ANTHONY JOSEPH
GAIL LETZTER

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Annales scientifiques de l'É.N.S. 4e série, tome 28, nº 4 (1995), p. 493-526
<http://www.numdam.org/item?id=ASENS_1995_4_28_4_493_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 28, 1995, p. 493 à 526.

VERMA MODULE ANNIHILATORS FOR QUANTIZED ENVELOPING ALGEBRAS*

BY ANTHONY JOSEPH AND GAIL LETZTER**

ABSTRACT. – Let $U_q(\mathfrak{g})$ be the well-known quantization of the enveloping algebra of a semisimple Lie algebra discovered independently by Drinfeld and Jimbo. The main result of this paper is that the "simply connected" version of $U_q(\mathfrak{g})$ has the property that the annihilator of any Verma module is generated by the intersection with the centre, whilst this fails for $U_q(\mathfrak{g})$ itself. As a consequence an equivalence between highest weight and Harish-Chandra categories is obtained and this gives as a corollary a Duflo theorem for $Prim\ U_q(\mathfrak{g})$. This relates $Prim\ U_q(\mathfrak{g})$ to $Prim\ U(\mathfrak{g})$. Finally it is noted that a natural analogue of Kostant's primeness result fails for $U_q(\mathfrak{g})$.

1. Introduction

The base field k is assumed of characteristic zero, with $K = k(q)$. The notation is that of [JL1, JL2]; but will be redefined where necessary.

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The main object of our study is the quantized enveloping algebra $U_q(\mathfrak{g})$ introduced in now well-known works of Drinfeld and Jimbo. It was quickly recognized, for q an indeterminate, that the representation theory of $U_q(\mathfrak{g})$ is similar to that of the enveloping algebra $U(\mathfrak{g})$. Thus complete reducibility and the Weyl character formula for irreducible finite dimensional modules were obtained independently by G. Lusztig [L, 4.12] and M. Rosso [R1, Sect. V] though the latter had an error occurring in loc. cit. p. 512, line -6, which was subsequently eliminated through the use of the Rosso form [R2]. In the sequel we refer to JL1 for the description of $U_q(\mathfrak{g})$ and these assertions, since not only is the notation the same; but some unnecessary complications were eliminated from the proofs. Since the time our manuscript was communicated the paper [Dr] by V. Drinfeld appeared in which these results are recovered via an isomorphism of completed algebras though it is not immediate how this can be applied to the finer results considered here. Again this isomorphism fails [Dr, Prob. 8.1] in the Kac-Moody case whilst the Weyl formula and complete irreducibility still hold. For each $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ denote the Verma module with highest weight λ. A basic question in the theory of enveloping algebras was to determine the annihilator of $M(\lambda)$ in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, and it is a well-known result that this is

* Work supported in part by United States-Israel Grant No. 88-00130.
** Supported in part by a National Science Foundation Postdoctoral Fellowship.

generated by its intersection with the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. The apparent ease of the original proof is misleading as it relied on a deep theorem of Kostant concerning the primeness of $gr(U(\mathfrak{g})Z_\chi)$ for $Z_\chi \in Max\ Z(\mathfrak{g})$. For the quantized enveloping algebra $U_q(\mathfrak{g})$ (or simply, U), we are denied this result and the question becomes very difficult. In [JL2, Sect. 8] it was resolved by specialization at $q = 1$; but this only applies to certain choices of highest weight albeit the most natural. Moreover, a careful analysis shows (4.3) that the annihilator of a Verma module need not be generated by its intersection with the centre.

1.2. In [JL2, Sect. 7] we showed that the "simply connected" version of $\check{U}_q(\mathfrak{g})$ (or simply, $\check{U}$), admitted a separation of variables decomposition whereas this can fail [JL2, Example 5.5] for $U_q(\mathfrak{g})$. A main result of this paper shows that the annihilator of a Verma module for $\check{U}_q(\mathfrak{g})$ is again generated by its intersection with the centre. The proof which is necessarily quite different to the classical case obtains from the quantum analogue of a determinant considered in [PRV], [K], [J2], a main observation here being that we do not need to know the Verma module annihilators or Kostant's primeness result to compute it. In this we obtain the quantum analogue of Hesselink's formula [He] for the generalized exponents, though we were not able to get so neat a result.

1.3. A further result (7.3, 7.5) of this paper is the quantum analogue of [D, 8.4 (ii)], which was a technical result crucial to the determination of Verma module annihilators. The latter used Kostant's primeness result in an essential way. This is no longer available here and so understandably our proof is quite hard. Unlike the classical case we were unable to use it to determine Verma module annihilators. However, it does apply to show that the analogue of Kostant's primeness result fails in the quantum case. More precisely if $Z(\check{U})$ denotes the centre of $\check{U}$ and we set $Y(\check{U}) = gr_{\mathcal{F}} Z(\check{U})$ with $\check{Y}_+$ its augmentation ideal, then $\check{Y}_+ gr_{\mathcal{F}}(\check{U})$ need not be prime (8.10). A further argument (8.9) shows that if $Z(U)Y_+$ is prime, then U admits a separation of variables and we recall [JL2, 5.5] that the latter can fail. On the other hand we show for generic $Y_\lambda \in Max\ Y(U)$ that $Y_\lambda gr_{\mathcal{F}}(U)$ is completely prime (8.4, 8.5).

1.4. A main consequence of our result on Verma module annihilators is the quantum version of Duflo's theorem (6.4) and the more precise equivalence (5.12) between the highest weight and Harish-Chandra categories. Although the proof follows roughly the classical case, there are many additional subtleties.

Let $\mathfrak{h}^*_{\mathbb{Q}}$ denote the $\mathbb{Q}$-linear span of the fundamental weights. For each $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ one may interpret (2.1) q^λ as a weight with values in $\overline{k(q)}$. Then after Drinfeld [Dr] the simple quotient of the Verma module with highest weight q^λ remains simple on specialization. For such weights the primitive spectra of $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$ can be identified.

1.5. Some fairly standard notation concerning the semisimple Lie algebra $\mathfrak{g}$ is retained. Thus $\pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ with $\ell = rk\ \mathfrak{g}$ denotes the set of simple roots relative to the triangular decomposition of 1.1 with $\{\omega_i\}_{i=1}^{\ell}$ the corresponding fundamental weights and $\Delta^+(\pi)$ the set of positive roots. Then $Q(\pi) := \sum \mathbb{Z}\alpha_i$, $P(\pi) := \sum \mathbb{Z}\omega_i$, $R(\pi) := 4P(\pi) \cap Q(\pi)$, $P^+(\pi) := \sum \mathbb{N}\omega_i$, $Q^+(\pi) := Q(\pi) \cap P^+(\pi)$, etc. We let s_i denote the reflection corresponding to the root α_i and $W :=< s_i : i = 1, 2, \cdots, \ell >$ the Weyl group. The map τ is an isomorphism from $P(\pi)$ to a multiplicative group $\check{T}$, so in particular $\tau(\mu + \lambda) = \tau(\mu)\tau(\lambda)$, $\forall\ \mu, \lambda \in P(\pi)$. The relations on the generators of U will not be redefined; but we remark that $t_i = \tau(\alpha_i)$ and x_i (resp. y_i) have $\check{T}$ weight α_i (resp. $-\alpha_i$).

The Verma module annihilator theorem was reported at a seminar in the Weizmann Institute in June 1991 and during a workshop on Group representations and Algebraic Groups held in Warwick during 8-12 July, 1991. The quantized Duflo theorem was presented at a meeting on Harmonic Analysis on Lie Groups held at the Sandbjerg Estate, Denmark, during 26-30 August, 1991. We would like to thank V. Hinich for useful discussions.

2. Generalized Exponents

2.1. The algebra $U = U_q(\mathfrak{g})$ is that defined in [JL2, 2.1 and 3.1] via generators x_i, y_i, t_i, t_i^{-1} for $i = 1, 2, \cdots, \ell$ and relations. Recall that T denotes the free abelian subgroup of U generated by the t_i. Let M be a U module. A weight Λ for M is an element of the character group T^* of T that is an ℓ-tuple $\Lambda_1, \Lambda_2, \cdots, \Lambda_\ell \in K^*$ such that for some $m \in M\backslash\{0\}$ one has $t_i m = \Lambda_i m$ for all i. We let $M_\Lambda = \{m \in M \,\big|\, t_i m = \Lambda_i m, \ \forall\ i\}$ denote the corresponding weight space of M and $\Omega(M)$ the set of weights of M. We shall say that M is a weight module if it is a direct sum of its weight subspaces. Analogous definitions apply to $\check{U}$ replacing T by $\check{T} =< \tau(\omega_i) \,\big|\, i = 1, 2, \cdots, \ell >$ and the $t_i = \tau(\alpha_i)$ by the $\tau(\omega_i)$. We shall say that Λ is linear weight for M if there exists $\lambda \in \mathfrak{h}^*$ such that $(\alpha_i, \lambda) \in \mathbb{Z}$ and $\Lambda_i = q^{(\alpha_i, \lambda)}$, for all i, and we write $\Lambda = q^\lambda$. By slight abuse of notation we set $M_\Lambda = M_\lambda$ in this case. We should then have $\tau(\mu)m = q^{(\mu,\lambda)}m$, for all $m \in M_\lambda$ and $\mu \in Q(\pi)$. This formula can be extended to $P(\pi)$ given $(\mu, \lambda) \in \mathbb{Z}$ for all $\mu \in P(\pi)$, that is if $\lambda \in Q(\pi)$.

We shall let $\mathcal{E}$ denote the set (of isomorphism classes) of finite dimensional simple U modules having *linear* weights. Every element $E \in \mathcal{E}$ is determined by its highest weight $\mu \in P^+(\pi)$ and then is denoted by $E(\mu)$. We let $\check{\mathcal{E}}$ denote the subset of $\mathcal{E}$ consisting of those $E(\mu)$ with $\mu \in Q(\pi)$. It is clear that $E \in \mathcal{E}$ extends uniquely to a $\check{U}$ module with linear weights if and only if $E \in \check{\mathcal{E}}$. Of course depending on the choice of k other weight modules for U may extend (not necessarily uniquely) to weight modules for $\check{U}$. Indeed over the algebraic closure $\bar{K}$ of K every weight module for U extends to a weight module for $\check{U}$. However, especially in discussing specialization at $q = 1$ (see also 4.3) it may be important to distinguish between linear weights (which specialize to the trivial weight at $q = 1$) and more general weights.

We use $\bigoplus \mathcal{E}$ (resp. $\bigoplus \check{\mathcal{E}}$) to denote a U (resp. $\check{U}$) module which is a finite direct sum of objects in $\mathcal{E}$ (resp. $\check{\mathcal{E}}$).

2.2. Let $F(U)$ (resp. $F(\check{U})$) denote the subspace of U (resp. $\check{U}$) on which $ad\ U$ acts locally finitely. We recall that the filtration $\mathcal{F}$ of U (resp. $\check{U}$) defined in [JL2, 2.2] is $ad\ U$ stable. Let $F_m(U)$, notation [JL2, 4.11], denote the space of homogeneous elements of degree m of $gr_{\mathcal{F}}F(U)$ with an analogous definition for $F_m(\check{U})$. We remark that the conclusion of [JL2, 4.10] applies to $\check{U}$ with $R^+(\pi)$ replaced by $4P^+(\pi)$. It is an elementary fact that the sum of the coefficients of $\lambda \in 4P^+(\pi)$ with respect to π is always a non-negative integer. Consequently $F_m(\check{U}) = 0$ unless $m \in \mathbb{N}$. Following [JL2, 5.4] we define for each $\mu \in P^+(\pi)$ the Poincaré series

$$\check{R}_\mu(z) := \sum_{m=0}^{\infty} [F_m(\check{U}) : E(\mu)] z^m \ .$$

As noted in [JL2, 5.4] this may be written in the form

$$\check{R}_\mu(z) = \check{P}_\mu(z) \check{R}_0(z)$$

where $\check{P}_\mu(z)$ is a polynomial in z. By [JL2, 7.6] one has $\check{R}_\mu(z) = 0$ unless $\mu \in Q(\pi)$, so we may as well assume $\mu \in Q^+(\pi)$.

2.3. Let ρ be the half sum of the positive roots and define the translated action of the Weyl group W on $\mathfrak{h}^*$ through $w \cdot \lambda = w(\lambda + \rho) - \rho, \quad \forall \ \lambda \in \mathfrak{h}^*$. Given $w \in W$, let $\ell(w)$ denote its reduced length.

For each $\lambda \in Q^+(\pi)$, we write $4\lambda = \sum_{i=1}^{\ell} k_i \alpha_i$ and set $deg\ \lambda = \sum_{i=1}^{\ell} k_i$. As noted above $deg\ \lambda \in \mathbb{N}$. Define the formal power series

$$\begin{aligned} \check{S}(z) : &= \sum_{\lambda \in P^+(\pi)} \sum_{y \in W} (-1)^{\ell(y)} e^{y \cdot \lambda - \lambda} z^{deg\ \lambda} \\ &= \sum_{y \in W} \frac{(-1)^{\ell(y)} e^{y \cdot 0}}{\prod_{i=1}^{\ell} (1 - z^{deg\ \omega_i} e^{y\omega_i - \omega_i})} \ . \end{aligned}$$

We further set

$$\check{Q}(z) = \sum_{y \in W} (-1)^{\ell(y)} e^{y \cdot 0} \prod_{i=1}^{\ell} \left(\frac{1 - z^{deg\ \omega_i}}{1 - z^{deg\ \omega_i} e^{y\omega_i - \omega_i}} \right) \ .$$

We shall soon see that $\check{S}$ and $\check{Q}$ are the generating functions for $\check{R}_\mu$ and $\check{P}_\mu$ respectively. The latter is the quantum analogue of the generating function $Q_C(z)$ in Hesselink's formula [He, Sect. 1]. One had

$$Q_C(z) = \frac{\mathbf{\Delta}}{\mathbf{\Delta}(z)} \ , \quad \text{where} \quad \mathbf{\Delta}(z) = \prod_{\alpha > 0} (1 - z e^{-\alpha}) \ , \quad \mathbf{\Delta} = \mathbf{\Delta}(1) \ ,$$

which made the derivative $Q'_C(1)$ of $Q_C(z)$ at $z = 1$ particularly simple to calculate [J2, 2.4]. Although $\check{Q}(z)$ does not seem to have a simple expression, one does have the

LEMMA. – $\check{Q}'(1) = \sum_{i=1}^{\ell} \left(\frac{e^{-\alpha_i}}{1 - e^{-\alpha_i}} \right) deg\ \omega_i$.

It suffices to observe that the only terms in the sum $y \in W$ which can contribute to $\check{Q}'(1)$ occur when $y = s_{\alpha_i}$ and this term gives the expression in the right hand side above.

2.4. Recall that by say [JL1, 5.10] the weight space decomposition of $E(\mu) : \mu \in P^+(\pi)$ is given by the Weyl character formula

$$ch\ E(\mu) = \mathbf{\Delta}^{-1} \sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \mu} .$$

PROPOSITION. – *For all $\mu \in Q^+(\pi)$,*
(i) *$\check{R}_\mu(z)$ is the coefficient of e^0 in $(ch\ E(\mu))\check{S}(z)$.*
(ii) *$\check{P}_\mu(z)$ is the coefficient of e^0 in $(ch\ E(\mu))\check{Q}(z)$.*

By [JL2, 3.5 and 4.10] we have

$$\check{R}_\mu(z) = \sum_{\lambda \in P^+(\pi)} [End_K E(\lambda) : E(\mu)] z^{deg\ \lambda} .$$

The expression in square brackets is completely determined by the Weyl character formula. It may therefore be computed as if $\check{U}$ were replaced by $U(\mathfrak{g})$. In particular, we may use the BGG resolution [D, 7.8.14] for $E(\lambda)$. This and [D, 7.6.14] gives in the Grothendieck group

$$\begin{aligned}[E(\lambda) \otimes E(-\lambda)] &= \sum_{w \in W} (-1)^{\ell(w)} [M(w \cdot \lambda) \otimes E(-\lambda)] \\ &= \sum_{w \in W} \sum_{\nu \in \Omega(E(-\lambda))} (-1)^{\ell(w)} [M(w \cdot \lambda + \nu)] .\end{aligned}$$

Consequently

$$\begin{aligned}[End_K E(\lambda) : E(\mu)] &= \sum_{w \in W} (-1)^{\ell(w)} dim\ E(-\lambda)_{\mu - w \cdot \lambda} \\ &= \sum_{w \in W} (-1)^{\ell(w)} dim\ E(\lambda)_{w \cdot \lambda - \mu} \\ &= \sum_{w, y \in W} (-1)^{\ell(y)+\ell(w)} P(y \cdot \lambda - w \cdot \lambda + \mu)\end{aligned}$$

where P is Kostant's partition function. Set $we^\gamma = e^{w\gamma}$ for $w \in W$ and extend this action of W linearly. We conclude that $\check{R}_\mu(z)$ is just the coefficient of $e^{-\mu}$ in

$$\begin{aligned}\check{R}(z) :&= \sum_{\beta \in \mathbb{Z}\pi} \sum_{\lambda \in P^+(\pi)} \sum_{y, w \in W} (-1)^{\ell(y)+\ell(w)} e^{-\beta} P(\beta) e^{-(w \cdot \lambda - y \cdot \lambda)} z^{deg\ \lambda} , \\ &= \mathbf{\Delta}^{-1} \sum_{\lambda \in P^+(\pi)} \Big(\sum_{y, w \in W} (-1)^{\ell(y)+\ell(w)} e^{-w(\lambda+\rho)+y(\lambda+\rho)} \Big) z^{deg\ \lambda} , \\ &= \mathbf{\Delta}^{-1} \sum_{\lambda \in P^+(\pi)} \Big\{ \sum_{w \in W} w \Big(\sum_{y \in W} (-1)^{\ell(y)} e^{y \cdot \lambda - \lambda} \Big) \Big\} z^{deg\ \lambda} , \\ &= \mathbf{\Delta}^{-1} \sum_{w \in W} w\check{S}(z) .\end{aligned}$$

Set $w_*\lambda = w(\lambda-\rho)+\rho$. Since $w_*(\mathbf{\Delta}^{-1}e^{\xi}) = (-1)^{\ell(w)}\mathbf{\Delta}^{-1}e^{w\xi}, \ \forall \ \xi \in P(\pi)$ we obtain

$$(*) \qquad \check{R}(z) = \sum_{w\in W} (-1)^{\ell(w)} w_*(\mathbf{\Delta}^{-1}\check{S}(z)) \ .$$

On the other hand the coefficient of e^0 in $ch\ E(\mu)\check{S}(z)$ is just the sum of each of the coefficients of $e^{-w\cdot\mu}$ in the term $\mathbf{\Delta}^{-1}(-1)^{\ell(w)}\check{S}(z)$. Since $w\cdot\mu = -w_*(-\mu)$, this is just the coefficient of $e^{-\mu}$ in $(-1)^{\ell(w)}w_*^{-1}(\mathbf{\Delta}^{-1}\check{S}(z))$. Comparison with (*) above gives (i). Then (ii) follows from (i) if we observe that

$$\check{R}_0(z) = \sum_{\lambda\in P^+(\pi)} z^{deg\ \lambda} = \prod_{i=1}^{\ell}(1 - z^{deg\ \omega_i})^{-1} \ .$$

2.5. We shall not need $\check{P}_\mu(z)$ but only its derivative at $z = 1$. This is given by the

COROLLARY. – $\check{P}'_\mu(1) = \frac{1}{2} \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} dim\ E(\mu)_{n\alpha} deg\ \alpha.$

Set $E = E(\mu)$. By 2.3 and 2.4(ii) we have just to show that

$$\sum_{i=1}^{\ell}(deg\ \omega_i) dim\ E_{n\alpha_i} = \frac{1}{2}\sum_{\alpha>0}(deg\ \alpha) dim\ E_{n\alpha}$$

for all $n \in \mathbb{N}^+$. Recall that $dim\ E_{n\alpha}$ is constant on roots of the same length and that deg is additive. It thus suffices to observe that the half sum of the positive short (resp. long) roots is just the sum of the fundamental weights corresponding to the short (resp. long) roots.

2.6. Set $T_< = \tau(-4P^+(\pi)\cap Q(\pi))$, $\check{T}_< = \tau(-4P^+(\pi))$. We shall eventually need the following

LEMMA. – $T_<^{-1}F(U)$, $\check{T}_<^{-1}F(\check{U})$ *are left and right noetherian rings.*

It is enough to prove the first assertion since the second ring is a finitely generated left (or right) module over the first.

Set $T_\diamond = T_<^{-1}T_<$ (noted T_0 in [JL1]). From [JL1, 6.4], we see that U is graded by the finite group $\Gamma := T/T_\diamond$. Indeed

$$U = \bigoplus_{\gamma\in\Gamma} U_\gamma$$

with $U_e = T_<^{-1}F(U)$. Let I be a left ideal of $F(U)$. Then $J = UI$ is a graded left ideal of U satisfying $J_e = I$. After McConnell [M, 4.9] U is left (and right) noetherian. Hence J is finitely generated over U and hence over $F(U)$, by the finiteness of Γ. Taking homogeneous components of generators it follows that each component of J is finitely generated over $F(U)$, in particular I is finitely generated over $F(U)$.

Remark. – One expects that $F(U)$ itself is noetherian. By [JL2, 4.11, 4.12(iii)] it is finitely generated which was not obvious from [JL1, 6.4].

2.7. Recall the gradation of 2.6. Observe that an ideal I of U is graded if and only if it satisfies $I = U(I \cap F(U))$. This will generally fail; but only in some "trivial" fashion. For the moment we just note the following fact leaving more complete details till 6.1.

Let K_Λ denote the one-dimensional U module given by the weight $\Lambda \in T^*$. By definition $t_i m = \Lambda_i m$, for all $m \in K_\Lambda$, whilst the conditions $U_+^+ m = U_+^- m = 0$ are satisfied if and only if $\Lambda_i^4 = 1$, for all i. Thus we have a bijection $\Lambda \to K_\Lambda$ of $(T/4T)^*$ onto the set of isomorphism classes of one-dimensional U modules. It is clear from [JL1, 6.4] that $K_\Lambda \big|_{F(U)}$ is the trivial $F(U)$ module if and only if $t(\Lambda) = 1$, for all $t \in T_\diamond$. Hence $(T/T_\diamond)^*$ identifies with the set of isomorphism classes of one dimensional U modules trivial on $F(U)$. If we take representatives t_i' for $i = 1, 2, \cdots, r$ from $T/T_\diamond$ then by finite group theory the characters $t_i' \mapsto t_i'(\Lambda^j)$ where $\Lambda^j \in (T/T_\diamond)^*$ are linearily independent. In particular

$$(*) \qquad det(t_i'(\Lambda^j))_{i,j=1}^r \neq 0 \ .$$

Of course the analogous result holds for $\check{U}$ with $(T/4T)^*$ replaced by $(\check{T}/4T)^*$ and $(T/T_\diamond)^*$ replaced by $(\check{T}/4\check{T})^*$.

3. Determinant of the basic form

3.1. We now use 2.5 to compute the quantum analogue of the determinant first considered in [PRV, Thm. 4.2] and given a more modern proof in [J2, Sect. 2] which we shall follow. Although the analysis is similar to [J2] it is complicated by two new features. First we have not yet determined the Verma module annihilators and secondly $\check{U}_q(\mathfrak{g})$ admits non-scalar invertible elements.

3.2. For $\alpha \in \Delta^+(\pi)$ set $d_\alpha = (\alpha, \alpha)/2$ and $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Identify $\mathfrak{h}_\mathbb{Q}^*$ with the $\mathbb{Q}$ linear span of the $\{\alpha_i\}_{i=1}^\ell$. For each integer $n > 0$ we set $\Lambda_{n,\alpha} = \{\lambda \in \mathfrak{h}_\mathbb{Q}^* \big| \ (\lambda + \rho, \alpha^\vee) = n\}$ and $\Lambda_{n,\alpha}^0 = \{\lambda \in \Lambda_{n,\alpha} | (\lambda + \rho, \beta^\vee) \notin \mathbb{N}^+, \ \forall \ \beta \in \Delta^+(\pi), \ \beta \neq \alpha\}$. Consider $q^{(\lambda,\mu)}$ for $\lambda \in \mathfrak{h}_\mathbb{Q}^*$, and $\mu \in P(\pi)$ as an element of $\bar{K}$ and let $M(\lambda) : \lambda \in \mathfrak{h}_\mathbb{Q}^*$ denote the Verma module for $\check{U}$ with highest weight $q^\lambda \mapsto q^{(\lambda,\omega_i)}$.

LEMMA. – *For all* $\lambda \in \Lambda_{n,\alpha}$ *one has*
(i) $M(s_\alpha \cdot \lambda)$ *is a submodule of* $M(\lambda)$.
(ii) *If* $\lambda \in \Lambda_{n,\alpha}^0$ *then* $M(s_\alpha \cdot \lambda)$ *is simple.*

Following the classical computation [D, 7.6.12] one checks that conclusion (i) is equivalent to a finite family of finite sums

$$F_m^\lambda(q) = \sum_{\mu \in P(\pi)} q^{(\lambda,\mu)} p_{\mu,m}(q)$$

being identically zero, where the $p_{\mu,m}(q)$ are polynomials independent of the choice of $\lambda \in \Lambda_{n,\alpha}$. The $F_m^\lambda(q)$ are analytic at $q = 1$ and their derivatives at $q = 1$ are polynomials in λ. We obtain their vanishing by the density argument of [D, 7.6.13]. By repeated application of [JL1, 5.6] conclusion (i) holds for $\lambda \in Q^+(\pi)$. Yet $Q^+(\pi)$ has a Zariski dense intersection with $\Lambda_{n,\alpha}$ and so (i) obtains. For (ii) observe that for $\lambda \in \Lambda^0_{n,\alpha}$, the module $M(s_\alpha.\lambda)$ has no weight of the form $w \cdot \lambda$ for $w \in W$ other than its highest weight $s_\alpha \cdot \lambda$. (In the usual terminology $s_\alpha \cdot \lambda$ is antidominant). Then (ii) follows from [JL1, 8.6] as in the classical case (i.e., apply 4.6(i) below).

Remark. – The proof of (i) in [DeC-K, 1.9(a)] is incomplete because the authors implicitly assume that the degeneracy of the contravariant form on $M(\lambda) : \lambda \in \Lambda_{n,\alpha}$ is implied by its degeneracy at $q = 1$. Nevertheless, their formula for the quantum Shapovalov form is ultimately correct and will be used in the sequel.

3.3. Fix $\lambda \in \Lambda^0_{n,\alpha}$ and let $\mathcal{O}_\lambda$ denote the category of weight modules of finite length whose simple factors take the form $L(\mu)$ for $\mu \in \Lambda + Q(\pi)$. By [JL1, 8.6] a standard argument shows that $M(\lambda)$, $M(s_\alpha \cdot \lambda) \in Ob\mathcal{O}_\lambda$ with $M(\lambda)$ projective. Set $N(\lambda) = M(\lambda)/M(s_\alpha \cdot \lambda)$. Recall the notation of [JL2, 6.10].

LEMMA. – *For each $E \in \mathcal{E}$ one has*

$$[F(N(\lambda), N(\lambda)) : E] = dim\ E_0 - dim\ E_{n\alpha}\ .$$

Let $d(M)$ denote the Gelfand-Kirillov dimension [KL, Chap. 5] of a $\check{U}$ module M. A standard reasoning [cf. JL2, 8.4(*) and 8.5] shows that $d(N(\lambda)) < d(M(s_\alpha \cdot \lambda))$. Since $M(s_\alpha \cdot \lambda)$ is simple it follows from [JL2, 8.4] that $F(M(s_\alpha \cdot \lambda), N(\lambda)) = 0$. Hence the natural injection $F(N(\lambda), N(\lambda)) \hookrightarrow F(M(\lambda), N(\lambda))$ is surjective. Since $M(\lambda)$ is projective the natural map $F(M(\lambda), M(\lambda)) \longrightarrow F(M(\lambda), N(\lambda))$ is surjective. Moreover this has kernel $F(M(\lambda), M(s_\alpha \cdot \lambda))$. Finally using again that $M(\lambda)$ is projective gives as in [JL2, 8.3] that $[F(M(\lambda), M(\lambda)) : E] = dim\ E_0$ and $[F(M(\lambda), M(s_\alpha \cdot \lambda)) : E] = dim\ E_{s_\alpha \cdot \lambda - \lambda} = dim\ E_{-n\alpha} = dim\ E_{n\alpha}$.

3.4. A basic result of [JL2] asserts that $F(\check{U})$ is a free module over its centre $Z(\check{U})$. More precisely [JL2, 7.4], there exists an $ad\ \check{U}$ invariant subspace $\mathbb{H}$ such that

$$F(\check{U}) = \mathbb{H} \otimes Z(\check{U})\ .$$

Moreover for each simple finite dimensional module E one has

$$[\mathbb{H} : E] = dim\ E_0 =: m\ .$$

In particular $[\mathbb{H} : E] \neq 0 \iff E \in \check{\mathcal{E}}$.

This above formula also means that we can choose $a_{ij} \in \mathbb{H}$ for $i, j = 1, 2, \cdots, m$ such that $\{a_{ij}\}_{i=1}^m$ forms a basis of E_0 for the j^{th} copy of E in $\mathbb{H}$. Let φ_E denote the matrix with entries $\{\varphi(a_{ij})\}_{i,j=1}^m$ where $\varphi : \check{U} \to \check{U}^0$ is the Harish-Chandra map [JL1, 8.1].

For each $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ we denote by $L(\lambda)$ the unique simple quotient of $M(\lambda)$. Recall the notation of 3.2.

LEMMA.

(i) *For any finite dimensional ad U submodule M of $F(\check{U})$ one has $ML(\lambda) = 0 \iff \varphi(M)(\lambda) = 0$.*

For each $E \in \check{\mathcal{E}}$ one has

(ii) $rk\ \varphi_E(\lambda) = [F(\check{U})/Ann_{F(\check{U})} L(\lambda) : E]$

(iii) *det* $\varphi_E \neq 0$.
(iv) λ *is a zero of det* φ_E *of order* $\geq [Ann_{\mathbb{H}} L(\lambda) : E]$.
(v) *If* $\lambda \in \Lambda^0_{n,\alpha}$, *then* λ *is a zero of det* φ_E *of order* $\geq$ *dim* $E_{n\alpha}$.

Assertion (i) is just [JL2, 3.2]. Assertion (ii) follows from (i) as in [J2, 2.2(ii)]. By [JL2, 5.2] we can choose $\lambda \in Q^+(\pi)$ so that $[F(\check{U})/Ann_{F(\check{U})}L(\lambda) : E] = dim\ E_0 = [\mathbb{H} : E_0]$ and so (ii) implies (iii). Again (iv) follows from (ii) as in [J2, 2.2(iii)]. Finally observe that the action of $F(\check{U})$ on $N(\lambda)$ defines a map $F(\check{U}) \longrightarrow F(N(\lambda), N(\lambda))$ with kernel $Ann_{F(\check{U})}N(\lambda) \subset Ann_{F(\check{U})}L(\lambda)$. We conclude from 3.3 that $[F(\check{U})/Ann_{F(\check{U})}L(\lambda) : E] \leq dim\ E_0 - dim\ E_{n\alpha}$. Since $\mathbb{H}/Ann_{\mathbb{H}}L(\lambda) \xrightarrow{\sim} F(\check{U})/Ann_{F(\check{U})}L(\lambda)$ and $[\mathbb{H} : E] = dim\ E_0$, we conclude that $[Ann_{F(\check{U})}L(\lambda) : E] \geq dim\ E_{n\alpha}$ and so (v) obtains from (iv).

Remark. – Statements (i) and (ii) hold for any weight $\Lambda \in T^*$. This will be used in Section 4.1.

3.5. Before going further let us analyse the general form that $det\ \varphi_E$ for $E \in \check{\mathcal{E}}$ can take. First different choices for the bases $\{a_{ij}\}$ only alters it by a non-zero element of K. In view of this and [JL2, 4.10] we can assume that the j^{th} copy of E in $\mathbb{H}$ lies in some $(ad\ U)\tau(\lambda_j)$ where $\lambda_j \in -4P^+(\pi)$. From the relations in $\check{U}$ it is trivial to check that $\varphi(a) \in K[t_i^4 : 1 \leq i \leq \ell]\tau(\lambda_j)$ for all $a \in (ad\ U)\tau(\lambda_j)$. We conclude that

$$(*) \qquad det\ \varphi_E \in K[t_i^4 : 1 \leq i \leq \ell] \prod_{j=1}^{m} \tau(\lambda_j) \ .$$

Lemma. – *Take* $E \in \check{E}$. *For* $\alpha \in \Delta^+(\pi)$ *and each* $n \in \mathbb{N}^+$ *the polynomial* $(\tau(\alpha)^4 - q^{4d_\alpha(n-(\rho,\alpha^\vee))})^{dim\ E_{n\alpha}}$ *divides det* φ_E.

Take $\lambda \in \Lambda^0_{n,\alpha}$. Then

$$\tau(\alpha)(\lambda) = q^{(\alpha,\lambda)} = q^{d_\alpha(\alpha^\vee,\lambda+\rho-\rho)} = q^{d_\alpha(n-(\rho,\alpha^\vee))} \ .$$

It follows by 3.4(v) as in [J2, 2.3] that $(\tau(\alpha) - q^{d_\alpha(n-(\rho,\alpha^\vee)})^{dim\ E_{n\alpha}}$ divides $det\ \varphi_E$. Taking (*) into account this proves the lemma.

3.6. We are now ready to compute $det\ \varphi_E$.

Proposition. – *Take* $E \in \check{\mathcal{E}}$. *For each simple finite dimensional* U *module one has*

$$det\ \varphi_E = \prod_{n\in\mathbb{N}^+} \prod_{\alpha>0} (\tau(\alpha)^2 - q^{4d_\alpha(n-(\rho,\alpha^\vee))}\tau(\alpha)^{-2})^{dim\ E_{n\alpha}} \ ,$$

up to a non-zero element of K.

We already know by 3.5 that the right hand side divides the left hand side and that its degree as a polynomial in the t_i equals

$$2 \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} dim\ E_{n\alpha} deg\ \tau(\alpha) = \frac{1}{2} \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} dim\ E_{n\alpha} deg\ \alpha \ .$$

This by 2.5, see also [J2, 2.4], is greater than or equal to the degree of $det\ \varphi_E$. Yet this is not enough to prove the assertion since $det\ \varphi_E$ could also be divisible by invertible elements of negative degree.

We refine 3.5(*) using [JL2, 4.16] to compute an upper bound on the largest possible degree of a monomial in the t_i for $i \in \{1,2,\cdots,\ell\}$ occurring in $det\ \varphi_E$. Let w_0 be the unique longest element in W. By [JL2, 4.16] the element of largest degree in $\varphi((ad\ U)\tau(\lambda_j))$ has degree equal to $deg\ \tau(w_0\lambda_j) = -deg\ \tau(\lambda_j)$, so this largest degree is just $-deg\ \tau(\lambda)$ where $\lambda = \sum_{j=1}^{m} \lambda_j$.

Now from 2.5 and the definition of the polynomial $P_\mu(z)$, taking account of [JL2, 4.5, 4.10] we obtain

$$deg\ \tau(\lambda) = \sum_{j=1}^{m} deg\ \tau(\lambda_j) = -\frac{1}{2} \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} dim\ E_{n\alpha} deg\ \alpha\ .$$

Now by 3.5 we can write

$$(*) \qquad det\ \varphi_E = p \prod_{n\in\mathbb{N}^+} \prod_{\alpha>0} (\tau(\alpha)^4 - q^{4d_\alpha(n-(\rho,\alpha^\vee))})^{dim\ E_{n\alpha}} \tau(\lambda)$$

for some $p \in K[t_i^4 : i \in \{1,2,\cdots,\ell\}]$. Hence

$$\begin{aligned} deg\ det\ \varphi_E &= deg\ p + \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} dim\ E_{n\alpha} deg\ \alpha + deg\ \tau(\lambda) \\ &= deg\ p - deg\ \tau(\lambda)\ . \end{aligned}$$

Then by our previous upper bound, namely $deg\ det\ \varphi_E \leq -deg\ \tau(\lambda)$, we conclude that $deg\ p \leq 0$ and so p is a (non-zero) element of K. The equality that this forces in the above, implies that the greatest possible degree term, namely $\tau(w_0\lambda)$, does occur in $det\ \varphi_E$. We conclude from (*) that $\lambda + 4\beta = w_0\lambda$ and

$$\beta = \sum_{n\in\mathbb{N}^+} \sum_{\alpha>0} \alpha\ dim\ E_{n\alpha}\ .$$

If one can show that $w_0\lambda = -\lambda$, then we must have $\lambda = -2\beta$ which on substitution in (*) gives the required result. To show that $w_0\lambda = -\lambda$ we can assume $\mathbb{H}$ is chosen so that

$$\mathbb{H} = \bigoplus_{\mu\in -4P^+(\pi)} \mathbb{H} \cap (ad\ U)\tau(\mu)\ .$$

Then it suffices to show that

$$(**) \qquad [\mathbb{H} \cap (ad\ U)\tau(\mu) : E] = [\mathbb{H} \cap (ad\ U)\tau(-w_0\mu) : E]$$

for all $\mu \in -4P^+(\pi)$. Of course it suffices to prove this for just H (which identifies with $gr_{\mathcal{F}}\mathbb{H}$).

By [JL2, 3.5] we have U module isomorphisms $(ad\ U)\tau(\mu) \cong E\left(-\frac{1}{4}\ \mu\right) \otimes E\left(\frac{1}{4}\ w_0\mu\right) \cong (ad\ U)\tau(-w_0\mu)$. By [JL2, 4.10] these extend to a U module isomorphism θ of $gr_{\mathcal{F}}F(\check{U})$ leaving $Y(\check{U})$ fixed. By [JL2, 4.12(iii)] it follows that θ is an algebra involution. Since H is defined to be any U stable complement to $Y_+\ gr_{\mathcal{F}}F(\check{U})$ itself θ stable, we can assume H to be also θ stable. Then (**) is immediate. This completes the proof of the proposition.

4. Verma module annihilators

4.1. Let $M(\Lambda)$ denote the Verma module with highest (not necessarily linear) weight Λ. Let $L(\Lambda)$ denote its unique simple quotient. We wish to show that $Ann_{F(\check{U})}M(\Lambda)$ is generated by its intersection with $Z(\check{U})$. This needs the following preliminary. Observe the remarkable similarity between $det\ \varphi_{E(\mu)}$ and the corresponding determinant det_η of the quantum Shapovalev form given in [DeC-K, 1.9]. Although these are not quite the same, they only differ by involving different powers of the same factors

$$(\tau(\alpha)^2 - q^{4d_\alpha(n-(\rho,\alpha^\vee))}\tau(\alpha)^{-2}) \quad \text{where} \quad n \in \mathbb{N}^+, \quad \text{and} \quad \alpha \quad \text{is a positive root,}$$

which is a positive power for μ, or for η sufficiently large. Since the kernel of the Shapovalev form is the kernel of the canonical projection $M(\Lambda) \longrightarrow L(\Lambda)$, we conclude from this observation and 3.4(i) the

LEMMA. – *The following two conditions are equivalent*
(i) $Ann_{\mathbb{H}}L(\Lambda) = 0$,
(ii) $M(\Lambda) \xrightarrow{\sim} L(\Lambda)$.

Remark. – The derivation of the Shapovalev form only requires 3.2 and some easy degree estimates. It implies that $M(w_0 \cdot \lambda)$ for $\lambda \in P^+(\pi)$ is a simple Verma module and moreover it is immediate that this result extends to the case when q is viewed as an element of k^* and which is not a root of unity. In [JL2, 6.7, Remark 2] it was pointed out that one could give a particularly simple determination of $Z(U)$ based on the existence of a simple Verma module and this is provided by the above argument.

4.2. By [JL1, 8.6] and a standard reasoning cf. [D, 7.6.1], any Verma module $M(\Lambda)$ has finite length and a further standard reasoning using [JL1, 8.6] shows that $M(\Lambda)$ admits a simple Verma submodule $M(\Lambda')$. Recall that the action of $F(\check{U})$ on $M(\Lambda)$ defines a map $\varphi_\Lambda : F(\check{U}) \longrightarrow F(M(\Lambda), M(\Lambda))$.

THEOREM.
(i) $Ann_{F(\check{U})}M(\Lambda) = F(\check{U})Ann_{Z(\check{U})}M(\Lambda) = ker\ \varphi_\Lambda$.
(ii) φ_Λ *is surjective.*

Clearly $Ann_{\mathbb{H}}M(\Lambda) \subset Ann_{\mathbb{H}}M(\Lambda') = 0$ by 4.1. On the other hand $Ann_{Z(\check{U})}M(\Lambda)$ has codimension 1 in $Z(\check{U})$. Thus (i) follows from [JL2, 7.4]. By the reasoning of [JL2, 8.3, 8.5] we obtain

$$[F(M(\Lambda), M(\Lambda)) : E] \leq [F(M(\Lambda'), M(\Lambda')) : E] = dim\ E_0\ .$$

Yet $[\mathbb{H} : E] = dim\ E_0$ by [JL2, 7.6] and so (i) implies (ii).

4.3. The reader may recall that we already proved 4.2 in the case when Λ is an integral (linear) weight by specialization [JL2, 6.12, 8.6] and this result further applies to U. That reasoning is still valid if Λ is a linear weight. To justify the considerable extra effort needed in the non-linear case we give an example where both assertions in 4.2 fail for U.

Example. – Take $U = U_q(\mathfrak{sl}(3))$ and define e_1, e_2 as in [JL2, 5.5] viewed as elements of $\check{U}$. Recall that e_1, e_2 form a basis for the highest weight space of the isotypical component of $\mathbb{H}$ corresponding to the "adjoint representation" E. Let $z_i : i = 1, 2$ be a basis vector for the trivial one-dimensional submodule of $(ad\ U)\tau(-4\omega_i)$. Then z_1, z_2 are generators for the polynomial algebra $Z(\check{U})$. Furthermore, $e_1 z_2,\ e_1 z_1^2,\ e_2 z_1,\ e_2 z_2^2 \in U$, and using (JL2, 7.4) one easily checks that these form a set of generators over $Z(U)$ for the highest weight space of the E isotypical component U_E of U. Now suppose we can find a Verma module $M(\Lambda)$ for $\check{U}$ satisfying $z_1, z_2 \in Ann_{Z(\check{U})} M(\Lambda)$. Then $U_E \subset Ann_{F(U)} M(\Lambda)$. This clearly excludes that the conclusions of 4.2 hold for U.

The condition we require is satisfied exactly when Λ is a common root of $\varphi(z_i)$ for $i = 1, 2$. By formula (6) in [JL1, 8.6] we find that

$$\varphi(z_i) = \hat{\tau}(-4\omega_i) := \sum_{y\in W} \tau(-4y\omega_i) q^{(\rho, -4y\omega_i)} .$$

Take $\Lambda = \gamma q^{-\rho}$ where γ is a primitive cube root of unity. Then

$$\varphi(z_i)(\Lambda) = \sum_{y\in W} \tau(-4y\omega_i)(\gamma) .$$

In the evaluation of the right hand side we must replace t_1, t_2 by γ. Since

$$\sum_{y\in W} \tau(-4y\omega_1) = t_1^{-8/3} t_2^{-4/3} + t_1^{4/3} t_2^{-4/3} + t_1^{4/3} t_2^{8/3}$$

the resulting evaluation is just $\gamma^{-4} + 1 + \gamma^4 = 0$. A similar result holds when $i = 2$. This proves the assertion.

One can also see that the specialization argument of [JL2, 6.11, 6.12] does not apply to this example. In the notation of [JL2, 6.11] we have

$$\frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}} = x_i y_i - y_i x_i \in \hat{U}$$

and so $t_i^4 = 1$ at the specialization $q = 1$. Thus for $M(\Lambda)$ to specialize to a Verma module of $U(\mathfrak{g})$ we must have $\Lambda_i^4(1) = 1\ mod < q - 1 >$, for all i which is not the case here.

4.4. After McConnell [M, 4.9] one has the quantum analogue of [D, 2.6.9], namely

LEMMA. – *Let M be a simple U module. Then $End_U M$ is an algebraic extension of K.*

Remark. – We do not know if this also holds for $F(U)$.

4.5. In the remainder of Sect. 4 we shall replace K by $\bar{K}$ setting $\overset{\triangledown}{U} := \check{U} \otimes_K \bar{K}$. It is quite trivial that McConnell's proof noted above carries over to $\overset{\triangledown}{U}$, so we have the

COROLLARY. – *If $I \in Prim\ \overset{\triangledown}{U}$, then $Z(\overset{\triangledown}{U}) \cap I$ has codimension one in $Z(\overset{\triangledown}{U})$.*

4.6. The following result justifies our use of not necessarily linear weights. From their definition it is immediate that they identify with the set $\check{T}^*$ of characters on $\check{T}$ (with values in $\bar{K}^*$). Through the isomorphism $\tau : P(\pi) \xrightarrow{\sim} \check{T}$ we deduce an action of W on $\check{T}$ and hence on $\check{T}^*$. Now recall that each $\Lambda \in \check{T}^*$ is by definition the ℓ-tuple $(\Lambda_1, \Lambda_2, \cdots, \Lambda_\ell)$ with $\Lambda_i = \tau(\omega_i)(\Lambda)$. We define an action of $\mathbb{Z}_4^\ell$ on $\check{T}^*$ by letting the generator of the i^{th}

copy of $\mathbb{Z}_4^\ell$ multiply Λ_i by a primitive fourth root of unity. Combined with the above we obtain an action of the extended Weyl group $\check{W} = \mathbb{Z}_4^\ell \rtimes W$ on $\check{T}^*$. Let χ denote the map $\Lambda \mapsto Ann_{Z(\overset{\triangledown}{U})} M(\Lambda)$ of $\check{T}^*$ into $Max Z(\overset{\triangledown}{U})$.

LEMMA.
(i) $\chi(\Lambda_1) = \chi(\Lambda_2) \iff \Lambda_1 q^\rho, \Lambda_2 q^\rho$ *are conjugate under* $\check{W}$.
(ii) χ *is surjective.*

As in the calculation given in 4.3, we deduce from [JL1, 8.6] that their exist (polynomial) generators $z_i' : i = 1, 2, \cdots, \ell$ of $Z(\overset{\triangledown}{U})$ satisfying

$$\varphi(z_i')(\Lambda) = \sum_{y \in W} \tau(-4y\omega_i)(\Lambda q^\rho) \ .$$

From this, (i) is an immediate consequence of [Bo, Chap. 5, Thm. 2.2.2] - see also remarks in the proof of (ii). Set $\Lambda_i' = \tau(-4\omega_i)(\Lambda q^\rho)$. For (ii) we must show that the system of (Laurent) polynomials

$$\Omega_i := \sum_{y \in W} y\Lambda_i'$$

can be given any set of values $\gamma_i \in \bar{K}$ by making suitable choices of the $\Lambda_i' \in \bar{K}^*$. Let A denote the Laurent polynomial algebra over $\bar{K}$ generated by the Λ_i' for $i = 1, 2, \cdots, \ell$ and B the subalgebra generated by the Ω_i for $i = 1, 2, \cdots, \ell$. It is clear from the above that A identifies with the group algebra (over $\bar{K}$) of $P(R)$ and B with $(\bar{K}P(R))^W$. By a result of Steinberg [S, Thm. 2.2] we conclude that A is a free B module on $rank|W|$ generators, say $A = B \otimes F$. Hence any maximal ideal $I := < \Omega_i - \gamma_i | i = 1, 2, \cdots, \ell >$ of B generates in A an ideal, namely $I \otimes F$, of non-zero (in fact $|W|$) codimension. Applying Hilbert's Nullstellensatz proves the required assertion. Notice that (i) results from the fact that any two ideals of A lying over a maximal ideal of B are conjugate under W. This holds for any finite group G action on any commutative ring A with $B = A^G$, [Bo, loc. cit.].

Remarks. – The corresponding result for $U(\mathfrak{g})$ can be proved in a similar fashion, though it is customary to give a more elementary argument based on the fact that W acts linearly on $\mathfrak{h}^*$.

It is clear that (i) holds also for $\check{U}$, it is only (ii) which needs the base field $\bar{K}$ to be algebraically closed. Recall (2.7) that the one dimensional $\check{U}$ modules are in bijection with $(\check{T}/4T)^*$. It is exactly that part, namely $(\check{T}/4\check{T})^*$, trivial on $F(\check{U})$ whose members cannot be distinguished by $Z(\check{U})$.

5. Equivalence of categories

5.1. We define a translated action of $\check{W}$ on $\check{T}^*$ by $w \cdot \Lambda = q^{-\rho} w(\Lambda q^\rho), \ \forall \ w \in W$. Then 4.6(i) translates to the assertion that $Ann_{Z(\check{U})} M(\Lambda) = Ann_{Z(\check{U})} M(\Lambda')$ if and only if $\Lambda \in \check{W} \cdot \Lambda'$.

Let Λ be a weight. We call Λ regular if $Stab_{W}.\Lambda = \{e\}$. We set $\check{W}_\Lambda = \{w \in \check{W} | w\Lambda = q^\beta \Lambda$ for some $\beta \in Q(\pi)\}$, which is a subgroup of $\check{W}$ called the integral subgroup with respect to Λ. Define an order relation $\leq$ on $\check{T}^*$ through $\Lambda \leq \Lambda'$ if $\Lambda = \Lambda' q^{-\beta}$ for some $\beta \in Q^+(\pi)$. We call $\Lambda \in \check{T}^*$ dominant if Λ is a maximal element of the set $\check{W} \cdot \Lambda$. By 4.6(i) it is natural to define $Max_1 Z(\check{U}) := \{Ann_{Z(\check{U})} M(\Lambda) : \Lambda \ dominant\}$.

5.2. Fix $\Lambda \in \check{T}^*$ dominant. Let $\mathcal{O}_\Lambda$ denote the category of all U modules M satisfying

(i) *U^+ acts locally finitely on M.*

(ii) *M is a weight module, has finite dimensional weight spaces, and each weight Λ' of M takes the form $\Lambda' = \Lambda q^\beta$ where $\beta \in Q(\pi)$.*

(iii) *$Ann_{Z(\check{U})} M$ has finite codimension in $Z(\check{U})$.*

One checks that the Verma modules $M(\Lambda')$ for $\Lambda' \in \Lambda q^{Q(\pi)}$ lie in $\mathcal{O}_\Lambda$. It is then immediate that the simple objects of $\mathcal{O}_\Lambda$ are the $L(\Lambda')$ where $\Lambda' \in \Lambda q^{Q(\pi)}$. A standard reasoning based on 4.6(i) and hypotheses (ii), (iii) above shows that each $M \in Ob\mathcal{O}_\Lambda$ has finite length. Similarly 4.6(i) and universality of Verma modules shows for $\Lambda' \in \Lambda q^{Q(\pi)}$ dominant that $M(\Lambda')$ is projective in $\mathcal{O}_\Lambda$. As noted in [JL1, 5.12] we have a duality functor δ on $\mathcal{O}_\Lambda$ with $\delta L \cong L$ for L simple. Finally $E \otimes M \in \mathcal{O}_\Lambda$ for all $M \in \mathcal{O}_\Lambda$, $E \in \check{\mathcal{E}}$.

5.3. Following [J1, 1.3.7] we should like to define a Harish-Chandra category for bimodules. Since U is not locally *ad* U finite one cannot take U bimodules, whilst taking $F(U)$ bimodules runs into the difficulty that $F(U)$ is not a Hopf or even a bi-subalgebra of U. We start with the following result. Let Δ denote the coproduct on U.

LEMMA. – *The subalgebra $F(U)$ is a left co-ideal of U, that is $\Delta(F(U)) \subset U \otimes F(U)$.*

Recall [JL2, 4.11]. Since $\Delta(t) = t \otimes t, \quad \forall \quad t \in T_<$ we see that it is enough to show, for each weight vector $a_\mu \in F(U)$, that the property $\Delta(a_\mu) \in U \otimes F(U)$ implies the corresponding result for $\Delta((ad\ x_i)a_\mu)$ and $\Delta((ad\ y_i)a_\mu)$.

Using the formulae in [JL1, 3.1] it follows that we can write in the usual Hopf algebra summation convention

$$\Delta(a_\mu) = a'_{\mu-\nu} \otimes a''_\nu$$

where the subscripts denote weights. On the other hand

$$(ad\ x_i)a_\mu = q^{-(\alpha_i,\mu)} x_i t_i a_\mu - q^{(\alpha_i,\mu)} a_\mu x_i t_i$$

and

$$\Delta(x_i t_i) = x_i t_i \otimes 1 + t_i^2 \otimes x_i t_i \quad .$$

We therefore obtain

$$\Delta((ad\ x_i)a_\mu) = (q^{-(\alpha_i,\mu)} x_i t_i a'_{\mu-\nu} - q^{(\alpha_i,\mu)} a'_{\mu-\nu} x_i t_i) \otimes a''_\nu$$
$$+ q^{-(\alpha_i,\mu)} t_i^2 a'_{\mu-\nu} \otimes x_i t_i a''_\nu - q^{(\alpha_i,\mu)} a'_{\mu-\nu} t_i^2 \otimes a''_\nu x_i t_i \ .$$

Our (induction) hypothesis means that we can assume the a''_ν to all lie in $F(U)$. Thus the first term above has the required form. Miraculously the second term can be rewritten as

$$q^{(\alpha_i,\nu-\mu)} t_i^2 a'_{\mu-\nu} \otimes (q^{-(\alpha_i,\nu)} x_i t_i a''_\nu - q^{(\alpha_i,\nu)} a''_\nu x_i t_i)$$
$$= q^{(\alpha_i,\nu-\mu)} t_i^2 a'_{\mu-\nu} \otimes (ad\ x_i) a''_\nu \quad ,$$

which again has the required form. A similar result holds for $\Delta((ad\ y_i)a_\mu)$. This proves the assertion.

Remarks. – It is always false that $\Delta(F(U)) \subset F(U) \otimes F(U)$. Finally the conclusion of the lemma extends to $\check{U}$ without difficulty.

5.4. Intuitively an object in the category of Harish-Chandra modules $\mathcal{H}$ should be an $F(U)$ bimodule V with a compatible locally finite $ad\ U$ action. Compatibility should at least mean

$$(c_1) \qquad (ad\ u)(f,f')v = ((ad\ u_1)f, (ad\ u_3)f')(ad\ u_2)v\ ,$$

$\forall\ \ u \in U,\ f, f' \in F(U),\ v \in V$ where $\Delta^2(u) = u_1 \otimes u_2 \otimes u_3$ in the Hopf algebra convention as defined in [JL1, 2.1], and $(f,f')v = fvf'$.

However (c_1) is not enough as we shall want to use the $ad\ U$ action to pull $F(U)$ across from left to right.

Recall that $\tau(\lambda) \in F(U)$ for all $\lambda \in -R^+(\pi)$. It therefore makes sense to consider

$$(c_2) \qquad [ad\ \tau(\lambda)v]\tau(\lambda) = \tau(\lambda)v$$

for all $\lambda \in -R^+(\pi),\ v \in V$. The above conditions (c_1), (c_2) are said to define a compatible $ad\ U$ action on a $F(U)$ bimodule V.

It is clear that (c_2) (and of course (c_1)) would hold if V is say the submodule of a U bimodule formed from its locally $ad\ U$ finite elements. In particular (c_1), (c_2) hold for the $F(U)$ bimodules $F(M,N)$ introduced [JL2, 6.10].

5.5. Let V be a $F(U)$ bimodule with a compatible $ad\ U$ action. Then by (c_2), replacing also v by $(ad\ \tau(-\lambda))v$, it follows that $V\tau(\lambda) = \tau(\lambda)V$. From (c_1) we obtain $(ad\ U)\ (V\tau(\lambda)) \subset V(ad\ U)\tau(\lambda)$. Moreover the reverse inclusion is easily established if we use the formulae in [JL1, 2.2, 3.1] and the fact that $ad\ t$ for $t \in T$ is bijective on V. Recalling [JL2, 4.11] we deduce the

LEMMA. – *Let V be a $F(U)$ bimodule with a compatible ad U action. Then for any ad U stable subspace E of V one has*

$$F(U)E = EF(U)\ \ .$$

5.6. Take V as in 5.5. Since $T_<$ is Ore in $F(U)$ we may form the localized ring $F(U)T_<^{-1}$ and the localized module $VT_<^{-1} := V \otimes_{F(U)} F(U)T_<^{-1}$. It is immediate for any U module M that

$$(*) \qquad VT_<^{-1} \otimes_{F(U)T_<^{-1}} M \xrightarrow{\sim} V \otimes_{F(U)} M\ .$$

Suppose V is generated by an $ad\ U$ stable subspace E as an $F(U)$ bimodule, that is $V = F(U)EF(U)$. Then by 5.5 we have $V = EF(U)$. Again suppose $VT_<^{-1}$ is finitely generated as a right $F(U)T_<^{-1}$ module. We can assume that the finite generating subspace E lies in V itself. Then if the action of $ad\ U$ on V is locally finite we can further assume that E is $ad\ U$ stable. In this case we obtain $VT_<^{-1} = EF(U)T_<^{-1}$.

Let V' be an $F(U)$ bimodule with a compatible *ad* U action, for example $F(U)T_<^{-1}$. Let E be a U module. Give $E \otimes_K V'$ the (obvious) right $F(U)$ module structure coming from right multiplication on V'. By 5.3, we may also give $E \otimes_K V'$ a left $F(U)$ module structure through the coproduct Δ. Again the coproduct gives $E \otimes_K V'$ an *ad* U module structure which is easily seen to be compatible. Returning to our previous situation we can therefore view $VT_<^{-1}$ as an image of $E \otimes_K F(U)T_<^{-1}$. Then for any U module M we have the U module maps

$$(**) \qquad E \otimes_K M = E \otimes_K F(U)T_<^{-1} \otimes_{F(U)T_<^{-1}} M \twoheadrightarrow V \otimes_{F(U)} M \ .$$

Again if E is finite dimensional (and it will be enough that $E \in \oplus\mathcal{E}$) we have for any U modules $M,\ N$, an isomorphism

$$(***) \qquad E \otimes_K F(N, M) \xrightarrow{\sim} F(N, E \otimes_K M) \ .$$

Notice that this could be rather uninteresting if V is right $T_<$ torsion for then $V \otimes_{F(U)} M = 0$. Obviously this won't happen if V is a submodule of a U bimodule.

All the above considerations go over when U is replaced by $\check{U}$ and T by $\check{T}$, except then we restrict the finite dimensional modules to lie in $\oplus\check{\mathcal{E}}$.

5.7. We define $V \in Ob\mathcal{H}$ to be an $F(\check{U})$ bimodule with a compatible *ad* $\check{U}$ action satisfying

(i) *As an ad* $\check{U}$ *module,* V *is a possibly infinite direct sum of simple finite dimensional modules* $E \in \check{\mathcal{E}}$ *each simple occurring with finite multiplicity.*

(ii) *As a left (or right)* $F(\check{U})$ *module,* $Ann_{Z(\check{U})}V$ *has finite codimension in* $Z(\check{U})$.

Now fix $\chi \in Max_1 Z(\check{U})$ and let $\mathcal{H}_\chi$ denote the full subcategory of $\mathcal{H}$ of all $F(\check{U})$ bimodules whose right annihilator contains χ. Fix $\Lambda \in \check{T}^*$ dominant such that $\chi = Ann_{Z(\check{U})}M(\Lambda)$.

LEMMA. – *For all* $N \in Ob\mathcal{O}_\Lambda$ *one has* $F(M(\Lambda), N) \in Ob\mathcal{H}_\chi$.

As discussed in 5.4 the bimodule structure and compatibility is evident. It is clear that the left (resp. right) annihilator of $F(M(\Lambda), N)$ contains $Ann_{Z(\check{U})}N$ (resp. $Ann_{Z(\check{U})}M(\Lambda) = \chi$). Since N has finite length (5.2) we conclude that $Ann_{Z(\check{U})}N$ has finite codimension. In view of complete reducibility, it remains to show that as an *ad* $\check{U}$ module only objects from $\check{\mathcal{E}}$ can occur in $F(M(\Lambda), N)$ and then with finite multiplicity.

By finiteness of lengths in $\mathcal{O}_\Lambda$ it is enough to assume N simple, say $N = L(\Lambda')$, and then $\Lambda' = \Lambda q^\beta$ for some $\beta \in Q(\pi)$. As in [JL2, 8.3] we have maps

$$F(M(\Lambda), L(\Lambda')) \hookrightarrow F(M(\Lambda), \delta M(\Lambda')) \xrightarrow{\sim} F(M(\Lambda') \otimes M(\Lambda))^*$$

whilst

$$[(M(\Lambda') \otimes M(\Lambda))^* : E] = dim\ E_\beta$$

for every simple finite dimensional U module E. This proves the required assertion.

5.8. Set $\mathbf{F}(\Lambda) = F(M(\Lambda), M(\Lambda))$. Since $M(\Lambda)$ is projective in $\mathcal{O}_\Lambda$ it follows that the functor $\mathcal{T} : N \to F(M(\Lambda), N)$ from $\mathcal{O}_\Lambda$ to $\mathcal{H}_\chi$ is exact. Frobenius reciprocity shows that it admits $\mathcal{T}' : V \mapsto V \otimes_{F(U)} M(\Lambda)$ as an adjoint functor. Suppose $V \in Ob\mathcal{H}_\chi$ is finitely generated as an $F(\check{U})$ bimodule. By the hypothesis 5.7(i) we can assume V is generated as a $F(\check{U})$ bimodule by a finite dimensional $ad\ \check{U}$ stable subspace $E \in \oplus\check{\mathcal{E}}$ and then by 5.6 we deduce that $\mathcal{T}'V$ is an image of $E \otimes M(\Lambda)$ and hence lies in $Ob\mathcal{O}_\Lambda$. Note also by 4.2(i) and 5.5 that $V = E\mathbf{F}(\Lambda)$. By adjointness we have a map $\theta_V : V \to \mathcal{T}\mathcal{T}'V$. Let V_0 be the right $\check{T}_<$ torsion submodule of V. Note that V_0 is not only an $F(U)$ bisubmodule; but is also $ad\ U$ stable. Again V_0 is also the left $\check{T}_<$ torsion submodule of V, so "right" may be omitted.

LEMMA. – *Let $V \in Ob\mathcal{H}_\chi$ be finitely generated as an $F(\check{U})$ bimodule. Then θ_V is bijective and $V_0 = 0$. Moreover every object in $\mathcal{H}_\chi$ is $\check{T}_<$ torsion-free.*

By 4.2(ii), $\theta_{\mathbf{F}(\Lambda)}$ is an isomorphism. By 5.6(***) so is $\theta_{E\otimes\mathbf{F}(\Lambda)}$, $\forall\ E \in \oplus\check{\xi}$. As in 5.6 we can choose $E_1 \in \oplus\check{\mathcal{E}}$ and a surjective map $\psi_1 : E_1 \otimes \mathbf{F}(\Lambda) \to V$. A fortiori its localization $\hat{\psi}_1 : E_1 \otimes \mathbf{F}(\Lambda)\check{T}_<^{-1} \to V\check{T}_<^{-1}$ is surjective. By 2.6, $ker\ \hat{\psi}_1$ is a finitely generated right $F(\check{U})\check{T}_<^{-1}$ module and we can choose a finite dimensional generating subspace E_2 to lie in $ker\ \psi_1$ and further by 5.7(i) we can assume E_2 to be $ad\ \check{U}$ stable. View E_2 as an element of $\oplus\check{\mathcal{E}}$. This gives a map $\psi_2 : E_2 \otimes \mathbf{F}(\Lambda) \to ker\ \psi_1$ whose localization $\hat{\psi}_2 : E_2 \otimes \mathbf{F}(\Lambda)\check{T}_<^{-1} \to ker\ \hat{\psi}_1$ is surjective. From 5.6(*) we deduce an exact sequence

$$E_2 \otimes M(\Lambda) \xrightarrow{\mathcal{T}'(\psi_2)} E_1 \otimes M(\Lambda) \xrightarrow{\mathcal{T}'(\psi_1)} \mathcal{T}'V \to 0$$

and applying $\mathcal{T}$, the commuting diagram

$$\begin{array}{ccccccc} E_2 \otimes \mathbf{F}(\Lambda) & \xrightarrow{\psi_2} & E_1 \otimes \mathbf{F}(\Lambda) & \xrightarrow{\psi_1} & V & & \\ {\scriptstyle\theta_2}\downarrow & & {\scriptstyle\theta_1}\downarrow & & {\scriptstyle\theta_V}\downarrow & & \\ E_2 \otimes \mathbf{F}(\Lambda) & \xrightarrow{\psi_2'} & E_1 \otimes \mathbf{F}(\Lambda) & \xrightarrow{\psi_1'} & \mathcal{T}\mathcal{T}'V & \to & 0 \end{array}$$

with the top row a complex, the bottom row exact, where $\theta_i = \theta_{E_i\otimes\mathbf{F}(\Lambda)}$ are isomorphisms and $\psi_i' = \mathcal{T}\mathcal{T}\psi_i$.

It is immediate that θ_V is surjective. Diagram chasing shows that $Im\ \psi_2 = ker\ \psi_1$ and then that $ker\ \theta_V \cap Im\ \psi_1 = 0$. Yet ψ_1 is surjective and so θ_V is bijective. Finally $\mathcal{T}\mathcal{T}'V$ has no $\check{T}_<$ torsion being a submodule of a U bimodule. Hence $V_0 = 0$. Finally take $V' \in Ob\mathcal{H}_\chi$ arbitrary. If $V_0' \neq 0$, then it contains a finite dimensional $ad\ U$ invariant subspace E and $V := EF(U) \in Ob\mathcal{H}_\chi$ is torsion and finitely generated, hence zero. This contradiction proves the last part.

5.9. Let $\mathcal{S}_\chi$ denote the set of simple objects in $\mathcal{H}_\chi$.

LEMMA. – *If $L \in Ob\mathcal{O}_\Lambda$ is simple, then $\mathcal{T}L$ is either zero or $\mathcal{T}L \in \mathcal{S}_\chi$. Moreover every $V \in \mathcal{S}_\chi$ takes the above form.*

Suppose $\mathcal{T}L \neq 0$ and not simple. Let V be a proper finitely generated submodule (in $\mathcal{H}_\chi$). Then $V \overset{i}{\hookrightarrow} \mathcal{T}L$ and adjointness gives $Hom(\mathcal{T}'V, L) \neq 0$ and so $\mathcal{T}'V$ surjects to L. Since $\mathcal{T}$ is exact, $\mathcal{T}\mathcal{T}'V$ surjects to $\mathcal{T}L$. By 5.8 the map $\theta_V : v \mapsto (m \mapsto v \otimes m)$ of V into $\mathcal{T}\mathcal{T}'V$ is an isomorphism. The composed map $V \xrightarrow{\sim} \mathcal{T}\mathcal{T}'V \twoheadrightarrow \mathcal{T}L$ is given by $v \mapsto (m \mapsto i(v)m)$ which is just the original embedding i. This contradiction proves that $\mathcal{T}L$ is simple.

Conversely suppose $V \in \mathcal{S}_\chi$. Then $V \xrightarrow{\sim} \mathcal{T}\mathcal{T}'V$ by 5.8, so in particular $\mathcal{T}'V \in Ob\mathcal{O}_\Lambda$ is non-zero. By finiteness of length in $\mathcal{O}_\Lambda$, we can choose L to be a simple quotient of $\mathcal{T}'V$. Then $Hom(V, \mathcal{T}L) \neq 0$ and by the first part $V \xrightarrow{\sim} \mathcal{T}L$ as required.

5.10. By 4.6 and 5.9 it follows that the number of simples in $\mathcal{S}_\chi$, in which $Z(\check{U})$ acts by a given scalar, is bounded by $|\check{W}|$. By the finite multiplicity hypothesis 5.7(i) and primary decomposition made possible by the hypothesis 5.7(ii), we deduce the

COROLLARY. – *Each $V \in Ob\mathcal{H}_\chi$ (resp. $Ob\mathcal{H}$) has finite length.*

5.11. Taking account of Frobenius reciprocity as used in 5.7, the argument in the last paragraph of [J1, 1.3.8] applies to give the

LEMMA. – *Assume Λ regular. Then for all $L(\Lambda') \in Ob\mathcal{O}_\Lambda$, the module $F(M(\Lambda), L(\Lambda')) \neq 0$ and is simple.*

5.12. Combining 5.8, 5.9, and 5.11 we obtain the

THEOREM. – *Assume Λ dominant and regular. Then $M \mapsto F(M(\Lambda), M)$ is an equivalence of categories from $\mathcal{O}_\Lambda$ to $\mathcal{H}_\chi$.*

Remarks. – Remarkably, every simple object V in $\mathcal{H}_\chi$ viewed as a left $Z(\check{U})$ module has annihilator in $Max_1 Z(\check{U})$. The above result may fail for U; but can be trivially extended to $\overset{\triangledown}{U}$. In this case $Max_1 Z(\overset{\triangledown}{U}) = Max\ Z(\overset{\triangledown}{U})$ by 4.6. Note that $\mathbf{F}(\Lambda) = F(U)/Ann\ M(\Lambda)$ is artinian for two-sided ideals which are $ad\ \check{U}$ invariant.

5.13. The two-sided ideals of $F(\Lambda)$ which are $ad\ \check{U}$ invariant, admit the following nice description.

LEMMA. – *Let J be an ad $\check{U}$ invariant ideal of $F(\Lambda)$. Then $J = Ann_{F(\check{U})}(M(\Lambda)/JM(\Lambda))$.*

Set $M = M(\Lambda)/JM(\Lambda)$. Clearly $JM = 0$ so $J \subset Ann_{F(\check{U})}M$. On the other hand M identifies with $F(\Lambda)/J \otimes_{F(\check{U})} M(\Lambda) = \mathcal{T}'(F(\Lambda)/J)$. Then $F(\Lambda)/J \xrightarrow{\sim} \mathcal{T}M = F(M(\Lambda), M)$ by 5.8. This give the opposite inclusion.

5.14. Let us use $(Spec\ F)^{\check{U}}$ to denote the $ad\ \check{U}$ invariant prime ideals of F where $F = F(\check{U}), \quad \mathbf{F}(\Lambda)$, etc.

COROLLARY. – *Each $J \in (Spec\ \mathbf{F}(\Lambda))^{\check{U}}$ takes the form $J = Ann_{F(\check{U})}L(\Lambda')$ for some $\Lambda' \in \Lambda q^{Q(\pi)} \cap \check{W} \cdot \Lambda$.*

Take $J \in (Spec\ \mathbf{F}(\Lambda))^{\check{U}}$. By 5.13 one has $J = Ann_{F(\check{U})}M$ for some quotient M of $M(\Lambda)$. Then the minimal primes over J take the form $Ann_{F(\check{U})}L(\Lambda')$ where $L(\Lambda')$ is a simple subquotient of M. Since J is assumed prime one of these must be J itself. Finally by 4.6 and the remarks in 5.2 it follows that Λ' has the required form.

5.15. Not every prime ideal P of $F(\check{U})$ is $ad\ \check{U}$ invariant. By [JL1, 6.4] we have for all i that $x_i t_i \tau(\lambda) \in F(U)$ for some $\lambda \in -R^+(\pi)$. From the formula for $(ad\ x_i)a_\mu$ in 5.3 we conclude that a two-sided ideal I of $F(\check{U})$ is $ad\ \check{U}$ invariant if for all $\lambda \in -4P^+(\pi)$ the image of $\tau(\lambda)$ in $F(\check{U})/I$ is regular. From the property $tF(\check{U}) = F(\check{U})t$ it is enough for a prime ideal P to require that the images of the $\tau(\omega_i)$ for $i \in \{1, 2, \cdots, \ell\}$ are regular in $F(\check{U})/P$. Conversely for any $ad\ U$ invariant ideal J of $F(\Lambda)$, it follows from 5.13 that the image of $\tau(\lambda)$ in $F(\Lambda)/J$ is regular for any $\lambda \in -4P^+(\pi)$.

Finally take $t \in \check{T}_<$ and $\Lambda \in \check{T}^*$. Then $t\mathbf{F}(\Lambda)$ is a two-sided ideal of $\mathbf{F}(\Lambda)$. If $t\mathbf{F}(\Lambda) = \mathbf{F}(\Lambda)$, then there exists $s \in \mathbf{F}(\Lambda)$ such that $ts = 1$. Now as in [JL2, 8.1] it follows by 4.2 that $\mathbf{F}(\Lambda)$ is an integral domain. Then as in [JL1, 9.1, 9.2] it follows that s is a weight vector (of zero weight) and $(ad\ x_i)t = (ad\ y_i)t = 0$ for all i. Consequently $t = 1$. We conclude that $J := t\mathbf{F}(\Lambda)$ for $t \neq 1$, is a proper two-sided ideal of $\mathbf{F}(\Lambda)$. However J cannot be $ad\ U$ invariant since this would contradict 5.13. Taking P a maximal two-sided ideal of $\mathbf{F}(\Lambda)$ over J we also deduce that $(Spec\ \mathbf{F}(\Lambda))^{\check{U}}$ is a proper subset of $Spec\ \mathbf{F}(\Lambda)$.

6. The Quantum Analogue of Duflo's theorem

6.1. Let us first elucidate the relation between ideals of U and of $F(U)$. Recall the notation of 2.7.

LEMMA. – *Let M be a U module. Then*

$$U\ Ann_{F(U)}M = \bigcap_{\Lambda \in (T/T_\diamond)^*} Ann_U(M \otimes_K K_\Lambda)\ .$$

By 5.3 we can view $M \otimes_K K_\Lambda$ as an $F(U)$ module. Since $u \in F(U)$ acts by $\varepsilon(u)$ on K_Λ and using $u = u_1\varepsilon(u_2)$ where $\Delta(u) = u_1 \otimes u_2$ in the conventions of [JL1, 2.1] we conclude that $Ann_{F(U)}(M \otimes_K K_\Lambda) = Ann_{F(U)}M$. This proves the inclusion $\subset$. Now take a in the right hand side. In the notation of 2.7 we can write (possibly replacing a by $\tau(\lambda)a$ with $-\lambda \in R^+(\pi)$ sufficiently large)

$$a = \sum_{i=1}^{r} t'_i a_i : a_i \in F(U)\ .$$

Then for each $b \otimes m \in M \otimes_K K_\Lambda$ we have

$$a(b \otimes m) = \sum_{i=1}^{r} t'_i a_i b \otimes t'_i m = 0$$

as in the calculation above. Since this holds for all $\Lambda \in (T/T_\diamond)^*$ we conclude from 2.7(*) that $t'_i a_i b = 0$ and so $a_i b = 0$ as required.

Remark. – Of course an analogous results holds for $\check{U}$ and for $\overset{\triangledown}{U}$.

6.2. To obtain the quantum analogue of Duflo's theorem, we must first appeal to 5.14 which applies to $\check{U}$ rather than U. Furthermore in discussing primitive ideals it is natural to assume the base field to be algebraically closed. We need first the following easy facts about Gelfand-Kirillov dimension d_A of an algebra A. (For definitions see [KL].)

LEMMA.

(i) $d_A(A) \geq d_B(B)$ *for any subalgebra B of A.*

(ii) $d_A(A) \geq d_A(\bar{A})$ *for any quotient $\bar{A}$ of A.*

(iii) *Suppose $d_A(A)$ is finite, A is prime Goldie, and $\bar{A}$ a proper quotient of A. Then* $d_A(A) > d_A(\bar{A})$.

Let M be a U module of finite length whose simple factors are highest weight modules. Then

(iv) $d_U(M) = d_{F(U)}M$.

(v) $d_U(U/Ann_U M) = 2d_U(M)$.

(v) $d_{F(U)}(F(U)/Ann_{F(U)}M) = 2d_{F(U)}M$.

Assertions (i) and (ii) can be found in [KL, 3.1] and (iii) in [B, 1.6.]. From [KL, 5.1(f)] it is enough to prove (iv), (v) and (vi) for simple highest weight modules. Note that (iv) holds because both sides are given by the growth rate [see BK, 1.7] relative to weight spaces. Assertions (v) and (vi) can be made to follow the proof in the enveloping algebra case. For this and what follows we remark that a simple weight module for U is also simple as an $F(U)$ module.

Remark. – The last three assertions hold for $\check{U}$ and $\overset{\triangledown}{U}$.

6.3. We now prove the main result of this section.

THEOREM. – *Every $I \in Prim\ \overset{\triangledown}{U}$ is the annihilator of some simple highest weight module. Moreover, $I \in Prim\ \overset{\triangledown}{U}$ if and only if $I \in Spec\ \overset{\triangledown}{U}$ and $I \cap Z(\overset{\triangledown}{U})$ has codimension one in $Z(\overset{\triangledown}{U})$.*

Suppose $I \in Prim\ \overset{\triangledown}{U}$. By 4.5 and 4.6 there exists $\Lambda \in \check{T}^*$ dominant such that $I \supset Ann_{Z(\overset{\triangledown}{U})} M(\Lambda)$. Set $J = I \cap F(\overset{\triangledown}{U})$. The image $\bar{J}$ of J in $\mathbf{F}(\Lambda)$ is an *ad* U invariant two-sided ideal. We conclude from 5.13 that $J = Ann_{F(\overset{\triangledown}{U})} M$ for some quotient M of $M(\Lambda)$. By 6.1

$$\overset{\triangledown}{U}J = \bigcap_{\Lambda \in (\check{T}/4\check{T})^*} Ann_{\overset{\triangledown}{U}}(M \otimes K_\Lambda)$$

and so the minimal primes over $\overset{\triangledown}{U}J$ are amongst the $Ann_{\overset{\triangledown}{U}}(L(\Lambda') \otimes K_\Lambda)$ with $L(\Lambda')$ a simple subquotient of M. Every such tensor product is of course a simple highest weight module. It remains to show that I is a minimal prime over $\overset{\triangledown}{U}J$. For this we shall use Gelfand-Kirillov dimension. Indeed by 6.2 (iii) it is enough to show that

$$d_{\overset{\triangledown}{U}}(\overset{\triangledown}{U}/I) = d_{\overset{\triangledown}{U}}(\overset{\triangledown}{U}/\overset{\triangledown}{U}J) .$$

By 6.2(ii) we have the inequality $\leq$. The reverse inequality (which is of course the main issue) results from

$$\begin{aligned} d_{\overset{\triangledown}{U}}(\overset{\triangledown}{U}/\overset{\triangledown}{U}J) &= 2d_{\overset{\triangledown}{U}}(M), \quad \text{by (v), 6.1 and (KL, 5.1(f))} \\ &= d_{F(\overset{\triangledown}{U})}(F(\overset{\triangledown}{U})/J), \quad \text{by (iv) and (vi)} \\ &\leq d_{\overset{\triangledown}{U}}(\overset{\triangledown}{U}/I), \quad \text{by (i)} , \end{aligned}$$

as required.

6.4. Recall that $\bar{U} := U \otimes_K \bar{K}$. By [M, 4,9], $\bar{U}$ is noetherian. Further $\overset{\triangledown}{U}$ is a finitely generated left (or right) module over $\bar{U}$, so in particular noetherian and $d_{\bar{U}} = d_{\overset{\triangledown}{U}}$ we denote this common dimension by d.

COROLLARY. – *Every $J \in Prim\ \bar{U}$ is the annihilator of some simple highest weight module.*

Set $I = \overset{\triangledown}{U} J$ and let $P_1, P_2, \cdots, P_k$, denote the set of minimal primes over I. Then

$$\begin{aligned} d(\bar{U}/J) &= d(\overset{\triangledown}{U}/I), \quad \text{by finiteness of } \Gamma \text{ and } [\text{KL}, 5.1(\text{f})]. \\ &= min_i\ d(\overset{\triangledown}{U}/P_i), \text{by } [\text{KL}, 3.3]\ . \end{aligned}$$

Choose i such that $d(\bar{U}/J) = d(\overset{\triangledown}{U}/P_i)$. From the embedding $\bar{U}/\bar{U} \cap P_i \hookrightarrow \overset{\triangledown}{U}/P_i$ and the finiteness of Γ we obtain

$$d(\bar{U}/J) = d(\overset{\triangledown}{U}/P_i) = d(\bar{U}/\bar{U} \cap P_i) \leq d(\bar{U}/J), \quad \text{by } 6.2(\text{ii}).$$

Since J is primitive by hypothesis, hence prime we conclude from 6.2(iii) that $J = \bar{U} \cap P_i$. If we can show that $P_i \cap Z(\overset{\triangledown}{U})$ has codimension 1, the corollary will follow from 6.3.

One has $P_i \cap Z(\bar{U}) = J \cap Z(\bar{U})$ and the latter is of codimension 1 in $Z(\bar{U})$ by 4.5 and the hypothesis that J is primitive. Yet it is clear from [JL2, 4.11] and the analogous assertion for $F(\check{U})$ that $Z(\overset{\triangledown}{U})$ is a finite module over $Z(U)$. We conclude that $P_i \cap Z(\overset{\triangledown}{U})$ has finite codimension over $Z(\overset{\triangledown}{U})$. Yet P_i is prime, so $P_i \cap Z(\overset{\triangledown}{U})$ is completely prime, from which the required assertion then follows.

Remark. – Recalling the remark in 4.1 it follows that the corresponding result holds for U when q is viewed as a non-zero element of k which is not a root of unity and k is assumed to be algebraically closed (and of characteristic zero).

7. Harmonic Elements

7.1. Define $\check{G} = gr_{\mathcal{F}}\check{U}$, $G(\check{U}) := gr_{\mathcal{F}}F(\check{U})$, $Y(\check{U}) := gr_{\mathcal{F}}Z(\check{U})$, $\check{Y}_+$ the augmentation ideal of $Y(\check{U})$ and $\check{J}_+ := G(\check{U})\check{Y}_+$. By [JL2, 7.3] we can write $G(\check{U}) = H \otimes_K Y(\check{U})$ with H identified with any graded complement of $\check{J}_+$ in $G(\check{U})$. Set $F(\lambda) = (ad\ U)\tau(\lambda)$. By complete reducibility [JL1, 5.2] and by [JL2, 4.10] we can choose H to be $ad\ U$ stable and to satisfy

$$(*) \qquad H = \bigoplus_{\lambda \in -4P^+(\pi)} H(\lambda) \quad \text{where} \quad H(\lambda) = H \cap F(\lambda)\ .$$

Given $\lambda \in -4P^+(\pi)$ we let y_λ denote the unique up to scalars element of $F(\lambda) \cap Y(\check{U})$. We recall that by [JL2, 3.5 and 4.10] the $y_\lambda : \lambda \in -4P^+(\pi)$ form a basis for $Y(\check{U})$. Now take $b \in F(\lambda)$. Then by (*) and freeness we can write b uniquely in the form

$$b = \sum_{\mu,\nu \in -4P^+(\pi)} h_{\mu,\nu} y_\mu \text{ with } h_{\mu,\nu} \in F(\nu) \ .$$

By [JL2, 4.12] we have $h_{\mu,\nu} y_\mu \in F(\mu+\nu)$. By the uniqueness there can be no cancellation of terms in this expression and so from [JL2, 4.10] we conclude that $\mu + \nu = \lambda$. For the same reason if b is a lowest weight vector of weight $-\beta$, then so is each $h_{\mu,\nu}$.

Finally recall [JL1, 6.4] and the fact that the elements of $\check{T}$ are homogeneous in $gr_{\mathcal{F}}\check{U}$. Identifying $\check{T}$ with its image in $gr_{\mathcal{F}}\check{U}$ we conclude that

$$\check{G} = \check{T}_<^{-1} H[\check{T}/4\check{T}] \otimes_K Y(\check{U}) \ .$$

7.2. Recall that $G := gr_{\mathcal{F}}U$, $\check{G} := gr_{\mathcal{F}}\check{U}$ and set $G^0 := K[T]$, $\check{G}^0 := K[\check{T}]$. Let G^+ (resp. G^-) denote the subalgebra of G (or $\check{G}$) generated by the $x_i t_i$ (resp. $y_i t_i$) for $i = 1, 2, \cdots, \ell$. As in [JL2, 4.6] we have a triangular decomposition for $\check{G}$, namely the multiplication map $g_- \otimes g_0 \otimes g_+ \mapsto g_- g_0 g_+$ is an isomorphism of $G^- \otimes \check{G}^0 \otimes G^+$ onto $\check{G}$. Setting $G(\lambda) = G^- \otimes K\tau(\lambda) \otimes G^+$, then $\check{G}$ is a direct sum of the $G(\lambda)$, for $\lambda \in P(\pi)$. Let G_+^+ denote the subspace of G^+ spanned by homogeneous monomials in the $x_i t_i$ for $i = 1, 2, \cdots, \ell$ of positive degree. Then $\check{G}$ is the direct sum $(G^- \otimes \check{G}^0 \otimes G_+^+) \oplus (G^- \otimes \check{G}^0)$. Let γ denote the projection of $\check{G}$ onto $G^- \otimes \check{G}^0$ which results. Clearly γ commutes with the action of T. Let $\check{G}^T$ (resp. $\check{G}^{U^-}$) denote the zero weight subspace (resp. the subspace of $ad\ U^-$ invariant elements) of $\check{G}$.

LEMMA.
(i) $\gamma(a) \in \check{G}^0 \ , \quad \forall \quad a \in \check{G}^T$.
(ii) $\gamma(ga) = \gamma(g)\gamma(a), \quad \forall \quad g \in \check{G} \ , \ a \in \check{G}^T$.
Suppose $a_\mu \in \check{G}^{U^-}$ *has weight* μ *and* $b_\nu \in G(\lambda)$ *has weight* ν, *then*
(iii) $\gamma(a_\mu \ b_\nu) = q^{(\mu, 2\nu-\lambda)}\gamma(b_\nu)\gamma(a_\mu)$.

Assertion (i) is obvious. For (ii), (iii) set $L = G^- \otimes \check{G}^0 \otimes G_+^+$ which identifies with a left ideal of $\check{G}$. Then (ii) results on observing that

$$\gamma(L\check{G}^T) = \gamma(L\check{G}^0) = \gamma(L) = 0 \ .$$

For (iii) observe that we can write $\gamma(b_\nu) = b_\nu^- \tau(\lambda)$ for some $b_\nu^- \in G^-$ of weight ν. Now

$$0 = (ad\ y_i)a_\mu = q^{-(\mu,\alpha_i)} y_i t_i a_\mu - q^{(\mu,\alpha_i)} a_\mu y_i t_i \ .$$

Consequently

$$q^{(\mu,\nu)} b_\nu^- a_\mu = q^{-(\mu,\nu)} a_\mu b_\nu^- \ .$$

Hence

$$\begin{aligned} a_\mu b_\nu &\in a_\mu(L + \gamma(b_\nu)) \ , \\ &= L + a_\mu b_\nu^- \tau(\lambda) \ , \\ &= L + q^{(\mu,2\nu-\lambda)} b_\nu^- \tau(\lambda) a_\mu \ , \\ &= L + q^{(\mu,2\nu-\lambda)} \gamma(b_\nu)\gamma(a_\mu) \ , \end{aligned}$$

from which (iii) results.

7.3. Take $\lambda \in -4P^+(\pi)$. We fix y_λ so that $\gamma(y_\lambda) = \tau(\lambda)$. Set $v_i = y_{-4\omega_i}$ for $i = 1, 2, \cdots, \ell$. Recall [JL2, 4.9] that there is a subspace $K(\lambda)^-$ (resp. $K(\lambda)^+$) of G^- (resp. G^+) such that $(ad\ U^-)\tau(\lambda) = K(\lambda)^-\tau(\lambda)$ (resp. $(ad\ U^+)\tau(\lambda) = K(\lambda)^+\tau(\lambda)$), which as a U module is isomorphic to $L\left(-\frac{1}{4}\lambda\right)$ $\left(\text{resp. } L\left(\frac{1}{4}w_0\lambda\right)\right)$ for the twisted action [JL2, 4.7] implemented by these identifications. Consequently as a T subspace of G the lowest (resp. highest) weight of $K(\lambda)^-$ (resp. $K(\lambda)^+$) is $\frac{1}{4}(\lambda - w_0\lambda)$ $\left(\text{resp. } -\frac{1}{4}(\lambda - w_0\lambda)\right)$.

For all $\mu \in Q^+(\pi)$, let G^+_μ denote the subspace of G^+ of vectors of weight μ. Extend the order relation $\geq$ on $Q(\pi)$ given by $\alpha \geq \beta$ if $\alpha - \beta \in Q^+(\pi)$, to a total ordering.

Proposition. – *Take $\lambda \in -4P^+(\pi)$. Let b be a lowest weight vector of weight $-\beta$ in $F(\lambda)$ such that*

$$\gamma(b) \in \sum_{i=1}^{\ell} K(\lambda + 4\omega_i)^-\tau(\lambda) .$$

Then there exist lowest weight vectors b_i of weight $-\beta$ in $F(\lambda+4\omega_i) : 1 \leq i \leq \ell$ such that

$$b = \sum_{i=1}^{\ell} b_i v_i .$$

Set $\lambda' = -\frac{1}{4}(\lambda - w_0\lambda) \in Q^+(\pi)$. We first show, by induction with respect to $\geq$, that for each $\eta \in Q(\pi)$, there exist elements

(1) $b_{i,\eta} \in \sum_{0 \leq \mu \leq \eta} K(\lambda + 4\omega_i)^-\tau(\lambda + 4\omega_i)G^+_\mu$ *of weight $-\beta$ such that*

(2) $b = \sum_{i=1}^{\ell} b_{i,\eta} v_i + \sum_{\eta < \mu \leq \lambda' - \beta} K(\lambda)^-\tau(\lambda)G^+_\mu$.

This holds for $\eta < 0$ taking $b_{i,\eta} = 0$ for all i. Take $\eta \geq 0$. Assume that we have found $b_{i,\eta}$ satisfying (1), (2). Let $\eta' \in Q^+(\pi)$ be minimal with the property that $\eta' > \eta$. The induction hypothesis means that we can find $f \in K(\lambda)^-\tau(\lambda)G^+_{\eta'}$ of weight $-\beta$ such that

$$b = \sum_{i=1}^{\ell} b_{i,\eta} v_i + f + \sum_{\eta' < \mu \leq \lambda' - \beta} K(\lambda)^-\tau(\lambda)G^+_\mu .$$

We show that

(3) $f \in \sum_{i=1}^{\ell} K(\lambda + 4\omega_i)^-\tau(\lambda)G^+_{\eta'}$.

If $\eta' = 0$, we have $f = \gamma(b)$ and so the assertion results from the hypothesis of the lemma. Assume $\eta' > 0$. Take $y \in U^-_{-\eta'}$, that is of weight $-\eta'$. Suppose $\mu > \eta'$. The expression $\mu - \eta'$ written as a sum of the α_i has at least one positive coefficient. Hence $(ad\ y)G^+_\mu \subset G^+_+$. Recalling that b is a lowest weight vector, we obtain via 7.2(ii) that

(4) $0 = \gamma((ad\ y)b) = \sum_{i=1}^{\ell} \gamma((ad\ y)b_{i,\eta})\gamma(v_i) + \gamma((ad\ y)f)$.

Now since G^+ is an $ad\ U$ submodule of G, it follows that $(ad\ y)G^+_\mu \subset G^+$, also when $0 \leq \mu \leq \eta$. We deduce from (1) that

$$\gamma((ad\ y)b_{i,\eta}) \in K(\lambda + 4\omega_i)^- \tau(\lambda + 4\omega_i) \quad .$$

Combined with (4) this gives

(5) $\gamma((ad\ y)f) \in \sum_{i=1}^{\ell} K(\lambda + 4\omega_i)^- \tau(\lambda)$.

Recall [JL2, 6.1] that the map $(a, c) \mapsto \gamma((ad\ a)c)$ is a non-degenerate pairing $U^-_{-\eta'} \times G^+_{\eta'} \longrightarrow K$. Since $f \in K(\lambda)^- \tau(\lambda) G^+_{\eta'}$ we conclude that (3) follows from (5).

By (3) we can write

$$f = \sum_{i=1}^{\ell} f_i \tau(-4\omega_i)$$

with $f_i \in K(\lambda + 4\omega_i)^- \tau(\lambda + 4\omega_i) G^+_{\eta'}$ of weight $-\beta$. By [JL2, 4.9] we have $v_i \in F(-4\omega_i) = K(-4\omega_i)^- \tau(-4\omega_i) K(-4\omega_i)^+$. Again v_i lies in the centre of $\check{G}$ and so further using [JL2, 4.12] we conclude that

$$f_i v_i \in K(\lambda)^- \tau(\lambda) K(-4\omega_i)^+ G^+_{\eta'} \quad .$$

Since $f_i v_i$ has weight $-\beta$ and $K(\lambda)^-$ has lowest weight $-\lambda'$ we may conclude that

$$f_i v_i \in \sum_{\eta' \leq \mu \leq \lambda' - \beta} K(\lambda)^- \tau(\lambda) G^+_\mu \ .$$

Since $\gamma(v_i) = \tau(-4\omega_i)$, the above calculation further shows that

$$f_i v_i - f_i \tau(-4\omega_i) \in \sum_{\eta' < \mu \leq \lambda' - \beta} K(\lambda)^- \tau(\lambda) G^+_\mu \quad .$$

Setting $b_{i,\eta'} = b_{i,\eta} + f_i$ gives (2) with η replaced by η' and so completes the induction. Taking $\eta \geq \lambda' - \beta$ in (2) we deduce that

$$b = \sum_{i=1}^{\ell} b_i v_i \ , \quad \text{where} \quad b_i \in \check{G} \ .$$

Since $b \in G(\check{U})$ we deduce from the uniqueness implied by the decomposition in 7.1(**) that we can assume $b_i \in G(\check{U})$ without loss of generality. Then it similarly follows from the discussion proceeding 7.1(**) that we can further assume that the b_i satisfy the conclusion of the proposition.

7.4. We may now deduce the quantum analogue of [D, 8.4(ii)]. This may be expressed as follows. Take $\lambda \in -4P^+(\pi)$ and set

$$K(\lambda)^{--} = K(\lambda)^- / \sum_{i=1}^{\ell} K(\lambda + 4\omega_i)^- .$$

The restriction γ_λ of γ to $F(\lambda)$ has image $K(\lambda)^-\tau(\lambda)$ which we identify with $K(\lambda)^-$. Let $\bar{\gamma}_\lambda$ denote the composition of γ_λ with canonical projection onto $K(\lambda)^{--}$. From 7.3 we deduce the

THEOREM. – *$\bar{\gamma}_\lambda$ restricts to an injection of $H(\lambda)^{U^-}$ into $K(\lambda)^{--}$.*

Remark. – One easily checks that $K(\lambda)^{--}_{\frac{1}{4}(\lambda - w_0\lambda)} \neq 0$ and moreover $H(\lambda)^{U^-}$ surjects to this (one-dimensional) subspace.

7.5. Take $\Lambda \in \check{T}^*$. Given $a = \sum a_i^- \tau(\lambda_i)$ with $a_i^- \in G^-$ we set $a(\Lambda) = \sum a_i^- \tau(\lambda_i)(\Lambda)$. Define a map $\gamma(\Lambda) : \check{G} \to G^-$ by setting $\gamma(\Lambda)(b) = \gamma(b)(\Lambda)$. We deduce from 7.4 the

COROLLARY. – *For all $\Lambda \in \check{T}^*$, $\gamma(\Lambda)$ restricts to an injection of H^{U^-} into G^-.*

Remark. – Let e_Λ denote a highest weight vector of the Verma module $M(\Lambda)$. Since $M(\Lambda)$ is a freely generated U^- module with generator e_Λ, we deduce a map $\gamma(\Lambda) : U \longrightarrow U^-$ given by $(\gamma(\Lambda)(u) - u)e_\Lambda = 0$. Consider U as a U module for the adjoint action and let V denote the subspace U^{U^-} of U^- invariant vectors. From [JL2, 8.1] we deduce that the restriction of $\gamma(\Lambda)$ to V is just $Ann_V M(\Lambda)$. By 4.2 we conclude that $\gamma(\Lambda)$ restricts to an injection of $\mathbb{H}^{U^-}$ into U^-. We were not able to deduce this result directly from the corollary above.

7.6. It is clear that every weight of $\gamma(\Lambda)(H^{U^-})$ lies in $-Q(\pi) \cap P^+(\pi)$. However this does not determine the image completely. It is therefore instructive to determine $\gamma(H^{U^-})$ in the special case when $U = U_q(\mathfrak{sl}(3))$.

Take $\pi = \{\alpha_1, \alpha_2\}$. Set $b_i = y_i t_i$, for $i = 1, 2$ and $f_1 = (ad\ y_2)b_1 = q^{-1}b_2b_1 - qb_1b_2$ and $f_2 = (ad\ y_1)b_2 = q^{-1}b_1b_2 - qb_2b_1$. One checks that $\{1, b_i, f_i\}$ is a basis for $K(-4\omega_i)^-$ for $i = 1, 2$. The quantized Serre relations are just

$$b_1^2 b_2 - (q^2 + q^{-2})b_1b_2b_1 + b_1b_2^2 = 0$$

$$b_2^2 b_1 - (q^2 + q^{-2})b_2b_1b_2 + b_2b_1^2 = 0$$

The first can be written as

(1) $qb_1f_2 - q^{-1}f_2b_1 = 0$ or $q^{-1}b_1f_1 - qf_1b_1 = 0$,

whilst the second can be written as

(2) $qb_2f_1 - q^{-1}f_1b_2 = 0$ or $q^{-1}b_2f_2 - qf_2b_2 = 0$.

In particular we conclude that

(3) $f_1f_2 - f_2f_1 = 0$.

Given $\lambda \in P^+(\pi) \cap Q(\pi)$, let $H[\lambda]$ denote the isotypical component of H of type $E(\lambda)$.

LEMMA. – *Take $U = U_q(\mathfrak{sl}(3))$. Then $\gamma(G(\check{U})^{U^-})$ is the subalgebra of $G^- \otimes \check{G}^0$ generated by $\tau(-4\omega_i)$, $f_i\tau(-4\omega_i) : i = 1, 2$ and $f_1b_2\tau(-4(\omega_1 + \omega_2))$, $f_2b_1\tau(-4(\omega_1 + \omega_2))$.*

Moreover setting $\lambda_{r,s} = 3r\omega_1 + s(\omega_1 + \omega_2)$, $\mu_{r,s} = 3r\omega_2 + s(\omega_1 + \omega_2)$, $\nu_{r,s,t} = -4[(s-t)\omega_1 + t\omega_2 + r(\omega_1 + \omega_2)] : r, s, t \in \mathbb{N}$ with $t \leq s$, we obtain

(i) $\gamma(H[\lambda_{r,s}]^{U^-}) = \sum_{t=0}^{s} K f_1^{s-t} f_2^t (f_2 b_1)^r \tau(\nu_{r,s,t})$.

(ii) $\gamma(H[\mu_{r,s}]^{U^-}) = \sum_{t=0}^{s} K f_1^{s-t} f_2^t (f_1 b_2)^r \tau(\nu_{r,s,t})$.

The fact that $A := \gamma(G(\check{U})^{U^-})$ is a subalgebra of $G^- \otimes \check{G}^0$ follows from 7.2(iii).

One has $\gamma(v_i) = \tau(-4\omega_i)$, whereas $f_i\tau(-4\omega_i) \in H^{U^-}$ for $i = 1, 2$. Hence these four elements lie in A.

We now show that $f_1 b_2 \tau(-4(\omega_1+\omega_2)) \in A$. Let V be the submodule of $F(-4(\omega_1+\omega_2))$ generated by the lowest weight vector $f_1 f_2 \tau(-4(\omega_1 + \omega_2))$. Then $V \cong E(2(\alpha_1 + \alpha_2))$ and so $dim\ V_{-\alpha_1 - 2\alpha_2} = 1$. Yet the corresponding weight space of $F(-4(\omega_1 + \omega_2))$ has basis consisting of $f_1 f_2 \tau(-4(\omega_1 + \omega_2)) x_1 t_1$ and $f_1 y_2 t_2 \tau(-4(\omega_1 + \omega_2))$. It follows that $F(-4(\omega_1+\omega_2))$ admits a submodule $V^{(2)}$ having lowest weight $-\alpha_1 - 2\alpha_2$. Since the first of these vectors is not $ad\, U^-$ invariant the linear combination $v^{(2)}$ which is $ad\, U^-$ invariant can be assumed to satisfy $\gamma(v^{(2)}) = f_1 b_2 \tau(-4(\omega_1 + \omega_2))$. Similarly $F(-4(\omega_1 + \omega_2))$ has a submodule $V^{(1)}$ of lowest weight $-2\alpha_1 - \alpha_2$ with lowest weight vector $v^{(1)}$ satisfying $\gamma(v^{(1)}) = f_2 b_1 \tau(-4(\omega_1 + \omega_2))$. We remark that

$$F(-4(\omega_1 + \omega_2)) = V \oplus V^{(1)} \oplus V^{(2)} \oplus F(-4\omega_1)y(-4\omega_2) \oplus F(-4\omega_2)y(-4\omega_1) \oplus Ky(-4(\omega_1 + \omega_2)) .$$

It is clear that $f_2 b_1$ and $f_1 b_2$ do not lie in $K(-4\omega_1)^- + K(-4\omega_2)^-$ and so we conclude from 7.3 that

$$H(-4(\omega_1 + \omega_2)) = V \bigoplus V^{(1)} \bigoplus V^{(2)} .$$

A similar argument gives the inclusion $\supset$ in (i), (ii). For example we claim that $u_{r,s,t} := f_1^{s-t} f_2^t (f_2 b_1)^r \tau(v_{r,s,t}) \in \gamma(H[\lambda_{r,s}]^{U^-})$. The first part shows that $u_{r,s,t} \in A$ and of course this vector has weight $-\lambda_{r,s}$. It remains to show that $u_{r,s,t} \in \gamma(H^{U^-})$. By 7.3 and the above, it is enough to show that $f_1^{s-t} f_2^{r+t} b_1^r$ lies in $K(\nu_{r,s,t})^-$ and has a non-zero image in $K(\nu_{r,s,t})^{--}$. This assertion is established below.

First from (1) - (3) one easily checks that G^- has basis

$$\{f_1^m f_2^n b_1^u\ ,\ f_1^m f_2^n b_2^v : m, n, u, v \in \mathbb{N}\} .$$

Now recall [JL2, 4.12(i)] that $K(\lambda)^- K(\mu)^- = K(\lambda + \mu)^-$ for all $\lambda, \mu \in -4P^+(\pi)$. It follows easily from the given bases for $K(-4\omega_i)^-$ and (1) - (3) again that $f_1^m f_2^n b_1^u \in K(-4(m+u)\omega_1 - 4n\omega_2)^-$. Moreover had this a zero image in $K(-4(m+u)\omega_1 - 4nw_2)^{--}$ then b_1^u would have had a zero image in $K(-4u\omega_1)^{--}$, which is absurd. Taking $m = s - t,\ n = t + r,\ u = r$ proves the required assertion and the claim.

It is well-known and easy to verify that $dim\ E(\lambda_{r,s})_0 = s + 1$. Taking account of [JL2, 7.4] this gives the inequality $\leq$ of dimensions in (i). In view of the opposite inclusion established above, this proves equality in (i).

A similar argument establishes equality in (ii).

It follows from (i), (ii) that $\gamma(H^{U^-})$ is contained in the subalgebra generated by the last four of the given generators, whereas $\gamma(Y(\check{U}))$ is generated by the first two. Taking account of the decomposition $G(\check{U})^{U^-} = H^{U^-} \otimes Y(\check{U})$ and the injectivity of γ on $Y(\check{U})$, it follows that these six elements generate A. This completes the proof of the lemma.

8. Complete Primeness

8.1. Let $Max_1 Y$ denote the ideals of codimension 1 in $Y := Y(U)$. Given $Y_\chi \in Max_1 Y$, we set $J_\chi = G(U)Y_\chi$. We show (8.4, 8.5) that generically J_χ is completely prime. On the other hand we show (8.9) that J_+ being prime, implies that $G(U)$ admits a separation of variables which may fail, for example for $U_q(\mathfrak{sl}(3))$ [JL2, 5.5]. Since $J_+ \subsetneq G(U) \cap \check{J}_+$ in this case, it does not followtneq that $\check{J}_+$ need fail to be completely prime. However we shall show that $\check{J}_+$ is not even prime in this case and in fact we find (8.10) that there are exactly two prime ideals over $\check{J}_+$.

8.2. Set $G_< := G^- T_< G^+$ which is both a graded subalgebra and U submodule of G. It contains $G(U)$ as a subalgebra and as a U submodule.

Let $T_<^*$ denote the character semigroup of $T_<$, that is the set of all multiplicative maps $\Lambda : T_< \longrightarrow K$. If $\Lambda \in T_<^*$ satisfies $\Lambda(t) \in K^*, \ \forall \ t \in T_<$, then Λ extends to an element of the character group $T_\diamond^*$ of $T_\diamond$ and every element is so obtained. Thus we may regard $T_\diamond^*$ as a subset of $T_<^*$.

To each $\Lambda \in T_<^*$ we may associate a one-dimensional $T_< G^+$ module, denoted K_Λ, with highest weight Λ and we set $N(\Lambda) = G_< \bigotimes_{T_< G^+} K_\Lambda$. Let v_Λ (or simply, v) denote the highest weight vector of N_Λ.

Lemma. – *For all $\Lambda \in T_<^*$, $N(\Lambda)$ is a simple $G_<$ module.*

Let M be a proper $G_<$ submodule of $N(\Lambda)$. Then M contains a highest weight vector which takes the form fv, with f a non-scalar weight vector of G^-. Then $(x_i t_i) fv = 0 = f(x_i t_i)v$. Recalling [JL2, 4.6] that G^- is an $(ad\ U)$ module, this equality implies that $(ad\ x_i)f \in Ann_{G^-} v_\Lambda$. Yet $N(\Lambda)$ is freely generated as a G^- module over v_Λ and so $(ad\ x_i)f = 0$, for all i. Yet [JL2, 4.7] we have $(G^-)^{U^+} = K$ and so f is a scalar. This contradiction proves the lemma.

Remark. – Unless $\Lambda \in T_\diamond^*$, it is false that $N(\Lambda)$ is simple as a $G(U)$ module.

8.3. Following say [JL1, 5.4] it is quite easy to construct a contravariant form on $N(\Lambda)$, which by 8.2 is non-degenerate. Then as in say [JL1, 8.3] one deduces the

Lemma. – *One has*

$$\bigcap_{\Lambda \in T_<^*} Ann_{G_<} N(\Lambda) = 0 \ .$$

Remark. – Recall (7.2) the definition of γ. By the lemma we deduce that the restriction of γ to G^{U^-} is injective. Noting that $a v_\Lambda = \gamma(a)(\Lambda) v_\Lambda = \gamma(\Lambda)(a) v_\Lambda$ this also follows from 7.5 but the present proof does not need the separation of variables theorem.

8.4. The analysis of [JL2, 8.1] can be easily adapted to show that

LEMMA. – *For all* $\Lambda \in T^*_<$, $Ann_{G(U)}N(\Lambda)$ *is completely prime.*

8.5. Take $\Lambda \in T^*_<$. It is clear that $Y_\Lambda := Ann_{Y(U)}N(\Lambda) \in Max_1 Y$. Set $J_\Lambda = G(U)Y_\Lambda$.

PROPOSITION. – *For all* $\Lambda \in T^*_\diamond$, *one has* $Ann_{G(U)}N(\Lambda) = J_\Lambda$.

It is enough to show that $a \in Ann_G N(\Lambda)$ implies $a \in J_\Lambda$ and furthermore we may assume that a is a lowest weight vector (with respect to $ad\ U$). By [JL2, 4.10] we may write $a = g_1 + g_2 + \cdots + g_n$ with $g_i \in (ad\ U)\tau(\lambda_i)$ a lowest weight vector and $\lambda_i \in R^+(\pi)$. We can write $\gamma(g_i) = g_i^- \tau(\lambda_i) : g_i^- \in G^-$. Choose $\lambda \in -R^+(\pi)$ so that $\lambda - \lambda_i \in -R^+(\pi)$, for all $i = 1, 2, \cdots, n$. By our hypothesis $\Lambda(\tau(\lambda_i - \lambda))$ is defined and we set

$$b = \sum_{i=1}^{n} g_i y_{\lambda-\lambda_i} \Lambda(\tau(\lambda_i - \lambda)) \ .$$

Then

$$\begin{aligned}\gamma(b) &= \tau(\lambda)\Lambda(\tau(-\lambda)) \sum_{i=1}^{n} g_i^- \Lambda(\tau(\lambda_i)) \\ &= \tau(\lambda)\Lambda(\tau(-\lambda))\gamma(a)(\Lambda) = 0\end{aligned}$$

by the hypothesis on a. From 8.3 we deduce that $b = 0$. Finally

$$a = a - b = \sum_{i=1}^{n} g_i[1 - y_{\lambda-\lambda_i}\Lambda(\tau(\lambda_i - \lambda))] \in J_\Lambda$$

since the terms in square brackets lie in Y_Λ.

Remarks. – A similar analysis gives the corresponding result for $G(\check{U})$. The latter also follows from 7.5; but this last proof uses the separation theorem. In the case $\Lambda = 0$ we have $Y_\Lambda = Y_+$ but then $Ann_{G(U)}N(\Lambda)$ has codimension 1 and so is very different from J_+.

8.6. We shall need the following dimensionality estimate. Set $d = d_{G(U)}$.

LEMMA. – *For each* $\Lambda \in T^*_<$ *one has*

$$d(G(U)/J_\Lambda) = 2|\Delta^+(\pi)| \ .$$

From [JL2, 4.8, 4.12] it follows that $G(U)$ is finitely generated as a K-algebra. Hence $G(U)/J_\Lambda$ is finitely generated. It follows by [BK, 1.7a] that $d(G(U)/J_\Lambda)$ is determined by the growth rate of $G(U)/J_\Lambda$ viewed as an algebra with filtration induced by the gradation on $G(U)$.

Let us show that

$$(*) \qquad d(G(U)/J_+) \leq 2|\Delta^+(\pi)| \ .$$

For this we first establish that

$$(**) \qquad G(\check{U})Y_+ \cap G(U) = G(U)Y_+ =: J_+ \ .$$

By [JL2, 4.8, 4.12] the family of subspaces $F(\lambda) : \lambda \in -4P^+(\pi)$ form a gradation of $G(\check{U})$ and both sides of (**) are graded subspaces. Thus it is enough to show that

$$G(\check{U})Y_+ \cap F(\lambda) \subset G(U)Y_+$$

for each $\lambda \in -R^+(\pi)$. Both sides are generated by their U^- invariant subspaces to which we can therefore restrict. Applying γ, using injectivity, (8.3) and 7.2(ii) we are reduced to showing that if $\tau(\lambda) \in K[\check{T}_<]T_<^+$, then $\tau(\lambda) \in K[T_<]T_<^+$, where $T_<^+ := \{\tau(\lambda) \mid \lambda \in -R^+(\pi) \setminus \{0\}\}$. Since each element of $K[\check{T}_<]T_<^+$ is a sum of elements of the form $\tau(\mu)\tau(\nu) : \mu \in -4P^+(\pi), \quad \nu \in -R^+(\pi) \setminus \{0\}$ we conclude that $\lambda = \mu+\nu$, so then $\mu = \nu - \lambda \in R(\pi) \cap -4P^+(\pi) = -R^+(\pi)$, as required. This proves (**).

From (**) and 6.2(i) we deduce the inequality

$$(***) \qquad d(G(U)/J_+) \leq d_{G(\check{U})}(G(\check{U})/G(\check{U})Y_+) .$$

We show that the right hand side equals $2|\Delta^+(\pi)| = dim\ \mathfrak{g} - rank\ \mathfrak{g}$ where $U = U_q(\mathfrak{g})$. This will establish (*).

As a graded vector space $G(\check{U})/G(\check{U})Y_+$ is isomorphic to $H \otimes Y(\check{U})/Y_+$. Since $Y(\check{U})$ is finite over $Y(U)$, the latter has the same growth rate as H. Since $G(\check{U}) = H \otimes Y(\check{U})$ and growth rates add under tensor product we deduce (using [BK,1.7a]) that the right hand side of (***) equals

$$d_{G(\check{U})}(G(\check{U})) - d_{Y(\check{U})}(Y(\check{U})) .$$

Now $Y(\check{U})$ is a polynomial ring on $rank\ \mathfrak{g}$ generators, so the second term is just $rank\ \mathfrak{g}$. The first term can be shown to be equal to $dim\ \mathfrak{g}$ by using a growth rate estimate based on [JL2, 3.5, 4.8, 4.12]. Alternatively from the embedding $G(\check{U}) \hookrightarrow \check{T}_<^{-1}G(\check{U})$ and the fact that the latter ring is a finite module over G, it is enough to show that $d_G(G) = dim\ \mathfrak{g}$. The last equality follows either by the methods of [M] or by a growth rate estimate based on triangular decomposition $G = G^- \otimes G^0 \otimes G^+$ and the formal characters of $G^\pm$. This completes the proof of (*).

Since $gr\ J_\Lambda \supset J_+$ we obtain

$$d(G(U)/J_+) \geq d(G(U)/J_\Lambda) .$$

It remains to show that

$$({}^{**}_{**}) \qquad d(G(U)/J_\Lambda) \geq 2|\Delta^+(\pi)| .$$

By [JL2, 4.8, 4.12] it follows that $G(\check{U})$ is a finite module over $G(U)$. Hence

$$d(G(U)/J_\Lambda) \geq d(G(U)/(G(\check{U})Y_\Lambda \cap G(U))) = d_{G(\check{U})}(G(\check{U})/G(\check{U})Y_\Lambda) = 2|\Delta^+(\pi)|$$

where the last step is obtained as in the proof of $(***)$.

Remarks. – For $\Lambda \in T_\diamond^*$, we obtain equality in $({}^{**}_{**})$ using 8.2, 8.5 and the considerations used in the proof of 6.2(v). One can ask if it is possible in the proof of the lemma to avoid the use of the separation theorem and just use growth rate estimates. Unfortunately this is rather tricky. For example to prove equality in (*) we could use that $G(U)/J_+$ is a direct sum of its isotypical components and that by [JL2, 5.3, 5.4] the multiplicity of the component of $G(U)/J_+$ of type $E(\mu)$ is at least $dim\ E(\mu)_0$. However we also have to estimate the maximum $m(\mu)$ and minimum $n(\mu)$ degrees in which this component occurs. It turns out that if we can show that $m(\mu)/n(\mu)$ is uniformly bounded then the required result obtains. Here we remark that the corresponding bound in $G(\check{U})/\check{J}_+$ can in principle be solved by the combinatorics of Sect. 2. The corresponding bound in for $U(\mathfrak{g})$ can be resolved using [D, 8.4(ii)] and one obtains the well-known fact that this bound (for $\mathfrak{g}$ simple) can be taken to be the order of the highest root.

8.7. We need the following technical result.

LEMMA. – $\check{T}_< \cap G(\check{U})\check{Y}_+ = \emptyset$.

Suppose $\tau(\mu) \in G(\check{U})\check{Y}_+$. Through the gradation of $G(\check{U})$ given by the $F(\lambda)$ we conclude that

$$\tau(\mu) \in \sum_{i=1}^{\ell} y_{-4\omega_i} F(\mu + 4\omega_i) .$$

Applying $ad\ U$ this gives

$$F(\mu) \subset \sum_{i=1}^{\ell} y_{-4\omega_i} F(\mu + 4\omega_i) .$$

Applying γ we deduce from the remark in 7.4 that

$$K(\mu)^- \subset \sum_{i=1}^{\ell} K(\mu + 4\omega_i)^- \subsetneq K(\mu)^- .$$

This contradiction proves the lemma.

8.8. Take $t \in T_<$. One has $tG(U) = G(U)t$ and this property passes to $G(U)/J_+$. We conclude that $T_<$ is Ore in both these algebras. Now assume that J_+ is a prime ideal. Then by 8.7 we conclude that (the image of) t is regular in $G(U)/J_+$. Consequently the natural maps give

$$\begin{aligned} G(U)/J_+ &\hookrightarrow T_<^{-1}(G(U)/J_+) \\ &\xrightarrow{\sim} T_<^{-1}G(U)/T_<^{-1}J_+ . \end{aligned}$$

Exactly as in 2.6 one shows that $T_<^{-1}G(U)$ is a noetherian ring. We conclude that $T_<^{-1}(G(U)/J_+)$ is prime noetherian and hence by say [H, Thm. 4.5] that $G(U)/J_+$ is a Goldie ring.

8.9. We can now prove a main result of this section.

THEOREM. – *Suppose that J_+ is a prime ideal. Then for any graded complement H to J_+ in $G(U)$ the map $h \otimes y \mapsto hy$ is an isomorphism of $H \otimes Y(U)$ onto $G(U)$.*

This will follow as in [JL2, 7.4] if we can show that $H \cap J_\Lambda = 0$ for all $\Lambda \in T^*_<$. Since H is graded it is enough to show that $H \cap gr\ J_\Lambda = 0$ and for this we must show that the inclusion $gr\ J_\Lambda \supset J_+$ is an equality. Now the hypothesis that J_+ is prime, implies via 8.8 that $G(U)/J_+$ is prime, Goldie. Then a strict inclusion would contradict 8.6 and the strict inequality in 6.2(iii). This proves the theorem.

Remarks. – Of course the point of the theorem is that we already know [JL2, 5.5] that $G(U)$ is *not* always free over $Y(U)$. Consequently J_+ will not always be prime. The analysis of [JL2, 7.4] shows that freeness holds if $R^+(\pi)$ is stable under the cap operation of [JL2, 4.14]; but we do not know if this condition is also necessary, nor do we know if this condition is sufficient for J_+ to be prime.

8.10. Surprisingly enough $\check{J}_+$ also fails to be prime in the example of [JL2, 5.5] namely for $U = U_q(\mathfrak{sl}(3))$. This does not obviously follow from 8.9 because for one thing $G(U) \cap \check{J}_+ \supsetneq J_+$ in this example. The proof is by explicit calculation in which we also determine the ideals prime over $\check{J}_+$.

In what follows $U = U_q(\mathfrak{sl}(3))$ and we retain the notation of 7.6. As in 8.3 one checks that γ restricted to $G(\check{U})^{U^-}$ is injective and so we identify $G(\check{U})^{U^-}$ with its image under γ.

Let B be the K-algebra with generators g_1, g_2, g_3, g_4 satisfying the relations

$$g_1g_2 = g_2g_1\ ,\quad g_3g_4 = g_4g_3 = 0$$

$$qg_ig_j = q^{-1}g_jg_i : i = \{1,2\}, j \in \{3,4\}\ .$$

Recall (7.3) that $v_i \in Y(\check{U})$ satisfies $\gamma(v_i) = \tau(-4\omega_i)$ for $i = 1,2$ and that $\gamma(v^{(1)}) = f_2b_1\tau(-4(\omega_1+\omega_2))$, $\gamma(v^{(2)}) = f_1b_2\tau(-4(\omega_1+\omega_2))$. One checks from (1) - (3) and the lemma of 7.6 that the map φ from $G(\check{U})^{U^-}$ into B sending v_i to zero, $f_i\tau(-4\omega_i)$ to g_i for $i = 1, 2, v^{(1)}$ to g_3 and $v^{(2)}$ to g_4 is an algebra epimorphism with kernel $\check{J}_+^{U^-}$. From the complete reducibility of finite dimensional U modules one easily sees that the natural map $G(\check{U})^{U^-} \longrightarrow (G(\check{U})/\check{J}_+)^{U^-}$ is surjective and hence φ factors to an isomorphism $\bar{\varphi}$ of $(G(\check{U})/\check{J}_+)^{U^-}$ onto B. It is clear from the above relations that Bg_3, Bg_4 are completely prime two-sided ideals of B and are exactly the prime ideals over $\{0\}$ in B.

Now let F (resp. F^*) denote the simple submodule of H with lowest weight vector $v^{(1)}$ (resp. $v^{(2)}$). By the analogue of 5.5 for $G(U)$ we have $I := G(\check{U})F = FG(\check{U})$, $I^* := G(\check{U})F^* = F^*G(U)$. Let $\bar{I}$, $\bar{I}^*$ denote their images in $G(\check{U})/\check{J}_+$.

We claim that FF^*, $F^*F \subset \check{J}_+$ and this will prove that $\bar{I}\bar{I}^* = \bar{I}^*\bar{I} = 0$. Consider FF^* which is an image of $F \otimes F^*$. Using say Weyl's character formula, one checks that $F \otimes F^*$ is a direst sum of 4 simple modules having highest weights $n(\omega_1+\omega_2) : n = 0, 1, 2, 3$. Let us rescale the grading of $G(\check{U})$ so that $F(-4\omega_1)$, $F(-4\omega_2)$ have degree 1. Then F, F^* have degree 2, so FF^* has degree 4. Yet by 7.6 (with $r = 0$, $s + t = n$) one immediately sees that the representations of highest weights $n(\omega_1 + \omega_2)$ all occur in H at degree n, which in our case ≤ 3. This proves the required assertion.

It is clear that $I^{U^-} \supset Bg_3$ (resp. $(I^*)^{U^-} \supset Bg_4$) and we claim that equality holds. Since the Dynkin diagram automorphism induces an automorphism of $G(U)$ which exchanges g_3, g_4 and I, I^* these assertions are equivalent. The proof uses the Gelfand-Kirillov d of an algebra. Since B/Bg_3 is a domain of GK dimension 3, a strict inequality would imply

$d(G(\check{U})/I)^{U^-} \leq 2$. Now by [BK, 1.7a] the relation between the GK dimensions of the finite generated graded algebras $(G(\check{U})/I)^{U^-}$ and $G(\check{U})/I$ is controlled by their relative growth rates and this in turn is controlled by the dimension behaviour of simple finite dimensional U modules with respect to their lowest weights. Since the Weyl dimension formula is the same for U as for $U(\mathfrak{g})$ modules, it is enough to compare the relative GK dimensions of $S^{\mathfrak{n}^-}$ and S for a graded $U(\mathfrak{g})$ ring S. (Specifically we could take S to be the S-specialization of $G(\check{U})/I$ defined in [JL2, 6.6]). Now $\mathfrak{n}^-$ acts by locally nilpotent derivations so by [J3, 2.1, 4.4] one has $d(S)-d(S^{\mathfrak{n}^-}) \leq dim\ \mathfrak{n}^- = |\Delta^+(\pi)|$. Consequently

$$d(G(\check{U})/I) \leq d(G(\check{U})/I)^{U^-} + |\Delta^+(\pi)| \, .$$

(This also follows by noting that the Weyl dimension formula is a polynomial of degree $|\Delta^+(\pi)|$ in the lowest weight). Thus a strict inclusion above would give

$$d(G(\check{U})/I) \leq 2+3 \leq 5$$

with a similar (equivalent) result for I^*. Yet by 8.6 we have

$$d(G(\check{U})/\check{J}_+) = 6$$

which is incompatible with $\bar{I}\bar{I}^* = 0$ and the previous inequalities. This proves our claim.

We conclude that $(G(\check{U})/I)^{U^-} \cong B/Bg_3$, which is a domain. By the same reasoning as in [JL2, 8.1] it follows that $G(\check{U})/I$ is domain and so I is completely prime. Similarly I^* is completely prime and then the inclusion $II^* = \check{J}_+$ implies that $I,\ I^*$ are the prime ideals over $\check{J}_+$. We have established the

LEMMA. – *Take $U = U_q(\mathfrak{sl}(3))$. Let I (resp. I^*) denote the ideal of $G(\check{U})$ generated over $\check{J}_+$ by the (single copy) of the simple submodule of H of lowest weight $-3\omega_1$ (resp. $-3\omega_2$). Then $I,\ I^*$ are completely prime and are the minimal prime ideals over $\check{J}_+$.*

Remark. – All this has an analogue in the S-specialization [JL2, 6.6] of $G(\check{U})/\check{J}_+$. Forgetting the gradation this can be viewed as the quotient of $S(\mathfrak{n}^-)\otimes S(\mathfrak{n}^+)$ by the ideal I_+ generated by the invariant elements $y^0_{-4\omega_i} : i = 1, 2, \cdots, \ell$ defined in [JL2, 6.2]. The same analysis as above proves that I_+ is not a prime ideal, unlike the situation encounted in Kostant's separation theorem for $S(\mathfrak{g})$.

Index of Notation

Symbols used frequently are given below where they are first defined.
See also indexes of notation in [JL1, JL2].

1. $k,\ K$.
1.1. $\mathfrak{g},\ \mathfrak{n}^+,\ \mathfrak{h},\ \mathfrak{n}^-,\ U(\mathfrak{g}),\ Z(\mathfrak{g})$.
1.3. $Z(\check{U}),\ Y(\check{U}),\ \check{Y}_+,\ Y_\lambda$.
1.5. $\alpha_i,\ \pi,\ \omega_i,\ \Delta^+(\pi),\ P(\pi),\ Q(\pi),\ R(\pi),\ P^+(\pi),\ s_i,\ W,\ \tau,\ t_i,\ x_i,\ y_i$.
2.1. $U,\ T,\ \check{U},\ \check{T},\ \mathcal{E},\ \check{\mathcal{E}},\ \oplus\mathcal{E},\ \oplus\check{\mathcal{E}}$.
2.2. $F(U),\ gr_{\mathcal{F}},\ \check{R}_\mu(z),\ \check{P}_\mu(z)$.

2.3. $\rho,\ W,\ w\cdot\lambda,\ \ell(w),\ deg\ \lambda,\ \check{S}(z),\ \boldsymbol{\Delta},\ \check{Q}(z)$.
2.6. $T_<,\ \check{T}_<,\ T_\diamond$.
2.7. K_Λ .
3.2. $d_\alpha,\ \alpha^\vee, \mathfrak{h}^*_{\mathbb{Q}},\ \Lambda_{n,\alpha},\ \Lambda^0_{n,\alpha},\ M(\lambda)$.
3.3. $\mathcal{O}_\lambda,\ d(M)$.
3.4. $\mathbb{H},\ \varphi_E,\ \varphi,\ L(\lambda)$.
4.1. $M(\Lambda),\ L(\Lambda)$.
4.5. $\overset{\triangledown}{\check{U}}$.
4.6. $\check{W},\ \boldsymbol{\chi}$.
5.2. $\mathcal{O}_\Lambda$.
5.4. $F(M,N)$.
5.6. $\mathcal{H},\ \mathcal{H}_\chi$.
5.7. $\mathbf{F}(\Lambda),\ \mathcal{T},\ \mathcal{T}',\ \theta_V$.
5.8. $\mathcal{S}_\chi$.
6.2. d_A .
7.1. $\check{G},\ G(\check{U}),\ \check{J}_+,\ F(\lambda),\ H(\lambda),\ y_\lambda$.
7.2. $G,\ G^0,\ \check{G}^0,\ G^+,\ G^-,\ \gamma$.
7.3. $K(\lambda)^-,\ G^+_\mu$.
7.4. $K(\lambda)^{--}$.
8.1. J_χ .
8.2. $T^*_<,\ T^*_\diamond$.

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(Manuscript received May 26, 1994;
revised June 24, 1994).

Anthony Joseph
The Donald Frey Professorial Chair,
Department of Theoretical Mathematics,
The Weizmann Institute of Science,
Rehovot 76100, Israel,
and
Laboratoire de Math. Fond.,
Université P. et M. Curie, France

Gail Letzter
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge,
Massachusetts 02139,
U.S.A.