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ON CONTACT SUB-RIEMANNIAN SYMMETRIC SPACES

By Elisha FALBEL and Claudio GORODSKI

ABSTRACT. – A *Sub-Riemannian manifold* is a smooth manifold which carries a metric defined only on a smooth distribution. There is a concept of *sub-Riemannian symmetric space*, it is the analogue of a Riemannian symmetric space in this context. In this paper we attempt to study sub-Riemannian symmetric spaces where the associated distribution is a codimension one contact distribution. We use a canonical connection defined on contact sub-Riemannian manifolds to give a characterization of contact sub-Riemannian symmetric spaces in terms of the curvature and torsion tensors. Furthermore, we linearize the sub-Riemannian symmetric structure and obtain a restricted classification theorem.

0. Introduction

Sub-Riemannian geometry is concerned with the study of a smooth manifold equipped with a smooth distribution which carries a metric, henceforth a *sub-Riemannian manifold*. *See* [10] for an introduction and references on the subject. We will restrict our attention to sub-Riemannian manifolds where the associated distribution is a codimension one contact distribution. This is the simplest interesting case in this geometry and has the advantage of having a canonical connection defined in [6, 5] which generalizes the pseudo-Hermitian connection of [13]. In this case there also exists a characteristic direction transversal to the distribution, and part of the torsion tensor, which we call *sub-torsion*, measures the rate of change of the metric along that direction.

It is worth mentioning here a relation with CR-structures: the invariants of CR geometry with non-degenerate Levi-form are invariants of the conformal geometry of contact sub-Riemannian manifolds (*see* [4]).

The analogue of a Riemannian symmetric space in the context of sub-Riemannian geometry, a *sub-Riemannian symmetric space*, or more briefly, *sub-symmetric space*, was introduced by Strichartz in [10]. It is a homogeneous sub-Riemannian manifold for which there exists an involutive isometry at each point which is a central symmetry when restricted to the distribution. Strichartz classified the three dimensional sub-symmetric spaces, they fall into six classes which include Lie groups of semisimple, nilpotent and solvable type. In this paper we pursue the concept further in the contact case and arbitrary dimension.

The first result of this paper is a *local* characterization of sub-symmetric spaces by means of the adapted connection, namely, *a sub-Riemannian manifold is locally sub-symmetric if and only if the curvature and torsion tensors of the adapted connection are parallel along the distribution* (*cf.* Theorem 2.1).

Furthermore, we linearize the structure of sub-symmetric spaces by means of a special class of involutive Lie algebras, the so called *sub orthogonal involutive Lie (sub-OIL) algebras*. The assumption that the sub-symmetric space is of contact type is very strong and prohibits a reasonable decomposition theorem preserving the contact structure as shown by the second example in Section 5. Therefore it is convenient to restrict the classification to the "irreducible" sub-symmetric spaces. We obtain that *every irreducible simply connected sub-symmetric space is a homogeneous manifold canonically fibered over an irreducible Hermitian symmetric space with fibers diffeomorphic to a circle and generated by the flow of the characteristic field. The distribution is then uniquely defined and the sub-Riemannian metric is also uniquely defined as the pull-back of the Riemannian metric on the base (and hence it has null sub-torsion) except in two cases, where there exists a two-parameter family of sub-Riemannian metrics, and a one-parameter sub-family of them has null sub-torsion* (*cf.* Theorem 5.1 and the ensuing example).

Finally, we distinguish a class of sub-OIL algebras associated to the Heisenberg group which play the role of the Euclidean algebras in the theory of Riemannian symmetric spaces and show that the only sub-symmetric space with a nilpotent isometry group is the Heisenberg group (*cf.* Theorems 4.1 and 4.2).

We now state some open problems and difficulties.

It is not known whether the homogeneity assumption in the definition of sub-symmetric spaces is essential, even with our assumption that sub-symmetric spaces are always of contact type.

The proof of the local characterization of sub-symmetric spaces makes use of geodesic coordinates of the adapted connection. It would be desirable, and more natural, to use sub-Riemannian geodesics (*see* [10]) and a notion of parallelism along them.

The failure of the decomposition theorem in the contact case shows that it is important to consider sub-symmetric spaces of non-contact type. The relation to affine symmetric spaces is relevant as non-orthogonal involutive Lie algebras may appear as sub-OIL algebras of those spaces. An analysis of the list of simple involutive Lie algebras compiled by Berger [1] will probably supply new examples in the general case. On the other hand sub-symmetric spaces may provide new geometric motivation for deeper study of involutive Lie algebras.

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1. Sub-Riemannian geometry

A *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, g)$ where M is an oriented manifold, $\mathcal{D}$ is an oriented smooth distribution on M and g is a smoothly varying positive definite symmetric bilinear form defined on $\mathcal{D}$.

In this paper we shall consider only the case in which $\mathcal{D}$ is a *contact* distribution on M. Let dV be the volume form on $\mathcal{D}$ and let $n = \frac{1}{2}\dim\mathcal{D}$. The (normalized) *contact form* is the 1-form θ on M such that

$$\ker\theta = \mathcal{D},$$
$$d\theta^n|_{\mathcal{D}} = n!\, 2^n\, dV.$$

Observe that it is uniquely defined when n is odd, but it is defined up to sign when n is even. To solve that ambiguity, we impose furthermore that the orientation of M is defined by $\theta \wedge d\theta^n$.

Since $d\theta$ has rank $2n$ and both M and $\mathcal{D}$ are oriented, there is a unique vector field ξ on M such that

$$\theta(\xi) = 1,$$
$$\iota_\xi d\theta = 0.$$

It is called the *characteristic vector field.*

Note that the sub-Riemannian metric g has a natural extension to a Riemannian metric on M by setting ξ to be orthonormal to $\mathcal{D}$.

A canonical connection analogous to the Levi-Civita connection in the case of Riemannian geometry is uniquely defined on M. Let $\underline{TM}$ and $\underline{\mathcal{D}}$ denote respectively the set of sections of TM and of $\mathcal{D}$.

THEOREM 1.1 [6]. – *There exists a unique connection* $\nabla : \underline{TM} \to \underline{TM}^* \otimes \underline{TM}$, *called the* adapted connection, *and a unique symmetric tensor* $\tau : \mathcal{D} \to \mathcal{D}$, *called the* sub-torsion, *with the following properties (T is the torsion tensor of the connection):*

a. $\nabla_U : \underline{\mathcal{D}} \to \underline{\mathcal{D}}$;

b. $\nabla \xi = 0$;

c. $\nabla g = 0$;

d. $T(X,Y) = d\theta(X,Y)\xi,$
$T(\xi,X) = \tau(X)$;

for X, $Y \in \underline{\mathcal{D}}$, $U \in \underline{TM}$.

Proof. – Let X, Y, $Z \in \underline{\mathcal{D}}$. As is Riemannian geometry, a., c. and d. uniquely define $\nabla_X Y$:

$$\begin{aligned} &X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \\ &\quad 2\langle \nabla_X Y, Z\rangle \\ &\quad + \langle Y,[X,Z]+T(X,Z)\rangle + \langle X,[Y,Z]+T(Y,Z)\rangle + \langle Z,[Y,X]+T(Y,X)\rangle \end{aligned}$$

Because of b., it remains only to define $\nabla_\xi X$. But $\nabla_\xi X - \nabla_X \xi = [\xi,X] + T(\xi,X)$, so

$$\nabla_\xi X = [\xi,X] + \tau(X).$$

Finally,

$$\begin{aligned} \xi\langle X,Y\rangle &= \langle \nabla_\xi X, Y\rangle + \langle X, \nabla_\xi Y\rangle \\ &= \langle [\xi,X]+\tau(X),Y\rangle + \langle X,[\xi,Y]+\tau(Y)\rangle \\ &= \langle [\xi,X],Y\rangle + \langle [\xi,Y],X\rangle + 2\langle \tau(X),Y\rangle \end{aligned}$$

determines $\tau(X)$ (note that

$$0 = d\theta(\xi,X) = \xi(\theta(X)) - X(\theta(\xi)) - \theta([\xi,X]) = -\theta([\xi,X]),$$

so $[\xi,X] \in \underline{\mathcal{D}}$). □

COROLLARY 1.1 [6]. – *The connection ∇ has the following properties:*

a. $L_\xi : \underline{\mathcal{D}} \to \underline{\mathcal{D}}$;

b. $d\theta(X,Y) = \theta(T(X,Y))$;

c. $\langle \tau(X), Y\rangle = \frac{1}{2}L_\xi g(X,Y)$;

for $X, Y \in \underline{\mathcal{D}}$.

COROLLARY 1.2 [6]. – *The characteristic vector field ξ is a Killing field on M if and only if $\tau = 0$.*

The curvature of this connection is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

From general theory of connections we have the Bianchi identity

$$\mathfrak{S}R(X,Y)Z = \mathfrak{S}T(T(X,Y),Z) + \mathfrak{S}(\nabla_X T)(Y,Z).$$

where $\mathfrak{S}$ denotes the cyclic summation in X, Y and Z. In the case of the adapted connection we get the following identities

$$\mathfrak{S}R(X,Y)Z = \mathfrak{S}d\theta(X,Y)\tau(Z),$$
$$R(\xi,Y)Z - R(\xi,Z)Y = (\nabla_Z\tau)(Y) - (\nabla_Y\tau)(Z).$$

where $X, Y, Z \in \underline{\mathcal{D}}$.

A *local isometry* between two sub-Riemannian manifolds $(M, \mathcal{D}, g)$ and $(M', \mathcal{D}', g')$ is a diffeomorphism between open sets $\psi : U \subset M \to U' \subset M'$ such that $\psi_*(\mathcal{D}) = \mathcal{D}'$ and $\psi^* g' = g$. In the contact case it follows that that $\psi^*\theta' = \pm\theta$ and $\psi_*\xi = \pm\xi'$ (and therefore ψ will be a local Riemannian isometry relative to the extended Riemannian metrics on M and on M'). If ψ is globally defined on M to M', we say simply that ψ is an *isometry*.

Observe that an isometry $\psi : M \to M'$ is affine with respect to the adapted connections, that is, $\nabla'_{\psi_* X}\psi_* Y = \psi_*(\nabla_X Y)$ for $X, Y \in \underline{TM}$.

2. Sub-Riemannian symmetric spaces

The definition of sub-symmetric space was given by Strichartz in [10]. Since we have restricted our investigation to contact distributions, we will use a simplified definition. A *sub-Riemannian symmetric space* (or *sub-symmetric space*) is an homogeneous sub-Riemannian manifold $(M, \mathcal{D}, g)$ such that for every point $x_0 \in M$ there is an isometry ψ, called the *sub-symmetry* at x_0, with $\psi(x_0) = x_0$ and $\psi_*|_{\mathcal{D}_{x_0}} = -1$ (in the contact case it follows that $\psi_*(\xi_{x_0}) = \xi_{x_0}$, where ξ is the characteristic field).

It is easy to see that the sub-symmetry at a point x_0 must be unique; in fact, it is given by $\exp_{x_0}(X) \mapsto \exp_{x_0}(\psi_{*x_0}X)$, where exp is the affine exponential map associated to the adapted connection. Moreover, by homogeneity it is enough to check the existence of the sub-symmetry at one single point of M.

The main result of this section (*see* Theorem 2.1) is a characterization of sub-symmetric spaces in terms of the curvature and torsion tensors of the underlying sub-Riemannian manifold, analogous to the Riemannian case. However, for that we will need only the weaker concept of a *locally sub-symmetric space*. It is a sub-Riemannian manifold $(M, \mathcal{D}, g)$ such that for every point $x_0 \in M$ there exists a local isometry ψ such that $\psi(x_0) = x_0$ and $\psi_*|_{\mathcal{D}_{x_0}} = -1$. Observe that, unlike the global case, we do not require homogeneity here.

THEOREM 2.1. – *A sub-Riemannian manifold is locally sub-symmetric if and only if the following conditions are verified:*

a. $\nabla_X T = 0$;

b. $\nabla_X R = 0$;

for all $X \in \mathcal{D}$.

Before proving the above theorem, we will recall one of Cartan's results (*cf.* [3], p. 238) as it is formulated in [14].

Let (M, ∇) and (M', ∇') be two affine manifolds. Let $p \in M$ and $p' \in M'$ and choose a linear isomorphism $\phi : TM_p \to TM'_{p'}$. Choose convex open subsets $V \subset TM_p$ and $V' \subset \phi(V)$ which determine normal coordinate neighborhoods $U = \exp_p(V)$ and $U = \exp_p(V)$. Define a diffeomorphism $f : U \to U'$ by $f(\exp_p Z) = \exp_{p'}(\phi Z)$ for $Z \in V$. Define also the linear isomorphisms $\phi_z : TM_z \to TM'_{f(z)}$ for $Z \in U$, given by $\phi(\tau Y) = \tau'(\phi Y)$ for all $Y \in TM_z$, where τ is parallel translation along the radial geodesic $\exp_p(tZ)$ from p to $z = \exp_p(Z)$, and τ' is parallel translation along the radial geodesic $\exp_{p'}(t\phi Z)$ from p' to $z = \exp_p(\phi Z)$.

THEOREM 2.2 [14]. – *Let R, T and R', T' denote the curvature and torsion tensors of M and M'. Suppose for every $z \in U$ that ϕ_z sends R_z to $R'_{f(z)}$ and T_z to $T'_{f(z)}$. Then $f : U \to U'$ is an affine diffeomorphism, $f_* : TM_z \to TM'_{f(z)}$ is just ϕ_z, and $f_* : TM_p \to TM'_{f(p)}$ is ϕ. Furthermore, f is the only affine diffeomorphism which induces ϕ on TM_p.*

Proof of Theorem 2.1. – 1) Suppose M is sub-symmetric. The sub-symmetry ψ is an affine map with respect to the adapted connection.

a. We compute for $Z \in \mathcal{D}$ and $X, Y \in \mathcal{D}$:

$$\psi_*(\nabla_Z T(X,Y)) = \nabla_{\psi_* Z} T(\psi_* X, \psi_* Y) = \nabla_{-Z} T(-X,-Y) = -\nabla_Z T(X,Y).$$

Observe that $T(X,Y)$ is parallel to ξ. This implies that $\psi_*(\nabla_Z T(X,Y)) = \nabla_Z T(X,Y)$. Comparing with the equality obtained above, we find

$$\nabla_Z T(X,Y) = 0.$$

We compute next

$$\psi_*(\nabla_Z T(X,\xi)) = \nabla_{\psi_* Z} T(\psi_* X, \psi_* \xi) = \nabla_{-Z} T(-X,\xi) = \nabla_Z T(X,\xi).$$

Now $T(X,\xi) \in \mathcal{D}$, so $\psi_*(\nabla_Z T(X,\xi)) = -\nabla_Z T(X,\xi)$, and together with the equality above we obtain

$$\nabla_Z T(X,\xi) = 0.$$

b. To analyze the covariant derivative of the curvature tensor we first observe that $R(X, Y)W \in \mathcal{D}$ for any vectors X, Y, W. We need to analyze both expressions,

$$\nabla_Z R(X, Y)W \quad \text{and} \quad \nabla_Z R(\xi, Y)W \qquad \text{for } X,\ Y,\ W \in \mathcal{D}.$$

In the first case we have

$$-\nabla_Z R(X, Y)W = \psi_* \nabla_Z R(X, Y)W = \nabla_{\psi_* Z}(\psi_* X, \psi_* Y)\psi_* W = \nabla_Z R(X, Y)W.$$

Therefore

$$\nabla_Z R(X, Y)W = 0.$$

The second case follows from the easily observed property

$$R(\xi, Y)W = 0.$$

2) We now suppose that conditions a. and b. are satisfied and proceed as in the proof of the Riemannian case making use of Theorem 2.2. The problem in this case is that the curvature and torsion tensors are not parallel along the geodesic rays, so the use of that theorem depends on describing the tensors along those rays. We will find differential equations satisfied by the curvature and torsion tensors along the geodesic rays.

Suppose $\{X_i\} = \{X_1, \ldots, X_{2n}, X_{2n+1} = \xi\}$ is an adapted frame at the point $p \in M$ where $d\theta(X_1, X_2) \neq 0$ and denote by the same symbols $\{X_i\}$ the frame obtained by parallel translation along geodesic rays. We have

$$\begin{aligned} R(X_i, X_j)X_l &= R^k_{lij} X_k, \\ T(X_i, X_j) &= T^k_{ij} X_k. \end{aligned}$$

Let $Z = a^j X_j$ be a direction at p. Then $Z = a^j X_j$ is the tangent along the geodesic ray in this direction. Write

$$\nabla_Z (R(X_i, X_j)X_l) = \dot{R}^k_{lij} X_k, \tag{1}$$

$$\nabla_Z (T(X_i, X_j)) = \dot{T}^k_{ij} X_k. \tag{2}$$

Write also $Z = Z' + a\xi$ where $Z' \in \mathcal{D}$. Using condition a. we get

$$\begin{aligned} \dot{R}^k_{lij} X_k &= \nabla_{Z' + a\xi}(R(X_i, X_j)X_l) \\ &= a \nabla_\xi (R(X_i, X_j)X_l) \\ &= a h^{-1} \nabla_{[X_1, X_2]}(R(X_i, X_j)X_l), \end{aligned}$$

and analogously for the torsion tensor, where $h = \theta([X_1, X_2])$ is a function to be determined.

We now claim that

$$(3)\qquad \begin{aligned}&(\nabla_{[X_1,X_2]}R)(X_i,X_j,X_k)\\ &\quad= R(R(X_1,X_2)X_i,X_j,X_k)+R(X_i,R(X_1,X_2)X_j,X_k)\\ &\qquad+R(X_i,X_j,R(X_1,X_2)X_k)-R(X_1,X_2)R(X_i,X_j,X_k),\end{aligned}$$

$$(4)\qquad \begin{aligned}&(\nabla_{[X_1,X_2]}T)(X_i,X_j)\\ &\quad= T(R(X_1,X_2)X_i,X_j)+T(X_i,R(X_1,X_2)X_j)\\ &\qquad-R(X_1,X_2)T(X_i,X_j).\end{aligned}$$

In fact, we first observe that

$$(5)\qquad \begin{aligned}&(\nabla_{[X_1,X_2]}R)(X_i,X_j,X_k)\\ &\quad= \nabla_{[X_1,X_2]}(R(X_i,X_j,X_k))-R(\nabla_{[X_1,X_2]}X_i,X_j,X_k)\\ &\qquad-R(X_i,\nabla_{[X_1,X_2]}X_j,X_k)-R(X_i,X_j,\nabla_{[X_1,X_2]}X_k).\end{aligned}$$

According to the definition $R(X,Y)Z=R(X,Y,Z)=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$, we also have the formula

$$(6)\qquad \begin{aligned}&\nabla_{[X_1,X_2]}(R(X_i,X_j,X_k))\\ &\quad= -R(X_1,X_2,R(X_i,X_j,X_k))\\ &\qquad+\nabla_{X_1}\nabla_{X_2}(R(X_i,X_j,X_k))-\nabla_{X_2}\nabla_{X_1}(R(X_i,X_j,X_k)).\end{aligned}$$

Putting together (5) and (6) we get

$$(7)\qquad \begin{aligned}&(\nabla_{[X_1,X_2]}R)(X_i,X_j,X_k)\\ &\quad= -R(X_1,X_2,R(X_i,X_j,X_k))+\nabla_{X_1}\nabla_{X_2}(R(X_i,X_j,X_k))\\ &\qquad-\nabla_{X_2}\nabla_{X_1}(R(X_i,X_j,X_k))-R(\nabla_{[X_1,X_2]}X_i,X_j,X_k)\\ &\qquad-R(X_i,\nabla_{[X_1,X_2]}X_j,X_k)-R(X_i,X_j,\nabla_{[X_1,X_2]}X_k).\end{aligned}$$

We next compute the term $\nabla_{X_1}\nabla_{X_2}(R(X_i,X_j,X_k))-\nabla_{X_2}\nabla_{X_1}(R(X_i,X_j,X_k))$ above. Using the fact that $\nabla_X R=0$ for $X\in\mathcal{D}$, we obtain

$$\begin{aligned}&\nabla_{X_1}\nabla_{X_2}(R(X_i,X_j,X_k))-\nabla_{X_2}\nabla_{X_1}(R(X_i,X_j,X_k))=\\ &+R(\nabla_{X_1}\nabla_{X_2}X_i,X_j,X_k)+R(\nabla_{X_2}X_i,\nabla_{X_1}X_j,X_k)+R(\nabla_{X_2}X_i,X_j,\nabla_{X_1}X_k)\\ &+R(\nabla_{X_1}X_i,\nabla_{X_2}X_j,X_k)+R(X_i,\nabla_{X_1}\nabla_{X_2}X_j,X_k)+R(X_i,\nabla_{X_2}X_j,\nabla_{X_1}X_k)\\ &+R(\nabla_{X_1}X_i,X_j,\nabla_{X_2}X_k)+R(X_i,\nabla_{X_1}X_j,\nabla_{X_2}X_k)+R(X_i,X_j,\nabla_{X_1}\nabla_{X_2}X_k)\\ &-R(\nabla_{X_2}\nabla_{X_1}X_i,X_j,X_k)-R(\nabla_{X_1}X_i,\nabla_{X_2}X_j,X_k)-R(\nabla_{X_1}X_i,X_j,\nabla_{X_2}X_k)\\ &-R(\nabla_{X_2}X_i,\nabla_{X_1}X_j,X_k)-R(X_i,\nabla_{X_2}\nabla_{X_1}X_j,X_k)-R(X_i,\nabla_{X_1}X_j,\nabla_{X_2}X_k)\\ &-R(\nabla_{X_2}X_i,X_j,\nabla_{X_1}X_k)-R(X_i,\nabla_{X_2}X_j,\nabla_{X_1}X_k)-R(X_i,X_j,\nabla_{X_2}\nabla_{X_1}X_k).\end{aligned}$$

We note that the terms with no double covariant differentiation cancel out, and substituting the remaining terms into (7) we finally obtain formula (3). Similarly for formula (4).

To find the function h along the geodesic ray determined by $Z = Z' + a\xi$, compute

$$\dot{h} = \nabla_Z \theta([X_1, X_2]) = -\nabla_Z(\theta(T(X_1, X_2))),$$

so that

$$\dot{h} = -\theta((\nabla_{\frac{a}{h}[X_1,X_2]}T)(X_1, X_2)).$$

Using equation (4) we find

$$(8)\quad \dot{h} = -\frac{a}{h}\theta(T(R(X_1, X_2)X_1, X_2) + T(X_1, R(X_1, X_2)X_2) - R(X_1, X_2)T(X_1, X_2)).$$

Now, combining equations (8), (1), (3), (2) and (4) we obtain a system of diferential equations for h, T and R along the geodesic ray determined by $Z = Z' + a\xi$ which has unique solutions for given initial conditions.

It is clear that this system is the same along the geodesic ray determined by $\phi(Z) = -Z' + a\xi$. This implies that we are in the hypothesis of Theorem 2.2 and completes the proof of the theorem. □

3. Sub-orthogonal involutive Lie algebras

We now associate a linear object to a sub-symmetric space.

PROPOSITION 3.1. – *Let $(M, \mathcal{D}, g)$ be a simply-connected sub-symmetric space, G the Lie group of all sub-Riemannian isometries of M. Choose $x_0 \in M$, let K be the isotropy subgroup at x_0 and let $\psi \in K$ be the sub-symmetry at x_0. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the respective Lie algebras of G and K and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of the involution $s = Ad_\psi$ of $\mathfrak{g}$. Then:*

a. M is represented as the coset space G/K;

b. the projection $\pi : G \to M$, given by $\pi(g) = g(x_0)$, has differential $\pi_* : \mathfrak{p} \cong \mathcal{D}_{x_0}$;

c. $\mathfrak{k}$ is a compact subalgebra of codimension one of $\mathfrak{h}$ which contains no nonzero ideal of $\mathfrak{g}$;

d. if $k \in K$ and $X \in \mathfrak{p}$, then $\pi_*(Ad_k X) = k_*(\pi_* X)$;

e. the inner product B on $\mathcal{D}_{x_0}$ lifted to $\mathfrak{p}$ by π_* is Ad_K-invariant;

f. the skew-symmetric bilinear form $\Theta : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{h}/\mathfrak{k}$ defined by setting $\Theta(X, Y) = [X, Y] \bmod \mathfrak{k}$ is non-degenerate;

Proof. – *a.* is obvious. K is a compact subgroup of G, since G is a group of Riemannian isometries relative to the canonical extended Riemannian metric on M. For any $k \in K$, Ad_k factors through a linear map on $\mathfrak{g}/\mathfrak{k}$, and because K is compact we can find a complementary Ad_K-invariant space $\mathfrak{m}$. Now π_* identifies $\mathfrak{m}$ with the tangent space $T_{x_0}M$ and is easily seen to be an equivalence between the Ad_K-action on $\mathfrak{m}$ and the K-action on $T_{x_0}M$. Define $\mathfrak{p}_0 \subset \mathfrak{m}$ to be the inverse image of $\mathcal{D}_{x_0}$ under π_*. Then $\mathfrak{p}_0 \subset \mathfrak{p}$ since $\psi_*|_{\mathcal{D}_{x_0}} = -1$. $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{g}$ because Ad_K is effective on $\mathfrak{m}$ (because K is effective on $T_{x_0}M$). In fact, Ad_K is effective on $\mathfrak{p}_0$ as $\mathfrak{g}/\mathfrak{k}$ is generated by $\mathfrak{p}_0$ modulo $\mathfrak{k}$ by the contact condition. Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}_0$. Then $[X - sX, Y] \in \mathfrak{p}_0$, so

$$-[X - sX, Y] = s[X - sX, Y] = [sX - X, sY] = [X - sX, Y]$$

and so $\mathrm{ad}_{X-sX}[\mathfrak{p}_0] = 0$. But the centralizer of $\mathfrak{p}_0$ in $\mathfrak{k}$ is zero since Ad_K is effective on $\mathfrak{p}_0$. Thus we have $X - sX = 0$ and $\mathfrak{k} \subset \mathfrak{h}$. Now $\mathfrak{g} = \mathfrak{k} + [\mathfrak{p}_0, \mathfrak{p}_0] + \mathfrak{p}_0$ where $\mathfrak{h} = \mathfrak{k} + [\mathfrak{p}_0, \mathfrak{p}_0]$ and $\mathfrak{p} = \mathfrak{p}_0$; b., c., d. are proved and e. is immediate. Finally, Θ on $\mathfrak{p}$ is equivalent to $d\theta_{x_0}$ on $\mathcal{D}_{x_0}$, so f. follows from the contact condition. □

To a point x_0 in our sub-symmetric space M we now have associated a quadruple $(\mathfrak{g}, s, \mathfrak{k}, B)$ where s is an involutive automorphism of $\mathfrak{g}$, $\mathfrak{k}$ is a codimension one subalgebra of the $+1$-eigenspace $\mathfrak{h}$ which does not contain a nonzero ideal of $\mathfrak{g}$ and B is an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product on the -1-eigenspace $\mathfrak{p}$. Furthermore, the skew-symmetric bilinear form Θ on $\mathfrak{p}$ is non-degenerate. The quadruple is called the *sub-orthogonal-involutive Lie (sub-OIL) algebra* of M at x_0. Since M is homogeneous, its sub-OIL algebra is the same at all points. Observe that $\mathfrak{g}/\mathfrak{k}$ and $\mathfrak{p}$ are oriented vector spaces. Note also that it follows from the contact condition that $\mathfrak{h} = \mathfrak{k} + [\mathfrak{p}, \mathfrak{p}]$.

An *abstract sub-orthogonal involutive Lie algebra* is defined to be a quadruple $(\mathfrak{g}, s, \mathfrak{k}, B)$ with the properties in the above paragraph. By a *sub-orthogonal involutive sub-algebra* of $(\mathfrak{g}, s, \mathfrak{k}, B)$ we mean a sub-OIL algebra $(\mathfrak{g}', s', \mathfrak{k}', B')$ such that $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$, s' is the restriction of s, $\mathfrak{k}'$ is a subalgebra of $\mathfrak{k}$, $\mathfrak{p}' = \mathfrak{p}$ and $B' = B$. A sub-OIL algebra is called *Heisenberguian* if $n = \frac{1}{2}\dim\mathfrak{p} \geq 2$ and $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{k} = 0$. It is called *irreducible* if it is not Heisenberguian and if $\mathrm{ad}_{\mathfrak{h}}$ is irreducible on $\mathfrak{p}$.

Given an abstract sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$ we can construct a sub-symmetric space as follows. Let $\tilde{G}$ be the simply-connected group with Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ the decomposition into ± 1-eigenspaces of s and $\tilde{K}$ the connected subgroup for $\mathfrak{k}$. Let M denote the simply-connected manifold $\tilde{G}/\tilde{K}$. In general $\tilde{G}$ does not act effectively on M; there is a discrete kernel $\tilde{Z}$. Now $M = G_0/K_0$ where $G_0 = \tilde{G}/\tilde{Z}$ and $K_0 = \tilde{K}/\tilde{Z}$ still have $\mathfrak{g}$ and $\mathfrak{k}$ as their Lie algebras. We have a projection $\pi : G_0 \to M$ by $\pi(g) = g(x_0)$ where x_0 is the coset K_0. As in Proposition 3.1, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ for an Ad_{K_0}-invariant space $\mathfrak{m}$, $\pi_* : \mathfrak{m} \cong T_{x_0}M$ and $\pi_*(\mathrm{Ad}_k X) = k_*\pi_* X$ for $k \in K_0$ and $X \in \mathfrak{p}$. Then B induces an inner product on $\mathcal{D}_{x_0}$ invariant under K_0. Hence $\mathcal{D}_{x_0}$ and B translate respectively to a G_0-invariant distribution $\mathcal{D}$ on M and to a G_0-invariant metric g on $\mathcal{D}$. The distribution $\mathcal{D}$ is contact because the skew-symmetric form Θ is non-degenerate. The automorphism s of $\mathfrak{g}$ induces an automorphism σ of G_0 which in turn gives a transformation ψ of M by the rule $\psi(g.x_0) = \sigma(g).x_0$ for $g \in G$. Then ψ is an isometry of M and $\psi_*(\pi_* X) = \pi_*(sX) = -\pi_* X$ for $X \in \mathfrak{p}$, so ψ fixes x_0 and induces -1 on $\mathcal{D}_{x_0}$. Thus ψ is the sub-symmetry at x_0 and $g\psi g^{-1}$ is the sub-symmetry at $g(x_0)$. We have proved that M is sub-symmetric. In other words:

PROPOSITION 3.2. – *Let $(\mathfrak{g}, s, \mathfrak{k}, B)$ be an abstract sub-OIL algebra. Let M be the simply-connected sub-symmetric space constructed above from $(\mathfrak{g}, s, \mathfrak{k}, B)$. Then $(\mathfrak{g}, s, \mathfrak{k}, B)$ is a sub-OIL subalgebra of the sub-OIL algebra associated to M.*

The significance of this result comes from

PROPOSITION 3.3. – *Let* $(\mathfrak{g}, s, \mathfrak{k}, B) \subset (\mathfrak{g}', s', \mathfrak{k}', B')$ *be sub-OIL algebras; let* M *and* M' *be the corresponding simply-connected sub-symetric spaces. Then* M *is isometric to* M'.

Proof. – We have $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{p}'$ under s and s'. Now $M = \tilde{G}/\tilde{K}$ and $M' = \tilde{G}'/\tilde{K}'$ as coset spaces of the simply-connected groups of $\mathfrak{g}$ and $\mathfrak{g}'$. Define $f : M \to M'$ by $f(g\tilde{K}) = g\tilde{K}'$, well defined because $\tilde{K} \subset \tilde{K}'$. f is onto because $M = (\exp \mathfrak{m})(x_0)$ and $M' = (\exp \mathfrak{m})(x_0')$ where $\mathfrak{m} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g} \subset \mathfrak{g}'$ and where x_0 and x_0' are the cosets $\tilde{K}$ and $\tilde{K}'$. Now $M' = \tilde{G}/\tilde{K}''$ where $\tilde{K}'' = \tilde{G} \cap \tilde{K}'$ and f is given by $g\tilde{K} \mapsto g\tilde{K}''$. We have

$$\dim \tilde{K} = \dim \tilde{G} - \dim M = \dim \tilde{G} - \dim \mathfrak{p} - 1 = \dim \tilde{G} - \dim M' = \dim \tilde{K}''$$

so f is a covering. Now simple-connectivity of M' implies that f is a diffeomorphism. Finally f is an isometry because $\mathfrak{p} = \mathfrak{p}'$ and $B = B'$. □

PROPOSITION 3.4. – *If* $(\mathfrak{g}, s, \mathfrak{k}, B)$ *is a sub-OIL algebra then it admits a sub-OIL subalgebra* $(\mathfrak{g}', s', \mathfrak{k}', B')$ *such that* $[\mathfrak{p}', \mathfrak{p}'] = \mathfrak{h}'$.

Proof. – Write $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ under s. Then $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a subalgebra (in fact, an ideal) of $\mathfrak{g}$, and we may take $\mathfrak{k}' = \mathfrak{k} \cap (\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]) = \mathfrak{k} \cap [\mathfrak{p}, \mathfrak{p}]$, $\mathfrak{p}' = \mathfrak{p}$, $s' = s|_{\mathfrak{g}}$ and $B' = B$. □

4. The contact structure of sub-symmetric spaces

Let $(M = G/K, \mathcal{D}, \langle , \rangle)$ be a sub-symmetric space, G the group of sub-Riemannian isometries of M. We are going to study its sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$ by exploring deeper the contact structure. Recall $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ under s. Let θ be the normalized contact form and ξ the characteristic vector field as in Section 1. Recall there is a projection $\pi : G \to M$.

LEMMA 4.1 ([2]). – *The pull-back* $\theta^* = \pi^*(\theta)$ *is a left-invariant* 1*-form on* G *such that:*

a. θ^* *is* Ad_K*-invariant;*

b. $\theta^*(\mathfrak{k}) = 0$;

c. $\theta^* \wedge (d\theta^*)^n \neq 0$;

d. $\mathfrak{h} = \{X \in \mathfrak{g} : d\theta^*(X, \mathfrak{g}) = 0\}$.

Proof. – *a.*, *b.* and *c.* are immediate. For *d.*, we have $s(\ker d\theta^*) = \ker d\theta^*$ because θ is invariant under the sub-symmetry of M. Now $\ker d\theta^* = (\ker d\theta^* \cap \mathfrak{h}) + (\ker d\theta^* \cap \mathfrak{p})$ where $\ker d\theta^* \cap \mathfrak{p} = \{0\}$ by the contact condition, so $\ker d\theta^* \subset \mathfrak{h}$. But

$$\dim \mathfrak{g} - \dim \ker d\theta^* = \text{rank } (d\theta^*) = 2n = \text{rank } (d\theta) = \dim \mathfrak{g}/\mathfrak{k} - 1$$

and so $\dim \ker d\theta^* = \dim \mathfrak{k} + 1 = \dim \mathfrak{h}$. □

LEMMA 4.2 ([2]). – *There is an element* $\xi^* \in \mathfrak{h}$ *such that* $\pi_*(\xi^*) = \xi_{x_0}$, i.e. *a left-invariant vector field on* G *which projects down to the transversal field* ξ.

Proof. – Since $\pi_* : \mathfrak{g} \longrightarrow T_{x_0}M$ is onto, we can find $\xi^* \in \mathfrak{g}$ such that $\pi_*(\xi^*) = \xi_{x_0}$ and then $\pi_*(\xi^*_g) = \xi_{\pi(g)}$, by G-invariance of ξ. As $d\theta^*(\xi^*, \mathfrak{g}) = d\theta(\xi, T_{x_0}M) = 0$, we get $\xi^* \in \mathfrak{h}$. □

Note that $\theta^*(\xi^*) = \theta(\xi) = 1$ implies $\xi^* \notin \mathfrak{k}$ and ξ^* is defined modulo $\mathfrak{k}$ only. In any case we may now write $\mathfrak{g} = \mathfrak{k} + \langle \xi^* \rangle + \mathfrak{p}$ where $\mathfrak{h} = \mathfrak{k} + \langle \xi^* \rangle$.

LEMMA 4.3 [2]. – *$\mathfrak{k}$ is an ideal in $\mathfrak{h}$.*

Proof. – Let $X \in \mathfrak{k}$, $Y \in \mathfrak{h}$ and $h = \exp tY$. Then

$$\theta^*(\mathrm{Ad}_h X) = \theta(\pi_* \mathrm{Ad}_h X) = \theta(h_* \pi_* X) = h^* \theta(\pi_* X) = \theta^*(X) = 0.$$

So if we write $\mathrm{Ad}_h(X) = \alpha\xi^* + Z$ for some $\alpha \in \mathbb{R}$, $Z \in \mathfrak{k}$, then $0 = \theta^*(\mathrm{Ad}_h X) = \alpha$. Hence, $\mathrm{Ad}_{\exp tY} X \in \mathfrak{k}$ for all $t \in \mathbb{R}$. By differentiation, $[Y, X] \in \mathfrak{k}$. □

Since $\mathfrak{k}$ is already a subalgebra of $\mathfrak{g}$, Lemma 4.3 really means $\mathrm{ad}_{\xi^*}[\mathfrak{k}] \subset \mathfrak{k}$. Next we show how ξ^* can be chosen so that ξ^* centralizes $\mathfrak{k}$.

LEMMA 4.4. – *There is $\bar{\xi} \in \mathfrak{h}$ such that $\pi_*(\bar{\xi}) = \xi$ and $ad_{\bar{\xi}}[\mathfrak{k}] = 0$.*

Proof. – Let ξ^* be as in Lemma 4.3 and consider the connected component K_0 of K. Note that any $k \in K_0$ is M- and $\mathcal{D}$-orientation preserving, and so it fixes ξ_{x_0}. Then

$$\xi_{x_0} = k_* \xi_{x_0} = k_* \pi_* \xi^* = \pi_*(\mathrm{Ad}_k \xi^*).$$

Also, $\mathrm{Ad}_k \xi^* \in \mathfrak{h}$. As K_0 is compact, there is a Haar measure dk on K_0 and we may define

$$\bar{\xi} = \frac{1}{\mathrm{vol}\ (K_0)} \int_{K_0} \mathrm{Ad}_k \xi^*\, dk.$$

We get $\bar{\xi} \in \mathfrak{h}$ with $\pi_*(\bar{\xi}) = \xi_{x_0}$. As $\mathrm{Ad}_k \bar{\xi} = \bar{\xi}$ for all $k \in K_0$, we get also $[\mathfrak{k}, \bar{\xi}] = 0$. □

From now on, unless otherwisely stated, we will assume that $\bar{\xi}$ centralizes $\mathfrak{k}$, but we will drop the bar on $\bar{\xi}$.

The sub-OIL algebra of a sub-symetric space M at a point x_0 determines the torsion tensor τ_{x_0}:

LEMMA 4.5. – *Let τ be the sub-torsion tensor of M. Let X', Y' be vector fields on the distribution $\mathcal{D}$ defined on open set U of M, let $x_0 \in U$, and let X, Y be the left-invariant vector fields on G such that $\pi_*(X) = X'$ and $\pi_*(Y) = Y'$ at x_0. Then*

$$\langle ad_\xi X, Y \rangle + \langle X, ad_\xi Y \rangle = -2\langle \tau(X'), Y' \rangle_{x_0} \tag{9}$$

In particular, the torsion vanishes if and only if B is ad_ξ-invariant (note that (9) is independent of the chosen ξ).

Proof. – The right side of (9) depends only on X'_{x_0} and Y'_{x_0}. Thus we assume X and Y G-invariant vector fields on $\mathcal{D}$ with those values at x_0. Now $\langle X', Y' \rangle$ is constant on M and Theorem 1.1 gives

$$\langle [\xi, X'], Y' \rangle_{x_0} + \langle X', [\xi, Y'] \rangle_{x_0} = -2\langle \tau(X'), Y' \rangle_{x_0}. \tag{10}$$

Finally we may pull-back the left side of (10) to $\mathfrak{g}$. □

A sub-OIL algebra for which B is $\mathrm{ad}_{\mathfrak{h}}$-invariant will be called *subtorsionless*.

Next we are going to show that there is a G-invariant Hermitian structure on $\mathcal{D}$.

LEMMA 4.6. – *The skew-symmetric bilinear form* Θ *on* $\mathfrak{p}$ *is* $ad_{\mathfrak{h}}$*-invariant.*

Proof. – Let $X,\ Y \in \mathfrak{p},\ Z \in \mathfrak{h}$. As ξ centralizes $\mathfrak{k}$, Jacobi gives

$$[\mathrm{ad}_Z X, Y] + [X, \mathrm{ad}_Z Y] = \mathrm{ad}_Z[X,Y] \in [\mathfrak{h},\mathfrak{h}] \in \mathfrak{k}.$$

Modulo $\mathfrak{k}$, that yields

$$\Theta(\mathrm{ad}_Z X, Y) + \Theta(X, \mathrm{ad}_Z Y) = 0.$$

$\square$

Now we turn Θ into a real form by choosing the vector $-\frac{1}{2}\xi + \mathfrak{k} \in \mathfrak{h}/\mathfrak{k}$. Then there is a positively oriented orthonormal basis of $\mathfrak{p}$, $\{X_1, Y_1, \dots, X_n, Y_n\}$, such that the matrix of Θ is

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_n \\ & & & -\lambda_n & 0 \end{pmatrix}$$

with $\Pi_{i=1}^n \lambda_i = 1$. Suppose that

$$\lambda_1 = \dots = \lambda_{n_1}, \quad \lambda_{n_1+1} = \dots = \lambda_{n_1+n_2}, \quad \dots, \quad \lambda_{n_1+\dots+n_{r-1}+1} = \dots = \lambda_n,$$

and let $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_r$ be the corresponding eigenspace decomposition. As $\mathrm{Ad}(K)$ acts on $\mathfrak{p}$ preserving a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ and a non-degenerate antisymmetric bilinear form Θ, we must have

LEMMA 4.7. – *Each* $\mathfrak{p}_i$ *is* K*-invariant; hence,* $K \subset U(n_1) \times \dots \times U(n_r)$.

COROLLARY 4.1. – *Let* $\{X_1, Y_1, \dots, X_n, Y_n\}$ *be a positively oriented orthonormal basis of* $\mathfrak{p}$ *which diagonalizes* Θ. *Then* $JX_i = Y_i$, $JY_i = -X_i$ *defines an* Ad_K*-invariant complex structure on* $\mathfrak{p}$ *which preserves* $\langle\ ,\ \rangle$. *Thus* J *translates to a* G*-invariant Hermitian structure on* $\mathcal{D}$.

We now look at the Heisenberg algebras. As it is, they represent the sub-Riemannian analogues of the Euclidean algebras for Riemannian symmetric spaces. Recall that the Heisenberg algebra of dimension $2n+1$ $\mathfrak{H}^{2n+1}$ is the nilpotent Lie algebra spanned by $X_1,\ Y_1, \dots,\ X_n,\ Y_n,\ Z$ where $[X_i, Y_i] = Z$ and all the other brackets are zero. We can define a sub-OIL algebra $(\mathfrak{H}(n), s, \mathfrak{k}, B)$ where $\mathfrak{k} = \mathfrak{u}(n_1) + \dots + \mathfrak{u}(n_r)$, $\xi = -\frac{1}{2}Z$, $\mathfrak{p} = \langle X_1, Y_1, \dots, X_n, Y_n \rangle$, $B(X_i, X_i) = B(Y_i, Y_i) = \lambda_i^{-1}$, $B(X,Y) = 0$ in all the other cases, where $\lambda_1 = \dots = \lambda_{n_1} > 0$, $\lambda_{n_1+1} = \dots = \lambda_{n_1+n_2} > 0$, $\lambda_{n_1+\dots+n_{r-1}+1} = \dots = \lambda_n > 0$, $\Pi_{i=1}^n \lambda_i = 1$ and $\mathfrak{H}(n) = \mathfrak{k} + \mathfrak{H}^{2n+1}$ semidirect sum where $\mathfrak{k}$ acts trivially on Z. Such a $(\mathfrak{H}(n), s, \mathfrak{k}, B)$ will be called the *standard Heisenberguian sub-OIL algebra of rank* n *and parameters* $\lambda_1, \dots, \lambda_n$.

THEOREM 4.1. – *Assume* $n \geq 2$. *Let* $\mathfrak{k}'$ *be a subalgebra of* $\mathfrak{u}(n_1) + \ldots \mathfrak{u}(n_r)$, $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{H}^{2n+1}$, s' *the restriction of* s *and* $B' = B$. *Then* $(\mathfrak{g}', s', \mathfrak{k}', B')$ *is a Heisenberguian sub-OIL algebra. On the other hand, every Heisenberguian sub-OIL algebra is a subalgebra of a standard Heisenberguian sub-OIL algebra.*

Proof. – The first assertion follows from the fact that $[\mathfrak{p}, \mathfrak{p}] = \langle Z \rangle$. Now let $(\mathfrak{g}', s', \mathfrak{k}', B')$ be Heisenberguian. Then $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{p}'$ under s'. Define $n = \frac{1}{2} \dim \mathfrak{p}'$. Choose an orthonormal basis of $\mathfrak{p}'$ which diagonalizes Θ as above. and renormalize it to get an orthogonal basis $\{X_1, Y_1, \ldots, X_n, Y_n\}$ such that $\Theta(X_i, Y_i) = 1$. Since $(\mathfrak{g}', s', \mathfrak{k}', B')$ is Heisenberguian, $[X, Y] = \Theta(X, Y)\xi'$ for X, $Y \in \mathfrak{p}'$ and for some $\xi' \in \mathfrak{h}'$, $\xi' \notin \mathfrak{k}'$. Then Jacobi and $n \geq 2$ imply that

$$[X_1, [X_2, Y_2]] + [X_2, [Y_2, X_1]] + [Y_2, [X_1, X_2]] = 0$$

so $[X_1, \xi'] = 0$. As $X_1 \in \mathfrak{p}'$ is arbitrary, ξ' centralizes $\mathfrak{p}'$. Also,

$$[\xi', \mathfrak{k}'] = [[X_1, Y_1], \mathfrak{k}'] \subset [[X_1, \mathfrak{k}'], Y_1] + [[Y_1, \mathfrak{k}'], X_1] \subset [\mathfrak{p}', Y_1] + [\mathfrak{p}', X_1] \subset [\mathfrak{p}', \mathfrak{p}']$$

and $[\xi', \mathfrak{k}'] \subset \mathfrak{k}'$ by Lemma 4.3. We get $[\xi', \mathfrak{k}'] = 0$ by the Heisenberguian condition. Now $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{H}^{2n+1}$ semidirect sum. $\mathfrak{k}' \subset \mathfrak{u}(n_1) + \ldots + \mathfrak{u}(n_r)$ because it is effective on $\mathfrak{p}'$. Thus $(\mathfrak{g}', s', \mathfrak{k}', B') \subset (\mathfrak{H}(n), s, \mathfrak{u}(n_1) + \ldots + \mathfrak{u}(n_r), B)$. □

LEMMA 4.8. – *Let* $(\mathfrak{g}, s, \mathfrak{k}, B)$ *be a sub-OIL algebra and let* β *be the Killing form of* $\mathfrak{g}$. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are* β*-orthogonal subspaces of* $\mathfrak{p}$ *such that* $\mathfrak{b}$ *is* $ad_{\mathfrak{k}}$*-invariant, then* $[\mathfrak{a}, \mathfrak{b}] \cap \mathfrak{k} = 0$.

Proof. – Similar to the proof of Lemma 8.2.1 of [14]. □

PROPOSITION 4.1. – *Assume* $n \geq 2$ *and let* $(\mathfrak{g}, s, \mathfrak{k}, B)$ *be a sub-OIL algebra such that* $\mathfrak{g}$ *is a nilpotent Lie algebra. Then* $\mathfrak{g}$ *is Heisenberguian and* $\mathfrak{k}$ *is a subalgebra of* $\mathfrak{u}(1) + \ldots + \mathfrak{u}(1)$.

Proof. – Since $\mathfrak{g}$ is nilpotent, its Killing form β is null. In particular, $\beta(\mathfrak{p}, \mathfrak{p}) = 0$. By Lemma 4.8 applied to $\mathfrak{a} = \mathfrak{b} = \mathfrak{p}$, $(\mathfrak{g}, s, \mathfrak{k}, B)$ is Heisenberguian. As $\mathfrak{g}$ is nilpotent, $\mathfrak{k}$ must be a subalgebra of $\mathfrak{u}(1) + \ldots + \mathfrak{u}(1)$. □

THEOREM 4.2. – *The only sub-symmetric space with a nilpotent group of isometries is the Heisenberg group.*

Proof. – If $n = 1$ this follows from the classification in dimension three (*see* [10]). For $n \geq 2$ we apply Proposition 4.1. □

5. On the classification of sub-symmetric spaces

We keep the notation from the previous sections and assume $n = \dfrac{1}{2} \dim \mathcal{D} \geq 2$ since the classification in dimension three has already been done in [10].

A simply-connected sub-symmetric space $M = G/K$ will be called *irreducible* if its sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$ is irreducible. Our aim in this section is to classify the irreducible simply-connected sub-symmetric spaces.

AN EXAMPLE

There is a canonical way of building up a sub-symmetric space from an Hermitian symmetric space.

LEMMA 5.1. – *Let* $(\mathfrak{g}, s, B)$ *be an irreducible Hermitian orthogonal involutive Lie (OIL) algebra and write* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *under* s. *Then* $(\mathfrak{g}, s, [\mathfrak{h}, \mathfrak{h}], B)$ *is an irreducible subtorsionless sub-OIL algebra.*

Proof. – Since $(\mathfrak{g}, s, B)$ is Hermitian, $\mathfrak{h}$ is not semisimple, its center is one-dimensional and generated by an element η such that $\mathrm{ad}_\eta|_{\mathfrak{p}} = J$, the complex structure on $\mathfrak{p}$. Now $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] + \langle \eta \rangle$. We check the contact condition. Let β be the Killing form of $\mathfrak{g}$. Then $\beta = a\langle\ ,\ \rangle$ for some scalar $a \neq 0$, by irreducibility. Fix an Hermitian basis $\{X_1, JX_1, \ldots, X_n, JX_n\}$ of $\mathfrak{p}$. Then

$$\beta(\eta, [X_i, JX_i]) = \beta([\eta, X_i], JX_i) = a\langle JX_i, JX_i \rangle = a,$$

$$\beta(\eta, [X_i, X_j]) = a\langle JX_i, X_j \rangle = 0, \quad \text{etc.}$$

and

$$\beta(\eta, \eta) = \text{trace } (\mathrm{ad}_\eta^2) = \text{trace } (\mathrm{ad}_\eta^2|_{\mathfrak{p}}) = \text{trace } (J^2) = -\text{trace } (\mathrm{id}_{\mathfrak{p}}) = -2n.$$

Now $\mathfrak{g} = [\mathfrak{h}, \mathfrak{h}] + \langle \eta \rangle + \mathfrak{p}$ is an orthogonal decomposition relative to β, so

$$[X_i, JX_i] \equiv -\frac{a}{2n}\eta \bmod [\mathfrak{h}, \mathfrak{h}],$$
$$[X_i, X_j] \equiv 0 \bmod [\mathfrak{h}, \mathfrak{h}], \quad \text{etc.},$$

and so Θ is non-degenerate. Thus we get a sub-OIL algebra $(\mathfrak{g}, s, [\mathfrak{h}, \mathfrak{h}], B)$. It is subtorsionless because of Lemma 4.5. □

COROLLARY 5.1. – *Let* G/H *be an irreducible simply connected Hermitian symmetric space. Then there is a canonical circle bundle over* G/H *which has the structure of an irreducible simply-connected subtorsionless sub-symmetric space.*

Proof. – Consider the Hermitian OIL-algebra $(\mathfrak{g}, s, B)$ where $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ under s and construct the associated sub-OIL algebra $(\mathfrak{g}, s, [\mathfrak{h}, \mathfrak{h}], B)$ as in the Lemma. The corresponding simply-connected sub-symmetric space is G/K where $K = \exp \mathfrak{k}$, $\mathfrak{k} = [\mathfrak{h}, \mathfrak{h}]$. Now $H/K \to G/K \to G/H$ is a circle bundle, the distribution on G/K is the G-invariant distribution determined at the base-point by the Ad_K-invariant complement to the fiber (or, equivalently, the distribution is the horizontal distribution relative to any Ad_K-invariant *Riemannian* metric on G/K) and the metric on the distribution is the pull-back of the metric on the base. □

We shall prove that every irreducible simply-connected subtorsionless sub-symmetric space is obtained as above. Moreover, we will also see that an irreducible simply-connected sub-symmetric space with arbitrary subtorsion only differs from the example above by a different choice of metric. To give an idea of what may happen if M is not irreducible, we will mention one example.

ANOTHER EXAMPLE

Let $(\bar{\mathfrak{g}}, \bar{s}, \bar{B})$ be any Hermitian OIL-algebra of semisimple type. Write $\bar{\mathfrak{g}} = \bar{\mathfrak{h}} + \bar{\mathfrak{p}}$ under $\bar{s}$ and let $\bar{J}$ be the $\mathrm{ad}_{\bar{\mathfrak{h}}}$-invariant complex structure on $\bar{\mathfrak{p}}$. Then $(\bar{\mathfrak{g}}, \bar{s}, [\bar{\mathfrak{h}}, \bar{\mathfrak{h}}], \bar{B})$ is a sub-OIL algebra (*cf.* Lemma 5.1). We shall construct a new sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$ which is *not* irreducible. Define the Lie algebra $\mathfrak{g}$ to be the semi-direct product of an abelian ideal $\mathfrak{g}_2 = \{(0, X) : X \in \mathfrak{g}\}$ and a subalgebra $\mathfrak{g}_1 = \{(X, 0) : X \in \mathfrak{g}\}$ naturally isomorphic with $\mathfrak{g}$, relative to the adjoint representation of $\mathfrak{g}_1$ on $\mathfrak{g}_2$. Then $\mathfrak{g}_2$ is the radical of $\mathfrak{g}$ and $\mathfrak{g}_1$ is a Levi subalgebra of $\mathfrak{g}$. Define an involutive automorphism s of $\mathfrak{g}$ by setting $s(X_1, X_2) = (\bar{s}(X_1), \bar{s}(X_2))$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ under s where $\mathfrak{h} = \bar{\mathfrak{h}} + \bar{\mathfrak{h}}$ and $\mathfrak{p} = \bar{\mathfrak{p}} + \bar{\mathfrak{p}}$. Take $\mathfrak{k} = \bar{\mathfrak{h}} + [\bar{\mathfrak{h}}, \bar{\mathfrak{h}}]$, $B = \bar{B} \oplus c\bar{B}$ (c any nonzero scalar) and fix an Hermitian basis $\{X_1, \bar{J}X_1, \ldots, X_n, \bar{J}X_n\}$ of $\bar{\mathfrak{p}}$. We check that $\{(X_1, 0), (0, \bar{J}X_1), \ldots, (X_n, 0), (0, \bar{J}X_n), (\bar{J}X_1, 0), (X_1, 0), \ldots, (\bar{J}X_n, 0), (X_n, 0)\}$ is a *symplectic* basis of $\mathfrak{p}$ for Θ. Thus the contact condition is verified and we obtain a sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$. It is not irreducible because $\mathfrak{p} \cap \mathfrak{g}_2$ is a non-trivial $\mathrm{ad}_{\mathfrak{h}}$-invariant subspace of $\mathfrak{p}$. Moreover, the restriction of Θ to $\mathfrak{p} \cap \mathfrak{g}_2$ is null, so $\mathfrak{p} \cap \mathfrak{g}_2$ is *not* a symplectic subspace of $\mathfrak{p}$. This example shows that the contact structure of a reducible sub-OIL algebra may not be inherited by the factors in a natural decomposition of it.

Now we take up the classification problem. Let $(M = G/K, \mathcal{D}, g)$ be an irreducible simply-connected sub-symmetric space with arbitrary sub-torsion and consider its associated sub-OIL algebra $(\mathfrak{g}, s, \mathfrak{k}, B)$. Write $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ under s. There are two cases to consider.

CASE 1: $\mathfrak{h}$ DOES NOT CONTAIN A NON-ZERO IDEAL OF $\mathfrak{g}$

In this case it follows from [8], volume 2, Proposition 7.5, p. 251, that either one of the following holds:

a. $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}'$ with $\mathfrak{g}'$ a simple Lie algebra, $\mathfrak{h}$ the diagonal in $\mathfrak{g}$ and $s(X, Y) = (Y, X)$ for X, $Y \in \mathfrak{g}'$;

b. $\mathfrak{g}$ is a simple Lie algebra;

c. $[\mathfrak{p}, \mathfrak{p}] = 0$.

We can have neither a. (since $\mathfrak{g}'$ would be isomorphic to $\mathfrak{h}$, but $\mathfrak{h}$ is not centerless) nor c. (because of the contact condition). We conclude $\mathfrak{g}$ is simple.

LEMMA 5.2. – *The connected Lie group of linear transformations of* $\mathfrak{p}$ *generated by ad*$[\mathfrak{h}]$ *is compact.*

Proof. – The result follows from Berger's list of simple involutive Lie algebras ([1]), but we shall give an independent proof based on [9].

Since $\mathrm{ad}_{\mathfrak{k}}$ leaves B invariant and ξ centralizes $\mathfrak{k}$, it is enough to check that ad_ξ generates a compact one-parameter group of linear transformations of $\mathfrak{p}$. Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ be a Cartan decomposition of $\mathfrak{g}$ such that the corresponding Cartan involution commutes with s. Then the center $\mathfrak{c}$ of $\mathfrak{h}$ decomposes as $\mathfrak{c} = \mathfrak{c} \cap \mathfrak{l} + \mathfrak{c} \cap \mathfrak{m}$ and we may write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathfrak{c} \cap \mathfrak{l}$ and $\xi_2 \in \mathfrak{c} \cap \mathfrak{m}$. We have that $\mathrm{ad}_{\mathfrak{h}}$ is irreducible on $\mathfrak{p}$ and $[\mathrm{ad}_{\xi_2}, \mathrm{ad}_{\mathfrak{h}}] = \mathrm{ad}_{[\xi_2, \mathfrak{h}]} = 0$. By Schur's Lemma, $\mathrm{ad}_{\xi_2} : \mathfrak{p} \to \mathfrak{p}$ is either an isomorphism or zero. The former cannot happen;

otherwise $\mathfrak{h}$ would be the centralizer of an element $\xi_2 \in \mathfrak{h} \cap \mathfrak{m}$ and then $\mathrm{ad}_{\mathfrak{h}}$ would be reducible on $\mathfrak{p}$ by [9], Theorem 4, p. 303. Therefore, $\mathrm{ad}_{\xi_2} = 0$. Since $\mathfrak{h}$ does not contain a non-zero ideal of $\mathfrak{g}$, we conclude that $\xi_2 = 0$. It follows that $\xi = \xi_1 \in \mathfrak{c} \cap \mathfrak{l}$ and so it generates a compact group of transformations of $\mathfrak{p}$ (in fact, an element of $\mathfrak{c} \cap \mathfrak{l}$ acts on $\mathfrak{p} \cap \mathfrak{l}$ and on $\mathfrak{p} \cap \mathfrak{m}$ preserving the Killing form β and β is definite in each of those spaces). □

Lemma 5.2 shows that $(\mathfrak{g}, s)$ is an orthogonal involutive Lie (OIL) algebra and it is also of Hermitian type as $\mathfrak{h}$ has a non-zero center. The only possibility for $\mathfrak{k}$ is to be equal to $[\mathfrak{h}, \mathfrak{h}]$. Finally we determine the possible choices of metric B on $\mathfrak{p}$.

Let η be in the center of $\mathfrak{h}$ so that $J' = \mathrm{ad}_\eta$ is the $\mathrm{ad}_{\mathfrak{h}}$-invariant complex structure on $\mathfrak{p}$ (note that η and ξ are linearly dependent, but J' does not need to be the complex structure J on $\mathfrak{p}$ constructed in Corollary 4.1).

There are two cases to consider. Either $\mathrm{ad}_{\mathfrak{k}}$ is irreducible on $\mathfrak{p}$ or it is not. In the first case, B is a multiple of the Killing form β of $\mathfrak{g}$, the sub-torsion is null, the complex structures J and J' coincide and the symplectic form Θ is equal to the Kähler form $\Omega(X,Y) = B(X, J'Y)$. In the second case, let $\mathfrak{p}_1 \subset \mathfrak{p}$ be an $\mathrm{ad}_{\mathfrak{k}}$-irreducible subspace. Then we must have $(\mathrm{ad}_{\mathfrak{k}}, \mathfrak{p}) = (\mathrm{ad}_{\mathfrak{k}}, \mathfrak{p}_1) \oplus (\mathrm{ad}_{\mathfrak{k}}, \mathfrak{p}_2)$ where $\mathfrak{p}_2 = J'\mathfrak{p}_1$, $(\mathrm{ad}_{\mathfrak{k}}, \mathfrak{p}_i)$ is irreducible and $B = a_i^{-1}\beta$ on $\mathfrak{p}_i$ for some $a_i \neq 0$ (here β is the Killing form of $\mathfrak{g}$). Observe that the subtorsion vanishes if and only if $a_1 = a_2$ (*cf.* Lemma 4.5).

CASE 2: $\mathfrak{h}$ DOES CONTAIN A NON-ZERO IDEAL $\mathfrak{a}$ OF $\mathfrak{g}$

Then $\mathfrak{a} \cap \mathfrak{k} \subset \mathfrak{k}$ is an ideal of $\mathfrak{g}$, so $\mathfrak{a} \cap \mathfrak{k} = 0$. This shows $\dim \mathfrak{a} = 1$. We claim

LEMMA 5.3. – *$\mathfrak{a}$ is exactly the centralizer $\mathfrak{z}$ of $\mathfrak{p}$ in $\mathfrak{h}$.*

Proof. – $\mathfrak{a} \subset \mathfrak{z}$ because $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{a} \cap [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{a} \cap \mathfrak{p} = 0$. Also, $\mathfrak{z}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. Now $\mathfrak{z} \cap \mathfrak{k} = 0$ implies $\dim \mathfrak{z} \leq 1$. Thus, $\mathfrak{a} = \mathfrak{z}$. □

Since $\mathfrak{z}$ is transversal to $\mathfrak{k}$ in $\mathfrak{h}$, we may choose $\xi^* \in \mathfrak{z}$ such that $\pi_*(\xi^*) = \xi_{x_0}$. Moreover $\mathfrak{z}$ is Ad_K-invariant, so our averaging method will give $\xi \in \mathfrak{z}$ which centralizes $\mathfrak{k}$. Now we have $\mathfrak{g} = \mathfrak{k} + \langle\xi\rangle + \mathfrak{p}$ and $\mathfrak{z} = \langle\xi\rangle$ is the center of $\mathfrak{g}$. In particular, the sub-torsion vanishes.

We may factor the center out and get $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}$. Since $s(\xi) = \xi$, s induces an involution $\bar{s}$ on $\bar{\mathfrak{g}}$. Now $\bar{\mathfrak{g}} = \mathfrak{k} + \mathfrak{p}$ under $\bar{s}$, B is an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product on $\mathfrak{p}$ and there is an $\mathrm{ad}_{\mathfrak{k}}$-invariant complex structure J on $\mathfrak{p}$ coming from Corollary 4.1. Again by irreducibility of $\mathrm{ad}_{\mathfrak{k}}$ on $\mathfrak{p}$ and Proposition 7.5 of [8], either $\mathfrak{g}$ is simple or $[\mathfrak{p}, \mathfrak{p}] = 0$ in $\bar{\mathfrak{g}}$. But the latter cannot happen as that would imply $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{k} = 0$ in $\mathfrak{g}$, *i.e.* $\mathfrak{g}$ Heisenberguian. Therefore $(\bar{\mathfrak{g}}, \bar{s}, B)$ is an irreducible Hermitian OIL algebra of simple type. Note that

$$0 \to \mathfrak{z} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\rho} \bar{\mathfrak{g}} \to 0$$

is a central extension of $\bar{\mathfrak{g}}$, where $\iota : \mathfrak{z} \to \mathfrak{g} = \mathfrak{k} + \mathfrak{z} + \mathfrak{p}$ is inclusion and $\rho : \mathfrak{g} = \mathfrak{k} + \mathfrak{z} + \mathfrak{p} \to \bar{\mathfrak{g}} = \mathfrak{k} + \mathfrak{p}$ is projection. Let $\nu : \bar{\mathfrak{g}} \to \mathfrak{g}$ be inclusion, so that $\rho\nu = 1$. Then the extension is characterized by the cohomology class $\omega \in H^2(\bar{\mathfrak{g}}, \mathfrak{z})$ given by

$$\omega(X,Y) = \iota^{-1}([\nu(X), \nu(Y)] - \nu[X,Y]) = \begin{cases} \xi & \text{if } Y = JX, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5.4. – *Let $(\bar{\mathfrak{g}}, \bar{s}, B)$ be an irreducible Hermitian OIL algebra and write $\bar{\mathfrak{g}} = \mathfrak{k} + \mathfrak{p}$ under $\bar{s}$. Let $(\mathfrak{g}, s, \mathfrak{k}, B)$ be the sub-OIL algebra which is a central extension by $\mathfrak{z}$ as above,* *and let* $(\bar{\mathfrak{g}}, \bar{s}, [\mathfrak{k}, \mathfrak{k}], B)$ *be the sub-OIL algebra which is constructed from* $(\bar{\mathfrak{g}}, \bar{s}, B)$ *as in Lemma* 5.1. *Then* $(\bar{\mathfrak{g}}, \bar{s}, [\mathfrak{k}, \mathfrak{k}], B)$ *is isomorphic to a sub-OIL subalgebra of* $(\mathfrak{g}, s, \mathfrak{k}, B)$.

We need

LEMMA 5.5. – *Consider the sub-OIL algebra* $(\mathfrak{g}, s, \mathfrak{k}, B)$ *in Lemma* 5.4. *Then* $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$.

Proof. – The Lemma follows from the following facts proved as in Lemma 5.1:

$$\begin{aligned} [X_i, JX_i] - [X_j, JX_j] &\in [\mathfrak{k}, \mathfrak{k}], \\ [X_i, X_j] &\in [\mathfrak{k}, \mathfrak{k}], \\ [X_i, JX_j] &\in [\mathfrak{k}, \mathfrak{k}], \\ [JX_i, JX_j] &\in [\mathfrak{k}, \mathfrak{k}], \end{aligned}$$

if $i \neq j$. □

Proof of Lemma 5.4. – First write $\mathfrak{g} = \mathfrak{k} + \langle \xi + c\eta \rangle + \mathfrak{p}$ where η is central in $\mathfrak{k}$ with $\mathrm{ad}_\eta|_{\mathfrak{p}}$ the complex structure J on $\mathfrak{p}$ and c is a yet to be determined non-zero scalar. Now apply Lemma 5.5 and Proposition 3.4 to get a sub-OIL subalgebra $(\mathfrak{g}', s', [\mathfrak{k}, \mathfrak{k}], B)$ where $\mathfrak{g}' = [\mathfrak{k}, \mathfrak{k}] + \langle \xi + c\eta \rangle + \mathfrak{p}$ and $s' = s|_{\mathfrak{g}'}$. We have $(\bar{\mathfrak{g}}, \bar{s}, [\mathfrak{k}, \mathfrak{k}], B) \cong (\mathfrak{g}', s', [\mathfrak{k}, \mathfrak{k}], B)$ as sub-OIL algebras. In fact, consider the map which takes $c\eta \in \bar{\mathfrak{g}}$ to $\xi + c\eta \in \mathfrak{g}'$ and is the identity on $[\mathfrak{k}, \mathfrak{k}] + \mathfrak{p}$. It is easily seen to preserve brackets; for instance, if $\{X_1, JX_1, \ldots, X_n, JX_n\}$ is an Hermitian basis of $\mathfrak{p}$, then, using $[,]$ and $[,]'$ to denote brackets in $\bar{\mathfrak{g}}$ and in $\mathfrak{g}'$, respectively, we get

$$[X_i, JX_i]' = [X_i, JX_i] + \xi = ([X_i, JX_i]^{[\mathfrak{k},\mathfrak{k}]} + c\eta) + \xi = [X_i, JX_i]^{[\mathfrak{k},\mathfrak{k}]} + (\xi + c\eta),$$

as the calculation from Lemma 5.1 shows that $[X_i, JX_i] = [X_i, JX_i]^{[\mathfrak{k},\mathfrak{k}]} + c\eta$ for a non-zero scalar c. □

Lemma 5.4 and Proposition 3.3 put together imply that in Case 2 we obtain exactly the irreducible simply-connected spaces with null sub-torsion of Case 1.

We summarize the above discussion in the following

THEOREM 5.1. – *Every irreducible, simply-connected sub-symmetric space* $(M, \mathcal{D}, g)$ *is a homogeneous manifold* G/K *canonically fibered over an irreducible Hermitian symmetric space* G/H *with fibers diffeomorphic to a circle* H/K *and generated by the flow of the characteristic field. The distribution* $\mathcal{D}$ *is the* G*-invariant distribution,* Ad_K*-invariant complement to the fibers. Either the* G*-invariant sub-Riemannian metric* g *is uniquely defined as the pull-back of the metric on the base and the sub-torsion is null, or there is a two-parameter family of such metrics* g *and a one-parameter sub-family of them is subtorsionless. In order to describe them explicitly, let* $(\mathfrak{g}, s, \mathfrak{k}, B)$ *be the sub-OIL algebra associated to* M. *Then* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *under* s, *and* g *at the base-point lifted to* $\mathfrak{p}$ *by the projection* $G \to M$ *is* B. *The two cases correspond respectively to* $ad_{\mathfrak{k}}$ *being irreducible or not on* $\mathfrak{p}$. *In the first case,* B *is a multiple of the Killing form of* $\mathfrak{g}$. *In the second case,* $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, $ad_{\mathfrak{k}}$*-irreducible decomposition, and* B *is a multiple of the Killing form of* $\mathfrak{g}$

on each one of the $\mathfrak{p}_i$. *From the classification of Hermitian symmetric spaces we list the diffeomorphic types of irreducible compact and non-compact sub-symmetric spaces:*

$$\begin{array}{ll} SU(p+q)/[SU(p)\times SU(q)] & SU(p,q)/[SU(p)\times SU(q)] \\ SO(n+2)/SO(n), \quad n\geq 3 & SO_0(n,2)/SO(n), \quad n\geq 3 \\ Sp(n)/SU(n) & Sp(n,\mathbb{R})/SU(n) \\ SO(2n)/SU(n) & SO^*(2n)/SU(n) \\ E_6/[Spin(10)/\mathbb{Z}^2] & E_6^*/[Spin(10)/\mathbb{Z}^2] \\ E_7/[E_6/\mathbb{Z}^3] & E_7^*/[E_6/\mathbb{Z}^3] \end{array}$$

(the second class of compact examples and the second class of non-compact examples are the only ones in the list which fall in the case of a reducible $ad_{\mathfrak{k}}$*-action on* $\mathfrak{p}$ *(cf. the next example)).*

THE SUB-SYMMETRIC SPACE $SO(n+2)/SO(n)$, $n\geq 3$

To illustrate some of the above results, we now describe in detail the sub-symmetric structure of $SO(n+2)/SO(n)$, $n\geq 3$. Let E^{ij} denote the $(n+2)\times(n+2)$ matrix which has the entry 1 in the (i,j)th slot and 0 elsewhere.

The Lie algebra $\mathfrak{g}=\mathfrak{so}(n+2)$ has a sub-OIL algebra structure given by

$$\mathfrak{so}(n+2)=\mathfrak{so}(n)+\langle\xi\rangle+\mathbb{R}^{2n}$$

where $\mathfrak{k}=\mathfrak{so}(n)$ is generated by $\{E^{ij}-E^{ji}: 4\leq i\leq n+2,\ 3\leq j\leq n+1,\ j<i\}$, ξ is a multiple of $\eta=E^{21}-E^{12}$ and $\mathfrak{p}=\mathbb{R}^{2n}$ is spanned by $X_k=E^{1,k+2}-E^{k+2,1}$, $Y_k=E^{2,k+2}-E^{k+2,2}$ for $k:1,\dots,n$.

It is an easy check now that $\{X_1,Y_1,\dots,X_n,Y_n\}$ is a symplectic basis for Θ and that

$$\mathfrak{p}=\langle X_1,\dots,X_n\rangle\oplus\langle Y_1,\dots,Y_n\rangle$$

is an $\mathrm{ad}_{\mathfrak{k}}$-irreducible decomposition of $\mathfrak{p}$. Therefore, B is such that $|X_k|=a_1$, $|Y_k|=a_2$ and $\langle X_k,Y_k\rangle=a_3$ for a_1, $a_2>0$, $|a_3|<a_1a_2$ and $k:1,\dots,n$. Let us assume, for simplicity, that $a_3=0$. The complex structure J on $\mathfrak{p}$ which is defined by Θ and B is given by

$$JX_k=\frac{a_1}{a_2}Y_k \quad\text{and}\quad JY_k=-\frac{a_2}{a_1}X_k,$$

for $k:1,\dots,n$, whereas the complex structure $J'=\mathrm{ad}_\eta$ coming from the Hermitian OIL-algebra structure of $\mathfrak{so}(n+2)$ maps X_k to Y_k and Y_k to $-X_k$.

From the normalization $(d\theta)^n|_{\mathcal{D}}=n!\,2^n\,dV$ and $\Theta=-\frac{1}{2}\theta^*([\cdot,\cdot])=\frac{1}{2}d\theta^*$ we get that $\Theta^n=n!\,dV_{\mathfrak{p}}$, $dV_{\mathfrak{p}}$ the volume form on $\mathfrak{p}$. Now $\Theta(X_k,Y_k)=-\frac{1}{2}\theta^*(\eta)$, so that $\theta^*(\eta)=-2a_1a_2$, $\xi=-\dfrac{1}{2a_1a_2}\eta$ and

$$\lambda_k=\Theta\left(\frac{1}{a_1}X_k,\frac{1}{a_2}Y_k\right)=-\frac{1}{2a_1a_2}\theta^*(\eta)=1$$

for $k:1,\dots,n$.

Finally, formula (9) shows that, with respect to the orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}\left(\frac{1}{a_1}X_1+\frac{1}{a_2}Y_1\right),\ldots,\frac{1}{\sqrt{2}}\left(\frac{1}{a_1}X_n+\frac{1}{a_2}Y_n\right),\right.$$
$$\left.\frac{1}{\sqrt{2}}\left(\frac{1}{a_1}X_1-\frac{1}{a_2}Y_1\right),\ldots,\frac{1}{\sqrt{2}}\left(\frac{1}{a_1}X_n-\frac{1}{a_2}Y_n\right)\right\}$$

of $\mathfrak{p}$, the sub-torsion τ is represented by the following symmetric matrix:

$$\frac{1}{4}\begin{pmatrix}\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right)I_n & 0\\ 0 & \left(\frac{1}{a_2^2}-\frac{1}{a_1^2}\right)I_n\end{pmatrix},$$

where I_n is an $n \times n$ identity block.

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