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LUSTERNIK-SCHNIRELMANN-CATEGORICAL SECTIONS

BY OCTAVIAN CORNEA

ABSTRACT. – Let X be a finite type, simply connected CW -complex. If the Lusternik-Schnirelmann category of the localizations of X at each prime is bounded from above by n , then the category of X is bounded from above by $2n + 1$; if X is finite, this upper bound can be improved to $2n$.

1. Introduction

We will work here inside the pointed category of finite type, simply connected spaces with the homotopy type of a CW -complex.

The starting point of this paper is a question of Charles McGibbon [11]: Is $\text{cat} X \leq n$ a generic property in the sense of the Mislin genus? Recall that the Mislin genus of a space is the set of all homotopy types of spaces which, when localized at any prime, coincide with the localization at that prime of the given space [13]. In particular, McGibbon conjectured that if the Lusternik-Schnirelmann category of the localization of a space at any prime is less or equal to n , then the category of the space is, at least, finite. It is this conjecture that we prove here.

If P is a set of primes we denote by P' the complementary set of primes; X_P is the localization of X at P ; $X_{(0)}$ is the rationalization of X .

The results of this paper imply that:

1. For a finite space X :

a. There is a finite set of primes P such that $\text{cat}(X_{P'}) = \text{cat}(X_{(0)})$

b. If $\text{cat}(X_{(p)}) \leq n$ for each prime p , then $\text{cat}(X) \leq 2n$.

2. For X of finite type if $\text{cat}(X_{(p)}) \leq n$ for each prime p , then $\text{cat}(X) \leq 2n + 1$.

The second point above is the main aim of the paper; it is deduced from the point 1 b with the help of the next result:

3. Let X be a CW -complex and let $X^{(n)}$ be its n -th dimensional skeleton. We have $\text{cat}(X^{(n)}) \leq \text{cat}(X) + 1$.

The tools for proving 1 b, which is clearly central to our approach, are provided by a close look at one of the basic constructions in the Lusternik-Schnirelmann theory. Recall first the definition of the L.S.-category of a space, X , [9] : $\text{cat}(X) \leq n$ iff there is a covering of X by $n + 1$ open subsets each contractible in X , equivalently, iff the diagonal $X \rightarrow X^{(n+1)}$ factors, up to homotopy, through the inclusion of the fat wedge $T^{n+1}X \rightarrow X^{(n+1)}$, or

once more, iff the n -th fibration in the Milnor classifying construction [12], applied to ΩX : $E_n \Omega X \longrightarrow B_n \Omega X \xrightarrow{p_n} X$ has a homotopy section. This type of section will be called a categorical section. The Milnor fibrations above appear as a particular case of the following construction due to Ganea [6]: given a fibration $F \longrightarrow E \longrightarrow B$ define inductively the fibrations $E_n(E, F) \longrightarrow G_n(E, F) \longrightarrow B$ by taking $E_0(E, F) = F$, $G_0(E, F) = E$; if C is the cofibre of the inclusion $E_n(E, F) \hookrightarrow G_n(E, F)$ transform the obvious map $C \longrightarrow B$ into a fibration $p_{n+1} : G_{n+1}(E, F) \longrightarrow B$ and let $E_{n+1}(E, F)$ be its fibre. A formula of Ganea shows that $E_{n+1}(E, F) \simeq E_n(E, F) * \Omega B$. The classical case is obtained for the path-loop fibration. Notice that we denote $E_n(*, \Omega B) = E_n \Omega B$ and $G_n(*, \Omega B) = B_n \Omega B$.

Our study of this construction was initiated by a natural question suggested to the author by John Moore: given two maps $s_1, s_2 : X \longrightarrow E$ such that when composed with the projection on B they become homotopic by a homotopy $h : X \times I \longrightarrow B$, what is the smallest k such that after composing s_1 and s_2 with the inclusion $i^k : E \longrightarrow G_k(E, F)$ they become homotopic by a homotopy that covers h ? As consequences of the results presented in the following we get:

4. *The number k as above is at most $\text{cat}(X)$.*
5. *There are examples where the least k is exactly $\text{cat}(X)$ even when the initial fibration is the path-loop fibration.*

Remark. – For the path-loop fibration and h constant to the identity, point 4 above follows also from a result of Ganea [6]. The line of proof that we use here is different from that of Ganea.

The natural context for analyzing this problematic is that suggested by some ideas of Israel Berstein (as they were communicated to the author by Peter Hilton, see also [1]): for our space X , instead of considering the L.S.-category simply as a numerical invariant, focus on the finer structure consisting of a categorical section for X .

In the second section we will formalize these ideas by introducing the notion of n -B(erstein)-structures and prove, in this setting, the results relevant for 1, 2, 3 and 4. In the third section we discuss some other properties of the n -B-structures with a special emphasis on their behaviour with respect to push out squares. This is applied to prove 5.

The original question of McGibbon remains open. We believe that 1 b is, in fact, valid for infinite complexes too.

It turns out that the notions of cone-length and cone-decomposition are very useful in this study. A space X is an n -cone if there is a sequence of spaces X_i , $0 \leq i \leq n$ such that $X_0 = *$, $X_{i+1} = X_i \cup CZ_i$ for $0 \leq i < n$ and some spaces Z_i (here CA is the cone over A). An n -cone decomposition of (the homotopy type of) X is a sequence of cofibration sequences $Z_i \longrightarrow X_i \longrightarrow X_{i+1}$ with $0 \leq i < n$, $X_0 \simeq *$, $X_n \simeq X$. The cone-length of X , $Cl(X)$, is the least n such that there is an n -cone homotopy equivalent to X . We will use here the techniques developed for handling cone decompositions in [3] and [5]. In particular, it was shown there that, by adding in the definition of the cone-length the requirement that the spaces Z_i are i -th order suspensions, the invariant does not change. We will see in the following that there is a very strong relation between n -B-structures and n -cone-decompositions.

Remark. – The notion of cone-length first appeared in [6] under the name of strong L.S.-category of X where it is shown to be also equal to the minimal number of self-contractible subcomplexes needed to cover some CW -complex of the homotopy type of X . It is proved by Ganea and Takens [14], that the strong L.S.-category is bigger than the usual one by at most one.

We will now discuss a rather surprising consequence of 2. We can think about the cone-length as a measure of how much a given homotopy type can be compressed.

Let X be a finite CW -complex and let $\{p_i\}_{1 \leq i \leq \infty}$ be a strictly increasing sequence of primes. Construct, starting from X , spaces X_i by means of cofibration sequences:

$$S^{u_i} \xrightarrow{t_i} X_{i-1} \longrightarrow X_i$$

with t_i being a $p_i^{k_i}$ -torsion homotopy class and such that the sequence u_i is strictly increasing. Let X' be the limit space. The points 1 b and 2 together with the inequality $cat(S) \leq Cl(S) \leq cat(S)+1$ for any space S , imply the curious fact that, independently of the choices involved in the construction, $Cl(X_i) \leq 2Cl(X)+3$ for all i . and $Cl(X') \leq 2Cl(X)+4$. We do not expect, *a priori*, to be able to compress in this fashion each such space.

Acknowledgements

My thanks go to Charles McGibbon for suggesting the original problem, to Steve Halperin for pointing out the last example in Section 3 and for numerous useful suggestions and to John Harper for a useful discussion.

2. Localization and n-B-structures

We start by defining the notion of Bernstein structure which provides the natural context for our study of the relation between localization and the L.S.-category.

DEFINITION 1. – An n -B-structure (n Bernstein structure) for X is a fibre-homotopy class of a section $s : X \longrightarrow B_n \Omega X$.

The $(n+k)$ -extension of an n -B-structure represented by s is the $(n+k)$ -B-structure represented by $i_n^k \circ s$ where $i_n^k : B_n \Omega X \longrightarrow B_{n+k} \Omega X$ is the inclusion.

A map of two n -B-structures is defined by the existence of representatives $s : X \longrightarrow B_n \Omega X$ and $r : X' \longrightarrow B_n \Omega X'$ that are related by a map $v : X \longrightarrow X'$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{v} & X' \\ s \downarrow & & \downarrow r \\ B_n \Omega X & \xrightarrow{B_n(v)} & B_n \Omega X' \\ p_n \downarrow & & \downarrow p_n \\ X & \xrightarrow{v} & X' \end{array}$$

If the map v above is a homotopy equivalence the two n -B-structures are equivalent.

Remark. – 1. For a fibration $F \rightarrow E \rightarrow B$, two sections that are homotopic as maps in E , are also fiber homotopic, [8].

2. In [1] a factorization of the diagonal through the fat wedge, as in the definition of $cat(-)$, is called a (categorical) structure map. It is clear that structure maps and categorical sections are equivalent notions. Also, a map v as above, is n -primitive in the sense of [1].

Our approach to the points 1,2 from the introduction is roughly the following: given, for each prime p , one “local” n -B-structure on $X_{(p)}$ we intend to construct an m -B-structure on X , with m as close to n as possible, and whose localization has the homotopy type, for each prime p , of the extension of the one given.

The construction is based on a fracture lemma [2] implying that, for a simply-connected, finite type space X , and for P a finite set of primes of complement P' , there is a weak homotopy equivalence from X to the homotopy pull-back of the maps $\prod_{p \in P'} X_{(p)} \rightarrow \prod_{p \in P'} X_{(0)}$ and $X_{P'} \xrightarrow{\delta} \prod_{p \in P'} X_{(0)}$. We will see (Proposition 2.4) that when X is finite we can indeed use only a finite product of “local” n -B-structures. However, the key to reassemble a “geometric” n -B-structure out of local data, is to synchronize the rationalizations of the local structures. In fact, the local structures agree rationally after a number of extensions. This number is bounded by the cone-length of X . This follows from a more general type of obstruction arguments based on attachments of cones instead of cells.

We get back to the L.S.-category by recalling that, by [3], if $cat(X) = n$, then there is a suspension Z such that $X \vee Z$ is of cone-length n .

To deal with the case when X is infinite we show (at the end of the section) that there is a strong relation between the n -B-structures of X and those of its skeleta. In particular, we prove the point 3 from the introduction.

PROPOSITION 2.1. – *Let X be a space as above. Let $F \rightarrow E \xrightarrow{p} B$ be a fibration. Let $s_{1,2} : X \rightarrow E$ be two maps such that there is a homotopy $h : X \times I \rightarrow B$ between $p \circ s_1$ and $p \circ s_2$. If X is an n -cone, then there is a homotopy $H : X \times I \rightarrow G_n(E, F)$ between $i_n \circ s_1$ and $i_n \circ s_2$ and such that $p_n \circ H = h$. Here $i_n : E \rightarrow G_n(E, F)$ is the inclusion and $p_n : G_n(E, F) \rightarrow B$ is the projection.*

In particular all n -B-structures on X extend to the same $2n$ -B-structure.

Proof. – Notice the following facts:

Let $Z \rightarrow A \xrightarrow{j} D$ be a cofibration sequence and let $U \rightarrow V \xrightarrow{p} W$ be a fibration.

a. Given a map $t : D \rightarrow W$ and one $k : A \rightarrow V$ such that $p \circ k = t \circ j$, the obstruction to extending k to a map $k' : D \rightarrow V$ such that $p \circ k' = t$ lies in $[Z, U]$.

b. Given two maps $g_{1,2} : D \rightarrow V$ together with homotopies $r : A \times I \rightarrow V$ between $g_1 \circ j$ and $g_2 \circ j$ and $r' : D \times I \rightarrow W$ between $p \circ g_1$ and $p \circ g_2$ such that $p \circ r = r' \circ (j \times 1_I)$, the obstruction to extending r to a homotopy $b : D \times I \rightarrow V$ between g_1 and g_2 , such that $p \circ b = r'$, lies in $[\Sigma Z, U]$.

Point a. is trivial; b. is an easy exercise. Recall that CZ is the cone over Z . We assume $D = A \cup CZ$. The key point is to lift r' over $CZ \times I$ taking into account the fact that r

gives us already a lift over $CZ \times \{0, 1\} \cup I \times Z = T$. If we were working with unreduced cones and suspensions, then T would be homeomorphic with $\Sigma'Z$ and it is simple to verify that $CZ \times I$ would be homeomorphic to $C\Sigma'Z$ where $\Sigma'Z$ is the unreduced suspension. Our result follows by recalling that we use pointed maps and homotopies.

LEMMA 2.2. – *In the context of the point a. above there is a map $t' : D \rightarrow G_1(V, U)$ such that $p_1 \circ t' = t$ and $t' \circ j = i_1 \circ k$. In the context of the point b. there is a homotopy $b' : D \times I \rightarrow G_1(V, U)$ such that $p_1 \circ b' = r'$ and $b' \circ (j \times I) = i_1 \circ r$. Here $p_1 : G_1(V, U) \rightarrow W$ is the projection and $i_1 : V \rightarrow V/U \simeq G_1(V, U)$ the collapsing map.*

Proof of the lemma. – Let $\alpha \in [Z, U]$ and $\beta \in [\Sigma Z, U]$ be the two obstructions given respectively by the points a and b. There is an obvious map of fibrations:

$$\begin{array}{ccccc} U & \longrightarrow & V & \xrightarrow{p} & W \\ i'_1 \downarrow & & i_1 \downarrow & & id \downarrow \\ E_1(V, U) & \xrightarrow{v} & G_1(V, U) & \xrightarrow{p_1} & W \end{array}$$

For the lemma it is enough to show that the images α' of α and β' of β in $[Z, E_1(V, U)]$ and, respectively, in $[\Sigma Z, E_1(V, U)]$ by the maps induced by i'_1 are null. Notice that, because $G_1(V, U) \simeq V/U$, the map $v \circ i'_1$ is nullhomotopic and, therefore the composites $v \circ \alpha'$ and $v \circ \beta'$ are null. The proof ends by remarking [6], that the connectant $\Omega W \rightarrow E_1(V, U)$ is null (indeed, it is identified with the natural map $\Omega W \rightarrow \Omega W * U \simeq E_1(V, U)$ which is null).

We turn now to the proposition. The proof is completed by applying inductively the lemma. Indeed as X is an n -cone there is a sequence of cofibration sequences $Z_i \rightarrow X_i \rightarrow X_{i+1}$ for $0 \leq i < n$ with $X_0 = *$ and $X_n = X$. We restrict all the maps and homotopies involved to the i -th level. Recall that $G_{i+1}(E, F) \simeq G_i(E, F)/E_i(E, F) \simeq G_1(G_i(E, F), E_i(E, F))$. The lemma shows how to extend a homotopy $H_i : X_i \times I \rightarrow G_i(E, F)$ to $H_{i+1} : X_{i+1} \times I \rightarrow G_{i+1}(E, F)$.

Remark. – 1. It's clear that, under the circumstances above, knowing that Z_i is an (iterated) suspension can be of interest. This points out one reason why the possibility to assume $Z_i = \Sigma^i P_i$ in the definition of the cone-length is meaningful. Indeed, if we use above this type of cone decomposition, at the i -th step the obstruction will lie in $\pi_{i+1}(P_i, U)$.

2. If H' is another homotopy constructed the same way as H in the proposition, a similar argument as above can be used to show that, after extension to $G_{n+1}(E, F)$, H and H' can be connected by a homotopy K defined on $(X \times I) \times I$ and which is constant to s_1 on $X \times \{0\} \times I$ and to s_2 on $X \times \{1\} \times I$ and which also covers h . We construct inductively $K_i : X_{i-1} \times I \times I \rightarrow G_i(E, F)$, a homotopy between $H_{i-1} \circ j_{i-1}^1$ and $H'_{i-1} \circ j_{i-1}^1$ where j_{i-1}^1 is the inclusion of $G_{i-1}(E, F)$ into $G_i(E, F)$.

COROLLARY 2.3. – *If $\text{cat}(X) \leq n$ replaces the condition of X being an n -cone in the above proposition, then the conclusions hold.*

Proof. – If $\text{cat}(X) \leq n$, by the results in [3], it follows that the space $Y = X \vee \Sigma \Omega E_n \Omega X$ is of cone-length bounded by n . Of course X is a retract of Y . Hence, by composing with the collapsing map $X \vee \Sigma \Omega E_n \Omega X \rightarrow X$, we reduce the problem for X to the same problem for Y . Here it can be solved after possibly changing Y , inside the same homotopy type, as to become an n -cone. Finally we compose again with the inclusion $X \rightarrow Y$. However, we start with s_1, s_2 and we constructed a homotopy between $s_1 \circ u$ and $s_2 \circ u$ covering $h \circ u$ where $u : X \rightarrow X$ is homotopic to the identity and appears because of the change of Y into an n -cone. The result follows by recalling that X has the homotopy type of a CW -complex and, hence, u can be inverted.

Before going further it is useful to note that, when applied to simply connected spaces, the functors $B_n \Omega(-)$ commute with localizations.

We start now the comparison between geometric and localized n -B-structures. The first result shows that, for finite CW -complexes, rational n -B-structures can be pulled back to geometric ones when inverting some finite number of primes.

PROPOSITION 2.4. – *Let X be a finite, simply-connected CW -complex. Given an n -B-structure s_0 on the rationalization $X_{(0)}$ of X , there is a finite set of primes P and an n -B-structure on $X_{P'}$ which, when rationalized, coincides with s_0 . In particular, $\text{cat}(X_{P'}) = \text{cat}(X_{(0)})$.*

Proof. – Consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ j \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

Let $Z \rightarrow N \xrightarrow{h} M$ be a cofibration sequence and consider maps $f' : M \rightarrow B$, $g' : M \rightarrow C$, $l : N \rightarrow A$ such that $f \circ f' \simeq g \circ g'$ and $k \circ l \simeq f' \circ h$, $j \circ l \simeq g' \circ h$. The obstruction to the construction of a map $s : M \rightarrow A$ such that $k \circ s \simeq f'$ and $j \circ s \simeq g'$ lies in $[Z, L]$ where L is the homotopy fibre of the obvious map of A into the homotopy pull back of f and g . A similar statement holds in the case of homotopies.

We are going to apply this remark to the following situation: let s' be a representative of s_0 . Let $X^{(i)}$ be the i -th skeleton of X . Suppose that we have constructed: $s_i : (X^{(i)})_{P'_i} \rightarrow B_n \Omega X_{P'_i}$, P_i a finite set of primes such that the rationalization of s_i is the restriction of s' to $(X^{(i)})_{(0)}$. We have the square:

$$\begin{array}{ccc} B_n \Omega X_{P'_i} & \longrightarrow & B_n \Omega X_{(0)} \\ \downarrow & & p_0 \downarrow \\ X_{P'_i} & \xrightarrow{l_i} & X_{(0)} \end{array}$$

Here the horizontal arrows are rationalizations and the vertical ones are the usual projections. Notice that the homotopy fibre of the obvious map from $B_n \Omega X_{P'_i}$ to the pull back of l_i and p_0 has only torsion homotopy groups. Recalling that X is finite it follows that, by inverting some other (finite number of) primes, we can kill all the

obstructions to the construction of s_{i+1} satisfying the induction hypothesis. At the end we are left with a section $s'' : X_{P'} \rightarrow B_n \Omega X_{P'}$ whose rationalization is s' ; P being the union of the P_i 's.

Remark. – 1. The same method as before can be used to "derationalize" homotopies.

2. The finiteness condition is necessary. Indeed, if p_i is an infinite sequence of primes and n_i is an increasing sequence of integers ≥ 3 , let T be the wedge of the Moore spaces $P^{n_i}(p_i)$. Clearly, the category of T is 1 and inverting any finite number of primes does not reduce it. However, rationally T is contractible.

THEOREM 2.5. – *Let X be simply connected and of finite type. Let s_0 be a rational n -B-structure on $X_{(0)}$ and let s_p be n -B-structures on $X_{(p)}$ for each prime p .*

a. If X is finite there is a $2n$ -B-structure on X which, when localized at p , for any prime p , is equivalent with the one extending s_p and when rationalized gives the extension of s_0 . In particular, $\text{cat}(X) \leq 2n$.

b. For X not necessarily finite, there is a $(2n+1)$ -B-structure with the analogous properties. In particular, $\text{cat}(X) \leq 2n + 1$.

Proof. – We begin with the finite case.

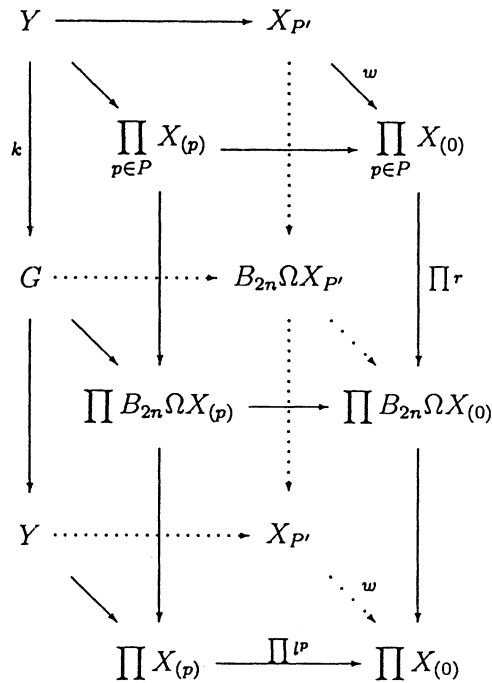
Let s' represent s_0 . By Proposition 2.4, there is a finite set of primes P such that there is a section $s'_{P'} : X_{P'} \rightarrow B_n \Omega X_{P'}$ whose rationalization is s' . Let u_P be the $2n$ -extension of $s'_{P'}$ and let r be the $2n$ -extension of s' . From now on all the primes involved belong to P .

Denote by $l^p : X_{(p)} \rightarrow X_{(0)}$ a rationalization. We denote the rationalizations at the level of the spaces $B_n \Omega X_{(p)}$ by Bl^p . We may assume Bl^p to be a fibration for all the p 's. Let $s'_p : X_{(p)} \rightarrow B_n \Omega X_{(p)}$ be a section that represents s_p . Denote by $s'_{p,0}$ the localization of this section. Let r_p be the composite $i_n^n \circ s'_{p,0}$ where $i_n^n : B_n \Omega X_{(0)} \rightarrow B_{2n} \Omega X_{(0)}$ is the inclusion. Also, let u_p be the $2n$ -extension of s'_p (r_p is the rationalization of u_p). By applying Proposition 2.1 and its corollary it follows that there are homotopies $H_p : X_{(p)} \times I \rightarrow B_{2n} \Omega X_{(0)}$ such that $H_p(0) = r \circ l^p$, $H_p(1) = r_p \circ l^p$ and $p_{2n,0} \circ H_p(t) = l^p$ for any fixed $t \in I$; $p_{2n,0} : B_{2n} \Omega X_{(0)} \rightarrow X_{(0)}$ being the projection. We have the square:

$$\begin{array}{ccc} B_{2n} \Omega X_{(p)} & \xrightarrow{Bl^p} & B_{2n} \Omega X_{(0)} \\ p_{2n} \downarrow & & p_{2n,0} \downarrow \\ X_{(p)} & \xrightarrow{l^p} & X_{(0)} \end{array}$$

It is easy to see that we can lift the homotopy H_p to $H'_p : X_{(p)} \times I \rightarrow B_{2n} \Omega X_{(p)}$ such that $Bl^p \circ H'_p = H_p$ and with the additional properties : $H'_p(1) = u_p$, $p_{2n} \circ H'_p(t) = id_X$ for each fixed t . Let $u'_p = H'_p(0)$. It is clear that u'_p is fiber homotopic with u_p . Moreover the rationalization of u'_p is r .

Let us now consider the following commutative double cube diagram:



Here w is the rationalization composed with the diagonal; in the bottom cube the vertical arrows are projections in the corresponding fibrations; in the top cube the vertical, non marked arrows are $\prod u'_{(p)}$ and $u_{P'}$; we assume the rationalizations $B l^p, l^p$ to be fibrations; all the horizontal squares are pull backs. In this case, by the fracture lemma mentioned at the beginning of the section, there is a weak homotopy equivalence $X \xrightarrow{v} Y$. Also, the space G comes with a weak homotopy equivalence with domain $B_{2n}\Omega X$. We obtain the map k which provides the $2n$ -B-structure that we are looking for. Indeed, to conclude the point a of the theorem notice that the localization of the n -B-structure represented by k coincides, trivially, for the primes in P with the extension of s_p and for the primes in P' it is the n -th extension of an n -B-structure (induced by the derationalization of s') hence, by Proposition 2.1 it again coincides with the extension of s_p .

Remark. – Another variant of the fracture lemmas [2] can be used to avoid the “derationalization” argument.

We will now pass to the infinite case. The reduction to the finite one is provided by the next result.

PROPOSITION 2.6. – *Let $f : Y \rightarrow X$ be a map of simply connected spaces with Y being of dimension m and let s be a n -B-structure on X . If $H_*(f)$ is an isomorphism for $* < m$ and an epimorphism for $* = m$, then there is an $(n+1)$ -B-structure on Y , s' , such that, f maps the $(n + 2)$ -extension of s' into the $(n + 2)$ -extension of s .*

In particular $\text{cat}(Y) \leq \text{cat}(X) + 1$.

Remark. – 1. A simple example when $\text{cat}(Y) = \text{cat}(X) + 1$ is provided by the trivial map $S^m \rightarrow *$ with $m \geq 2$.

2. It would be interesting to know if we can find a *CW*-decomposition for X such that for each m $\text{cat}(X^{(m)}) \leq \text{cat}(X)$. Rationally this is true.

Proof. – Let r be a representative of s . We have the following commutative square:

$$\begin{array}{ccc} B_n \Omega Y & \xrightarrow{B_n(f)} & B_n \Omega X \\ p_n^Y \downarrow & & \downarrow p_n^X \\ Y & \xrightarrow{f} & X \end{array}$$

Let H be the pull back of f and p_n^X . Let $g : B_n \Omega Y \rightarrow H$ be the map induced by $B_n(f)$ and p_n^Y . Let $k : Y \rightarrow B_n \Omega X$ be the map induced by id_X and $r \circ f$. If k could be lifted over g , then that lift would provide a section with all the required properties (it would give, in fact, even an n -B-structure !). Hence, it is natural to see what are the obstructions to such a lift. Let F be the homotopy fibre of g .

LEMMA 2.7. – *With the notations above, F is $(m-2)$ -connected.*

Proof of the lemma. – It is easy to see that the map $H_*(B_n(f))$ is an isomorphism in dimensions less or equal to $m - 1$ and an epimorphism in dimension m . Indeed, this follows from the corresponding property for f and the fact that the fibre of $E_n \Omega Y = (\Omega Y)^{*(n+1)} \rightarrow (\Omega X)^{*(n+1)} = E_n \Omega X$ is $n(m - 1)$ connected.

Obviously, the map $u : H \rightarrow B_n \Omega X$ has the same connectivity as f . As $u \circ g = B_n(f)$, it follows that $H_*(g)$ is an isomorphism for $* \leq m - 1$. However, it might not be an epimorphism when $* = m$. This means that F is $(m - 2)$ -connected and, in general, it is not more connected than that.

We return now to the proof of the proposition. The lemma implies that when trying to construct a lift of k over g we do not encounter any obstacles to the construction of a map $v : Y^{(m-1)} \rightarrow B_n \Omega Y$ which is a lift of the inclusion $j : Y^{(m-1)} \rightarrow Y$ and such that $r \circ f \circ j = B_n(f) \circ v$. Of course, Y can be obtained from $Y^{(m-1)}$ by attaching a number of m -cells. By applying now Lemma 2.2 we get a section $r' : Y \rightarrow B_{n+1} \Omega Y$ which extends $i_m^1 \circ v$. The fibre homotopy of $B_{n+2}(f) \circ i_{n+1}^1 \circ r'$ and $i_n^2 \circ r \circ f$ follows by noticing that the needed commutativity already exists on $Y^{(m-1)}$ and applying again the Lemma 2.2.

We are now in good shape for proving the point b of the theorem. We will construct the needed section by constructing coherent lifts of the inclusions $j^m : X^{(m)} \hookrightarrow X$ into $B_{2n+1} \Omega X$.

The first step is to notice that we can build a double cube similar to that above only that the top square will be replaced by its restriction to the m -th skeleton $X^{(m)}$. More precisely, if we represent s_p by s'_p we can restrict these sections to $(X^{(m)})_{(p)}$ and similarly for s_0 . Moreover we may assume that if the diagram corresponding to $X^{(m)}$ is constructed with respect to the finite set of primes P^m (which takes the place of P), then we have the inclusions $P^m \subset P^{m+1}$. Denote by s_p^m the restriction to $(X^{(m)})_{(p)}$ of the $(2n+1)$ -extension of s'_p . Notice that, for each p , the s_p^m 's form a coherent system of lifts in the sense that s_p^{m+1} composed with the localization of the inclusion $j_1^m : X^{(m)} \hookrightarrow X^{(m+1)}$ gives s_p^m . Remark

also that s_p^m is the $(2n+1)$ -extension of the restriction of s'_p to $(X^{(m)})_{(p)}$. The proposition above shows that the category of $(X^{(m)})_{(p)}$ and $(X^{(m)})_{(0)}$ is bounded by $n+1$. Hence, by Proposition 2.1 all the rationalizations of the s_p^m 's coincide. The derationalization argument works also. As at the point a this implies the existence of lifts $k^m : X^{(m)} \rightarrow B_{2n+1}\Omega X$ of j^m which have the property that the p-localization of k_m has the fiber homotopy type of s_p^m . Moreover, as $P^m \subset P^{m+1}$ and $s_p^{m+1} \circ (j_1^m)_{(p)} = s_p^m$ for all primes p we obtain that k^m is homotopic to $k_{m+1} \circ j_1^m$ by a homotopy that covers j_1^m .

An infinite telescope argument shows this to be enough for the construction of the needed section.

Remark. – A more direct but inefficient way to approach the point b of the theorem is to use a result of Hardie [7, 9] which shows that the category of a direct limit of spaces of category less than n is bounded from above by $2n$. Applying this together with the point a of the theorem and the last proposition we would get an upper bound for the category of X in point b of the order of $4n$.

In fact Hardie's result can be restated in a somewhat stronger form in terms of n-B-structures: given n-B-structures on a direct system of spaces there is a $2n$ -B-structure on the direct limit whose restriction to the spaces in the system coincides with the extension of the original B-structures. This is an immediate consequence of Proposition 2.1.

3. Cone-length and n-B-structures

It is clear from the previous section that there is a strong relation between n-B-structures and cone-decompositions. We will try to make it now more explicit. In [4] a map of cone-decompositions (level-preserving map), $f : X \rightarrow X'$, is defined by the diagrams:

$$\begin{array}{ccccc} Z_i & \longrightarrow & X_i & \longrightarrow & X_{i+1} \\ & & \downarrow & f_n \downarrow & f_{n+1} \downarrow \\ & & Z'_i & \longrightarrow & X'_i \longrightarrow X'_{i+1} \end{array}$$

Here $0 \leq i < n$, both horizontal sequences are cofibration sequences defining cone-decompositions for $X_n = X$ and, respectively, $X'_n = X'$; $f = f_n$. Two cone-decompositions are equivalent if they are related by a map of cone-decompositions which is also a homotopy equivalence. Any n cone-decomposition can be trivially extended to one of length $n+1$ by defining $Z_n = *$, $X_{n+1} = X_n$.

Remarks. – 1. Each n-B-structure on a space X induces an equivalence class of cone-decompositions of length n on $X \vee \Sigma\Omega E_n\Omega X$ and an equivalence class of cone-decompositions of length $n+1$ on X .

Conversely, any cone-decomposition of length n on X induces a n-B-structure on X . The $(n+1)$ -extensions of the n-B-structures induced by equivalent cone-decompositions coincide.

Moreover, the two constructions are inverse at the level $n+1$.

2. There is a result similar to the one above concerning the relation between maps of cone-decompositions and maps of n-B-structures.

3. There is some asymmetry in the statement at 1. The reason is immediate: n -B-structures behave well with respect to retracts. In particular an n -B-structure on $X \vee \Sigma\Omega E_n \Omega X$ induces one on X . Also any n -B-structure on X can be extended to one on $X \vee \Sigma\Omega E_n \Omega X$. The same type of arguments do not work for cone-decompositions.

In some circumstances it is easier to work with cone-decompositions in others with n -B-structures. The results mentioned in the remarks above show that from many points of view it does not make too much difference which variant one chooses. In particular the following lemma will be stated in terms of cone-decompositions. It immediately implies an obvious analogue in terms of n -B-structures (compare with Theorem 3.4 in [1]).

LEMMA 3.1. – *A map of n -cone-decompositions, $f : X \rightarrow Y$ induces on the cofibre Y/X an n -cone-decomposition such that the collapsing map is a level preserving map.*

Proof. – Let us take a look at the diagrams given by f :

$$\begin{array}{ccccc} Z_i & \longrightarrow & X_i & \longrightarrow & X_{i+1} \\ & & \downarrow & f_i \downarrow & f_{i+1} \downarrow \\ & & Z'_i & \longrightarrow & Y_i & \longrightarrow & Y_{i+1} \end{array}$$

with $f = f_n$. By collapsing the top row into the bottom one we get a new sequence of cofibration sequences: $Z'_i/Z_i \rightarrow Y_i/X_i \rightarrow Y_{i+1}/X_{i+1}$.

Remark. – Obviously, in general (that is, when there are no restrictions on f) there is a map of $(n+1)$ -cone-decompositions between the trivial extension of the decomposition on Y and the $(n+1)$ -cone-decomposition on Y/X obtained by first building up Y and then adding the cone over X .

Another interesting fact that is worth mentioning here is the following:

LEMMA 3.2. – *If $F \rightarrow E \rightarrow B$ is a fibration then for any n -B-structure on B there is an n -B-structure on E/F such that the projection $E/F \rightarrow B$ is a map of n -B-structures.*

Proof. – Recall from the introduction that one alternative definition for the Lusternik-Schnirelmann category is in terms of open coverings. A covering with all sets contractible in X and with only $cat(X)$ sets is called a categorical covering for X . Categorical coverings and categorical sections are equivalent notions [9] the relation between them being established by remarking that they are both equivalent to factorizations of the diagonal. Moreover, the equivalences are functorial. In [3] it is shown that for a given categorical covering of B there is one of E/F , with the same number of sets, and such that the projection $E/F \rightarrow B$ is a map of coverings. This implies the statement.

Remark. – The cone-length version of the above result is described in detail in [3].

Having the above results at our disposal we intend to discuss (and prove) the proposition:

PROPOSITION 3.3. – *There is a space X that supports two t -B-structures with different $(t+k)$ -extensions for all $k < t$.*

Proof. – To start consider the following construction: assume we are given the homotopy push out square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \\ D & \longrightarrow & C \end{array}$$

Assume also that we have n -B-structures on B and D represented respectively by sections s and r . Let F be the homotopy fibre of f . It is not hard to see that we also have a homotopy push out square:

$$\begin{array}{ccc} A/F & \xrightarrow{f'} & B \\ g' \downarrow & & \downarrow \\ D/F & \longrightarrow & C \end{array}$$

Here f' and g' are induced by f and g ; the map $F \rightarrow D$ used in the collapsing is the composite of g and the inclusion of F . Now on A/F there is an n -B-structure represented by a section s' inherited from s . On D/F there is an $(n+1)$ -B-structure represented by r' that is induced by r . Let F' be the homotopy fibre of g' . We have a homotopy push out square:

$$\begin{array}{ccc} (A/F)/F' & \xrightarrow{f''} & B/F' \\ g'' \downarrow & & \downarrow \\ D/F & \longrightarrow & C \end{array}$$

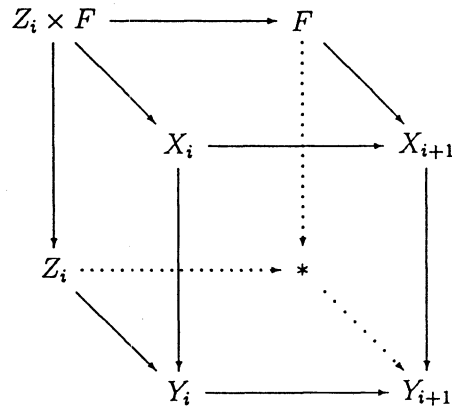
It is constructed in a similar fashion with the last one. We also get two $(n+1)$ -B-structures on $(A/F)/F'$ and one on B/F' represented respectively by sections r'' inherited from r' , s'' induced by s' and s_1 induced by s . Moreover, f'' is a map of $(n+1)$ -B-structures between s'' and s_1 and g'' is a map of $(n+1)$ -B-structures between r'' and r' . Suppose now that the $(n+1+k)$ -extensions of s'' and r'' coincide. It is easy to see that we can assemble the $(n+1+k)$ -extensions of s'' , s_1 and r' to obtain a $(n+1+k)$ -B-structure on C . This provides a method to show that, for k small, $(n+1+k)$ -extensions of $(n+1)$ -B-structures do not generally coincide. Indeed the above argument shows, in particular, that $\text{cat}(C) \leq n + 1 + k$. However, it is well-known that the result that is valid in general is $\text{cat}(C) \leq 2n + 1$. An example when $\text{cat}(C) = 2n + 1$ would show that $k \geq n$ in general. However, as $\text{cat}(A/F)/F' \leq n + 1$ this type of result is not enough for the proposition (we would need $k \geq n + 1$).

In the following we will present a modification of the construction and a relevant example in order to prove the proposition.

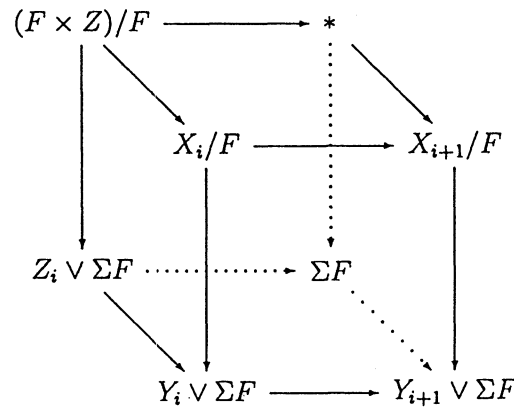
LEMMA 3.4. – Consider a map $f : X \rightarrow Y$. Let F be the homotopy fibre of f . If $\text{Cl}(Y) \leq n$, then there is map of $(n+1)$ -cone-decompositions, $f' : X/F \rightarrow Y \vee \Sigma F$, which, up to homotopy is induced by f after collapsing F in both X and Y .

Remark. – This result allows us to replace any map by a level preserving one (in $n+1$ -stages, though) such that the cofibre of the replacement is the same as that of the original.

Proof of the lemma. – We assume f to be a fibration. Let $Z_i \rightarrow Y_i \rightarrow Y_{i+1}$ be a sequence of cofibration sequences with $0 \leq i < n$, $X_0 = *$, $Y_n \simeq Y$. We pull back these cofibration sequences, thus getting the cubes:



Here the vertical faces are pull backs and, hence, the top square is a push out, [9]. By collapsing F in each of the spaces appearing in the cube we get:



We remark that for $i \geq 1$ we may assume Z_i to be a suspension. Also for $i = 0$ in the bottom square the two arrows with domain $Z_0 \vee \Sigma F$ are equal. This makes it possible to use a technique discussed in [5] to transform the bottom push out square into a cofibration sequence: $Z_i \vee \Sigma F \rightarrow Y_i \vee \Sigma F \vee \Sigma F \rightarrow Y_{i+1} \vee \Sigma F$. It is easy to see that this cofibration sequence can be transformed into: $Z_i \rightarrow Y_i \vee \Sigma F \rightarrow Y_{i+1} \vee \Sigma F$ into which maps the top cofibration sequence appearing in the cube: $(F \times Z)/F \rightarrow X_i/F \rightarrow X_{i+1}/F$.

Now let $f : X \rightarrow Y, g : X \rightarrow K$ be two maps that are related by homotopy equivalences in the sense that there is a commutative square:

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ g \downarrow & & f \downarrow \\ K & \longrightarrow & Y \end{array}$$

with all horizontal arrows homotopy equivalences. Given a n -cone-decomposition on Y the lemma above shows how we can replace f by $f' : X/F \rightarrow Y \vee \Sigma F$ (F is the homotopy fibre of f and also of g) with f' a map of $(n+1)$ -cone decompositions. Given another n -cone-decomposition on K there is an induced cone-decomposition on X/F and an induced map of n -cone-decompositions: $g' : X/F \rightarrow K$. Clearly we can trivially assume g' to be a map of $(n+1)$ -cone-decompositions.

The corresponding statement for n -B-structures is that, after fixing a n -B-structure on Y and one on K , we may construct $(n+1)$ -B-structures such that f' and g' become maps of $(n+1)$ -B-structures.

It is easy to see that the push out of f' and g' is, up to homotopy, the same as that of f and g . Let us denote by T this push out.

Suppose that the two different $(n+1)$ -B-structures on X/F have the same $n + 1 + k$ extension, then we can construct on T an $(n+1+k)$ -B-structure by pasting the $(n+1+k)$ -structures constructed on K and $Y \vee \Sigma F$. In particular, the category of T is bounded from above by $n + 1 + k$.

Notice that the category of X/F is less or equal to n (by lemma 3.3). The proposition is proved by constructing an example such that $cat(T) = 2n + 1$.

Example. – In the $(2n+2)$ -dimensional complex space \mathbb{C}^{2n+2} consider two orthogonal $(n+1)$ -dimensional complex subspaces $C_1 = \mathbb{C}^{n+1}, C_2 = \mathbb{C}^{n+1}$ such that C_1 corresponds to points with all the last $(n+1)$ coordinates zero and C_2 to points with all the first $(n+1)$ coordinates zero. Let U_j be the set of all lines in \mathbb{C}^{2n+2} which are not contained in $C_j, j = 1, 2$. It is easy to see that $U_j \simeq \mathbb{C}P^n$ and, of course $U_1 \cup U_2 = \mathbb{C}P^{2n+1}$. Recall that $cat(\mathbb{C}P^k) = k$. The only thing that remains to be shown is that $l_j : U_1 \cap U_2 \hookrightarrow U_j, j = 1, 2$ are related by homotopy equivalences. This follows from the fact that the following square is commutative:

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{w} & U_1 \cap U_2 \\ j_1 \downarrow & & j_2 \downarrow \\ U_1 & \xrightarrow{w} & U_2 \end{array}$$

Where w is a homeomorphism that corresponds to the restriction of the map on \mathbb{C}^{2n+1} which permutes the i -th coordinate and the $i + n + 1$ one for all $1 \leq i \leq n + 1$.

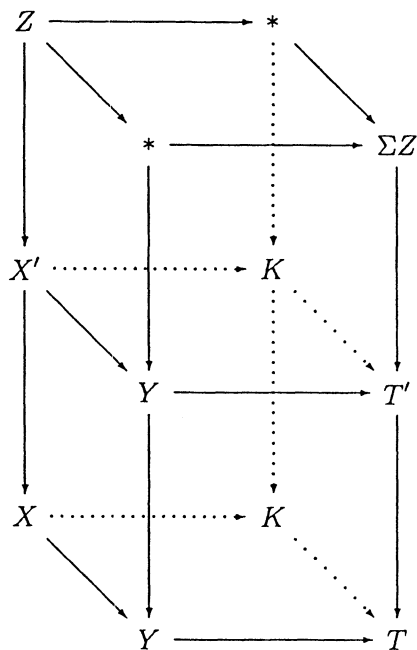
Remark. – The behaviour of the L.S.-category with respect to push outs is prescribed by the following inequality: let T be the homotopy push out of $f : X \rightarrow Y$ and $g : X \rightarrow K$. We have [7, 9]:

$$cat(T) \leq \max\{cat(Y) + cat(K) + 1, cat(Y) + cat(X), cat(K) + cat(X)\}$$

The cone-length variant of this inequality is also true. To underline the usefulness of cone-decompositions we will present here a short proof for:

$$Cl(T) \leq \max\{Cl(Y) + Cl(X), Cl(K) + Cl(X)\}.$$

Indeed if X is a point this is obvious. The general case follows by induction on $Cl(X)$. Let $Z \rightarrow X' \rightarrow X$ be a cofibration sequence with $Cl(X') < Cl(X)$. It is enough to inspect the following double cube where all horizontal squares are push outs and all columns are cofibration sequences:



Here the maps from X' into Y and K are the composites of f and, respectively g with the inclusion $X' \hookrightarrow X$.

REFERENCES

- [1] I. BERSTEIN and P. J. HILTON, *Category and generalized Hopf invariants* (*Illinois J. Math.*, Vol. 4, 1960, pp. 437-451).
- [2] A. K. BOUSFIELD and D. M. KAN, *Homotopy Limits, Completions and Localizations* (*Lecture Notes in Math.*, Vol. 304, 1972, Springer).
- [3] O. CORNEA, *Cone-length and Lusternik Schnirelmann category* (*Topology*, Vol. 33, 1994, pp. 95-111).
- [4] O. CORNEA, *There is just one rational cone-length* (To appear in *Trans. of the A.M.S.*).
- [5] O. CORNEA, *Strong L.S.-category equals cone-length* (To appear in *Topology*).
- [6] T. GANEA, *Lusternik-Schnirelmann category and strong category* (*Ill. J. of Math.*, Vol. 11, 1967, pp. 417-427).
- [7] K. A. HARDIE, *On the category of the double mapping cylinder* (*Tohoku Math. J.*, Vol. 25, 1973, pp. 355-358).

- [8] J. R. HARPER, *A proof of Gray's Conjecture and Co-H-Maps to Spheres* (Preprint Univ. of Rochester 1989).
- [9] I. M. JAMES, *On category in the sense of Lusternik and Schnirelmann* (*Topology*, Vol. 17, 1978, pp. 331-348).
- [10] M. MATHER, *Pull-backs in Homotopy Theory* (*Can. J. Math.*, Vol. 28, 1976, pp. 225-263).
- [11] C. A. MCGIBBON, *The Mislin genus of a space* (To appear in *Proc. Hilton 70-th birthday conf.*, Montreal 93).
- [12] J. W. MILNOR, *Construction of universal bundles II* (*Ann. of Math.*, Vol. 63, 1956, pp. 430-436).
- [13] G. MISLIN, *The genus of an H-space* (*Springer Lecture Notes in Math.*, Vol. 249, 1971, pp. 75-83).
- [14] F. TAKENS, *The Lusternik-Schnirelmann categories of a product space* (*Comp. Math.*, Vol. 22, 1970, pp. 175-180).

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