FIONA MURNAGHAN

Characters of supercuspidal representations of classical groups

Annales scientifiques de l'É.N.S. 4e série, tome 29, nº 1 (1996), p. 49-105

<http://www.numdam.org/item?id=ASENS_1996_4_29_1_49_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 29, 1996, p. 49 à 105.

CHARACTERS OF SUPERCUSPIDAL REPRESENTATIONS OF CLASSICAL GROUPS

BY FIONA MURNAGHAN (*)

ABSTRACT. – In this paper we derive a Kirillov type character formula for irreducible supercuspidal representations of p-adic orthogonal, symplectic and unitary groups. This formula is valid on a neighbourhood of zero in the Lie algebra. It expresses the composition of the character of a supercuspidal representation with the exponential map as the formal degree of the representation times the Fourier transform of an invariant measure on a certain elliptic adjoint orbit. The orbit is defined using the data from the construction of the supercuspidal representation as a representation induced from a finite dimensional representation of an open compact subgroup. As a consequence of the above character formula, the coefficients in Harish-Chandra's local character expansion of a supercuspidal representation are given as multiples of values of Shalika germs.

1. Introduction

Let $\mathbf{G}$ be a reductive algebraic group defined over a p-adic field F_0 of characteristic zero. Given a supercuspidal representation π of the F_0-points $G = \mathbf{G}(F_0)$ of $\mathbf{G}$, let $d(\pi)$ and Θ_π denote the formal degree and the character of π, respectively. In [Mu1-3], it was shown that, under certain conditions on the residual characteristic of F_0, if $G = \mathbf{GL}_n(F_0)$, $\mathbf{SL}_n(F_0)$ or the 3 by 3 unramified unitary group, for many irreducible supercuspidal representations π of G there exists a Kirillov type character formula relating $d(\pi)^{-1}\Theta_\pi \circ \exp$ to the Fourier transform of a G-invariant measure on some elliptic $\mathrm{Ad}\, G$-orbit. More precisely, there exists a regular elliptic element X_π in $\mathfrak{g}$ and an open neighbourhood V_π of zero such that

$$d(\pi)^{-1}\Theta_\pi(\exp X) = \widehat{\mu}_{\mathcal{O}(X_\pi)}(X), \qquad X \in V_\pi \cap \mathfrak{g}_{reg}. \tag{1.1}$$

Here, $\widehat{\mu}_{\mathcal{O}(X_\pi)}$ is the Fourier transform of a G-invariant measure $\mu_{\mathcal{O}(X_\pi)}$ on the $\mathrm{Ad}\, G$-orbit $\mathcal{O}(X_\pi)$ and $\mathfrak{g}_{reg}$ is the set of regular elements in $\mathfrak{g}$. In this paper, we investigate whether (1.1) holds for those supercuspidal representations of p-adic classical groups which have been constructed by Morris ([Mor2-3]).

Our methods depend on existence of constructions of supercuspidal representations as representations induced from finite dimensional representations of open, compact modulo centre subgroups of G. The inducing data for supercuspidal representations (in cases where

1992 Mathematics Subject Classifications: 22 E 50, 22 E 35.
(*) Research supported in part by NSERC.

such data has been found) involves certain filtrations (nested sequences of open normal subgroups) of parahoric subgroups which are attached to elliptic Cartan subgroups of G. There is a canonical way, using height functions on affine roots, to define filtrations of parahoric subgroups ([PR]). However, filtrations which are not canonical may occur in inducing data ([Mor2]).

If $G = \mathbf{GL}_n(F_0)$, due to the fact that an elliptic Cartan subgroup T is isomorphic to the multipicative group of a degree n extension E of F_0, the filtrations arising are associated to powers of the Jacobson radical of the hereditary order which stabilizes the powers of the prime ideal in the ring of integers of E. These filtrations are canonical.

More generally, an elliptic Cartan subgroup T of a classical group G is a subgroup of the product of the multiplicative groups of finitely many extensions of F_0 (*see* [Mor1], §1 for a description of the classification of elliptic Cartan subgroups). Consequently, filtrations other than those arising for general linear groups appear in the inducing data for supercuspidal representations of G. Some filtrations are not canonical. Moreover, in the construction of one supercuspidal representation, it may be necessary to work with several different filtrations, where the relations between the groups belonging to two distinct filtrations may not be explicit. Hence the construction of supercuspidal representations of G is more technical than that for $\mathbf{GL}_n(F_0)$. The inducing data is built up inductively out of smaller pieces of data which are concentrated on certain subgroups. On each of these subgroups the construction is much like that for general linear groups (*see* the introduction and 6.2-3 of [Mor3]). As a result, parts of the proof of (1.1) for a supercuspidal representation of G are like the proof for general linear groups. However, the parts of the proof which deal with the technicalities arising from the way these smaller pieces are fitted together are different.

In [Mu2], it was determined that, with the exception of one family of representations (those associated to a cuspidal unipotent representation of the group over the finite field), (1.1) holds for supercuspidal representations of the 3 by 3 unramified unitary group $\mathbf{U}_3$. This was done using Jabon's inducing data ([J]), which was obtained from Moy's characterization of the irreducible admissible representations via nondegenerate representations ([Moy]). The results of this paper yield an alternate proof of (1.1) for those supercuspidal representations of $\mathbf{U}_3(F_0)$ constructed by Morris.

We now summarize the contents of this paper. The main results appear in §§7-10 and are proved under the assumption that the residual characteristic of F_0 is sufficiently large (*see* §2).

Some notation is defined in §2.

§3 begins with background from [Mor2] concerning lattice chains, parahoric subgroups and filtrations attached to tamely ramified elliptic Cartan subgroups. At the end of the section, we prove some lemmas about lattice chains which allow us to define a truncated exponential map on certain elements in the Lie algebra.

Let T be a tamely ramified elliptic Cartan subgroup of G. A *T-cuspidal datum* $\underline{\Psi}$ ([Mor3]) consists of a collection of objects, including elements in the Lie algebra of T, linear characters on the stabilizers of these elements in certain subgroups of G, and integers which describe the size of the conductors of the linear characters. §4 and §5 begin with elementary lemmas regarding the values of linear characters of unitary groups. After recalling the definition of cuspidal datum, these lemmas are applied to the linear characters occurring in $\underline{\Psi}$ to produce an element $c_{\underline{\Psi}}$ in the Lie algebra of T. The element $c_{\underline{\Psi}}$ is then shown to be regular. §4 deals with the case where the fields in the commutator algebra of T are unramified (over a particular extension of F_0), and §5 deals with the general case.

The T-cuspidal datum $\underline{\Psi}$ is used to construct an open compact subgroup $P_{\underline{\Psi}}$ of G, and an irreducible representation $\rho_{\underline{\Psi}}$ of $P_{\underline{\Psi}}$ ([Mor3]). The datum $\underline{\Psi}$ is defined inductively in terms of cuspidal data associated to subgroups of T. §6 consists of a few lemmas describing basic properties of $P_{\underline{\Psi}}$ and $\rho_{\underline{\Psi}}$ which follow from the inductive definition of $\underline{\Psi}$.

The topic of §7 is properties of adjoint orbits of linear functionals on the Lie algebra $\mathfrak{g}$. The particular linear functionals considered here occur in the integral formula for the Fourier transform of a G-invariant measure on the adjoint orbit of $c_{\underline{\Psi}}$.

The representation $\rho_{\underline{\Psi}}$ is a tensor product of various representations of $P_{\underline{\Psi}}$, and $c_{\underline{\Psi}}$ is a sum of elements in the Lie algebra of T. There is a one to one correspondence between the representations occurring in the tensor product defining $\rho_{\underline{\Psi}}$ and the semisimple elements occurring in the sum defining $c_{\underline{\Psi}}$. In §8, we derive a relation between the character of one of these representations and the $\mathrm{Ad}\, P_{\underline{\Psi}}$-orbit of a particular linear functional which is defined in terms of the corresponding semisimple element.

As shown in [Mor3], the representation $\pi_{\underline{\Psi}}$ of G which is induced from the representation $\rho_{\underline{\Psi}}$ of $P_{\underline{\Psi}}$ is an irreducible supercuspidal representation of G. If we define a function on G by giving it the same values as the character of $\rho_{\underline{\Psi}}$ at points in $P_{\underline{\Psi}}$ and zero at other points, we obtain a finite sum of matrix coefficients of $\pi_{\underline{\Psi}}$. One of the main results of this paper is Proposition 9.1, which can be stated as follows. The value of this function at a unipotent element in G coincides, up to multiplication by its value at the identity element, with the value of the $\mathrm{Ad}\, P_{\underline{\Psi}}$-orbit of a linear functional defined in terms of $c_{\underline{\Psi}}$ at the corresponding nilpotent element in the Lie algebra. In proving the proposition, we assume that the T-cuspidal datum $\underline{\Psi}$ is uniform: that is, $\underline{\Psi}$ satisfies certain mild restrictions on the size of some conductoral exponents. This uniformity condition, which most cuspidal data satisfy, was introduced in §7 in order to deal with cases where the relations between the distinct filtrations attached to cuspidal data are not explicit (*see* the proof of Lemma 7.9). It is possible that further investigation of the properties of these filtrations will allow us to modify the proof so that Proposition 9.1 holds for all $\underline{\Psi}$.

Theorem 10.1, which states that if $\pi = \pi_{\underline{\Psi}}$ and $\underline{\Psi}$ is uniform, then (1.1) holds with $X_\pi = c_{\underline{\Psi}}$, is a consequence of Proposition 9.1 and Harish-Chandra's integral formulas for Θ_π and $\widehat{\mu}_{\mathcal{O}(X_\pi)}$. Corollary 10.3 then gives the coefficients in the local character expansion of π at the identity as multiples of values of Shalika germs at $c_{\underline{\Psi}}$.

In defining $c_{\underline{\Psi}}$ and proving the results of §§7-10, we assume that if there are cuspidal representations of reductive groups over finite fields occurring in $\underline{\Psi}$, then these representations are associated via the construction of Deligne and Lusztig to a character of a minisotropic torus. In §11, we discuss some examples where this assumption is dropped. For some of these examples, (1.1) does not hold. In some cases, there exists a generalization of (1.1), in which the single Fourier transform is replaced by a finite linear combination of Fourier transforms.

At the end of [Mor3], Morris outlines a method of combining the results of [Mor1] with those in the main body of [Mor2-3] to construct more general types of cuspidal data. Some of the supercuspidal representations associated to this more general data are not equivalent to any of those obtained via the earlier construction. As long as any representations of finite classical groups which occur satisfy the assumption mentioned above, and (analogues of) the results of [Mor2-3] are valid in this setting, it should be possible to define an element $c_{\underline{\Psi}}$ such that (1.1) is satisfied (with $X_\pi = c_{\underline{\Psi}}$ and $\pi = \pi_{\underline{\Psi}}$). We make some comments on this in §12.

2. Notation

Let F be a p-adic field of characteristic zero, O the ring of integers in F, $\mathfrak{p}$ the maximal prime ideal in O, and ϖ a generator of $\mathfrak{p}$. Suppose that σ_0 is an involution on F, with fixed field F_0. O_0 and $\mathfrak{p}_0$ will denote the ring of integers and maximal prime ideal in F_0. If σ_0 is nontrivial, set

$$F^1 = \{ x \in F \mid x\,\sigma_0(x) = 1 \}$$
$$F^1_j = F^1 \cap (1 + \mathfrak{p}^j), \qquad j \in \mathbf{Z}\ j \geq 1.$$

Let V be a finite-dimensional vector space over F. Throughout this paper, the residual characteristic p of F will be assumed to be strictly greater than the dimension of V over F. Suppose that V is equipped with a non degenerate (ϵ, σ_0)-sesquilinear form f, where $\epsilon = \pm 1$ is fixed:

$$f(u,v) = \epsilon\,\sigma_0 f(v,u), \qquad u,\, v \in V$$
$$f(\lambda u, v) = \lambda\, f(u,v), \qquad \lambda \in F, \quad u, v \in V.$$

Let $\mathbf{G} = \mathbf{U}(f)$ denote the corresponding classical group (defined over F_0). Then

$$G = \mathbf{G}(F_0) = \{ g \in \mathbf{GL}(V) \mid f(gu, gv) = f(u,v),\, \forall\, v, w \in V \}.$$

If σ_0 is trivial ($F = F_0$) and $\epsilon = 1$, resp. -1, then f is symmetric, resp. alternating, and G is an orthogonal, resp. symplectic group. In the case σ_0 is nontrivial, f is hermitian or skew-hermitian, according to whether $\epsilon = 1$ or -1, and we shall say that G is a genuine unitary group.

Let σ be the involution on $\mathrm{End}_F(V)$ induced by the form f:

$$f(Tu, v) = f(u, \sigma(T)v), \qquad u, v \in V \quad T \in \mathrm{End}_F(V).$$

Then

$$G = \{ g \in \mathbf{GL}(V) \mid g\,\sigma(g) = 1 \},$$

and the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \{ X \in \mathrm{End}_F(V) \mid X + \sigma(X) = 0 \}.$$

If X is an element of $\mathrm{End}_F(V)$ such that $\det(1+X) \neq 0$, the *Cayley transform* $C(X)$ of X is defined to be $(1-X)(1+X)^{-1}$. It is easily verified that if $\det(1+X) \neq 0$, then X is in $\mathfrak{g}$ if and only if $C(X)$ belongs to G.

Given a σ-stable subset B of $\mathrm{End}_F(V)$, define

$$B^+ = \{ X \in B \mid \sigma(X) = X \}$$
$$B^- = \{ X \in B \mid \sigma(X) = -X \}.$$

Let tr be the trace map on $\mathrm{End}_F(V)$, and let $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}$, where $\mathrm{tr}_{F/F_0}(x) = x + \sigma_0(x)$, $x \in F$. If A is a subspace of $\mathrm{End}_F(V)$, set

$$A^\perp = \{ X \in \mathrm{End}_F(V) \mid \mathrm{tr}(XY) = 0 \ \forall Y \in A \}.$$

If $X \in \mathrm{End}_F(V)$, the notation $\mathcal{C}[X]$ will be used for the commuting algebra of X in $\mathrm{End}_F(V)$.

3. Lattice chains and filtrations associated to elliptic Cartan subgroups

Let f, V, and G be as in §2. In [Mor2], Morris constructed parahoric subgroups and filtrations associated to tamely ramified elliptic Cartan subgroups of classical groups. The section begins with a summary of definitions and results from [Mor2] which will be used in this paper. The last part of the section is concerned with relations between elements of $\mathfrak{g}$ and G which are used in later sections when evaluating linear characters of various subgroups of G.

By a *lattice chain* in V, we mean a family $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ of O-lattices in V which is stable under multiplication by elements of $F^\times$ and such that $L_{i+1} \subset L_i$. The *period* of $\mathcal{L}$ is the smallest integer e such that $\varpi L_i = L_{i+e}$ for all i in $\mathbf{Z}$. $\mathcal{L}$ is the sequence of lattices

$$\cdots \varpi^{-1} L_{e-1} \supset L_0 \supset L_1 \supset \cdots \supset L_{e-1} \supset \varpi L_0 \supset \cdots$$

Given a lattice L in V, the *dual* (or *complementary*) *lattice* $L^\sharp$ is defined by

$$L^\sharp = \{ v \in V \mid f(v, L) \subset O \}.$$

$\mathcal{L}$ is said to be *self dual* if $L^\sharp \in \mathcal{L}$ whenever $L \in \mathcal{L}$.

Let $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ be a lattice chain in V. Then, as shown in [BF],

(3.1) $$\mathcal{A} \stackrel{\text{def}}{=} \{ X \in \mathrm{End}_F(V) \mid X L_i \subset L_i \ \forall i \}$$

is a hereditary order in $\mathrm{End}_F(V)$ and

(3.2) $$\mathcal{B} \stackrel{\text{def}}{=} \{ X \in \mathrm{End}_F(V) \mid X L_i \subset L_{i+1} \ \forall i \}$$

is the Jacobson radical of $\mathcal{A}$. The *parahoric subgroup* determined by $\mathcal{A}$ is defined to be $P = \mathcal{A} \cap G$ ([Mor3], p. 714).

A lattice chain $\{\mathcal{B}_i\}_{i \in \mathbf{Z}}$ in $\mathrm{End}_F(V)$ will be called self dual if it is self dual with respect to the trace map tr on $\mathrm{End}_F(V)$. That is, for each i, the lattice

$$\mathcal{B}_i^* = \{ X \in \mathrm{End}_F(V) \mid \mathrm{tr}(X \mathcal{B}_i) \subset O \}$$

belongs to the lattice chain. Since $\mathcal{B}_i$ is an O-lattice in $\mathrm{End}_F(V)$, $\mathcal{B}_i$ is also an O_0-lattice in $\mathrm{End}_F(V)$. Its *dual (complementary)* lattice $\mathcal{B}_i^\sharp$ is defined by

$$\mathcal{B}_i^\sharp = \{\, X \in \mathrm{End}_F(V) \mid \mathrm{tr}_0(X\mathcal{B}_i) \subset O_0 \,\}.$$

If e_0 is the ramification degree of F over F_0, then ([Mor3], p. 20)

$$\mathcal{B}_i^\sharp = \varpi^{1-e_0}\,\mathcal{B}_i^*.$$

Thus $\mathcal{B}_i^*$ belongs to the lattice chain if and only if $\mathcal{B}_i^\sharp$ belongs to the lattice chain.

Suppose that $V = V_1 \oplus V_2$ where V_i, $i = 1, 2$, is a σ-stable subspace of V. Then the restriction f_i of f to $V_i \times V_i$ is a non degenerate (ϵ, σ_0)-sesquilinear form on V_i. Suppose that $\mathcal{L} = \{L_i\}$, resp. $\mathcal{M} = \{M_i\}$, is a self dual lattice chain in V_1, resp. V_2. Morris ([Mor2], §2) defines a self dual lattice chain, denoted $\mathcal{L} \oplus \mathcal{M}$ and called the sum of $\mathcal{L}$ and $\mathcal{M}$, in V. In addition, Morris constructs a periodic self dual lattice chain in $\mathrm{End}_F(V)$ ([Mor2], 2.10). The hereditary order associated to $\mathcal{L} \oplus \mathcal{M}$ ((3.1)) and its Jacobson radical ((3.2)) belong to this lattice chain. Also, each lattice in the chain is σ-stable.

The lattice chains in V which are associated to elliptic Cartan subgroups are produced by taking sums of canonical lattice chains in certain extensions of F ([Mor2, §3]). The related lattice chains in $\mathrm{End}_F(V)$ then give rise to filtrations of the parahoric subgroups determined by the lattice chains in V.

Let T be an elliptic Cartan subgroup of G. The assumption $p > \dim V$ implies that T is tamely ramified. The commuting algebra A of T,

$$A = \{\, x \in \mathrm{End}_F(V) \mid xy = yx \;\forall\, y \in T \,\},$$

is isomorphic to a finite direct sum of separable field extensions of F. We write $A = \bigoplus_{1\le i\le r} E_i$. Since T is elliptic, $\sigma(E_i) = E_i$ for every i, and either the restriction of σ to every E_i is non-trivial for all i, or the restriction of σ to exactly one E_{i_0} is trivial, and in this case $E_{i_0} = F$. As in §3.1 of [Mor2], choose $\mu = (\mu_1, \ldots, \mu_r) \in A$ with $0 \neq \mu_i \in E_i$ and $\sigma(\mu) = \epsilon\mu$. Define

$$\begin{aligned} f_A :& A \times A \to F \\ & (x, y) \mapsto \mathrm{tr}\,(\mu\, x\, \sigma(y)). \end{aligned}$$

Then f_A is a non degenerate (ϵ, σ_0)-sesquilinear form on A. Assuming that $\mathbf{U}(f, V) \simeq \mathbf{U}(f_A, A)$, we identify f with f_A and V with A. Then

$$T = \{\, x \in V \mid x\,\sigma(x) = 1 \,\}.$$

For the moment, assume that $A = E_1 = E$, that is, $r = 1$. Let $e = e(E/F)$ be the ramification degree of E over F, v the E-valuation of μ, and $\mathfrak{p}_E$ the prime ideal in E. Then, ([Mor3], 3.3), for any integer i,

$$(\mathfrak{p}_E^i)^\sharp = \mathfrak{p}_E^{1-e-v-i}$$

(where the dual is with respect to $f = f_A$). Thus $\mathcal{L} = \{\mathfrak{p}_E^i\}_{i\in\mathbf{Z}}$ is a self-dual lattice chain of period e in $V = E$. This lattice chain will be referred to as the canonical lattice chain in E. Let $\mathcal{A}$, $\mathcal{B}$ and P be the corresponding hereditary order ((3.1)), Jacobson radical ((3.2)), and parahoric subgroup. From $\varpi\,\mathfrak{p}_E^i = \mathfrak{p}_E^{i+e}$, it follows that $\varpi\mathcal{B}^i = \mathcal{B}^{i+e}$. The filtration $\{\mathcal{B}^i\}_{i\in\mathbf{Z}}$ is a lattice chain (with period e) in $\mathrm{End}_F(V)$. Each $\mathcal{B}^i$ is σ-stable, and $\mathcal{B}^i\mathcal{B}^j \subset \mathcal{B}^{i+j}$ for all integers i and j.

The next case to consider is that where $A = \bigoplus_{1\le i\le r} E_i$ is such that every E_i is an unramified extension of F. As shown in §3.4 of [Mor2], summing the canonical lattice chains (of period 1) in the E_i results in a lattice chain $\mathcal{L}_u$ of period 1 or 2, a hereditary order $\mathcal{A}_u$, and a filtration of $\mathcal{A}_u$ arising from powers of the Jacobson radical $\mathcal{B}$ of $\mathcal{A}_u$. Moreover, $\mathcal{L}_u$ is independent of the order of summation of the canonical lattice chains. Following the conventions of [Mor3], if $\mathcal{L}_u$ has period 2, set $\mathcal{B}_i = \mathcal{B}^i$, and if $\mathcal{L}_u$ has period 1, set $\mathcal{B}_{2i} = \mathcal{B}_{2i-1} = \mathcal{B}^i$.

Finally, suppose that $A = \bigoplus_{1\le i\le r} E_i$ is such that at least one E_i is ramified over F. As in §3.5 of [Mor2], reordering summands if necessary, we assume that there exists ℓ, $0 \le \ell \le r-1$, such that E_i is unramified over F for $i \le \ell$, and E_i is ramified over F ($e(E_i/F) > 1$) for $i > \ell$. Set $A_u = \bigoplus_{1\le i\le \ell} E_i$. Let $\mathcal{L}_u$ be the lattice chain in A_u obtained by summing the canonical lattice chains in the E_i, $1 \le i \le \ell$. Set $P_u = \mathcal{A}_u \cap \mathbf{U}(f_u, A_u)$, where f_u is the restriction of f to $A_u \times A_u$. For $\ell+1 \le j \le r$, let $\mathcal{M}_j$ be the canonical lattice chain in E_j. Working inductively, define $\mathcal{L}_j = \mathcal{L}_{j-1} \oplus \mathcal{M}_j$, where $\mathcal{L}_\ell = \mathcal{L}_u$. The notation $\mathcal{A}^{(r-j)}$ will be used to denote the hereditary order associated to $\mathcal{L}_j$. The parahoric subgroup determined by $\mathcal{A}^{(r-j)}$ is

$$P^{(r-j)} = \mathcal{A}^{(r-j)} \cap \mathbf{U}(f_u \oplus f_{\ell+1} \oplus \cdots \oplus f_j, A_u \oplus E_{\ell+1} \oplus \cdots \oplus E_j).$$

Here, f_i is the restriction of f to $E_i \times E_i$. Let $\mathcal{B}_i^{(r-j)}$, $i \in \mathbf{Z}$, be the lattice chain in $\mathrm{End}_F\left(A_u \oplus \sum_{i=\ell}^{j} E_i\right)$ constructed using $\mathcal{L}_j$. Then $\mathcal{A}^{(r-j)} = \mathcal{B}_0^{(r-j)}$ and the filtration of $P^{(r-j)}$ arising from $\{\mathcal{B}_i^{(r-j)}\}_{i\ge 1}$ is

$$P_i^{(r-j)} = P^{(r-j)} \cap (1 + \mathcal{B}_i^{(r-j)}), \qquad i \ge 1.$$

Each $P_i^{(r-j)}$ is an open compact normal subgroup of $P^{(r-j)}$.

LEMMA 3.3. – *Fix $j \ge \ell+1$. Let $\mathcal{L} = \mathcal{L}_{j-1}$ and $\mathcal{M} = \mathcal{M}_j$. Set $\mathcal{A}_{\mathcal{L}} = \mathcal{A}_u$ if $j = \ell+1$, and $\mathcal{A}_{\mathcal{L}} = \mathcal{A}^{(r-j+1)}$ if $j > \ell+1$. $\mathcal{B}_{\mathcal{L}}$ denotes the Jacobson radical of $\mathcal{A}_{\mathcal{L}}$. Let $V_2 = E_j$ and*

$$V_1 = \begin{cases} A_u \oplus E_{\ell+1} \oplus \cdots \oplus E_{j-1}, & \text{if } j > \ell+1,\\ A_u, & \text{if } j = \ell+1.\end{cases}$$

Set $\mathcal{B}_i = \mathcal{B}_i^{(r-j)}$, $\mathcal{A} = \mathcal{A}^{(r-j)}$ and $P = P^{(r-j)}$. Let $\mathcal{A}_{\mathcal{M}} \subset \mathrm{End}_F(V_2)$ be the hereditary order associated to $\mathcal{M}$. Set $P_{\mathcal{L}} = \mathcal{A}_{\mathcal{L}} \cap \mathbf{U}(\tilde{f}_1, V_1)$ and $P_{\mathcal{M}} = \mathcal{A}_{\mathcal{M}} \cap \mathbf{U}(\tilde{f}_2, V_2)$, where $\tilde{f}_h$ is the restriction of f to $V_h \times V_h$, $h = 1, 2$. Then

(1) $\mathcal{B}_i\mathcal{B}_k \subset \mathcal{B}_{i+k}$, $i, k \in \mathbf{Z}$.

(2) *$\mathcal{B}_1$ is the Jacobson radical of $\mathcal{A}$.*

(3) *Let* e_j *be the ramification degree of* E_j *over* F. *Then* $\varpi\mathcal{B}_i = \mathcal{B}_{i+2e_j}$, $i \in \mathbf{Z}$.

(4) $\mathcal{B}_i \cap (\mathrm{End}_F(V_1) \oplus \mathrm{End}_F(V_2)) = (\mathcal{B}_i \cap \mathrm{End}_F(V_1)) \oplus (\mathcal{B}_i \cap \mathrm{End}_F(V_2)), \ i \in \mathbf{Z}$

(5) $P_{\mathcal{L}} \subset P \cap \mathbf{U}(\tilde{f}_1, V_1)$.

(6) $\mathcal{A}_{\mathcal{L}} \subset \mathcal{A} \cap \mathrm{End}_F(V_1)$.

(7) $\mathcal{B}_1 \cap \mathrm{End}_F(V_1) \subset \mathcal{B}_{\mathcal{L}}$.

(8) $\mathcal{A} \cap \mathrm{End}_F(V_2) = \mathcal{A}_{\mathcal{M}}$ *and* $P \cap \mathbf{U}(\tilde{f}_2, V_2) = P_{\mathcal{M}}$. *The filtration defined by intersecting the* $\mathcal{B}_i$*'s with* $\mathrm{End}_F(V_2)$ *coincides with the filtration from powers of the Jacobson radical* $\mathcal{B}_{\mathcal{M}}$ *of* $\mathcal{A}_{\mathcal{M}}$:

$$\mathcal{B}^i_{\mathcal{M}} = \mathcal{B}_{2i-1} \cap \mathrm{End}_F(V_2) = \mathcal{B}_{2i} \cap \mathrm{End}_F(V_2), \qquad i \in \mathbf{Z}.$$

Proof. – (1)-(6), and (8) are results from Proposition 2.7, Lemma 2.10, and Propositions 3.11 and 3.14 of [Mor2].

(7) By (6), the Jacobson radical of $\mathcal{A}_{\mathcal{L}}$ contains the Jacobson radical of $\mathcal{A} \cap \mathrm{End}_F(V_1)$. By Lemma 3.13 of [Mor2], the Jacobson radical of $\mathcal{A} \cap \mathrm{End}_F(V_1)$ is $\mathcal{B}_1 \cap \mathrm{End}_F(V_1)$. □

For the remainder of the section assume that $\mathcal{A}$ is a hereditary order, and the sequence

$$\ldots \supset \mathcal{B}_{-1} \supset \mathcal{B}_0 = \mathcal{A} \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_i \supset \ldots$$

is a lattice chain in $\mathrm{End}_F(V)$ such that

(i) $\mathcal{B}_i$ is σ-stable

(ii) $\mathcal{B}_i\mathcal{B}_j \subset \mathcal{B}_{i+j}$, for all i, $j \in \mathbf{Z}$.

For $i \geq 1$, define

$$P_i = \{\, x \in G \mid x - 1 \in \mathcal{B}_i \,\}.$$

LEMMA 3.4. – *Given* $X \in \mathcal{B}_i^+$, $i \geq 1$, *there exists* $Y \in \mathcal{B}_i$ *such that* $Y + \sigma(Y) + Y\sigma(Y) = X$.

Proof. – Since $X \in \mathcal{B}_i^+$, there exists $Y_1 \in \mathcal{B}_i$ such that $Y_1 + \sigma(Y_1) = X$ (for example, since $2 \in \mathcal{O}^\times$, Y_1 could be taken to be $X/2$). Then

$$X - (Y_1 + \sigma(Y_1) + Y_1\sigma(Y_1)) = -Y_1\sigma(Y_1),$$

is σ-stable, and, by (i) and (ii), lies in $\mathcal{B}_{2i}$. Suppose $Y_1, Y_2 \ldots, Y_k \in \mathcal{B}_i$ are such that

$$Y_r - Y_{r-1} \in \mathcal{B}_{ri}$$
$$X - (Y_r + \sigma(Y_r) + Y_r\sigma(Y_r)) \in \mathcal{B}^+_{(r+1)i}$$

for $2 \leq r \leq k$. Since $X - (Y_k + \sigma(Y_k) + Y_k\sigma(Y_k)) \in \mathcal{B}^+_{(k+1)i}$, there exists $W_{k+1} \in \mathcal{B}_{(k+1)i}$ such that

$$W_{k+1} + \sigma(W_{k+1}) = X - (Y_k + \sigma(Y_k) + Y_k\sigma(Y_k)).$$

Set $Y_{k+1} = Y_k + W_{k+1}$. Then

$$X - (Y_{k+1} + \sigma(Y_{k+1} + Y_{k+1}\sigma(Y_{k+1})) = Y_k\sigma(W_{k+1}) + W_{k+1}\sigma(Y_k) + W_{k+1}\sigma(W_{k+1}),$$

is σ-stable and, by (i) and (ii), is an element of $\mathcal{B}_{(k+2)i}$. The $\mathcal{B}_i$, $i \geq 1$, form a neighbourhood base of zero in $\mathrm{End}_F(V)$ (for $j \geq 1$, $\varpi^j\mathcal{A} = \mathcal{B}_i$, some i). Thus the

Y_k's converge to an element Y satisfying the conclusion of the lemma. Note that $Y \in \mathcal{B}_i$, since each $Y_k \in \mathcal{B}_i$. □

LEMMA 3.5. – *Let* $X \in \mathcal{A}$ *be such that* $\det(1+X) \neq 0$ *and* $X + \sigma(X) + X\sigma(X) \in \mathcal{B}_i$ *for some* $i \geq 1$. *Then there exists* $Y \in \mathcal{B}_i$ *such that* $1 + X + Y \in G$.

Proof. – Since $\det(1+X) \neq 0$, $\det(1+\sigma(X)) = \det(\sigma(1+X)) \neq 0$. So

$$X + \sigma(X) + X\sigma(X) \in \mathcal{B}_i \ \text{ implies } \ ((1+\sigma(X))(1+X))^{-1} \in 1 + \mathcal{B}_i.$$

Let $R = 1 - ((1+\sigma(X))(1+X))^{-1}$. As $\sigma(R) = R$ and $R \in \mathcal{B}_i$, we can apply Lemma 3.4 to obtain $W \in \mathcal{B}_i$ such that

$$W + \sigma(W) + W\sigma(W) = R.$$

Then

$$(1+W)(1+\sigma(W)) = ((1+\sigma(X))(1+X))^{-1},$$

which implies that $(1+X)(1+W) \in G$. Set $Y = W + XW$. Then $(1+X)(1+W) = 1 + X + Y$. Since $W \in \mathcal{B}_i$ and $X \in \mathcal{A} = \mathcal{B}_0$, it follows that $Y \in \mathcal{B}_i$. □

Suppose $2 \leq m \leq p$. Given $X \in \operatorname{End}_F(V)$, set

$$e_m(X) = \sum_{h=0}^{m-1} \frac{X^h}{h!} \qquad \text{and} \qquad \ell_m(X) = \sum_{h=1}^{m-1} (-1)^h \frac{(X-1)^h}{h}.$$

Assume X is in $\mathcal{A} \cap \mathfrak{g}$ and X^m is in $\mathcal{B}_i$ for some $i \geq 1$. We define a coset $p_{(m,i)}(X)$ of $P_i = P \cap (1 + \mathcal{B}_i)$ in P as follows:

$$p_{(m,i)}(X) = ((e_m(X) + \mathcal{B}_i) \cap P)P_i. \tag{3.6}$$

The assumption $m \leq p$ implies $X^h/h! \in \mathcal{A}$ for $1 \leq h \leq m-1$. Thus $e_m(X) \in \mathcal{A}$. It is a simple matter to check that X occurs with exponent at least m in any nonzero term in $e_m(X)e_m(-X) - 1$, and the coefficient of each nonzero term lies in $\mathcal{O}$. Thus $e_m(X)e_m(-X) \in 1 + \mathcal{B}_i$. It follows that $\det(e_m(X)) \neq 0$. Since $X \in \mathfrak{g}$, $\sigma(X) = -X$, which implies that $\sigma(e_m(X)) = e_m(-X)$. This fact, together with $X^m \in \mathcal{B}_i$, implies

$$e_m(X) - 1 + \sigma(e_m(X) - 1) + (e_m(X) - 1)\sigma(e_m(X) - 1) \in \mathcal{B}_i.$$

Thus, by Lemma 3.5, $p_{(m,i)}(X)$ is well defined. That is, $e_m(X) + \mathcal{B}_i$ does intersect P. To see that $p_{(m,i)}(X)$ is just one coset of P_i in P, suppose that $e_m(X) + Y$ and $e_m(X) + Y'$ belong to P for some $Y, Y' \in \mathcal{B}_i$. Then

$$\begin{aligned}(e_m(X) + Y)(e_m(X) + Y')^{-1} &= (e_m(X) + Y)(e_m(-X) + \sigma(Y')) \\ &\in (e_m(X)e_m(-X) + \mathcal{B}_i) \cap G \subset (1 + \mathcal{B}_i) \cap G = P_i.\end{aligned}$$

LEMMA 3.7. – *Suppose* $2 \leq m \leq p$ *and* $X \in \mathcal{A} \cap \mathfrak{g}$ *is such that* $X^m \in \mathcal{B}_i$. *Let* $x \in p_{(m,i)}(X)$. *Then*

(1) $x^{-1} \in p_{(m,i)}(-X)$
(2) $X \in \ell_m(x) + \mathcal{B}_i$

Proof. – (1) $x^{-1} = \sigma(x) \in (\sigma(e_m(X)) + \mathcal{B}_i)P_i = (e_m(-X) + \mathcal{B}_i)P_i$.

(2) Write $x = (e_m(X) + Y)(1 + Y')$, with $Y \in \mathcal{B}_i$ and $Y' \in \mathcal{B}_i^-$. Then $e_m(X) \in \mathcal{A}$ implies $x - 1 \in e_m(X) - 1 + \mathcal{B}_i$ and $e_m(X) - 1 \in \mathcal{A}$. Thus

$$\ell_m(x) \in \ell_m(e_m(X)) + \mathcal{B}_i = \sum_{h=1}^{m-1} \frac{(-1)^{h+1}}{h} \left(\sum_{j=1}^{m-1} \frac{X^j}{j!} \right)^h + \mathcal{B}_i.$$

If $1 \leq n \leq m - 1$, the coefficient of X^n coincides with the coefficient of t^n in the power series expansion for $\log(e^t) = \log(1 + (e^t - 1)) = t$. That is, the coefficient of X equals one, and the coefficient of X^n, $2 \leq n \leq m - 1$, equals zero. If $n \geq m$, the coefficient of X^n belongs to O_0 and $X^n = X^{n-m}X^m \in \mathcal{B}_i$. Therefore

$$\sum_{h=1}^{m-1} \frac{(-1)^{h+1}}{h} \left(\sum_{j=1}^{m-1} \frac{X^j}{j!} \right)^h \in X + \mathcal{B}_i. \qquad \square$$

4. Linear characters and cuspidal data: the unramified case

Throughout this section, T denotes a fixed elliptic Cartan subgroup of G such that every E_i in the direct sum $A = \bigoplus_{1 \leq i \leq r} E_i$ is unramified over F. We prove some elementary results about linear characters and we state Morris' definition of a cuspidal datum associated to T, with a minor modification. At the end of the section, a semisimple element in the Lie algebra of T is associated to a cuspidal datum and this element is shown to be regular.

Let $\mathcal{A} = \mathcal{A}_u$ and $P = P_u$, and define $\mathcal{B}$ and $\mathcal{B}_i$ as in §3. For $i \geq 1$, set $T_i = T \cap (1 + \mathcal{B}_i)$, and for $i \in \mathbf{Z}$, $\mathcal{T}_i = \mathcal{T} \cap \mathcal{B}_i$, where $\mathcal{T}$ denotes the Lie algebra of T, that is, $\mathcal{T} = A^-$. If

$$(\mathcal{B}_i^-)^\sharp = \{ X \in \mathrm{End}_F(V)^- \mid \mathrm{tr}_0(X\mathcal{B}_i^-) \subset O_0 \},$$

then ([Mor3], 4.13) $(\mathcal{B}_i^-)^\sharp = (\mathcal{B}_i^\sharp)^-$. Define a linear function λ from $\mathbf{Z}$ to $\mathbf{Z}$ by

$$\mathcal{B}_i^\sharp = \mathcal{B}_{\lambda(i)}$$
$$(\mathcal{B}_i^-)^\sharp = \mathcal{B}_{\lambda(i)}^-.$$

Then $\lambda(i + 1) = \lambda(i) - 1$ ([Mor3], 4.18). As in [Mor3], the notation $\mathcal{A}^-(\lambda(i))$ may be used in place of $\mathcal{B}_{\lambda(i)}^-$.

For $1 \leq i \leq r$, let B_i be the group of roots of unity in $E_i^\times$ of order prime to p. Set $\varpi_A = (\varpi_{E_1}, \ldots, \varpi_{E_r})$, where ϖ_{E_i} is a uniformizer in E_i. Let C_A be the subgroup of $A^\times$ generated by ϖ_A and $\prod_{i=1}^{r} B_i$.

If $F \neq F_0$, by a linear character of G we mean a character of G of the form $\chi \circ \det$, where χ is a character of F^1. The following two lemmas concern the relation between values of linear characters on certain subsets of G and elements of the centre of $\mathfrak{g}$.

Let Ω be a character of F_0 having conductor equal to O_0.

LEMMA 4.1. – *Suppose that $F \neq F_0$ and $1 \leq m \leq p$. Let $f > 1$. If ψ is a linear character of G such that $\psi \mid P_{f-1}$ is nontrivial and $\psi \mid P_f$ is trivial, then there exists c' in the centre of $\mathfrak{g}$ such that $c' \in \mathcal{A}^-(\lambda(f)) - \mathcal{A}^-(\lambda(f-1))$ and*

$$\psi(p_{(m,f)}(X)) = \Omega(\mathrm{tr}_0(c'X)),$$

for all $X \in \mathcal{A} \cap \mathfrak{g}$ such that $X^m \in \mathcal{B}_f$.

Remark. – The coset $p_{(m,f)}(X)$ defined by (3.6) (with $i = f$) is not an element of P but a coset of P_f. However, since ψ is trivial on P_f, $\psi \mid P$ can be viewed as a character of P/P_f.

Proof. – Let e be the period of $\mathcal{L}_u$. By definition of $\mathcal{B}_i$,

$$\mathcal{B}_i = \mathcal{B}^{a(i)}, \quad \text{where } \ a(i) = [(i-1)/(3-e)] + 1.$$

Here, $[\ \cdot\ ]$ is the greatest integer function.

The centre Z of G is

$$Z = \{\, \beta \cdot 1 \mid \beta \in F^1 \,\}$$

where 1 denotes the $t \times t$ identity matrix ($t = \dim V$). For $i \geq 1$, set $Z_i = Z \cap P_i$. Via the above identification of Z with F^1, Z_i can be identified with $F^1_{[(i+1)/2]}$ (*see* §2). Thus $\det(Z_i)$ is the group of t^{th} powers in $F^1_{[(i+1)/2]}$. Since p and t are relatively prime, the t^{th} powers in $F^1_{[(i+1)/2]}$ coincide with $F^1_{[(i+1)/2]}$. As $\mathcal{B}$ is the Jacobson radical of $\mathcal{A}$, $\mathrm{tr}\,(\mathcal{B}) = \mathfrak{p}$ ([BF]). By periodicity of $\{\mathcal{B}^i\}_{i \in \mathbf{Z}}$, we see that $\mathrm{tr}\,(\mathcal{B}^i) = \mathfrak{p}^{[(i-1)/e]+1}$. Hence

$$\det(1 + \mathcal{B}^i) \subset 1 + \mathfrak{p}^{[(i-1)/e]+1} \quad \text{and} \quad \det(1 + \mathcal{B}_i) \subset 1 + \mathfrak{p}^{[(i+1)/2]}.$$

This implies that $\det(P_i) \subset F^1_{[(i+1)/2]}$. Since $\det(Z_i) = F^1_{[(i+1)/2]}$ and $Z_i \subset P_i$, $\det(P_i) = F^1_{[(i+1)/2]}$, $i \geq 1$. We are assuming that ψ is nontrivial on P_{f-1}, so $\det(P_{f-1}) \neq \det(P_f)$. Therefore $[f/2] < [(f+1)/2]$. That is, f is odd. Set $\ell = (f+1)/2$.

Let χ be a character of F^1 such that

$$\psi(x) = \chi(\det x), \qquad x \in G. \tag{4.2}$$

Because ψ is trivial on P_f, resp. nontrivial on P_{f-1}, and $\det(Z_f) = \det(P_f) = F^1_\ell$, resp. $\det(Z_{f-1}) = \det(P_{f-1}) = F^1_{\ell-1}$, the conductor of χ is F^1_ℓ.

Assume $X \in \mathcal{A} \cap \mathfrak{g}$ is such that $X^m \in \mathcal{B}_f$. Let L be a finite extension of F containing the eigenvalues $\alpha_1, \ldots, \alpha_n$ of X. Extend $|\cdot|$ from F to a norm $|\cdot|_L$ on L. $\mathcal{A}$ is conjugate (by a matrix in $\mathbf{GL}_t(F)$) to a subset of the $t \times t$ matrices with entries in O ([BF]). It follows from

$$X^m \in \mathcal{B}^{a(f)} \ \Rightarrow\ X^{em} \in \mathcal{B}^{e\,a(f)} \Rightarrow \varpi^{-a(f)} X^{em} \in \mathcal{A}$$

that

$$q^{a(f)}|\alpha_i|_L^{em} \leq 1, \qquad 1 \leq i \leq n.$$

Thus

$$|\alpha_i|_L^m \leq q^{[(-a(f)-1)/e]+1} = q^{-[(f+1)/2]} = q^{-\ell}, \qquad 1 \leq i \leq n.$$

This implies

$$(\operatorname{tr} X)^m \in \mathfrak{p}^\ell.$$

Set $j = [(\ell + m - 1)/m]$. Then

$$(\operatorname{tr} X)^m \in \mathfrak{p}^\ell \Rightarrow \operatorname{tr} X \in \mathfrak{p}^j.$$

Let $(\mathfrak{p}^k)^-$ be the set of elements $x \in \mathfrak{p}^k$ such that $\operatorname{tr}_{F/F_0} x = 0$. Define a map

$$\begin{aligned} \phi : (\mathfrak{p}^j)^-/(\mathfrak{p}^\ell)^- &\to F_j^1/F_\ell^1 \\ x + (\mathfrak{p}^\ell)^- &\mapsto \left\{ \left(\sum_{s=0}^{m-1} x^s/s! + \mathfrak{p}^\ell \right) \cap F_j^1 \right\} F_\ell^1. \end{aligned}$$

Applying remarks preceding Lemma 3.5 to the element $\sum_{s=1}^{m-1} x^s/s!$ of O (with $\mathcal{B}_i = \mathfrak{p}^i$, $\sigma = \sigma_0$, and $G = F^1$), we see that $\phi(x + (\mathfrak{p}^\ell)^-)$ is a coset of F_ℓ^1 in F_j^1. Note that

$$\begin{aligned} &e_m(x_1)e_m(x_2) \in e_m(x_1 + x_2) + \mathfrak{p}^\ell, \quad x_1, x_2 \in \mathfrak{p}^\ell. \\ &y \in \phi(x + (\mathfrak{p}^\ell)^-) \Longleftrightarrow y^{-1} \in \phi(-x + (\mathfrak{p}^\ell)^-) \quad \text{by Lemma 3.5(1)} \\ &\phi(x + (\mathfrak{p}^\ell)^-) = F_\ell^1 \Longrightarrow x \in \ell_m(F_\ell^1) \subset \mathfrak{p}^\ell \quad \text{by Lemma 3.5(2).} \end{aligned}$$

Thus ϕ factors to an isomorphism between $(\mathfrak{p}^j)^-/(\mathfrak{p}^\ell)^-$ and F_j^1/F_ℓ^1. This isomorphism will be used to identify $\chi \,|\, F_j^1$ with a character of $(\mathfrak{p}^j)^-/(\mathfrak{p}^\ell)^-$.

We also have an isomorphism between the group of characters of $(\mathfrak{p}^j)^-/(\mathfrak{p}^\ell)^-$ and $(\mathfrak{p}^{-\ell})^-/(\mathfrak{p}^{-j})^-$. Given $\tau \in (\mathfrak{p}^{-\ell})^-$ the coset $\tau + (\mathfrak{p}^{-j})^-$ maps to the character

$$x \mapsto \Omega(\operatorname{tr}_{F/F_0}(\varpi^{1-e_0}\tau x)),$$

where e_0 is the ramification degree of F over F_0.

Let c' be such that $\varpi^{-1+e_0}c'$ belongs to the coset $\tau + (\mathfrak{p}^{-j})^-$ which is associated to χ via the above two isomorphisms. From $X \in \mathfrak{g}$ and $\operatorname{tr} X \in \mathfrak{p}^j$, it follows that $\operatorname{tr} X \in (\mathfrak{p}^j)^-$. Therefore,

$$\chi(\phi(\operatorname{tr} X)) = \Omega(\operatorname{tr}_{F/F_0}(c' \operatorname{tr} X)) = \Omega(\operatorname{tr}_0(c'X)). \tag{4.3}$$

Let $W \in \mathcal{B}_f$ such that $e_m(X) + W \in G$. That is, $(e_m(X) + W)P_f = p_{(m,f)}(X)$. Then

$$\det(e_m(X) + W) = \det e_m(X) \det(1 + e_m(X)^{-1}W) \in \det e_m(X)(1 + \mathfrak{p}^\ell).$$

By definition of j, $|\alpha_i|_F^m \leq q^{-\ell}$ implies $|\alpha_i|_F \leq q^{-j}$, $1 \leq i \leq n$. This can be used to show (*see* Lemma 3.16 of [Mu1])

$$\det e_m(X) \in \left(\sum_{s=0}^{m-1} \frac{(\operatorname{tr} X)^s}{s!}\right)(1+\mathfrak{p}^\ell).$$

Thus

$$\det(e_m(X)+W) \in \left(\sum_{s=0}^{m-1} \frac{(\operatorname{tr} X)^s}{s!}\right)(1+\mathfrak{p}^\ell).$$

Since

$$e_m(X)+W \in G \;\Rightarrow\; \det(e_m(X)+W) \in F^1,$$

we actually have

$$\det(e_m(X)+W) \in \phi(\operatorname{tr} X).$$

Using (4.2) and (4.3),

$$\psi(p_{(m,f)}(X)) = \chi(\det(e_m(X)+W)) = \chi(\phi(\operatorname{tr} X)) = \Omega(\operatorname{tr}_0(c'X)).$$

If $X \in \mathcal{B}_f^-$, then $p_{(m,f)}(X) \in P_f$. Therefore $\psi(p_{(m,f)}(X)) = 1$ for all $X \in \mathcal{B}_f^-$, which implies that $c' \in \mathcal{A}^-(\lambda(f))$. Similarly, $c' \neq \mathcal{A}^-(\lambda(f-1))$ (because $\psi(p_{(m,f)}(X)) \neq 1$ for at least one $X \in \mathcal{B}_{f-1}^-$). □

LEMMA 4.4. – *Suppose that* $F \neq F_0$ *and* $1 \leq m \leq p$. *Let* $f > 1$ *and* $c \in C_A^- \cap (\mathcal{A}^-(\lambda(f)) - \mathcal{A}^-(\lambda(f-1)))$. *Assume that* c *is in the centre of* $\mathfrak{g}$. *Let* ψ *be a linear character of* G *such that*

$$\psi(\gamma) = \Omega(\operatorname{tr}_0(c(\gamma-1)), \qquad \gamma \in T_{f-1}.$$

Then there exists c' *in the centre of* $\mathfrak{g}$ *such that* $c - c' \in \mathcal{A}^-(\lambda(f-1))$ *and*

$$\psi(p_{(m,f)}(X)) = \Omega(\operatorname{tr}_0(c'X)),$$

for all $X \in \mathcal{A} \cap \mathfrak{g}$ *such that* $X^m \in \mathcal{B}_f$.

Proof. – Because it is assumed that $\mathcal{A}^-(\lambda(f)) \neq \mathcal{A}^-(\lambda(f-1)))$, it must be the case that $\lambda(f)$ is even ([Mor3], 4.13) and f is odd ([Mor3], 4.18).

Since $c \neq \mathcal{A}^-(\lambda(f-1))$, there exists $W \in \mathcal{B}_{f-1}^-$ such that $\Omega(\operatorname{tr}_0(cW)) \neq 1$. Because $\operatorname{tr} W \in (\mathfrak{p}^{[(f-1)/2]})^- = (\mathfrak{p}^{\ell-1})^-$ (*see* the proof of Lemma 4.1) and 2, $t = \dim_F(V) \in O^\times$, we can choose $\beta \in (\mathfrak{p}^{\ell-1})^-$ such that $\beta = -(\operatorname{tr} W)/(2t)$. Set $\gamma = C(\beta \cdot 1)$, where 1 denotes the $t \times t$ identity matrix. Then $\gamma \in T_{f-1}$ and

$$\begin{aligned}\psi(\gamma) &= \Omega(\operatorname{tr}_0(c(\gamma-1))) = \Omega(\operatorname{tr}_0(c(C(\beta\cdot 1)-1)) = \Omega(\operatorname{tr}_0(c(-2\beta\cdot 1)))\\ &= \Omega(\operatorname{tr}_{F/F_0}(-2tc\beta)) = \Omega(\operatorname{tr}_{F/F_0}(c\operatorname{tr} W)) = \Omega(\operatorname{tr}_0(cW)) \neq 1.\end{aligned}$$

Thus ψ is nontrivial on T_{f-1}, hence nontrivial on P_{f-1}.

Let χ be a character of F^1 such that $\psi = \chi \circ \det$. The equality $\det(P_f) = \det(Z_f) = F_\ell^1$ (*see* the proof of Lemma 4.1) implies $\det(T_f) = F_\ell^1$. Since $\psi \,|\, T_f$ is trivial, we conclude that $\psi \,|\, P_f$ is trivial. Let c' be as in Lemma 4.1.

Let $\gamma \in T_{f-1}$. Then

$$\gamma + 1 \in 2 + \mathcal{B}_{f-1} = 2(1 + \mathcal{B}_{f-1}) \quad (2 \in O^\times)$$
$$\Longrightarrow \gamma + 1 \text{ is invertible and } (\gamma+1)^{-1} \in (1/2)(1 + \mathcal{B}_{f-1}) \subset \mathcal{A}.$$

Set $Y = 2(\gamma+1)^{-1}(\gamma - 1)$. Then $\gamma - 1 \in Y + \mathcal{B}_f$, $Y \in \mathcal{T}_{f-1}$, and $\gamma \in e_m(Y)T_f$ $(Y^2 \in \mathcal{B}_f)$. Thus

$$\psi(\gamma) = \Omega(\mathrm{tr}_0(c'Y)) = \Omega(\mathrm{tr}_0(c'(\gamma - 1)))$$

and ψ and $t \mapsto \Omega(\mathrm{tr}_0(c'(\gamma - 1)))$ coincide on T_{f-1}. Let W be any element of $\mathcal{B}_{f-1}^-$. Choose β as above and again set $\gamma = C(\beta \cdot 1)$. Then

$$\Omega(\mathrm{tr}_{F/F_0}((c-c')\mathrm{tr}\, W)) = \Omega(\mathrm{tr}_0((c-c')(-2\beta)\cdot 1)) = \Omega(\mathrm{tr}_0(c-c')(\gamma-1)) = 1, \quad W \in \mathcal{B}_{f-1}^-,$$

which implies that $c - c' \in \mathcal{A}^-(\lambda(f-1))$. □

Given $c \in A = \bigoplus_{i=1}^{r} E_i$, let F_i be the subfield of E_i generated by the ith component of c, $1 \le i \le r$. Set $A_c = \bigoplus_{i=1}^{r} F_i$. If $X \in \mathfrak{g}$, $Z_G(X)$ denotes the stabilizer of X in G.

A *cuspidal datum of rank n* (associated with T, $\mathcal{A}$, P) consists of a set of objects as follows:

(a) A sequence $f_1 > f_2 > \cdots > f_n$ of positive integers

(b) If $f_n > 1$ a sequence $c_1, c_2, \ldots, c_n \in C_A^-$ such that $F \subset A_{c_1} \subset \cdots \subset A_{c_n} \subset A$ and linear characters ψ_j of $G(j) = Z_{G(j-1)}(c_j)$, $G(0) = G$, $1 \le j \le n$, such that

$$\psi_j(\gamma) = \Omega(\mathrm{tr}_0(c_j(\gamma - 1))), \qquad \gamma \in T_{f_j - 1},$$

and

$$c_j \in \mathcal{A}^-(\lambda(f_j)) - \mathcal{A}^-(\lambda(f_{j-1})).$$

In addition, $Z_{G(n-1)}(c_n) = T$.

(c) In the case $f_n = 1$, a sequence $c_1, \ldots, c_{n-1} \in C_A^-$ as in (b). The Cartan subgroup $T \subset P(n-1) = P \cap G(n-1)$ can be taken to fix a unique vertex in the affine building associated to $G(n-1)$ ([Mor3], §5). If $\overline{P(n-1)^0}$ is as in §3.17 of [Mor1], and $\overline{T} = T/(T \cap P_1)$, then $\overline{T} \cap \overline{P(n-1)^0}$ is a minisotropic torus in $\overline{P(n-1)^0}$. Let ψ_n be a character of $\overline{T} \cap \overline{P(n-1)^0}$ in general position such that the irreducible cuspidal representation σ_n of $\overline{P(n-1)^0}$ associated to ψ_n is fixed by no element of $P(n-1)/P(n-1)^0$. (Here $P(n-1)^0$ denotes the inverse image of $\overline{P(n-1)^0}$ in $P(n-1)$.)

(d) For each linear character ψ_j of $G(j)$ as above, an element c_j' in the centre of the Lie algebra $\mathfrak{g}(j)$ of $G(j)$ such that

$$\psi_j(p_{(d,f_j)}(X)) = \Omega(\mathrm{tr}_0(c_j' X)), \tag{4.5}$$

for $X \in \mathcal{A} \cap \mathfrak{g}(j)$ such that $X^d \in \mathcal{B}_{f_j}$. Here, $d = 2[\dim_F(V)/2] + 1$ and $p_{(d,f_j)}(X)$ is defined by (3.6).

Remarks. – (1) As discussed in [Mor3] Remark 3.5, the groups $G(j)$ are products of (genuine) unitary groups over field extensions of F. A linear character of $G(j)$ is simply a product of linear characters of these unitary groups. The group $G(j)$ acts on the vector space V and $\mathcal{L}_u$ is an O_{c_j}-chain, where O_{c_j} is the unique maximal order in A_{c_j}, and $G(j) \cap P$ inherits the arithmetic structure arising from T, so properties of $(G(j), T, \cdots)$ are the same as those of $(G, T, \cdots)$. Thus, given c_j as in (b), Lemma 4.4 (applied to each of the unitary groups occurring in $G(j)$) establishes the existence of c'_j satisfying (4.5).

(2) Recall that f_j is always odd (see the proof of Lemma 4.4). Suppose $f_j > 1$. Set $i_j = (f_j - 1)/2$ and $i'_j = (f_j + 1)/2$. Morris' definition of a cuspidal datum of rank n ([Mor3], 3.5) differs from the above in that the condition (4.5) on c'_j in (d) is replaced by the weaker condition

$$\psi_j(x) = \Omega(\mathrm{tr}_0(c'_j(x-1))), \qquad x \in P_{i'_j} \cap G(j).$$

However, since existence of c'_j as in (d) is guaranteed by Lemma 4.4 (*see* Remark(1)), the above definition of cuspidal datum is actually equivalent to Morris' definition.

Suppose for the moment that $f_n = 1$. The representation σ_n (as in (c) above) is the irreducible representation of $\overline{P(n-1)^0}$ associated by Deligne and Lusztig ([DL]) to $\psi_n \,|\, \overline{T} \cap \overline{P(n-1)^0}$. Fix $c_n = c'_n \in \left(\varpi^{-e_0} \prod_{i=1}^{r} B_i\right) \cap T$ such that $Z_{G(n-1)}(c'_n) = T$. The Lie algebra of the finite reductive group $\overline{P(n-1)}$ is

$$\overline{\mathcal{P}(n-1)} = (\mathcal{A} \cap \mathfrak{g}(n-1))/(\mathcal{B} \cap \mathfrak{g}(n-1))$$

and the image $\overline{c}_n$ of $\varpi^{e_0} c_n$ in $\overline{\mathcal{P}(n-1)}$ is regular and belongs to the Lie algebra of $\overline{T}$. By our assumption on p, the exponential and logarithm maps are defined on the nilpotent subset and unipotent subset of $\overline{\mathcal{P}(n-1)}$ and $\overline{P(n-1)^0}$, respectively, and the Campbell-Hausdorff formula holds. Therefore, results of Kazhdan ([K1]) regarding the character of σ_n may be applied.

Let $\overline{\Omega}$ be a nontrivial character of the residue class field of F_0. There is no loss of generality in assuming that $\overline{\Omega}(\overline{\varpi^{e_0}\tau}) = \Omega(\tau)$, for $\tau \in \varpi^{-e_0} O_0$. (Here, if $w \in O_0$, $\overline{w}$ denotes the image of w in the residue class field of F_0). Given a nilpotent element $\overline{X}$ of $\overline{\mathcal{P}(n-1)^0}$, set $\overline{x} = \exp(\overline{X})$. If χ_{σ_n} is the character of σ_n, then ([K1])

$$\frac{\chi_{\sigma_n}(\overline{x})}{\chi_{\sigma_n}(1)} = \#(\overline{P(n-1)^0}) \sum_{\overline{y} \in \overline{P(n-1)^0}} \overline{\Omega}(\overline{\mathrm{tr}}_0(\overline{c}_n \mathrm{Ad}\, \overline{y}^{-1}(\overline{X}))).$$

Here we have used $\overline{\mathrm{tr}}_0$ to denote the trace over the finite field, and $\#$ for cardinality. Conditions on σ_n ((c) above) guarantee that σ_n induces to an irreducible representation $\tilde{\sigma}_n$ of $\overline{P(n-1)}$. From $\tilde{\sigma}_n$, we obtain a representation ρ_n of $P(n-1)$ which is trivial on $P_1(n-1)$. Let $X \in \mathcal{A} \cap \mathfrak{g}(n-1)$ be such that $X^d \in \mathcal{B}_1$. Then the image $\overline{X}$ of X in $\overline{\mathcal{P}(n-1)}$ is nilpotent, and hence $\exp \overline{X}$ belongs to $\overline{P(n-1)^0}$. Also, the image of $p_{(d,1)}(X)$ in $\overline{P(n-1)}$ equals $\exp \overline{X}$. The Frobenius formula for the character of an induced representation and the above expression for χ_{σ_n} combine to yield

$$\frac{\chi_n(p_{(d,1)}(X))}{\chi_n(1)} = \int_{P(n-1)} \Omega(\mathrm{tr}_0(c'_n \mathrm{Ad}\, h^{-1}(X)))\, dh, \tag{4.6}$$

where χ_n denotes the character of ρ_n.

In general, there exist irreducible cuspidal representations of $\overline{P(n-1)^0}$ which do not correspond to characters in general position, and these representations do not give rise to a relation of the form (4.6). As remarked in [Mor3], cuspidal data involving such representations may be associated to supercuspidal representations of G. For more discussion of these cases, *see* §11.

Let $\Psi = (T, (\psi_1, c_1, c'_1, f_1), (\psi_2, c_2, c'_2, f_2), \ldots)$ be a cuspidal datum of rank n (and drop the assumption that $f_n = 1$). Set

$$c_\Psi = \sum_{j=1}^{n} c'_j.$$

By definition, $c_\Psi \in \mathcal{T}$. Let $\mathfrak{g}_{reg}$ denote the set of regular elements in $\mathfrak{g}$. The following lemma is a special case of a result of Morris ([Mor3], Lemma 4.21).

LEMMA 4.7. – *Let* $c \in C_A^- \cap (\mathcal{A}^-(\lambda(i)) - \mathcal{A}^-(\lambda(i-1)))$ *for some integer* i. *If* $X \in (c + \mathcal{A}^-(\lambda(i-1))) \cap \mathcal{C}[c]$, *then* $Z_G(X) \subset Z_G(c)$.

LEMMA 4.8. – *For* $1 \leq j \leq n$, $T = Z_{G(j-1)}(c'_j + \cdots + c'_n)$. *In particular,* $c_\Psi \in \mathfrak{g}_{reg}$.

Proof. – By definition, $c'_j + \cdots + c'_n \in \mathcal{T}$. Thus $T \subset Z_{G(j-1)}(c'_j + \cdots + c'_n)$ and it suffices to show that $Z_{G(j-1)}(c'_j + \cdots + c'_n) \subset T$.

If $f_n = 1$, by definition $Z_{G(n-1)}(c'_n) = T$. If $f_n > 1$, by definition ((b) above) $Z_{G(n-1)}(c_n) = T$. By (b) and (d),

$$c'_n \in c_n + (\mathcal{A}^-(\lambda(f_n - 1)) \cap A),$$

Applying Lemma 4.7 to $G(n-1)$,

$$\mathrm{Ad}\, x(c'_n) = c'_n \Longrightarrow x \in Z_{G(n-1}(c_n) = T.$$

Thus $Z_{G(n-1)}(c'_n) = T$.

Let $1 \leq j \leq n-1$. Arguing as above, except with T replaced by $G(j)$ and A replaced by $\mathcal{C}[c_j]$, we see that $G(j) = Z_{G(j-1)}(c'_j)$. Because $f_j > f_{j+1} > \ldots > f_n$ and $c'_{j+1}, \ldots, c'_n \subset \mathcal{T} \subset \mathcal{C}[c_j]$,

$$\begin{aligned} c'_j + \ldots + c'_n &\in c'_j + (\mathcal{A}^-(\lambda(f_j - 1)) \cap \mathcal{C}[c_j]) \\ &\subset c_j + (\mathcal{A}^-(\lambda(f_j - 1)) \cap \mathcal{C}[c_j]). \end{aligned}$$

By Lemma 4.7, applied to $G(j-1)$,

$$Z_{G(j-1)}(c'_j + \ldots + c'_n) \subset Z_{G(j)}(c'_j + \ldots + c'_n) = Z_{G(j)}(c'_{j+1} + \ldots + c'_n).$$

By induction, $Z_{G(j)}(c'_{j+1} + \cdots + c'_n) = T$. □

5. Linear characters and cuspidal data: the ramified case

Let T be an elliptic Cartan subgroup of G having the property that the ramification degree $e(E_i/F)$ is greater than one for $\ell < i \leq r$ and E_i is unramified over F for $1 \leq i \leq \ell$, where $0 \leq \ell \leq r-1$. That is, at least one of the extensions E_i is ramified over F. Let the lattice chains $\mathcal{L}_u$ and $\mathcal{L}_{r-t}$, orders $\mathcal{A}_u$ and $\mathcal{A}^{(t)}$, and parahoric subgroups P_u and $P^{(t)}$, $0 \leq t \leq r-\ell-1$, be as in §3.

For $0 \leq t \leq r-\ell-1$, let σ_{r-t} denote the (non-trivial) involution on $E_{r-t}^{\times}$ which corresponds to the restriction f_{r-t} of f to $E_{r-t} \times E_{r-t}$. Set

$$T^{(t)} = T \cap U(f_t, E_t) = \{\, x \in E_{r-t}^{\times} \mid x\,\sigma_{r-t}(x) = 1 \,\}.$$

Similarly, if $\ell > 0$, let f_u be the restriction of f to $A_u \times A_u$, and set

$$T_u = T \cap U(f_u, A_u) = \{\, x \in A_u^{\times} \mid x\,\sigma_u(x) = 1 \,\},$$

where σ_u corresponds to f_u.

By Lemma 3.3(8), if $\ell \neq 0$ or $t \neq r-1$, $\mathcal{A}^{(t)} \cap \mathrm{End}_F(E_{r-t})$ is the hereditary order associated to the canonical lattice chain $\mathcal{M}_{r-t}$ (*see* §3) in E_{r-t}:

$$\mathcal{B}^{i}_{\mathcal{M}_{r-t}} = \mathcal{B}^{(t)}_{2i-1} \cap \mathrm{End}_F(E_{r-t}) = \mathcal{B}^{(t)}_{2i} \cap \mathrm{End}_F(E_{r-t}), \qquad i \in \mathbf{Z}.$$

Here, $\mathcal{B}_{\mathcal{M}_{r-t}}$ is the Jacobson radical of $\mathcal{A}^{(t)} \cap \mathrm{End}_F(E_{r-t})$. If $\ell = 0$ and $t = r-1$, then, in order to make the notation consistent with the general case, set $\mathcal{B}^{(t)}_i = \mathcal{B}^{[(i+1)/2]}_{\mathcal{M}_1}$. As in [Mor3], the notation $\mathcal{A}^{(t)}(i)$ may be used in place of $\mathcal{B}^{(t)}_i$. Let $C^{(t)}$ be the group generated by a uniformizer $\varpi_{E_{r-t}}$ in E_{r-t} and the roots of unity in E_{r-t} of order prime to p.

Analogues of Lemmas 4.1 and 4.4 are proved below in the case $\ell = 0$ and $r = 1$. Let the map λ be defined as in the unramified case:

$$(\mathcal{B}^{(0)}_i)^{\sharp} = \mathcal{B}^{(0)}_{\lambda(i)}$$
$$(\mathcal{B}^{(0)-}_i)^{\sharp} = \mathcal{B}^{(0)-}_{\lambda(i)}.$$

Suppose $1 \leq m \leq p$. If $X \in \mathcal{A}^{(0)} \cap \mathfrak{g}$ and X^m is in $\mathcal{B}^{(0)}_f$ for some $f \geq 1$, define $p_{(m,f)}(X)$ by (3.6).

LEMMA 5.1. – *Suppose $r = 1$, $\ell = 0$ and $F \neq F_0$. Set $\mathcal{A} = \mathcal{A}^{(0)}$, $\mathcal{B}_i = \mathcal{B}^{(0)}_i$, $i \in \mathbf{Z}$, and $P_i = P^{(0)}_i$, $i \geq 1$. Let $1 \leq m \leq p$ and $f > 1$. If ψ is a linear character of G such that $\psi \mid P_{f-1}$ is nontrivial and $\psi \mid P_f$ is trivial, then there exists c' in the centre of $\mathfrak{g}$ such that $c' \in \mathcal{A}^-(\lambda(f)) - \mathcal{A}^-(\lambda(f-1))$ and*

$$\psi(p_{(m,f)}(X)) = \Omega(\mathrm{tr}_0(c'X)),$$

for all $X \in \mathcal{A} \cap \mathfrak{g}$ such that $X^m \in \mathcal{B}_f$.

Proof. – The proof is very similar to that of Lemma 4.1. We indicate a few of the differences. As remarked above, if $\mathcal{B}$ is the Jacobson radical of $\mathcal{A}$, then

$$\mathcal{B}_i = \mathcal{B}^{[(i+1)/2]}, \qquad i \in \mathbf{Z}.$$

Since it is assumed that $P_f \neq P_{f-1}$, f must be odd. Let e be the ramification degree of E_1 over F. From $\mathrm{tr}\,(\mathcal{B}) = \mathfrak{p}$ and $\varpi\mathcal{B}^i = \mathcal{B}^{e+i}$, it follows that

$$\mathrm{tr}\,(\mathcal{B}_i) \subset \mathfrak{p}^{\left[[(i-1)/2]/e\right]+1}.$$

For $i \geq 1$, set $Z_i = Z \cap P_i$, where Z is the centre of G. Then, via the identification of Z with F^1, $Z \cap (1+\mathcal{B}^i)$ is identified with $F^1_{[(i-1)/e]+1}$, so Z_i is identified with $F^1_{[[(i-1)/2]/e]]+1}$. Arguing as in the proof of Lemma 4.1, we see that

$$\det(P_i) = F^1_{\left[[(i-1)/2]/e\right]+1}, \qquad i \geq 1.$$

Since the assumptions on ψ require that $\det(P_{f-1}) \neq \det(P_f)$, e must divide $[(f-1)/2] = (f-1)/2$, and

$$\det(P_{f-1}) = F^1_{(f-1)/(2e)} \text{ and } \det(P_f) = F^1_{(f-1)/(2e)+1}.$$

Set $\ell = (f-1)/(2e) + 1$ and proceed as in the proof of Lemma 4.1 (replacing $a(f)$ by $(f+1)/2$). □

The proof of the next lemma is omitted as it is the same as the proof of Lemma 4.4, except that Lemma 5.1 is used in place of Lemma 4.1.

LEMMA 5.2. – *Suppose $r = 1$, $\ell = 0$ and $F \neq F_0$. Let the notation be as in Lemma 5.1. In addition, set $C = C^{(0)}$ and $T_i = T \cap P_i$, $i \geq 1$. Let $f > 1$ and $c \in C^- \cap (\mathcal{A}^-(\lambda(f)) - \mathcal{A}^-(\lambda(f-1))$. Assume that c is in the centre of $\mathfrak{g}$. Let ψ be a linear character of G such that*

$$\psi(\gamma) = \Omega(\mathrm{tr}_0(c(\gamma - 1))), \qquad \gamma \in T_{f-1}.$$

Then there exists c' in the centre of $\mathfrak{g}$ such that $c - c' \in \mathcal{A}^-(\lambda(f-1))$ and

$$\psi(p_{(m,f)}(X)) = \Omega(\mathrm{tr}_0(c'X)),$$

for all $X \in \mathcal{A} \cap \mathfrak{g}$ such that $X^m \in \mathcal{B}_f$.

Write $T = {}^{(r-1)}T \times T^{(0)}$, where ${}^{(r-1)}T$ is an elliptic Cartan subgroup of $U\left(\bigoplus_{i=1}^{r-1} f_i, \bigoplus_{i=1}^{r-1} E_i\right)$. A T-cuspidal datum is defined inductively, in terms of a $T^{(0)}$-cuspidal datum and a ${}^{(r-1)}T$-cuspidal datum.

A $T^{(0)}$*-cuspidal datum* $\Psi^{(0)}$ (relative to $(P^{(0)}, \{P_i^{(0)}\}_{i\geq 1})$) of rank n_0, is defined ([Mor3], §6) to be a set of objects as follows:

(a) A sequence $f_1^{(0)} > f_2^{(0)} > \cdots > f_{n_0}^{(0)}$ of integers.

(b) If $f_{n_0}^{(0)} > 1$, a sequence $c_1^{(0)}, \ldots, c_{n_0}^{(0)} \in C^{(0)-}$. Define $F_1^{(0)} = F(c_1^{(0)})$ and $F_i^{(0)} = F_{i-1}^{(0)}(c_i^{(0)})$, $1 < i \leq n_0$. Assume $E_r = F_{n_0}^{(0)}$. Define

$$G^{(0)}(0) = U(f_r, E_r)$$
$$G^{(0)}(j) = Z_{G^{(0)}(j-1)}(c_j^{(0)}), \qquad 1 \leq j \leq n_0$$

and assume that $T^{(0)} = G^{(0)}(n_0)$. Suppose there exist linear characters $\psi_j^{(0)}$ of $G^{(0)}(j)$ ([Mor3, p. 256) such that

$$\psi_j^{(0)}(\gamma) = \Omega(\mathrm{tr}_0(c_j^{(0)}(\gamma - 1))), \qquad \gamma \in T_{f_j^{(0)}-1},$$

where f_j is defined by $c_j^{(0)} \in \mathcal{A}^{(0)-}(\lambda(f_j^{(0)})) - \mathcal{A}^{(0)-}(\lambda(f_j^{(0)} - 1))$. Here $\mathcal{A}^{(0)-}(i) = \mathcal{A}^{(0)}(i) \cap \mathrm{End}_F(E_r)^-$.

The notation $\mathfrak{g}^{(0)}(j)$ will be used for the Lie algebra of $G^{(0)}(j)$. For each $\psi_j^{(0)}$ as above, assume there exists $c_j'^{(0)} \in E_r^-$ such that

$$\psi_j^{(0)}(p_{(d,f_j^{(0)})}(X)) = \Omega(\mathrm{tr}_0(c_j'^{(0)} X)), \tag{5.3}$$

for $X \in \mathcal{A}^{(0)} \cap \mathfrak{g}^{(0)}(j)$ such that $X^d \in \mathcal{B}^{(0)}_{f_j^{(0)}}$. Here, $d = 2[\dim_F(V)/2] + 1$ and $p_{(d,f_j^{(0)})}$ is defined by (3.6).

(c) Suppose $f_{n_0}^{(0)} = 1$. Take a sequence $c_1^{(0)}, \ldots, c_{n_0-1}^{(0)}, c_1'^{(0)}, \ldots, c_{n_0-1}'^{(0)}, \psi_1^{(0)}, \ldots, \psi_{n_0-1}^{(0)}$ as in (b). Assume that $P^{(0)}(n_0 - 1) = P^{(0)} \cap G^{(0)}(n_0 - 1)$ fixes a unique vertex in the affine building associated to $G^{(0)}(n_0 - 1)$. The Levi component $\overline{P}^{(0)}(n_0 - 1)$ is a group of rational points of a reductive group defined over the residue class field of $F(c_{n_0-1}^{(0)})$. Let $\overline{P}^{(0)\,0}(n_0 - 1)$ be its identity component, and let $P^{(0)\,0}(n_0 - 1)$ be the inverse image of this identity component in $P^{(0)}(n_0 - 1)$. Set $\overline{T}^{(0)} = T^{(0)}/(T^{(0)} \cap P^{(0)}(n_0 - 1))$. Take a character $\psi_{n_0}^{(0)}$ of the minisotropic torus $\overline{T}^{(0)} \cap \overline{P}^{(0)\,0}(n_0 - 1)$ in general position and such that the associated irreducible cuspidal representation $\sigma_{n_0}^{(0)}$ of $\overline{P}^{(0)\,0}(n_0 - 1)$ is fixed by no element of $P^{(0)}(n_0 - 1)/P^{(0)\,0}(n_0 - 1)$.

Remarks. – (1) Because each $G^{(0)}(j)$ is a product of (genuine) unitary groups over field extensions of F, Lemma 5.2 can be applied to each of those unitary groups and the resulting elements in their centres summed to obtain the $c_j'^{(0)}$ of (5.3).

(2) As in the unramified case, Morris' definition of $T^{(0)}$-cuspidal datum differs from the above in that a condition weaker than that of (5.3) is imposed upon the $c_j'^{(0)}$'s. However, as remarked in (1), the existence of $c_j'^{(0)}$ satisfying (5.3) is guaranteed by Lemma 5.2.

(3) As in the unramified case, the condition $\mathcal{A}^{(0)-}(\lambda(f_j^{(0)})) \neq \mathcal{A}^{(0)-}(\lambda(f_j^{(0)} - 1))$ implies that $f_j^{(0)}$ is odd.

(4) Morris includes more general representations $\sigma_{n_0}^{(0)}$ than we have specified in (c) (*see* §11 for comments on this).

If $f_{n_0}^{(0)} = 1$, let $c_{n_0}'^{(0)} \in E_r$ be such that $\varpi^{e_0} c_{n_0}'$ is a product of roots of unity of order prime to p, and $Z_{G^{(0)}(n_0-1)}(c_{n_0}'^{(0)}) = T^{(0)}$. Let $\chi_{n_0}^{(0)}$ be the character of the representation $\rho_{n_0}^{(0)}$ of $P^{(0)}(n_0 - 1)$ obtained by inflating the representation of $\overline{P}^{(0)}(n_0 - 1)$ which is induced from $\sigma_{n_0}^{(0)}$. As was seen in §4, if $X \in \mathcal{A}^{(0)} \cap \mathfrak{g}^{(0)}(n_0 - 1)$ is such that $X^d \in \mathcal{B}_1^{(0)}$, then

$$\frac{\chi_{n_0}^{(0)}(p_{(d,1)}(X))}{\chi_{n_0}^{(0)}(1)} = \int_{P^{(0)}(n_0-1)} \Omega(\mathrm{tr}_0(c_{n_0}'^{(0)} \mathrm{Ad}\, h^{-1}(X)))\, dh. \tag{5.4}$$

Note that $f_1^{(0)} > 1$. Indeed, $f_1^{(0)} = 1$ implies that $n_0 = 1$ and E_r is generated (over F) by $c_1^{(0)}$. This is impossible because $\varpi^{e_0} c_1^{(0)}$ is a product of roots of unity of order prime to p and $e(E_r/F) \geq 2$.

A T-*cuspidal datum* $\underline{\Psi}$ (relative to $(P^{(0)}, \{P_n^{(0)}\}_{n\geq 1})$) consists of ([Mor3], §6):

(a) A $T^{(0)}$-cuspidal datum $\Psi^{(0)}$ (relative to $(P^{(0)}$, $\{P_n^{(0)}\}_{n\geq 1})$) of rank n_0.

(b) If $r > \ell + 1$, a ${}^{(r-1)}T$-cuspidal datum relative to $(P^{(1)}$, $\{P_n^{(1)}\}_{n\geq 1})$ such that if $i_1^{(0)} = (f_1^{(0)} - 1)/2$ and $i_1'^{(1)} = (f_1^{(1)} + 1)/2$,

(i) $\ker(\Omega(\mathrm{tr}_0(c_1'^{(1)} \cdot))) \supseteq P_{f_1^{(1)}}^{(1)} \supseteq P_{f_1^{(0)}-1}^{(0)} \cap U\left(\bigoplus_{s=1}^{r-1} f_s, \bigoplus_{s=1}^{r-1} E_s\right)$

(ii) $\mathcal{B}_{i_1'^{(1)}}^{(1)-} \supseteq \mathcal{B}_{i_1^{(0)}}^{(0)-} \cap \mathrm{End}_F(E_1 \oplus \cdots \oplus E_{r-1})$.

(c) If $r = \ell+1$ and $\ell > 0$, a cuspidal datum of rank n_{r-1} for the torus $T_u = T \cap \mathbf{U}(f_u, A_u)$ as in §4 and such that

(i) If $f_1^u > 1$, then the appropriate modifications of (i) and (ii) as in (b) hold (with (1) replaced by u).

(ii) If $f_1^u = 1$, then $P_{u,f_1^u} \supseteq P_{i_1^{(0)}}^{(0)} \cap U(f_u, A_u)$.

(d) We write the T-cuspidal datum as

$$\underline{\Psi} = \{\, \Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(r-\max\{1,\ell\})} \,\},$$

where $\Psi^{(r-\max\{1,\ell\})} = \Psi_u$ if $\ell > 0$, and each $\Psi^{(t)}$ is a $T^{(t)}$-cuspidal datum of rank n_t relative to $(P^{(t)}$, $\{P_n^{(t)}\}_{n\geq 1})$ satisfying (b) and (c) (with (0),(1) replaced by (t),(t+1), respectively, in (b) if $t < r - \ell - 1$, and with (0) replaced by (t) in (c) if $t = r - \ell - 1$ and $\ell > 0$). For a given $t < r - \ell$, or $t = r - \ell, \ell > 0$, and $n_u = n_{r-\ell} = n_t > 1$ the final condition is:

$$P_{i_1^{(t)}}^{(t)} \subseteq P_1^{(s)} \cap \mathbf{U}(f_1 \oplus \cdots \oplus f_{r-t}, E_1 \oplus \cdots \oplus E_{r-t})$$

for all $s < t$.

Remarks. – (1) In (b)(i) above, $\Omega(\mathrm{tr}_0(c_1'^{(1)} \cdot)$ denotes the character of $P_{f_1^{(1)}-1}^{(1)}$ defined by

$$x \mapsto \Omega(\mathrm{tr}_0(c_1'^{(1)}(x-1))).$$

(2) When defining $c_j'^{(t)}$, resp. $c_j'^{u}$, (*see* (5.3) and (4.5)), we take $d = 2[\dim_F(V)/2] + 1$ (rather than $2[\dim_F\left(\bigoplus_{i=1}^{r-t} E_i\right)/2]+1$, resp. $2[\dim_F(A_u)/2]+1$). This choice of d is dictated by two conditions: that $d \geq \dim_F(V)$ and that d be odd. We require $d \geq \dim_F(V)$ in order that $X^d = 0$ for all nilpotent $X \in \mathrm{End}_F(V)$.

In the proof of Proposition 9.1, it is necessary that $X \in \mathrm{End}_F\left(\sum_{s=1}^{r-1} E_s\right)^-$ and $X^d \in \mathcal{B}_{f_1^{(0)}-1}^{(0)}$ imply $X^d \in \mathcal{B}_{f_1^{(1)}}^{(1)}$. If d is odd, then $X^d \in \mathfrak{g}$, that is, $X^d \in \mathcal{B}_{f_1^{(0)}-1}^{(0)-}$

Note that condition (b)(i) above is equivalent to

$$\mathcal{B}^{(0)-}_{f_1^{(0)}-1} \cap \operatorname{End}_F\left(\sum_{s=1}^{r-1} E_s\right) \subset \mathcal{B}^{(1)-}_{f_1^{(1)}}.$$

Next we define a regular elliptic element $c_{\underline{\Psi}}$ associated to $\underline{\Psi}$. Set

$$c_{\Psi^{(0)}} = \sum_{j=1}^{n_0} c_j'^{(0)}.$$

For $1 \leq t \leq r-\ell-1$, $c_{\Psi^{(t)}}$ is defined similarly . If $\ell > 0$, $c_{\Psi^{(r-\ell)}} = c_{\Psi_u}$ is defined as in §4.

Given an element X of $\operatorname{End}_F(E_{r-t})$, $0 \leq t \leq r-\ell-1$, or of $\operatorname{End}_F(A_u)$, whenever convenient that element will be identified with its image $(0,\cdots,X,\cdots,0)$ in $\operatorname{End}_F(V)$, without any change in notation.

LEMMA 5.5 ([Mor3], Lemma 4.21). – *Let $c \in \mathcal{A}^-(\lambda(i)) - \mathcal{A}^-(\lambda(i-1))$ for some integer i. Assume that $c \in C^{(0)}$. If $X \in (c + \mathcal{A}^-(\lambda(i-1))) \cap \mathcal{C}[c]$, then $Z_G(X) \subset Z_G(c)$.*

LEMMA 5.6. – *For $1 \leq j \leq n_0$, $T^{(0)} = Z_{G^{(0)}(j-1)}(c_j'^{(0)} + \cdots + c_{n_0}'^{(0)})$.*

Proof. – The proof is the same as that of Lemma 4.8. In particular, Lemma 5.5 is applied to the group $G^{(0)}(j-1)$, and the proof is by induction. □

To $\underline{\Psi}$, we associate the following element of the Lie algebra $\mathcal{T}$ of T:

$$c_{\underline{\Psi}} = \sum_{t=0}^{r-\max\{1,\ell\}} c_{\Psi^{(t)}}. \tag{5.7}$$

LEMMA 5.8. – $Z_G(c_{\underline{\Psi}}) = T$.

Proof. – Conditions (b) and (c) in the definition of cuspidal datum imply that $c_{\underline{\Psi}} - c_{\Psi^{(0)}} \in \mathcal{A}^-(\lambda(f_1^{(0)}-1))$. Also, $c_{\Psi^{(0)}} \in c_1^{(0)} + \mathcal{A}^-(\lambda(f_1^{(0)}-1))$. Thus

$$c_{\underline{\Psi}} \in c_1^{(0)} + (\mathcal{A}^-(\lambda(f_1^{(0)}-1)) \cap \mathcal{T}).$$

Since $\mathcal{T} \subset \mathcal{C}[c_1^{(0)}]$, Lemma 5.5 implies that $Z_G(c_{\underline{\Psi}}) \subset Z_G(c_1^{(0)})$. Set $H = \mathbf{U}\left(\bigoplus_{s=1}^{r-1} f_s, \bigoplus_{s=1}^{r-1} E_s\right)$. As Morris showed in §4 of [Mor3], $Z_G(c_1^{(0)}) = H \times Z_{G^{(0)}(0)}(c_1^{(0)})$. It follows that

$$Z_G(c_{\underline{\Psi}}) = Z_H\left(\sum_{t=1}^{r-\max\{1,\ell\}} c_{\Psi^{(t)}}\right) \times Z_{G^{(0)}(0)}(c_{\Psi^{(0)}}).$$

By Lemma 5.6, $Z_{G^{(0)}(0)}(c_{\Psi^{(0)}}) = T^{(0)}$. By induction, since $\{\Psi^{(1)},\ldots,\Psi^{(r-\max\{1,\ell\})}\}$ is a cuspidal datum (relative to $(P^{(1)},\{P_n^{(1)}\}_{n\geq 1})$) for the group H, $Z_H\left(\sum_{t=1}^{r-\max\{1,\ell\}} c_{\Psi^{(t)}}\right) = {}^{(r-1)}T$. □

6. Cuspidal data and inducing representations

Let $\underline{\Psi} = \{ \Psi^{(0)}, \cdots, \Psi^{(r-\max\{1,\ell\})} \}$ be a T-cuspidal datum. The assumption $r > \ell$ of §5 is now dropped. $r + 1 - \max\{1, \ell\}$ will be called the *length* of $\underline{\Psi}$.

Morris defines subgroups $P_{\Psi^{(t)}}$ and $\widetilde{P}_{\Psi^{(t)}}$, $0 \leq t \leq r - \max\{1, \ell\}$, as follows ([Mor3], §§5.4, 5.9, 6.6)

$$P_{\Psi^{(t)}} = \begin{cases} (T^{(t)} \cap P_1^{(t)})P^{(t)}_{i^{(t)}_{n_t}}(n_t - 1) \cdots P^{(t)}_{i^{(t)}_2}(1)P^{(t)}_{i^{(t)}_1}, & \text{if } f^{(t)}_{n_t} > 1, \\ P_1^{(t)}(n_t - 1)P^{(t)}_{i_{n_{t-1}}}(n_t - 2) \cdots P^{(t)}_{i^{(t)}_2}(1)P^{(t)}_{i^{(t)}_1}, & \text{if } f^{(t)}_{n_t} = 1 \text{ and } n_t > 1, \\ P_1^{(0)}, & \text{if } t = 0,\ r = \ell, \\ & n_0 = 1 \text{ and } f_1^{(0)} = 1. \end{cases}$$

$$\widetilde{P}_{\Psi^{(t)}} = \begin{cases} T^{(t)}P_{\Psi^{(t)}}, & \text{if } f^{(t)}_{n_t} > 1, \\ P^{(t)}(n_t - 1)P_{\Psi^{(t)}}, & \text{if } f^{(t)}_{n_t} = 1. \end{cases}$$

As a consequence of the definition of $P^{(t)}$ and Lemma 3.3,

$$P^{(t)} \subset P^{(0)} \cap \mathbf{GL}(E_1 \oplus \cdots \oplus E_{r-t}).$$

Since $\widetilde{P}_{\Psi^{(t)}} \subset P^{(t)}$, and $\widetilde{P}_{\Psi^{(t+1)}}$ normalizes $\widetilde{P}_{\Psi^{(t)}}$ ([Mor3], §6.6),

$$P_{\underline{\Psi}} = \widetilde{P}_{\Psi^{(0)}}\widetilde{P}_{\Psi^{(1)}} \cdots \widetilde{P}_{\Psi^{(r-\max\{1,\ell\})}}$$

is a compact open subgroup of $P^{(0)}$.

Morris constructs a finite dimensional representation $\rho_{\underline{\Psi}}$ of $P_{\underline{\Psi}}$ and proves that the induced representation $\pi_{\underline{\Psi}} = Ind^G_{P_{\underline{\Psi}}}\, \rho_{\underline{\Psi}}$ is an irreducible supercuspidal representation of G ([Mor3] §7). $\rho_{\underline{\Psi}}$ is a tensor product

$$\rho_{\underline{\Psi}} = \rho_{\Psi^{(0)}} \otimes \cdots \otimes \rho_{\Psi^{(r-\max\{1,\ell\})}},$$

where $\rho_{\Psi^{(t)}}$, $0 \leq t \leq r - \max\{1, \ell\}$, is a representation of $P_{\underline{\Psi}}$ constructed from the $T^{(t)}$-cuspidal datum $\Psi^{(t)}$. Each of the representations $\rho_{\Psi^{(t)}}$ is a tensor product

$$\rho_{\Psi^{(t)}} = \rho_1^{(t)} \otimes \cdots \otimes \rho^{(t)}_{n_t}.$$

If $f_j^{(t)} > 1$, the construction of $\rho_j^{(t)}$ involves the linear character $\psi_j^{(t)}$ of $G^{(t)}(j)$ appearing in the definition of $\Psi^{(t)}$. If $f^{(t)}_{n_t} = 1$, $\rho^{(t)}_{n_t}$ arises from the cuspidal representation $\sigma^{(t)}_{n_t}$. For complete details, *see* [Mor3]. In this paper, some information regarding the construction of the various representations can be found in Lemmas 6.2 and 8.2. Let $\chi_{\Psi^{(t)}}$ be the character of $\Psi^{(t)}$, $0 \leq t \leq r - \max\{1, \ell\}$, and $\chi_j^{(t)}$ the character of $\rho_j^{(t)}$, $1 \leq j \leq n_t$.

If the length of $\underline{\Psi}$ is greater than one, then

$$\underline{\Psi}' = \{ \Psi^{(1)}, \ldots, \Psi^{(r-\max\{1,\ell\})} \}$$

is a ${}^{(r-1)}T$-cuspidal datum (relative to $(P^{(1)}, \{P_i^{(1)}\}_{i \geq 1})$) of length one less than $\underline{\Psi}$. The notation V' will be used for $E_1 \oplus \cdots \oplus E_{r-1}$, viewed as a vector space over F. Let $P_{\underline{\Psi}'}$, $\rho_{\underline{\Psi}'}$, and $\chi_{\underline{\Psi}'}$ be the group, representation and character attached to $\underline{\Psi}'$. The group $P_{\underline{\Psi}'}$ is viewed as a subgroup of G in the same way that $\mathbf{GL}(V')$ is viewed as a subgroup of $\mathbf{GL}(V)$.

If the rank n_0 of $\Psi^{(0)}$ is greater than one (and the length of $\underline{\Psi}$ is arbitrary), then

$$\begin{aligned}\underline{\Psi}'' =& \{ f_j^{(0)}, c_j^{(0)}, c_j'^{(0)}, \psi_j^{(0)} \}_{2\le j\le n_0} \\ \text{or} \quad & \{ \{ f_j^{(0)}, c_j^{(0)}, c_j'^{(0)}, \psi_j^{(0)} \}_{2\le j\le n_0-1}, \{\psi_{n_0}^{(0)}, \sigma_{n_0}^{(0)}\} \}\end{aligned}$$

is a $T^{(0)}$-cuspidal datum (relative to $(P^{(0)}(1), \{P_i^{(0)}(1)\}_{i\ge 1})$) of rank $n_0 - 1$, and length one. If $r > \ell$, set $V'' = E_r$, and if $r = \ell$, set $V'' = A_u$. Let $P_{\underline{\Psi}''}$, $\rho_{\underline{\Psi}''}$, and $\chi_{\underline{\Psi}''}$ be the various objects attached to the $T^{(0)}$-cuspidal datum $\underline{\Psi}''$. The group $P_{\underline{\Psi}''}$ will viewed as a subgroup of G.

LEMMA 6.1.

(1)

$$P_{\underline{\Psi}} \cap Z_G(c_1^{(0)}) = \begin{cases} P_{\underline{\Psi}'} \times P_{\underline{\Psi}''}, & \text{if } r > \max\{1,\ell\} \text{ and } n_0 > 1, \\ P_{\underline{\Psi}'} \times T^{(0)}, & \text{if } r > \max\{1,\ell\} \text{ and } n_0 = 1, \\ P_{\underline{\Psi}''}, & \text{if } r = \max\{1,\ell\} \text{ and } n_0 > 1. \end{cases}$$

(2) *If* $f_1^{(0)} > 1$, *then* $P_{\underline{\Psi}} = (P_{\underline{\Psi}} \cap Z_G(c_1^{(0)}))P^{(0)}_{i_1^{(0)}}$.

Proof. – The only case not mentioned in (1) is $r = \max\{1,\ell\}$ and $n_0 = 1$. By definition of $c_1^{(0)}$ and $P_{\underline{\Psi}}$, $Z_G(c_1^{(0)}) = T^{(0)} = P_{\underline{\Psi}} \cap T^{(0)}$, so (2) holds in this case.

Recall that, if the length of $\underline{\Psi}$ is greater than one, then ([Mor3], §4)

$$\mathcal{C}[c_1^{(0)}] = \mathrm{End}_F(V') \oplus (\mathcal{C}[c_1^{(0)}] \cap \mathrm{End}_F(V'')).$$

Thus

$$P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}) = \begin{cases} (P^{(0)}_{i_1^{(0)}} \cap \mathbf{GL}(V')) \times P^{(0)}_{i_1^{(0)}}(1), & \text{if } r > \max\{1,\ell\}, \\ P^{(0)}_{i_1^{(0)}}(1), & \text{if } r = \max\{1,\ell\}. \end{cases}$$

The inequality $i_2^{(0)} < i_1^{(0)}$ implies $P^{(0)}_{i_1^{(0)}}(1) \subset P^{(0)}_{i_2^{(0)}}(1)$. If $n_0 = 1$, then $P^{(0)}_{i_2^{(0)}}(1) \subset T^{(0)}$, and if $n_0 > 1$, then $P^{(0)}_{i_2^{(0)}}(1) \subset P_{\underline{\Psi}''}$. If $r > \max\{1,\ell\}$, then $P^{(0)}_{i_1^{(0)}} \cap \mathbf{GL}(V') \subset P^{(1)}_{i_1^{(1)}}$ is a consequence of condition (b)(ii) if $r > \ell + 1$, and (c)(i),(ii) if $r = \ell + 1$. By definition of $P_{\underline{\Psi}'}$, $P^{(1)}_{i_1^{(1)}} \subset P_{\underline{\Psi}'}$. Therefore,

$$P^{(0)}_{i_1^{(0)}} \cap Z_G(c_1^{(0)}) \subset \begin{cases} P_{\underline{\Psi}'} \times P_{\underline{\Psi}''}, & \text{if } r > \max\{1,\ell\} \text{ and } n_0 > 1, \\ P_{\underline{\Psi}'} \times T^{(0)}, & \text{if } r > \max\{\ell,1\} \text{ and } n_0 = 1, \\ P_{\underline{\Psi}''}, & \text{if } r = \max\{1,\ell\} \text{ and } n_0 > 1. \end{cases}$$

By definition,

$$P_{\underline{\Psi}} = \begin{cases} (P_{\underline{\Psi}'} \times P_{\underline{\Psi}''})P^{(0)}_{i_1^{(0)}}, & \text{if } r > \max\{1,\ell\} \text{ and } n_0 > 1, \\ (P_{\underline{\Psi}'} \times T^{(0)})P_{i_1^{(0)}}, & \text{if } r > \max\{1,\ell\} \text{ and } n_0 = 1, \\ P_{\underline{\Psi}''}P^{(0)}_{i_1^{(0)}}, & \text{if } r = \max\{1,\ell\} \text{ and } n_0 > 1. \end{cases}$$

Since $P_{\underline{\Psi}'}$, $P_{\underline{\Psi}''}$ and $T^{(0)}$ are all subsets of $Z_G(c_1^{(0)})$, the lemma is immediate given the above inclusion for $P_{i_1^{(0)}}^{(0)} \cap Z_G(c_1^{(0)})$. □

The next two lemmas concern the values of $\rho_j^{(0)}$, $j \geq 2$, and $\rho_{\Psi^{(t)}}$, $t \geq 1$, on certain subgroups of $P_{\underline{\Psi}}$.

LEMMA 6.2. – *Suppose* $n_0 > 1$. *Then*
(1) $(\rho_2^{(0)} \otimes \cdots \otimes \rho_{n_0}^{(0)}) \,|\, P_{\underline{\Psi}''} = \rho_{\underline{\Psi}''}$
(2) $\rho_j^{(0)}(x) = \Omega(\mathrm{tr}_0(c_j'^{(0)}(x-1)))\rho_j^{(0)}(1), \quad x \in P_{i_1^{(0)}}^{(0)}, \quad 2 \leq j \leq n_0$
(3) *If* $r > 1$ *and* $r > \ell$, *then* $\rho_j^{(0)} \,|\, P_{\underline{\Psi}'} \equiv \rho_j^{(0)}(1), \quad 2 \leq j \leq n_0$.

Proof. – To simplify the notation, the superscript (0) will be dropped from almost all groups and indices. Fix j, $2 \leq j \leq n_0$. Let ρ_j'' be the representation of $P_{\underline{\Psi}''}$ associated to the linear character ψ_j.

Suppose $f_j \neq 1$ and $P_{i_j}(j-1) = P_{i_j'}(j-1)$. According to Morris' definitions ([Mor3, §5.5, §6.8), if $f_{n_0} > 1$,

$$\begin{aligned}
&\rho_j(x) = \rho_j''(x) = \psi_j(x),\\
&\qquad\qquad x \in TP_{i_{n_0}}(n_0-1)\cdots P_{i_{j+1}}(j) = \widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(j) = P_{\underline{\Psi}''} \cap G^{(0)}(j)\\
&\rho_j''(x) = \Omega(\mathrm{tr}_0(c_j'(x-1))), \qquad x \in P_{i_j}(j-1)\cdots P_{i_2}(1)\\
&\rho_j(x) = \Omega(\mathrm{tr}_0(c_j'(x-1))), \qquad x \in P_{i_j}(j-1)\cdots P_{i_2}(1)P_{i_1}.
\end{aligned}$$

Note that $f_{n_0} = 1$ is allowed (as long as $j < n_0$, since we've assumed $f_j \neq 1$). If $j < n_0$ and $f_{n_0} = 1$, in the above, $TP_{i_{n_0}}(n_0-1)$ should be replaced by $P(n_0-1)$.

Suppose $f_j \neq 1$ and $P_{i_j}(j-1) \neq P_{i_j'}(j-1)$. In this case, the same Heisenberg construction is used to produce ρ_j and ρ_j'' on the group $TP_{i_{n_0}}(n_0-1)\ldots P_{i_j}(j-1)$ ($f_{n_0} > 1$), or $P(n_0-1)P_{i_{n_0-1}}(n_0-2)\cdots P_{i_j}(j-1)$ ($f_{n_0} = 1$ and $j < n_0$) ([Mor3], §§5.6-5.9, 6.10-6.13, 6.16). Then ρ_j and ρ_j'' are extended to representations of $\widetilde{P}_{\Psi^{(0)}}$ and $P_{\underline{\Psi}''}$, respectively, by

$$\begin{aligned}
\rho_j(x) &= \Omega(\mathrm{tr}_0(c_j'(x-1)))\rho_j(1), \qquad x \in P_{i_{j-1}}(j-2)\cdots P_{i_2}(1)P_{i_1}\\
\rho_j''(x) &= \Omega(\mathrm{tr}_0(c_j'(x-1)))\rho_j''(1), \qquad x \in P_{i_{j-1}}(j-2)\cdots P_{i_2}(1).
\end{aligned}$$

Next, if $f_{n_0} = 1$ and $j = n_0$, the irreducible cuspidal representation σ_{n_0} of a finite reductive group which is given in the definition of cuspidal datum is used to produce an irreducible representation of $P(n_0-1)$ (*see* §§4 and 5). This representation is then extended trivially across $P_{\Psi^{(0)}}$, respectively $P_{\underline{\Psi}''} \cap P_1$, ([Mor3], §§5.9, 6.16) to obtain ρ_{n_0} and ρ_{n_0}''. Note that $\Omega(\mathrm{tr}_0(c_{n_0}'(x-1))) = 1$ for all $x \in P_{i_1}$, because $i_1 \geq 1$.

As seen above, (2) holds for all $j \geq 2$. Also, since $P_{\underline{\Psi}''} = \widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1)$ (Lemma 6.1),

$$\rho_j \,|\, P_{\underline{\Psi}''} = \rho_j'', \qquad 2 \leq j \leq n_0.$$

By definition $\rho_{\underline{\Psi}''} = \rho_2'' \otimes \cdots \otimes \rho_{n_0}''$, so (1) holds.

(3) Assume that $r > 1$ and $r > \ell$. After ρ_j is defined on $\widetilde{P}_{\Psi^{(0)}}$, ρ_j must be extended to a representation of $P_{\underline{\Psi}}$. Note that ρ_j is already defined on P_{i_1}, hence on $P_{\underline{\Psi}'} \cap P_{i_1}$. Since $c_j' \in \mathrm{End}_F(V'')$ and $x - 1 \in \mathrm{End}_F(V')$ for $x \in P_{\underline{\Psi}'} \cap P_{i_1}$, it follows from (2) that ρ_j is

trivial on $P_{\underline{\Psi}'} \cap P_{i_1}$. Because $P_{\underline{\Psi}'} \subset \mathbf{GL}(V')$, $P_{\underline{\Psi}'}$ commutes with $\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1)$. Thus the commutator of $P_{\underline{\Psi}'}$ and $\widetilde{P}_{\Psi^{(0)}} = (\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1))P_{i_1}$ equals the commutator of $P_{\underline{\Psi}'}$ and P_{i_1}. Let $y \in P_{\underline{\Psi}'}$ and $z = C(Z) \in P_{i_1}$. Since $P_{\underline{\Psi}'} \subset P$, we have $y^{-1}z^{-1}y \in P_{i_1}$. Thus

$$y^{-1}z^{-1}yz = C(-\mathrm{Ad}\, y^{-1}(Z))C(Z) \in C(Z - \mathrm{Ad}\, y^{-1}(Z))P_{2i_1}.$$

The character $x \mapsto \Omega(\mathrm{tr}_0(c_j'(x-1)))$ is trivial on P_{f_j}, and $f_1 > f_j$, so the character is trivial on $P_{f_1 - 1} = P_{2i_1}$. It follows that

$$\rho_j(y^{-1}z^{-1}yz) = \Omega(\mathrm{tr}_0(c_j'(Z - \mathrm{Ad}\, y^{-1}(Z)))).$$

Because $y \in \mathbf{GL}(V')$, $Z - \mathrm{Ad}\, y^{-1}(Z) \in \mathfrak{g}^{(0)}(0)^{\perp}$. By definition, $c_j' \in \mathfrak{g}^{(0)}(j) \subset \mathfrak{g}^{(0)}(0)$, so it follows that $\rho_j(y^{-1}z^{-1}yz) = \rho_j(1)$. That is, ρ_j is trivial on the commutator of $P_{\underline{\Psi}'}$ and $\widetilde{P}_{\Psi^{(0)}}$. Therefore the representation ρ_j can be extended trivially accros $P_{\underline{\Psi}'}$ to obtain a representation of $P_{\underline{\Psi}}$ ([Mor3], §§6.14, 6.16). □

LEMMA 6.3. – *Suppose $r > 1$ and $r > \ell$. Then*
(1) $(\rho_{\Psi^{(1)}} \otimes \cdots \otimes \rho_{\Psi^{(r-\max\{1,\ell\})}}) \,|\, P_{\underline{\Psi}'} = \rho_{\underline{\Psi}'}$
(2) *If $t > 0$, then $\rho_{\Psi^{(t)}}(x) = \Omega(\mathrm{tr}_0(c_{\Psi^{(t)}}(x-1)))\rho_{\Psi^{(t)}}(1)$, for $x \in P^{(0)}_{i_1^{(0)}}$.*

Proof. – As shown in §§6.15-6.17 of [Mor3], after using $\Psi^{(t)}$ to define $\rho_j^{(t)}$, $1 \le j \le n_t$, on $\widetilde{P}_{\Psi^{(t)}} \cdots \widetilde{P}_{\Psi^{(r-\max\{1,\ell\})}}$, $\rho_j^{(t)}$ is extended to $\widetilde{P}_{\Psi^{(0)}} \cdots \widetilde{P}_{\Psi^{(t-1)}}$ via $x \mapsto \Omega(\mathrm{tr}_0(c_j'^{(t)}(x-1)))$. By definition, $P_{\underline{\Psi}'} = \widetilde{P}_{\Psi^{(1)}} \cdots \widetilde{P}_{\Psi^{(r-\max\{1,\ell\})}}$, and so $\rho_{\underline{\Psi}'} = (\rho_{\Psi^{(1)}} \otimes \cdots \otimes \rho_{\underline{\Psi}^{(r-\max\{1,\ell\})}}) \,|\, P_{\underline{\Psi}'}$. Since $\rho_{\Psi^{(t)}} = \rho_1^{(t)} \otimes \cdots \otimes \rho_{n_t}^{(t)}$, (2) is immediate given the way the representations $\rho_j^{(t)}$ have been extended to $P_{\underline{\Psi}}$. □

7. Adjoint orbits of linear functionals

Given a compact subgroup K of G, let dx be Haar measure on K, normalized so that K has volume one. If ω is an open subset of K, set

$$\mathcal{I}(X,Y;\omega) = \int_{\omega} \Omega(\mathrm{tr}_0(X \mathrm{Ad}\, x^{-1}(Y)))\, dx, \qquad X, Y \in \mathfrak{g}. \tag{7.1}$$

We are integrating the linear functional $\Omega(\mathrm{tr}_0(X\cdot))$ over a subset of its $\mathrm{Ad}\, K$-orbit and then evaluating the resulting linear functional at a point in $\mathfrak{g}$. In §§8 and 9, relations between integrals of the form $\mathcal{I}(X,Y;K)$, and character values of $\rho_1^{(0)}$ and $\rho_{\underline{\Psi}}$ will be derived. This section contains results concerning the equality of $\mathcal{I}(X,Y;K)$ and $\mathcal{I}(X,Y;\omega)$ for various X, K and ω.

The following lemma is a collection of results from §4 of [Mor3].

LEMMA 7.2. – *If $\ell = r$, let $c = \varpi^m b \in C^-_{A_u}$, where $b \in \prod_{i=1}^{r} B_i$. If $\ell < r$, let $c = \varpi_{E_r}^m b \in C^{(0)-}$, where b is a root of unity. Then*
(1) $\mathcal{B}_i^{(0)} = (\mathcal{B}_i^{(0)} \cap \mathcal{C}[c]) \oplus (\mathcal{B}_i^{(0)} \cap \mathcal{C}[c]^{\perp})$, $i \in \mathbf{Z}$. (Here, $\mathcal{B}_0^{(0)} = \mathcal{A}^{(0)}$).

(2) $\mathcal{C}[c]$ *and* $\mathcal{C}[c]^{\perp}$ *are* σ*-stable.*
(3) *If* $c' \in c + (\mathcal{B}_{2m+1}^{(0)-} \cap \mathcal{C}[c])$, *then*

$$\operatorname{ad} c' : (\mathcal{B}_i^{(0)} \cap \mathcal{C}[c]^{\perp})^- \longrightarrow (\mathcal{B}_{i+2m}^{(0)} \cap \mathcal{C}[c]^{\perp})^-$$

is an isomorphism of O*-modules.*

If $\mathcal{S}$ is a subset of $\mathfrak{g}$, and H a subgroup of G, define

$$H(X, \mathcal{S}) = \{ x \in H \mid \operatorname{Ad} x^{-1}(X) \in \mathcal{S} \} \qquad X \in \mathfrak{g}.$$

LEMMA 7.3. – *Let* c *and* c' *be as in Lemma 7.2. Assume* $2m$ *is of the form* $2m = \lambda(f)$ *for some odd integer* $f \geq 1$. *Let* $j \in \mathbf{Z}$, $j \leq 1$ *if* $f > 1$, *and* $j \leq -1$ *if* $f = 1$. *Define*

$$\mathcal{S}_j = (\mathcal{B}_j^{(0)-} \cap \mathcal{C}[c]) + \mathcal{B}_{[(f+j)/2]}^{(0)-}.$$

Let K *be a subgroup of* $P^{(0)}$ *containing* $P_{(f-1)/2}^{(0)}$. *Then*

$$\mathcal{I}(c', X; K) = \mathcal{I}(c', X; K(X, \mathcal{S}_j)), \qquad X \in \mathcal{B}_j^{(0)-}.$$

Proof. – To simplify the notation, the superscript (0) will be dropped. Set

$$t = [(f - j + 1)/2].$$

The following are easily checked and will be used later in the proof:

$$2t + j \geq f \tag{7.4 i}$$

$$w - f \geq -t \Longrightarrow w \geq \left[\frac{f+j}{2}\right] \tag{7.4 ii}$$

$$t + j \geq \left[\frac{f+j}{2}\right]. \tag{7.4 iii}$$

The conditions on j in the statement of the lemma imply that $t \geq (f-1)/2$, so $P_t \subset P_{(f-1)/2} \subset K$. Introducing an extra integration over P_t, and changing the order of integration results in

$$\begin{aligned} \mathcal{I}(c', X; K) &= \int_K \left\{ \int_{P_t} \Omega(\operatorname{tr}_0(c' \operatorname{Ad}(xh)^{-1}(X)))\, dh \right\} dx \\ &= \int_K \mathcal{I}(c', \operatorname{Ad} x^{-1}(X); P_t)\, dx. \end{aligned} \tag{7.5}$$

Fix $x \in K$ and set $Y = \operatorname{Ad} x^{-1}(X)$. Then $X \in \mathcal{B}_j^-$ and $x \in P$ imply $Y \in \mathcal{B}_j^-$. The Cayley transform is a bijection from $\mathcal{B}_t^-$ to P_t ([Mor2], §2.12), so for each $h \in P_t$ there exists a unique $H \in \mathcal{B}_t^-$ such that $h = C(H)$. It is easily verified that

$$\operatorname{Ad} h^{-1}(Y) \in Y - 2[Y, H] + \mathcal{B}_{2t+j}^-.$$

Thus it follows from $c' \in \mathcal{B}^-_{\lambda(f)}$, (7.4 i), and the above inclusion, that

$$\mathcal{I}(c', Y; P_t) = \Omega(\mathrm{tr}_0(c'Y)) \int_{\mathcal{B}_t^-} \Omega(-2\,\mathrm{tr}_0(c'[Y, H]))\, dH,$$

where dH is the measure on $\mathcal{B}_t^-$ corresponding to the measure dh on P_t. It follows from $\mathrm{tr}_0(c'[Y, H]) = \mathrm{tr}_0([c', Y]H)$ that

$$\mathcal{I}(c', Y; P_t) = \Omega(\mathrm{tr}_0(c'Y)) \int_{\mathcal{B}_t^-} \Omega(-2\,\mathrm{tr}_0([c', Y]H))\, dH.$$

The map $H \mapsto \Omega(-2\mathrm{tr}_0([c', Y]H))$ is a character of $\mathcal{B}_t^-$. Therefore in order for $\mathcal{I}(c', Y; P_t)$ to be nonvanishing, it is necessary that this character is trivial. That is,

$$[c', Y] \in (\mathcal{B}_t^-)^\sharp = \mathcal{B}^-_{\lambda(t)}. \tag{7.6}$$

By Lemma 7.2(1), Y decomposes (uniquely) as

$$Y = Y_1 + Y_2, \qquad Y_1 \in \mathcal{B}_j \cap \mathcal{C}[c], \qquad Y_2 \in \mathcal{B}_j \cap \mathcal{C}[c]^\perp.$$

It follows from Lemma 7.2(1), (2) and $\sigma(Y) = -Y$ that $\sigma(Y_i) = -Y_i$, $i = 1, 2$. Because c' and Y_1 both belong to $\mathcal{C}[c]$, $[c', Y_1]$ also belongs to $\mathcal{C}[c]$. It is immediate from the definition of $\mathcal{C}[c]^\perp$ that $\mathcal{C}[c]^\perp$ is invariant under multiplication on both the left and the right by elements of $\mathcal{C}[c]$. Therefore, $[c', Y_2] \in \mathcal{C}[c]^\perp$. Applying Lemma 7.2 to $[c', Y]$, we see that $[c', Y_1]$ and $[c', Y_2]$ are the $\mathcal{C}[c]$ and $\mathcal{C}[c]^\perp$ components, respectively, of $[c', Y]$. In order for (7.6) to hold, both of these components must belong to $\mathcal{B}^-_{\lambda(t)}$. In particular, if (7.6) holds, then

$$[c', Y_2] \in \mathcal{B}^-_{\lambda(t)}. \tag{7.7}$$

Let w be such that

$$Y_2 \in (\mathcal{B}_w \cap \mathcal{C}[c]^\perp)^- - (\mathcal{B}_{w+1} \cap \mathcal{C}[c]^\perp)^-.$$

By Lemma 7.2(3),

$$[c', Y_2] \in (\mathcal{B}_{w+\lambda(f)} \cap \mathcal{C}[c]^\perp)^- - (\mathcal{B}_{w+\lambda(f)+1} \cap \mathcal{C}[c]^\perp)^-. \tag{7.8}$$

We remark that, in order to conclude (7.8) from Lemma 7.2(3), we must have $\mathcal{B}_{w+\lambda(f)} \neq \mathcal{B}_{w+\lambda(f)+1}$. In some cases, pairs of successive lattices in $\{\mathcal{B}_i\}$ can coincide (for example, when $r = \ell$ and $\mathcal{L}_u$ has period one). However, in such cases $\mathcal{B}_i = \mathcal{B}_{i+1}$ only when i is odd. In this situation, $\mathcal{B}_w \neq \mathcal{B}_{w+1}$ implies w is even. By assumption, $\lambda(f) = 2m$, so $w + \lambda(f)$ is even, which guarantees that $\mathcal{B}_{w+\lambda(f)} \neq \mathcal{B}_{w+\lambda(f)+1}$.

In view of (7.8), (7.7) implies $\mathcal{B}^-_{w+\lambda(f)} \subset \mathcal{B}^-_{\lambda(t)}$. If $\mathcal{B}_{\lambda(t)} \neq \mathcal{B}_{\lambda(t)-1}$, then $w+\lambda(f) \geq \lambda(t)$. Otherwise, $\lambda(t)$ is even (*see* above) and $\mathcal{B}_{\lambda(t)-2} \neq \mathcal{B}_{\lambda(t)-1}$ imply that $w+\lambda(f) \geq \lambda(t)-1$. If this is the case, since $\lambda(t)-1$ is odd and $w+\lambda(f)$ is even, we must have $w+\lambda(f)\geq\lambda(t)$. By Lemma 4.18 of [Mor3],

$$\lambda(i)=1-i-2ee_0,\qquad i\in\mathbf{Z},$$

where e_0 is the ramification degree of F over F_0, $e=1$ if $r=\ell$, and e is the ramification degree of E_r over F if $r>\ell$. Thus, in order for (7.7) to hold, we require

$$w+\lambda(f)=w+1-f-2ee_0\geq 1-t-2ee_0=\lambda(t).$$

By (7.4 ii), this last condition implies that $w\geq[(f+j)/2]$, which is equivalent to $Y\in\mathcal{S}_j$. We have shown that if $\mathcal{I}(c',\mathrm{Ad}\,x^{-1}(X);P_t)$ does not vanish, then $x\in K(X,\mathcal{S}_j)$. Thus (*see* (7.5))

$$\mathcal{I}(c',X;K)=\int_{K(X,\mathcal{S}_j)}\int_{P_t}\Omega(\mathrm{tr}_0(c'\mathrm{Ad}\,(xh)^{-1}(X)))\,dh\,dx.$$

Suppose $x\in K(X,\mathcal{S}_j)$ and $h=C(H)\in P_t$. It follows from $\mathrm{Ad}\,x^{-1}(X)\in\mathcal{B}_j^-$, $H\in\mathcal{B}_t^-$, and (7.4 iii), that

$$[H,\mathrm{Ad}\,x^{-1}(X)]\in\mathcal{B}_{j+t}^-\subset\mathcal{B}_{[(f+j)/2]}^-.$$

Therefore, since $\mathcal{B}_{[(f+j)/2]}^-$ and $\mathcal{B}_f^-$ are subsets of $\mathcal{S}_j$ and we are assuming that $\mathrm{Ad}\,x^{-1}(X)\in\mathcal{S}_j$,

$$\mathrm{Ad}\,(xh)^{-1}(X)\in\mathrm{Ad}\,x^{-1}(X)-2[\mathrm{Ad}\,x^{-1}(X),H]+\mathcal{B}_f^-\subset\mathcal{S}_j.$$

Thus $K(X,\mathcal{S}_j)$ is invariant under right translation by elements of P_t. To finish the proof, reverse the order of integration above and then absorb the P_t integral into the $K(X,\mathcal{S}_j)$ integral. □

COROLLARY 7.9. – *Suppose* $f_1^{(0)}>1$. *Let* $X\in\mathcal{A}^{(0)-}$. *If* $X\neq(\mathcal{A}^{(0)-}\cap\mathcal{C}[c_1^{(0)}])+\mathcal{B}_{i_1^{(0)}}^{(0)-}$, *then* $\mathcal{I}(c_{\underline{\Psi}},X;P_{\underline{\Psi}})=0$.

Proof. – Let $\mathcal{S}_0$ be as in Lemma 7.3 (where $c=c_1^{(0)}$). Recall that $i_1^{(0)}=[f_1^{(0)}/2]$. Hence the conclusion of the corollary can be stated as follows: if $X\in\mathcal{A}^{(0)-}$ and $X\neq\mathcal{S}_0$, then $\mathcal{I}(c_{\underline{\Psi}},X;P_{\underline{\Psi}})=0$. Since

$$\mathrm{Ad}\,h^{-1}(Y)-Y\in\mathcal{B}_{[f_1^{(0)}/2]}^{(0)-},\qquad Y\in\mathcal{A}^{(0)-},\quad h\in P_{i_1^{(0)}}^{(0)},$$

it follows that $\mathrm{Ad}\,P_{i_1^{(0)}}^{(0)}(\mathcal{S}_0)=\mathcal{S}_0$. Also, by definition $P_{\underline{\Psi}}\subset P^{(0)}$, and it is easily checked that $\mathrm{Ad}\,(P^{(0)}\cap Z_G(c_1^{(0)}))(\mathcal{S}_0)=\mathcal{S}_0$. Thus, by Lemma 6.1(2), $\mathrm{Ad}\,P_{\underline{\Psi}}(\mathcal{S}_0)=\mathcal{S}_0$. Therefore, to prove the corollary it suffices to show that Lemma 7.3 applies with $K=P_{\underline{\Psi}}$, $c=c_1^{(0)}$, $c'=c_{\underline{\Psi}}$, and $j=0$ or 1. By definition (§§4, 5) $c_1^{(0)}\in C^{(0)-}$, and $c_1^{(0)}\in\mathcal{B}_{\lambda(f_1^{(0)})}^{(0)-}$. It remains to verify that

$$c_{\underline{\Psi}}\in c_1^{(0)}+\mathcal{B}_{\lambda(f_1^{(0)})+1}^{(0)-}.$$

In particular, (*see* (5.7)), we will show that

$$c_{\Psi^{(0)}} \in c_1^{(0)} + \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)})+1}$$
$$c_{\Psi^{(t)}} \in \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)})+1}, \qquad 1 \leq t \leq r - \ell.$$

From the definitions of $c_1^{(0)}$ and $c_1'^{(0)}$ (§4 if $r = \ell$, and §5 if $r > \ell$), it follows that $c_1'^{(0)} - c_1^{(0)} \in \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)}-1)} = \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)})+1}$. For details, *see* Lemmas 4.4 and 5.2.

Suppose $n_0 > 1$. Let $2 \leq j \leq n_0$. By definition, $c_j'^{(0)} \in \mathcal{B}^{(0)-}_{\lambda(f_j^{(0)})}$. Since $f_j^{(0)} < f_1^{(0)}$ and λ is order reversing ([Mor3], §4.18), $\mathcal{B}^{(0)-}_{\lambda(f_j^{(0)})} \subset \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)})+1}$.

Because $c_{\Psi}^{(0)} = \sum_{j=1}^{n_0} c_j'^{(0)}$, the proof is complete if the length of $\underline{\Psi}$ equals one.

Assume that the length of $\underline{\Psi}$ is greater than one. By induction on the length, since $\underline{\Psi}'$ has length one less than $\underline{\Psi}$, we may assume that

$$c_{\underline{\Psi}'} = c_{\Psi^{(1)}} + \cdots + c_{\Psi^{(r-\ell)}} \in c_1^{(1)} + \mathcal{B}^{(1)-}_{\lambda(f_1^{(1)})+1}.$$

By definition, $c_1^{(1)} \in \mathcal{B}^{(1)-}_{\lambda(f_1^{(1)})}$. Thus $c_{\underline{\Psi}'} \in \mathcal{B}^{(1)-}_{\lambda(f_1^{(1)})}$. By condition (b)(i) $r > \ell + 1$, or (c)(i), $r = \ell + 1$, in the definition of $\underline{\Psi}$,

$$P^{(0)}_{f_1^{(0)}-1} \cap \mathbf{GL}(V') \subset P^{(1)}_{f_1^{(1)}},$$

which is equivalent to

$$\mathcal{B}^{(0)-}_{f_1^{(0)}-1} \cap \operatorname{End}_F(V') \subset \mathcal{B}^{(1)-}_{f_1^{(1)}}.$$

Hence

$$\mathcal{B}^{(0)-}_{\lambda(f_1^{(0)}-1)} \supset \mathcal{B}^{(0)-}_{\lambda(f_1^{(0)}-1)} \cap \operatorname{End}_F(V') \supset \mathcal{B}^{(1)-}_{\lambda(f_1^{(1)})} \ni c_{\underline{\Psi}'}.$$

We have shown above that $c_{\Psi^{(0)}} \in c_1^{(0)} + \mathcal{B}_{\lambda(f_1^{(0)})+1}$. The desired result now follows from $c_{\underline{\Psi}} = c_{\Psi^{(0)}} + c_{\underline{\Psi}'}$. □

We will say that the cuspidal datum $\underline{\Psi}$ is *uniform* if $f_1^{(t)} \geq 2e(E_{r-t}/F) + 1$ for every $t \leq r - \ell - 2$. Note that if $\underline{\Psi}$ is uniform and $r \geq \max\{1, \ell\} + 1$, then $\underline{\Psi}'$ is also uniform. If $r \leq \ell + 1$, then every cuspidal datum attached to T is uniform.

Define $\nu_0 : \operatorname{End}_F(V) \to \mathbf{Z}$ by

$$\nu_0(X) = j \iff X \in \mathcal{B}_j^{(0)} - \mathcal{B}_{j+1}^{(0)}.$$

LEMMA 7.10. – *Assume that $\underline{\Psi}$ is uniform. Let e be the ramification degree of E_r over F if $\ell < r$ and $e = 1$ if $\ell = r$. Suppose that $X \in \mathfrak{g}$ and $2em \leq \nu_0(X) < 2e(m+1)$ for some integer $m < 0$. If $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) \neq 0$, then $\lim_{t\to\infty} \nu_0(\varpi^{-t(m+1)} X^t) = -\infty$.*

Proof. – Assume that $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) \neq 0$. For $j = \nu_0(X)$, let $\mathcal{S}_j$ be as in Lemma 7.3, with $c = c_1^{(0)}$ and $f = f_1^{(0)}$. As shown in the proof of Corollary 7.9, Lemma 7.3 may be applied with $c' = c_{\underline{\Psi}}$ and $K = P_{\underline{\Psi}}$. Thus $\mathrm{Ad}\, w^{-1}(X) \in \mathcal{S}_j$ for some $w \in P_{\underline{\Psi}}$. Note that $P_{\underline{\Psi}} \subset P^{(0)}$ implies $\nu_0(X) = \nu(\mathrm{Ad}\, w^{-1}(X))$ for all $w \in P_{\underline{\Psi}}$. The set $P_{\underline{\Psi}}(X, \mathcal{S}_j)$ is invariant under right multiplication by $P_{\underline{\Psi}} \cap Z_G(c_1^{(0)})$ because $\mathcal{C}[c_1^{(0)}]^{\perp -}$ is $\mathrm{Ad}\, Z_G(c_1^{(0)})$-invariant. Introducing an integration over $P_{\underline{\Psi}}(X, \mathcal{S}_j)$ into $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}})$, and then reversing the order of integration results in:

$$\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) = \int_{P_{\underline{\Psi}}(X, \mathcal{S}_j)} \Omega(\mathrm{tr}_0(c_1'^{(0)} \mathrm{Ad}\, w^{-1}(X))) \\ \times \mathcal{I}(c_{\underline{\Psi}} - c_1'^{(0)}, \mathrm{Ad}\, w^{-1}(X); P_{\underline{\Psi}} \cap Z_G(c_1^{(0)}))\, dw.$$

Fix $w \in P_{\underline{\Psi}}(X, \mathcal{S}_j)$ such that the inner integral above is nonzero. Set $\tilde{X} = \mathrm{Ad}\, w^{-1}(X)$. By definition of $\mathcal{S}_j$,

$$\tilde{X} - \tilde{U} \in \mathcal{B}^{(0)-}_{[(f_1^{(0)}+j)/2]} \quad \text{for some } \tilde{U} \in \mathcal{C}[c_1^{(0)}] \cap \mathcal{B}_j^{(0)-}.$$

Apply Lemma 7.2 with $i = [(f_1^{(0)} + j)/2]$ and $c = c_1^{(0)}$, to conclude that

$$\tilde{X} - \tilde{U} = \tilde{W} + \tilde{Z}, \quad \text{where } \tilde{W} \in \mathcal{B}^{(0)-}_{[(f_1^{(0)}+j)/2]} \cap \mathcal{C}[c_1^{(0)}], \ \ \tilde{Z} \in \mathcal{B}^{(0)-}_{[(f_1^{(0)}+j)/2]} \cap \mathcal{C}[c_1^{(0)}]^{\perp}.$$

Set $\tilde{Y} = \tilde{U} + \tilde{W}$. Then $\tilde{X} = \tilde{Y} + \tilde{Z}$. Since $\nu_0(\tilde{X}) = j$ and $\nu_0(\tilde{Z}) \geq [(f_1^{(0)} + j)/2] \geq j + 1$ (recall that $f_1^{(0)} \geq 1$ and $j < 0$ by assumption), it follows that $\nu_0(\tilde{Y}) = j$. As a consequence of the $\mathrm{Ad}\, Z_G(c_1^{(0)})$-invariance of $\mathcal{C}[c_1^{(0)}]^{\perp}$ and $c_{\underline{\Psi}} - c_1'^{(0)} \in \mathcal{C}[c_1^{(0)}]$, we may replace $\tilde{X}$ by $\tilde{Y}$ in the inner integral. That is,

$$\mathcal{I}(c_{\underline{\Psi}} - c_1'^{(0)}, \tilde{X}; P_{\underline{\Psi}} \cap Z_G(c_1^{(0)})) = \mathcal{I}(c_{\underline{\Psi}} - c_1'^{(0)}, \tilde{Y}; P_{\underline{\Psi}} \cap Z_G(c_1^{(0)})) \neq 0. \tag{7.11}$$

Suppose that the length of $\underline{\Psi}$ is one. We will prove by induction on the rank n_0 of $\underline{\Psi} = \Psi^{(0)}$ that $\nu_0(X^t) = tj$ for all integers $t \geq 1$, where $j = \nu_0(X)$.

If $n_0 = 1$, then $\mathcal{C}[c_1^{(0)}] = \mathcal{T}^{(0)}$. The equality $\mathcal{T}^{(0)} \cap \mathcal{B}^{(0)}_{2i-1} = \mathcal{T}^{(0)} \cap \mathcal{B}^{(0)}_{2i}$, $i \in \mathbf{Z}$, ([Mor3], §4.21), implies that j is even. Thus $\tilde{Y} = \eta^{j/2}\zeta + W$, where $W \in \mathcal{B}^{(0)}_{j+1}$, $\eta = \varpi$ if $r = \ell$ ([Mor3, §4.6), and $\eta = \varpi_{E_1}$ if $r = 1$ and $\ell = 0$ ([Mor3], §4.1). If $r = 1$ and $\ell = 0$, ζ is a root of unity in $\mathcal{T}^{(0)}$. Otherwise, $\zeta = \sum_{s=1}^{\ell} \zeta_s$ is a nonzero element of $\mathcal{T}^{(0)}$ such that each nonzero ζ_s is a root of unity in E_s. Observe that

$$\nu_0(\eta^{(tj)/2}\zeta^n) = tj, \ \ \nu_0(W) \geq j + 1, \ \ \nu_0(\tilde{Z}) \geq j + 1 \\ \Longrightarrow \quad \nu_0(\eta^{(tj)/2}\zeta^t - \tilde{X}^t) \geq (t-1)j + (j+1) = tj + 1.$$

Thus $\nu(X^t) = \nu(\tilde{X}^t) = tj$ when $n_0 = 1$.

If $n_0 > 1$, then $c_{\underline{\Psi}} - c_1'^{(0)} = c_{\underline{\Psi}''}$. By Lemma 6.1, $P_{\underline{\Psi}} \cap Z_G(c_1^{(0)}) = P_{\underline{\Psi}''}$. Thus the integral in (7.11) is $\mathcal{I}(c_{\underline{\Psi}''}, \tilde{Y}, P_{\underline{\Psi}''})$. Note that $\underline{\Psi}''$ is a cuspidal datum of length one and rank

n_0-1. By induction, $\nu_0(\tilde{Y}^t)=tj$ for all $t\geq 1$. Along with $\nu_0(\tilde{Z})\geq[(f_1^{(0)}+j)/2]\geq j+1$, this implies $\nu_0(\tilde{X}^t-\tilde{Y}^t)\geq(t-1)j+j+1=tj+1$. Therefore, $\nu_0(\tilde{X}^t)=\nu_0(X^t)=tj$.

We can now conclude that if the length of $\underline{\Psi}$ is one, then

$$\nu_0(\varpi^{-t(m+1)}X^t)=2et(-m-1)+tj\leq -t,$$

and the lemma follows.

Turning to the general case, assume that the lemma holds for all cuspidal data of length strictly less than that of $\underline{\Psi}$. Since ([Mor3], §4)

$$\mathcal{C}[c_1^{(0)}]=\mathrm{End}_F(V')\oplus(\mathcal{C}[c_1^{(0)}]\cap\mathrm{End}_F(V'')),$$

we can write $\tilde{Y}=\tilde{Y}_1+\tilde{Y}_2$ with $\tilde{Y}_1\in\mathrm{End}_F(V')^-$ and $\tilde{Y}_2\in(\mathcal{C}[c_1^{(0)}]\cap\mathrm{End}_F(V''))^-$. Applying Lemma 6.1 and the fact that $\mathrm{End}_F(V')$ and $\mathrm{End}_F(V'')$ commute and are orthogonal with respect to trace results in

$$\mathcal{I}(c_{\underline{\Psi}}-c_1^{\prime(0)},\tilde{Y};P_{\underline{\Psi}}\cap Z_G(c_1^{(0)}))=\begin{cases}\mathcal{I}(c_{\underline{\Psi}'},\tilde{Y}_1;P_{\underline{\Psi}'})\mathcal{I}(c_{\underline{\Psi}''},\tilde{Y}_2,P_{\underline{\Psi}''}), & \text{if } n_0>1\\ \mathcal{I}(c_{\underline{\Psi}'},\tilde{Y}_2,P_{\underline{\Psi}'}), & \text{if } n_0=1.\end{cases}$$

By (7.11), all of the above integrals are nonzero.

If $\nu_0(\tilde{Y}_2)=j$, then the arguments in the length one case imply that $\nu_0(\tilde{Y}_2^t)=tj$.

Otherwise, $\nu_0(\tilde{Y}_2)\geq j+1$ and $\nu_0(\tilde{Y})=j$ imply that $\nu_0(\tilde{Y}_1)=j$. Let the integer valued function ν_1 on $\mathrm{End}_F(V')$ be defined in the same manner as ν_0, but relative to $\{\mathcal{B}_i^{(1)}\}_{i\in\mathbf{Z}}$. The inequality $j<2e(m+1)$ is equivalent to $\tilde{Y}_1\neq\varpi^{m+1}\mathcal{A}^{(0)}$. Also $\mathcal{A}^{(1)}\subset\mathcal{A}^{(0)}\cap\mathrm{End}_F(V')$ (Lemma 3.3(6)). Thus $\tilde{Y}_1\notin\varpi^{m+1}\mathcal{A}^{(1)}$. That is, $\nu_1(\tilde{Y}_1)<2e'(m+1)$. It follows that

$$2e'm'\leq\nu_1(\tilde{Y}_1)<2e'(m'+1),\qquad\text{for some } m'\leq m,\tag{7.12}$$

where e' is the ramification degree of E_{r-1} over F if $r>\ell+1$, and $e'=1$ if $r=\ell+1$.

For the moment, assume that $r=\ell+1$. The length of $\underline{\Psi}'=\Psi_u$ is one, so the nonvanishing of $\mathcal{I}(c_{\underline{\Psi}'},\tilde{Y}_1;P_{\underline{\Psi}'})$ implies that $\nu_1(\tilde{Y}_1^t)=\nu_1(\tilde{Y}_1)t$. The argument in the rank one case also implies that $\nu_1(\tilde{Y}_1)$ is even. Since $e'=1$, from (7.12) we have $\nu_1(\tilde{Y}_1)=2m'$. The equality $\nu_1(\tilde{Y}_1^t)=2m't$ is equivalent to $\tilde{Y}_1^t\in\varpi^{tm'}(\mathcal{A}^{(1)}-\mathcal{B}_1^{(1)})$. Recall that $\mathcal{A}^{(1)}\subset\mathcal{A}^{(0)}$ and $\mathcal{B}_1^{(1)}\supset\mathcal{B}_1^{(0)}\cap\mathrm{End}_F(V')$ (Lemma 3.3(7)). Thus $\tilde{Y}_1^t\in\varpi^{tm'}(\mathcal{A}^{(0)}-\mathcal{B}_1^{(0)})$. That is, $\nu_0(\tilde{Y}_1^t)=etm'$. Assumptions on j in the case $t=1$ then force $j=em'$ and $m'=m$. So $\nu_0(\tilde{Y}_1^t)=tj$ (when $\nu_0(\tilde{Y}_2)\geq j+1$).

We can now conclude that if $r=\ell+1$, then $\nu_0(\tilde{Y}_i^t)=tj$, $t\geq 1$, for at least one $i\in\{1,2\}$. Because $\tilde{Y}_1\in\mathrm{End}_F(V')$ and $\tilde{Y}_2\in\mathrm{End}_F(V'')$, this implies $\nu_0(\tilde{Y}^t)=tj$, $t\geq 1$. By the same argument as in the length one case, $\nu_0(\tilde{X}^t)=\nu_0(X^t)=tj$. Thus $\nu_0(\varpi^{-t(m+1)}X^t)=2et(-m-1)+tj\leq -t$ and the lemma is proved in the case $r=\ell+1$.

Suppose that $r>\ell+1$ and $\nu_0(\tilde{Y}_2)\geq j+1$. Thus $\nu_0(\tilde{Y}_1)=j$. In this case, $e'>1$, and we do not have an explicit relation between $\{\mathcal{B}_i^{(1)}\}_{i\in\mathbf{Z}}$ and $\{\mathcal{B}_i^{(0)}\cap\mathrm{End}_F(V')\}_{i\in\mathbf{Z}}$, so it is not possible to determine $\nu_1(\tilde{Y}_1)$ precisely. The length of $\underline{\Psi}'$ is one less than the length of $\underline{\Psi}$, $\underline{\Psi}'$ is uniform, and $\mathcal{I}(c'_{\underline{\Psi}},\tilde{Y}_1,P_{\underline{\Psi}'})\neq 0$. By induction, $\lim_{t\to\infty}\nu_1(\varpi^{-t(m'+1)}\tilde{Y}_1^t)=-\infty$, where m' is as in (7.12). That implies $\lim_{t\to\infty} \nu_0(\varpi^{-t(m'+1)}\tilde{Y}_1^t) = -\infty$. If $m' < m$, then $2e(m'+1) \le 2em \le \nu_0(\tilde{Y}_1)$ implies that $\varpi^{-(m'+1)}\tilde{Y}_1 \in \mathcal{A}^{(0)}$. This leads to a contradiction because $W \in \mathcal{A}^{(0)}$ implies $W^t \in \mathcal{A}^{(0)}$ and hence $\nu_0(W^t) \ge 0$. Thus $m' = m$ and $\lim_{t\to\infty} \nu_0(\varpi^{-t(m+1)}\tilde{Y}_1) = -\infty$.

On the other hand, if $\nu_0(\tilde{Y}_2) = j$, then, as remarked above, $\nu_0(\tilde{Y}_2^t) = tj$. This implies that $\nu_0(\varpi^{-t(m+1)}\tilde{Y}_2^t) \le -t$.

We have shown that $\lim_{t\to\infty} \nu_0(\varpi^{-t(m+1)}\tilde{Y}_i^t) = -\infty$ for at least one $i \in \{1,2\}$. Therefore $\lim_{t\to\infty} \nu_0(\varpi^{-t(m+1)}\tilde{Y}^t) = -\infty$. Recall that $\nu_0(\tilde{Z}) \ge [(f_1^{(0)} + j)/2]$. Since $\underline{\Psi}$ is uniform, $f_1^{(0)} = 2i_1^{(0)} + 1 \ge 2e + 1$. Also $j \ge 2em$, and $m \le -1$. Therefore $\nu_0(\varpi^{-m-1}\tilde{Z}) \ge 0$. That is, $\varpi^{-m-1}\tilde{X} - \varpi^{-m-1}\tilde{Y} \in \mathcal{A}^{(0)}$. The lemma now follows from the result for $\tilde{Y}$ and from $\nu_0(\tilde{X}^t) = \nu_0(X^t)$. □

COROLLARY 7.13. – *Suppose that $\underline{\Psi}$ is uniform. Let $d = 2[\dim_F(V)/2] + 1$. If $X \in \mathfrak{g}$ is such that $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}}$ and $X \ne \mathcal{A}^{(0)}$, then $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) = 0$.*

Proof. – By Lemma 7.9, if $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) \ne 0$, then $\nu_0(X^t) < 0$ for all $t \ge 1$. However $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}}$ and $f_1^{(0)} \ge 1$ imply that $\nu_0(X^t) > 0$ for t large. □

Remark. – It is possible that the conclusion of Corollary 7.13 holds if $\underline{\Psi}$ is not uniform. However, the proof of Lemma 7.9 requires that $\underline{\Psi}$ be uniform. Hopefully further analysis of the lattice chains will allow us to drop this condition on $\underline{\Psi}$ in the future.

8. The character of $\rho_1^{(0)}$

The content of the main result of this section, Lemma 8.3, is as follows. If the linear functional $\Omega(\mathrm{tr}_0(c_1'^{(0)}\cdot))$ is integrated over its $\mathrm{Ad}\, P^{(0)}_{i_1^{(0)}}$-orbit to produce a new linear functional which is then evaluated at a certain type of point in $\mathfrak{g}$, the value obtained coincides, up to the degree of the representation $\rho_1^{(0)}$, with the value of the character $\chi_1^{(0)}$ of $\rho_1^{(0)}$ at a related point in $P_{\underline{\Psi}}$. It follows from Lemma 6.1 that the $\mathrm{Ad}\, P^{(0)}_{i_1^{(0)}}$-orbit of $\Omega(\mathrm{tr}_0(c_1'^{(0)}\cdot))$ is the same as its $\mathrm{Ad}\, P_{\underline{\Psi}}$-orbit. This relation between $\chi_1^{(0)}$ and the $\mathrm{Ad}\, P_{\underline{\Psi}}$-orbit of a linear functional resembles the relations between characters and orbits of linear functionals which appear in Howe's Kirillov theory for compact p-adic groups ([H2], Theorem 1.1). However, here the elements of $\mathfrak{g}$ and G are related via a truncated exponential map ((3.6)) rather than via the exponential map. Furthermore, our result does not hold for the values of $\chi_1^{(0)}$ at arbitrary points in $P_{\underline{\Psi}}$.

We begin with a preliminary lemma concerning the decomposition of elements in $\mathfrak{g}$ and G relative to $\mathcal{C}[c_1^{(0)}]$.

LEMMA 8.1. – *Let $X \in \mathcal{C}[c_1^{(0)}] + \mathcal{B}^{(0)}_{i_1^{(0)}}$ be such that $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}}$, where $d = 2[\dim_F(V)/2] + 1$. Assume $f_1^{(0)} > 1$. Choose $j \in \mathbf{Z}$ such that $X \in \mathcal{B}_j^{(0)}$.*

(1) *Then there exist Y, $Z \in \mathcal{B}_j^{(0)}$ such that*

$$X = Y + Z, \qquad Y \in \mathcal{C}[c_1^{(0)}], \qquad Z \in \mathcal{C}[c_1^{(0)}]^\perp \cap \mathcal{B}_s^{(0)},$$

where $s \in \{i_1^{(0)}, i_1'^{(0)}\}$. *If* $X \in \mathfrak{g}$, *then* $Y,\ Z \in \mathfrak{g}$. *Furthermore,* $X \in \mathcal{A}^{(0)}$ *implies* $Y^d \in \mathcal{B}^{(0)}_{s+i_1^{(0)}}$.

(2) *If* $X \in \mathcal{A}^{(0)-}$ *and* $x \in p_{(d,f_1^{(0)})}(X)$, *then there exist* y *and* z *such that*

$$x = yz, \quad y \in p_{(d,s+i_1^{(0)})}(Y) \cap Z_G(c_1^{(0)}), \quad z \in P_s^{(0)}, \quad z - 1 \in \mathcal{C}[c_1^{(0)}]^{\perp} + \mathcal{B}^{(0)}_{s+i_1^{(0)}},$$

where Y *is as in (1). If* $s = i_1'^{(0)}$, *then* $\Omega(\mathrm{tr}_0(c_1'^{(0)}(z-1))) = 1$. *If* $x \in P_{\underline{\Psi}}$, *then* $y \in P_{\underline{\Psi}} \cap Z_G(c_1^{(0)})$.

Proof. – (1) By Lemma 7.2(1),

$$X = Y + Z, \qquad \text{for some } Y \in \mathcal{C}[c_1^{(0)}] \cap \mathcal{B}_j^{(0)},\ Z \in \mathcal{C}[c_1^{(0)}]^{\perp} \cap \mathcal{B}_j^{(0)}.$$

Since $X \in \mathcal{C}[c_1^{(0)}] + \mathcal{B}^{(0)}_{i_1^{(0)}}$ and $Y \in \mathcal{C}[c_1^{(0)}]$,

$$Z = X - Y \in (\mathcal{C}[c_1^{(0)}] + \mathcal{B}^{(0)}_{i_1^{(0)}}) \cap \mathcal{C}[c_1^{(0)}]^{\perp} = \mathcal{B}^{(0)}_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]^{\perp}.$$

If $Z \neq \mathcal{B}^{(0)}_{i_1'^{(0)}}$, let $s = i_1^{(0)}$. Otherwise, let $s = i_1'^{(0)}$.

Suppose $X \in \mathfrak{g}$. By Lemma 7.2(1) and (2), $-X = -Y - Z = \sigma(Y) + \sigma(Z)$ implies $-Y = \sigma(Y)$ and $-Z = \sigma(Z)$. That is, $Y,\ Z \in \mathfrak{g}$.

Suppose that $X \in \mathcal{A}^{(0)}$. Because $\mathcal{C}[c_1^{(0)}]^{\perp}$ is invariant under multiplication on the left and right by elements of $\mathcal{C}[c_1^{(0)}]$, any monomial of the form $Y^j Z Y^{d-1-j}$, $0 \le j \le d-1$, belongs to $\mathcal{C}[c_1^{(0)}]^{\perp}$. If Z occurs at least twice in a monomial, then, since $Y \in \mathcal{A}^{(0)}$ and $Z \in \mathcal{B}_s^{(0)}$, that monomial lies in $\mathcal{B}_{2s}^{(0)}$. Writing X^d as a sum of monomials involving products of powers of Y and Z, and decomposing the monomials into sums of elements in $\mathcal{C}[c_1^{(0)}]$ and $\mathcal{C}[c_1^{(0)}]^{\perp}$, it follows that the $\mathcal{C}[c_1^{(0)}]$-component of X^d belongs to $Y^d + \mathcal{B}_{2s}^{(0)}$. Then $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}} \subset \mathcal{B}^{(0)}_{s+i_1^{(0)}}$ and $2s \ge s + i_1^{(0)}$ imply that $Y^d \in \mathcal{B}^{(0)}_{s+i_1^{(0)}}$. This finishes the proof of (1).

(2) Assume that $X \in \mathcal{A}^{(0)-}$ and let $x \in p_{(d,f_1^{(0)})}(X)$ (our assumption $p > \dim_F(V)$ implies $p \ge d$, so $p_{(d,f_1^{(0)})}(X)$ is defined, cf. §3). Let Y and Z be as in part (1). Because $Y \in \mathcal{C}[c_1^{(0)}]$ and $Y^d \in \mathcal{B}^{(0)}_{s+i_1^{(0)}}$, there exists $R \in \mathcal{C}[c_1^{(0)}] \cap \mathcal{B}^{(0)}_{s+i_1^{(0)}}$ such that $e_d(Y) + R \in G$. That is, $p_{(d,s+i_1^{(0)})}(Y)$ intersects $\mathcal{C}[c_1^{(0)}]$. Set $y = e_d(Y) + R$ and $z = y^{-1}x$.

$$\frac{X^j}{j!} \in \frac{Y^j}{j!} + \mathcal{B}_{2s}^{(0)} + (\mathcal{C}[c_1^{(0)}]^{\perp} \cap \mathcal{B}_s^{(0)}), \qquad 1 \le j \le d-1.$$

This was shown above in the case $j = d$, but without dividing by $j!$ If $j \le d-1$, then $|j!| = 1$ (recall $d \le p$), so $(j!)^{-1}\mathcal{B}_i^{(0)} = \mathcal{B}_i^{(0)}$. By definition, $x \in e_d(X) + \mathcal{B}^{(0)}_{f_1^{(0)}}$. Combining this with the above information about $X^j/(j!)$, we obtain

$$x \in e_d(Y) + \mathcal{B}^{(0)}_{s+i_1^{(0)}} + (\mathcal{C}[c_1^{(0)}]^{\perp} \cap \mathcal{B}_s^{(0)}).$$

Because $y^{-1} \in \mathcal{A}^{(0)} \cap \mathcal{C}[c_1^{(0)}]$, $\mathcal{B}^{(0)}_{s+i_1^{(0)}}$ and $\mathcal{C}[c_1^{(0)}] \cap \mathcal{B}_s^{(0)}$ are invariant under left multiplication by y^{-1}. Thus

$$z = y^{-1}x \in y^{-1}e_d(Y) + \mathcal{B}^{(0)}_{s+i_1^{(0)}} + (\mathcal{C}[c_1^{(0)}]^{\perp} \cap \mathcal{B}_s^{(0)}).$$

It follows from $y^{-1} \in e_d(-Y) + \mathcal{B}^{(0)}_{s+i_1^{(0)}}$ (Lemma 3.7) that $y^{-1}e_d(Y) \in 1 + \mathcal{B}^{(0)}_{s+i_1^{(0)}}$.

Finally, if $s = i_1'^{(0)}$,

$$c_1'^{(0)} \in \mathcal{C}[c_1^{(0)}] \cap \mathcal{B}^{(0)}_{\lambda(f_1^{(0)})} \quad \text{and} \quad z - 1 \in \mathcal{C}[c_1^{(0)}]^{\perp} + \mathcal{B}^{(0)}_{f_1^{(0)}} \Longrightarrow \mathrm{tr}_0(c_1'^{(0)}(z-1)) \in O_0.$$

For the final assertion of the lemma, note that $z \in P_{i_1^{(0)}} \subset P_{\underline{\Psi}}$. Thus $x \in P_{\underline{\Psi}}$ implies $y = xz^{-1} \in P_{\underline{\Psi}}$. By definition, $y \in Z_G(c_1^{(0)})$. □

As in §6, if the length of $\underline{\Psi}$ is greater than one, set $V' = E_1 \oplus \cdots \oplus E_{r-1}$ and $V'' = E_r$. If the length of $\underline{\Psi}$ equals one, then $V'' = E_1$ if $\ell = 0$, and $V'' = A_u$ if $\ell = r$. Recall from the definition of $T^{(0)}$-cuspidal datum in §5 that $G^{(0)}(1)$ denotes the centralizer of $c_1^{(0)}$ in $G \cap \mathbf{GL}(V'')$ and $P_j^{(0)}(1) = P_j^{(0)} \cap G^{(0)}(1)$. The notation $\mathfrak{g}^{(0)}(1)$ is used for the Lie algebra of $G^{(0)}(1)$. The next lemma shows how elements of $\mathcal{C}[c_1^{(0)}]^-$ and $Z_G(c_1^{(0)})$ decompose relative to $\mathrm{End}_F(V')$ and $\mathrm{End}_F(V'')$. The second part of the lemma will be used in §9.

LEMMA 8.2. – *Let X, Y, y, d and s be as in Lemma 8.1. Assume that $X \in \mathcal{A}^{(0)}$.*

(1) *Suppose that $r > \max\{1, \ell\}$. Then*

$$Y = Y_1 + Y_2 \quad \textit{for some} \quad Y_1 \in \mathcal{A}^{(0)-} \cap \mathit{End}_F(V'),\ Y_2 \in \mathcal{A}^{(0)} \cap \mathfrak{g}^{(0)}(1).$$

and $y = y_1 y_2$ where

$$y_1 \in p_{(d, s+i_1^{(0)})}(Y_1) \cap \mathbf{GL}(V') \quad \textit{and} \quad y_2 \in p_{(d, s+i_1^{(0)})}(Y_2) \cap G^{(0)}(1).$$

Furthermore, if $X \in \mathcal{B}_1^{(0)}$, then Y_1, $Y_2 \in \mathcal{B}_1$ and y_1, $y_2 \in P_1^{(0)}$.

(2) *Assume that the rank n_0 of $\Psi^{(0)}$ is strictly greater than one. If $r = \max\{1, \ell\}$, let $Y_2 = Y$. Otherwise, let Y_2 be as in (1). Then*

$$(p_{(d, s+i_1^{(0)})}(Y_2) \cap P^{(0)}(1)) P^{(0)}_{f_2^{(0)}}(1) = p_{(d, f_2^{(0)})}(Y_2) \cap P^{(0)}(1).$$

Proof. – Let $\mathcal{B}_j = \mathcal{B}_j^{(0)}$, $j \in \mathbf{Z}$, and $P_j = P_j^{(0)}$, $j \geq 1$.

(1) Apply ([Mor3], §4)

$$\mathcal{C}[c_1^{(0)}] = \mathrm{End}_F(V') \oplus (\mathcal{C}[c_1^{(0)}] \cap \mathrm{End}_F(V'')),$$

and Lemma 3.3(4) to obtain Y_1 and Y_2. Choose $R \in \mathcal{C}[c_1^{(0)}] \cap \mathcal{B}_{s+i_1^{(0)}}$ such that $y = e_d(Y) + R$ (as in the proof of Lemma 8.1). Arguing as for Y, we can write

$$R = R_1 + R_2 \text{ for some } R_1 \in \mathrm{End}_F(V') \cap \mathcal{B}_{s+i_1^{(0)}} \text{ and } R_2 \in \mathcal{B}_{s+i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}] \cap \mathrm{End}_F(V'').$$

Let $y_1 = e_d(Y_1) + R_1$ and $y_2 = e_d(Y_2) + R_2$. Note that $y_1 - 1 \in \mathrm{End}_F(V')$ and $y_2 - 1 \in \mathrm{End}_F(V'')$. Using the fact that the product of an element of $\mathrm{End}_F(V')$ with any element of $\mathrm{End}_F(V'')$ equals zero, we obtain

$$y_1 y_2 = (e_d(Y_1) + R_1)(e_d(Y_2) + R_2) = e_d(Y_1)e_d(Y_2) + R_1 + R_2 = e_d(Y_1 + Y_2) + R = y.$$

Because σ preserves $\mathbf{GL}(V')$ and $\mathbf{GL}(V'')$, it follows that $\sigma(y) = y^{-1}$ implies $\sigma(y_n) = y_n^{-1}$, $n = 1, 2$. Since Y_1 and Y_2 lie in different blocks, we have $Y^d = Y_1^d + Y_2^d \in \mathcal{B}_{s+i_1^{(0)}}$ implies $Y_1^d, Y_2^d \in \mathcal{B}_{s+i_1^{(0)}}$. Also $Y \in \mathcal{A}^{(0)}$ (by Lemma 8.1) implies $Y_1, Y_2 \in \mathcal{A}^{(0)}$. Therefore $p_{(d,s+i_1^{(0)})}(Y_n)$ is defined and $y_n \in p_{(d,s+i_1^{(0)})}(Y_n)$, $n = 1, 2$. By definition, $y_1 \in \mathbf{GL}(V')$ and $y_2 \in G^{(0)}(1)$.

Finally, if $X \in \mathcal{B}_1$, then $Y \in \mathcal{B}_1$ by Lemma 8.1. By Lemma 3.3(4), we have Y_1, $Y_2 \in \mathcal{B}_1$. That y_1, $y_2 \in P_1$ now follows from their definitions.

(2) Observe that $s + i_1^{(0)} \geq 2i_1^{(0)} = f_1^{(0)} - 1 \geq f_2^{(0)}$, which implies $P_{f_2^{(0)}} \supset P_{s+i_1^{(0)}}$ and $Y_2^d \in \mathcal{B}_{f_2^{(0)}}$. Suppose that $R_2 \in \mathcal{B}_{s+i_1^{(0)}} \cap \mathfrak{g}^{(0)}(1)$ and $\hat{R}_2 \in \mathcal{B}_{f_2^{(0)}} \cap \mathfrak{g}^{(0)}(1)$ are such that

$$y_2 \stackrel{def}{=} e_d(Y_2) + R_2 \in p_{(d,s+i_1^{(0)})}(Y_2) \cap P^{(0)}(1)$$
$$\hat{y}_2 \stackrel{def}{=} e_d(Y_2) + \hat{R}_2 \in p_{(d,f_2^{(0)})}(Y_2) \cap P^{(0)}(1).$$

Then

$$Y_2^d \in \mathcal{B}_{f_2^{(0)}},\quad R_2 \in \mathcal{B}_{f_2^{(0)}} \cap \mathfrak{g}^{(0)}(1),\quad y_2 \in P^{(0)}(1) \quad \Longrightarrow \quad y_2 \in p_{(d,f_2^{(0)})}(Y_2) \cap P^{(0)}(1).$$

This implies $y_2^{-1}\hat{y}_2 \in P_{f_2^{(0)}}(1)$. Thus

$$\hat{y}_2 \in y_2 P_{f_2^{(0)}} \subset (p_{(d,s+i_1^{(0)})}(Y_2) \cap G^{(0)}(1))P_{f_2^{(0)}}.$$

Note that both sets in (2) contain $P_{f_2^{(0)}}(1)$. As we have shown that y_2 and $\hat{y}_2$ belong to both sets, the desired equality holds. □

LEMMA 8.3. – *Suppose $X \in \mathcal{A}^{(0)-}$ is such that $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}}$ and $p_{(d,f_1^{(0)})}(X) \subset P_{\underline{\Psi}}$. Then, if $x \in p_{(d,f_1^{(0)})}(X)$,*

$$\frac{\chi_1^{(0)}(x)}{\chi_1^{(0)}(1)} = \begin{cases} \mathcal{I}(c_1'^{(0)}, X; P^{(0)}_{i_1^{(0)}}), & \text{if } f_1^{(0)} > 1, \\ \mathcal{I}(c_1'^{(0)}, X; P^{(0)}), & \text{if } f_1^{(0)} = 1. \end{cases}$$

Proof. – Set $\mathcal{A} = \mathcal{A}^{(0)}$, $\mathcal{B}_j = \mathcal{B}_j^{(0)}$, $j \in \mathbf{Z}$, and $P_j = P_j^{(0)}$, $j \geq 0$.

Suppose $n_0 = 1$ and $f_1^{(0)} = 1$. Then $r = \ell$ and $P_{\underline{\Psi}} = P(0) = P$. Since $x \in p_{(d,1)}(X)$, by (4.6) and $c_{\underline{\Psi}} = c_1'^{(0)}$,

$$\chi_1^{(0)}(x) = \chi_1^{(0)}(1) \int_P \Omega(\mathrm{tr}_0(c_1'^{(0)} \mathrm{Ad}\, h^{-1}(X)))\, dh = \mathcal{I}(c_{\underline{\Psi}}, X; P).$$

Henceforth we assume that $f_1^{(0)} > 1$. The proof is broken up into several steps.

Step 1. – Let $\psi_1^{(0)}$ be the linear character of $G^{(0)}(1)$ given in the definition of $\Psi^{(0)}$.

In the first step, $\psi_1^{(0)}$ is used to define a linear character of $(P_{\underline{\Psi}} \cap Z_G(c_1^{(0)}))P_{i_1'^{(0)}}$. In §6.10 of [Mor3], Morris defines a character $\phi_1^{(0)}$ of $(\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1))P_{i_1'^{(0)}}$ by:

$$\phi_1^{(0)} \mid (\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1)) = \psi_1^{(0)}$$
$$\phi_1^{(0)}(w) = \Omega(\mathrm{tr}_0(c_1'^{(0)}(w-1))), \qquad w \in P_{i_1'^{(0)}}.$$

If $r > \max\{1, \ell\}$, set $H = P_{\underline{\Psi}} \cap \mathbf{GL}(V')$ and $H_1 = H \cap P_1$. Because $H \subset P$, H normalizes $P_{i_1'^{(0)}}$. Also, $H \subset \mathbf{GL}(V')$ and $c_1'^{(0)} \in \mathrm{End}_F(V'')$ imply that $\mathrm{Ad}\, h(c_1'^{(0)}) = c_1'^{(0)}$. Thus,

$$\phi_1^{(0)}(h^{-1}wh) = \Omega(\mathrm{tr}_0(c_1'^{(0)}(h^{-1}wh-1))) = \Omega(\mathrm{tr}_0(c_1'^{(0)}(w-1))) = \phi_1^{(0)}(w),$$
$$h \in H,\ w \in P_{i_1'^{(0)}}.$$

As $H \subset \mathbf{GL}(V')$ and $G^{(0)}(1) \subset \mathbf{GL}(V'')$, it follows that H commutes with $G^{(0)}(1)$. Thus H normalizes $(\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1))P_{i_1'^{(0)}}$, and $\phi_1^{(0)}$ is trivial on the commutator and on the intersection of these two groups. This means that $\phi_1^{(0)}$ can be extended trivially across H to a character of $H(\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1))P_{i_1'^{(0)}}$. Set

$$L = \begin{cases} H \times (\widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1)), & \text{if } r > \max\{1, \ell\}, \\ \widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1), & \text{if } r = \max\{1, \ell\}, \end{cases}$$

and $L_1 = L \cap P_1$. By Lemma 6.1, $L = P_{\underline{\Psi}} \cap Z_G(c_1^{(0)})$ and $P_{\underline{\Psi}} = LP_{i_1^{(0)}}$. The character $\phi_1^{(0)}$ is defined on $LP_{i_1'^{(0)}}$.

The restriction of $\rho_1^{(0)}$ to $L_1 P_{i_1'^{(0)}}$ is a multiple of $\phi_1^{(0)}$. There are two cases to be considered. In the first case, $P_{i_1^{(0)}} = P_{i_1'^{(0)}}$ and $\rho_1^{(0)} = \phi_1^{(0)}$. In the second case, $P_{i_1^{(0)}} \neq P_{i_1'^{(0)}}$ and a Heisenberg construction is used to produce $\rho_1^{(0)}$. The next remarks show that both cases can occur.

Recall that $f_1^{(0)}$ is odd (*see* Remark(3) following the definition of $\Psi^{(0)}$, resp. Ψ_u, in §5 if $r > \ell$, resp. in §4 if $r = \ell$).

In certain cases, $\mathcal{B}_{2j-1} = \mathcal{B}_{2j} \neq \mathcal{B}_{2j+1}$, $j \in \mathbf{Z}$. For example, this happens if $r = \ell$ and the period $e(\mathcal{L}_u)$ of $\mathcal{L}_u$ is one, or if $\ell = 0$ and $r = 1$. In this situation, $\mathcal{B}_{i_1^{(0)}} = \mathcal{B}_{i_1'^{(0)}}$ is equivalent to $i_1^{(0)} = (f_1^{(0)} - 1)/2$ odd, that is, to $f_1^{(0)}$ congruent to 3 modulo 4. Otherwise, $\mathcal{B}_{i_1^{(0)}} \neq \mathcal{B}_{i_1'^{(0)}}$.

If $r = \ell$ and $e(\mathcal{L}_u) = 2$, or $r > \max\{1, \ell\}$, then in most cases $\mathcal{B}_{j-1} \neq \mathcal{B}_j$, $j \in \mathbf{Z}$. If so, $f_1^{(0)}$ odd implies $i_1^{(0)} = i_1'^{(0)} - 1$, so $\mathcal{B}_{i_1^{(0)}} \neq \mathcal{B}_{i_1'^{(0)}}$.

Step 2. – Let X and x be as in the statement of the lemma. By Lemma 6.1, $X \in \mathcal{A}$ and $x \in P_{\underline{\Psi}}$ imply that $x - 1 \in (\mathcal{A} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{B}_{i_1^{(0)}}$. Applying Lemma 3.7(2), and the fact that $|j!| = 1$ for $j \leq d-1$ (recall $d \leq p$), we see that $X \in (\mathcal{A} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{B}_{i_1^{(0)}}$. Thus we may apply Lemma 8.1.

Let Y, Z, s, y and z be as in Lemma 8.1. If $r > \max\{1, \ell\}$, apply Lemma 8.2(1) to write

$$\begin{aligned} &Y = Y_1 + Y_2, \quad Y_1 \in \mathrm{End}_F(V') \cap \mathcal{A}^-, \quad Y_2 \in \mathfrak{g}^{(0)}(1) \cap \mathcal{A}, \\ &y = y_1 y_2, \quad y_1 \in p_{(d, s+i_1^{(0)})}(Y_1) \cap \mathbf{GL}(V'), \quad y_2 \in p_{(d, s+i_1^{(0)})}(Y_2) \cap G^{(0)}(1). \end{aligned}$$

Also,

$$x = yz \in LP_{i_1^{(0)}},\ z \in P_{i_1^{(0)}},\ y \in Z_G(c_1^{(0)}) \Longrightarrow y \in LP_{i_1^{(0)}} \cap Z_G(c_1^{(0)}) = L.\ \text{(Lemma 6.1)}$$

Thus $y_1 \in L \cap \mathbf{GL}(V')$ and $y_2 \in L \cap G^{(0)}(1) = \widetilde{P}_{\Psi^{(0)}} \cap G^{(0)}(1)$. If $r = \max\{1, \ell\}$, set $Y_2 = Y$ and $y_2 = y$. The above statement that $y \in L$ is still valid.

Step 3. – Consider the case $P_{i_1^{(0)}} = P_{i_1'^{(0)}}$. Because $\mathcal{B}_{i_1^{(0)}} = \mathcal{B}_{i_1'^{(0)}}$, we have $Z \in \mathcal{B}_{i_1'^{(0)}}$. That is, $s = i_1'^{(0)}$. Recall from step 2 that $y_2 \in p_{(d, f_1^{(0)})}(Y_2)$. This implies that $\phi_1^{(0)}(y_2) = \Omega(\mathrm{tr}_0(c_1'^{(0)} Y_2))$ ((4.5) or (5.3)). Also, $z - 1 \in \mathcal{C}[c_1^{(0)}]^\perp + \mathcal{B}_{f_1^{(0)}}$ implies $\phi_1^{(0)}(z) = \Omega(\mathrm{tr}_0(c_1'^{(0)}(z-1))) = 1$. Since $\phi_1^{(0)}$ is trivial on H, $\phi_1^{(0)}(y_1) = 1$. In this case, $\rho_1^{(0)} = \phi_1^{(0)}$. Therefore $\rho_1^{(0)}(x) = \Omega(\mathrm{tr}_0(c_1'^{(0)} Y_2))$. Note that

$$\begin{aligned} &c_1'^{(0)} \in \mathrm{End}_F(V''),\ Y_1 \in \mathrm{End}_F(V') \Longrightarrow \mathrm{tr}_0(c_1'^{(0)} Y_1) = 0 \\ &c_1'^{(0)} \in \mathcal{C}[c_1^{(0)}],\ Z \in \mathcal{C}[c_1^{(0)}]^\perp \Longrightarrow \mathrm{tr}_0(c_1'^{(0)} Z) = 0. \end{aligned}$$

Thus $\mathrm{tr}_0(c_1'^{(0)} Y_2) = \mathrm{tr}_0(c_1'^{(0)} X)$. It follows that

$$\frac{\chi_1^{(0)}(x)}{\chi_1^{(0)}(1)} = \phi_1^{(0)}(x) = \Omega(\mathrm{tr}_0(c_1'^{(0)} X)).$$

Recall that ((7.1))

$$\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}}) = \int_{P_{i_1^{(0)}}} \Omega(\mathrm{tr}_0(c_1'^{(0)} \mathrm{Ad}\, h^{-1}(X)))\, dh.$$

Conjugation by $P_{i_1^{(0)}} = P_{i_1'^{(0)}}$ has no effect on $\chi_1^{(0)} = \rho_1^{(0)}$, and thus $\Omega(\mathrm{tr}_0(c_1'^{(0)} X)) = \mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}})$.

Step 4. – For the remainder of the proof, we assume that $P_{i_1^{(0)}} \neq P_{i_1'^{(0)}}$. In this case, a Heisenberg construction must be used to produce $\rho_1^{(0)}$ on all of $P_{\underline{\Psi}} = LP_{i_1^{(0)}}$.

Let $\ker \phi_1^{(0)}$ be the intersection of the kernel of $\phi_1^{(0)}$ with L_1. Morris ([Mor3], Corollary 6.11 and §6.16) showed that $(P_{\Psi^{(0)}} \cap G^{(0)}(1)) P_{i_1'^{(0)}} / \ker \phi_1^{(0)}$ is central in $P_{\Psi^{(0)}} / \ker \phi_1^{(0)}$. If $r > \max\{1, \ell\}$, then $H_1 \subset \ker \phi_1^{(0)}$. Hence

$$L_1 P_{i_1'^{(0)}} / \ker \phi_1^{(0)} = (P_{\Psi^{(0)}} \cap G^{(0)}(1) P_{i_1'^{(0)}} / \ker \phi_1^{(0)},$$

and

$$H_1 P_{\Psi^{(0)}} / \ker \phi_1^{(0)} = L_1 P_{i_1^{(0)}} / \ker \phi_1^{(0)} = P_{\Psi^{(0)}} / \ker \phi_1^{(0)}.$$

Therefore, $L_1P_{i_1'^{(0)}}/\ker\phi_1^{(0)}$ is central in $L_1P_{i_1^{(0)}}/\ker\phi_1^{(0)}$. From properties of the Heisenberg construction, it follows that $\rho_1^{(0)}\mid L_1P_{i_1^{(0)}}$ is the unique irreducible component of the representation $\delta_1^{(0)}$ of $L_1P_{i_1^{(0)}}$ induced from $\phi_1^{(0)}\mid L_1P_{i_1'^{(0)}}$.

$\chi_1^{(0)}\mid L_1P_{i_1^{(0)}}$ is therefore a positive multiple of the character of $\delta_1^{(0)}$. In particular,

$$\frac{\chi_1^{(0)}(w)}{\chi_1^{(0)}(1)}=\begin{cases}\phi_1^{(0)}(w), & \text{if } w\in L_1P_{i_1'^{(0)}},\\ 0 & \text{if } w\in L_1P_{i_1^{(0)}}-L_1P_{i_1'^{(0)}}.\end{cases}\tag{8.4}$$

To obtain $\rho_1^{(0)}$ on all of $LP_{i_1^{(0)}}$, the theory of the Weil representation is used to produce a unique extension $\rho_1^{(0)}$ of $\delta_1^{(0)}$. This will be discussed in step 5.

Suppose that $x\in L_1P_{i_1^{(0)}}$. Since $x\in P_1$, it is a consequence of Lemma 3.7(2) that $X\in\mathcal{B}_1$, and $y_1,\,y_2\in P_1$. As seen in step 2, $y\in L$. Thus $y_1,\,y_2\in L_1$. If $s=i_1^{(0)}$ (that is, $Z\neq\mathcal{B}_{i_1'^{(0)}}$), then $z\neq P_{i_1'^{(0)}}$, so $x\neq L_1P_{i_1'^{(0)}}$. By (8.4), $\chi_1^{(0)}(x)=0$. Applying Lemma 7.3 with $j=1$, $c'=c_1'^{(0)}$ and $K=P_{i_1^{(0)}}$, we get

$$\mathcal{I}(c_1'^{(0)},X;P_{i_1^{(0)}})=\mathcal{I}(c_1'^{(0)},X;P_{i_1^{(0)}}(X,\mathcal{S}_1)).$$

Because $X\in\mathcal{B}_1$, it follows that $\mathrm{Ad}\,w^{-1}(X)\in X+\mathcal{B}_{i_1'^{(0)}}$ for all $w\in P_{i_1^{(0)}}$. But $s=i_1^{(0)}$ implies that $X\neq\mathcal{S}_1$. The fact that $\mathcal{S}_1$ is invariant under addition of elements of $\mathcal{B}_{i_1'^{(0)}}$ then guarantees that $P_{i_1^{(0)}}(X,\mathcal{S}_1)=\emptyset$. Therefore, $\mathcal{I}(c_1'^{(0)},X;P_{i_1^{(0)}})=0$. This proves the lemma for $x\in L_1P_{i_1^{(0)}}-L_1P_{i_1'^{(0)}}$.

If $s=i_1'^{(0)}$, then $x\in L_1P_{i_1'^{(0)}}$ and exactly as in step 3, $\phi_1^{(0)}(x)=\Omega(\mathrm{tr}_0(c_1'^{(0)}X))$. Applying (8.4) and $\phi_1^{(0)}(h^{-1}xh)=\phi_1^{(0)}(x)$ for all $h\in P_{i_1^{(0)}}$, the lemma now follows for $x\in L_1P_{i_1'^{(0)}}$.

Step 5. – It remains to prove the lemma for $x\in LP_{i_1^{(0)}}-L_1P_{i_1'^{(0)}}$. (Recall that we are assuming that $P_{i_1^{(0)}}\neq P_{i_1^{(0)}}$.) The group L acts by conjugation on the Heisenberg group associated to $L_1P_{i_1^{(0)}}/\ker\phi_1^{(0)}$ and this action factors through the action of the symplectic group (over the finite field $\mathbf{F}_p$) associated to the Heisenberg group. Because $y\in p_{(d,s+i_1^{(0)})}(Y)$ and $Y^d\in\mathcal{B}_{s+i_1^{(0)}}\subset\mathcal{B}_1$, the image of y in this finite symplectic group is unipotent. Let U be the group generated by y and by L_1. The image of U in the finite symplectic group is the cyclic group generated by the image of the element y. Remarks on page 295 of [H1] imply that

$$\rho_1^{(0)}\mid UP_{i_1^{(0)}}=Ind_{UJ}^{UP_{i_1^{(0)}}}(\bar{\phi}_1^{(0)}),$$

where J is a subgroup of $P_{i_1^{(0)}}$ containing $P_{i_1'^{(0)}}$, and the image of L_1J in $L_1P_{i_1^{(0)}}/\ker\phi_1^{(0)}$ is a maximal abelian subgroup which is fixed under the action induced by conjugation by y. Also, $\bar{\phi}_1^{(0)}$ is any character of UJ which coincides with $\phi_1^{(0)}$ on $UP_{i_1'^{(0)}}$. Set

$$\bar{\phi}_1^{(0)}(C(W))=\Omega(-2\,\mathrm{tr}_0(c_1'^{(0)}W)),\qquad W\in C^{-1}(J).$$

Recall that C is the Cayley transform. After checking that $\bar{\phi}_1^{(0)}\,|\,U\cap J = \phi_1^{(0)}\,|\,U\cap J$, (Lemma 8.5(2)) $\bar{\phi}_1^{(0)}$ can be extended to all of UJ by taking $\bar{\phi}_1^{(0)}\,|\,U = \phi_1^{(0)}\,|\,U$. As will be shown in Lemma 8.5(3), $\bar{\phi}_1^{(0)}$ is a representation of UJ.

Set $\mathcal{F}(x) = \{\, w \in UP_{i_1^{(0)}} \mid w^{-1}xw \in UJ \,\}$. From the Frobenius formula for the character of an induced representation,

$$\frac{\chi_1^{(0)}(x)}{\chi_1^{(0)}(1)} = \int_{\mathcal{F}(x)} \bar{\phi}_1^{(0)}(w^{-1}xw)\, dw.$$

Since conjugation by U fixes $\bar{\phi}_1^{(0)}$ and U normalizes $P_{i_1^{(0)}}$, $\mathcal{F}(x)$ may be replaced by $\mathcal{F}_1(x) = \mathcal{F}(x) \cap P_{i_1^{(0)}}$. The choice of the subgroups U and J depends only on y modulo L_1, so if x is replaced by $w^{-1}xw$, $w \in \mathcal{F}_1(x)$, nothing is changed. Thus there is no loss of generality in assuming that $x \in UJ$. In Lemma 8.7, we will prove that

$$\bar{\phi}_1^{(0)}(w^{-1}xw) = \bar{\phi}_1^{(0)}(x)\Omega(\mathrm{tr}_0(c_1'^{(0)}(\mathrm{Ad}\, w^{-1}(X) - X))), \qquad w \in \mathcal{F}_1(x)$$

and also

$$\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}}) = \mathcal{I}(c_1'^{(0)}, X; \mathcal{F}_1(x)).$$

Hence

$$\frac{\chi_1^{(0)}(x)}{\chi_1^{(0)}(1)} = \bar{\phi}_1^{(0)}(x)\Omega(-\mathrm{tr}_0(c_1'^{(0)}X))\,\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}}).$$

Howe ([H1]) proved that $\chi_1^{(0)}$ is supported on conjugacy classes in $UP_{i_1^{(0)}}$ which intersect $UP_{i_1'^{(0)}}$. Replacing x by $w^{-1}xw$ for some $w \in P_{i_1^{(0)}}$ such that $w^{-1}xw \in UP_{i_1'^{(0)}}$ if necessary, we see that to complete the proof, it suffices to show that $\bar{\phi}_1^{(0)}(x) = \Omega(\mathrm{tr}_0(c_1'^{(0)}X))$ when $x \in UP_{i_1'^{(0)}}$. Recall that $\bar{\phi}_1^{(0)}\,|\,UP_{i_1'^{(0)}} = \phi_1^{(0)}\,|\,UP_{i_1'^{(0)}}$. From $s = i_1'^{(0)}$ (Lemma 8.1), we conclude that

$$\phi_1^{(0)}(y_2) = \psi_1^{(0)}(y_2) = \Omega(\mathrm{tr}_0(c_1'^{(0)}Y_2))$$
$$\phi_1^{(0)}(z) = \Omega(\mathrm{tr}_0(c_1'^{(0)}(z-1))) = 1.$$

It follows that

$$\bar{\phi}_1^{(0)}(x) = \phi_1^{(0)}(y_2)\phi_1^{(0)}(z) = \Omega(\mathrm{tr}_0(c_1'^{(0)}Y_2)) = \Omega(\mathrm{tr}_0(c_1'^{(0)}X)). \qquad \square$$

LEMMA 8.5. – *Let U, J, L_1, $\phi_1^{(0)}$ and $\bar{\phi}_1^{(0)}$ be as in the proof of Lemma 8.3. Set $\mathcal{J} = C^{-1}(J)$. Then*

(1) *$\frac{1}{m}\mathcal{J} = \mathcal{J}$ for any positive integer m which is not divisible by p.*

(2) *$\bar{\phi}_1^{(0)}$ coincides with $\phi_1^{(0)}$ on $U \cap J$.*

(3) *$\bar{\phi}_1^{(0)}$ is a representation of UJ.*

Proof. – (1) Note that $m\mathcal{J} \subset \mathcal{J}$ (J is a group) implies $\mathcal{J} = (1/m)(m\mathcal{J}) \subset (1/m)\mathcal{J}$. Thus

$$\mathcal{J} \subset (1/m)\mathcal{J} \subset (1/m)^s\mathcal{J}, \qquad s \geq 1.$$

Since $1/m \in O^\times$, there exists s_0 such that $(1/m)^{s_0} \in 1+\mathfrak{p}$. From $\mathcal{J} \subset \mathcal{B}_{i_1^{(0)}}$, it follows that $(1+\mathfrak{p})\mathcal{J} = \mathcal{J}$. Hence $(1/m)^{s_0}\mathcal{J} = \mathcal{J}$. Setting $s = s_0$ above, we obtain (1).

(2) By definition of U, $Z_G(c_1^{(0)}) \cap P_{i_1^{(0)}} \subset U \subset Z_G(c_1^{(0)})$. Also, $J \subset P_{i_1^{(0)}}$. Thus $U \cap J = Z_G(c_1^{(0)}) \cap J$. Let $w \in U \cap J$. Set $W = C^{-1}(w)$. Since $\mathcal{J} \subset \mathcal{B}_{i_1^{(0)}}$, it follows that $W^3 \in \mathcal{B}_{3i_1^{(0)}} \subset \mathcal{B}_{f_1^{(0)}}$. This implies that $w = C(W) \in p_{(d,f_1^{(0)})}(-2W)$. By the same type of argument as for Lemma 8.2(1), if $r > \max\{1,\ell\}$, we can write

$$\begin{aligned} &W = W_1 + W_2, \quad W_1 \in \operatorname{End}_F(V'), \quad W_2 \in \mathfrak{g}^{(0)}(1),\\ &w = w_1 w_2 \qquad w_1 \in p_{(d,f_1^{(0)})}(-2W_1) \cap \mathbf{GL}(V') \quad w_2 \in p_{(d,f_1^{(0)})}(-2W_2) \cap G^{(0)}(1). \end{aligned}$$

Then $\phi_1^{(0)}(w) = \phi_1^{(0)}(w_2) = \Omega(-2\mathrm{tr}_0(c_1'^{(0)}W_2))$ by (4.5) or (5.3). If $r = \max\{1,\ell\}$, then (4.5) or (5.3) may be applied directly. In any case, $\phi_1^{(0)}(w) = \Omega(-2\mathrm{tr}_0(c_1'^{(0)}W))$, for $w = C(W) \in U \cap J$. Recall that $\bar{\phi}_1^{(0)}(c(W)) = \Omega(\mathrm{tr}_0(-2c_1'^{(0)}W))$, $W \in \mathcal{J}$.

(3) By (2), it suffices to show that $\bar{\phi}_1^{(0)} \mid J$ is a representation. Let $w_1 = c(W_1)$, $w_2 = c(W_2) \in J$. Then

$$\begin{aligned} &\bar{\phi}_1^{(0)}(w_1w_2)\bar{\phi}_1^{(0)}(w_1)^{-1}\bar{\phi}_1^{(0)}(w_2)^{-1}\\ &\quad = \bar{\phi}_1^{(0)}(C(W_1+W_2-[W_1,W_2]))\Omega(-2\,\mathrm{tr}_0(c_1'^{(0)}(W_1+W_2)))^{-1}\\ &\quad = \Omega(2\,\mathrm{tr}_0(c_1'^{(0)}[W_1,W_2])) = \phi_1^{(0)}(C(-[W_1,W_2])). \end{aligned}$$

Set $\tilde{w}_1 = C(W_1/2)$. By (1), $\tilde{w}_1 \in J$. By definition of J, the element $\tilde{w}_1^{-1}w_2^{-1}\tilde{w}_1w_2$ belongs to the kernel of $\phi_1^{(0)}$. It is easily checked that

$$\tilde{w}_1^{-1}w_2^{-1}\tilde{w}_1w_2 \in C(-[W_1,W_2])P_{f_1^{(0)}}.$$

Therefore $\phi_1^{(0)}(C(-[W_1,W_2])) = 1$, which implies (*see* above) that $\bar{\phi}_1^{(0)}$ is a representation. □

The next lemma will be used in the proof of Lemma 8.7, which itself consists of a proof of facts used in the proof of Lemma 8.3. Lemmas 8.6 and 8.7 are similar to Lemmas 3.17 and 3.20 of [Mu1]. The main difference is that in Lemma 8.7, use of Lemma 8.1 is avoided where it is not necessary, which makes some of the formulas look simpler than those in the proof of Lemma 3.20 of [Mu1].

LEMMA 8.6. – *Assume* $f_1^{(0)} > 1$. *Let* $X \in \mathcal{A}^{(0)-} \cap (\mathcal{C}[c_1^{(0)}] + \mathcal{B}_{i_1^{(0)}})$, *and let* Y, Z, x, y, *and* z *be as in Lemma 8.1. Assume* $x \in P_{\underline{\Psi}}$. *Let* $\mathcal{B}_j = \mathcal{B}_j^{(0)}$, $j \in \mathbf{Z}$. *Set* $\mathcal{D} = (\mathcal{B}_{i_1^{(0)}} \cap \operatorname{End}_F(V')) + \mathcal{B}_{i_1'^{(0)}}$ *if* $r > \max\{1,\ell\}$, *and* $\mathcal{D} = \mathcal{B}_{i_1'^{(0)}}$ *otherwise. Then*

$$(1) \qquad \operatorname{Ad} x^{-1}(R) \in \sum_{m=0}^{d-1} \frac{(-1)^m}{m!}(\operatorname{ad} X)^m(R) + \mathcal{D}^-, \qquad R \in \mathcal{B}_{i_1^{(0)}}^-$$

$$(2)\qquad [X,R]\in\sum_{m=1}^{d-1}\frac{(-1)^m}{m}(\operatorname{Ad}x^{-1}-1)^m(R)+\mathcal{D}^-,\qquad R\in\mathcal{B}^-_{i_1^{(0)}}$$

$$(3)\qquad z\in C\left(\frac{-1}{2}\sum_{m=0}^{d-2}\frac{(-1)^{m+1}}{(m+1)!}(\operatorname{ad}X)^m(Z)+\mathcal{B}^-_{i_1'^{(0)}}\right)$$

$$(4)\qquad Z\in-2\sum_{m=0}^{d-2}\frac{(-1)^m}{m+1}(\operatorname{Ad}x^{-1}-1)^m(C^{-1}(z))+\mathcal{B}^-_{i_1'^{(0)}}.$$

Remarks. – (a) In (1)-(4), x and X may be replaced by y and Y, respectively. This is immediate from $y\in xP_1$ and $Y-X\in\mathcal{B}_1^-$.

(b) Suppose $r>\max\{1,\ell\}$. In (3) and (4), there is no $\mathcal{B}_{i_1^{(0)}}\cap\operatorname{End}_F(V')$-component because $\operatorname{End}_F(V')\subset\mathcal{C}[c_1^{(0)}]$ and Z, $z-1\in\mathcal{C}[c_1^{(0)}]^\perp+\mathcal{B}_{i_1'^{(0)}}$. Thus $\mathcal{B}^-_{i_1'^{(0)}}$ appears, not $\mathcal{D}^-$.

Proof. – Analogues of these results were proved in [Mu1] with y and Y replacing x and X. The difference here is that r may exceed $\max\{1,\ell\}$. (For the general linear group, $r=1$.) The proofs differ very little from those in [Mu1], so we include only the proof of (1). Let $R\in\mathcal{B}^-_{i_1^{(0)}}$. Since $x\in p_{(d,f_1^{(0)})}(X)$,

$$\begin{aligned}\operatorname{Ad}x^{-1}(R)&\in\left(\sum_{t=0}^{d-1}\frac{(-X)^t}{t!}\right)R\left(\sum_{u=0}^{d-1}\frac{X^u}{u!}\right)+\mathcal{B}_{i_1^{(0)}+f_1^{(0)}}\\&=\sum_{u=0}^{d-1}\sum_{t=0}^{d-1}\frac{(-X)^tRX^u}{t!u!}+\mathcal{B}_{i_1^{(0)}+f_1^{(0)}}.\end{aligned}$$

Isolating those terms with $t+u\le d-1$, we get

$$\begin{aligned}\sum_{t=0}^{d-1}\sum_{u=0}^{d-1-t}\frac{(-X)^tRX^u}{t!u!}&=\sum_{m=0}^{d-1}\sum_{n=0}^{m}\frac{(-1)^n}{n!(m-n)!}X^nRX^{m-n}\\&=\sum_{m=0}^{d-1}\frac{(-1)^m}{m!}\left((-1)^m\sum_{n=0}^{m}(-1)^n\binom{m}{n}X^nRX^{m-n}\right)\\&=\sum_{m=0}^{d-1}\frac{(-1)^m}{m!}(\operatorname{ad}X)^m(R).\end{aligned}$$

To complete the proof of (1), it remains to show that the sum S of the terms $(-X)^tRX^u/(t!u!)$ over $1\le t,u\le d-1$ and $t+u\ge d$, belongs to $\mathcal{D}$. As $|t!u!|=1$ for $1\le t,u\le d-1$, we need only prove that $X^tRX^u\in\mathcal{D}$ for $t+u\ge d$. At this point, it is convenient to work with Y rather than X. Because $Y-X\in\mathcal{B}_1$, we have $X^tRX^u-Y^tRY^u\in\mathcal{B}_{i_1^{(0)}+1}=\mathcal{B}_{i_1'^{(0)}}$. We will show that $Y^tRY^u\in\mathcal{D}$.

If $r > \max\{1, \ell\}$, by Lemma 8.2(1),

$$Y = Y_1 + Y_2, \quad Y_1 \in \mathcal{A}^{(0)-} \cap \mathrm{End}_F(V'), \quad Y_2 \in \mathcal{A}^{(0)-} \cap \mathfrak{g}^{(0)}(1).$$

Let $d_1 = \dim V'$ and $d_2 = \dim V''$. Otherwise, set $Y_2 = Y$, $d_2 = d$ and $d_1 = 0$.

Let $F_1 = F(c_1^{(0)})$ and $a = [F_1 : F]$. By definition, $Y_2 \in \mathcal{C}[c_1^{(0)}] \cap \mathrm{End}_F(V'') = \mathrm{End}_{F_1}(V'')$. By Lemma 3.3(8), $\mathcal{A}^{(0)} \cap \mathrm{End}_{F_1}(V'')$ is a principal order in $\mathrm{End}_{F_1}(V'')$ with Jacobson radical $\mathcal{B}_1 \cap \mathrm{End}_{k_1}(V'')$. Therefore ([BF]) the quotient $(\mathcal{A}^{(0)} \cap \mathrm{End}_{F_1}(V''))/(\mathcal{B}_1 \cap \mathrm{End}_{F_1}(V''))$ is isomorphic to a subalgebra of the algebra $\mathcal{M}$ of d_2/a by d_2/a matrices with entries in the residue class field of F_1. Since $Y_2^d \in \mathcal{B}_{f_1^{(0)}-1} \subset \mathcal{B}_1$, the image of Y_2 in this quotient is nilpotent. Any nilpotent element of $\mathcal{M}$ has order at most d_2/a. Thus $Y_2^{d_2/a} \in \mathcal{B}_1$. As $a \geq 2$, we have $[d_2/2] \geq d_2/a$, and so $Y_2^{[d_2/2]} \in \mathcal{B}_1$.

If $d_1 > 0$, then, since $Y_1^d \in \mathcal{B}_{f_1^{(0)}-1} \subset \mathcal{B}_1$, the image of Y_1 in the quotient $(\mathcal{A}^{(0)} \cap \mathrm{End}_F(V'))/(\mathcal{B}_1 \cap \mathrm{End}_F(V'))$ is nilpotent. This quotient is isomorphic to a subalgebra of d_1 by d_1 matrices with entries in the residue class field of F ([Mor2], 2.13). Hence $Y_1^{d_1} \in \mathcal{B}_1$.

Suppose $d_1 = 0$. The inequality $t + u \geq d = d_2$ implies that at least one of t and u is greater than or equal to $[d_2/2]$, so at least one of Y^t and Y^u belongs to $\mathcal{B}_1^{(0)}$. Thus $Y^t R Y^u \in \mathcal{B}_{i_1^{(0)}+1} = \mathcal{B}_{i_1'^{(0)}} = \mathcal{D}$.

Suppose $d_1 > 0$. If $t < [d_2/2]$, then $d \leq t + u < [d_2/2] + u$ implies that $u \geq d_1 + [(d_2+1)/2]$, that is, $Y^u \in \mathcal{B}_1$. Similarly, $u < [d_2/2]$ implies $Y^t \in \mathcal{B}_1$. Therefore, if one of u and t is less than $[d_2/2]$, $Y^t R Y^u \in \mathcal{B}_{i_1'^{(0)}} \subset \mathcal{D}$. Finally, if t, $u \geq [d_2/2]$, then Y_2^t, $Y_2^u \in \mathcal{B}_1$ implies that $Y^t R Y^u \in Y_1^t R Y_1^u + \mathcal{B}_{i_1'^{(0)}}$. Since Y_1^t and Y_1^u belong to $\mathcal{A}^{(0)} \cap \mathrm{End}_F(V')$, we have

$$Y_1^t R Y_1^u \in \mathcal{B}_{i_1^{(0)}} \cap \mathrm{End}_F(V') \subset \mathcal{D}, \qquad R \in \mathcal{B}_{i_1^{(0)}}. \qquad \square$$

LEMMA 8.7. – *Let X and x be as in Lemma 8.3. Suppose that $x \in P_{\underline{\Psi}} - L_1 P_{i_1^{(0)}}$. Let U, J, $\bar{\phi}_1^{(0)}$, $\mathcal{F}_1(x)$, etc. be as in the proof of step 5 of Lemma 8.3. Assume that $x \in UJ$. Then*

(1) $\bar{\phi}_1^{(0)}(w^{-1}xw) = \bar{\phi}_1^{(0)}(x)\Omega(\mathrm{tr}_0(c_1'^{(0)}(\mathrm{Ad}\, w^{-1}(X) - X))), \qquad w \in \mathcal{F}_1(x)$.

(2) $\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}}) = \mathcal{I}(c_1'^{(0)}, X; \mathcal{F}_1(x))$.

Proof. – We start with some preliminary results.

Let $u \in U$ and $w \in J$ be such that $uw \in P_{i_1^{(0)}}$. By definition of J, resp. U, $w \in P_{i_1^{(0)}}$, resp. $u \in Z_G(c_1^{(0)})$. Thus $u \in P_{i_1^{(0)}} \cap Z_G(c_1^{(0)})$. On the other hand, $P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}) \subset L_1 \subset U$. Therefore

$$(8.8) \qquad (UJ) \cap P_{i_1^{(0)}} = (UL_1) \cap P_{i_1^{(0)}} = (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J.$$

Recall that the image of $L_1 J$ in $L_1 P_{i_1^{(0)}}/\ker \phi_1^{(0)}$ is a maximal abelian subgroup which is fixed under the action induced by conjugation by y. Since $x \in yP_1$, it follows that x normalizes $(P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J$. Thus

$$(8.9\text{ i}) \qquad \mathrm{Ad}\, x^{-1}((\mathcal{B}_{i_1^{(0)}}^- \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}) \subset (\mathcal{B}_{i_1^{(0)}}^- \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}.$$

Let $\mathcal{D}$ be as in Lemma 8.6. By definition, $\mathcal{D} \subset \mathcal{C}[c_1^{(0)}] + \mathcal{B}_{i_1'^{(0)}}$. Applying Lemma 8.5(1), and Lemma 8.6(2), we obtain

$$\text{(8.9 ii)} \qquad \operatorname{ad} X((\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}) \subset (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}.$$

Let $w \in \mathcal{F}_1(x)$. Recall that this means $w \in P_{i_1^{(0)}}$ and $wxw^{-1} \in UJ$. By assumption, $x \in UJ$. Since $x \in P^{(0)}$ and $w \in P_{i_1^{(0)}}$, $x^{-1}wx \in P_{i_1^{(0)}}$. Thus $x^{-1}w^{-1}xw \in UJ \cap P_{i_1^{(0)}} = (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J$ (by (8.8)). Indeed,

$$\text{(8.10)} \qquad \begin{aligned} x^{-1}w^{-1}xw &= C(-\operatorname{Ad} x^{-1}(W))C(W) \\ &\in C(W - \operatorname{Ad} x^{-1}(W) + [\operatorname{Ad} x^{-1}(W), W])P_{f_1^{(0)}}. \end{aligned}$$

Note that

$$x^{-1}w^{-1}xw \in (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J \Longrightarrow W - \operatorname{Ad} x^{-1}(W) \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}.$$

Combining this with Lemma 8.5(1) and Lemma 8.6(2) results in

$$\text{(8.11)} \qquad [X, W] \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}.$$

The next step is to prove that

$$\text{(8.12)} \qquad \Omega(\operatorname{tr}_0(c_1'^{(0)}\,[(\operatorname{ad} X)^m(R), W])) = 1, \qquad m \geq 1,\ R \in (\mathcal{C}[c_1^{(0)}] \cap \mathcal{B}_{i_1^{(0)}}) + \mathcal{J}.$$

Because $L_1 J$ is abelian modulo the kernel of $\phi_1^{(0)}$, it follows that

(8.13) $\Omega(\operatorname{tr}_0(c_1'^{(0)}\cdot))$ is trivial on the commutator of $(\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}$.

Suppose that m and R are as in (8.12). It follows from (8.9 ii) and (8.13) that

$$\Omega\Big(\operatorname{tr}_0(c_1'^{(0)}[(\operatorname{ad} X)^{m-1}(R), [W, X]])\Big) = 1.$$

Furthermore, writing $X = Y + Z$, $Y \in \mathcal{C}[c_1^{(0)}]$, $Z \in \mathcal{B}^-_{i_1^{(0)}}$, as in Lemma 8.1,

$$\begin{aligned} &\operatorname{tr}_0(c_1'^{(0)}[X, [(\operatorname{ad} X)^{m-1}(R), W]]) \\ &\quad \in \operatorname{tr}_0(c_1'^{(0)}[Y, [(\operatorname{ad} Y)^{m-1}(R), W]]) + \operatorname{tr}_0(c_1'\mathcal{B}^-_{f_1^{(0)}}) \subset \mathcal{O}_0. \end{aligned}$$

Here we have used $\operatorname{tr}_0(c_1'^{(0)}[Y, X']) = \operatorname{tr}_0([c_1'^{(0)}, Y]X') = 0$ for all $X' \in \mathfrak{g}$. Note that

$$[(\operatorname{ad} X)^m(R), W] = [X, [(\operatorname{ad} X)^{m-1}(R), W]] + [(\operatorname{ad} X)^{m-1}(R), [W, X]].$$

Combining this with the above remarks yields (8.12).

By definition of U, $y \in U$. Let y and z be as in Lemma 8.1. Therefore $x = yz \in UJ$ implies that $z \in UJ$. By Lemma 8.1, $z \in P_{i_1^{(0)}}$. Thus $z \in (UJ) \cap P_{i_1^{(0)}} = (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J$, by (8.8). This implies $C^{-1}(z) \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}$. Combining this with Lemma 8.5(1), (8.9 i), and Lemma 8.6(4), we get

$$Z \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}. \tag{8.14}$$

Fix $w = C(W) \in \mathcal{F}_1(x)$. Recall that $y \in Z_G(c_1^{(0)})$ and $c_1'^{(0)}$ belongs to the centre of $Z_G(c_1^{(0)})$. Thus $\operatorname{Ad} y(c_1'^{(0)}) = c_1'^{(0)}$. Together with properties of tr_0, this implies that

$$\begin{aligned}\operatorname{tr}_0(c_1'^{(0)}(W - \operatorname{Ad} x^{-1}(W))) &= \operatorname{tr}_0(c_1'^{(0)}(\operatorname{Ad} z(W) - \operatorname{Ad} y^{-1}(W)))\\ &= \operatorname{tr}_0(c_1'^{(0)}(\operatorname{Ad} z(W) - W)).\end{aligned}$$

Let $Z_1 = C^{-1}(z)$. Then $Z_1 \in \mathcal{B}^-_{i_1^{(0)}}$ implies $\operatorname{Ad} z(W) - W \in -2[Z_1, W] + \mathcal{B}_{f_1^{(0)}}$. Therefore

$$\begin{aligned}\Omega(-2\operatorname{tr}_0(c_1'^{(0)}(W - \operatorname{Ad} x^{-1}(W)))) &= \Omega(4\operatorname{tr}_0(c_1'^{(0)}([Z_1, W])))\\ &= \prod_{m=0}^{d-2} \Omega\left(\operatorname{tr}_0\left(c_1'^{(0)}\left[(\operatorname{ad} X)^m\left(\frac{(-1)^m 2Z}{(m+1)!}\right), W\right]\right)\right).\end{aligned}$$

For the second equality above, we have used Lemma 8.6(3) and the fact that $W \in \mathcal{B}^-_{i_1^{(0)}}$ implies $[\mathcal{B}^-_{i_1'^{(0)}}, W] \in \mathcal{B}^-_{i_1^{(0)}+i_1'^{(0)}} = \mathcal{B}^-_{f_1^{(0)}}$. By (8.14) and Lemma 8.5(1),

$$\frac{(-1)^m 2Z}{(m+1)!} \in (\mathcal{B}^-_{i_1^{(0)}} \cap C[c_1^{(0)}]) \cap \mathcal{J}, \qquad 1 \le m \le d-2.$$

Applying (8.12) (with $R = (-1)^m 2Z/(m+1)!$), we conclude that

$$\begin{aligned}\Omega(-2\operatorname{tr}_0(c_1'^{(0)}(W - \operatorname{Ad} x^{-1}(W))) &= \Omega(-2\operatorname{tr}_0(c_1'^{(0)}([Z, W])))\\ &= \Omega(-2\operatorname{tr}_0(c_1'^{(0)}[X, W])).\end{aligned} \tag{8.15}$$

Here we have used $\operatorname{tr}_0(c_1'^{(0)}[Y, W]) = \operatorname{tr}_0([c_1'^{(0)}, Y]W) = 0$ (recall $[c_1'^{(0)}, Y] = 0$).

From the definition of $\mathcal{D}$, and $\operatorname{End}_F(V') \subset \mathcal{C}[c_1^{(0)}]$,

$$\operatorname{tr}_0(c_1'^{(0)}[\mathcal{D}^-, W]) = \operatorname{tr}_0([c_1', \mathcal{D}^-]W) \subset \operatorname{tr}_0([c_1^{(0)}, \mathcal{B}^-_{i_1'^{(0)}}]\mathcal{B}^-_{i_1^{(0)}}) \subset \operatorname{tr}_0(c_1'\mathcal{B}^-_{f_1^{(0)}}) \subset O_0,$$

Applying Lemma 8.6(1), we obtain

$$\begin{aligned}\Omega(-2\operatorname{tr}_0(c_1'^{(0)}[\operatorname{Ad} x^{-1}(W), W])) &= \Omega(-2\operatorname{tr}_0(c_1'^{(0)}[\operatorname{Ad} x^{-1}(W) - W, W]))\\ &= \prod_{m=1}^{d-1} \Omega\left(-2\operatorname{tr}_0\left(c_1'^{(0)}\left[(\operatorname{ad} X)^m\left(\frac{(-1)^m W}{m!}\right), W\right]\right)\right).\end{aligned}$$

Let $R = (-1)^m \operatorname{ad} X(W)/m!$. By (8.11) and Lemma 8.5(1), $R \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}$. Applying (8.12) with $m - 1$ instead of m, we see that the terms corresponding to $2 \le m \le d-1$ in the above product are equal to one. Thus

$$\Omega(-2\operatorname{tr}_0(c_1'^{(0)}[\operatorname{Ad} x^{-1}(W), W])) = \Omega(-2\operatorname{tr}_0(c_1'^{(0)}[W, [X, W]])). \tag{8.16}$$

Proof of (1). – By definition of $\bar{\phi}_1^{(0)}$, and (8.10),

$$\bar{\phi}_1^{(0)}(x^{-1}w^{-1}xw) = \Omega(-2\,\mathrm{tr}_0(c_1'^{(0)}(W - \mathrm{Ad}\,x^{-1}(W) + [\mathrm{Ad}\,x^{-1}(W), W]))).$$

Using (8.15) and (8.16) this can be rewritten as

$$\bar{\phi}_1^{(0)}(x^{-1}w^{-1}xw) = \Omega(-2\,\mathrm{tr}_0(c_1'^{(0)}([X,W] + [W,[X,W]]))).$$

Putting this together with

$$\mathrm{Ad}\,w^{-1}(X) \in X + 2[W,X] + 2[W,[W,X]] + \mathcal{B}^-_{f_1^{(0)}}$$

we obtain (1).

Proof of (2). – Introducing an integration over J into $\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}})$ results in

$$\begin{aligned}\mathcal{I}(c_1'^{(0)}, X; P_{i_1^{(0)}}) &= \int_{P_{i_1^{(0)}}} \left(\int_J \Omega(\mathrm{tr}_0(c_1'^{(0)}\mathrm{Ad}\,w^{-1}\mathrm{Ad}\,h^{-1}(X)))\,dw \right) dh \\ &= \int_{P_{i_1^{(0)}}} \mathcal{I}(c_1'^{(0)}, \mathrm{Ad}\,h^{-1}(X); J)\,dh.\end{aligned}$$

Fix $h \in P_{i_1^{(0)}}$ and set $\widetilde{X} = \mathrm{Ad}\,h^{-1}(X)$. We will show that if $\mathcal{I}(c_1'^{(0)}, \widetilde{X}; J) \neq 0$, then $h \in \mathcal{F}_1(x)$. Let $W = C^{-1}(w)$, $w \in J$. Then

$$\mathcal{I}(c_1'^{(0)}, \widetilde{X}; J) = \Omega(\mathrm{tr}_0(c_1'^{(0)}\widetilde{X}) \int_{\mathcal{J}} \Omega(\mathrm{tr}_0(c_1'^{(0)}(2[W,\widetilde{X}] + 2[W,[W,\widetilde{X}]])))\,dw.$$

By definition, $\widetilde{X} \in X + \mathcal{B}^-_{i_1^{(0)}}$. It follows from (8.9 ii) and (8.13) that

$$\Omega(\mathrm{tr}_0(c_1'^{(0)}2[W,[W,\widetilde{X}]])) = \Omega(\mathrm{tr}_0(c_1'^{(0)}2[W,[W,X]])) = 1, \qquad W \in \mathcal{J}.$$

Write $\widetilde{X} = \widetilde{Y} + \widetilde{Z}$, with $\widetilde{Y}$ and $\widetilde{Z}$ playing the same role relative to $\widetilde{X}$ that Y and Z do relative to X (*see* Lemma 8.1). Similarly, write $\widetilde{x} = \widetilde{y}\widetilde{z}$ for $\widetilde{x} \in p_{(d,f_1^{(0)})}(\widetilde{X})$. Since $\widetilde{Y} \in \mathcal{C}[c_1^{(0)}]$, $\mathrm{tr}_0(c_1'^{(0)}[W,\widetilde{X}]) = \mathrm{tr}_0(c_1'^{(0)}[W,\widetilde{Z}])$. We have

$$\mathcal{I}(c_1'^{(0)}, \widetilde{X}; J) = \Omega(\mathrm{tr}_0(c_1'^{(0)}\widetilde{X})) \int_{\mathcal{J}} \Omega(\mathrm{tr}_0(c_1'^{(0)}2[W,\widetilde{Z}]))\,dW.$$

Suppose that the above integral is nonvanishing. Then $\Omega(\mathrm{tr}_0(c_1'^{(0)}2[W,\widetilde{Z}])$ must equal one for all $W \in \mathcal{J}$. However, it is easily checked that

$$\phi_1^{(0)}(C(-\widetilde{Z})w^{-1}C(\widetilde{Z})w) = \Omega(\mathrm{tr}_0(c_1'^{(0)}2[W,\widetilde{Z}])), \qquad w = C(W) \in J.$$

By definition of J, L_1J is a maximal abelian subgroup modulo the kernel of $\phi_1^{(0)}$. Thus we must have $C(\widetilde{Z}) \in L_1J \cap P_{i_1^{(0)}} = (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J$ (by 8.8). This implies that $\widetilde{Z} \in (\mathcal{B}^-_{i_1^{(0)}} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{J}$. Applying Lemma 8.5(1), (8.9 ii) and Lemma 8.6(3), we conclude that $\widetilde{z} \in (P_{i_1^{(0)}} \cap Z_G(c_1^{(0)}))J = (L_1J) \cap P_{i_1^{(0)}}$. From

$$\widetilde{y} = \widetilde{x}\widetilde{z}^{-1} = y(y^{-1}h^{-1}yh)(h^{-1}zh)\widetilde{z}^{-1} \in yP_{i_1^{(0)}},$$

and $y, \widetilde{y} \in Z_G(c_1^{(0)})$, we get $\widetilde{y} \in y(P_{i_1^{(0)}} \cap Z_G(c_1^{(0)})) \subset U$. We have shown that if $\mathcal{I}(c_1'^{(0)}, \widetilde{X}; J) \neq 0$, then $h^{-1}xh = \widetilde{x} = \widetilde{y}\widetilde{z} \in UJ$. That is, $h \in \mathcal{F}_1(x)$. □

9. The character of $\rho_{\underline{\Psi}}$

In this section, we show that, on the nilpotent set, the composition of the character $\chi_{\underline{\Psi}}$ of $\rho_{\underline{\Psi}}$ with the exponential map, agrees (up to degree) with the $\operatorname{Ad} P_{\underline{\Psi}}$-orbit of the linear functional $\Omega(\operatorname{tr}_0(c_{\underline{\Psi}}\cdot))$.

If χ is the character of a representation of $P_{\underline{\Psi}}$, let $\dot{\chi}$ be the function on G which is equal to χ on $P_{\underline{\Psi}}$ and vanishes elsewhere.

Recall that $\underline{\Psi}$ is said to be uniform if $r \le \ell + 1$, or $f_1^{(t)} \ge 2\,e(E_{r-t}/F) + 1$ for all $t \le r - \ell - 2$.

PROPOSITION 9.1. – *Assume that $\underline{\Psi}$ is uniform. Suppose $X \in \mathcal{N}$ and $x = \exp X$. Then*

$$\dot{\chi}_{\underline{\Psi}}(x) = \chi_{\underline{\Psi}}(1)\,\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}).$$

Remark. – For $x = \exp X \in P_{\underline{\Psi}}$ and $X \in \mathcal{N}$, the above relation between $\chi_{\underline{\Psi}}(x)$ and the value of the $\operatorname{Ad} P_{\underline{\Psi}}$-orbit of $\Omega(\operatorname{tr}_0(c_{\underline{\Psi}}\cdot))$ at X is similar to Howe's Kirillov theory for compact groups ([H2], Theorem 1.1).

Proof. – Let $d = 2[\dim_F(V)/2] + 1$. Recall that $d \le p$. Since $X \in \mathcal{N}$ is equivalent to $X^d = 0$, it follows that $\exp X = e_d(X)$ and $\exp X \in p_{(d,i)}(X)$ for any $i \ge 1$. Thus (*see* Lemma 3.7(2)), $X \in \mathcal{A}^{(0)}$ is equivalent to $x \in P^{(0)}$. If $X \ne \mathcal{A}^{(0)}$, by Corollary 7.13, $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) = 0$. Also, $x \ne P^{(0)}$ implies $x \ne P_{\underline{\Psi}}$. Therefore, both sides of the above equality vanish if $X \ne \mathcal{A}^{(0)}$.

Because it suffices to prove the proposition for $X \in \mathcal{N} \cap \mathcal{A}^{(0)}$, and $X \in \mathcal{N}$ implies $X^d = 0 \in \mathcal{B}^{(0)}_{f_1^{(0)}}$, the proposition is implied by the following statement:

(9.2) *Suppose $X \in \mathcal{A}^{(0)-}$ is such that $X^d \in \mathcal{B}^{(0)}_{f_1^{(0)}}$. Then $x \in p_{(d,f_1^{(0)})}(X)$ implies $\dot{\chi}_{\underline{\Psi}}(x) = \chi_{\underline{\Psi}}(1)\,\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}})$.*

We shall prove (9.2). Let X and x be as in (9.2). Set

$$\mathcal{S} = (\mathcal{A}^{(0)-} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{B}^{(0)-}_{i_1^{(0)}}.$$

If $x \in (P^{(0)} \cap Z_G(c_1^{(0)}))P^{(0)}_{i_1^{(0)}}$, then $x-1 \in (\mathcal{A}^{(0)} \cap \mathcal{C}[c_1^{(0)}]) + \mathcal{B}^{(0)}_{i_1^{(0)}}$, which, by Lemma 3.7(2), implies $X \in \mathcal{S}$. Conversely, if $X \in \mathcal{S}$, by Lemma 8.1, $x \in (P^{(0)} \cap Z_G(c_1^{(0)}))P^{(0)}_{i_1^{(0)}}$. As $\dot{\chi}_{\underline{\Psi}}(x) = 0$ if $x \ne (P^{(0)} \cap Z_G(c_1^{(0)}))P^{(0)}_{i_1^{(0)}}$ (Lemma 6.1(2)) and $\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}) = 0$ if $X \ne \mathcal{S}$ (Corollary 7.9 if $f_1^{(0)} > 1$), there is no loss of generality in assuming that $X \in \mathcal{S}$.

Suppose $n_0 = 1$ and the length of $\underline{\Psi}$ is one. If $f_1^{(0)} = 1$, then (9.2) is equivalent to Lemma 8.2. Assume $f_1^{(0)} > 1$. Then $c_{\underline{\Psi}} = c_1'^{(0)}$, $\rho_{\underline{\Psi}} = \rho_1^{(0)}$ and $P_{\underline{\Psi}} = T^{(0)}P^{(0)}_{i_1^{(0)}} = Z_G(c_1^{(0)})P^{(0)}_{i_1^{(0)}}$, so (9.2) is again equivalent to Lemma 8.2.

From now on we assume that if $n_0 = 1$ the length of $\underline{\Psi}$ exceeds one. If $n_0 > 1$, $\dot{\chi}_2^{(0)}$ is supported on $P_{\underline{\Psi}}$, and if $r > \max\{1, \ell\}$, $\dot{\chi}_{\Psi^{(1)}}$ is supported on $P_{\underline{\Psi}}$. Thus, applying Lemma 8.2,

$$\dot{\chi}_{\underline{\Psi}}(x) = \chi_1^{(0)}(1)\,\mathcal{I}(c_1'^{(0)}, X; P^{(0)}_{i_1^{(0)}})\,(\dot{\chi}_2^{(0)} \cdots \dot{\chi}_{n_0}^{(0)} \dot{\chi}_{\Psi^{(1)}} \cdots \dot{\chi}_{\Psi^{(r-\ell)}})(x).$$

Here, it is understood that if $n_0 = 1$, then there are no terms $\dot\rho_j^{(0)}$, $j \geq 2$, and if $r = \ell$, then there are no terms $\dot\chi_{\Psi^{(j)}}$, $j \geq 1$. Because $P^{(0)}_{i_1^{(0)}}$ is a subgroup of $P_{\underline{\Psi}}$ and characters are class functions, the value of $\dot\chi_j^{(0)}$, $j \geq 2$, or $\dot\chi_{\Psi^{(t)}}$, $t \geq 1$ at x is the same as at $w^{-1}xw$, $w \in P^{(0)}_{i_1^{(0)}}$. Therefore, the integration over $P^{(0)}_{i_1^{(0)}}$ above extends to give

$$(9.3)\quad \dot\chi_{\underline{\Psi}}(x) = \chi_1^{(0)}(1) \int_{P^{(0)}_{i_1^{(0)}}} \Omega(\mathrm{tr}_0(c_1'^{(0)} \mathrm{Ad}\, w^{-1}(X)))\, (\dot\chi_2^{(0)} \cdots \dot\chi_{n_0}^{(0)}) \times (\dot\chi_{\Psi^{(1)}} \cdots \dot\chi_{\Psi^{(r-\ell)}})(w^{-1}xw)\, dw.$$

For the remainder of the proof, assume that $2 \leq j \leq n_0$ and $1 \leq t \leq r - \max\{1, \ell\}$.

Fix $w \in P^{(0)}_{i_1^{(0)}}$, and set $\widetilde{x} = w^{-1}xw$. Let $\widetilde{X}$, $\widetilde{Z}$, $\widetilde{y}$, $\widetilde{z}$ and s be as in Lemma 8.1. As was shown in Lemmas 6.2(1) and 6.3(2),

$$\rho_j^{(0)}(\widetilde{z}) = \Omega(\mathrm{tr}_0(c_j'^{(0)}(\widetilde{z} - 1)))\rho_j^{(0)}(1),$$
$$\rho_{\Psi^{(t)}}(\widetilde{z}) = \Omega(\mathrm{tr}_0(c_{\Psi^{(t)}}(\widetilde{z} - 1)))\rho_{\Psi^{(t)}}(1).$$

By Lemma 8.1(1),

$$\widetilde{z} - 1 \in \mathcal{C}[c_1^{(0)}]^{\perp} + \mathcal{B}^{(0)}_{f_1^{(0)}-1}.$$

Recall that $f_j^{(0)} \leq f_1^{(0)} - 1$. Thus

$$c_j'^{(0)} \in \mathcal{C}[c_1^{(0)}] \quad \text{and} \quad \mathrm{tr}_0(c_j'^{(0)}\mathcal{B}^{(0)}_{f_1^{(0)}-1}) \subset \mathrm{tr}_0(c_j'^{(0)}\mathcal{B}^{(0)}_{f_j^{(0)}}) \subset O_0,$$

imply that $\rho_j^{(0)}(\widetilde{z}) = \rho_j^{(0)}(1)$.

Given the form of $\widetilde{z} - 1$ (*see* above),

$$c_{\Psi^{(t)}} \in \mathcal{C}[c_1^{(0)}], \quad \mathrm{tr}_0(c_{\Psi^{(t)}}V'') = 0, \quad \mathcal{B}^{(0)-}_{f_1^{(0)}-1} \cap \mathrm{End}_F(V') \subset \mathcal{B}^{(1)-}_{f_1^{(1)}}$$

and orthogonality of $\mathrm{End}_F(V)^-$ and $\mathrm{End}_F(V)^+$ with respect to tr_0 ([Mor3], 4.17) imply that

$$\mathrm{tr}_0(c_{\Psi^{(t)}}(\widetilde{z} - 1)) \in \mathrm{tr}_0(c_{\Psi^{(t)}}(\mathcal{B}^{(1)-}_{f_1^{(1)}} + \mathrm{End}_F(V')^+)) \subset O_0.$$

Therefore $\rho_{\Psi^{(t)}}(\widetilde{z}) = \rho_{\Psi^{(t)}}(1)$.

Because $\widetilde{z} \in P^{(0)}_{i_1^{(0)}}$ and $\widetilde{y} \in Z_G(c_1^{(0)})$, by Lemma 6.1,

$$(9.4)\qquad \widetilde{x} \in P_{\underline{\Psi}} \iff \widetilde{y} \in \begin{cases} P_{\underline{\Psi}'} \times P_{\underline{\Psi}''}, & \text{if } r > \max\{1, \ell\} \text{ and } n_0 > 1, \\ P_{\underline{\Psi}'} \times T^{(0)}, & \text{if } r > \max\{1, \ell\} \text{ and } n_0 = 1, \\ P_{\underline{\Psi}''}, & \text{if } r = \max\{1, \ell\} \text{ and } n_0 > 1. \end{cases}$$

Combining this with the fact that $\rho_j^{(0)}$ and $\rho_{\Psi^{(t)}}$ are trivial at $\widetilde{z}$, results in

$$(9.5)\qquad \dot\chi_j^{(0)}(\widetilde{x}) = \dot\chi_j^{(0)}(\widetilde{y}),$$
$$\dot\chi_{\Psi^{(t)}}(\widetilde{x}) = \dot\chi_{\Psi^{(t)}}(\widetilde{y}).$$

If $r > \max\{1,\ell\}$, applying Lemma 8.2(1) allows us to write

$$\widetilde{Y} = \widetilde{Y}_1 + \widetilde{Y}_2, \quad \widetilde{Y}_1 \in \mathrm{End}_F(V') \cap \mathcal{A}^{(0)-}, \quad \widetilde{Y}_2 \in \mathcal{A}^{(0)} \cap \mathfrak{g}^{(0)}(1),$$
$$\widetilde{y} = \widetilde{y}_1\widetilde{y}_2, \quad \widetilde{y}_1 \in p_{(d,s+i_1^{(0)})}(\widetilde{Y}_1) \cap \mathbf{GL}(V'), \quad \widetilde{y}_2 \in p_{(d,s+i_1^{(0)})}(\widetilde{Y}_2) \cap P^{(0)}(1).$$

If $r = \max\{1,\ell\}$, set $\widetilde{y}_2 = \widetilde{y}$ and $\widetilde{Y}_2 = \widetilde{Y}$. If $r > \max\{1,\ell\}$ and $n_0 > 1$, $\rho_j^{(0)} \mid P_{\underline{\Psi}'} = \rho_j^{(0)}(1)$ (Lemma 6.2(3)). Thus if $\widetilde{y} \in P_{\underline{\Psi}}$, we have

$$\rho_j^{(0)}(\widetilde{y}) = \rho_j^{(0)}(\widetilde{y}_2).$$

If $r > \max\{1,\ell\}$, then, applying Lemma 6.3(2), if $\widetilde{y} \in P_{\underline{\Psi}}$,

$$\rho_{\Psi^{(t)}}(\widetilde{y}) = \rho_{\Psi^{(t)}}(\widetilde{y}_1)\Omega(\mathrm{tr}_0(c_{\Psi^{(t)}}(\widetilde{y}_2 - 1))) = \rho_{\Psi^{(t)}}(\widetilde{y}_1),$$

the final equality resulting from $\mathrm{tr}_0(c_{\Psi^{(t)}}V'') = \{0\}$.

We may now write

$$(9.6\ \mathrm{i}) \qquad (\dot{\chi}_2^{(0)} \cdots \dot{\chi}_{n_0}^{(0)})(\widetilde{x}) = (\dot{\chi}_2^{(0)} \cdots \dot{\chi}_{n_0}^{(0)})(\widetilde{y}_2) = \dot{\chi}_{\underline{\Psi}''}(\widetilde{y}_2)$$
$$(9.6\ \mathrm{ii}) \qquad (\dot{\chi}_{\Psi^{(1)}} \cdots \dot{\chi}_{\Psi^{(r-\ell)}})(\widetilde{x}) = (\dot{\chi}_{\Psi^{(1)}} \cdots \dot{\chi}_{\Psi^{(r-\ell)}})(\widetilde{y}_1) = \dot{\chi}_{\underline{\Psi}'}(\widetilde{y}_1).$$

In obtaining (9.6), we have used (9.5), (9.4) and Lemmas 6.2(1) and 6.3(1).

Assume $n_0 > 1$. By Lemma 8.2(2), $p_{(d,f_2^{(0)})}(\widetilde{Y}_2)$ is defined, and

$$(p_{(d,s+i_1^{(0)})}(\widetilde{Y}_2) \cap G^{(0)}(1))P^{(0)}_{f_2^{(0)}}(1) = p_{(d,f_2^{(0)})}(\widetilde{Y}_2) \cap P^{(0)}(1).$$

Thus $\widetilde{y}_2 \in p_{(d,f_2^{(0)})}(\widetilde{Y}_2)$. This means that the hypotheses of (9.2) are satisfied by $\widetilde{Y}_2$ and $\widetilde{y}_2$, relative to the cuspidal datum $\underline{\Psi}''$. Note that $\underline{\Psi}''$, being of length one, is uniform. The rank of $\underline{\Psi}''$ is $n_0 - 1$. Recall that (9.2) has already been proved for rank one. By induction on rank, we may assume that

$$(9.7) \qquad \dot{\chi}_{\underline{\Psi}''}(\widetilde{y}_2) = \chi_{\underline{\Psi}''}(1)\mathcal{I}(c_{\underline{\Psi}''}, \widetilde{Y}_2; P_{\underline{\Psi}''}).$$

From $P_{\underline{\Psi}''} \subset Z_G(c_1^{(0)}) \cap \mathbf{GL}(V'')$, it follows that $\mathrm{Ad}\, P_{\underline{\Psi}''}$ stabilizes both $\mathcal{C}[c_1^{(0)}]^{\perp}$ and $\mathrm{End}_F(V')$. Also, $\mathrm{tr}_0(c_{\underline{\Psi}''}\mathcal{C}[c_1^{(0)}]^{\perp}) = \{0\}$ and $\mathrm{tr}_0(c_{\underline{\Psi}''}\mathrm{End}_F(V')) = \{0\}$. Thus

$$\mathrm{tr}_0(c_{\underline{\Psi}''}(\mathrm{Ad}\, u^{-1}(\widetilde{Y}_2))) = \mathrm{tr}_0(c_{\underline{\Psi}''}(\mathrm{Ad}\, u^{-1}(\widetilde{X}))), \qquad u \in P_{\underline{\Psi}''}.$$

Putting this together with (9.6 i) and (9.7), we obtain

$$(9.8) \qquad (\dot{\chi}_2^{(0)} \cdots \dot{\chi}_{n_0}^{(0)})(\widetilde{x}) = \chi_{\underline{\Psi}''}(1)\mathcal{I}(c_{\underline{\Psi}''}, \widetilde{X}; P_{\underline{\Psi}''}).$$

Suppose that $\underline{\Psi}$ has length one, that is, $r = \max\{1,\ell\}$. We still assume that $n_0 > 1$. In view of (9.8), (9.3) becomes

$$\dot{\chi}_{\underline{\Psi}}(x) = \chi_1^{(0)}(1)\chi_{\underline{\Psi}''}(1)\int_{P^{(0)}_{i_1^{(0)}}} \Omega(\mathrm{tr}_0(c_1'^{(0)}\mathrm{Ad}\, w^{-1}(X)))\mathcal{I}(c_{\underline{\Psi}''}, \mathrm{Ad}\, w^{-1}(X); P_{\underline{\Psi}''})\, dw$$
$$= \chi_{\underline{\Psi}}(1)\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}}).$$

Here, we have used the facts $P_{\underline{\Psi}''} \subset Z_G(c_1^{(0)})$, $c_1'^{(0)}$ belongs to the centre of $Z_G(c_1^{(0)})$, $c_{\underline{\Psi}} = c_1'^{(0)} + c_{\underline{\Psi}''}$, and Lemma 6.1(1). Hence (9.2) holds for a cuspidal datum of length one.

We will now complete the proof of (9.2) via induction on length. Assume that the length of $\underline{\Psi}$ exceeds one ($r > \max\{1,\ell\}$), and that (9.2) holds for any uniform cuspidal datum of length less than $r+1-\max\{1,\ell\}$. By definition of uniform, $\underline{\Psi}$ uniform implies $\underline{\Psi}'$ is uniform. Note that n_0 may equal one. Assume $X \in \mathcal{S}$. Fix $w \in P^{(0)}_{i_1^{(0)}}$, and define $\widetilde{x}$, $\widetilde{X}$, etc. as before. By Lemma 3.7(2), $\widetilde{y}_1 \in P^{(1)}$ is equivalent to $\widetilde{Y}_1 \in \mathcal{A}^{(1)}$. If $\widetilde{Y}_1 \not\in \mathcal{A}^{(1)}$, then d odd implies that $\widetilde{Y}_1^d \in \mathfrak{g}$, and from

$$\widetilde{Y}_1^d \in (\mathcal{B}^{(0)-}_{f_1^{(0)}-1} \cap \operatorname{End}_F(V')) \subset \mathcal{B}^{(1)-}_{f_1^{(1)}}$$

and Corollary 7.13, applied to $c_{\underline{\Psi}'}$, $\widetilde{Y}_1$, and $P_{\underline{\Psi}'}$, we get $\mathcal{I}(c_{\underline{\Psi}'}, \widetilde{Y}_1; P_{\underline{\Psi}'}) = 0$. Furthermore, as $\widetilde{y}_1 \not\in P^{(1)}$ and $P_{\underline{\Psi}'} \subset P^{(1)}$, we have $\dot{\chi}_{\underline{\Psi}'}(\widetilde{y}_1) = 0$. If $\widetilde{Y}_1 \in \mathcal{A}^{(1)}$, then $\widetilde{Y}_1^d \in \mathcal{B}^{(1)-}_{f_1^{(1)}}$ and $P^{(1)}_{f_1^{(1)}} \subset P_{\underline{\Psi}'}$ imply that $\widetilde{y}_1 \in p_{(d,f_1^{(1)})}(\widetilde{Y}_1) \cap P^{(1)}$. By induction on length, we may assume that (9.2) holds when applied to $\dot{\chi}_{\underline{\Psi}'}(\widetilde{y}_1)$.

We conclude that, regardless of whether or not $\widetilde{Y}_1 \in \mathcal{A}^{(1)}$,

$$\dot{\chi}_{\underline{\Psi}'}(\widetilde{y}_1) = \chi_{\underline{\Psi}'}(1)\mathcal{I}(c_{\underline{\Psi}'}, \widetilde{Y}_1; P_{\underline{\Psi}}). \tag{9.9}$$

Because $P_{\underline{\Psi}'} \subset \operatorname{End}_F(V')$, $\operatorname{Ad} P_{\underline{\Psi}'}$ stabilizes $\mathcal{C}[c_1^{(0)}]^\perp$ and $\operatorname{End}_F(V'')$. Also,

$$c_{\underline{\Psi}'} \in \mathcal{C}[c_1^{(0)}] \quad \text{and} \quad c_{\underline{\Psi}'} \in \operatorname{End}_F(V') \subset \operatorname{End}_F(V'')^\perp.$$

Thus

$$\operatorname{tr}_0(c_{\underline{\Psi}'}(\operatorname{Ad} u^{-1}(\widetilde{Y}_1))) = \operatorname{tr}_0(c_{\underline{\Psi}'}(\operatorname{Ad} u^{-1}(\widetilde{X}))), \qquad u \in P_{\underline{\Psi}'}.$$

Along with (9.6 ii) and (9.9), this yields

$$(\dot{\chi}_{\Psi^{(1)}} \cdots \dot{\chi}_{\Psi^{(r-\ell)}})(\widetilde{x}) = \chi_{\underline{\Psi}'}(1)\mathcal{I}(c_{\underline{\Psi}'}, \widetilde{X}; P_{\underline{\Psi}'}).$$

It follows that (*see* also (9.8) if $n_0 > 1$), (9.3) reduces to

$$\begin{aligned}\dot{\chi}_{\underline{\Psi}}(x) =& \chi_1^{(0)}(1)\chi_{\underline{\Psi}''}(1)\chi_{\underline{\Psi}'}(1) \int_{P^{(0)}_{i_1^{(0)}}} \Omega(\operatorname{tr}_0(c_1'^{(0)} \operatorname{Ad} w^{-1}(X))) \\ &\times \mathcal{I}(c_{\underline{\Psi}'}, \operatorname{Ad} w^{-1}(X); P_{\underline{\Psi}'})\mathcal{I}(c_{\underline{\Psi}''}, \operatorname{Ad} w^{-1}(X); P_{\underline{\Psi}''})\, dw,\end{aligned} \tag{9.10}$$

where it should be understood that if $n_0 = 1$, no terms involving $\underline{\Psi}''$ appear. $P_{\underline{\Psi}'}$ and $P_{\underline{\Psi}''}$ commute with $c_1'^{(0)}$, so both of these integrations can be extended to include the part of the integrand involving $\Omega(\operatorname{tr}_0(c_1'^{(0)} \cdot))$.

If $n_0 = 1$ (recall $f_1^{(0)} > 1$), then $c_{\underline{\Psi}} = c_1'^{(0)} + c_{\underline{\Psi}'}$ and we have

$$\dot{\chi}_{\underline{\Psi}}(x) = \chi_{\underline{\Psi}}(1)\mathcal{I}(c_{\underline{\Psi}}, X; P_{\underline{\Psi}'}P^{(0)}_{i_1^{(0)}}).$$

To finish the proof in this case, it suffices to recall that $T^{(0)}$ commutes with $c_{\underline{\Psi}}$, allowing us to replace $P_{\underline{\Psi}'}P^{(0)}_{i_1^{(0)}}$ by $P_{\underline{\Psi}}$ (Lemma 6.1).

Suppose $n_0 > 1$. Since $P_{\underline{\Psi}''}$, resp. $P_{\underline{\Psi}'}$, commutes with $c_{\underline{\Psi}'}$, resp. $c_{\underline{\Psi}''}$, we have

$$\mathcal{I}(c_{\underline{\Psi}'}, \operatorname{Ad} w^{-1}(X); P_{\underline{\Psi}'})\mathcal{I}(c_{\underline{\Psi}''}, \operatorname{Ad} w^{-1}(X); P_{\underline{\Psi}''}) = \mathcal{I}(c_{\underline{\Psi}'}+c_{\underline{\Psi}''}, \operatorname{Ad} w^{-1}(X); P_{\underline{\Psi}'}\times P_{\underline{\Psi}''})$$

for all $w \in P^{(0)}_{i_1^{(0)}}$. Combining this with (9.10) and comments following it, Lemma 6.1, and

$$c_{\underline{\Psi}} = c_{\underline{\Psi}'} + c_{\underline{\Psi}''} + c_1'^{(0)},$$

we obtain (9.2). □

10. Main Theorem

The character Θ_π of an irreducible supercuspidal representation π as a locally integrable function on G which is locally constant on the regular set. This was proved by Harish-Chandra ([HC2]) for G connected, and generalized to non-connected G by Clozel ([C]). Let $G^0 = \mathbf{G}^0(F_0)$, where $\mathbf{G}^0$ is the identity component of $\mathbf{G}$.

Let $\underline{\Psi}$ be a cuspidal datum as in §5, and let $\rho_{\underline{\Psi}}$ be the representation of $P_{\underline{\Psi}}$ constructed by Morris. As shown in §7 of [Mor3], $\pi_{\underline{\Psi}} = Ind^G_{P_{\underline{\Psi}}}\rho_{\underline{\Psi}}$ is irreducible and supercuspidal. Recall (§2) that we are assuming $p > \dim_F(V)$.

Since G has compact centre, a Haar measure on G yields a G-invariant measure on the $\operatorname{Ad} G$-orbit of a regular elliptic element in $\mathfrak{g}$. We assume that all measures on regular elliptic $\operatorname{Ad} G$-orbits, and also the formal degrees of supercuspidal representations of G, are defined relative to the same Haar measure.

THEOREM 10.1. – *Let $\pi = \pi_{\underline{\Psi}}$. Assume that $\underline{\Psi}$ is uniform. Then there exists an open neighbourhood $\mathcal{V}_\pi$ of zero in $\mathfrak{g}$ and a regular elliptic element $X_\pi \in \mathfrak{g}$ such that*

$$\Theta_\pi(\exp X) = d(\pi)\,\widehat{\mu}_{\mathcal{O}(X_\pi)}(X), \qquad X \in \mathcal{V}_\pi \cap \mathfrak{g}_{reg}.$$

X_π can be taken equal to $c_{\underline{\Psi}}$, where $c_{\underline{\Psi}}$ is as in (5.7). Here, $d(\pi)$ denotes the formal degree of π.

Remarks. – (1) As can be seen from the proof, $\mathcal{V}_\pi$ is taken small enough that the exponential map is defined on $\mathcal{V}_\pi$.

(2) The requirement that $\underline{\Psi}$ be uniform (that is, $f_1^{(t)} \geq 2e(E_{r-t}/F)+1$, $0 \leq t \leq r-\ell-2$) is necessary because the proof of Proposition 9.1 depends on Corollary 7.13, which was proved only for uniform $\underline{\Psi}$. If the conclusion of Corollary 7.13 can be verified for each of the cuspidal data $\{\Psi^{(t)}, \ldots, \Psi^{(r-\max\{1,\ell\})}\}$, $0 \leq t \leq r-\ell-2$, then Proposition 9.1 and hence Theorem 10.1, hold without the assumption that $\underline{\Psi}$ is uniform.

Proof. – Suppose for the moment that G is not connected. The restriction π_0 of π to G^0 is a finite direct sum of irreducible supercuspidal representations of G^0. Recall that the function $\dot{\chi}_{\underline{\Psi}}$ equals $\chi_{\underline{\Psi}}$ on $P_{\underline{\Psi}}$ and zero elsewhere in G. The restriction of $\dot{\chi}_{\underline{\Psi}}$ to

G^0 is a finite sum of matrix coefficients of π_0. Harish-Chandra's integral formula for the character of supercuspidal representation on the regular set ([HC1], p. 60) applies to a finite direct sum of supercuspidal representations of G^0, hence to π_0. Because $\pi_0^g = \pi_0$ for all $g \in G$, it is easy to see that, the integral formula for Θ_{π_0} is actually an integral formula for the restriction of Θ_π to G^0.

Let K_0 be an open compact subgroup of G. If G is not connected, take K_0 to be a subgroup of G^0. Then if $x \in G^0$ and x is regular, Harish-Chandra's formula for $\Theta_\pi(x)$ is as follows:

$$\Theta_\pi(x) = \frac{d(\pi)}{\chi_{\underline{\Psi}}(1)} \int_G \int_{K_0} \dot{\chi}_{\underline{\Psi}}(y^{-1}h^{-1}xhy)\, dh\, dy.$$

Let $\mathfrak{g}^*$ be an open and closed $\operatorname{Ad} G$-invariant subset of $\mathfrak{g}$ containing zero, such that $\exp : \mathfrak{g}^* \to G$ is a homeomorphism of $\mathfrak{g}^*$ onto an open subset of G, and

$$\exp(\operatorname{Ad} x(X)) = x(\exp X)x^{-1}, \qquad x \in G, \quad X \in \mathfrak{g}^*.$$

Let log denote the inverse of exp. Recall that $\mathcal{N}$ denotes the set of nilpotent elements in $\mathfrak{g}$. Let $\mathcal{V}_0$ be an open neighbourhood of zero in $\mathfrak{g}$ satisfying:

(i) $\mathcal{V}_0 \subset \mathcal{B}^{(0)-}_{f_1^{(0)}}$

(ii) $(\mathcal{A}^{(0)} \cap \mathcal{N}) + \mathcal{V}_0 \subset \mathfrak{g}^*$ and $N \in \mathcal{A}^{(0)} \cap \mathcal{N} \Longrightarrow \exp(N + \mathcal{V}_0) \subset (\exp N)P^{(0)}_{f_1^{(0)}}$.

Clearly (i) holds for any sufficently small neighbourhood of zero. Because $\mathfrak{g}^*$ contains zero, $\mathfrak{g}^*$ is open and $\operatorname{Ad} G$-invariant, and zero belongs to the closure of every nilpotent $\operatorname{Ad} G$-orbit, it follows that $\mathcal{N} \subset \mathfrak{g}^*$. Since $\mathcal{A}^{(0)} \cap \mathcal{N}$ is compact and exp is continuous, if $\mathcal{V}_0$ is small enough, (ii) holds. Take $\mathcal{V}_0$ to be the intersection of two open neighbourhoods of zero satisfying (i) and (ii), respectively.

Next, we take $\mathcal{V}_1$ to be an open neighbourhood of zero such that:

(iii) $\mathcal{V}_1 \subset \mathfrak{g}^*$

(iv) $\operatorname{Ad} G(\mathcal{V}_1) \subset \mathcal{N} + \mathcal{V}_0$

(v) $\log(P^{(0)} \cap \exp(\operatorname{Ad} G(\mathcal{V}_1))) \subset \mathcal{A}^{(0)}$.

We indicate why such a $\mathcal{V}_1$ exists. By Lemma 13 of [C], there exists a lattice L such that $\operatorname{Ad} G(\mathcal{V}_0) \subset \mathcal{N} + L$. Let ϖ_0 be a prime element in $\mathfrak{p}_0$. Choose $m \geq 0$ such that $\varpi_0^m L \subset \mathcal{V}_0$. Note that $\varpi_0^m \mathcal{N} = \mathcal{N}$. Thus, if we assume that $\mathcal{V}_1 \subset \varpi_0^m \mathcal{V}_0$, then (iv) holds. Let $\widetilde{\mathcal{N}}$ be the nilpotent subset of $\operatorname{End}_F(V)$. Since $d \leq \dim_F(V) + 1 \leq p$, we have $\log(1 + \widetilde{\mathcal{N}} \cap \mathcal{A}^{(0)}) \subset \mathcal{A}^{(0)}$. This implies that, if $\widetilde{\mathcal{V}}_0$ is a small enough neighbourhood of zero in $\operatorname{End}_F(V)$, then

$$\log(\mathcal{A}^{(0)} \cap (1 + \widetilde{\mathcal{N}} + \widetilde{\mathcal{V}}_0)) \subset \mathcal{A}^{(0)}.$$

Arguing as above, we choose an open neighbourhood $\widetilde{\mathcal{V}}_1$ of zero in $\operatorname{End}_F(V)$ such that

$$\operatorname{Ad} \mathbf{GL}(V)(\widetilde{\mathcal{V}}_1) \subset \widetilde{\mathcal{N}} + \widetilde{\mathcal{V}}_0.$$

If $\mathcal{V}_1$ is taken so that $\exp \mathcal{V}_1 \subset 1 + \widetilde{\mathcal{V}}_1$, then, by definition of $\widetilde{\mathcal{V}}_1$,

$$P^{(0)} \cap \exp(\operatorname{Ad} G(\mathcal{V}_1)) \subset \mathcal{A}^{(0)} \cap (1 + \widetilde{\mathcal{N}} + \widetilde{\mathcal{V}}_0).$$

By definition of $\widetilde{\mathcal{V}}_0$, this is enough to guarantee that $\mathcal{V}_1$ satisfies (v).

Fix $X \in \mathcal{V}_1 \cap \mathfrak{g}_{reg}$, $y \in G$, and $h \in K_0$. Let $x = \exp X$. By condition (iv), $\mathrm{Ad}\,(hy)^{-1}(X) = N + X_0$, where $N \in \mathcal{N}$ and $X_0 \in \mathcal{V}_0$. Recall that $P^{(0)}_{f_1^{(0)}}$ is contained in the kernel of $\rho_{\underline{\Psi}}$. Therefore, by (ii) and Proposition 9.1,

$$N \in \mathcal{A}^{(0)} \Longrightarrow \dot{\chi}_{\underline{\Psi}}(y^{-1}h^{-1}xhy) = \dot{\chi}_{\underline{\Psi}}(\exp(\mathrm{Ad}\,(hy)^{-1}(X))) \\ = \dot{\chi}_{\underline{\Psi}}(\exp N) = \chi_{\underline{\Psi}}(1)\,\mathcal{I}(c_{\underline{\Psi}}, N; P_{\underline{\Psi}}).$$

If $N \neq \mathcal{A}^{(0)}$, then (v) implies that $y^{-1}h^{-1}xhy \neq P^{(0)}$. Since $P^{(0)} \supset P_{\underline{\Psi}}$, we have $\dot{\chi}_{\underline{\Psi}}(y^{-1}h^{-1}xhy) = 0$. Since $N \in \mathcal{N}$ and $\underline{\Psi}$ is uniform, Corollary 7.13 applies, so $\mathcal{I}(c_{\underline{\Psi}}, N; P_{\underline{\Psi}}) = 0$. Therefore,

$$\dot{\chi}_{\underline{\Psi}}(y^{-1}h^{-1}xhy) = \chi_{\underline{\Psi}}(1)\,\mathcal{I}(c_{\underline{\Psi}}, N; P_{\underline{\Psi}}) = \chi_{\underline{\Psi}}(1)\,\mathcal{I}(c_{\underline{\Psi}}, \mathrm{Ad}\,(hy)^{-1}(X); P_{\underline{\Psi}}),$$

the second equality holding because, $\mathcal{B}^{(0)-}_{f_1^{(0)}}$ is $\mathrm{Ad}\,P_{\underline{\Psi}}$-stable, $\mathrm{tr}_0(c_{\underline{\Psi}}\mathcal{B}^{(0)-}_{f_1^{(0)}}) \subset O_0$, and, by (i), $X_0 \in \mathcal{B}^{(0)-}_{f_1^{(0)}}$. It now follows that the integral formula above can be rewritten, for $X \in \mathcal{V}_1 \cap \mathfrak{g}_{reg}$, as

$$\Theta_\pi(\exp X) = d(\pi) \int_G \int_{K_0} \int_{P_{\underline{\Psi}}} \Omega(\mathrm{tr}_0(c_{\underline{\Psi}}\mathrm{Ad}\,(hyw)^{-1}(X)))\,dw\,dh\,dy.$$

As observed in Lemma 4.1 of [Mu1], the order of the $P_{\underline{\Psi}}$ and K_0 integrations can reversed, and the $P_{\underline{\Psi}}$ integration may be absorbed into the G integration, giving

$$\Theta_\pi(\exp X) = d(\pi) \int_G \int_{K_0} \Omega(\mathrm{tr}_0(c_{\underline{\Psi}}\mathrm{Ad}\,(hy)^{-1}(X)))\,dh\,dy.$$

This last expression coincides with Harish-Chandra's integral formula for $\widehat{\mu}_{\mathcal{O}(c_{\underline{\Psi}})}$ ([HC2], Lemma 19). We remark that Harish-Chandra derived the formula for G connected, but it is straightforward to show that it holds in this setting if G is not connected. Therefore, we conclude that the theorem holds for $X_\pi = c_{\underline{\Psi}}$ and $\mathcal{V}_\pi = \mathcal{V}_1$. □

As each regular semisimple $\mathrm{Ad}\,G$-orbit is a union of finitely many regular semisimple $\mathrm{Ad}\,G^0$-orbits, and Harish-Chandra proved the existence of a Shalika germ expansion for regular semisimple $\mathrm{Ad}\,G^0$-orbital integrals ([HC2]), we obtain a Shalika germ expansion for an $\mathrm{Ad}\,G$-orbital integral simply by adding the germ expansions for the various $\mathrm{Ad}\,G^0$-orbital integrals. That is, there exist functions $\Gamma_{\mathcal{O}}$ on $\mathfrak{g}_{reg}$, one for each nilpotent $\mathrm{Ad}\,G^0$-orbit $\mathcal{O}$ such that, given a locally constant compactly supported function f on $\mathfrak{g}$, there exists a neighbourhood $\mathcal{V}_f$ of zero in $\mathfrak{g}$ such that

$$\mu_{\mathcal{O}(Y)}(f) = \sum_{\mathcal{O} \in (\mathcal{N})^0} \mu_{\mathcal{O}}(f)\,\Gamma_{\mathcal{O}}(Y), \qquad Y \in \mathcal{V}_f \cap \mathfrak{g}_{reg}.$$

Here, $(\mathcal{N})^0$ denotes the set of nilpotent $\mathrm{Ad}\,G^0$-orbits, and $\mu_{\mathcal{O}}$ is a G^0-invariant measure on such an orbit $\mathcal{O}$. $\mathcal{O}(Y)$ is the full $\mathrm{Ad}\,G$-orbit of Y. As in [HC2], Lemma 18, there is an analogous expansion for Fourier transforms:

$$\widehat{\mu}_{\mathcal{O}(Y)}(X) = \sum_{\mathcal{O} \in (\mathcal{N})^0} \Gamma_{\mathcal{O}}(Y)\,\widehat{\mu}_{\mathcal{O}}(X) \tag{10.2}$$

where, if X is a fixed regular element, the equality holds for regular Y in some neighbourhood of zero.

The local character expansion of π at the identity expresses $\Theta_\pi \circ \exp$ as a linear combination of Fourier transforms of nilpotent $\mathrm{Ad}\, G^0$-orbits on a neighbourhood of zero ([HC2], [C]):

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O}\in(\mathcal{N})^0} c_{\mathcal{O}}(\pi)\,\widehat{\mu}_{\mathcal{O}}(X),$$

if $X \in \mathfrak{g}_{reg}$ is sufficiently close to zero.

On any open neighbourhood of zero intersected with $\mathfrak{g}_{reg}$, the collection of functions $\{\,\widehat{\mu}_{\mathcal{O}} \mid \mathcal{O} \in (\mathcal{N})^0\,\}$ is linearly independent ([HC2]). Therefore, a comparison of the local character expansion of π at the identity with the expansion obtained by combining (10.2) (with $Y = X_\pi$) and Theorem 10.1, yields the following:

COROLLARY 10.3. – *Let $\pi = \pi_{\underline{\Psi}}$ and $X_\pi = c_{\underline{\Psi}}$. Assume that $\underline{\Psi}$ is uniform. Then*

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(X_\pi), \qquad \mathcal{O} \in (\mathcal{N})^0.$$

Remarks. – (1) Waldspurger ([W]) has proved a result giving a set $\mathcal{S}_\pi$ on which the local character expansion of an irreducible admissible representation π at the identity is valid. This set is defined in terms of the size of a particular compact subgroup K such that the space of $\pi(K)$-fixed vectors is nonzero. If π is as above and if it is possible to compute $d(\pi)$, $\Gamma_{\mathcal{O}}(X_\pi)$ and the functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})^0$, then Corollary 10.3 combines with Waldspurger's result to give the values of Θ_π on $\mathcal{S}_\pi$.

(2) In the special cases where Howe's Kirillov theory for compact groups ([H2]) applies to the inducing representation $\rho_{\underline{\Psi}}$, it may be possible to simplify the proof of Theorem 10.1, using an argument similar to that used by Kazhdan in the beginning of the appendix to [K2]. However, in general, Kirillov theory does not apply to $\rho_{\underline{\Psi}}$.

11. Cuspidal data involving representations of finite classical groups

A supercuspidal representation for which the conclusion of Theorem 10.1 is valid will be called a *Kirillov* representation.

As remarked in [Mor 3], one way to produce supercuspidal representations other than those arising from $\underline{\Psi}$ as in §5 is to modify the definition of the cuspidal datum $\underline{\Psi}$ slightly to allow representations of reductive groups over finite fields which do not correspond to characters of minisotropic tori. This section consists of a brief discussion of examples of this type.

Suppose $0 \le t \le r - \max\{1,\ell\}$ and $f_{n_t}^{(t)} = 1$. In the definition of $\Psi^{(t)}$ (*see* §§4 and 5), in order that $\rho_{n_t}^{(t)}$ be irreducible, it is necessary that the cuspidal representation $\sigma_{n_t}^{(t)}$ of $\overline{P}^{(t)\,0}(n_t-1)$ be irreducible and fixed by no element of $P^{(t)}(n_t-1)/P^{(t)\,0}(n_t-1)$. In addition, we required that:

(11.1) *$\sigma_{n_t}^{(t)}$ is associated (via the construction of Deligne and Lusztig* [DL]*) to a character $\psi_{n_t}^{(t)}$ of the minisotropic torus $\overline{T}^{(t)} \cap \overline{P}^{(t)\,0}(n_t-1)$.*

In §§4-5, the requirement that $\psi^{(t)}_{n_t}$ be in general position guarantees that $\sigma^{(t)}_{n_t}$ is irreducible. The condition (11.1) allows us to use Kazhdan's proof of Springer's hypothesis to define $c'^{(t)}_{n_t}$ so that (4.6) or (5.4) holds. In general, not all irreducible cuspidal representations of $\overline{P}^{(t)\,0}(n_t - 1)$ will satisfy (11.1). According to Morris, ([Mor3], §§5.10, 6.5), if the condition (11.1) is dropped, the cuspidal datum $\underline{\Psi}$ still gives rise to an irreducible supercuspidal representation $\pi_{\underline{\Psi}}$ of G. Without (11.1), it is not clear whether $\pi_{\underline{\Psi}}$ is a Kirillov representation.

Cuspidal unipotent representations of reductive groups over finite fields do not satisfy (11.1). Let $\mathbf{G}$ be an unramified unitary group. That is, the form f is hermitian and F is an unramified quadratic extension of F_0. Then, if $\dim_F(V) = (n^2 + n)/2$ for some positive integer n, there exists a cuspidal unipotent representation σ of $\overline{G} = \mathbf{G}(O_0/\mathfrak{p}_0)$ ([L]). If $\overline{T}$ is a minisotropic torus in $\overline{G}$, let $R^1_{\overline{T}}$ be the virtual representation attached to the trivial character of $\overline{T}$. The character χ_σ of σ is a linear combination of the characters $\chi^1_{\overline{T}}$ of the $R^1_{\overline{T}}$'s, as $\overline{T}$ runs through the minisotropic tori ([L]). Kazhdan's result ([K]) can be applied to express the restriction of $\chi^1_{\overline{T}}$ to the unipotent subset of $\overline{G}$ as a certain trigonometric sum. This was done in §4 in the case of a nontrivial character of $\overline{T}$, but it works the same way here because the restriction to the unipotent set is independent of the choice of character of the torus. Thus, on the unipotent set, χ_σ is a linear combination of the various trigonometric sums attached to the tori $\overline{T}$. Suppose ρ is the representation of $\mathbf{G}(O_0)$ obtained by inflating σ. Then we have a generalization of (4.6) which gives values of χ_ρ at certain points in $\mathbf{G}(O_0)$ as a finite linear combination of integrals of the type appearing in (4.6). From this it follows that, if $\pi = Ind^G_{\mathbf{G}(O_0)}\rho$, on a neighbourhood of zero, $\Theta_\pi \circ \exp$ can be expressed as a linear combination of Fourier transforms $\widehat{\mu}_{\mathcal{O}_i}$, where $\{\mathcal{O}_i\}$ is a finite collection of regular elliptic adjoint orbits in $\mathfrak{g}$. The details of this when $\dim_F(V) = 3$ appear in [Mu2] (in this case, there are two orbits involved).

If $\mathbf{G} = \mathbf{Sp}_4$ ($\dim_F(V) = 4$) or $\mathbf{GSp}_4$, the same type of result as for the 3×3 unitary group is valid for the supercuspidal representation induced from the inflation of the cuspidal unipotent respresentation of $\mathbf{G}(O/\mathfrak{p})$ (again, there are two orbits).

Suppose $\mathbf{G}$ is symplectic and $\dim_F(V) = 2$, that is, $G = \mathbf{SL}_2(F)$. Let σ_i, $i = 1, 2$ be the two irreducible components of the cuspidal representation σ of $\overline{G} = \mathbf{G}(O/\mathfrak{p})$ associated to a character of a minisotropic torus, where the character is not in general position. All irreducible cuspidal representations of $\overline{G}$ which do not satisfy (11.1) are of this form (and they are not unipotent representations). If π_i is the supercuspidal representation of G obtained by induction from the inflation of σ_i to $\mathbf{G}(O)$, then, as was shown in [Mu3], π_i is not a Kirillov representation. Moreover, the restriction of χ_{σ_i} to the unipotent subset of $\overline{G}$ does not lie in the span of the set of all Deligne and Lusztig virtual characters, so the procedure outlined above for unitary groups cannot be applied here. However, the reducible representation $\pi = \pi_1 \oplus \pi_2$ is induced from the inflation of σ to $\mathbf{G}(O)$, and is a Kirillov representation.

We remark that all of the supercuspidal representations of $\mathbf{SL}_n(F)$, $n \geq 2$, which are not Kirillov representations arise in a similar manner (as shown in [Mu3]). That is, some part (not necessarily all) of their inducing data involves an irreducible component of some reducible cuspidal representation associated to a character of a minisotropic torus in a reductive group over a finite field. Again, certain finite direct sums of these supercuspidal representations are Kirillov representations. (The set of representations occurring in each such direct sum is a subset of an L-packet.)

Summarizing, if the condition (11.1) is dropped, in some cases there exists a generalization of Theorem 10.1 expressing $\Theta_{\pi_{\underline{\Psi}}}$ in terms of an explicit linear combination of Fourier transforms of measures on elliptic adjoint orbits. In certain other cases, there are supercuspidal representations which are not Kirillov representations (and for which no similar generalization of Theorem 10.1 is known), but certain finite direct sums of these representations are Kirillov representations.

12. Other supercuspidal representations

The representations $\pi_{\underline{\Psi}}$, $\underline{\Psi}$ as in §6 of [Mor3] (*see* §5 and §11), do not exhaust the irreducible supercuspidal representations of G. In the afterword (§8) of [Mor3], Morris outlines a method for combining the constructions of [Mor1] with those of [Mor2-3] to obtain a more general method of constructing supercuspidal representations of classical groups. Morris gives an example ([Mor3], 8.6) which shows that this more general construction yields supercuspidal representations of the 4 by 4 symplectic group which do not arise via the earlier construction.

Let T be an elliptic Cartan subgroup of G with commuting algebra $A = \bigoplus_{i=1}^{r} E_i$. As before, we assume that E_i, $i \leq \ell$ is unramified over F and each E_i, $\ell < i \leq r$ has some ramification over F. In [Mor2-3], the lattices and cuspidal data are built up inductively from lattices and cuspidal data attached to the tori $T^{(t)}$, $0 \leq t \leq r - \max\{1, \ell\}$. Recall that the commuting algebra of $T^{(t)}$ is $A_u = \sum_{i=1}^{\ell} E_i$ if $t = r - \ell$ and $\ell > 0$, and E_{r-t} otherwise. All of the unramified extensions are grouped together, and the other extensions are taken one at a time. If the ramified extensions are taken in a different order, then the lattices and cuspidal data obtained will change ([Mor3] 6.2), but the general procedure by which they are constructed does not change.

The idea behind the modifications of [Mor 3] §8 is that, under certain conditions on T ([Mor3], 8.1-2), several of the fields E_i having the same ramification degree over F may be grouped together, rather than taken separately, in the definition of cuspidal datum. This is where the results of [Mor1] come in. The definition of cuspidal datum in the unramified case is based on definitions in [Mor1]. Also considered in [Mor1] are data arising from tori whose commuting algebras consist of fields having the same ramification degree. Suppose that E_i, $m \leq i \leq n$ ($\ell < m < n \leq r$) have the same ramification degree over F, and that the condition (8.2) of [Mor3] is satisfied. Let $A_{m,n} = \bigoplus_{i=m}^{n} E_i$. Then, as described in [Mor3], 8.3-4, a self dual lattice chain $\mathcal{L}_{m,n}$ in $\operatorname{End}_F(A_{m,n})$, a hereditary order, a filtration of the associated parahoric subgroup, the notion of principal element in $A_{m,n}$, and a group $C_{A_{m,n}}$, may be defined in such a way that the results of [Mor1] yield analogues of the results of [Mor2], §3 and of [Mor3], §4.

The procedure of summing lattice chains ([Mor2]) can be carried out with lattices of the form $\mathcal{L}_{m,n}$, and according to [Mor3], 8.5, the results of [Mor2], §3 regarding the associated hereditary orders and filtrations of parahoric subgroups remain valid.

If definition 3.18 of [Mor1] is used to define a cuspidal datum attached to each of the tori of the form $T \cap A_{m,n}$ (and the usual definition (§4) is used in the unramified case), then a cuspidal datum $\underline{\Psi}$ attached to T may be defined inductively, as in definition 6.5 of [Mor3]. Using $\underline{\Psi}$, an irreducible supercuspidal representation $\pi_{\underline{\Psi}}$ of G is constructed along the same lines as in [Mor3], §§6-7 ([Mor3], 8.6).

Suppose that $\underline{\Psi}$ has the property that all of the cuspidal representations of finite classical groups occurring in $\underline{\Psi}$ satisfy (11.1). We define a regular elliptic element $c_{\underline{\Psi}}$ in the same manner as in §5, that is, by adding the elements arising from the cuspidal data attached to the tori $T \cap A_{m,n}$. The results of [Mor2-3] which have been used in this paper generalize to this setting ([Mor3], §8). Consequently, it is likely that if the analogue of Corollary 7.13 holds, then these results of Morris can be used to prove that Theorem 10.1 and Corollary 10.3 are valid (with $X_{\pi_{\underline{\Psi}}} = c_{\underline{\Psi}}$).

REFERENCES

[BF] C. J. BUSHNELL and A. FROHLICH, *Non-abelian congruence Gauss sums and p-adic simple algebras* (*Proc. London Math. Soc.*, Vol. 50, 1985, pp. 207-264).

[C] L. CLOZEL, *Characters of non-connected reductive p-adic groups* (*Canad. J. Math.*, Vol. 39, 1987, pp. 149-167).

[DL] P. DELIGNE and G. LUSZTIG, *Representations of reductive groups over finite fields* (*Ann. Math.*, Vol. 103, pp. 103-161).

[HC1] HARISH-CHANDRA, *Harmonic analysis on reductive p-adic groups* (*Lecture Notes in Math.*, Vol. 162, Springer-Verlag, Berlin, 1970).

[HC2] HARISH-CHANDRA, *Admissible distributions on reductive p-adic groups*, in *Lie Theories and Their Applications* (*Queen's Papers in Pure and Applied Mathematics*, Vol. 48, 1978, pp. 281-347).

[H1] R. HOWE, *On the character of Weil's representation* (*Trans. A.M.S.*, Vol. 117, 1973, pp. 287-298).

[H2] R. HOWE, *Kirillov theory for compact p-adic groups* (*Pacific J. Math.*, Vol. 73, 1977, pp. 365-381).

[J] D. JABON, *The supercuspidal representations of* $\mathbf{U}(2,1)$ *and* $\mathbf{GSp}_4$ *over a local field* (*thesis*, University of Chicago, 1989).

[K1] D. KAZHDAN, *Proof of Springer's Hypothesis* (*Israel J. Math.*, Vol. 28, 1977, pp. 272-286).

[K2] D. KAZHDAN, *Cuspidal geometry of p-adic groups* (*J. Analyse Math.*, Vol. 47, 1986, pp. 1-36).

[L] G. LUSZTIG, *Irreducible representations of finite classical groups* (*Invent. Math.*, Vol. 43, 1977, pp. 125-175).

[Mor1] L. MORRIS, *Some tamely ramified supercuspidal representations of symplectic groups* (*Proc. London Math. Soc.*, Vol. 63, 1991, pp. 519-551).

[Mor2] L. MORRIS, *Tamely ramified supercuspidal representations of classical groups I. Filtrations* (*Ann. scient. Ec. Norm. Sup.*, 4e série, t. 24, 1991, pp. 705-738).

[Mor3] L. MORRIS, *Tamely ramified supercuspidal representations of classical groups II: Representation theory* (*Ann. scient. Ec. Norm. Sup.*, 4e série. t. 25, 1992, pp. 233-274).

[Moy] A. MOY, *Representations of* $\mathbf{U}(2,1)$ *over a p-adic field* (*J. fur die reine u. angewandte Math.*, Vol. 372, 1986, pp. 178-208).

[Mu1] F. MURNAGHAN, *Local character expansions and Shalika germs for* $\mathbf{GL}(n)$ (*Math. Ann.*, to appear).

[Mu2] F. MURNAGHAN, *Local character expansions for supercuspidal representations of* $\mathbf{U}(3)$ (*Canadian J. Math.*, to appear).

[Mu3] F. MURNAGHAN, *Characters of supercuspidal representations of* $\mathbf{SL}(n)$ (*Pacific J. Math.*, to appear).

[PR] G. PRASAD and M. RAGHUNATHAN, *Topological central extensions of semisimple groups over local fields I* (*Ann. Math.*, Vol. 119, 1984, pp. 143-201).

[W] J.-L. WALDSPURGER, *Homogénéité de certaines distributions sur les groupes p-adiques*, preprint.

(Manuscript received July 5, 1994;
revised April 14, 1995.)

F. MURNAGHAN
Department of Mathematics,
University of Toronto,
Toronto, Canada M5S 1A1.