J.-M. FONTAINE, Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III*, Astérisque **65**, Soc. Math. France, 1979, 3–80.
- [17] A. HUBER, G. KINGS, Degeneration of l -adic Eisenstein classes and of the elliptic polylog, *Invent. Math.* **135** (1999), 545–594.
- [18] A. HUBER, J. WILDESHAUS, Classical motivic polylogarithm according to Beilinson and Deligne, *Doc. Math.* **3** (1998), 27–133; correction: *idem*, 297–299.
- [19] N. M. KATZ, p -adic interpolation of real analytic Eisenstein series, *Ann. of Math.* **104** (1976), 459–571.
- [20] A. LEVIN, Elliptic polylogarithms: an analytic theory, *Compositio Math.* **106** (1997), 267–282.
- [21] A. LEVIN, G. RACINET, Towards multiple elliptic polylogarithm, preprint [arXiv:math/0703237](https://arxiv.org/abs/math/0703237).
- [22] J. I. MANIN, M. M. VIŠIK, p -adic Hecke series of imaginary quadratic fields, *Mat. Sb. (N.S.)* **95** (1974), 357–383.
- [23] P. MONSKY, G. WASHNITZER, Formal cohomology. I, *Ann. of Math.* **88** (1968), 181–217.

- [24] B. PERRIN-RIOU, Fonctions L p -adiques des représentations p -adiques, *Astérisque* **229** (1995).
- [25] P. SCHNEIDER, J. TEITELBAUM, p -adic Fourier theory, *Doc. Math.* **6** (2001), 447–481.
- [26] E. DE SHALIT, *Iwasawa theory of elliptic curves with complex multiplication*, Perspectives in Mathematics **3**, Academic Press Inc., 1987.
- [27] A. SHIHO, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, *J. Math. Sci. Univ. Tokyo* **9** (2002), 1–163.
- [28] N. SOLOMON, p -adic elliptic polylogarithms and arithmetic applications, Ph.D. Thesis, Ben-Gurion University, 2009.
- [29] N. TSUZUKI, On base change theorem and coherence in rigid cohomology, *Doc. Math.* extra vol. (2003), 891–918.
- [30] A. WEIL, *Elliptic functions according to Eisenstein and Kronecker*, *Ergebn. Math. Grenz.* **88**, Springer, 1976.
- [31] E. WEISSTEIN, Weierstrass sigma function, <http://mathworld.wolfram.com/WeierstrassSigmaFunction.html>.
- [32] J. WILDESHAUS, *Realizations of polylogarithms*, Lecture Notes in Math. **1650**, Springer, 1997.
- [33] S. YAMAMOTO, *On p -adic L -functions for CM elliptic curves at supersingular primes*, Master's Thesis, University of Tokyo, 2002.

(Manuscrit reçu le 13 décembre 2007 ;
 accepté, après révision, le 12 octobre 2009.)

Kenichi BANNAI
 Department of Mathematics
 Keio University
 3-14-1 Hiyoshi, Kouhoku-ku
 Yokohama 223-8522, Japan
 E-mail: bannai@math.keio.ac.jp

Shinichi KOBAYASHI
 Mathematical Institute
 Tohoku University
 Aoba 6-3, Aramaki-aza-Aoba, Aoba-ku
 Sendai, 980-8578 Japan
 E-mail: shinichi@math.tohoku.ac.jp

Takeshi TSUJI
 Graduate School of Mathematics
 University of Tokyo
 Komaba, Meguro-ku
 Tokyo 153-8914, Japan
 E-mail: t-tsuji@ms.u-tokyo.ac.jp