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# A LEFSCHETZ TRACE FORMULA FOR EQUIVARIANT COHOMOLOGY

By Minhyong KIM

Abstract. – This paper studies the effect, on the equivariant cohomology of a compact manifold $X$ with a compact Lie group action $G$, of *an equivariant pair* $F = (f, \phi)$ of maps, *i.e.*, a smooth map $f : X \to X$ and a Lie group homomorphism $\phi : G \to G$ such that $f(gx) = \phi(g)f(x)$. Such a pair mimics the Frobenius map for varieties in characteristic $p$, and induces a graded map $F^*$ on equivariant cohomology. Under certain transversality conditions (again mimicing the behaviour of the Frobenius map), we can define a 'regularized' Lefschetz number and prove a trace formula relating this number to local fixed point data. This fixed point data is extracted from the *fixed-point groupoid* associated to the pair $F$. That is, $F$ induces a functor of the groupoid defined by $X$ and $G$ and the fixed-point groupoid is the two-product of the graph of this functor and the diagonal functor. When the group action is trivial, this formula reduces to the usual Lefschetz trace formula for transverse maps.

## 1

Let $G$ be a compact Lie group acting smoothly on a differentiable, orientable manifold $X$. Let $f : X \to X$ be differentiable and let $\phi : G \to G$ be a Lie group homomorphism. We say $(f, \phi)$ is an *equivariant pair* if $f(gx) = \phi(g)f(x)$ for all $g \in G, x \in X$. Such a pair gives rise to a self-map $F = F(f, \phi)$ of the *Borel construction* $X_{hG} := EG \times_G X$ of the group action, and hence, a graded endomorphism $F^*$ of the *equivariant cohomology* $H^*_G(X) := H^*_{\text{sing}}(X_{hG}, \mathbf{C})$. Recall that the (usually infinite-dimensional) complex vector space $H^*_G(X)$ is a finitely-generated graded module over the graded algebra $H^*(BG)$. In particular, each graded piece$H^n_G(X)$ is a finite-dimensional vector space. Define a power series in the indeterminate $t$ by

$$\text{Tr}(F, t) = \sum_0^\infty \text{Tr}(F^* | H^n_G(X)) t^n.$$

Then

Theorem 1. – *For any $f$ and $\phi$, Tr$(F(f, \phi), t)$ is a rational function of $t$.*

Given this theorem, it is tempting to attempt to define a 'global Lefschetz number' of the map $F$ by evaluating $\text{Tr}(F, t)$ at $t = -1$. Unfortunately, this rational function often has a pole at $-1$. This gives rise to

Problem: When can one extract a global Lefschetz number (GLN) out of $\mathrm{Tr}(F, t)$?

Condition (*) below is a simple-minded case, namely, causing $\mathrm{Tr}(F, t)$ to be regular at $-1$.

When Problem admits a solution, it is natural to look for a 'local Lefschetz number (LLN)' computed in terms of the geometry of the pair $(f, \phi)$, which should equal the GLN, in analogy with the classical case. The most natural approach turns out to involve a 'fixed-point groupoid' associated to $(f, \phi)$ which degenerates to the fixed-point set in the classical case when there is no group action. The groupoid formulation also suggests a generalization, to be outlined below, to general *compact differentiable groupoids*. Recall that groupoids arise in differential geometry in essentially the same manner as algebraic stacks in algebraic geometry, namely, as parameter 'spaces' which do not exist naturally as manifolds: for example, the 'space of leaves' of a non-fibrating foliation, as in A. Connes' approach to the Atiyah-Singer Index Theorem for foliations.

Recall that a *groupoid* is a category such that the morphisms form a set and, furthermore, are all isomorphisms.

One also defines a *groupoid object* in any category $\mathcal{C}$, in which products exist, as a groupoid $\mathcal{X}$ such that the objects and the morphisms, $\mathrm{Ob}(\mathcal{X})$ and $\mathrm{Mor}(\mathcal{X})$, are objects of $\mathcal{C}$, and such that all the structure maps in the definition of a groupoid (source and target maps, composition, etc) are morphisms of $\mathcal{C}$. For a precise formulation, *see* [G].

A topological groupoid $\mathcal{X}$, *i.e.*, a groupoid object in the category of topological spaces, gives rise to a simplicial space, its *classifying space* $B\mathcal{X}$, in a standard fashion [S]. We can use this to define the cohomology of the groupoid $\mathcal{X}$ with (constant) coefficient $K$ to be the singular cohomology of the geometric realization of $B\mathcal{X}$: $H^i(\mathcal{X}, K) := H^i_{\mathrm{sing}}(|B\mathcal{X}|, K)$.

By a *differentiable groupoid* $\mathcal{X}$, we mean a topological groupoid such that $\mathrm{Ob}(\mathcal{X})$ and $\mathrm{Mor}(\mathcal{X})$ are both equipped with differentiable manifold structures.

Now, given differentiable groupoids $\mathcal{X}$ and $\mathcal{Y}$, a functor $F : \mathcal{X} \to \mathcal{Y}$, is called a *differentiable map of groupoids* if the induced maps $\mathrm{Ob}(\mathcal{X}) \longrightarrow \mathrm{Ob}(\mathcal{Y})$ and $\mathrm{Mor}(\mathcal{X}) \longrightarrow \mathrm{Mor}(\mathcal{Y})$ are both differentiable. The *graph* $\Gamma(F)$ of $F$, the functor

$$(Id \times F) \circ \Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{Y},$$

is then also differentiable, where $\Delta$ denotes the obvious diagonal functor. If $F$ in fact maps $\mathcal{X}$ to $\mathcal{X}$, then we define the fixed-point groupoid $(\Gamma(F) \cdot \Delta)$ to be the 2-product of $\mathcal{X}$ with itself with respect to the maps $\Gamma(F)$ and $\Delta$ from $\mathcal{X}$ to $\mathcal{X} \times \mathcal{X}$:

$$\begin{array}{ccc} (\Gamma(F) \cdot \Delta) & \longrightarrow & \mathcal{X} \\ \downarrow & 2 & \downarrow \Delta \\ \mathcal{X} & \xrightarrow{\Gamma(F)} & \mathcal{X} \times \mathcal{X} \end{array}$$

(for 2-products, *see* [G-Z]) The 2-product is a natural construction in the 2-category of groupoids analogous to the homotopy pull-back in topology. It reduces to the ordinary product when the groupoid is a set. Note that $(\Gamma(F) \cdot \Delta)$ may be only a topological groupoid.

The problem that arises in this context has two parts: for a suitable class of maps $F$,

(1) define a natural measure on $\mathrm{Ob}(\Gamma(F)\cdot\Delta)$;

(2) find a cohomological 'trace' formula for the integral of a multiplicity function on $\mathrm{Ob}((\Gamma(F)\cdot\Delta))$. (LLN=GLN)

The main example of this paper will hopefully clarify the meaning of these two problems. We first need a few more definitions. Given a groupoid $\mathcal{Y}$, we will simply write $\pi_0(\mathcal{Y})$ for the set of its isomorphism classes, which can be identified with the set of connected components of $|B\mathcal{Y}|$. We will say the groupoid $\mathcal{Y}$ is *essentially finite* if $\pi_0(\mathcal{Y})$ is finite and each object has only finitely many automorphisms. For a finite set $S$, we denote by $||S||$ the number of elements in $S$.

DEFINITION. – The *natural measure* $\mu_{\mathcal{Y}}$ of an essentially finite groupoid $\mathcal{Y}$ assigns the measure $1/||\mathrm{Aut}(\xi)||$ to each isomorphism class $\xi \in \mathrm{Ob}(\mathcal{Y})$.

Note that the number of automorphisms is constant over an isomorphism class so that the expression $||Aut(\xi)||$ makes sense. The natural measure on a finite set, for example, considered as a discrete groupoid is just the counting measure, while the single object in the groupoid defined by a finite group $G$ inherits the measure $1/||G||$.

The natural measure of an essentially finite groupoid allows us to integrate *locally constant* functions on $\mathrm{Ob}(\mathcal{Y})$, *i.e.*, functions which are constant on each isomorphism class.

We can formulate now another (hopefully easier) version of the problems above as follows:

$(1')$ Give conditions on $F$ such that $(\Gamma(F)\cdot\Delta)$ is an essentially finite groupoid;

$(2')$ define a locally constant multiplicity function on $\mathrm{Ob}((\Gamma(F)\cdot\Delta))$ and find a cohomological trace formula for its integral.

In the context of (equivalence classes of) groupoid objects in the category of algebraic spaces, viz. algebraic stacks, over a finite field, these problems were solved in many interesting cases by K. Behrend [B1]. His trace is that of the arithmetic Frobenius acting on the smooth cohomology of an algebraic stack. By taking a presentation of the stack and viewing it as a groupoid, we can again apply a classifying space construction to obtain a simplicial scheme in the same manner as that outlined for a differentiable groupoid. Then the smooth cohomology of the algebraic stack is seen to be isomorphic to the étale cohomology of this simplicial scheme [F], leading to the analogy with the differentiable situation. This paper arose from an attempt to translate this into a topological setting, in the spirit of [A-M].

We return to our example from above where the group action gives rise to a differentiable groupoid $\mathcal{X}$ such that $\mathrm{Ob}(\mathcal{X}) = X$ and $\mathrm{Mor}(\mathcal{X}) = G \times X$ and the structure maps are given by the group action and the projection. Thus for $x, y \in X$, $\mathrm{Mor}(x, y)$ consists of those $g \in G$ such that $gx = y$.

In this case, $|B\mathcal{X}|$ is homotopy equivalent to $X_{hG}$, and hence, $H^i(\mathcal{X}) \simeq H^i_G(X)$.

An equivariant pair $(f, \phi)$ gives rise to a differentiable self-map $F = F(f, \phi) : \mathcal{X} \longrightarrow \mathcal{X}$ which, in turn, gives rise to the self-map of $X_{hG}$, denoted by the same $F(f, \phi)$ above, by the functorial nature of the classifying space construction.

Denote by $C$ the group of connected components of $G$ and by $\pi : G \longrightarrow C$, the natural projection. For any $g \in G$, $c(g) : G \longrightarrow G$ denotes the inner automorphism defined by $g$.

Consider the following conditions on $f$, $\phi$, and the action $m : G{\times}X \longrightarrow X$:

(*)

(a) There exists a lift $R = \{\tilde{\gamma} \in G\}_{\gamma \in C}$ of $C$ such that the homomorphisms $c(\tilde{\gamma}^{-1}) \circ \phi, \quad \tilde{\gamma} \in R$ are tranverse to the diagonal.

(b) There exists a lift $S$ of $C$ such that the maps $\tilde{\gamma}^{-1} \circ f, \quad \tilde{\gamma} \in S$ are transverse to the diagonal.

For example, if $G$ is a finite group, the first condition is vacuous and the second condition just says that each $g^{-1} \circ f$ is transverse to the diagonal. In the general case, these conditions (especially (a)) are easily seen to be quite stringent, motivated as they are by analogy with the behaviour of the Frobenius morphism in positive characteristic. However, they give rise to various natural analogues of theorems in positive characteristic, clarifying their nature.

THEOREM 2. – *If $f$, $\phi$, and $m$ satisfy* (*) *then the groupoid* $(\Gamma(F) \cdot \Delta)$ *is essentially finite.*

Under condition (*), we can also define a multiplicity function on $\mathrm{Ob}(\gamma(F) \cdot \Delta)$ with values in $\{\pm 1\}$ The desired trace formula then takes the following form:

THEOREM 3. – *Under condition* (*), $\mathrm{Tr}(F(f, \phi), t)$ *is regular at* $t = -1$, *and*

$$\int_{\mathrm{Ob}(\gamma(F) \cdot \Delta)} \nu d\mu_{(\Gamma(F) \cdot \Delta)} \ \text{(LLN)} = \mathrm{Tr}(F(f, \phi), -1) \ \text{(GLN)}.$$

It should be emphasized that even in this example of a group action, there seems to be no natural way to define a LLN without using the language of groupoids, whereas the GLN is obtained in a more mundane manner.

In a future work, I hope to deal with a considerably weakened version of the condition (*), especially for the homomorphism $\phi$. In fact, I hope to formulate a 'cleannness' condition on the map $F$ that would allow a trace formula to hold.

Other possible directions include the case of a general compact differentiable groupoid outlined above as well as the inclusion of additional structures, such as a riemannian structure, on the groupoid. It would be interesting also to consider applications to the theory of toric varieties, which provide a natural class of equivariant pairs $(f, \phi)$.

For the case of a general compact differentiable groupoid, *i.e.*, a differentiable groupoid whose morphisms form a compact manifold, one immediate question of interest would be whether the analogue of Theorem 1 holds. This would fit in with the general philosophy arising from the case of varieties over finite fields, whereby the rationality of its zeta-function can be understood in terms of the fact that the points of the variety lie inside some compact space and thereby should be describable in terms of an object involving a finite amount of information. (I owe this observation to A. Beilinson.) Thus, even though the classifying space of a compact groupoid will, in general, have cohomology in infinitely many degrees, it may not be unreasonable to expect the sort of finiteness as would be expressed by a version of Theorem 1.

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## 2

Recall that given groupoids $\mathcal{X}$, $\mathcal{Y}$, and $Z$, and functors

$$F:\mathcal{X}\longrightarrow\mathcal{Z},\qquad G:\mathcal{Y}\longrightarrow\mathcal{Z},$$

the *2-product* of $\mathcal{X}$ and $\mathcal{Y}$ with respect to $\mathcal{Z}$ is the groupoid whose objects are triples $(x,y,\alpha)$, where $(x,y)\in \mathrm{Ob}(\mathcal{X}\times\mathcal{Y})$ and $\alpha : F(x)\longrightarrow G(y)$ is a morphism of $\mathcal{Z}$, and whose morphisms $(x,y,\alpha)\longrightarrow(w,z,\beta)$ are given by pairs $(\gamma,\delta)\in \mathrm{Mor}(\mathcal{X})\times\mathrm{Mor}(\mathcal{Y})$ such that the following diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha} & G(y)\\ {\scriptstyle F(\gamma)}\downarrow & & \downarrow{\scriptstyle G(\delta)}\\ F(w) & \xrightarrow{\beta} & G(z). \end{array}$$

We indicate this construction by the 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}\times_{\mathcal{Z}}\mathcal{Y} & \longrightarrow & \mathcal{Y}\\ \downarrow & 2 & \downarrow{\scriptstyle G}\\ \mathcal{X} & \xrightarrow{F} & Z. \end{array}$$

The 2-product gives the natural product structure on the 2-category of groupoids, in the sense that if $H:\mathcal{W}\longrightarrow\mathcal{X}$ and $H':\mathcal{W}\longrightarrow\mathcal{Y}$ are functors such that $F\circ H$ is equivalent to $G\circ H'$, then, given an equivalence, we can construct a functor $\tilde{H}:\mathcal{W}\longrightarrow\mathcal{X}\times_{\mathcal{Z}}\mathcal{Y}$ which projects to $H$ and $H'$. Concerning our groupoid $\mathcal{X}$ from the introduction equipped with the functor $F$, we first prove the following

LEMMA 1. – $(\Gamma(F)\cdot\Delta)$ *is equivalent to the groupoid* $(\Gamma(F)\cdot\Delta)_0$ *whose objects are given by pairs* $(y,g)\in X\times G$ *such that* $f(y)=gy$ *and whose morphisms* $(y,g)\longrightarrow(y',g')$ *are elements* $h\in G$ *such that* $y'=hy$ *and* $g'=\phi(h)gh^{-1}$.

*Proof.* – Note that if $f(y)=gy$, then

$$f(hy)=\phi(h)f(y)=\phi(h)gy=(\phi(h)gh^{-1})hy,$$

so that the above assignment does indeed define a morphism.

Now, if $(x,y,g\times h)\in \mathrm{Ob}(\gamma(F)\cdot\Delta)$, then $\Delta(x)\xrightarrow{g\times h}\Gamma(F)(y)$, that is, $y=gx$ and $f(y)=hx$. Thus $f(y)=hg^{-1}y$. This allows us to construct the commutative diagram

$$\begin{array}{ccc} (x,x) & \xrightarrow{g\times h} & (y,f(y))\\ {\scriptstyle g\times g}\downarrow & & \downarrow{\scriptstyle e\times e}\\ (y,y) & \xrightarrow{e\times hg^{-1}} & (y,f(y)), \end{array}$$

so that $(g,e)$ ($e$ denotes the identity element of the group) defines an isomorphism

$$(x,y,g\times h)\longrightarrow(y,y,e\times hg^{-1}).$$

The functor

$$\begin{gathered}(\Gamma(F)\cdot\Delta)_1 \longrightarrow (\Gamma(F)\cdot\Delta)\\ (y,g)\mapsto (y,y,e\times g)\\ ((y,g)\xrightarrow{h}(y',g'))\mapsto ((y,y,e\times g)\xrightarrow{(h,h)}(y',y',e\times g'))\end{gathered}$$

is well defined because of the diagram

$$\begin{array}{ccc}(y,y) & \xrightarrow{e\times g} & (y,f(y))\\ {\scriptstyle h\times h}\downarrow & & \downarrow{\scriptstyle h\times\phi(h)}\\ (y',y') & \xrightarrow{e\times g'} & (y',f(y')),\end{array}$$

and is essentially surjective by the argument in the previous paragraph. It is clearly faithful.

On the other hand, if $(y,y,e\times g)\xrightarrow{(h,k)}(y',y',e\times g')$ is a morphism, then the commutative diagram

$$\begin{array}{ccc}(y,y) & \xrightarrow{e\times g} & (y,f(y))\\ {\scriptstyle h\times h}\downarrow & & \downarrow{\scriptstyle k\times\phi(k)}\\ (y',y') & \xrightarrow{e\times g'} & (y',f(y'))\end{array}$$

tells us that $h=k$ and $\phi(k)g=g'h$, that is, $\phi(h)gh^{-1}=g'$.

This shows that the above functor is full, thus an equivalence of categories by [M], Theorem IV.4.1.

To proceed further, we need the following characteristic zero analogue of Lang's theorem for connected algebraic groups over finite fields:

LEMMA 2. – *If $\phi : G^0 \longrightarrow G^0$ is transverse to the diagonal, then the Lang map*

$$\mathcal{L}(\phi): G^0 \longrightarrow G^0,$$

$$h\mapsto \phi(h^{-1})h$$

*is surjective.*

*Proof.* – $\mathcal{L}(\phi)(e)=e$ and $d\mathcal{L}(\phi)(e)=\mathrm{Id}-d\phi(e)$ is an isomorphism so that $\mathcal{L}(\phi)$ has maximal rank at $e$. Denote by $R_g$ and $L_g$ the right and left translation operator defined by $g\in G$. Now, note that

$$\begin{aligned}\mathcal{L}(\phi)\circ R_g(h) &= \phi(hg)^{-1}hg\\ &= \phi(g^{-1})\phi(h^{-1})hg\\ &= L_{\phi(g^{-1})}\circ R_g\circ\mathcal{L}(\phi)(h).\end{aligned}$$

Thus,

$$d\mathcal{L}(\phi)(g)\circ dR_g(e)=dL_{\phi(g^{-1})}(g)\circ dR_g(e)\circ d\mathcal{L}(\phi)(e),$$

showing that $d\mathcal{L}(\phi)(g)$ has maximal rank for any $g\in G^0$. Therefore, $\mathcal{L}(\phi)$ is a submersion; in particular, it is an open map. On the other hand, its image is compact, and hence, closed. So the image must be all of $G^0$.

The following formulations will also be useful:

COROLLARY 1. – *If $g \in G$ is such that $c(g^{-1}) \circ \phi : G^0 \longrightarrow G^0$ is tranverse to the diagonal, then*

$$\mathcal{L}(\phi, g) : G^0 \longrightarrow G^0,$$
$$h \mapsto g^{-1}\phi(h^{-1})gh$$

*is surjective.*

COROLLARY 2. – *If $g \in G$ and $\phi$ are as in Corollary 1 and $g'$ satisfies $\pi(g) = \pi(g')$, then there exists $h \in G^0$ such that*

$$g' = \phi(h)gh^{-1}.$$

*Proof.* – We have $g' = gg_0$ for some $g_0 \in G^0$. But by Corollary 1, there exists $h \in G^0$ such that $g_0 = g^{-1}\phi(h^{-1})gh$.

The following is then an obvious consequence of Corollary 2:

COROLLARY 3. – *If there exists $R$ satisfying* (*) (*a*), *then for any $g_1, g_2 \in G$ lying in the same component, there exists an $h \in G^0$ such that $\phi(h)g_1h^{-1} = g_2$.*

Given any lifting $R := \{\tilde{\gamma}\}_{\gamma \in C}$ of $C$, we denote by $(\Gamma(F) \cdot \Delta)_R$ the full subcategory of $(\Gamma(F) \cdot \Delta)_0$ whose objects are given by

$$\mathrm{Ob}(\Gamma(F) \cdot \Delta)_R := \{(y, \tilde{\gamma}) \in \mathrm{Ob}(\Gamma(F) \cdot \Delta)_0 \mid \tilde{\gamma} \in R\}.$$

THEOREM 0. – *For $R$ as in (*) (a),*

$$(\Gamma(F) \cdot \Delta)_R \hookrightarrow (\Gamma(F) \cdot \Delta)_0$$

*is an equivalence of categories.*

*Proof.* – We need only show that this inclusion functor is essentially surjective. But for $(y, g) \in \mathrm{Ob}(\Gamma(F) \cdot \Delta)_0$, if $\pi(g) = \gamma$, then there exists $h \in G^0$ such that $g = \phi(h^{-1})\tilde{\gamma}h$. Thus, $(y, g) \longrightarrow^h (hy, \tilde{\gamma}) \in \mathrm{Ob}(\Gamma(F) \cdot \Delta)_R$.

COROLLARY 0. – *If there exists an $R$ satisfying the condition* (*) (*a*), *then for any lifting $S$ of $C$, the inclusion*

$$(\Gamma(F) \cdot \Delta)_S \hookrightarrow (\Gamma(F) \cdot \Delta)_0$$

*is an equivalence of categories.*

*Proof.* – This follows from Theorem 0 and Corollary 3.

PROPOSITION 1. – *If there exists an $R$ satisfying condition* (*) (*a*), *then $Aut(\xi)$ is finite for any $\xi \in (\Gamma(F) \cdot \Delta)$.*

*Proof.* – We need only check this for an object $(y, \tilde{\gamma})$ of $(\Gamma(F) \cdot \Delta)_R$. But an automorphism $h \in G$ in that case satisfies $hy = y, \phi(h)\tilde{\gamma}h^{-1} = \tilde{\gamma}$, so $c(\tilde{\gamma}^{-1}) \circ \phi(h) = h$, that is, $h$ is a fixed-point of $c(\tilde{\gamma}^{-1})\circ\phi$. By our assumption on $R$, and since $G^0$ is a compact manifold, there are only finitely many such $h \in G^0$. But then there are only finitely many possibilities for such $h$ on all of $G$, since the difference of any two fixed-points lying in the same component of $G$ will be a fixed-point lying in $G^0$ and $G$ has only finitely many components.

COROLLARY 4. – *If there exists an $R$ satisfying (*) (a), and an $S$ such that $g^{-1}\circ f$ has finitely many fixed points for each $g \in S$, then $(\Gamma(F)\cdot\Delta)$ is essentially finite.*

*Proof.* – By the above, we know that

$$(\Gamma(F)\cdot\Delta)\simeq(\Gamma(F)\cdot\Delta)_0\simeq(\Gamma(F)\cdot\Delta)_R\simeq(\Gamma(F)\cdot\Delta)_S$$

and every object in $(\Gamma(F)\cdot\Delta)$ has a finite number of automorphisms. However, if $(y,g)$ is an object of $(\Gamma(F)\cdot\Delta)_S$, then $g\in S$ and $f(y)=gy$, *i.e.*, $y$ is a fixed point of $g^{-1}\circ f$. Therefore, $\mathrm{Ob}(\Gamma(F)\cdot\Delta)_S$ is a finite set.

*Proof of Theorem 1.* – This is now clear from Corollary 4 and the fact that on a compact manifold, a map transverse to the diagonal has a finite fixed point set.

*Remark.* – It is clear from the proof that essential finiteness of the fixed-point groupoid will hold whenever there exists a $R$ satisfying (*) (a) and an $S$ such that $\mathrm{Ob}(\Gamma(F)\cdot\Delta)_S$ is finite. Theorem 1 gives one particular condition implying the existence of such an $S$.

Another interesting case of essential finiteness arises as follows:

PROPOSITION 2. – *Suppose $f^n$ has finitely many fixed points for all $n$. (As in [A-M].) Then there exists an $S$ such that $Ob(\Gamma(F)\cdot\Delta)_S$ is finite.*

*Proof.* – We need the following fact from elementary Lie theory: *every component of $G$ contains a point of finite order.* To see this, Take any $g\in G$ and consider the centralizer $Z(g)$, a closed Lie subgroup of $G$. Since $g\in Z(g)$, some power $g^n$ lies in the connected component $Z(g)^0\subset G^0$. But then, we can find $h\in Z(g)^0\subset G^0$ such that $h^n=g^n$. Hence $gh^{-1}$ has finite order and lies inside the same component as $g$.

Now, choose $S$ so that every element of $S$ has finite order. Then if $(y,g)\in \mathrm{Ob}(\Gamma(F)\cdot\Delta)_S$, we get $f(y)=gy$, so $f^{ord(g)}(y)=y$. By assumption, there are only finitely many such $y$.

Now, for any $g\in G$, denote by $(\Gamma(F)\cdot\Delta)_g$ the full subcategory of $(\Gamma(F)\cdot\Delta)_0$ such that $\mathrm{Ob}(\Gamma(F)\cdot\Delta)_g=\{(y,g)\in\mathrm{Ob}(\Gamma(F)\cdot\Delta)_0\}$, that is, objects whose second component is $g$. The following is now clear from the definitions:

PROPOSITION 3. – *The morphisms of $(\Gamma(F)\cdot\Delta)_g$ are the elements of $G^{c(g^{-1})\circ\phi}$, the fixed point set of $c(g^{-1})\circ\phi$ on $G$. So $(\Gamma(F)\cdot\Delta)_g$ is equivalent to the category whose objects are the fixed points $X^{g^{-1}\circ f}$ of $g^{-1}\circ f$ and whose morphisms are the elements of $G^{c(g^{-1})\circ\phi}$ acting via $g:y\mapsto gy$.*

Note that if $g'=\phi(h)gh^{-1}$, we have the following useful formula:

$$\begin{aligned}(1)\qquad h^{-1}\circ g'^{-1}\circ f\circ h &= h^{-1}\circ g'^{-1}\circ\phi(h)\circ f\\ &= h^{-1}(hg^{-1}\phi(h^{-1}))\phi(h)\circ f\\ &= g^{-1}\circ f\end{aligned}$$

and

$$c(g'^{-1}) \circ \phi = c(h) \circ c(g^{-1}) \circ \phi \circ c(h^{-1}). \tag{2}$$

This implies in particular

PROPOSITION 4. – *If there exists $R$ and $S$ as in (*) then $g^{-1} \circ f$ and $c(g^{-1}) \circ \phi$ are tranverse to the diagonal for every $g \in G$.*

*Proof.* – Clear from (1), (2) and Corollary 3.

Denote by $\bar{\phi}$ the homomorphism induced by $\phi$ on the group of connected components $C$.

LEMMA 3. – *Suppose there exists $R$ as in (*) (a). For $g, g' \in G$, suppose there exists $\gamma \in C$ such that $\pi(g') = \bar{\phi}(\gamma)\pi(g)\gamma^{-1}$. Then there exists a lift $\tilde{\gamma}$ such that $\phi(\tilde{\gamma})g\tilde{\gamma}^{-1} = g'$. In particular, if $\bar{\phi}(\gamma)\pi(g)\gamma^{-1} = \pi(g)$ then there exists a lift $\tilde{\gamma}$ such that $\phi(\tilde{\gamma})g\tilde{\gamma}^{-1} = g$.*

*Proof.* – First take $\tilde{\gamma}$ to be any lift. Then $\phi(\tilde{\gamma})g\tilde{\gamma}^{-1} = g'g_0$ for some $g_0 \in G^0$. But by Corollary 3, there exists $h \in G^0$ such that $\phi(h^{-1})g'h = g'g_0$. Then $h\tilde{\gamma}$ does what we want.

Thus, when $R$ as above exists, we get, for each $g \in G$, an exact sequence:

$$0 \longrightarrow (G^0)^{c(g^{-1}) \circ \phi} \longrightarrow G^{c(g^{-1}) \circ \phi} \longrightarrow C^{c(\pi(g) \circ \bar{\phi}} \longrightarrow 0.$$

Denote by $\mathcal{C}$ the groupoid whose objects are the elements of $C$ and whose morphisms are the elements of $C$ acting via $\gamma : x \mapsto \bar{\phi}(\gamma)x\gamma^{-1}$. Let $\mathcal{O} = \{\xi\}$ be a set of representatives for the isomorphism classes of $\mathcal{C}$.

Now, the following is clear from Lemma 3 and the exact sequence following it:

PROPOSITION 5. – *Suppose there exists $R$ as in (*) (a). Then for any lift $S$ of $C$, there is a decomposition into a disjoint union*

$$(\Gamma(F) \cdot \Delta)_S \simeq \coprod_{g \in S, \pi(g) \in \mathcal{O}} (\Gamma(F) \cdot \Delta)_g.$$

Now, let $(y, g) \in \text{Ob}(\Gamma(F) \cdot \Delta)_0$. Then we have the sign of the local determinant $d(y, g) := \text{sign}(\det(I - d(g^{-1} \circ f)(y)))$. By formula (1), we have $d(y, g) = d(y', g')$ if $(y, g)$ is isomorphic to $(y', g')$. Consider also the function $\tau(g) = \tau(\phi, g) := \text{sign}(\det(I - d(c(g^{-1}) \circ \phi)(e)))$. Again, if $g' = \phi(h)gh^{-1}$, then by (2)

$$\det(I - d(c(g^{-1}) \circ \phi)(e)) = \det(I - d(c(g'^{-1}) \circ \phi)(e)).$$

This allows us to formulate the following

DEFINITION. – If there exist $R$ and $S$ satisfying (*), then the *multiplicity function* on $\text{Ob}(\gamma(F) \cdot \Delta)$ is the locally constant function defined by the formula

$$\nu(y, g) := d(y, g)/\tau(g)$$

on $\text{Ob}(\gamma(F) \cdot \Delta)_0$.

Note that this definition makes sense since by Corollary 2 and Proposition 4, all the determinants involved are non-zero.

Now, for $g \in G$ denote by $X_+^{g^{-1}\circ f}$ and $X_-^{g^{-1}\circ f}$ the fixed points $y$ such that $d(y,g)$ is positive and negative, respectively. Both sets are easily seen to be preserved by the action of $G^{c(g^{-1})\circ\phi}$.

PROPOSITION 6. – *Suppose $R$ as in (*) (a) and $S$ as in (*) (b) exist so that any lift $R = \{\tilde{\gamma}\}$ satisfies (*) (a) (b) (by Proposition 4). Let $\mathcal{O}$ be as in Proposition 5. Then*

$$\int_{\mathrm{Ob}(\gamma(F)\cdot\Delta)} \nu d\mu_{(\Gamma(F)\cdot\Delta)} = \sum_{\tilde{\gamma}\in R, \gamma\in\mathcal{O}} (\tau(\tilde{\gamma}))(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/\|G^{c(\tilde{\gamma}^{-1})\circ\phi}\|$$
$$= \sum_{\tilde{\gamma}\in R, \gamma\in\mathcal{O}} (\tau(\tilde{\gamma}))(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/(\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|\|(C)^{c(\gamma^{-1})\circ\bar{\phi}}\|).$$

*Proof.* – Recall that the orbit formula says that for a finite group $H$ acting on a finite set $S$, $\|S\| = \sum[H : H_\xi]$, where the sum runs over the orbits $\xi$ on $S$ of the $H$-action, and $[H : H_\xi]$ is the index of the isotropy subgroup of any representative for the orbit $\xi$. That is, $\|S\|/\|H\| = \sum_{\text{orbits } \xi} 1/\|H_\xi\|$.

The first equality follows then from Proposition 3, Proposition 5, and the orbit formula applied to $G^{c(\tilde{\gamma}^{-1})\circ\phi}$ acting on $X_+^{\tilde{\gamma}^{-1}\circ f}$ and $X_-^{\tilde{\gamma}^{-1}\circ f}$. Note that the automorphism groups in these cases are just the isotropy groups inside $G^{c(\tilde{\gamma}^{-1})\circ\phi}$ of a point in $X_\pm^{\tilde{\gamma}^{-1}\circ f}$ . The second equality comes from the exact sequence following Lemma 3.

Actually, if $\gamma_1, \gamma_2$ are isomorphic in $\mathcal{C}$, that is, if there exists $\gamma$ such that $\bar{\phi}(\gamma)\gamma_1\gamma^{-1} = \gamma_2$, then by formulas (1) and (2) together with Lemma 3 and the exact sequence following it, we get

$$\|X_\pm^{\tilde{\gamma}_1^{-1}\circ f}\| = \|X_\pm^{\tilde{\gamma}_2^{-1}\circ f}\|$$

and

$$\|(G^0)^{c(\tilde{\gamma}_1^{-1})\circ\phi}\| = \|(G^0)^{c(\tilde{\gamma}_2^{-1})\circ\phi}\|,$$

as well as $\tau(\tilde{\gamma}_1^{-1}) = \tau(\tilde{\gamma}_2^{-1})$. So we get

$$(3)\ (1/\|C\|)\sum_{\gamma} \tau(\phi,\tilde{\gamma})(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|$$
$$= (1/\|C\|)\sum_{\gamma\in\mathcal{O}} [C : C^{c(\gamma^{-1})\circ\bar{\phi})}]\tau(\phi,\tilde{\gamma})(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|$$
$$= \sum_{\gamma\in\mathcal{O}} \tau(\phi,\tilde{\gamma})(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/(\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|\|C^{c(\gamma^{-1})\circ\bar{\phi})}\|)$$

giving another formula for the integral of the multiplicity function.

## 3

We begin this section by recalling some facts from elementary linear algebra. If $A$ is an endomorphism of the finite-dimensional complex vector space $V$, then $A$ possesses a unique Jordan decomposition $A = A_{ss} + A_n$ where the components are characterized by the fact that $A_{ss}$ is semi-simple, $A_n$ is nilpotent and $A_{ss}$ and $A_n$ commute with each other. In fact, both the nilpotent part and the semi-simple part are polynomials in $A$. In particular, any operator that commutes with $A$ commutes also with the semi-simple part. The operation of associating to an endomorphism its semi-simple part is compatible with subquotients: that is, if $W \subset V$ is a subspace stable under $A$, then it is stable under $A_{ss}$ and $A_{ss}|W = (A|W)_{ss}$. There is a similar statement for quotients. Also, if $A$ and $B$ are operators on $V_1$ and $V_2$, respectively, then $A \otimes B : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ satisfies

$$A_{ss} \otimes B_{ss} = (A \otimes B)_{ss}.$$

This can be seen by writing

$$\begin{aligned}(A \otimes B) &= (A_{ss} + A_n) \otimes (B_{ss} + B_n)\\ &= A_{ss} \otimes B_{ss} + A_{ss} \otimes B_n + A_n \otimes B_{ss} + A_n \otimes B_n\end{aligned}$$

and noting that the last three terms are nilpotent, and that all the terms commute with each other.

These statements together imply that if $S^i(A)$ is the operator induced by $A$ on the $i$-th symmetric power of $V$, then $S^i(A)_{ss} = S^i(A_{ss})$. Also, recall the obvious fact that $\mathrm{Tr}(A) = \mathrm{Tr}(A_{ss})$.

Given an LFD graded $\mathbf{C}$-vector space $M$ (or an *LFD space* for short) as in the introduction, which we assume henceforward to be $\mathbf{Z}_{\geq 0}$-graded, an endomorphism (which we always assume to be of degree 0 unless stated otherwise) $F$ is said to be semi-simple if all the $F|M^n$ are semi-simple. Equivalently, $F$ is semi-simple when restricted to any finite dimensional stable subspace. Denote by $F_{ss}$ the operator such that $F_{ss}|M^n = (F|M^n)_{ss}$ for all $n$ and call it the semi-simple part of $F$. Then clearly

$$\mathrm{Tr}(F, t) = \mathrm{Tr}(F_{ss}, t).$$

Another obvious fact concerning these traces is that if $F$ and $H$ are endomorphisms of ($\mathbf{Z}_{\geq 0}$-graded) LFD spaces $M$ and $N$, resp., then $M \otimes N$ has a natural structure of an LFD space:

$$(M \otimes N)^n = \oplus_{i+j=n} M^i \otimes N^j,$$

and

$$\mathrm{Tr}(F \otimes H, t) = \mathrm{Tr}(F, t)\mathrm{Tr}(H, t)$$

as formal power series.

Denote by $R = \oplus_{n=0}^{\infty} R^n$ the polynomial algebra $\mathbf{C}[X_1, ..., X_d]$ in d-variables, which has a natural structure of a LFD space. Let $R^+ = \oplus_1^{\infty} R^n$, the ideal $(X_1, ..., X_d)$, a LFD subspace of $R$. Fix $\rho : R \longrightarrow R$, a ring homomorphism which is $\mathbf{C}$-linear and of degree zero as a map of graded vector spaces. If $M$ is an $R$-module, then $F : M \longrightarrow M$ is said to be *$\rho$-semilinear* (or just *semi-linear*, if the reference to $\rho$ is clear) if $F(ax) = \rho(a)F(x)$ for all $a \in R, x \in M$. Henceforward, $M = \oplus_0^\infty M^n$ will be an LFD space with the structure of a graded $R$-module-an *LFD $R$-module*. Then $R^+M \subset M$ is a graded LFD submodule and we have an exact sequence of LFD $R$-modules:

$$0 \longrightarrow R^+M \longrightarrow M \longrightarrow M/R^+M \longrightarrow 0,$$

where the last term is graded in a standard fashion according to

$$(M/R^+M)^n = M^n/(M^n \cap R^+M).$$

Notice also that $M/R^+M$ is actually an $R/R^+ \simeq \mathbf{C}$-module.

We will be concerned only with finitely generated $R$-modules, which are actually Noetherian, since $R$ is Noetherian. In this case $M/R^+M$ is a finite-dimensional $\mathbf{C}$-vector space.

Assume now that $M$ is free as an $R$-module. By Nakayama's Lemma ([L1], Lemma VI.6.3), if $\{v_i(n)\}$ is a $\mathbf{C} = R/R^+$-basis for $(M/R^+M)^n$, so that $\cup_n\{v_i(n)\}$ is a basis for $M/R^+M$ consisting of homogeneous elements, then for any choice of liftings $\tilde{v}_i(n) \in M^n$, $\cup_n\{\tilde{v}_i(n)\}$ forms an $R$-basis for $M$ consisting, again, of homogeneous elements. Stated differently, if $V^n \subset M^n$ is a $\mathbf{C}$-subspace which is a $\mathbf{C}$-complement to $(R^+M)^n = (R^+M) \cap M^n$, *i.e.*,

$$M^n = (R^+M)^n \oplus V^n,$$

then, viewing $V^n$ with the $R$-module structure given by the projection $R \longrightarrow \mathbf{C}$, we get

$$M \simeq R \otimes_{\mathbf{C}} V := R \otimes_{\mathbf{C}} (\oplus_n V^n)$$

as an LFD (graded) $R$-module.

With these preliminaries, we can now state the

LEMMA. – *Let $M = \oplus_0^\infty M^n$ be a graded $R$-module which is finitely generated (and therefore, LFD) and free. Let $F : M \longrightarrow M$ be a semi-simple, semi-linear endomorphism of degree zero. Then there exists a finite dimensional graded $\mathbf{C}$-subspace $V \subset M$ such that*

$$M \simeq R \otimes_{\mathbf{C}} V$$

*in the sense mentioned above, and $V$ is $F$-stable, so that*

$$F = \rho \otimes (F|V)$$

*with respect to this isomorphism.*

*Proof.* – We have already stated how to obtain a tensor product decomposition for $M$ of the above sort. We need only show that $V$ can be chosen to be $F$-stable. Then the equality

$$F = \rho \otimes (F|V)$$

for $F$ will follow from the semi-linearity. But if $a \in R^+$ and $x \in M$, then the semi-linearity of $F$, $F(ax) = \rho(a)F(x)$, together with the fact that $\rho$ is of degree zero implies that $R^+M$ is $F$-stable. Also, each $M^n$ is $F$-stable by the assumption that $F$ is of degree zero. Therefore, each $(R^+M)^n = R^+M \cap M^n$ is an $F$-stable $\mathbf{C}$-subspace of $M^n$. But then, since $F$ is semi-simple, we can find an $F$-stable complement $V^n$. According to the discussion preceding the lemma, $V = \oplus_n V^n$ does the trick.

PROPOSITION. – *Let $M = \oplus_0^\infty M^n$ be a finitely-generated graded R-module, and let $F : M \longrightarrow M$ be a $\rho$-semi-linear endomorphism of degree zero. Then*

$$Tr(F, t) = Tr(\rho, t)P(t)$$

*for some finite degree polynomial $P(t)$.*

*Proof.* – We find an $F$-compatible resolution of $M$ as follows: Let $V_0 = \oplus_0^N M^n$ for some $N$ large enough so that the natural map

$$M_0 := R \otimes_{\mathbf{C}} V_0 \longrightarrow M$$

is an epimorphism of graded $R$-modules. $M_0$ carries the semi-linear operator $F_0 := \rho \otimes (F|V_0)$, which makes the preceding surjection equivariant with respect to $F$ and $F_0$. Define $M_1$ as the kernel of the above map:

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0.$$

Note that $M_1$ is then stable under the action of $F_0$, so that we can define $F_1 := F_0|M_1$, equipping each of the LFD R-modules in the exact sequence with a semi-linear operator in a compatible way. Continuing in this way, we get a graded resolution,

$$0 \longrightarrow M_d \longrightarrow M_{d-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

where each $M_j$ is equipped with an operator $F_j$ making the resolution equivariant. Furthermore, we have

$$M_j \simeq R \otimes V_j, \qquad F_j = \rho \otimes (F_j|V_j)$$

for each $j \leq d-1$, where the $V_j$'s are finite-dimensional stable subspaces. The last term $M_d$ is then just the kernel of $M_{d-1} \longrightarrow M_{d-2}$, while $F_d = F_{d-1}|M_d$. However, since $R$ is of homological dimension $d$, we get that $M_d$ must be projective and, therefore, free. (This also follows from Hilbert's Syzygy Theorem.) Now we can replace each operator by its semi-simple part and the resoluion above will remain equivariant. But

$$(F_j)_{ss} = \rho_{ss} \otimes (F_j|V_j)_{ss}, \qquad j \leq d-1.$$

Therefore, all the semi-simple parts will be $\rho_{ss}$-semi-linear, including $(F_d)_{ss}$ which is the restriction of $(F_{d-1})_{ss}$. Thus, according to the preceding Lemma, $M_d$ also has a tensor product decomposition $M_d \simeq R \otimes V_d$ for some finite-dimensional $F$-stable $V_d$ so that

$$(F_d)_{ss} = \rho_{ss} \otimes (F_d|V_d)_{ss}.$$

By the additivity of the trace, we conclude that

$$\begin{aligned}\mathrm{Tr}(F,t) &= \sum_i (-1)^i \mathrm{Tr}(F_i,t)\\ &= \sum_i (-1)^i \mathrm{Tr}((F_i)_{ss},t)\\ &= \mathrm{Tr}(\rho_{ss},t)\sum_i (-1)^i \mathrm{Tr}((F_i|V_i)_{ss},t)\\ &= \mathrm{Tr}(\rho,t)P(t)\end{aligned}$$

as desired.

## 4

To proceed towards the proof of Theorems 2 and 3, we recall some spectral sequences associated to equivariant cohomology. Namely, by considering the fibration

$$\begin{array}{c} X_{hG} \\ \downarrow \\ BG \end{array}$$

with fibre $X$, we get a spectral sequence converging to $H^*_G(X)$ whose $E_2$ term is given by

$$E_2^{p,q} = H^p(BG, H^q(X)).$$

As the spectral sequence for several different groups occur below, we will sometimes keep track by building the group into the notation as $E_r^{p,q}(G)$.

In particular, in the connected case $G = G^0$, since $\pi_1(BG) = 0$, we get

$$E_2^{p,q} = H^p(BG) \otimes H^q(X) \;\Rightarrow H_G^{p+q}(X).$$

We give now an explicit description of $F(f,\phi)$. Consider the principal $G$ bundle $EG{\times}_G\phi(G)$ associated to $EG$ by the action of $G$ on itself via $\phi{:}g \cdot x = \phi(g)x$. This gives rise to the Cartesian diagram

$$\begin{array}{ccc} EG{\times}_G\phi(G) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ BG & \xrightarrow{B\phi^*} & BG, \end{array}$$

whence

$$\begin{array}{ccc} EG{\times}_G\phi(G){\times}_G X & \longrightarrow & X_{hG} \\ \downarrow & & \downarrow \\ BG & \xrightarrow{B\phi^*} & BG. \end{array}$$

On the other hand, $f$ gives rise to a bundle map

$$X_{hG} = EG{\times}_G X \longrightarrow EG{\times}_G\phi(G){\times}_G X$$

induced by $(p,x) \mapsto (p,e,f(x))$. $F = F(f,\phi) : X_{hG} \longrightarrow X_{hG}$ is the composition of this bundle map with the upper arrow of the previous diagram. It is readily checked that for two equivariant pairs $(f,\phi)$ and $(h,\psi)$, we get

$$F(h \circ f, \psi \circ \phi) = F(h,\psi) \circ F(f,\phi),$$

which can also be deduced, from the simplicial point of view, from the equality on the level of functors. In particular, when $G$ is connected, we see that the map $B\phi^* \otimes f^*$ acting on the $E_2$ term of the above mentioned spectral sequence induces on $E_\infty$ a map compatible with $F^*$ acting on $H_G^*(X)$. That is, the $F^*$-action on $H_G^*(X)$ induces on the graded quotients $E_\infty^*$ of $H_G^*(X)$ an operator agreeing with that induced by $B\phi^* \otimes f^*$. Turn each term $E_r$ of the spectral sequence into a single $\mathbf{Z}_{\geq 0}$ graded complex $E_r^*$ in the usual way

$$E_r^n := \oplus_{p+q=n} E_r^{p,q}$$

with the differentials $d_r$ acting as degree one operators and endomorphisms $F_r = F_r(f,\phi)$ of degree zero, commuting with the differentials, induced by $B\phi^* \otimes f^*$.

A particular case arises when a Lie subgroup $K \subset G$ is normalized by $g \in G$. Then we get a homomorphism $c(g) : K \longrightarrow K$ and $(g, c(g))$ forms an equivariant pair for the action of $K$ on $X$: $g(kx) = c(g)(k)gx$, for $k \in K$ and $x \in X$. Thus we get a map

$$F(g, c(g)) : H_K^*(X) \longrightarrow H_K^*(X)$$

which, when $K$ is connected, is compatible with the map on the spectral sequence for $H_K^*(X)$ arising from the action of $Bc(g)^* \otimes g^*$ on the $E_2$-term $H^*(BK) \otimes H^*(X)$. When $G$ is connected and $g \in G$, then $F(g,c(g)) : H_G^*(X) \longrightarrow H_G^*(X)$ is just the identity, and the same is true for the maps $F_r(g,c(g))$ on the terms $E_r(G)$ of the spectral sequence since all the relevant topological maps lie in the connected component of the identity. Thus, for a general compact group $G$, if $C$ denotes as before the group of connected components, then $C$ acts on $H_{G^0}^*(X)$ as well as on the $E_r(G^0)$, giving rise to the well known isomorphisms

$$H_G^*(X) \simeq H_{G^0}^*(X)^C, \quad E_r^*(G) \simeq E_r^*(G^0)^C$$

induced by the natural inclusions.

Also, by the same reasoning, if $T \subset G^0$ is a maximal torus and $W = N(T)/T$ its Weyl group, then we have natural isomorphisms

$$H_{G^0}^*(X) \simeq H_T^*(X)^W, \quad E_r^*(G^0) \simeq E_r^*(T)^W$$

as is again well-known.

## 5

*Proof of Theorem 2.* – As mentioned previously, replacing $(f,\phi)$ by $(g\circ f, c(g)\circ\phi)$ for some $g \in G^0$ doesn't affect the trace on $H^*_G(X)$ or $H^*_{G^0}(X)$ so we may assume from the beginning that $\phi$ preserves a maximal torus. Then

$$
\begin{aligned}
(4)\quad \mathrm{Tr}(F(f,\phi)^*|H^*_{G^0}(X),t) &= \mathrm{Tr}(F(f,\phi)^*|H^*_T(X)^W,t)\\
&= (1/\|W\|)\sum_{w\in W}\mathrm{Tr}(F(f,\phi)^*\circ F(w,c(w))^*|H^*_T(X),t)\\
&= (1/\|W\|)\sum_{w\in W}\mathrm{Tr}(F(w\circ f,c(w)\circ\phi)^*|H^*_T(X),t),
\end{aligned}
$$

reducing the case of a connected group to the case of a torus. But also, when $G$ is disconnected with component group $C$, we have, in a similar manner,

$$
(5)\qquad \mathrm{Tr}(F(f,\phi)^*|H^*_G(X),t) = (1/\|C\|)\sum_{\gamma\in C}\mathrm{Tr}(F(\tilde\gamma^{-1}\circ f, c(\tilde\gamma^{-1})\circ\phi)^*|H^*_{G^0}(X),t),
$$

where $\tilde\gamma$ is any lifting to $G$ of $\gamma \in C$. Thus it suffices to prove the theorem for the case of a torus.

Recall that

$$
H^*(BT) \simeq S^*(H^2(BT)) \simeq \mathbf{C}[X_1,...,X_d],
$$

a polynomial algebra in $d$-variables where $d$ is the dimension of $T$. Furthermore, the map $B\phi^*$ on $H^*(BT)$ is just given by the symmetric powers of $B\phi^*|H^2(BT)$. Thus

$$
\mathrm{Tr}(B\phi^*|H^*(BT),t) = \prod(1-\lambda_i t^2)^{-1},
$$

where the $\lambda_i$ are the eigenvalues of $B\phi^*|H^2(BT)$.

On the other hand, the diagonal embedding

$$
H^*(BT) \hookrightarrow H^*(BT)\otimes H^0(X) = E_2^{*,0},
$$

together with the product structure on the spectral sequence, induces a $H^*(BT)$-module structure on each $E_r$ making it into a LFD graded $H^*(BT)$-module. Each of the operators $F_r$ on $E^*_r(T)$ is then evidently $B\phi^*$-semi-linear since this is true of $F_2$. Also, since $E_2 = H^*(BT)\otimes H^*(X)$ is finitely generated as a $H^*(BT)$-module, the same is true of each $E_r$, begin subquotients of $E_2$. In particular, these facts are true of $F_\infty$ and $E_\infty$ which actually occurs at a finite level, since $E_2$ is vertically bounded. Therefore, by the Proposition of Section 3, we get that

$$
\begin{aligned}
(6)\qquad \mathrm{Tr}(F^*|H^*_T(X),t) &= \mathrm{Tr}(F^*|E_\infty,t)\\
&= \mathrm{Tr}(B\phi^*|H^*(BT),t)P(t)\\
&= P(t)\prod(1-\lambda_i t^2)^{-1}
\end{aligned}
$$

which is a rational function.

*Proof of Theorem 3.* – Since $g^{-1}\circ f$ and $c(g^{-1})\circ\phi$ are transverse (to the diagonal) for all $g \in G$, we may assume that $R$ as in Proposition 6 of Section 2 has been chosen so that each $c(\tilde{\gamma}^{-1})\circ\phi$ preserves a fixed maximal torus and are transverse to the diagonal of $T$. We have already discussed why this does not affect the trace side. Then for any $w \in W$, the Weyl group of $T$ in $G^0$, each $c(w)\circ c(\tilde{\gamma}^{-1})\circ\phi$ is also transverse.

Now, given an equivariant pair $(h, \psi)$ for the action of $T$ on $X$, such that $\psi : T \longrightarrow T$ is transverse, notice that $\psi^*|H^1(T)$ is dual to $d\psi(e) : \mathrm{Lie}(T) \longrightarrow \mathrm{Lie}(T)$. We also have the transgression isomorphism $H^1(T) \simeq H^2(BT)$ which takes the action of $\psi^*$ to the action of $B\psi^*$. Hence, $B\psi^*|H^2(BT)$ does not have 1 as an eigenvalue. Thus, in the formula (6) obtained at the end of the previous section, $t = -1$ is a regular value of $\mathrm{Tr}(F(h,\psi)^*|H^*_T(X), t)$.

Then by the preceding paragraph and (4), (5) of Section 5, we get that $-1$ is a regular value of $\mathrm{Tr}(F(f,\phi)|H^*_{G^0}(X), t)$ as well as of $\mathrm{Tr}(F(f,\phi)|H^*_G(X), t)$.

To compute this last trace, notice that the arguments of Section 5 actually tell us that

$$\begin{aligned}&\mathrm{Tr}(F_r(\tilde{\gamma}^{-1}\circ f, c(\tilde{\gamma}^{-1})\circ\phi)|E_r(G^0), t)\\ &\quad = (1/\|W\|)\sum_{w\in W}\mathrm{Tr}(F_r(w\circ\tilde{\gamma}^{-1}\circ f, c(w)\circ c(\tilde{\gamma}^{-1})\circ\phi)^*|E_r(T), t)\end{aligned}$$

for each $r \geq 2$, allowing us to compare traces for $G^0$ and $T$ on each term of the corresponding spectral sequences.

Keeping this in mind, we argue again for a general equivariant pair with respect to the $T$-action: An easy computation shows us that

$$\begin{aligned}&\mathrm{Tr}(F_r(h,\psi)^*|E_r(T), t)\\ &\quad = \mathrm{Tr}(F_{r+1}(h,\psi)^*|E_{r+1}(T), t) + (t+1)\mathrm{Tr}(F_r(h,\psi)^*|I_r(T), t),\end{aligned}$$

where $I_r(T) \subset E_r(T)$ is the LFD sub-module such that $I^n_r(T) = d_r(E^n_r(T))$. But $I_r$, being a submodule of $E_r$, is also finitely generated as an $H^*(BT)-$module. Thus we conclude that $\mathrm{Tr}(F_r(h,\psi)^*|I_r(T), t)$ is also a rational function regular at $t = -1$. Therefore, we get

$$\mathrm{Tr}(F_r(h,\psi)^*|E_r(T), -1) = \mathrm{Tr}(F_{r+1}(h,\psi)^*|E_{r+1}(T), -1)$$

for each $r \geq 2$ and, in particular,

$$\begin{aligned}\mathrm{Tr}(F_r(h,\psi)^*|H^*_T(X), -1) &= \mathrm{Tr}(F_{r+1}(h,\psi)^*|E_\infty(T), -1)\\ &= \mathrm{Tr}(F_2(h,\psi)^*|E_2(T), -1).\end{aligned}$$

So , by (4), (5), for our $f, \phi$ and any $\tilde{\gamma} \in R$,

$$\begin{aligned}\mathrm{Tr}(F_r(\tilde{\gamma}^{-1}\circ f), c(\tilde{\gamma}^{-1})\circ\phi)^*|H^*_{G^0}(X), -1) &= \mathrm{Tr}(F_2(\tilde{\gamma}^{-1}\circ f), c(\tilde{\gamma}^{-1})\circ\phi)^*|E_2(G^0), -1)\\ &= \mathrm{Tr}((\tilde{\gamma}^{-1}\circ f)^*|H^*(X), -1)\mathrm{Tr}(B(c(\tilde{\gamma}^{-1})\circ\phi)^*|H^*(BG^0), -1).\end{aligned}$$

We have

$$\mathrm{Tr}((\tilde{\gamma}^{-1}\circ f)^*|H^*(X),-1) = \|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|$$

by the usual trace formula on $X$.

Now, according to [B2], we can relate the cohomology of $G^0$ and $BG^0$ as follows:

$$H^*(G^0) \simeq \wedge^*(V_1) \otimes \cdots \otimes \wedge^*(V_k),$$

for subspaces $V_i \subset H^*(X)$ of odd degree $d_i$ and

$$H^*(BG^0) \simeq S^*(W_1) \otimes \cdots \otimes S^*(W_k),$$

for subspaces $W_i \subset H^*(BG^0)$ of degree $d_i + 1$. In this presentation, $W_i$ is the image of $V_i$ under the transgression homomorphism arising from the spectral sequence for the fibration

$$\begin{array}{c} EG^0 \\ \downarrow \\ BG^0 \end{array}$$

which, furthermore, sends $(c(\tilde{\gamma}^{-1})\circ\phi)^*|V_i$ to $B(c(\tilde{\gamma}^{-1})\circ\phi)^*|W_i$. Thus, as an easy consequence, we get

$$\mathrm{Tr}((c(\tilde{\gamma}^{-1})\circ\phi)^*,-1) = \mathrm{Tr}(B(c(\tilde{\gamma}^{-1})\circ\phi)^*,-1).$$

We can now apply the trace formula on $G^0$ to get

$$\mathrm{Tr}(B(c(\tilde{\gamma}^{-1})\circ\phi)^*,-1) = \tau(\tilde{\gamma})\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|,$$

since the multiplicity of $c(\tilde{\gamma}^{-1})\circ\phi$ at any fixed point is equal to the multiplicity at the origin, namely, $\tau(\tilde{\gamma})$. This follows from the fact that if $\psi : G \longrightarrow G$ is any homomorphism and $g$ is a fixed point, then $\psi\circ L_g = L_{\psi(g)}\circ\psi = L_g\circ\psi$.

So we finally arrive at the formula

$$\mathrm{Tr}(F_r(\tilde{\gamma}^{-1}\circ f), c(\tilde{\gamma}^{-1})\circ\phi)^*|H^*_{G^0}(X),-1) = \tau(\tilde{\gamma})(\|X_+^{\tilde{\gamma}^{-1}\circ f}\| - \|X_-^{\tilde{\gamma}^{-1}\circ f}\|)/\|(G^0)^{c(\tilde{\gamma}^{-1})\circ\phi}\|$$

which, when combined with (5) and Proposition 6 of Section 2, yields the trace formula.

## 7

We give two elementary examples where the trace formula can be checked directly.

Let our groupoid be given by the finite group $G$ acting on the finite set $S$ and let $f : S \longrightarrow S, \phi : G \longrightarrow G$ satisfy $f(gs) = \phi(g)f(s)$. In fact, for any map $h : S \longrightarrow S$, we have the trace formula for finite sets

$$\|S^h\| = \mathrm{Tr}(h^*|\mathbf{C}[S])$$

where $\mathbf{C}[S]$ is the complex vector space of functions on $S$ to which the $h$-action extends naturally as a linear map. $\mathbf{C}[S]$ becomes also a complex linear representation of $G$. The complex coefficient equivariant cohomology of $S$ exists only in degree 0 and, in fact,

$$H_G^0(S,\mathbf{C}) = \mathbf{C}[S]^G.$$

Again, if $\mathcal{O}$ denotes a set of orbit representatives for the action of $G$ on itself via $g : x \mapsto \phi(g)xg^{-1}$, we can verify the trace formula directly starting from Proposition 6:

$$\begin{aligned}\sum_{g\in\mathcal{O}} \|S^{g^{-1}\circ F}\|/\|G^{c(g^{-1})\circ\phi}\| &= \sum_{g\in\mathcal{O}} \mathrm{Tr}(f^*\circ(g^{-1})^*|\mathbf{C}[S])/\|G^{c(g^{-1})\circ\phi}\|\\ &= (1/\|G\|)\sum_{g\in\mathcal{O}}[G : G^{c(g^{-1})\circ\phi}]\mathrm{Tr}(f^*\circ(g^{-1})^*|\mathbf{C}[S])\\ &= (1/\|G\|)\sum_{g\in G}\mathrm{Tr}(f^*\circ(g^{-1})^*|\mathbf{C}[S])\\ &= \mathrm{Tr}(f^*|\mathbf{C}[S]^G)\end{aligned}$$

We use here the fact that if $h = \phi(\gamma)g\gamma^{-1}$, then

$$\begin{aligned}\mathrm{Tr}(f^*\circ(h^{-1})^*) &= \mathrm{Tr}(f^*\circ\phi(\gamma^{-1})^*\circ(g^{-1})^*\gamma^*)\\ &= \mathrm{Tr}((\gamma^{-1})^*\circ f^*\circ(g^{-1})^*\gamma^*)\\ &= \mathrm{Tr}(f^*\circ(g^{-1})^*).\end{aligned}$$

Another simple case arises when $G = S^1$ acting on $X$ = point, and $\phi : z \longrightarrow z^n$, $n \neq \pm 1$, while $f$ is the identity. In this case, $X_{hG} = BS^1$ and the map $F(f,\phi)$ is just $B\phi = Bn$. The fixed points of $\phi$ on $S^1$ are the $(n-1)$-th roots of unity if $n \geq 0$ and the $(1-n)$-th roots of unity if $n \leq 0$. It is readily checked that $B\phi^* : H^2(BS^1)\longrightarrow H^2(BS^1)$ is multiplication by $n$ so that $B\phi^*|H^{2i}(BS^1) = (n)^i$). Thus,

$$\mathrm{Tr}(B\phi^*|H^*_{S^1}(\cdot),t) = \sum_0^\infty (t^2n)^i = 1/(1-t^2n),$$

giving

$$\mathrm{Tr}(B\phi^*|H^*_{S^1}(\cdot),-1) = 1/(1-n)$$

Meanwhile, since $d\phi(e) = n$, the index of each object $(\text{point},\zeta_k)$ is seen to be just $\mathrm{sign}(1-n)$.

So the trace formula says that

$$\mathrm{sign}(1-n)/(S^1)^\phi = 1/(1-n),$$

which is correct for both positive and negative $n$.

## REFERENCES


[A-M] M. Artin and B. Mazur, *On periodic points* (*Ann. of Math.*, (2), Vol. 81, 1965, pp. 82-99).

[B1] K. Behrend, *The Lefschetz trace formula for algebraic stacks* (*Invent. Math.*, Vol. 111, 1993, pp. 1-33).

[B2] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compact* (*Ann. of Math.*, Vol. 57, 1953, pp. 115-207).

[G-Z] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory* (*Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 35, Springer-Verlag, Berlin-Heidelberg-New York, 1967).

[G] H. Gillet, *Intersection theory on algebraic stacks and Q-varieties* (*J. Pure Appl. Algebra*, Vol. 34, 1984, pp. 193-240).

[L1] S. Lang, *Algebra*, 2nd. edition, Addison-Wesley, Reading, Massachusetts, 1984.

[L2] S. Lang, *Algebraic groups over finite fields* (*Amer. J. Math.*, Vol. 78, no. 3, 1956, pp. 555-563).

[M] S. MacLane, *Categories for the working mathematician* (*Graduate Texts in Mathematics*, Springer-Verlag, New York-Heidelberg-Berlin, 1971).

[S] G. Segal, *Classifying spaces and spectral sequences* (*Publ. Math. I.H.E.S.*, Vol. 34, 1966, pp. 105-112).


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