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SMALLNESS PROBLEM FOR QUANTUM AFFINE ALGEBRAS AND QUIVER VARIETIES

BY DAVID HERNANDEZ

ABSTRACT. – The geometric small property (Borho-MacPherson [2]) of projective morphisms implies a description of their singularities in terms of intersection homology. In this paper we solve the smallness problem raised by Nakajima [37, 35] for certain resolutions of quiver varieties [37] (analogs of the Springer resolution): for Kirillov-Reshetikhin modules of simply-laced quantum affine algebras, we characterize explicitly the Drinfeld polynomials corresponding to the small resolutions. We use an elimination theorem for monomials of Frenkel-Reshetikhin q-characters that we establish for non necessarily simply-laced quantum affine algebras. We also refine results of [21] and extend the main result to general simply-laced quantum affinizations, in particular to quantum toroidal algebras (double affine quantum algebras).

RÉSUMÉ. – La propriété géométrique de petitesse (Borho-MacPherson [2]) des morphismes projectifs implique une description de leurs singularités en termes d'homologie d'intersection. Dans cet article nous résolvons le problème de petitesse posé par Nakajima [37, 35] pour certaines résolutions de variétés carquois [37] (analogues de la résolution de Springer) : pour les modules de Kirillov-Reshetikhin des algèbres affines quantiques simplement lacées, nous caractérisons explicitement les polynômes de Drinfeld correspondant aux résolutions petites. Nous utilisons un théorème d'élimination pour les monômes des q-caractères de Frenkel-Reshetikhin, que nous établissons pour les algèbres affines quantiques non nécessairement simplement lacées. Nous raffinons également des résultats de [21] et étendons le résultat principal aux affinisées quantiques générales simplement lacées, en particulier aux algèbres toroïdales quantiques (algèbres quantiques doublement affines).

1. Introduction

Borho and MacPherson [2, Section 1.1] introduced remarkable geometric properties (smallness and semi-smallness) for a proper algebraic map $\pi : Z \to X$ where Z, X are irreducible complex algebraic varieties: for a finite stratification of X into irreducible smooth subvarieties, π is said to be semi-small if the dimension of the inverse image of a point in a

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stratum is at most half the codimension of the stratum, and π is said to be small if moreover the equality holds only if the stratum is dense. These properties do not depend on the stratification.

This geometric situation is of particular interest as the Beilinson-Bernstein-Deligne-Gabber decomposition Theorem [1] is simplified [2, Section 1.5] and provides an elegant description of the singularities of such maps in terms of intersection homology sheaves [15, 16]. A fundamental example of a semi-small morphism is given by the Springer resolution of the nilpotent cone of a complex simple Lie algebra, and the corresponding partial resolutions [2]. Nakajima [30, 31] defined important and intensively studied varieties called quiver varieties which depend on a quiver Q. They come with a resolution which is semi-small [31, Corollary 10.11] for a finite Dynkin diagram (see [34, Section 5.2]).

The graded version of quiver varieties is also of particular importance, for example for their deep relations with representations of quantum affine algebras (see [37]; the precise definition is reminded below). They also come with resolutions. A natural problem addressed in the present paper is to study the small property of these resolutions: we address [37, Conjecture 10.4] (see also [35]). Our study relies on the representation theory of quantum affine algebras: let us explain the context for our study.

In this paper $q \in \mathbb{C}^*$ is fixed and is not a root of unity. Affine Kac-Moody algebras $\hat{\mathfrak{g}}$ are infinite dimensional analogs of semi-simple Lie algebras $\mathfrak{g}$, and have remarkable applications in several branches of mathematics and physics (see [25]). Their quantizations $\mathcal{U}_q(\hat{\mathfrak{g}})$, called quantum affine algebras, have a very rich representation theory which has been intensively studied (see [7, 10] for references). In particular Drinfeld [12] discovered that they can also be realized as quantum affinization of usual quantum groups $\mathcal{U}_q(\mathfrak{g})$. By using this new realization, Chari-Pressley [7] classified their finite dimensional representations: they are parametrized by Drinfeld polynomials $(P_i(u))_{1\leq i\leq n}$ where n is the rank of $\mathfrak{g}$ and $P_i(u) \in \mathbb{C}[u]$ satisfies $P_i(0) = 1$.

A particular class of finite dimensional representations, called special modules, attracted much attention as Frenkel-Mukhin [13] proposed an algorithm which gives their q-character (analogs of usual characters adapted to the Drinfeld presentation of quantum affine algebras introduced by Frenkel-Reshetikhin [14]). Let us give some examples: for $k > 0, i \in I, a \in \mathbb{C}^*$, the Kirillov-Reshetikhin module $W_{k,a}^{(i)}$ is the simple module with Drinfeld polynomials

$$\begin{cases} P_j(u) = 1 \text{ for } j \neq i, \\ P_i(u) = (1 - uaq_i^{k-1})(1 - uaq_i^{k-3}) \cdots (1 - uaq_i^{1-k}). \end{cases}$$

(The q_i are certain powers of q, see section 3.) The $V_i(a) = W_{1,a}^{(i)}$ are called fundamental representations. The fundamental representations [13], and the Kirillov-Reshetikhin modules [36, 21] are special modules (this is the crucial point for the proof of the Kirillov-Reshetikhin conjecture). The corresponding standard module

$$M(X_{k,a}^{(i)}) = V_i(aq_i^{1-k}) \otimes V_i(aq_i^{3-k}) \otimes \cdots \otimes V_i(aq_i^{k-1})$$

is not special in general.

The breakthrough geometric approach of Nakajima [32, 37] to q-characters of representations of simply-laced quantum affine algebras via (graded) quiver varieties led to remarkable advances in the description of finite dimensional representations: for example this approach provides an algorithm [37] which computes the q-characters of *any* simple finite dimensional representations. Although in general the algorithm is very complicated, in some situations it provides a powerful tool to study these representations (for instance see [36]).

From the geometric point of view, the natural notion of small modules appeared in the following way: the small property of modules [37] is the representation theoretical interpretation of the smallness of certain resolutions of (graded) quiver varieties mentioned above.

A small module is special (but the converse is false in general). The representation theoretical interest of this notion is that all simple modules occurring in the Jordan-Hölder series of a small module are special, and so can be described by using the Frenkel-Mukhin algorithm.

A natural question is to characterize these small modules, and so the corresponding small resolutions. In particular, Nakajima ([37, Conjecture 10.4], [35]) raised the problem of characterizing the small standard modules corresponding to Kirillov-Reshetikhin modules.

In this paper we solve this problem by giving explicitly the corresponding Drinfeld polynomials.

The crucial point for our proof is an elimination theorem for monomials of q-characters, that we establish by refining our results of [21]. Indeed it is easy to produce monomials that occur in a certain q-character (for example see remark 3.16 below). But in general it is not clear if a given monomial *does not* occur in a q-character. The elimination theorem gives a criterion which implies that a monomial can be eliminated from the q-character of a simple module. Beyond the main result of the present paper (answer to the geometric smallness problem), we hope that this elimination theorem will be useful for other open problems in representation theory of quantum affine algebras. We already used it in a weak (non explicitly stated) form to prove the Kirillov-Reshetikhin conjecture [21]. Moreover it is used in [23] to study minimal affinizations of representations of quantum groups.

Let us state the main result of this paper. It can be stated in a simple compact way by using the following elementary definitions ($I = \{1, \dots, n\}$ is the set of vertices of the Dynkin diagram of $\mathfrak{g}$):

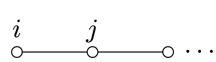
Definition 1.1. – *A node $i \in \{1, \dots, n\}$ is said to be extremal (resp. special) if there is a unique $j \in I$ (resp. three distinct $j, k, l \in I$) such that $C_{i,j} < 0$ (resp. $C_{i,j} < 0$, $C_{i,k} < 0$ and $C_{i,l} < 0$).*

For $i \in I$, we denote by d_i the minimal $d \geq 1$ such that there are distinct $i_1, \dots, i_d \in I$ satisfying $C_{i_j, i_{j+1}} < 0$ and i_d is special (if there are no special vertices, we set $d_i = +\infty$ for all $i \in I$).

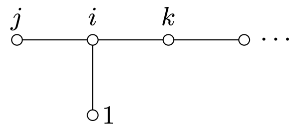
For example for $\mathfrak{g}$ of type A, we have $d_i = +\infty$ for all $i \in I$.

For an illustration, examples are given on the following pictures:

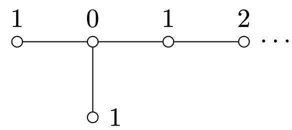
Extremal node i:



Special node i:



Distance d to a special node:



THEOREM 1.2 (Smallness problem). – *Let $k > 0, i \in I, a \in \mathbb{C}^*$. Then $M(X_{k,a}^{(i)})$ is small if and only if $k \leq 2$ or (i is extremal and $k \leq d_i + 1$).*

Remark: the condition is independent of the parameter $a \in \mathbb{C}^*$.

In particular for $\mathfrak{g} = sl_2$ or $\mathfrak{g} = sl_3$, all $M(X_{k,a}^{(i)})$ are small (it proves the corresponding [37, Conjecture 10.4]). In general it gives an explicit criterion so that the smallness holds. On the geometric side, it characterizes the small resolutions mentioned above.

Besides the statement of Theorem 1.2 is also satisfied for all simply-laced quantum affinizations $\mathcal{U}_q(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ is an arbitrary simply-laced Kac-Moody algebra, not necessarily semi-simple), in particular for quantum toroidal algebras (double affine quantum algebras).

The general idea of the proof is first to establish the result for type A by using the elimination strategy of monomials explained above. We prove by induction on the highest weight that representations in a certain class are special. Then an argument allows to use the type A to prove the result for general types.

Let us describe the organization of this paper. In section 2 we explain the geometric context of our results. In section 3 we give some background on finite dimensional representations of quantum affine algebras and q-characters. In section 4 we recall from [37] the definition of small modules and the geometric characterization (Theorem 4.3). We refine a Theorem of [37] and give a more representation theoretical characterization (Theorem 4.8). However this last result is not enough to prove Theorem 1.2, and technical work is needed in the next sections. The first point is the (representation theoretical) elimination Theorem (Theorem 5.1) which is proved in section 5: it gives a condition which implies that a monomial *does not* appear in the q-character of a simple module. Additional technical results are also proved in section 5: in particular the notion of thin modules (with l-weight spaces of dimension 1) is introduced and studied. In section 6, we complete the proof of Theorem 1.2: first type A is discussed, and then the general case is treated. The proof of the result for general simply-laced quantum affinizations is also discussed.

Acknowledgments

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2. The geometric problem: small property and graded quiver varieties

The geometric motivations and context of the results of the present paper have been explained at the beginning of the introduction. In this section we develop this discussion and define more precisely the involved geometric objects.

2.1. Small property

Let us recall the notion of semi-small and small morphism maps in the sense of Borho-MacPherson [2] for a proper algebraic map $\pi : Z \to X$ where Z, X are irreducible complex algebraic varieties.

We consider a finite stratification $X = \bigsqcup_i X_i$ into irreducible smooth subvarieties such that $\pi_{|\pi^{-1}(X_i)}$ is a topological fibration with base X_i and fiber $\pi^{-1}(x_i)$ where $x_i \in X_i$ is a distinguished base point.

DEFINITION 2.1 ([2]). – *The map π is said to be semi-small if for all i,*

$$2\,\dim(\pi^{-1}(x_i)) \leq \dim(X) - \dim(X_i).$$

π is said to be small if π is semi-small and if

$$(2\,\dim(\pi^{-1}(x_i)) = \dim(X) - \dim(X_i) \Rightarrow \dim(X) = \dim(X_i)).$$

In this case X_i is said to be relevant.

Note that stratification $X = \bigsqcup_i X_i$ exists ([17, 39]) and that the property of being semi-small or small does not depend on the stratification.

When π is projective and Z is rationally smooth, this geometric situation is of particular interest as there is a very elegant description [2, Section 1.5] of the singularities of such maps in terms of intersection homology sheaves [15, 16]: by using [2, Section 1.7] the decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber [1], for $u \in X$, the cohomology groups $H^i(\pi^{-1}(u), \mathbb{Q})$ of the fiber $\pi^{-1}(u)$ are given by explicit formula involving the intersection homology of the closures $\overline{X_i}$ of strata such that $u \in \overline{X_i}$. The formula [2, Section 1.5] can be expressed as a sum indexed by certain pairs (X_i, ϕ) where:

- X_i is a relevant stratum,
- $u \in \overline{X_i}$,
- ϕ is an irreducible representation of the fundamental group $\pi_1(X_i)$ of X_i,
- ϕ occurs in the decomposition of the representation of $\pi_1(X_i)$ on $H^{2\dim(\pi^{-1}(x_i))}(\pi^{-1}(x_i), \mathbb{Q})$ by monodromy.

The case of small resolutions is remarkable, as the formula reduces to a single summand (and in this case the result is essentially given in [16]).

A fundamental example of semi-small morphism is given by the Springer resolution $T^*\mathcal{B} \to \mathcal{N}$ of the nilpotent cone of a complex simple Lie algebra, and the corresponding partial resolutions [2].

Nakajima [30, 31] defined important and intensively studied varieties $\mathfrak{M}(v, w)$, $\mathfrak{M}_0(v, w)$ called quiver varieties which depend on a quiver Q (see [34, 38] for recent reviews). They come with a resolution

$$\pi\colon \mathfrak{M}(v, w) \to \mathfrak{M}_0(v, w),$$

which gives an analog of the Springer resolution. It is proved in [31, Corollary 10.11] for Q a finite Dynkin diagram type that π is semi-small (see [34, Section 5.2]).

The graded version of quiver varieties $\mathfrak{M}^\bullet(V,W), \mathfrak{M}_0^\bullet(V,W)$ are also of particular importance, for example for their deep relations with representations of quantum affine algebras (see [37]).

Let us recall the definition of these varieties:

2.2. (Graded) Quiver varieties

This section is essentially contained in [37].

Fix a Dynkin diagram and an orientation on this diagram. Let H be the set of oriented edges of the Dynkin diagram. For $h \in H$, $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) is the incoming (resp. outgoing) vertex of h, and $\overline{h}$ is the same edge as h with the reverse orientation. We fix $q : H \to \{1,-1\}$ such that $q(h) = -q(\overline{h})$ for any $h \in H$.

Let $V = \bigoplus_{i\in I, a\in\mathbb{C}^*} V_{i,a}$ (resp. $W = \bigoplus_{i\in I, a\in\mathbb{C}^*} W_{i,a}$) be a $I \times \mathbb{C}^*$-graded vector space such that the $V_{i,a}$ (resp. $W_{i,a}$) are finite dimensional and for at most finitely many $i \times a$, $V_{i,a} \neq 0$ (resp. $W_{i,a} \neq 0$). Consider for $n \in \mathbb{Z}$:

$$\mathrm{L}^\bullet(V,W)^{[n]} = \bigoplus_{i\in I, a\in\mathbb{C}^*} \mathrm{Hom}(V_{i,a}, W_{i,aq^n}),$$

$$\mathrm{E}^\bullet(V,W)^{[n]} = \bigoplus_{h\in H, a\in\mathbb{C}^*} \mathrm{Hom}(V_{\mathrm{out}(h),a}, W_{\mathrm{in}(h),aq^n}),$$

$$\mathbf{M}^\bullet(V,W) = \mathrm{E}^\bullet(V,V)^{[-1]} \oplus \mathrm{L}^\bullet(W,V)^{[-1]} \oplus \mathrm{L}^\bullet(V,W)^{[-1]}.$$

The above three components for an element of $\mathbf{M}^\bullet(V,W)$ are denoted by B, α, β respectively, the $\mathrm{Hom}(V_{\mathrm{out}(h),a}, V_{\mathrm{in}(h),aq^{-1}})$-component of B is denoted by $B_{h,a}$ and we denote by $\alpha_{i,a}$, $\beta_{i,a}$ the components of α, β. Consider the map

$$\mu\colon \mathbf{M}^\bullet(V,W) \to \mathrm{L}^\bullet(V,V)^{[-2]}$$

defined by

$$\mu_{i,a}(B,\alpha,\beta) = \sum_{\mathrm{in}(h)=i} q(h) B_{h,aq^{-1}} B_{\overline{h},a} + \alpha_{i,aq^{-1}}\beta_{i,a},$$

where $\mu_{i,a}$ is the (i,a)-component of μ. We have an action of $G_V = \prod_{i,a} \mathrm{GL}(V_{i,a})$ on $\mathbf{M}^\bullet(V,W)$ defined by

$$(B,\alpha,\beta) \mapsto g\cdot(B,\alpha,\beta) = (g_{\mathrm{in}(h),aq^{-1}} B_{h,a} g^{-1}_{\mathrm{out}(h),a},\ g_{i,aq^{-1}}\alpha_{i,a},\ \beta_{i,a} g^{-1}_{i,a}).$$

The subvariety $\mu^{-1}(0)$ in $\mathbf{M}^\bullet(V,W)$ is stable under the action.

Let us denote by $\mu^{-1}(0)^{\mathrm{s}}$ the set of stable points $(B,\alpha,\beta) \in \mu^{-1}(0)$, that is to say satisfying the condition: if an $I \times \mathbb{C}^*$-graded subspace S of V is B-invariant and contained in $\mathrm{Ker}(\beta)$, then $S = 0$. The stability condition is invariant under the action of G_V, so we may say an orbit is stable or not.

Consider the following quotient spaces of $\mu^{-1}(0)$:

$$\mathfrak{M}_0^\bullet(V,W) = \mu^{-1}(0) /\!/ G_V, \qquad \mathfrak{M}^\bullet(V,W) = \mu^{-1}(0)^{\mathrm{s}}/G_V.$$

Here // is the affine algebro-geometric quotient, the second one is the set-theoretical quotient. By [31, 3.18], there exists a natural projective morphism

$$\pi: \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W).$$

For $x \in \mu^{-1}(0)^{\mathrm{s}}$, $\pi(G_V.x)$ is the unique closed orbit contained in the closure of $G_V.x$.

Here $\mathfrak{M}^\bullet(V,W)$ is non singular and π can be considered as an analog of the Springer resolution.

A natural problem addressed in the present paper is to study the small property of such resolutions π: in the present paper we address [37, Conjecture 10.4] (see also [35]).

As our proof relies on the representation theory of quantum affine algebras, let us give some background about this subject:

3. Quantum affine algebras and their representations

In this section we recall definitions and results about the representation theory of quantum affine algebras.

3.1. Cartan matrix and quantized Cartan matrix

Let $C = (C_{i,j})_{1\leq i,j\leq n}$ be a Cartan matrix of finite type. We denote $I = \{1,\dots,n\}$. C is symmetrizable: there is a matrix $D = \mathrm{diag}(r_1,\dots,r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric. In particular if C is symmetric then $D = I_n$ (simply-laced case).

We consider a realization $(\mathfrak{h},\Pi,\Pi^\vee)$ of C (see [3, 25]): $\mathfrak{h}$ is a n dimensional $\mathbb{Q}$-vector space, $\Pi = \{\alpha_1,\dots,\alpha_n\} \subset \mathfrak{h}^*$ (set of the simple roots) and $\Pi^\vee = \{\alpha_1^\vee,\dots,\alpha_n^\vee\} \subset \mathfrak{h}$ (set of simple coroots) are set such that for $1 \leq i,j \leq n$, $\alpha_j(\alpha_i^\vee) = C_{i,j}$. Let $\Lambda_1,\dots,\Lambda_n \in \mathfrak{h}^*$ (resp. $\Lambda_1^\vee,\dots,\Lambda_n^\vee \in \mathfrak{h}$) be the fundamental weights (resp. coweights): $\Lambda_i(\alpha_j^\vee) = \alpha_i(\Lambda_j^\vee) = \delta_{i,j}$ where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. Denote $P = \{\lambda \in \mathfrak{h}^* \mid \forall i \in I, \lambda(\alpha_i^\vee) \in \mathbb{Z}\}$ the set of weights and $P^+ = \{\lambda \in P \mid \forall i \in I, \lambda(\alpha_i^\vee) \geq 0\}$ the set of dominant weights. For example we have $\alpha_1,\dots,\alpha_n \in P$ and $\Lambda_1,\dots,\Lambda_n \in P^+$. Denote $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i \subset P$ the root lattice and $Q^+ = \sum_{i\in I}\mathbb{N}\alpha_i \subset Q$. For $\lambda,\mu \in \mathfrak{h}^*$, denote $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. Let $\nu : \mathfrak{h}^* \to \mathfrak{h}$ linear such that for all $i \in I$, we have $\nu(\alpha_i) = r_i\alpha_i^\vee$. For $\lambda,\mu \in \mathfrak{h}^*$, $\lambda(\nu(\mu)) = \mu(\nu(\lambda))$. We use the enumeration of vertices of [25].

We denote $q_i = q^{r_i}$ and for $l \in \mathbb{Z}, r \geq 0, m \geq m' \geq 0$ we define in $\mathbb{Z}[q^\pm]$:

$$[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}}\,,\ [r]_q! = [r]_q[r-1]_q\cdots[1]_q\,,\ \begin{bmatrix} m \\ m' \end{bmatrix}_q = \frac{[m]_q!}{[m-m']_q![m']_q!}.$$

For $a,b \in \mathbb{Z}$, we denote $q^{a+b\mathbb{Z}} = \{q^{a+br} \mid r \in \mathbb{Z}\}$ and $q^{a+b\mathbb{N}} = \{q^{a+br} \mid r \in \mathbb{Z}, r \geq 0\}$.

Let $C(z)$ be the quantized Cartan matrix defined by ($i \neq j \in I$):

$$C_{i,i}(z) = z_i + z_i^{-1}\,,\ C_{i,j}(z) = [C_{i,j}]_z.$$

$C(z)$ is invertible (see [14]). We denote by $\tilde{C}(z)$ the inverse matrix of $C(z)$ and by $D(z)$ the diagonal matrix such that for $i,j \in I$, $D_{i,j}(z) = \delta_{i,j}[r_i]_z$.

3.2. Quantum algebras

3.2.1. *Quantum groups*

DEFINITION 3.1. – *The quantum group $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators $k_i^{\pm 1}$, $x_i^{\pm}$ ($i \in I$) and relations:*

$$k_i k_j = k_j k_i \, , \, k_i x_j^{\pm} = q_i^{\pm C_{i,j}} x_j^{\pm} k_i,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0\cdots 1-C_{i,j}} (-1)^r \begin{bmatrix} 1-C_{i,j} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^{1-C_{i,j}-r} x_j^{\pm} (x_i^{\pm})^r = 0 \text{ (for } i \neq j).$$

This algebra was introduced independently by Drinfeld [11] and Jimbo [24]. It is remarkable that one can define a Hopf algebra structure on $\mathcal{U}_q(\mathfrak{g})$ by:

$$\Delta(k_i) = k_i \otimes k_i,$$

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+ \, , \, \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-,$$

$$S(k_i) = k_i^{-1} \, , \, S(x_i^+) = -x_i^+ k_i^{-1} \, , \, S(x_i^-) = -k_i x_i^-,$$

$$\epsilon(k_i) = 1 \, , \, \epsilon(x_i^+) = \epsilon(x_i^-) = 0.$$

Let $\mathcal{U}_q(\mathfrak{h})$ be the commutative subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the $k_i^{\pm 1}$ ($i \in I$).

For V a $\mathcal{U}_q(\mathfrak{h})$-module and $\omega \in P$ we denote by V_ω the weight space of weight ω:

$$V_\omega = \{v \in V \mid \forall i \in I, k_i.v = q_i^{\omega(\alpha_i^\vee)} v\}.$$

In particular we have $x_i^{\pm}.V_\omega \subset V_{\omega \pm \alpha_i}$. We say that V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable if $V = \bigoplus_{\omega \in P} V_\omega$ (in particular V is of type 1).

3.2.2. *Quantum loop algebras*. – We will use the second realization (Drinfeld realization) of the quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ (subquotient of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$):

DEFINITION 3.2. – *$\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is the algebra with generators $x_{i,r}^{\pm}$ ($i \in I, r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,m}$ ($i \in I, m \in \mathbb{Z} - \{0\}$) and the following relations ($i, j \in I, r, r' \in \mathbb{Z}, m, m' \in \mathbb{Z} - \{0\}$):*

$$[k_i, k_j] = [k_i, h_{j,m}] = [h_{i,m}, h_{j,m'}] = 0,$$

$$k_i x_{j,r}^{\pm} = q_i^{\pm C_{i,j}} x_{j,r}^{\pm} k_i,$$

$$[h_{i,m}, x_{j,r}^{\pm}] = \pm \frac{1}{m} [m B_{i,j}]_q x_{j,m+r}^{\pm},$$

$$[x_{i,r}^+, x_{j,r'}^-] = \delta_{i,j} \frac{\phi_{i,r+r'}^+ - \phi_{i,r+r'}^-}{q_i - q_i^{-1}},$$

$$x_{i,r+1}^{\pm} x_{j,r'}^{\pm} - q^{\pm B_{i,j}} x_{j,r'}^{\pm} x_{i,r+1}^{\pm} = q^{\pm B_{i,j}} x_{i,r}^{\pm} x_{j,r'+1}^{\pm} - x_{j,r'+1}^{\pm} x_{i,r}^{\pm},$$

$$\sum_{\pi \in \Sigma_s} \sum_{k=0\cdots s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_{i,r_{\pi(1)}}^{\pm} \cdots x_{i,r_{\pi(k)}}^{\pm} x_{j,r'}^{\pm} x_{i,r_{\pi(k+1)}}^{\pm} \cdots x_{i,r_{\pi(s)}}^{\pm} = 0,$$

where the last relation holds for all $i \neq j$, $s = 1 - C_{i,j}$, all sequences of integers $r_1, \dots, r_s$. Σ_s is the symmetric group on s letters. For $i \in I$ and $m \in \mathbb{Z}$, $\phi^{\pm}_{i,m} \in \mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is determined by the formal power series in $\mathcal{U}_q(\mathcal{L}\mathfrak{g})[[z]]$ (resp. in $\mathcal{U}_q(\mathcal{L}\mathfrak{g})[[z^{-1}]]$):

$$\sum_{m\geq 0}\phi^{\pm}_{i,\pm m}z^{\pm m} = k_i^{\pm}\exp(\pm(q-q^{-1})\sum_{m'\geq 1}h_{i,\pm m'}z^{\pm m'}),$$

and $\phi^{\pm}_{i,\mp m} = 0$ for $m > 0$.

$\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ has a Hopf algebra structure (from the Hopf algebra structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$).

For $J \subset I$ we denote by $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$ the subalgebra generated by the $x^{\pm}_{i,m}$, $h_{i,m}$, $k_i^{\pm 1}$ for $i \in J$. $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$ is a quantum loop algebra associated to the semi-simple Lie algebra $\mathfrak{g}_J$ of Cartan matrix $(C_{i,j})_{i,j\in J}$. For example for $i \in I$, we denote $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i) = \mathcal{U}_q(\mathcal{L}\mathfrak{g}_{\{i\}}) \simeq \mathcal{U}_{q_i}(\mathcal{L}sl_2)$.

The subalgebra of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ generated by the $h_{i,m}, k_i^{\pm 1}$ (resp. by the $x^{\pm}_{i,r}$) is denoted by $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$ (resp. $\mathcal{U}_q(\mathcal{L}\mathfrak{g})^{\pm}$).

3.3. Finite dimensional representations of quantum loop algebras

Denote by $\mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}))$ the Grothendieck ring of (type 1) finite dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

3.3.1. *Monomials and q-characters.* – Let V be a representation in $\mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}))$. The subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is commutative, so we have:

$$V = \bigoplus_{\gamma=(\gamma^{\pm}_{i,\pm m})_{i\in I, m\geq 0}} V_\gamma,$$

where: $V_\gamma = \{v \in V \mid \exists p \geq 0, \forall i \in I, m \geq 0, (\phi^{\pm}_{i,\pm m} - \gamma^{\pm}_{i,\pm m})^p.v = 0\}$.

The $\gamma = (\gamma^{\pm}_{i,\pm m})_{i\in I, m\geq 0}$ are called l-weights (or pseudo-weights) and the $V_\gamma \neq \{0\}$ are called l-weight spaces (or pseudo-weight spaces) of V. One can prove [14] that γ is necessarily of the form:

$$\sum_{m\geq 0}\gamma^{\pm}_{i,\pm m}u^{\pm m} = q_i^{\deg(Q_i)-\deg(R_i)}\frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})},$$

where $Q_i, R_i \in \mathbb{C}(u)$ satisfy $Q_i(0) = R_i(0) = 1$.

Consider the ring $\mathcal{Y} = \mathbb{Z}[Y^{\pm}_{i,a}]_{i\in I, a\in\mathbb{C}^*}$. The Frenkel-Reshetikhin q-characters morphism χ_q [14] encodes the l-weights γ (see also [27]). It is an injective ring morphism:

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g})) \to \mathcal{Y}$$

defined by

$$\chi_q(V) = \sum_\gamma \dim(V_\gamma)m_\gamma,$$

where:

$$m_\gamma = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{q_{i,a}-r_{i,a}},$$

$$Q_i(u) = \prod_{a\in\mathbb{C}^*}(1-ua)^{q_{i,a}}\ ,\ R_i(u) = \prod_{a\in\mathbb{C}^*}(1-ua)^{r_{i,a}}.$$

The m_γ are called monomials (they are analogs of weight). We denote by A the set of monomials of $\mathbb{Z}[Y^{\pm}_{i,a}]_{i\in I, a\in\mathbb{C}^*}$. For an l-weight γ, we set $V_\gamma = V_{m_\gamma}$. We will also use the notation $i_r^p = Y^p_{i,q^r}$ for $i \in I$ and $r, p \in \mathbb{Z}$.

For $J \subset I$, χ_q^J is the morphism of q-characters for $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$. For an m monomial we denote $u_{i,a}(m) \in \mathbb{Z}$ such that $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}$. We also denote

$$\omega(m) = \sum_{i\in I, a\in\mathbb{C}^*} u_{i,a}(m)\Lambda_i \;,\; u_i(m) = \sum_{a\in\mathbb{C}^*} u_{i,a}(m) \;,\; u(m) = \sum_{i\in I} u_i(m).$$

m is said to be J-dominant if for all $j \in J, a \in \mathbb{C}^*$ we have $u_{j,a}(m) \geq 0$. An I-dominant monomial is said to be dominant.

Observe that χ_q, χ_q^J can also be defined for finite dimensional $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-modules in the same way.

In the following for V a finite dimensional $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-module, we denote by $\mathcal{M}(V)$ the set of monomials occurring in $\chi_q(V)$.

For $i \in I, a \in \mathbb{C}^*$, consider the analogs of simple roots for monomials:

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{\{j|C_{j,i}=-1\}} Y_{j,a}^{-1} \prod_{\{j|C_{j,i}=-2\}} Y_{j,aq^{-1}}^{-1} Y_{j,aq}^{-1} \prod_{\{j|C_{j,i}=-3\}} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1}.$$

As the $A_{i,a}$ are algebraically independent [14] (because $C(z)$ is invertible), for M a product of $A_{i,a}^{\pm 1}$ we can define $v_{i,a}(M) \in \mathbb{Z}$ by $M = \prod_{i\in I, a\in\mathbb{C}^*} A_{i,a}^{-v_{i,a}(m)}$. We put $v_i(M) = \sum_{a\in\mathbb{C}^*} v_{i,a}(M)$ and $v(M) = \sum_{i\in I} v_i(M)$.

For $\lambda \in Q^+$ we set $v(\lambda) = -\lambda(\Lambda_1^\vee + \cdots + \Lambda_n^\vee)$. For M a product of $A_{i,a}^{\pm 1}$, we have $v(M) = v(\omega(\lambda))$.

For m, m' two monomials, we write $m' \leq m$ if $m'm^{-1}$ is a product of $A_{i,a}^{-1}$.

Definition 3.3 ([13]). – *A monomial $m \in A - \{1\}$ is said to be right-negative if for all $a \in \mathbb{C}^*$, for $L = \max\{l \in \mathbb{Z} \mid \exists i \in I, u_{i,aq^l}(m) \neq 0\}$, we have $\forall j \in I$, $u_{j,aq^L}(m) \leq 0$.*

Observe that a right-negative monomial is not dominant.

Lemma 3.4 ([13]). – 1) *For $i \in I, a \in \mathbb{C}^*$, $A_{i,a}^{-1}$ is right-negative.*
2) *A product of right-negative monomials is right-negative.*
3) *If m is right-negative, then $m' \leq m$ implies that m' is right-negative.*

For $J \subset I$ and $Z \in \mathcal{Y}$, we denote by $Z^{\to J}$ the element of $\mathcal{Y}$ obtained from Z by putting $Y_{j,a}^{\pm 1} = 1$ for $j \notin J$.

3.3.2. *l-highest weight representations.* – The irreducible finite dimensional $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-modules have been classified by Chari-Pressley. They are parameterized by dominant monomials:

Definition 3.5. – *A $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module V is said to be of l-highest weight $m \in A$ if there is $v \in V_m$ such that $V = \mathcal{U}_q(\mathcal{L}\mathfrak{g})^-.v$ and $\forall i \in I, r \in \mathbb{Z}, x_{i,r}^+.v = 0$.*

For $m \in A$, there is a unique simple module $L(m)$ of l-highest weight m.

Theorem 3.6 ([7, Theorem 12.2.6]). – *The dimension of $L(m)$ is finite if and only if m is dominant.*

For $i \in I$, $a \in \mathbb{C}^*$, $k \geq 0$ we set $X_{k,a}^{(i)} = \prod_{k'\in\{1,\ldots,k\}} Y_{i,aq_i^{k-2k'+1}}$.

Definition 3.7. – *The Kirillov-Reshetikhin modules are the $W_{k,a}^{(i)} = L(X_{k,a}^{(i)})$.*

For $i \in I$ and $a \in \mathbb{C}^*$, $W_{1,a}^{(i)}$ is called a fundamental representation and is denoted by $V_i(a)$ (in the case $\mathfrak{g} = sl_2$ we simply write $W_{k,a}$ and $V(a)$).

For $m \in \mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^*}$ a dominant monomial, the standard module $M(m)$ is defined [32, 40] as the tensor product:

$$M(m) = \bigotimes_{a \in (\mathbb{C}^*/q^{\mathbb{Z}})} \left(\cdots \otimes \left(\bigotimes_{i \in I} V_i(aq)^{\otimes u_{i,aq}(m)} \right) \otimes \left(\bigotimes_{i \in I} V_i(aq^2)^{\otimes u_{i,aq^2}(m)} \right) \otimes \cdots \right).$$

It is well-defined as for $i, j \in I$ and $a \in \mathbb{C}^*$ we have $V_i(a) \otimes V_j(a) \simeq V_j(a) \otimes V_i(a)$ and for $a' \notin aq^{\mathbb{Z}}$, we have $V_i(a) \otimes V_j(a') \simeq V_j(a') \otimes V_j(a)$. Observe that fundamental representations are particular cases of standard modules.

Let $\mathfrak{g} = sl_2$. The monomials $m_1 = X_{k_1,a_1}$, $m_2 = X_{k_2,a_2}$ are said to be in special position if the monomial $m_3 = \prod_{a \in \mathbb{C}^*} Y_a^{\max(u_a(m_1), u_a(m_2))}$ is of the form $m_3 = X_{k_3,a_3}$ and $m_3 \neq m_1, m_3 \neq m_2$. A normal writing of a dominant monomial m is a product decomposition $m = \prod_{i=1,\ldots,L} X_{k_l,a_l}$ such that for $l \neq l'$, X_{k_l,a_l}, $X_{k_{l'},a_{l'}}$ are not in special position. Any dominant monomial has a unique normal writing up to permuting the monomials (see [7, Section 12.2]). It follows from the study of the representations of $\mathcal{U}_q(\mathcal{L}sl_2)$ in [6, 8, 14] that:

Proposition 3.8. – *Suppose that $\mathfrak{g} = sl_2$.*

(1) *$W_{k,a}$ is of dimension $k+1$ and:*

$$\chi_q(W_{k,a}) = X_{k,a}(1 + A_{aq^k}^{-1}(1 + A_{aq^{k-2}}^{-1}(1 + \cdots (1 + A_{aq^{2-k}}^{-1})) \cdots).$$

(2) *$V(aq^{1-k}) \otimes V(aq^{3-k}) \otimes \cdots \otimes V(aq^{k-1})$ is of q-character:*

$$X_{k,a}(1 + A_{aq^k}^{-1})(1 + A_{aq^{k-2}}^{-1}) \cdots (1 + A_{aq^{2-k}}^{-1}).$$

In particular all l-weight spaces of the tensor product are of dimension 1.

(3) *For m a dominant monomial and $m = X_{k_1,a_1} \cdots X_{k_l,a_l}$ a normal writing we have:*

$$L(m) \simeq W_{k_1,a_1} \otimes \cdots \otimes W_{k_l,a_l}.$$

3.3.3. *Special modules and complementary reminders.* – Let us consider analogs of cones of weights (for example used to define category $\mathcal{O}$ for affine Kac-Moody algebras) adapted to monomials:

Definition 3.9. – *For $m \in A$, $D(m)$ is the set of monomials $m' \in A$ such that there are $m_0 = m, m_1, \ldots, m_N = m' \in A$ satisfying for all $j \in \{1, \ldots, N\}$:*

1. *$m_j = m_{j-1} A_{i_j,a_1 q_{i_j}}^{-1} \cdots A_{i_j,a_{r_j} q_{i_j}}^{-1}$ where $i_j \in I$, $r_j \geq 1$ and $a_1, \ldots, a_{r_j} \in \mathbb{C}^*$,*
2. *for $1 \leq r \leq r_j$, $u_{i_j,a_r}(m_{j-1}) \geq |\{r' \in \{1, \ldots, r_j\} | a_{r'} = a_r\}|$ where r_j, i_j, a_r are as in condition (1).*

The motivation for this definition comes from the two simple facts:

for all $m' \in D(m)$, $m' \leq m$,

if $m' \in D(m)$, then $(D(m') \subset D(m))$,

and from the following result which gives a strong condition for a monomial to appear in a q-character:

Theorem 3.10 ([22, Theorem 5.21]). – *For V a finite dimensional l-highest weight module of highest monomial m, we have $\mathcal{M}(V) \subset D(m)$.*

In particular for all $m' \in \mathcal{M}(V)$, we have $m' \leq m$ and the $v_{i,a}(m'm^{-1}), v(m'm^{-1}) \geq 0$ are well-defined. As a direct consequence of Theorem 3.10, we also have:

LEMMA 3.11. – *For $i \in I, a \in \mathbb{C}^*$, we have $(\chi_q(V_i(a)) - Y_{i,a}) \in \mathbb{Z}[Y^{\pm}_{j,aq^l}]_{j \in I, l > 0}$.*

This last result was first proved in [13, Lemma 6.1, Remark 6.2].

The notion of special module was introduced in [37]:

DEFINITION 3.12. – *A $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module is said to be special if its q-character has a unique dominant monomial.*

This notion is of particular importance because an algorithm of Frenkel-Mukhin [13] gives the q-character of special modules. Observe that a special module is a simple l-highest weight module (as each simple module occurring in the Jordan-Hölder series of a representation contributes with at least one dominant monomial in the q-character). But in general all simple l-highest weight modules are not special.

The following result was proved in [37, 36] for simply laced types, and in full generality in [21] (see [13] for previous results). It gives a remarkable example of a family of special modules and is the crucial point for the proof of the Kirillov-Reshetikhin conjecture:

THEOREM 3.13 ([21, Theorem 4.1, Lemma 4.4]). – *The Kirillov-Reshetikhin modules are special. Moreover for $m \in \mathcal{M}(W^{(i)}_{k,a})$, $m \neq X^{(i)}_{k,a}$ implies $m \leq X^{(i)}_{k,a} A^{-1}_{i,aq_i^k}$.*

Now let us recall a decomposition result of q-characters relatively to sub-Dynkin diagrams corresponding to $J \subset I$ (Proposition 3.14). This is the analog at the level of q-character of the decomposition of a simple representation in simple representations for the subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$. This result will be intensively used in the following.

Define

$$\mu^I_J : \mathbb{Z}[(A^{\pm}_{j,a})^{\rightarrow(J)}]_{j \in J, a \in \mathbb{C}^*} \rightarrow \mathbb{Z}[A^{\pm}_{j,a}]_{j \in J, a \in \mathbb{C}^*},$$

the ring morphism such that $\mu^I_J((A^{\pm}_{j,a})^{\rightarrow(J)}) = A^{\pm}_{j,a}$. For m J-dominant, denote by $L^J(m^{\rightarrow(J)})$ the simple $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$-module of l-highest weight $m^{\rightarrow(J)}$. Define:

$$L_J(m) = m\mu^I_J((m^{\rightarrow(J)})^{-1}\chi^J_q(L^J(m^{\rightarrow(J)}))).$$

(Observe that from Proposition 3.8, we have explicit formulas for the $L_{\{i\}}(m)$ for $i \in I$.)

PROPOSITION 3.14 ([19, Proposition 3.1]). – *For a representation $V \in \mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}))$ and $J \subset I$, there is unique decomposition in a finite sum:*

$$\chi_q(V) = \sum_{m' \text{ } J\text{-dominant}} \lambda_J(m') L_J(m'). \tag{1}$$

Moreover for all m' J-dominant we have $\lambda_J(m') \geq 0$.

(In [19] the $\lambda_J(m') \geq 0$ were assumed, but the proof of the uniqueness does not depend on it.)

As a consequence:

COROLLARY 3.15. – *Let m be a dominant monomial and m' such that*

(i) *$m' \in \mathcal{M}(L(m))$,*

(ii) *m' is J-dominant monomial,*

(iii) *there are no $m'' > m'$ satisfying $m'' \in \mathcal{M}(m)$ and m' appears in $L_J(m'')$.*

Then the monomials of $L_J(m')$ are in $\mathcal{M}(L(m))$.

Proof. – From the last condition $L_J(m')$ occurs in the decomposition of Proposition 3.14. As the coefficients in this decomposition are positive, all monomials on $L_J(m')$ occur in $\chi_q(L(m))$. □

REMARK 3.16. – *In Corollary 3.15, we can start with $m' = m$, and then we use for m' monomials in $L_J(m)$, and so on. This process gives inductively from m a set of monomial occurring in $\chi_q(L(m))$.*

4. Representation theoretical interpretation of the small property

In this section $\mathfrak{g}$ is simply laced.

Originally the notion of small modules was given in terms of q,t-characters [37]. We recall this definition and the relation [37] with the geometric small property of Section 2 (Theorem 4.3).

Although the representation theoretical meaning of q,t-character is not totally understood (see [33, Conjecture 3.1.1]), the notion of small modules can be purely algebraically formulated: we give an additional representation theoretical interpretation of the notion (Theorem 4.8) by refining a proof of [37] (this provides an additional algebraic motivation for the study of the small modules).

We also comment the main result of the present paper (Theorem 1.2).

4.1. Definition of small modules and q,t-characters

The notion of small modules is related to the notion of q,t-characters defined in [33, 37]. There are t-deformations of q-characters which can be purely algebraically defined (see [18] for non-simply laced cases with a different approach including a purely algebraic proof of the existence). They are a very powerful tool, as Nakajima proved they provide an algorithm which allows to compute the q-character of any simple representation.

Consider the commutative ring $\hat{\mathcal{Y}}_t = \mathbb{Z}[V_{i,a}, W_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$. A monomial of $\hat{\mathcal{Y}}_t$ is a product of $V_{i,a}$, $W_{i,a}$. One says $m' \leq m$ if $m'm^{-1}$ is a product of $V_{i,a}$. The q,t-characters map $\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g})) \to \hat{\mathcal{Y}}_t$ is a $\mathbb{Z}$-linear map defined by three axioms in [37]:

1) the data of the image of $\chi_{q,t}$,
2) a compatibility property of the tensor product with a certain twisted product on $\hat{\mathcal{Y}}_t$,
3) for $m \in \mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^*}$ a dominant monomial of $\mathcal{Y}$, the relation:
$$\chi_{q,t}(M(m)) \in M_0 + \sum_{m'<M_0} \mathbb{Z}[t^{\pm}]m' \text{ where } M_0 = \prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{u_{i,a}(m)}.$$
(Only the last axiom will be explicitly used in the following, and so we refer to [37] for the details of the first two axioms).

Let m be a monomial of $\hat{\mathcal{Y}}_t$. For $i \in I$, $a \in \mathbb{C}^*$, one defines $w_{i,a}(m), v_{i,a}(m) \geq 0$ by $m = \prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{w_{i,a}(m)} V_{i,a}^{v_{i,a}(m)}$, and:

$$u_{i,a}(m) = w_{i,a}(m) - v_{i,aq^{-1}}(m) - v_{i,aq}(m) + \sum_{j\in I} C_{i,j} v_{j,a}(m),$$
$$d(m) = \sum_{i\in I, a\in\mathbb{C}^*} (v_{i,aq}(m)u_{i,a}(m) + w_{i,aq}(m)v_{i,a}(m)).$$

We define a $\mathbb{Z}$-linear map $\hat{\Pi} : \hat{\mathcal{Y}}_t \to \mathcal{Y}$ by (m is a monomial):

$$\hat{\Pi}(m) = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}\ ,\ \hat{\Pi}(t) = 1.$$

It is clear that $\hat{\Pi}$ is a ring morphism.

A monomial m of $\hat{\mathcal{Y}}_t$ is said to be dominant if $\hat{\Pi}(m)$ is dominant. For m a dominant monomial of $\hat{\mathcal{Y}}_t$, one defines $M_t(m) \in \hat{\mathcal{Y}}_t$ by:

$$M_t(m) = t^{d(m)} m \Big(\prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{-u_{i,a}(m)}\Big) \chi_{q,t}\Big(M\Big(\prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}\Big)\Big) \in \hat{\mathcal{Y}}_t.$$

Lemma 4.1. – *For m a dominant monomial, we have $\hat{\Pi}(M_t(m)) = \chi_q(M(\hat{\Pi}(m)))$.*

Proof. – From the defining axioms of q,t-characters, the evaluation at $t = 1$ gives q-characters [37], that is to say:

$$\hat{\Pi}\Big(\chi_{q,t}\Big(M\Big(\prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}\Big)\Big)\Big) = \chi_q\Big(M\Big(\prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}\Big)\Big) = \chi_q(M(\hat{\Pi}(m))).$$

As $\hat{\Pi}\left(t^{d(m)} m \left(\prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{-u_{i,a}(m)}\right)\right) = 1$, the result is clear. □

For m a dominant monomial and $m' \leq m$ a monomial, $c_{m,m'}(t) \in \mathbb{Z}[t^{\pm}]$ is defined by:

$$M_t(m) = \sum_{m'\leq m} c_{m,m'}(t) t^{d(m')} m'.$$

Definition 4.2 ([37]). – *Let m be a dominant monomial of $\mathcal{Y}$. The standard module $M(m)$ is said to be small if for all dominant monomials $m', m'' \leq m$, we have $c_{m',m''}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.*

Remark: Observe that in general there is no hope to have $c_{m',m''}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ without assuming that m'' is dominant. For example for $\mathfrak{g} = sl_2$, we have

$$\chi_{q,t}(M(W_a)) = W_a + W_a V_{aq}\ ,\ d(W_a V_{aq}) = 0\ ,\ c_{W_a, W_a V_{aq}} = 1 \notin t^{-1}\mathbb{Z}[t^{-1}].$$

However $M(W_a)$ is small (see Proposition 6.2 below).

4.2. Geometric characterization

The motivation for this Definition 4.2 comes from geometry [37] and from the relation to the small property of Section 2:

Consider the monomials $m_W, m_V \in \hat{\mathcal{Y}}_t$ defined by

$$M_W = \prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{\dim(W_{i,a})}, m_V = \prod_{i\in I, a\in\mathbb{C}^*} V_{i,a}^{\dim(V_{i,a})}.$$

As a consequence of the geometric construction of representations of quantum affine algebras, we have the following geometric characterization of small standard modules (see [37, Remark 10.2]):

THEOREM 4.3 ([37]). – *Let m be a dominant monomial of $\hat{\mathcal{Y}}_t$ and $W = \bigoplus_{i\in I, a\in\mathbb{C}^*} W_{i,a}$ be the graded space satisfying* $\dim(W_{i,a}) = u_{i,a}(m)$. *The standard module $M(m)$ is small if and only if for all V such that $M_W m_V$ is dominant, the resolution $\pi\colon \mathfrak{M}^\bullet(V,W) \to \mathfrak{M}_0^\bullet(V,W)$ is small.*

4.3. Representation theoretical characterization

Let us give another characterization of small modules.

Consider the $\mathbb{Z}$-linear involution of $\hat{\mathcal{Y}}_t$ defined by $\overline{m} = t^{2d(m)}m$, $\overline{t} = t^{-1}$. Observe that for m a monomial of $\hat{\mathcal{Y}}_t$, $t^{d(m)}m$ is invariant by the involution.

In [37] Nakajima constructed a family $\mathcal{L}(m) \in \hat{\mathcal{Y}}_t$, indexed by the set of dominant monomials m of $\hat{\mathcal{Y}}_t$, characterized by the properties:

i) $\overline{\mathcal{L}(m)} = \mathcal{L}(m)$,
ii) $\mathcal{L}(m) \in M_t(m) + \sum_{\{m' \text{dominant} | m' < m\}} t^{-1}\mathbb{Z}[t^{-1}]M_t(m')$.

They are analogs of canonical bases in $\hat{\mathcal{Y}}$ for the bar involution, and the transition coefficients to the basis $(M_t(m))_m$ are analogs of Kazhdan-Lusztig polynomials.

Nakajima proved [37] the following deep result:

THEOREM 4.4 ([37]). – *For any dominant monomial m of $\hat{\mathcal{Y}}_t$, we have*

$$\hat{\Pi}(\mathcal{L}(m)) = \chi_q(L(\hat{\Pi}(m))).$$

In particular this provides an algorithm to compute the q-characters of simple modules. It is very complicated in general, and it is difficult to get explicit formulas from it, but it provides applications in situations where the algorithm can be simplified (for example see [36]).

As a consequence of this result, we have:

THEOREM 4.5 ([37]). – *Let m be a dominant monomial of $\mathcal{Y}$. If $M(m)$ is small, then for all dominant monomial $m' \leq m$, $L(m')$ is special.*

In fact the converse is true by using the following two results:

THEOREM 4.6 ([37, Theorem 3.5 (6)]). – *For any dominant monomial m of $\hat{\mathcal{Y}}_t$, the coefficient of a monomial occurring in $M_t(m)$ is a Laurent polynomial with nonnegative coefficients.*

LEMMA 4.7. – *For M a dominant monomial of $\hat{\mathcal{Y}}_t$, the set*

$$\{M' \mid M' \leq M \text{ and } M' \text{ is dominant}\}$$

is finite.

Proof. – We can suppose that $M \in \mathbb{Z}[Y_{i,aq^r}]_{i\in I, r\in\mathbb{Z}}$ where $a \in \mathbb{C}^*$. Let $K = \max\{r \in \mathbb{Z} \mid \exists i \in I, u_{i,aq^r}(M) \neq 0\}$. For $M' < M$, $M'M^{-1}$ is right-negative so M' dominant implies $\sum_{i\in I, r\geq K} v_{i,aq^r}(M'M^{-1}) = 0$. It is proved in [18, Lemma 3.14] that the set

$$\{M' = MA^{-1}_{i_1,aq^{l_1}} \cdots A^{-1}_{i_R,aq^{l_R}} \mid R \geq 0\,, l_1, \ldots, l_R \leq K, M' \text{ is dominant}\}$$

is finite, and so we can conclude (note that in [18], $l_1, \ldots, l_R \leq K$ is replaced by $l_1, \ldots, l_R \geq K$, but the proof is the same). □

By using a slight modification of the proof of Theorem 4.5 in [37], we get the following characterization:

THEOREM 4.8. – *Let m be a dominant monomial of $\mathcal{Y}$. $M(m)$ is small if and only if for all dominant monomial $m' \leq m$, $L(m')$ is special.*

Observe that it is a purely representation theoretical characterization of small modules involving q-characters, without q,t-characters. This provides an additional algebraic motivation for the study of the small modules: all simple modules which could appear in the "cone of monomial" of a small module are special, and so can be described by using the Frenkel-Mukhin algorithm.

Proof. – The only if part is the statement of Theorem 4.4. Let us prove the if part.

For m, m' dominant monomials of $\hat{\mathcal{Y}}_t$, we consider $Z_{m,m'}(t) \in \mathbb{Z}[t^{\pm}]$ defined by

$$M_t(m) = \sum_{m' \text{dominant}} Z_{m,m'}(t)\mathcal{L}(m').$$

By definition of $\mathcal{L}(m')$ we have $Z_{m,m}(t) = 1$ and $Z_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ for $m' < m$. If $m' \not\leq m$, we have $Z_{m,m'}(t) = 0$.

As $M = m(\prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{u_{i,a}(m)})^{-1}$ satisfies $u_{i,a}(M) = 0$ for any $i \in I, a \in \mathbb{C}^*$, we can suppose that $m = \prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{u_{i,a}(m)}$. Suppose that for all dominant monomial $m' \leq m$, $L(\hat{\Pi}(m'))$ is special. From Lemma 4.7, there is a finite number of dominant monomial $m' \leq m$. Choose a numbering $m_1, m_2, \ldots, m_N = m$ of these monomials such that $m_r < m_{r'}$ implies $r < r'$. Denote by $Z'_{m_r,m_{r'}}(t)$ the coefficients of m'_r in $\mathcal{L}(m_r)$. As $L(\hat{\Pi}(m_r))$ is special, it follows from Theorem 4.5 that $Z'_{m_r,m_{r'}}(1) = 0$. Let us prove by induction r that

$$\forall r'' < r' \leq r\,, c_{m_{r'},m_{r''}}(t) \in t^{-1}\mathbb{Z}[t^{-1}] \text{ and } Z'_{m_{r'},m_{r''}}(t) = 0.$$

For $r = 1$, we have $\mathcal{L}(m_1) = M_t(m_1)$. Now we consider $r > 1$. By the induction hypothesis, for all $r' < r$, $\mathcal{L}(m_{r'})$ has no dominant monomial except $m_{r'}$. So for $r'' < r$,

$$c_{m_r,m_{r''}}(t) = Z_{m_r,m_{r''}}(t) + Z'_{m_r,m_{r''}}(t).$$

From Theorem 4.6, we have

$$c_{m_r,m_{r''}}(t) = tP^+(t) + \alpha + t^{-1}P^-(t^{-1})$$

where $\alpha \geq 0$, $P^+, P^- \in \mathbb{N}[t]$. As $Z_{m_r,m_{r''}}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ and $Z'_{m_r,m_{r''}}(t) = Z'_{m_r,m_{r''}}(t^{-1})$, we have

$$Z'_{m_r,m_{r''}}(t) = tP^+(t) + \alpha + t^{-1}P^+(t^{-1}).$$

Hence $Z'_{m_r,m_{r''}}(t)$ has positive coefficients, so $Z'_{m_r,m_{r''}}(1) = 0$ implies $Z'_{m_r,m_{r''}}(t) = 0$. Therefore $c_{m_r,m_{r''}}(t) = Z_{m_r,m_{r''}}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$. As a conclusion, $M(m)$ is small. □

4.4. Main result

A natural question is to characterize small modules and so the corresponding small resolutions. In particular, Nakajima [37, Conjecture 10.4], [35] raised the problem of characterizing the Drinfeld polynomials of small standard modules corresponding to Kirillov-Reshetikhin modules.

The main result of this paper is an explicit answer to this question (Theorem 1.2). First let us note that in general the standard modules corresponding to Kirillov-Reshetikhin modules are not necessarily small:

REMARK 4.9. – *Let $\mathfrak{g} = sl_4$ and $m = Y_{2,1}Y_{2,q^2}Y_{2,q^4}$. Consider $m' = mA_{2,q}^{-1} = Y_{1,q}Y_{3,q}Y_{2,q^4}$. Then by using the process described in Remark 3.16, the monomials $Y_{1,q^3}^{-1}Y_{3,q^3}^{-1}Y_{2,q^2}^{2}Y_{2,q^4} = m'A_{1,q^2}^{-1}A_{3,q^2}^{-1}$ and $Y_{2,q^2} = m'A_{1,q^2}^{-1}A_{3,q^2}^{-1}A_{2,q^3}^{-1}$ occur in $\chi_q(L(m'))$ and $L(m')$ is not special. So $M(m)$ is not small.*

A crucial step for the proof of Theorem 1.2 is the elimination theorem proved in the next section.

5. Elimination theorem and preliminary results

In this section $\mathfrak{g}$ is an arbitrary semi-simple Lie algebra. We prove several preliminary results so that we can prove Theorem 1.2 in the last section of the paper.

5.1. Elimination Theorem

We have seen a (combinatorial) procedure which allows to produce monomials occurring in a q-character (Remark 3.16). We first prove in this section a (representation theoretical) theorem (Theorem 5.1) which gives a criterion so that a monomial m' does not occur in the q-character of a simple module $L(m)$.

This theorem is used in [23] to study minimal affinizations of representations of quantum groups.

5.1.1. *Statement*

THEOREM 5.1. – *Let $V = L(m)$ be a $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module simple module. Let $m' < m$ and $i \in I$ satisfying the following conditions*

(i) *there is a unique i-dominant $M \in (\mathcal{M}(V) \cap m'\mathbb{Z}[A_{i,a}]_{a\in\mathbb{C}^*}) - \{m'\}$ and its coefficient is* 1*,*

(ii) $\sum_{r\in\mathbb{Z}} x^+_{i,r}(V_M) = \{0\}$*,*

(iii) *m' is not a monomial of $L_i(M)$,*

(iv) *if $m'' \in \mathcal{M}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i).V_M)$ is i-dominant, then $v(m''m^{-1}) \geq v(m'm^{-1})$,*

(v) *for all $j \neq i$, $\{m'' \in \mathcal{M}(V) \mid v(m''m^{-1}) < v(m'm^{-1})\} \cap m'\mathbb{Z}[A^{\pm 1}_{j,a}]_{a\in\mathbb{C}^*} = \varnothing$.*

Then $m' \notin \mathcal{M}(V)$.

To prove this result, we first need some preliminary lemmas.

5.1.2. *Technical lemmas.* – First let us consider a refined version of the operators τ_j of [14] which allows to study "independently" the subalgebras $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$ of the quantum loop algebra.

Let $i \in I$, $\mathfrak{h}_i^\perp = \{\mu \in \mathfrak{h} \mid \alpha_i(\mu) = 0\}$ and let $A^{(i)}$ be the commutative group of monomials generated by variables $Y^\pm_{i,a}$ ($a \in \mathbb{C}^*$), k_μ ($\mu \in \mathfrak{h}_i^\perp$), $Z^\pm_{j,c}$ ($j \neq i$, $c \in \mathbb{C}^*$). Let

$$\tau_i : A \to A^{(i)}$$

be the group morphism defined by ($j \in I$, $a \in \mathbb{C}^*$):

$$\tau_i(Y_{j,a}) = Y_{j,a}^{\delta_{j,i}} \prod_{k\neq i, r\in\mathbb{Z}} Z_{k,aq^r}^{p_{j,k}(r)} k_{\nu(\Lambda_j)-\delta_{j,i}r_i\alpha_i^\vee/2}.$$

The $p_{j,k}(r) \in \mathbb{Z}$ are defined in the following way: we write $\tilde{C}(z) = \frac{\tilde{C}'(z)}{d(z)}$ where $d(z)$, $\tilde{C}'_{j,k}(z) \in \mathbb{Z}[z^\pm]$ and $(D(z)\tilde{C}'(z))_{j,k} = \sum_{r\in\mathbb{Z}} p_{j,k}(r) z^r$.

Observe that we have $\nu(\Lambda_j) - \delta_{j,i}r_i\alpha_i^\vee/2 \in \mathfrak{h}_i^\perp$ because $\alpha_i(\nu(\Lambda_j) - \delta_{j,i}r_i\alpha_i^\vee/2) = \Lambda_j(r_i\alpha_i^\vee) - r_i\delta_{i,j} = 0$.

This morphism τ_i was first defined [13], and then refined in [21] with the terms k which will be used in the following. Moreover it is proved in [13, Lemma 3.5] (in [20, Lemma 20] with the term k_0) that:

LEMMA 5.2. – *For $j \in I$, $a \in \mathbb{C}^*$, we have $\tau_j(A_{j,a}) = Y_{j,aq_j^{-1}} Y_{j,aq_j} k_0$.*

This result indicates that the root monomials $A_{j,a}$ are sent to their analogs of type sl_2, as announced above.

The following result was proved in [13, Lemma 3.4] without the term k_μ, and in [20, Lemma 21] the proof was extended for the terms k_μ. It gives a decomposition of a q-character "compatible" with the action of the subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$:

LEMMA 5.3. – *Let $V \in \mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}))$ and consider a decomposition $\tau_i(\chi_q(V)) = \sum_r P_r Q_r$ where $P_r \in \mathbb{Z}[Y^\pm_{i,a}]_{a\in\mathbb{C}^*}$, Q_r is a monomial in $\mathbb{Z}[Z^\pm_{j,c}, k_\lambda]_{j\neq i, c\in\mathbb{C}^*, \lambda\in\mathfrak{h}_i^\perp}$ and all monomials Q_r are distinct. Then the $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$-module V is isomorphic to a direct sum $\bigoplus_r V_r$ where $\chi_q^i(V_r) = P_r$.*

The following result gives information on a cyclic $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-submodule of a $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module:

LEMMA 5.4. – *Let $V \in \mathrm{Rep}(\mathcal{U}_q(\mathcal{L}\mathfrak{g}))$ be a $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module, $m \in \mathcal{M}(L(m))$ and $v \in V_m$. Then for $j \in I$, $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v$ is a sub-$\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-module of V and $\chi_q(\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v) \in m\mathbb{Z}[A_{j,a}^{\pm}]_{a\in\mathbb{C}^*}$.*

Proof. – From the relation 3.2, $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v$ is a sub-$\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-module of V. Consider the decomposition $\tau_j(\chi_q(V)) = \sum_r P_r Q_r$ of Lemma 5.3 and the decomposition of V as a $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-module: $V = \bigoplus_r V_r$. Then there is R such that $\tau_j(m)$ is a monomial of $P_R Q_R$, and so $v \in V_R$. We have $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v \subset V_R$. Let us write $\tau_j(m) = m_R Q_R$. It follows from [4, Theorem 7.2] for $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j) \simeq \mathcal{U}_{q_j}(\mathcal{L}sl_2)$, that the q-character of the $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-module $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v$ is included in $m_R\mathbb{Z}[(Y_{j,aq_j^{-1}}Y_{aq_j})^{\pm}]_{a\in\mathbb{C}^*}$. From Lemma 5.2, the q-character of $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).v$ viewed as a $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-module belongs to $m\mathbb{Z}[A_{j,a}^{\pm}]_{a\in\mathbb{C}^*}$. □

In the sl_2-case, the following lemma produces a dominant monomial higher than a given monomial in a q-character (note that a weak version was proved in [19, Lemma 3.2 (ii)]):

LEMMA 5.5. – *Let L be a finite dimensional $\mathcal{U}_q(\mathcal{L}sl_2)$-module. For $p \in \mathbb{Z}$, let $L_p = \sum_{\{\lambda\in P^* \mid \lambda(\Lambda^\vee)\geq p\}} L_\lambda$ and $L'_p = \sum_{r\in\mathbb{Z}} x_r^-.L_p$. Then for $m' \in \mathcal{M}(L'_p)$ there is $m \in \mathcal{M}(L_p)$ such that*

(i) *m is dominant,*
(ii) *$m' \leq m$,*
(iii) $(\mathcal{U}_q(\mathcal{L}sl_2).L_m) \cap L_{m'} \neq \{0\}$.

Proof. – Let $m' \in \mathcal{M}(L'_p)$. Let us prove the result by induction on $\dim(L_p)$: if $L_p = \{0\}$ we have $L'_p = \{0\}$. In general let v be an l-highest weight vector of L_p (it exists, see for example the proof of [20, Proposition 15]) and denote by M the corresponding monomial. Consider $V = \mathcal{U}_q(\hat{\mathfrak{g}}).v$. It is an l-highest weight module and so it follows from Theorem 3.10 that $(V_m \neq \{0\} \Rightarrow m \leq M)$. If $V_{m'} \neq \{0\}$ the result is clear with $m = M$. Otherwise consider $L^{(1)} = L/V$. Observe that $\chi_q(L) = \chi_q(V) + \chi_q(L^{(1)})$. We use the induction hypothesis with $L^{(1)}$ and we get $m \in \mathcal{M}((L^{(1)})_p) \subset \mathcal{M}(L_p)$ such that $m \geq m'$ and $(\mathcal{U}_q(\mathcal{L}sl_2).(L^{(1)})_m) \cap (L^{(1)})_{m'} \neq \{0\}$. Let $v \in (L^{(1)})_m$ and $\alpha \in \mathcal{U}_q(\mathcal{L}sl_2)$ such that $\alpha v \in (L^{(1)})_{m'} - \{0\}$. Let $w \in v + V$ and consider the decomposition $w = w_m + w'$ where $w_m \in L_m$ and $w' \in \bigoplus_{m''\neq m} L_{m''}$. Consider $v \in (L^{(1)})_m$; we have $w' \in V$ and $w_m \in v + V$. Then $\alpha w_m = v' + v'' \in L_{m'} \oplus V$ where $v' \neq 0$. As $V'_m = \{0\}$, there is $h \in \mathcal{U}_q(\mathfrak{h})$ such that $h\alpha w_m = hv' \neq 0$ and so we get the result. □

An analog result is available for general type:

LEMMA 5.6. – *Let $V = L(m)$ be a $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module simple module and $m' < m$ in $\mathcal{M}(L(m))$. Then there are $j \in I$ and $M' \in \mathcal{M}(V)$ such that*

(i) *M' is j-dominant,*
(ii) *$M' > m'$,*
(iii) *$M' \in m'\mathbb{Z}[A_{j,b}]_{b\in\mathbb{C}^*}$,*
(iv) $((\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).V_{M'}) \cap (V)_{m'}) \neq \{0\}$.

(A weak version of the following lemma was proved in the proof of [21, Lemma 4.4] with different notations).

To prove this result, we need the following additional notations: for $M \in A^{(i)}$, we define $\mu(M) \in \mathfrak{h}_i^{\perp}$, $u_{i,a}(M) \in \mathbb{Z}$, by:

$$M \in k_{\mu(M)} \prod_{a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(M)} \mathbb{Z}[Z_{j,c}^{\pm}]_{j\neq i, c\in\mathbb{C}^*}.$$

We also set $u_i(M) = \sum_{a\in\mathbb{C}^*} u_{i,a}(M)$. Observe that for $m \in A$ and $a \in \mathbb{C}^*$ we have $u_{i,a}(m) = u_{i,a}(\tau_i(m))$ and:

$$\nu(\omega(m)) = \mu(\tau_i(m)) + u_i(m) r_i \alpha_i^{\vee}/2 = \mu(\tau_i(m)) + u_i(\tau_i(m)) r_i \alpha_i^{\vee}/2,$$

or equivalently

$$\mu(\tau_i(m)) = \nu(\omega(m)) - \alpha_i(\nu(\omega(m)))\alpha_i^{\vee}/2.$$

(See the definition of [20, Section 5.5].) Now let us prove Lemma 5.6:

Proof. – For $m'' \in \mathcal{M}(V)$ define $w(m'') = v(\omega(m'') - \omega(m))$. Let

$$W = \bigoplus_{\{m'' | w(m'') < w(m')\}} V_{m''}.$$

As V is an l-highest weight module, there is $j \in I$ such that $(\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).W)_{m'} \neq \{0\}$. Consider the decomposition $\tau_j(\chi_q(V)) = \sum_r P_r Q_r$ of Lemma 5.3 and the decomposition of V as a $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-module: $V = \bigoplus_r V_r$.

For a given r, consider $M_r \in \mathcal{M}(V)$ such that $\tau_j(M_r)$ appears in $P_r Q_r$. For another such M, we have $\mu(\tau_j(M)) = \mu(\tau_j(M_r))$ and so

$$\omega(M M_r^{-1}) = u_j(\tau_j(M M_r^{-1}))\alpha_j^{\vee}/2,$$

and

$$u_j(\tau_j(M)) = u_j(\tau_j(M_r)) - 2w(M) + 2w(M_r) = 2(p - w(M)) + p_r,$$

where $p_r = -2p + 2w(M_r) + u_j(\tau_j(M_r))$ (it does not depend on M). So we have $w(M) \leq p \Leftrightarrow u_j(\tau_j(M)) \geq p_r$. So $W = \bigoplus_r ((V_r)_{\geq p_r}) = \bigoplus_r (V_r \cap W)$. As V_r is a sub $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-module of V, we have $W_j = \bigoplus_r (V_r \cap W_j)$. Let $M \in \mathcal{M}(W_j)$ and let R be such that $\tau_j(M)$ is a monomial of $P_R Q_R$. We can apply Lemma 5.5 to the $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j)$-module V_R with $p = p_R$ and the monomial $Q_R^{-1}\tau_j(m)$: we get $m'' \in \mathcal{M}(V_R)$ dominant such that $Q_R^{-1}\tau_j(M) \in m''\mathbb{Z}[(Y_{j,a}Y_{j,aq_j^2})^{-1}]_{a\in\mathbb{C}^*}$ and $((\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).(V_R)_{m''}) \cap (V_R)_{Q_R^{-1}\tau_j(M)}) \neq \{0\}$. Let us translate this result in terms of monomials of $\chi_q(V)$. Consider the j-dominant monomial $M' = \tau_j^{-1}(Q_R m')$. Then $M' \in \mathcal{M}(W)$ and $(\mathcal{U}_q(\mathcal{L}\mathfrak{g}_j).V_{M'}) \cap V_M \neq \{0\}$. From Lemma 5.2 we have $M \in M'\mathbb{Z}[A_{j,b}^{-1}]_{b\in\mathbb{C}^*}$. □

5.1.3. *Proof of Theorem 5.1*. – Suppose that $m' \in \mathcal{M}(V)$. Let

$$W = \bigoplus_{\{M' \leq m | v(M'm^{-1}) < v(m'm^{-1})\}} V_{M'}.$$

As V is an l-highest weight module, there is $k \in I$ such that $\sum_{r\in\mathbb{Z}}(x_{k,r}^{-}.W)_{m'} \neq \{0\}$. From condition (v) and Lemma 5.4, we have $k = i$. From Lemma 5.6 and condition (i), we have $(V_{m'} \cap \mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)V_M) \neq \{0\}$. Consider $u \in V_M$ and $x \in (\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i).u \cap V_{m'})$ such that $x \neq 0$. From condition (ii), u is a highest weight vector for $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$, so $x \in \sum_{r\in\mathbb{Z}} \mathbb{C}x_{i,r}^{-}.u$. By condition (iii), x is in the maximal proper $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$-submodule of $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i).x$. By condition (iv), $v(m'M^{-1})$ is maximal for this condition. So for all $r \in \mathbb{Z}$, we have $x_{i,r}^+(x) = 0$. For $j \neq i$, $r \in \mathbb{Z}$, it follows from Lemma 5.4 that $x_{j,r}^+(x) \in \bigoplus_{m'' \in m'\mathbb{Z}[A_{j,a}^{\pm 1}]_{a \in \mathbb{C}^*}} V_{m''}$, and so from condition (v) we have $x_{j,r}^+(x) = 0$. So $\mathcal{U}_q(\mathcal{L}\mathfrak{g}).x$ is a proper submodule of V, contradiction. □

5.2. Other preliminary results

In this section, $\mathfrak{g}$ is an arbitrary semi-simple Lie algebra. We prove additional preliminary results.

5.2.1. *q-characters of simple modules*

LEMMA 5.7. – *Let $L(m_1), L(m_2)$ be two simple modules. Then $L(m_1 m_2)$ is a subquotient of $L(m_1) \otimes L(m_2)$. In particular $\mathcal{M}(L(m_1m_2)) \subset \mathcal{M}(L(m_1))\mathcal{M}(L(m_2))$.*

This first part of the lemma is proved in [7], and the second part is direct from [7, 14].

As a direct consequence of Theorem 3.10, we have:

LEMMA 5.8. – *Let $a \in \mathbb{C}^*$ and m be a monomial of $\mathbb{Z}[Y_{i,aq^r}]_{i \in I, r \geq 0}$. Then for $m' \in \mathcal{M}(L(m))$ and $b \in \mathbb{C}^*$, $(v_{i,b}(m'm^{-1}) \neq 0 \Rightarrow b \in aq^{r_i + \mathbb{N}})$.*

(Observe that it is also a direct consequence of Lemma 3.11 since a simple module is a subquotient of a tensor product of fundamental representations.)

The following result gives information on the sub $\mathcal{U}_q(\mathcal{L}\mathfrak{g}^J)$-module generated by a highest weight vector (the definition of $L_J(m)$ and $L^J(m^{\to(J)})$ has been given in section 3.3.3):

LEMMA 5.9. – *Let m be a dominant monomial and $J \subset I$. Let v be a highest weight vector of $L(m)$ and let $L' \subset L(m)$ be the $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$-submodule of $L(m)$ generated by v. Then L' is a $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-submodule of $L(m)$ and $\chi_q(L') = L_J(m)$.*

In particular for $\mu \in \omega(m) - \sum_{j \in J} \mathbb{N}\alpha_j$, we have

$$\dim((L(m))_\mu) = \dim((L^J(m^{\to(J)}))_{\mu^{\to(J)}}),$$

where $\mu^{\to(J)} = \sum_{j \in J} \mu(\alpha_j^\vee)\omega_j$.

Proof. – It is clear that $L' = \bigoplus_{\mu \in \omega(m) - \sum_{j \in J} \mathbb{N}.\alpha_j} (L(m))_\mu$. So it is a $\mathcal{U}_q(\mathcal{L}\mathfrak{h})$-submodule of $L(m)$ and $\chi_q(L')$ makes sense. Moreover $\chi_q(L') \in m\mathbb{Z}[A_{j,a}^{\pm}]_{j \in J, a \in \mathbb{C}^*}$ and $m' \in \mathcal{M}(L')$ is uniquely determined by $(m')^{\to(J)}$. So it suffices to prove that $L' \simeq L^J(m^{\to(J)})$ as $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$-module. As L' is a highest weight $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_J)$-module of highest weight monomial $m^{\to(J)}$, it suffices to prove that L' is simple. If it is not simple, there is $w \in L' \cap (L(m))_\mu$ where $\mu < \omega(m)$ and such that for all $j \in J$, $m \in \mathbb{Z}$, $x_{j,m}^+.w = 0$. But as $L(m)$ is a highest weight module and the weights of L' are in $\omega(m) - \sum_{j \in J} \mathbb{N}\alpha_j$, for weight reason we have:

$$\forall j \in (I - J)\,, \forall m \in \mathbb{Z}\,, x_{j,m}^+(L') = \{0\}.$$

So $\mathcal{U}_q(\mathcal{L}\mathfrak{g}).w$ is a proper submodule of $L(m)$, contradiction. □

5.2.2. *Thin modules and thin monomials.* – Let us introduce the notion of thin module:

DEFINITION 5.10. – *A $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module V is said to be thin if its l-weight spaces are of dimension* 1.

In [19, Theorem 3.2], we proved that for $\mathfrak{g}$ of type A, B, C, all fundamental representations are thin (this result was also proved later by a different method in [5]. It should also be possible to check this result directly from the formulas in [28]). We will discuss the notion of thin modules in more details in [23], but let us give some results that will be used in the present paper.

LEMMA 5.11. – *Let V be a $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module and $m' \in \mathcal{M}(V)$ such that there is $i \in I$ satisfying* $\mathrm{Min}\{u_{i,a}(m') \mid a \in \mathbb{C}^*\} \leq -2$. *Then there is $M \in \mathcal{M}(V)$ such that*

- $M > m'$,
- *M is i-dominant,*
- $\mathrm{Max}\{u_{i,b}(M) \mid b \in \mathbb{C}^*\} \geq 2$.

Proof. – Consider $L_i(M)$ occurring in the decomposition of $\chi_q(V)$ described in Proposition 3.14 and such that m' is a monomial of $L_i(M)$. $L_i(M)$ corresponds to the q-character $\chi_q^i(W)$ where W is a $\mathcal{U}_q(\mathcal{L}\mathfrak{g}_i)$-simple module, so subquotient of a standard module. In particular m' appears in

$$M \prod_{a \in \mathbb{C}^*} (Y_{i,a}^{-1}(1 + A_{i,aq_i}^{-1}))^{u_{i,a}(M)}. \tag{2}$$

By hypothesis there is $b \in \mathbb{C}^*$ such that $u_{i,b}(m') \leq -2$. As m' appears in the formula (2), necessarily $(1 + A_{i,aq_i}^{-1})$ appears at least twice in (2), and so $u_{i,bq_i^{-1}}(M) \geq 2$. Moreover by construction $M > m'$ and M is i-dominant. □

DEFINITION 5.12. – *A monomial m is said to be thin if* $\mathrm{Max}_{i \in I, a \in \mathbb{C}^*} |u_{i,a}(m)| \leq 1$.

LEMMA 5.13. – *Let V be a special module such that*

$$\mathrm{Max}\{u_{i,a}(m) \mid m \in \mathcal{M}(V), i \in I, a \in \mathbb{C}^*\} \leq 1.$$

Then V is thin. Moreover all $m \in \mathcal{M}(V)$ are thin.

Proof. – In [19, Proposition 3.3], the first statement is proved for fundamental representations. The proof of the first statement of the lemma is the same ($\chi_q(V)$ is given by the Frenkel-Mukhin algorithm, and so the property is proved by induction on the weight of monomials, see the proof of [19, Proposition 3.3] for details).

Now consider $m' \in \mathcal{M}(V)$. If m' is not thin, there are $i \in I$ and $a \in \mathbb{C}^*$ such that $u_{i,a}(m') \leq -2$. From Lemma 5.11, there are another monomial $M \in \mathcal{M}(V)$ and $b \in \mathbb{C}^*$ such that $u_{i,b}(M) \geq 2$, contradiction with the hypothesis on V, so m' is thin. □

PROPOSITION 5.14. – *If V is thin then all $m \in \mathcal{M}(V)$ are thin. If V is special and all $m \in \mathcal{M}(V)$ are thin, then V is thin.*

Proof. – If V is special and all $m \in \mathcal{M}(V)$ are thin, then the hypotheses of Lemma 5.13 are satisfied and so V is thin.

For the first statement, suppose that V is thin and that there is a monomial of $\mathcal{M}(V)$ which is not thin. We can suppose there are $m \in \mathcal{M}(V)$, $i \in I$, $a \in \mathbb{C}^*$ such that $u_{i,a}(m) \geq 2$ (in the case $u_{i,a}(m) \leq -2$ it follows from Lemma 5.11 that there is another monomial satisfying the condition with ≥ 2). Consider $L_i(M)$ occurring in the decomposition of $\chi_q(V)$ described in Proposition 3.14 and such that m is a monomial of $L_i(M)$. We can see as in the proof of Lemma 5.11 that there is $b \in \mathbb{C}^*$ satisfying $u_{i,b}(M) \geq 2$. From the explicit description of simple modules in Proposition 3.8 in the case sl_2, the monomial $MA_{i,bq_i}^{-1} \prod_{r>0} A_{i,b(q_i)^{r+1}}^{-u_{i,bq_i^r}(M)}$ occurs with multiplicity at least 2 in the q-character of the $\mathcal{U}_{q_i}(\mathcal{L}sl_2)$-module $L(M^{\rightarrow(i)})$, and so it is not a thin module. As the coefficients in the decomposition of Proposition 3.14 are positive, there is an l-weight space of V of dimension at least 2, and so V is not thin. $\square$

LEMMA 5.15. – *Let $L(m)$ be a simple $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module and $(m', i) \in \mathcal{M}(L(m)) \times I$ such that*

- *any $m'' \in \mathcal{M}(L(m))$ satisfying $v(m''m^{-1}) < v(m'm^{-1})$ is thin,*
- *m' is not i-dominant.*

Then there is $a \in \mathbb{C}^*$ such that $u_{i,a}(m') < 0$ and $m'A_{i,aq_i^{-1}} \in \mathcal{M}(L(m))$.

Proof. – Consider $L_i(M)$ occurring in the decomposition of $\chi_q(V)$ described in Proposition 3.14 and such that m is a monomial of $L_i(M)$. From the first hypothesis M is thin. If $L_i(M)$ corresponds to a Kirillov-Reshetikhin module of type sl_2, the result follows from the explicit formula of Proposition 3.8 (1). In general $L_i(M)$ is also known from the explicit description of q-characters of simple modules in the sl_2-case in Proposition 3.8 (3), and $L_i(M)$ corresponds to a product of Kirillov-Reshetikhin modules $L_i(M) = \prod_k W_k$. As M is thin we have moreover the following property: for m_1 appearing in W_k and m_2 appearing in $W_{k'}$, we have

$$u_{i,a}(m_1) \neq 0 \text{ and } u_{i,a}(m_2) \neq 0 \Rightarrow k = k'.$$

And so the result can be reduced to the case of Kirillov-Reshetikhin modules. $\square$

LEMMA 5.16. – *Suppose that $\mathfrak{g} = sl_{n+1}$ and $L(m)$ be a simple $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module. Let $(m', i, a) \in \mathcal{M}(L(m)) \times I \times \mathbb{C}^*$ such that:*

- *any $m'' \in \mathcal{M}(L(m))$ satisfying $v(m''m^{-1}) < v(m'm^{-1})$ is thin,*
- *$u_{i,a}(m') = -1$,*
- *$m'Y_{i,a}$ is dominant.*

Then there is $M \in \mathcal{M}(L(m))$ dominant such that $M > m'$ and $v_n(m'M^{-1}) \leq 1$, $v_1(m'M^{-1}) \leq 1$.

Proof. – By using Lemma 5.15, we construct inductively a sequence of monomials of $\mathcal{M}(L(m))$ starting with m'. Indeed as $u_{i,a}(m') = -1$ we first get $m'A_{i,aq^{-1}} \in \mathcal{M}(L(m))$. Then from the property $m'Y_{i,a}$ dominant we have

$$(u_{i-1,b}(m'A_{i,aq^{-1}}) < 0 \text{ or } u_{i+1,b}(m'A_{i,aq^{-1}}) < 0) \Rightarrow b = aq^{-1}.$$

Then we use again Lemma 5.15 $(i-1, aq^{-1})$ and $(i+1, aq^{-1})$ when it is possible. We get a monomial and we apply Lemma 5.15 with $(i-2, aq^{-2})$ and $(i+2, aq^{-2})$ when it is possible. We continue by induction until this is not possible, and we get a monomial:

$$\begin{aligned} m_1 = m'(&A_{i,aq^{-1}} A_{i-1,aq^{-2}} \cdots A_{i-\alpha,aq^{-1-\alpha}}) \\ &\times (A_{i+1,aq^{-2}} A_{i+2,aq^{-3}} \cdots A_{i+\beta,aq^{-1-\beta}}) \in \mathcal{M}(L(m)), \end{aligned}$$

where $\alpha, \beta \geq 0$, $i-\alpha \geq 1$, $i+\beta \leq n$. By construction m_1 is $(I-\{i\})$-dominant and we have $(u_{i,b}(m_1) < 0 \Rightarrow b = aq^{-2})$. If $\alpha = 0$ or $\beta = 0$, m_1 is dominant and we take $M = m_1$. Otherwise, we can suppose $\alpha \geq \beta$ (the case $\beta \geq \alpha$ can be treated in the same way). As at each step we get by construction thin monomials, we continue by induction, and for $2 \leq r \leq \beta + 1$, we have

$$\begin{aligned} m_r = m_{r-1}(&A_{i,aq^{1-2r}} A_{i-1,aq^{-2r}} \cdots A_{i-\alpha+r-1,aq^{-\alpha-r}}) \\ &\times (A_{i+1,aq^{-2r}} A_{i+2,aq^{-2r-1}} \cdots A_{i+\beta-r+1,aq^{-r-\beta}}) \in \mathcal{M}(L(m)), \end{aligned}$$

and m_r is $(I-\{i\})$ dominant. Moreover $m_{\beta+1}$ is dominant, so we take $M = m_{\beta+1}$. By construction we have $M > m'$ and $v_n(m'M^{-1}) \leq 1$, $v_1(m'M^{-1}) \leq 1$. □

LEMMA 5.17. – *Let $\mathfrak{g} = sl_{n+1}$ and let $L(m)$ be a simple $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-module. Let $(m', j) \in \mathcal{M}(L(m)) \times I$ such that*

- *any $m'' \in \mathcal{M}(L(m))$ satisfying $v(m''m^{-1}) < v(m'm^{-1})$ is thin,*
- *m' is $(I-\{j\})$-dominant,*
- *if $j \leq n-1$, then for all $a \in \mathbb{C}^*$, $(u_{j,a}(m') < 0 \Rightarrow u_{j+1,aq^{-1}}(m') > 0)$.*

Then there is $M \in \mathcal{M}(L(m))$ dominant of the form

$$M = m' \prod_{\{a \in \mathbb{C}^* | u_{j,a}(m') < 0\}} (A_{j,aq^{-1}} A_{j-1,aq^{-3}} \cdots A_{i_a,aq^{i_a-j-1}}),$$

where for $a \in \mathbb{C}^*$, $1 \leq i_a \leq j$.

Proof. – If $j < n$, the additional hypothesis $(u_{j,a}(m') < 0 \Rightarrow u_{j+1,aq^{-1}}(m') > 0)$ allows us to use the result for $\mathfrak{g}_{\{1,\ldots,j\}}$. So we can suppose that $j = n$. We prove the result by induction on n. For $n = 1$ the result is clear. In general, by using Proposition 3.14 we get $m_1 \in \mathcal{M}(L(m))$ n-dominant such that m' is a monomial of $L_n(m_1)$. As $m_1 > m'$ and $v(m_1m^{-1}) \leq P$, we have by the explicit description of $L_n(m_1)$ in Proposition 3.8:

$$m'(m_1)^{-1} = \prod_{\{a \in \mathbb{C}^* | u_{n,a}(m') < 0\}} A_{n,aq^{-1}}.$$

Moreover by construction:

- m_1 is $\{1, \ldots, n-2\}$-dominant,
- $\forall a \in \mathbb{C}^*, (u_{n-1,a}(m_1) < 0 \Rightarrow (u_{n,aq^{-1}}(m_1) = 1$ and $u_{n-1,a}(m_1) = -1))$.

By Lemma 5.15 there is $m_2 \in \mathcal{M}(L(m))$ which is $\{1, \ldots, n-1\}$-dominant and such that m_1 is a monomial of $L_{\{1,\ldots,n-1\}}(m_2)$. Then by using the induction property for $\mathfrak{g}_{\{1,\ldots,n-1\}}$ on m_1 monomial of $L_{\{1,\ldots,n-1\}}(m_2)$, we get the monomial M. □

6. Proof of Theorem 1.2

In this section, $\mathfrak{g}$ is simply-laced. We complete the proof of Theorem 1.2: after a technical lemma on dominant monomials (Lemma 6.1), fundamental representations (Proposition 6.2) and standard modules of the form $M(X_{2,a}^{(i)})$ (Proposition 6.3) are studied. Then the type A is discussed (Proposition 6.4), and finally we give the proof of Theorem 1.2 for the general case.

6.1. Dominant monomials

First let us prove some properties of dominant monomials lower than a monomial $X_{k,a}^{(i)}$. To do this, let us define the following number attached to the structure of the Dynkin diagram: for $i, j \in I$, we denote by $d(i,j)$ the minimal d such that there is a sequence $(i_1, \dots, i_d) \in I^d$ satisfying $i = i_1$, $j = i_d$ and for all $k \in \{1, \dots, d-1\}$, $C_{i_k, i_{k+1}} = -1$.

LEMMA 6.1. – *Let $i \in I$, $a \in \mathbb{C}^*$, $k \geq 0$ and $m = X_{k,a}^{(i)}$. Let $m' \leq m$ dominant. Then we have:*

- $m'm^{-1} \in \mathbb{Z}[A_{j,aq^l}^{-1}]_{j\in I, l\in\mathbb{Z}}$,
- $\forall j \in I, l \in \mathbb{Z}$, $v_{j,aq^l}(m'm^{-1}) > 0 \Rightarrow (d(i,j) + 1 - k \leq l \leq k - 1 - d(i,j))$,
- $\forall j \in I$, $v_j(m'm^{-1}) > 0 \Rightarrow d(i,j) \leq k - 1$.

Proof. – The last statement is a direct consequence of the second statement.

Let us prove that for any $j \in I$, $b \in \mathbb{C}^*$ we have:

$$v_{j,b}(m'm^{-1}) \neq 0 \Rightarrow b \in aq^{k-1-d(i,j)-\mathbb{N}}.$$

We prove this statement by induction on $d(i,j)$.

For $d(i,j) = 1$, we have $j = i$. Suppose that there is $b \in (\mathbb{C}^* - aq^{k-2-\mathbb{N}})$ such that $v_{i,b}(m'm^{-1}) > 0$. Let $L \in \mathbb{Z}$ maximal such that there is $p \in I$ satisfying $v_{p,bq^L}(m'm^{-1}) > 0$. We have $bq^L \notin aq^{k-2-\mathbb{N}}$. As $m' \neq m$, we have $m' < m$ and $m'm^{-1}$ is right negative. So $u_{p,bq^{L+1}}(m'm^{-1}) < 0$. As moreover $u_{p,c}(m) > 0$ implies $c \in aq^{k-1-\mathbb{N}}$, we have $u_{p,bq^{L+1}}(m') = u_{p,bq^{L+1}}(m'm^{-1}) < 0$. So m' is not dominant, contradiction.

In general suppose that $d(i,j) \geq 2$ and that there is $b \in (\mathbb{C}^* - aq^{k-d(i,j)-1-\mathbb{N}})$ such that $v_{j,b}(m'm^{-1}) > 0$. If $b \notin aq^{\mathbb{Z}}$, we can prove as in the previous case that m' is not dominant, contradiction. Otherwise let L maximal such that

$$\sum_{\{p\in I | d(i,p) \geq d(i,j)\}} v_{p,aq^L}(m'm^{-1}) > 0.$$

As $v_{j,b}(m'm^{-1}) > 0$, we have $L > k - 1 - d(i,j)$. Let $P \in I$ such that $d(i,P) \geq d(i,j)$ and $v_{P,aq^L}(m'm^{-1}) > 0$. We have

$$u_{P,aq^{L+1}}\Big(\prod_{\{p\in I | d(i,p)\geq d(i,j)\}} \prod_{c\in\mathbb{C}^*} A_{p,c}^{-1}\Big) < 0.$$

As $u_{P,aq^{L+1}}(m) = 0$ and m' is dominant, there is j' satisfying $d(i,j') = d(i,j) - 1$ and $v_{j',aq^{L+1}}(m'm^{-1}) > 0$. But $L + 1 \geq k + 1 - d(i,j) = k - d(i,j')$, contradiction with the induction hypothesis.

In the same way we can prove that for any $j \in I$, $b \in \mathbb{C}^*$:

$$v_{j,b}(m'm^{-1}) \neq 0 \Rightarrow b \in aq^{-k+1+d(i,j)+\mathbb{N}}.$$

This implies the first two statements of the lemma. □

6.2. Fundamental representations and $k = 2$ case

PROPOSITION 6.2. – *All fundamental representations are small.*

Proof. – Let $i \in I$ and $a \in \mathbb{C}^*$. Then from Lemma 6.1, a monomial satisfying $m' < Y_{i,a}$ is not dominant. So $V_i(a)$ is small. □

PROPOSITION 6.3. – *Let $i \in I$, $a \in \mathbb{C}^*$. Then $M(X_{2,a}^{(i)})$ is small.*

Proof. – From Lemma 6.1, a dominant monomial $m' < X_{2,a}^{(i)}$ is equal to

$$m' = X_{2,a}^{(i)} A_{i,a}^{-1} = \prod_{j \in I | C_{i,j} = -1} Y_{j,a}.$$

Consider a monomial $m'' < m'$.

Suppose that there are $j \in I$, $b \in (\mathbb{C}^* - aq^{\mathbb{Z}})$ such that $v_{j,b}(m''(m')^{-1}) > 0$. Let $L \in \mathbb{Z}$ maximal such that there is $p \in I$ satisfying $v_{p,bq^L}(m''(m')^{-1}) > 0$. We have $u_{p,bq^{L+1}}(m'') = u_{p,bq^{L+1}}(m''(m')^{-1}) < 0$ and so m'' is not dominant.

Otherwise let $L \in \mathbb{Z}$ maximal such that there is $p \in I$ satisfying $v_{p,aq^L}(m''(m')^{-1}) > 0$. If $L \geq 0$, we can prove as in the previous case that m'' is not dominant. Otherwise let $L' < 0$ minimal such that there is $p \in I$ satisfying $v_{p,aq^{L'}}(m''(m')^{-1}) > 0$. We have $u_{p,bq^{L'-1}}(m'') = u_{p,bq^{L'-1}}(m''(m')^{-1}) < 0$, and so m'' is not dominant.

So $L(m')$ is special and $M(X_{2,a}^{(i)})$ is small. □

6.3. Type A

In this section $\mathfrak{g}$ is of type A.

PROPOSITION 6.4. – *Let $k \geq 1, i \in I, a \in \mathbb{C}^*$. Then $M(X_{k,a}^{(i)})$ is small if and only if ($i = 1$ or $i = n$ or $k \leq 2$).*

In particular for $\mathfrak{g} = sl_2$ or $\mathfrak{g} = sl_3$, all $M(X_{k,a}^{(i)})$ are small.

We prove this proposition in three steps:

(1) we determine the dominant monomials m' such that $m' \leq X_{k,a}^{(1)}$ (Lemma 6.5),
(2) we prove that the corresponding simple modules are special (Proposition 6.6),
(3) we study the remaining cases (Lemma 6.8).

LEMMA 6.5. – *Let $k \geq 1, a \in \mathbb{C}^*$ and $m' \leq X_{k,a}^{(1)}$ dominant. Then m' is of the form*

$$m' = Y_{i_1,aq^{l_1}} Y_{i_2,aq^{l_2}} \cdots Y_{i_R,aq^{l_R}},$$

where $R \geq 0$, $i_1, i_2, \ldots, i_R \in I$, $l_1, l_2, \ldots, l_R \in \mathbb{Z}$ satisfy for all $1 \leq r \leq R-1$:

$$l_{r+1} - l_r \geq i_r + i_{r+1}.$$

Proof. – Let $m = Y_{1,a}Y_{1,aq^2}\cdots Y_{1,aq^{2(k-1)}}$ and $m' \leq m$ dominant. For $i \in I, l \in \mathbb{Z}$, let us define $v_{i,l} = v_{i,aq^l}(m'm^{-1})$ and $u_{i,l} = u_{i,aq^l}(m')$. We set $v_{n+1,l} = 0$. As m' is dominant, we have for $2 \leq i \leq n$ and $l \in \mathbb{Z}$:

$$v_{i,l-1} + v_{i,l+1} \leq v_{i-1,l} + v_{i+1,l},$$
$$v_{1,l-1} + v_{1,l+1} \leq 1 + v_{2,l}.$$

From Lemma 6.1, for $l \leq i-1$ or $l \geq 2k-i-2$, we have $v_{i,l} = 0$. Let us prove that for all $i \in I$, $(v_{i,l} \neq 0 \Rightarrow (l \in i + 2\mathbb{Z}))$. Indeed $m'' = \prod_{i\in I, l\in i+1+2\mathbb{Z}} A_{i,aq^l}^{-v_{i,l}}$ is right negative and for all $i \in I$, $l \in i+2\mathbb{Z}$, $u_{i,l}(m'') = u_{i,l}(m')$. So $m'' = 1$.

Let us prove that for all $l \in \mathbb{Z}$ we have $v_{1,l} \leq 1$, and for all $n \geq i \geq 2$, $l \in \mathbb{Z}$ we have $v_{i,l} \leq v_{i-1,l-1}$. We prove the result by induction on $d = l - i \geq 0$. First suppose that $d = 0$. Then we have $v_{1,1} \leq -v_{1,-1} + 1 + v_{2,1} = 1$. For $i \geq 2$, $v_{i,i} \leq -v_{i,i-2} + v_{i-1,i-1} + v_{i+1,i-1} = v_{i-1,i-1}$. Now consider a general $d > 0$. First we have $v_{1,1+d} \leq 1 + v_{2,d} - v_{1,d-1}$. But by the induction hypothesis, $v_{2,d} \leq v_{1,d-1}$. So $v_{1,1+d} \leq 1$. For $i \geq 2$, $v_{i,i+d} \leq (v_{i+1,d+i-1} - v_{i,d+i-2}) + v_{i-1,d+i-1}$. But by the induction hypothesis, $v_{i+1,d+i-1} - v_{i,d+i-2} \leq 0$, and so $v_{i,i+d} \leq v_{i-1,d+i-1}$.

In particular for all $i \in I, l \in \mathbb{Z}$, $v_{i,l} \leq 1$.

In the same way, for all $n \geq i \geq 2$, $v_{i,l} \leq v_{i-1,l+1}$. Let $n \geq i \geq 2$. We have proved $v_{i,l} \leq \mathrm{Min}\{v_{i-1,l-1}, v_{i-1,l+1}, 1\}$. In particular

$$(v_{i,l} = 1 \Rightarrow v_{i-1,l-1} = v_{i-1,l+1} = 1).$$

Moreover if $v_{i,l-1} = v_{i,l+1} = 1$, we have $2 = v_{i,l-1} + v_{i,l+1} \leq v_{i+1,l} + v_{i-1,l}$ and so $v_{i+1,l} = v_{i-1,l} = 1$. So

$$(v_{i,l} = 1 \Leftrightarrow v_{i-1,l-1} = v_{i-1,l+1} = 1).$$

As a conclusion, this can be rewritten in the following way. $m'm^{-1}$ is of the form:

$$m'm^{-1} = B_{p_1,f_1}B_{p_2,f_2}\cdots B_{p_R,f_R},$$

where $R \geq 0$, $n-1 \geq p_1, \ldots, p_R \geq 0$, $f_1, \ldots, f_R \in \mathbb{Z}$,

$$\begin{aligned} B_{p,f} = {} & (A_{1,aq^{f-p}}A_{1,aq^{f+2-p}}\cdots A_{1,aq^{f+p}}) \\ & \times (A_{2,aq^{f+1-p}}A_{2,aq^{f+3-p}}\cdots A_{2,aq^{f+p-1}})\cdots(A_{p+1,aq^f}), \end{aligned}$$

$f_i - p_i \in 1 + 2\mathbb{Z}$, $f_1 - p_1 \geq 1$, $f_R + p_R \leq 2k-3$ and $f_i + p_i + 4 \leq f_{i+1} - p_{i+1}$.

If $p \leq n-2$, we have

$$B_{p,l} = (Y^{-1}_{1,q^{f-p-1}}Y^{-1}_{1,q^{f-p+1}}\cdots Y^{-1}_{1,q^{f+p+1}})Y_{p+2,aq^f},$$

and we have

$$B_{n-1,l} = Y^{-1}_{1,q^{f-n}}Y^{-1}_{1,q^{f-n+2}}\cdots Y^{-1}_{1,q^{f+n}}.$$

So we get the result. □

Proposition 6.6. – *Let $m = Y_{i_1,aq^{l_1}}Y_{i_2,aq^{l_2}}\cdots Y_{i_R,aq^{l_R}}$ where $R \geq 0$, $i_1, i_2, \ldots, i_R \in I$, $l_1, l_2, \ldots, l_R \in \mathbb{Z}$ satisfying for all $1 \leq r \leq R-1$, $l_{r+1} - l_r \geq i_r + i_{r+1}$. Then:*

(1) *For $m' \in \mathcal{M}(L(m))$, if $v_{i_R,aq^{l_R-1}}(m') \geq 1$ then $v_{i_R,aq^{l_R+1}}(m') \geq 1$.*
(2) *$L(m)$ is special.*
(3) *$L(m)$ is thin.*

To prove this proposition, we will need the following direct consequence of the results in [13]:

LEMMA 6.7. – *Let V be a fundamental representation of a quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ and let $Y_{i,a}$ (resp. $Y_{j,b}^{-1}$) be the highest (resp. lowest) weight monomial of $\chi_q(V)$. Then we have:*

$$\chi_q(V) \in Y_{i,a}\left(1 + A_{i,aq}^{-1}\left(1 + \sum_{\{k\in I | C_{i,k}=-1\}} A_{k,bq^2}^{-1}.\mathbb{Z}[A_{l,d}^{-1}]_{l\in I, d\in\mathbb{C}^*}\right)\right),$$
$$\chi_q(V) \in Y_{j,b}^{-1}\left(1 + A_{j,bq^{-1}}\left(1 + \sum_{\{k\in I | C_{j,k}=-1\}} A_{k,bq^{-2}}.\mathbb{Z}[A_{l,d}]_{l\in I, d\in\mathbb{C}^*}\right)\right).$$

Proof. – As V is special, we can use the algorithm proposed by Frenkel-Mukhin [13] to compute $\chi_q(V)$ (see [13, Section 5.5] for details): we start with $Y_{i,a}$. Then we get $Y_{i,a}A_{i,aq}^{-1}$ with multiplicity 1 as $L_i(Y_{i,a}) = Y_{i,a} + Y_{i,a}A_{i,aq}^{-1}$. As

$$Y_{i,a}A_{i,aq}^{-1} = Y_{i,aq^2}^{-1} \prod_{\{k\in I | C_{i,k}=-1\}} Y_{k,aq},$$

the next step of the algorithm gives the monomials $Y_{i,a}A_{i,aq}^{-1}A_{k,aq^2}^{-1}$ with multiplicity one, and then inductively the other monomials occurring in $\chi_q(V)$ are lower than these monomials.

The second statement is obtained by the duality stated in [13, Proposition 6.18] (by replacing the Y_{i,aq^n} by $Y_{i,aq^{-n}}^{-1}$, we get the q-character of a fundamental representation). □

Now let us prove Proposition 6.6:

Proof. – Let us denote by (1_R) (resp. (2_R), (3_R)) the condition that the statement (1) (resp. (2), (3)) of the proposition is satisfied for any $R' \leq R$. We prove by induction on R simultaneously that (1_R), (2_R) and (3_R) are satisfied. For $R = 0$ this is clear.

Now we prove the following for $R \geq 1$:

- $((1_{R-1})$ and (2_{R-1}) and $(3_{R-1}))$ implies (1_R),
- $((1_R)$ and (2_{R-1}) and $(3_{R-1}))$ implies (2_R),
- $((1_R)$ and (2_R) and $(3_{R-1}))$ implies (3_R).

Let us start with: $((1_R)$ and (2_{R-1}) and $(3_{R-1}))$ implies (2_R).

By Lemma 5.7

$$\mathcal{M}(L(m)) \subset (mY_{i_R,aq^{l_R}}^{-1}\mathcal{M}(V_{i_L}(aq^{l_R}))) \cup (\mathcal{M}(L(mY_{i_R,aq^{l_R}}^{-1}))Y_{i_R,aq^{l_R}}).$$

As all monomials of $mY_{i_R,aq^{l_R}}^{-1}(\chi_q(V_{i_R}(aq^{l_R})) - Y_{i_R,aq^{l_R}})$ are lower than $mA_{i_R,aq^{l_R+1}}^{-1}$ (Theorem 3.13) which is right-negative, they are not dominant. Consider m' in the set $\mathcal{M}(L(mY_{i_R,aq^{l_R}}^{-1}))Y_{i_R,aq^{l_R}} - \{m\}$. If $v_{i_R,aq^{l_R+1}}(m'm^{-1}) \geq 1$ or $v_{i_R,aq^{l_R-1}}(m'm^{-1}) \geq 1$, it follows from the (1_R) that m' is lower than $mA_{i_R,aq^{l_R+1}}^{-1}$ which is right-negative, so m' is not dominant. We suppose that

$$v_{i_R,aq^{l_R-1}}(m'm^{-1}) = v_{i_R,aq^{l_R+1}}(m'm^{-1}) = 0.$$

So we have $u_{i_R,aq^{l_R}}(m'Y_{i_R,aq^{l_R}}^{-1}) \geq 0$. By (2_{R-1}), the monomial $m'Y_{i_R,aq^{l_R}}^{-1} \in \mathcal{M}(L(mY_{i_R,aq^{l_R}}^{-1}))$ is not dominant. So there are $i \in I$, $b \in \mathbb{C}^*$, such that $(i,b) \neq (i_R, aq^{l_R})$

and $u_{i,b}(m'Y^{-1}_{i_R,aq^{l_R}}) < 0$. As $u_{i,b}(m'Y^{-1}_{i_R,aq^{l_R}}) = u_{i,b}(m')$, m' is not dominant. So (2_R) is satisfied.

Now let us prove: ((1_R) and (2_R) and (3_{R-1})) implies (3_R).

From property (2_R) and Proposition 5.14, it suffices to prove that all monomials of $\mathcal{M}(L(m))$ are thin. Suppose that there is a monomial in $\mathcal{M}(L(m))$ which is not thin. From Lemma 5.11, we can suppose that there is $m' \in \mathcal{M}(L(m))$ such that there are $i \in I, a \in \mathbb{C}^*$ satisfying $u_{i,a}(m') = 2$ and such that any m'' satisfying $v(m''m^{-1}) < v(m'm^{-1})$ is thin. In particular from Proposition 3.14:

- m' is $(\{1, \dots, i-2\} \cup \{i\} \cup \{i+2, \dots, n\})$-dominant,
- $(u_{i-1,b}(m') < 0 \Rightarrow b = aq)$,
- $(u_{i+1,b}(m') < 0 \Rightarrow b = aq)$.

(Otherwise we could construct $m'' \in \mathcal{M}(L(m))$ not thin such that $v(m''m^{-1}) < v(m'm^{-1})$.) We can apply Lemma 5.17 for $\mathfrak{g}_{\{1,\dots,i-1\}}$ of type A_{i-1} and then for $\mathfrak{g}_{\{i+1,\dots,n\}}$ of type A_{n-i}. We get a monomial $M \in \mathcal{M}(L(m))$, and by construction

- M is $I - \{i\}$-dominant,
- $u_{j_1,aq^{j_1-i}}(M) \geq 1$ with $j_1 \leq i$,
- $u_{j_2,aq^{i-j_2}}(M) \geq 1$ with $j_1 < j_2, i \leq j_2$.

Moreover as $u_{i,a}(m') = 2$, by construction M is dominant. From property (2_R) we have $m = M$. In particular there is $r < r'$ such that $(i_r, l_r) = (j_1, j_1 - i)$ and $(i_{r'}, l_{r'}) = (j_2, i - j_2)$. We have

$$l_{r'} - l_r = 2i - j_2 - j_1 = i_r + i_{r'} + 2(i - j_2) - 2j_1 < i_r + i_{r'}.$$

But we have

$$l_{r'} - l_r = (l_{r'} - l_{r'-1}) + \cdots + (l_{r+1} - l_r) \geq i_{r'} + 2(i_{r'-1} + \cdots + i_{r+1}) + i_r \geq i_{r'} + i_r,$$

contradiction. So (3_R) is satisfied.

Finally we prove: ((1_{R-1}) and (2_{R-1}) and (3_{R-1})) implies (1_R).

We prove (1_R) by induction on $v(m'm^{-1}) \geq 0$. For $v(m'm^{-1}) = 0$ we have $m' = m$ and the result is clear. In general consider a monomial $m' < m$ such that for m'' satisfying $v(m''m^{-1}) < v(m'm^{-1})$, the property (1_R) is satisfied. We suppose that moreover the property is not satisfied for m', that is to say that $v_{i_R,aq^{l_R-1}}(m'm^{-1}) \geq 1$ and $v_{i_R,aq^{l_R+1}}(m'm^{-1}) = 0$. It follows from Proposition 3.14 and the induction hypothesis on v that m' is $(I - \{i_R\})$-dominant (otherwise we could construct m'' such that $v(m''m^{-1}) < v(m'm^{-1})$ and the property is not satisfied for m'').

If m' is not dominant, m' is not i_R-dominant and so it follows from Proposition 3.14 that there is $m'' \in \mathcal{M}(L(m))$ i_R-dominant such that $m'' > m'$ and m' is a monomial of $L_{i_r}(m'')$. Moreover there is $b \in \mathbb{C}^*$ such that $m' \leq m'A_{i_R,b} \leq m''$, and $m'A_{i_R,b}$ is a monomial of $L_{i_R}(m'')$ and so in $\mathcal{M}(L(m))$. By the induction hypothesis on v, $m'A_{i_R,b}$ satisfied the property (1_R), and so we have $b = aq^{l_R-1}$. So $v_{i_R,aq^{l_R-1}}(m''m^{-1}) = v_{i_R,aq^{l_R+1}}(m''m^{-1}) = 0$. In particular $u_{i_R,aq^{l_R}}(m'') \geq u_{i_R,aq^{l_R}}(m) \geq 1$. By Lemma 5.7, we have $m' \in \mathcal{M}(L(Y_{i_R,aq^{l_R}}))\mathcal{M}(L(mY^{-1}_{i_R,aq^{l_R}}))$. But by Theorem 3.13 the monomials of $\mathcal{M}(L(Y_{i_R,aq^{l_R}}))$ not equal to $Y_{i_R,aq^{l_R}}$ are lower than $Y_{i_R,aq^{l_R}}A^{-1}_{i_R,aq^{L_R+1}}$. So we have $m' \in Y_{i_R,aq^{l_R}}\mathcal{M}(L(mY^{-1}_{i_R,aq^{l_R}}))$. By the properties (2_{R-1}) and (3_{R-1}), $L(mY^{-1}_{i_R,aq^{l_R}})$ is special and thin. In particular $u_{i_R,aq^{l_R-2}}(m'') \leq 1$, and so by Proposition 3.8, m' is not a monomial of $L_{i_R}(m'')$, contradiction. So m' is dominant.

As $L(mY^{-1}_{i_R,aq^{l_R}})$ is special, the monomial $m'Y^{-1}_{i_R,aq^{l_R}}$ is not dominant. So

- $u_{i_R,aq^{l_R}}(m'Y^{-1}_{i_R,aq^{l_R}}) = -1$,
- $u_{j,b}(m'Y^{-1}_{i_R,aq^{l_R}}) < 0 \Rightarrow (j = i_R \text{ and } b = aq^{l_R})$.

So we can use Lemma 5.16 for the thin module $L(m'Y^{-1}_{i_R,aq^{l_R}})$. Let α, β as in the proof of Lemma 5.16. Let $j = i_R + \beta - \alpha$ and $b = aq^{l_R-\alpha-\beta-2}$. By construction of m from m' in the proof of Lemma 5.16, we have $u_{j,b}(mY^{-1}_{i_R,aq^{l_R}}) \geq 1$ and $m' \in mY^{-1}_{j,b}\mathcal{M}(V_j(b))$. Moreover there is $R' < R$ such that $j = i_{R'}$ and $l_R - \alpha - \beta - 2 = l_{R'}$. We have

$$\begin{aligned}\alpha + \beta + 2 = l_R - l_{R'} &\geq i_R + 2i_{R-1} + \cdots + 2i_{R'+1} + i_{R'}\\ &\geq 2(i_R + \cdots + i_{R'+1}) + \beta - \alpha.\end{aligned}$$

So $i_R + \cdots + i_{R'+1} \leq \alpha + 1$ and $(i_R - \alpha) + i_{R-1} + \cdots + i_{R'+1} \leq 1$. As $i_R - \alpha \geq 1$, we have $i_{R-1} + \cdots + i_{R'+1} = 0$, $R' = R - 1$ and $i_R - \alpha = 1$.

By construction, we have $m'm^{-1} \in \mathbb{Z}[A^{-1}_{i,aq^r}]_{i \leq i_R+\beta, r\in\mathbb{Z}}$. So from Lemma 5.9 we can suppose that $i_R + \beta = n$.

We have $i_R = n + 1 - i_{R-1}$. As

$$\begin{aligned}\omega(m(m')^{-1}) &= (\alpha_1 + \cdots + \alpha_n) + (\alpha_2 + \cdots + \alpha_{n-1})\\ &\quad + \cdots + (\alpha_{i_{R-1}} + \cdots + \alpha_{n+1-i_{R-1}})\\ &= \alpha_1 + \alpha_n + 2(\alpha_2 + \alpha_{n-2}) + \cdots + i_{R-1}(\alpha_{i_{R-1}} + \alpha_{n+1-i_{R-1}})\\ &\quad + i_{R-1}(\alpha_{i_{R-1}+1} + \cdots + \alpha_{n-i_{R-1}}),\end{aligned}$$

the monomial $m'(mY^{-1}_{i_{R-1},aq^{l_{R-1}}})^{-1}$ is the lowest monomial of $\mathcal{M}(V_{i_{R-1}}(aq^{l_{R-1}}))$ (the weight of the lowest weight of fundamental representations has been computed in [13, Lemma 6.8]).

Let us prove that

$$\mathcal{M}(L(m)) \cap m'\mathbb{Z}[A_{i_R,d}]_{d\in\mathbb{C}^*} \subset \{m', m'A_{i_R,aq^{l_R-1}}\}. \tag{3}$$

Let $m'' \in (\mathcal{M}(L(m)) \cap m'\mathbb{Z}[A_{i_R,d}]_{d\in\mathbb{C}^*})$ different from m'. In particular $m \geq m'' > m'$. By construction of m' from m, as $R' = R-1$, we have for $k \neq i_{R-1}$, $v_{k,aq^{l_k+1}}(m'm^{-1}) = 0$. So by Theorem 3.13 (for fundamental representations, that is to say the particular case proved in [13]), $m', m'' \in mY^{-1}_{i_{R-1},aq^{l_{R-1}}}\mathcal{M}(V_{i_{R-1}}(aq^{l_{R-1}}))$. As $m'(mY^{-1}_{i_{R-1},aq^{l_{R-1}}})^{-1}$ is the lowest monomial of $\mathcal{M}(V_{i_{R-1}}(aq^{l_{R-1}}))$, Lemma 6.7 gives:

$$\begin{aligned}(\chi_q(V_{i_{R-1}}(aq^{l_{R-1}}))Y_{i_R,aq^{L_R}} - 1)A^{-1}_{i_R,aq^{l_R-1}} \in 1 &+ A_{i_R+1,aq^{l_R-2}}\mathbb{Z}[A_{k,d}]_{k\in I,d\in\mathbb{C}^*}\\ &+ A_{i_R-1,aq^{l_R-2}}\mathbb{Z}[A_{k,d}]_{k\in I,d\in\mathbb{C}^*}.\end{aligned}$$

As by hypothesis $v_{i_R-1}(m'(m'')^{-1}) = v_{i_R+1}(m'(m'')^{-1}) = 0$, we get:

$$m''(mY^{-1}_{i_{R-1},aq^{l_{R-1}}})^{-1} = m'A_{i_R,aq^{l_R-1}}(mY^{-1}_{i_{R-1},aq^{l_{R-1}}})^{-1}.$$

Let us prove that

$$m'' \in (\mathcal{M}(L(m)) \cap m'\mathbb{Z}[A^{\pm 1}_{i_R,d}]_{d\in\mathbb{C}^*}) - \{m'A_{i_R,b}\} \Rightarrow v_{i_R}(m''(m')^{-1}) \geq 0. \tag{4}$$

Consider a monomial m'' satisfying the left property of (4). By Lemma 5.8, for $k \neq R-1$ we have $v_{i_k,aq^{l_k+1}}(m'm^{-1}) = 0$. So

$$m'' \in m(Y^{-1}_{i_{R-1},aq^{l_{R-1}}}\mathcal{M}(V_{i_{R-1}}(aq^{l_{R-1}})))\Big(\prod_{\{k|i_k=i_R\}} Y_{i_R,aq^{l_k}}\Big)^{-1}\prod_{\{k|i_k=i_R\}}\mathcal{M}(V_{i_R}(aq^{l_k}).$$

Let us write this decomposition

$$m'' = mY^{-1}_{i_{R-1},aq^{l_{R-1}}}(m'')_{R-1}\Big(\prod_{\{k|i_k=i_R\}} Y_{i_R,aq^{l_k}}\Big)^{-1}\prod_{\{k|i_k=i_R\}}(m'')_k.$$

(If $i_{R-1} = i_R$ we put $(m'')_{R-1}$ only once.) Let $k \neq R-1$ satisfying $i_k = i_R$. Observe that for $R_1 < R_2$, we have $l_{R_2} - l_{R_1} \geq i_{R_1} + i_{R_2} \geq 2$. So by Lemma 5.8, $v_{i_R+1,aq^{l_k+2}}(m'm^{-1}) = v_{i_R-1,aq^{l_k+2}}(m'm^{-1}) = 0$. So $(m'')_k = Y_{i_R,aq^{l_k}}$ or $(m'')_k = Y_{i_R,aq^{l_k}}A^{-1}_{i_R,aq^{l_k+1}}$. As a consequence, $(m'')_{R-1} = Y^{-1}_{i_R,aq^{l_R}}$ or $(m'')_{R-1} = Y^{-1}_{i_R,aq^{l_R}}A_{i_R,aq^{l_{R-1}}}$ (Lemma 5.8). So

$$\begin{aligned}v_{i_R}(m''(m')^{-1}) &= v_{i_R}((m'')_{R-1}Y_{i_R,aq^{l_R}}) + \sum_{\{k\neq R-1|i_k=i_R\}} v_{i_R}((m'')_kY^{-1}_{i_R,aq^{l_k}})\\ &\geq v_{i_R}((m'')_{R-1}Y_{i_R,aq^{l_R}}) \geq -1.\end{aligned}$$

If $v_{i_R}(m''(m')^{-1}) = -1$, then for all k satisfying $i_k = i_R$ we have $(m'')_k = Y_{i_R,aq^{l_k}}$ and $(m'')_{R-1} = Y^{-1}_{i_R,aq^{l_R}}A_{i_R,aq^{l_{R-1}}}$. So $m'' = m'A_{i_R,aq^{l_{R-1}}}$ and we can conclude the proof of (4).

Now it suffices to prove that the conditions of Theorem 5.1 with $i = i_R$ are satisfied for m'.

Condition (i) of Theorem 5.1. – The uniqueness follows from the statement (3) above. For the existence, it suffices to prove that $M = m'A_{i_R,aq^{l_{R-1}}}$ is in $\mathcal{M}(L(m))$. By Lemma 5.6, there is $j \in I$, $M' \in \mathcal{M}(L(m))$ j-dominant such that $M' > m'$ and $M' \in m'\mathbb{Z}[A_{j,a}]_{a\in\mathbb{C}^*}$. By the induction hypothesis on v we have $j = i_R$, and so by uniqueness $M' = M$.

Condition (ii) of Theorem 5.1. – We have by Lemma 5.4

$$\sum_{r\in\mathbb{Z}} x^+_{i,r}(V_M) \subset \sum_{m'\in m\mathbb{Z}[A^{\pm}_{i,d}]_{d\in\mathbb{C}^*}}(L(m))_{m'},$$

and so the result follows from the statement (4) above.

Condition (iii) of Theorem 5.1. – By Lemma 5.7, we have

$$M \in \mathcal{M}(L(Y_{i_R,aq^{l_R}}))\mathcal{M}(L(mY^{-1}_{i_R,aq^{l_R}})).$$

But by Theorem 3.13 the monomials of $\mathcal{M}(L(Y_{i_R,aq^{l_R}}))$ not equal to $Y_{i_R,aq^{l_R}}$ are lower than $Y_{i_R,aq^{l_R}}A^{-1}_{i_R,aq^{L_R+1}}$. So we have $M \in Y_{i_R,aq^{l_R}}\mathcal{M}(L(mY^{-1}_{i_R,aq^{l_R}}))$. By (3_{R-1}), the module $L(mY^{-1}_{i_R,aq^{l_R}})$ is thin and so $u_{i_R,aq^{l_R-2}}(M) \leq 1$. Moreover by the induction hypothesis on v, $v_{i_R,aq^{l_{R-1}}}(Mm^{-1}) = v_{i_R,aq^{l_R+1}}(Mm^{-1}) = 0$. So $u_{i_R,aq^{l_R}}(M) \geq 1$. So by Proposition 3.8, m' is not a monomial of $\mathcal{M}(L_{i_R}(M))$.

Condition (iv) of Theorem 5.1. – The result follows from the statement (4) above.

Condition (v) of Theorem 5.1. – Clear by the induction hypothesis on v. $\square$

The case of standard modules $M(X_{k,a}^{(n)})$ can be studied in the same way by replacing i by $\bar{i} = n - i + 1$.

We can conclude the proof of Proposition 6.4 with Proposition 6.2, Proposition 6.3 and the following counter-examples:

LEMMA 6.8. – *We suppose that $n \geq 3$. Let $k \geq 3, a \in \mathbb{C}^*$ and $1 < i < n$. Then $M(X_{k,a}^{(i)})$ is not small.*

Proof. – Consider $m' = X_{k,a}^{(i)} A_{i,aq^{-2k+2}}^{-1} \leq X_{k,a}^{(i)}$. Then m' is dominant. As $\mathfrak{g}_{\{i-1,i,i+1\}}$ is of type sl_4, by using Lemma 5.9, we can check as in Remark 4.9 that $L(m')$ is not special, and so $M(X_{k,a}^{(i)})$ is not small. $\square$

6.4. End of the proof of Theorem 1.2

In general for $\mathfrak{g}$ not of type A, i extremal does not imply that $M(X_{k,a}^{(i)})$ is small. For example:

REMARK 6.9. – *Let $\mathfrak{g}$ be of type D_4 and $m = Y_{1,q^3} Y_{1,q^5} Y_{2,1}$. By using the process described in Remark 3.16, the following monomials occur in $\chi_q(L(m))$: $1_3 1_5 2_1$, $1_1 1_3 1_5 2_2^{-1} 3_1 4_1$, $1_1 1_3 1_5 2_2 3_3^{-1} 4_3^{-1}$, $1_1 1_3^2 1_5 2_4^{-1}$, $1_1 1_3$. So $L(m)$ is not special. As $Y_{1,q^3} Y_{1,q^5} Y_{2,1} = X_{4,q^2}^{(1)} A_{1,1}^{-1} \in \mathcal{M}(M(X_{4,q^2}^{(1)}))$, $M(X_{4,q^2}^{(1)})$ is not small.*

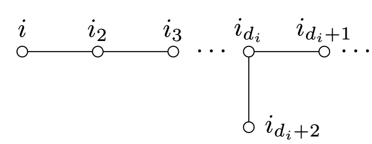
Let us end the proof of Theorem 1.2:

The case $k = 1$ follows from Lemma 6.2. The case $k = 2$ follows from Lemma 6.3. In the rest of the proof we suppose that $k \geq 3$.

Suppose that i is not extremal. There is $j \neq j'$ such that $C_{i,j} = C_{i,j'} = -1$. Consider $m' = X_{k,a}^{(i)} A_{i,aq^{-2k+2}}^{-1} \leq X_{k,a}^{(i)}$. Then m' is dominant. Let $J = \{i, j, j'\}$. $\mathfrak{g}_J$ is of type A_3 and so by using Lemma 5.9, we can check as in Remark 4.9 that $L(m')$ is not special.

Suppose that i is extremal. Let i_2 be the unique element of I satisfying $C_{i,i_2} = -1$. Let $i_3, \ldots, i_{d_i}$ such that for $2 \leq r \leq d_i - 1$, $C_{i_r, i_{r+1}} = -1$ and i_{d_i} is special. Let $i_{d_i+1} \neq i_{d_i+2}$ such that $C_{i_{d_i}, i_{d_i+1}} = C_{i_{d_i}, i_{d_i+2}} = -1$ and $i_{d_i-1}, i_{d_i+1}, i_{d_i+2}$ are distinct.

For an illustration an example is given on the following picture:



Suppose that $k \geq d_i + 2$. Let $m' = X_{k,a}^{(i)} A_{i,aq^{2-k}}^{-1} = Y_{i_2,aq^{2-k}} X_{k-1,aq}^{(i)}$. By Remark 3.16,

$$\begin{aligned} m'' =& m'(A_{i_2,aq^{3-k}}^{-1} A_{i_3,aq^{4-k}}^{-1} \cdots A_{i_{d_i+1},aq^{d_i+2-k}}^{-1}) \\ & \times (A_{i_{d_i+2},aq^{d_i+2-k}}^{-1} A_{i_{d_i},aq^{d_i+3-k}}^{-1} A_{i_{d_i-1},aq^{d_i+4-k}}^{-1} \cdots A_{i_2,aq^{2d_i+1-k}}^{-1}) \in \mathcal{M}(L(m')). \end{aligned}$$

But

$$\begin{aligned} (m'')^{\to(i)} &= X_{k-1,aq}^{(i)} (A_{i_2,aq^{3-k}}^{-1} A_{i_2,aq^{2d_i+1-k}}^{-1})^{\to(i)} \\ &= Y_{i,aq^{3-k}} Y_{i,aq^{5-k}} \cdots Y_{i,aq^{k-1}} Y_{i,aq^{2d_i+1-k}}, \end{aligned}$$

and $3-k \leq 2d_i+1-k \leq k-3$. So $m'' A^{-1}_{i,aq^{2d_i+2-k}}$ is dominant and occurs in $\chi_q(L(m''))$. So $L(m'')$ is not special and $M(X^{(i)}_{k,a})$ is not small.

Suppose that $k \leq d_i+1$ and that there is a dominant monomial $m' < X^{(i)}_{k,a}$. By Lemma 6.1, $(v_j(m'm^{-1}) \neq 0 \Rightarrow j \in \{i_1, \dots, i_{d_i}\})$. So from Lemma 5.9, we can work with $\mathfrak{g}_{\{i_1,\dots,i_{d_i}\}}$ of type A_{d_i}. So it follows from Proposition 6.4 that $M(X^{(i)}_{k,a})$ is small. □

6.5. General simply laced quantum affinizations

The notion of quantum affinization can be extended beyond quantum affine algebras: the quantum affinization $\mathcal{U}_q(\hat{\mathfrak{g}})$ of a quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$ is defined with the same generators and relations as the Drinfeld realization of quantum affine algebras, but by using the generalized symmetrizable Cartan matrix of $\mathfrak{g}$ instead of a Cartan matrix of finite type. The quantum affine algebras, quantum affinizations of usual quantum groups, are the simplest examples of quantum affinizations and have the particular property of being also quantum Kac-Moody algebras. The quantum affinization of a quantum affine algebra is called a quantum toroidal algebra (or double affine quantum algebra). It is not a quantum Kac-Moody algebra, but it is also of particular interest, in particular in relation to double affine Hecke algebras (Cherednik algebras).

In [29, 32, 20], the category $\mathcal{O}$ of integrable representations is studied. One can define for general quantum affinizations analogs of Kirillov-Reshetikhin modules (these representations are not finite dimensional in general). We can also define the notion of small modules by using the characterization in Theorem 4.8.

The statement of Theorem 1.2 is satisfied for all simply-laced quantum affinizations, by using exactly the same proof, except that in the end of the proof of Theorem 1.2 (subsection 6.4), for $J = \{i,j,j'\}$, $\mathfrak{g}_J$ may be of type A_3 or of type $A_2^{(1)}$ (in the second case we have $C_{i,j} = C_{i,j'} = C_{j,j'} = -1$). In this case, we can check as in the following remark that for m' as in subsection 6.4, $L(m')$ is not small.

Remark 6.10. – *Let $\mathfrak{g}$ be of $A_2^{(1)}$, consider $m = Y_{2,1}Y_{2,q^2}Y_{2,q^4}$, $m' = mA^{-1}_{2,q} = Y_{1,q}Y_{0,q}Y_{2,q^4}$. Then by using the process described in Remark 3.16, the monomials*

$$Y^{-1}_{1,q^3}Y^{-1}_{0,q^3}Y_{1,q^2}Y_{0,q^2}Y^2_{2,q^2}Y_{2,q^4} = m'A^{-1}_{1,q^2}A^{-1}_{0,q^2},$$

$$Y_{2,q^2}Y_{1,q^2}Y_{0,q^2} = m'A^{-1}_{1,q^2}A^{-1}_{0,q^2}A^{-1}_{2,q^3},$$

occur in $\chi_q(L(m'))$ and $L(m')$ is not special. So $M(m)$ is not small.

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