ANNALES SCIENTIFIQUES DE L'É.N.S.

WOLFGANG SOERGEL

The Hochschild cohomology ring of regular maximal primitive quotients of enveloping algebras of semisimple Lie algebras

Annales scientifiques de l'É.N.S. 4e série, tome 29, nº 4 (1996), p. 535-538

<http://www.numdam.org/item?id=ASENS_1996_4_29_4_535_0>

Ann. scient. Éc. Norm. Sup.,
4ᵉ série, t. 29, 1996, p. 535 à 538.

THE HOCHSCHILD COHOMOLOGY RING OF REGULAR MAXIMAL PRIMITIVE QUOTIENTS OF ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

BY WOLFGANG SOERGEL

ABSTRACT. – Let U be the enveloping algebra of a semisimple complex Lie algebra, χ a regular maximal ideal of its center. We show that the Hochschild cohomology ring of $U/\chi U$ is just the coinvariant algebra of the Weyl group. This turns out to be almost immediate if one uses localization.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $U = U(\mathfrak{g}) \supset Z$ the enveloping algebra with its center. Let $S = S(\mathfrak{h}^*)$ be the regular functions on $\mathfrak{h}$ and $S^+ \subset S$ the maximal ideal of all functions vanishing at $0 \in \mathfrak{h}$.

The Weyl group $\mathcal{W}$ acts on S and we consider the coinvariant algebra $C = S/(S^+)^{\mathcal{W}} S$, the quotient of S by the ideal generated by all $\mathcal{W}$-invariant functions vanishing at zero.

THEOREM 1. – *Let* $\chi \subset Z$ *be a regular maximal ideal,* i.e. *the kernel of a regular central character. Then the Hochschild cohomology ring of* $U/\chi U$ *is the coinvariant algebra with its obvious grading doubled,*

$$HH^\bullet(U/\chi U) = C^{2\bullet}.$$

Remark. – I thank Patrick Polo for telling me this problem along with the expected answer, and for his helpful comments on a first version.

Proof. – By the Hochschild cohomology ring of an associative k-algebra A we mean just the ring $HH^\bullet(A) = \mathrm{Ext}^\bullet_{A\otimes A^{\mathrm{opp}}}(A, A)$ of self-extensions of A considered as an $A \otimes_k A^{\mathrm{opp}}$-module.

I now give an outline of the proof. Let X be the variety of all Borel subalgebras of $\mathfrak{g}$. Under localization [BB81] the bimodule $U/\chi U$ becomes a $\mathcal{D}$-module $\mathcal{M}$ for some

1991 *Mathematics Subject Classification.* Primary 17B20; Secondary 22E46.

sheaf of twisted differential operators $\mathcal{D}$ on $X \times X$ such that $X \times X$ is $\mathcal{D}$-affine, and we just need to compute the selfextensions of this $\mathcal{D}$-module $\mathcal{M}$. Now choosing $\mathcal{D}$ correctly we can assume that

1. the restriction of $\mathcal{D}$ to the open orbit $Y \subset X \times X$ for the diagonal action of G is just the standard sheaf of differential operators $\mathcal{D}_Y$ on Y and

2. the $\mathcal{D}$-module $\mathcal{M}$ is just the standard module (with unique simple quotient) $i_! \mathcal{O}_Y$, where i is the inclusion of Y into $X \times X$.

Once we know that, we find quickly

$$\mathrm{Ext}^\bullet_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) = \mathrm{Ext}^\bullet_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) = H^\bullet(Y, \mathbb{C}),$$

the first step since $i_!$ is fully faithful even on derived categories, the last step by the Riemann-Hilbert-correspondence. Since Y is fibered over X with fibers just affine spaces, the cohomology ring of Y coincides with the cohomology ring of the flag manifold X which is known to be the ring C of coinvariants (with its obvious grading doubled).

Certainly 1 and 2 are known in much greater generality, they are just a special case of the correspondence between the classifications of Beilinson-Bernstein and Langlands. However, I want to avoid these heavy arguments and provide a more simple minded approach. So let us now start the work and find the sheaf of twisted differential operators (tdo for short) $\mathcal{D}$. We fix some notations concerning homogeneous twisted differential operators. I will follow [HMSW87], Appendix A.

For a complex algebraic group G and a closed subgroup $B \subset G$ and a B-invariant linear form λ on Lie$B$ we have a sheaf $\mathcal{D}_\lambda$ of homogeneous twisted differential operators on G/B. Let $\mathfrak{g} \supset \mathfrak{b}$ be the Lie algebras of $G \supset B$. The geometric stalk of $\mathcal{D}_\lambda$ at the point $x = B$ of G/B is given as

$$\mathcal{D}_\lambda / \mathcal{D}_\lambda \mathfrak{m}_x = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes \bigwedge^{\max} (\mathfrak{g}/\mathfrak{b})).$$

The ordinary differential operators are $\mathcal{D}_0$. Suppose now $H \subset G$ is a closed subgroup and let $f : H/H \cap B \hookrightarrow G/B$ be the inclusion. Pulling back the tdo $\mathcal{D}_\lambda$ via f we find $\mathcal{D}_\lambda^f = \mathcal{D}_\mu$ where μ is the restriction of f to Lie$(H \cap B)$.

Let us return now to our situation. We fix a semisimple connected complex algebraic group G with Lie$G = \mathfrak{g}$, let $T \subset G$ be the maximal torus with Lie$T = \mathfrak{h}$ and fix two Borel subgroups B^+, $B^- \subset G$ containing T which are opposite, *i.e.* $B^+ \cap B^- = T$. Their Lie algebras will be denoted $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. Any $\lambda \in \mathfrak{h}^*$ determines a homogeneous tdo $\mathcal{D}_\lambda^+$ on $X = G/B^+$ whose stalk at B^+ is the Verma module $M^+(\lambda) = U \otimes_{U(\mathfrak{b}^+)} (\mathbb{C}_\lambda \otimes \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b}^+))$. If we put $\xi^+(\lambda) = \mathrm{Ann}_Z M^+(\lambda)$, then $\Gamma(\mathcal{D}_\lambda^+) = U/\xi^+(\lambda)U$. The same statements hold with $+$ replaced by $-$, and since the $\mathfrak{h}$-finite vectors in the (ordinary) dual of $M^+(\lambda)$ have the same $\mathfrak{h}$-weights as $M^-(-\lambda)$, we see directly that the principal antiautomorphism of $\mathfrak{g}$ induces an isomorphism $U/\xi^+(\lambda)U \to (U/\xi^-(-\lambda)U)^{\mathrm{opp}}$. This way we associate to $\lambda \in \mathfrak{h}^*$ a tdo $\mathcal{D}_{(\lambda)} = \mathcal{D}_\lambda^+ \boxtimes \mathcal{D}_{-\lambda}^-$ on $X \times X$ with global sections $\Gamma(\mathcal{D}_{(\lambda)}) = U/\xi^+(\lambda)U \otimes (U/\xi^+(\lambda)U)^{\mathrm{opp}}$. We leave it to the interested reader to verify that $\mathcal{D}_{(\lambda)}$ and this identification of the global sections depend only on the linear form λ on the "abstract Cartan subgroup".

The localization theorem [BB81] says that for a homogeneous tdo $\mathcal{D}$ on X the functor of global sections is an equivalence of categories

$$\Gamma \ : \ \mathcal{D}\text{-mod} \ \rightarrow \ \Gamma(\mathcal{D})\text{-mod}$$

if and only if for one (equivalently any) point $x \in X$ the geometric stalk $\mathcal{D}/\mathcal{D}\mathfrak{m}_x$ at x is a simple $\mathfrak{g}$-module with regular central character. In this case X is called $\mathcal{D}$-affine. It is clear that for any regular maximal ideal $\chi \subset Z$ we can find $\lambda \in \mathfrak{h}^*$ such that $\xi^+(\lambda) = \chi$ and that $X \times X$ is $\mathcal{D}_{(\lambda)}$-affine. This homogeneous tdo $\mathcal{D}_{(\lambda)}$ is the $\mathcal{D}$ we were looking for.

To see this, remark first that by definition of λ the functor

$$\Gamma \ : \ \mathcal{D}_{(\lambda)}\text{-mod} \ \rightarrow \ U/\chi U\text{-mod-}U/\chi U$$

is an equivalence of categories. Furthermore, if $i : Y \hookrightarrow X \times X$ denotes the inclusion of the open G-orbit $Y = G(B^+, B^-) = G/T$, it is plain that $\mathcal{D}_{(\lambda)}$ pulls back to the ordinary differential operators $\mathcal{D}^i_{(\lambda)} = \mathcal{D}_Y$ on Y.

We have to check that $\Gamma(i_! \mathcal{O}_Y) = U/\chi U$. Remark first that since i is an open embedding, the direct and inverse images in the category of sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules coincide. We thus have an adjoint pair of functors (i^*, i_*) between sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules for any tdo $\mathcal{D}$ on $X \times X$. Certainly i^* is exact, and since i is affine i_* is exact, too. Now $\Gamma(i_* \mathcal{O}_Y)$ coincides with $\Gamma(\mathcal{O}_Y)$ and it is clear that every nonzero $\mathcal{D}_{(\lambda)}$-submodule of $i_* \mathcal{O}_Y$ contains the global section $\mathbf{1}_Y$ on Y. Thus $i_* \mathcal{O}_Y$ has a simple socle. Let us define the adjoint $\mathfrak{g}$-action on a U-bimodule by the formula

$$(\text{ad}A)(m) = Am - mA.$$

It is then clear that the socle of the U-bimodule $\Gamma(i_* \mathcal{O}_Y)$ is simple and generated by an $(\text{ad}\mathfrak{g})$-invariant line.

Now we study $i_! \mathcal{O}_Y$. Recall its definition from [Bei83]. If $\mathcal{D}$ is a tdo on a smooth variety of dimension n and $\mathcal{M}$ a holonomic $\mathcal{D}$-module, one defines its dual $*\mathcal{M} = \mathcal{R}\mathcal{H}om^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$. This is a holonomic $\mathcal{D}^{\text{opp}}$-module, and $* * \mathcal{M} = \mathcal{M}$ naturally. By definition $i_! \mathcal{O}_Y = *i_* * \mathcal{O}_Y$. It is clear that in our case (i an open immersion) the duality commutes with i^*, whence an adjoint pair $(i_!, i^*)$ of functors between holonomic modules. It is known in general (and clear in our case) that $i_! \mathcal{O}_Y$ has as its unique simple quotient the socle of $i_* \mathcal{O}_Y$. Passing to global sections we see that $\Gamma(i_! \mathcal{O}_Y)$ is generated by an $(\text{ad}\mathfrak{g})$-invariant line. From this we already find a surjection of bimodules

$$U/\chi U \twoheadrightarrow \Gamma(i_! \mathcal{O}_Y).$$

Now let us go to the category ${}_\chi\mathcal{H}_\chi$ of all locally $(\text{ad}\mathfrak{g})$-finite $U/\chi U$-bimodules. Under localization this corresponds to the category of all G-equivariant $\mathcal{D}_{(\lambda)}$-modules. All objects of this category are mapped under i^* to a finite direct sum of some copies of $\mathcal{O}_Y$. Hence $i_! \mathcal{O}_Y$ is a projective object in our category, whence $\Gamma(i_! \mathcal{O}_Y)$ is a projective object of ${}_\chi\mathcal{H}_\chi$. But this means that the surjection $U/\chi U \twoheadrightarrow \Gamma(i_! \mathcal{O}_Y)$ is split, and since $U/\chi U$ is indecomposable it has to be an isomorphism. □

REFERENCES

[BB81] A. A. BEILINSON and J. N. BERNSTEIN, *Localisation de $\mathfrak{g}$-modules* (*C. R. Acad. Sci. Paris*, T. 292, Series A, 1981, pp. 15-18).

[Bei83] A. A. BEILINSON, *Localization of representations of reductive Lie algebras (Proceedings of the ICM Warszawa*, 1983, pp. 699-709).

[HMSW87] H. HECHT, D. MILIČIĆ, W. SCHMID and J. A. WOLF, *Localization and standard modules for real semisimple Lie groups I: The duality theorem (Inventiones*, Vol. 90, 1987, pp. 297-332).

(Manuscript received September 11, 1995.)

W. SOERGEL
Universität Freiburg, Mathematisches Institut,
Albertstraße 23b, D-79104 Freiburg, Germany
E-mail address:
soergel@sun1.mathematik.uni-freiburg.de