ANNALES SCIENTIFIQUES DE L'É.N.S.

DAVID SOUDRY

On the archimedean theory of Rankin-Selberg convolutions for $SO_{2l+1} \times GL_n$

Annales scientifiques de l'É.N.S. 4e série, tome 28, nº 2 (1995), p. 161-224

<http://www.numdam.org/item?id=ASENS_1995_4_28_2_161_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 28, 1995, p. 161 à 224.

ON THE ARCHIMEDEAN THEORY OF RANKIN-SELBERG CONVOLUTIONS FOR $SO_{2\ell+1}\times GL_n$

BY DAVID SOUDRY

ABSTRACT. – In this paper, we study the local theory over an archimedean field F of certain Rankin-Selberg convolutions for pairs of generic representations (π, τ) of $SO_{2l+1}(F)$ and $GL_n(F)$. The corresponding local integrals involve Whittaker functions of π and sections of the representation $\rho_{\tau,s}$ of $SO_{2n}(F)$, induced from $\tau\otimes|\det\cdot|^{s-1/2}$, viewed as a representation of the "Siegel" parabolic subgroup. The integrals converge absolutely for Re(s) large enough and are shown to have a meromorphic continuation in s to the whole plane, to a continuous bilinear form on $\pi\times\rho_{\tau,s}$, which satisfies certain equivariance properties. These properties determine such bilinear forms in an essentially unique way. An important ingredient here is an application of Wallach's results on asymptotics of matrix coefficients (and variations). Using all this, we compute the corresponding gamma factors which turn to be, by results of Shahidi, the Artin gamma factors.

0. Introduction

In this paper we study the local theory over an archimedean field of certain Rankin-Selberg convolutions for $SO_{2\ell+1}\times GL_n$. The initial steps were already taken in [S], where the analogous theory over a nonarchimedean field is presented in great detail. Let F be a local field, and let π and τ be finitely generated admissible representations of $G_\ell = SO_{2\ell+1}(F)$ and $GL_n(F)$ respectively, each assumed to be generic, *i.e.* with a unique Whittaker model. Let $s\in\mathbb{C}$ and $\rho_{\tau,s} = \mathrm{Ind}_{Q_n}^{H_n}\tau_s$, where $H_n = SO_{2n}(F)$, Q_n is the Siegel parabolic subgroup and $\tau_s\begin{pmatrix} m & * \\ 0 & m^*\end{pmatrix} = |\det m|^{s-1/2}\tau(m)$. (The induction is unitary. Here $m\in GL_n(F)$ and $m^* = J_n{}^t m^{-1}J_n$, where $J_n = \begin{pmatrix} & & & 1\\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$.) In [S], we studied certain bilinear forms $A(W,\xi_{\tau,s})$, where W is in the Whittaker model

([1]) This research was supported by the Basic Research Foundation administered by the Israel Academy of Sciences and Humanities.

of π and $\xi_{\tau,s}$ is a section in $\rho_{\tau,s}$. These are defined as certain absolutely convergent integrals for $\mathrm{Re}(s)$ large enough (depending on π and τ only) and appear as local factors of global Rankin-Selberg convolutions in case π and τ come from automorphic, cuspidal representations. The definition of $A(W, \xi_{\tau,s})$ depends on whether $\ell \geq n$ or $\ell < n$. Let us list some of the main results proved in [S], in case F is nonarchimedean. The integrals defining $A(W, \xi_{\tau,s})$ have a meromorphic continuation to the whole plane, and actually are rational functions in q^{-s}, where q is the number of elements in the residual field. These bilinear forms satisfy a functional equation

$$\Gamma(\pi \times \tau, s, \psi) A(W, \xi_{\tau,s}) = \widetilde{A}(W, \xi_{\tau,s}) \ , \tag{0.1}$$

where $\widetilde{A}$ is essentially obtained from $A(W, \xi_{\tau,s})$ by applying an intertwining operator to $\xi_{\tau,s}$ (*see* [S]), ψ is a nontrivial additive character of F, and W lies in $W(\pi, \psi)$ – the standard Whittaker model of π with respect to ψ. $\Gamma(\pi \times \tau, s, \psi)$ is a rational function in q^{-s}. In case $\pi = \mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma$, and $\ell < n$, we proved that

$$\gamma(\tau, \Lambda^2, 2s-1, \psi) \Gamma(\pi \times \tau, s, \psi) = w_\tau(-1)^\ell \gamma(\sigma \times \tau, s, \psi) \gamma(\hat{\sigma} \times \tau, s, \psi) \ . \tag{0.2}$$

$\overline{P}_\ell$ is the opposite to the standard parabolic subgroup of G_ℓ, which has $\mathrm{GL}_\ell(F)$ as Levi part. σ is a generic representation of $\mathrm{GL}_\ell(F)$. $\gamma(\tau, \Lambda^2, 2s-1, \psi)$ is the Shahidi local coefficient, obtained from $\rho_{\tau,s}$ ([Sh1]). Its precise definition is given in Section 6. $\gamma(\sigma \times \tau, s, \psi), \gamma(\widehat{\sigma} \times \tau, s, \psi)$ are the gamma factors for $\mathrm{GL}_\ell \times \mathrm{GL}_n$ of Jacquet, Piatetski-Shapiro and Shalika. ω_τ is the central character of τ. (In [S] we proved a more general multiplicativity property than (0.2).) Our purpose here is to prove the meromorphic continuation of $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$, the functional equation (0.1) and the multiplicativity (0.2), in case F is archimedean and for any ℓ, n. (If $\ell \geq n$, (0.2) is slightly modified. In [S] we have already seen that in case $\ell \geq n$, $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$ admit a meromorphic continuation to the whole plane.) Note that, by (0.2), $\gamma(\tau, \Lambda^2, 2s-1, \psi)\Gamma(\pi \times \tau, s, \psi)$ equals up to a sign which depends on $\omega_\tau(-1), \ell, n$ and ψ, the local coefficient associated by Shahidi, [Sh1], to $\pi \otimes \tau$ which, by Shahidi's work [Sh2], is, up to a power of i, which depends on ℓ, n, ψ, the Artin gamma factor of $\pi \otimes \tau$, *i.e.* the gamma factor defined on the Weil group side. The methods that we use here are essentially the same as those in the nonarchimedean case. However, in order to have the same technique work for us in the archimedean case, we have to overcome several technical obstacles which are not present in the nonarchimedean case, in particular the asymptotic expansion of Whittaker functions of a given representation along the center of the Levi part of an arbitrary parabolic subgroup, with continuous coefficients, with respect to the Fréchet topology of the space of the given representation. I owe this to the results of Wallach [W3, chapter 15], where all the ideas and ingredients are present for the results of section 4 of this paper, results which are crucial for the proof of continuity (meromorphic continuation, meromorphic dependence on parameters of representations) of the bilinear forms $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$. I take this opportunity to express my gratitude to Nolan Wallach for his patient explanations over several telephone conversations, and for sending me chapters of his new book "Real Reductive Groups II" [W3], before it appeared in press.

In Section 2, we state the main results of this paper and say some words about the proofs. In Section 3, we prove certain uniqueness theorems (in analogy with [S, Section 8]) which imply the proportionality of $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$, once their meromorphic continuations and continuity are established. The results of Section 4 on the asymptotic expansion of Whittaker functions are used in Sections 5,6,7, in order to prove the meromorphic continuation and continuity of $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$, and the multiplicativity property of the gamma factors.

Apart from proving the results above, another goal of this paper is to highlight the "passage" from the nonarchimedean theory (as presented in [S]) to the archimedean theory. This passage is based on considerations of a general nature, and in this sense, we hope that this paper will be useful.

This work was done during the academic year 1991–1992, while I was a guest of the Department of Mathematics of The Ohio-State University, Columbus, Ohio. I thank the department for its hospitability. Special thanks are due to Steve Rallis for his encouragement and many helpful conversations, full of inspiration, ideas, answers and information.

Finally, I remind the reader that this work is part of a large scale project, whose architect is Ilya Piatetski-Shapiro. The goal of the project is to prove the existence of lifting of automorphic forms on $\mathrm{SO}_{2\ell+1}$ to automorphic forms on $\mathrm{GL}_{2\ell}$, by use of the converse theorem. I thank Ilya for inviting me to participate in this wonderful program, together with Jim Cogdell, Steve Gelbart, David Ginzburg and Steve Rallis, to whom I am indebted for fruitful discussions and happy times spent together on this project.

1. Notation

We mainly use the notation in [S].

$F = \mathbb{R}, \mathbb{C}$.

$$J_m = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}, \quad (m \times m \quad \text{matrix}).$$

$\mathrm{SO}_m = \{g \in \mathrm{SL}_m \,|\, {}^t g J_m g = J_m\}$.

$G_\ell = \mathrm{SO}_{2\ell+1}(F),\ H_n = \mathrm{SO}_{2n}(F)$.

P_ℓ = standard parabolic subgroup of G_ℓ, which preserves an ℓ-dimensional isotropic subspace. Its Levi decomposition is

$$P_\ell = M_\ell \ltimes Y_\ell\ ,$$

$$M_\ell = \left\{ \widehat{a} = \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix} \middle| a \in \mathrm{GL}_\ell(F) \right\}, \quad (a^* = J_\ell {}^t a^{-1} J_\ell),$$

$$Y_\ell = \left\{ y(x,z) = \begin{pmatrix} I_\ell & x & z \\ & 1 & x' \\ & & I_\ell \end{pmatrix} \in G_\ell \right\}, \quad (x' = -J_\ell {}^t x J_\ell),$$

$\overline{P}_\ell$ = the opposite to P_ℓ. Its Levi decomposition is

$$\overline{P}_\ell = M_\ell \ltimes \overline{Y}_\ell \ .$$

$$\overline{Y}_\ell = \left\{ \overline{y}(x,z) = \begin{pmatrix} I_\ell & & \\ x & 1 & \\ z & x' & I_\ell \end{pmatrix} \in G_\ell \right\}.$$

For a subgroup $B \subset \mathrm{GL}_\ell(F)$, we denote
$\widehat{B} = \left\{ \widehat{b} \mid b \in \mathrm{GL}_\ell(F) \right\}$.
A_ℓ = the diagonal subgroup of $\mathrm{GL}_\ell(F)$.
Z_ℓ = the standard maximal unipotent subgroup of $\mathrm{GL}_\ell(F)$.
N_ℓ = the standard maximal unipotent subgroup of G_ℓ,

$$N_\ell = \widehat{Z}_\ell \cdot Y_\ell \ .$$

Q_n = the Siegel parabolic subgroup of H_n. Its Levi decomposition is

$$Q_n = L_n \ltimes U_n \ ,$$

$$L_n = \left\{ m(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix} \middle| a \in \mathrm{GL}_n(F) \right\},$$
$$U_n = \left\{ u(x) = \begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \middle| x = x' \right\}.$$
$$\overline{U}_n = \left\{ \overline{u}(x) = \begin{pmatrix} I_n & \\ x & I_n \end{pmatrix} \middle| x = x' \right\}.$$

For a subgroup $B \subset \mathrm{GL}_n(F)$, we denote
$m(B) = \{m(b) \mid b \in B\}$.
V_n = the standard maximal unipotent subgroup of H_n.

$$V_n = m(Z_n) \cdot U_n \ .$$

ψ denotes a nontrivial additive character of F. We also denote by ψ the standard nondegenerate character it defines on Z_m, N_ℓ, V_n. Given a representation π which admits a unique Whittaker model, we denote its standard Whittaker model with respect to ψ, by $W(\pi,\psi)$.

For $\ell < n$, $i_{\ell,n}$ denotes the embedding of G_ℓ in H_n given by

$$i_{\ell,n}(G_\ell) = \left\{ \begin{pmatrix} A & & B \\ & I_{2r} & \\ C & & D \end{pmatrix} \in H_n \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} e_0 = e_0 \right\},$$

where $r = n - \ell - 1$ and e_0 is the column vector in $F^{2\ell+2}$ with 1 at its $\ell+1$ coordinate, -1 at its $\ell+2$ coordinate and zero elsewhere.

For $\ell \geq n$, $j_{n,\ell}$ denotes the embedding of H_n in G_ℓ given by

$$j_{n,\ell}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \begin{pmatrix} A & & B \\ & I_{2(\ell-n)+1} & \\ C & & D \end{pmatrix}.$$

$K_{\mathrm{GL}_n}, K_{G_\ell}, K_{H_n}$ denote the standard maximal compact subgroups of $\mathrm{GL}_n(F), G_\ell, H_n$ respectively.

Induction of representations is always assumed to be in normalized form. For a representation π, we denote by V_π a vector space realization of the action of π. If π has a central character, we denote it by ω_π. If V and W are two continuous modules over a topological group L, we denote by $Bi\ell_L(V,W)$ the space of all separately continuous bilinear forms on $V \times W$, which are L-invariant.

2. Statement of the Main Results and Sketch of Proofs

Let π and τ be representations of G_ℓ and $\mathrm{GL}_n(F)$ respectively, on Fréchet spaces V_π, V_τ, both assumed to be smooth (differentiable), of moderate growth, and so that the subspace of K_{G_ℓ}-finite vectors (K_{GL_n}-finite vectors resp.) is a Harish-Chandra module (*i.e.* admissible and finitely generated). We also assume that π and τ are generic. Let $s \in \mathbb{C}$. Put $\rho_{\tau,s} = \mathrm{Ind}_{Q_n}^{H_n} \tau_s$ – the smooth induced representation acting in the space $V_{\rho_{\tau,s}}$ of smooth functions $\xi_{\tau,s}$ on H_n which take values in the Whittaker model $W(\tau, \psi^{-1})$ and (regarding $\xi_{\tau,s}$ as a function of two variables) satisfy

$$\xi_{\tau,s}(m(a)u(b)h, x) = |\det a|^{s+\frac{n-2}{2}} \xi_{\tau,s}(h, xa) \quad , \quad h \in H_n, \; x \in \mathrm{GL}_n(F) \; .$$

Let $f_{\xi_{\tau,s}}(h) = \xi_{\tau,s}(h, I_n)$. The integrals defined in [S], for $W \in W(\pi, \psi)$ and $\xi_{\tau,s} \in V_{\rho_{\tau_s}}$, which are absolutely convergent in a right half plane are as follows.

THE CASE $\ell < n$

$$A(W, \xi_{\tau,s}) = \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} f_{\xi_{\tau,s}}(\overline{x}\beta_{\ell,n} i_{\ell,n}(g)) \psi_a(\overline{x}) d\overline{x} dg \; .$$

Here

$$\beta_{\ell,n} = \begin{cases} \left(\begin{array}{c|c|c} I_{\ell+1} & & \\ \hline & & I_r \\ & I_r & \\ \hline & & I_{\ell+1} \end{array}\right) , & r = n - \ell - 1 \text{ even} \\ \left(\begin{array}{c|ccc|c} I_\ell & & & & \\ \hline & & & 1 & \\ & & I_r & & \\ & I_r & & & \\ & 1 & & & \\ \hline & & & & I_\ell \end{array}\right) , & r \text{ odd} \end{cases}$$

$$\overline{X}^{(\ell,n)} = \left\{ \overline{x} = \overline{u}\begin{pmatrix} v & z \\ 0 & v' \end{pmatrix} \in H_n \,\middle|\, v \in M_{r \times (\ell+1)}(F) \right\} ,$$

$$\psi_a(\overline{x}) = \psi(v_{r,\ell+1}) \; .$$

THE CASE $\ell \geq n$

$$A(W,\xi_{\tau,s}) = \int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))f_{\xi_{\tau,s}}(h)d\overline{x}dh\ .$$

Here

$$\overline{X}_{(n,\ell)} = \left\{ \begin{pmatrix} I_n & \\ y & I_{\ell-n}\end{pmatrix}^{\wedge} \middle| y \in M_{(\ell-n)\times n}(F)\right\}.$$

Let

$$w_n := \begin{cases} \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}, & n \text{ even} \\ \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}\begin{pmatrix} & & 1 \\ & I_{2n-2} & \\ 1 & & \end{pmatrix}, & n \text{ odd}\end{cases}$$

and consider the intertwining operator $M(w_n,\xi_{\tau,s})$ of $\rho_{\tau,s}$ corresponding to w_n. In [S] we also considered $\widetilde{A}(W,\xi_{\tau,s})$, obtained (roughly) from $A(W,\xi_{\tau,s})$ by applying the intertwining operator to $\xi_{\tau,s}$. These are defined as follows.

THE CASE $\ell < n$, n EVEN

$$\widetilde{A}(W,\xi_{\tau,s}) = \int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}} M(w_n,\xi_{\tau,s})(\overline{x}\beta_{\ell,n}i_{\ell,n}(g),b_n^*)\psi_a(\overline{x})d\overline{x}dg\ .$$

Here $b_n = \begin{pmatrix} 1 & & & & & & \\ & -1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & -1 \end{pmatrix}.$

THE CASE $\ell < n$, n ODD

$$\widetilde{A}(W,\xi_{\tau,s}) = \int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}} \xi^*_{\tau^*,1-s}(\overline{x}\eta_{\ell,n}m(\varepsilon_{\ell,n})i_{\ell,n}(g),I_n)\psi_a^{-1}(\overline{x})d\overline{x}dg\ .$$

Here the notation is a bit complicated. Put $\omega = \begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1}\end{pmatrix}$ and $h^\omega = \omega^{-1}h\omega$.

Then

$$\xi^*_{\tau^*,1-s}(h,c) = M(w_n,\xi_{\tau,s})(h^\omega, b^*_{\ell,n}c^*)\ ,$$

where

$$b_{\ell,n} = \begin{pmatrix} 1 &&&&&& \\ & -1 &&&&& \\ && 1 &&&& \\ &&& -1 &&& \\ &&&& \ddots && \\ &&&&& -1 & \\ &&&&&& 1 \end{pmatrix} \begin{pmatrix} I_{\ell+1} & \\ & -I_r \end{pmatrix} ,$$

$$\varepsilon_{\ell,n} = \begin{cases} \left(\begin{array}{c|c} I_{\ell+1} & \\ \hline & 1 \\ & I_{r-1} \end{array}\right) & , r \geq 2 \\ I_n & , r = 0, 1 , \end{cases}$$

$$\eta_{\ell,n} = \begin{cases} \left(\begin{array}{c|c|c} I_{\ell+2} & & \\ \hline & I_{r-1} & \\ & I_{r-1} & \\ \hline & & I_{\ell+2} \end{array}\right) , & r \text{ odd}, r \geq 3 \\ \left(\begin{array}{c|c|c|c|c} I_\ell & & & & \\ \hline & & & 1 & \\ \hline & & 1 & & \\ & & I_{r-1} & & \\ & & I_{r-1} & & \\ & & 1 & & \\ \hline & 1 & & & \\ \hline & & & & I_\ell \end{array}\right) , & r \text{ even}, r \geq 2 \\ I_{2n} & r = 0, 1 . \end{cases}$$

THE CASE $\ell \geqslant n$, n EVEN

$$\widetilde{A}(W, \xi_{\tau,s}) = \int_{V_n \backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} j_{n,\ell}(h)) M(w_n, \xi_{\tau,s})(h, b_n^*) d\overline{x} dh .$$

THE CASE $\ell \geqslant n$, n ODD

$$\widetilde{A}(W, \xi_{\tau,s}) = \int_{V_n \backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\widehat{c}_{n,\ell} \overline{x} j_{n,\ell}(h) \delta_0') M(w_n, \xi_{\tau,s})(h^\omega, b_n) d\overline{x} dh .$$

Here

$$\delta_0' = \begin{pmatrix} I_\ell && \\ & -1 & \\ && I_\ell \end{pmatrix} j_{n,\ell}(\omega), \; b_n = \begin{pmatrix} 1 &&&&& \\ & -1 &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& \ddots & \\ &&&&& 1 \end{pmatrix}, \; c_{n,\ell} = \begin{pmatrix} I_n & \\ & -I_{\ell-n} \end{pmatrix}.$$

Now we are ready to state the main results of our paper. For Theorem A, write in what appears above, W_v instead of W $(v \in V_\pi)$ so that $v \mapsto W_v(I)$ is a Whittaker functional on V_π.

THEOREM A. – *The integrals $A(W_v, \xi_{\tau,s})$ and $\widetilde{A}(W_v, \xi_{\tau,s})$ admit a meromorphic continuation to the whole plane. As such, $A(W_v, \xi_{\tau,s})$ and $\widetilde{A}(W_v, \xi_{\tau,s})$ are continuous on $V_\pi \times V_{\rho_{\tau,s}}$. Moreover, if π is a quotient of the representation $I(\sigma_1, \ldots, \sigma_k; \sigma; s_1, \ldots, s_k)$, induced from $\mathrm{GL}_{t_1}(F), \ldots, \mathrm{GL}_{t_k}(F)$, $G_{\ell-(t_1+\cdots+t_k)}$ respectively, then $A(W_v, \xi_{\tau,s})$ and $\widetilde{A}(W_v, \xi_{\tau,s})$ are meromorphic in $(s_1, \ldots, s_k)$, if $W_v = W_{v;s_1,\ldots,s_k}$ is the analytic continuation of the Jacquet integral on $I(\sigma_1, \ldots, \sigma_k; \sigma'; s_1, \ldots, s_k)$.*

THEOREM B. – *There is a meromorphic function $\Gamma(\pi\times, \tau, s, \psi)$, such that*

$$\Gamma(\pi \times \tau, s, \psi) A(W, \xi_{\tau,s}) = \widetilde{A}(W, \xi_{\tau,s}) \ ,$$

for all $W \in W(\pi, \psi)$, $\xi_{\tau,s} \in V_{\rho_{\tau,s}}$.

THEOREM C. – *Let π be a quotient of $\mathrm{Ind}_{P_\ell}^{G_\ell} \sigma$, where σ is a generic representation of $\mathrm{GL}_\ell(F)$. Then*

$$\Gamma(\pi \times \tau, s, \psi) = \omega_\tau(-1)^a \frac{\gamma(\sigma \times \tau, s, \psi)\gamma(\widehat{\sigma} \times \tau, s, \psi)}{\gamma(\tau, \Lambda^2, 2s-1, \widetilde{\psi})} \ .$$

Here $a = \ell$ or $a = \ell + n$, according to whether $\ell < n$ or $\ell \geq n$ respectively, $\gamma(\sigma\times, \tau, s, \psi)$ and $\gamma(\hat{\sigma} \times \tau, s, \psi)$ are the $\mathrm{GL}_\ell \times \mathrm{GL}_n$-gamma factors of Jacquet, Piatetski-Shapiro, Shalika. $\gamma(\tau, \Lambda^2, 2s-1, \widetilde{\psi})$ is the Shahidi local coefficient. Its precise definition (together with that of $\widetilde{\psi}$) is given in Section 6 $(\ell < n)$ and Section 7 $(\ell \geq n)$.

On the Proofs. – In [S] we proved Theorem B by noting that both $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$ satisfy the following equivariance property.

IN CASE $\ell < n$,

$$A(\pi(g)W, \rho_{\tau,s}(i_{\ell,n}(g)y)\xi_{\tau,s}) = \psi_\alpha(y) A(W, \xi_{\tau,s}) \ . \tag{2.1}$$

Here $g \in G_\ell$, $y \in Y^{(\ell,n)} = Z'_r Y'^{(\ell,n)}$,

$$Y'^{(\ell,n)} = \left\{ y' = \begin{pmatrix} I_{\ell+1} & 0 & x'_2 & 0 \\ x_1 & I_r & b & x_2 \\ & & I_r & 0 \\ & & x'_1 & I_{\ell+1} \end{pmatrix} \in H_n \right\} ,$$

$$Z'_r = \left\{ z' = \begin{pmatrix} I_{\ell+1} & & & \\ & z & & \\ & & z^* & \\ & & & I_{\ell+1} \end{pmatrix} \middle| z \in Z_r \right\} ,$$

$$\psi_\alpha(y') = \psi\big((-1)^{r+1}((x_1)_{r,\ell+1} - (x_2)_{r,1})\big) ,$$
$$\psi_\alpha(z') = \psi(z_{12} + z_{23} + \cdots + z_{r-1,r}) .$$

IN CASE $\ell \geq n$,

$$A(\pi(j_{n,\ell}(h)y)W, \rho_{\tau,s}(h)\xi_{\tau,s}) = \psi_\alpha(y)A(W;\xi_{\tau,s}) . \tag{2.2}$$

Here $h \in H_n$, $y \in Y_{(n,\ell)} = Z'_{\ell-n}Y'_{(n,\ell)}$,

$$Y'_{(n,\ell)} = \left\{ y' = \begin{pmatrix} I_n & 0 & 0 & x'_2 & 0 \\ x_1 & I_{\ell-n} & x & b & x_2 \\ & & 1 & x' & 0 \\ & & & I_{\ell-n} & 0 \\ & & & x'_1 & I_n \end{pmatrix} \in G_\ell \right\} ,$$

$$Z'_{\ell-n} = \left\{ z' = \begin{pmatrix} I_n & & & & \\ & z & & & \\ & & 1 & & \\ & & & z^* & \\ & & & & I_n \end{pmatrix} \middle| z \in Z_{\ell-n} \right\} ,$$

$$\psi_\alpha(y') = \psi(x_{\ell-n}) ,$$
$$\psi_\alpha(z') = \psi(z_{12} + z_{23} + \cdots + z_{\ell-n-1,\ell-n}) .$$

The theorems in [S, Section 8] state (in case F is nonarchimedean) that except for a finite set of values of q^{-s}, the space of bilinear forms on $V_\pi \times V_{\rho_{\tau,s}}$ satisfying the equivariance property (2.1), in case $\ell < n$, or (2.2) in case $\ell \geq n$, is at most one dimensional. We prove here the analogous theorem, further requiring that our bilinear forms are *continuous*. (Note that since V_π and $V_{\rho_{\tau,s}}$ are Fréchet spaces, the notions of continuity and separate continuity coincide.) In Section 3, we prove, using Bruhat theory,

UNIQUENESS THEOREM. – *Except for a discrete set of values of s, the space of* continuous *bilinear forms on $V_\pi \times V_{\rho_{\tau,s}}$ which satisfy the equivariance property (2.1), in case $\ell < n$ (resp. (2.2), in case $(\ell \geq n)$), is at most one dimensional.*

In order to prove the functional equation of Theorem B, we have to prove that $A(W, \xi_{\tau,s})$ and $\widetilde{A}(W, \xi_{\tau,s})$ are bilinear forms as in the Uniqueness Theorem. Thus Theorem B follows from this theorem and Theorem A. The continuity assertion in Theorem A is not so easy to prove, and it is a very crucial point in the "passage" of proofs from the nonarchimedean case in [S] to the archimedean case. Fortunately, we can now benefit from the results of Wallach on the asymptotic expansion of Whittaker functions [W3, Sec. 15.2]. In Section 4, we write the asymptotic expansion of $W_v(a)$ for $v \in V_\pi$ and $a = \widehat{a}'$, $a' = \operatorname{diag}(a_1a_2 \cdot \ldots \cdot a_k,\ a_2 \cdot \ldots \cdot a_k, \ldots, a_{k-1}a_k, a_k, 1, \ldots, 1)$, as $(a_1, \ldots, a_k)$ tends to zero. We show that the coefficients of the expansion are continuous in v, and we control their growth. Similar properties hold for the difference of $W_v(a)$ and a finite sum taken from the asymptotic expansion. The methods and proofs are contained in [W3, Section 15.2], although not stated in the form suitable to us. In particular, we have to generalize the asymptotic expansion given there for $k = 1$ to any $k \geq 1$. The details are given in Section 4, where the treatment clearly generalizes to any real reductive group.

The results on the asymptotic expansion of the Whittaker functions imply the proof of Theorem A in case $\ell \geq n$ (Section 5). The proof of Theorem A, in this case, can be reduced to proving the same assertions for integrals similar to $A(W, \xi_{\tau,s})$, but without the unipotent integration $d\bar{x}$, *i.e.* $V_n \backslash H_n$ is replaced by $\widehat{A}_n$. This is done in Section 5. We note that in [S,5.4], we already obtained the meromorphic continuation of $A(W, \xi_{\tau,s})$ in case $\ell \geq n$, but the proof gave no information on the questions of continuity or holomorphic dependence on parameters. The case $\ell < n$ is quite intricate and involved. Here we do not know how to obtain for Theorem A a reduction similar to the previous case (*i.e.* "get rid" of the unipotent integration in $A(W, \xi_{\tau,s})$). It is interesting to note that the proofs of Theorem A and Theorem C are related. As a matter of fact, the proofs of these theorems are tied together, and we prove them both at the same time, using in a crucial way the Uniqueness Theorem. It may seem odd, but we prove Theorem B as a result of Theorem C. The main ingredient of the proof is the following. Assume that $\pi = \pi_{\sigma,\zeta} = Ind_{\overline{P}_\ell}^{G_\ell} \sigma_{-\zeta}$ where $\sigma_{-\zeta} = \sigma \cdot |\det \cdot|^{-\zeta}$ and σ is a generic representation of $\mathrm{GL}_\ell(F)$. Let $\phi_{\sigma,\zeta}$ be a section in $\pi_{\sigma,\zeta}$ which takes values in $W(\sigma, \psi)$. Then for the Whittaker function $W_{\phi_{\sigma,\xi}}$, given by the standard Whittaker integral for $\mathrm{Re}(\zeta) \gg 0$ (and similarly we define $W_{\xi_{\tau,s}}$), we have the following identity (6.10), which holds also in the nonarchimedean case,

$$\omega_\sigma(-1)^{n-1}\gamma(\sigma \times \tau, s-\zeta, \psi^{-1})A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$$
$$= \int_{\widetilde{N}_\ell \backslash G_\ell} \int_{M_{r\times \ell}(F)} W_{\xi_{\tau,s}}\left(m\begin{pmatrix} I_\ell & & \\ y & I_r & \\ 0 & 0 & 1\end{pmatrix}\beta'_{\ell,n} i_{\ell,n}(g)\right)\phi_{\sigma,\zeta}(g, I)dydg\ .$$

$\widetilde{N}_\ell$ is a maximal unipotent subgroup of G_ℓ (not the standard one) and $\beta'_{\ell,n}$ is a certain Weyl element. The r.h.s. of this identity is a "local integral" for $\mathrm{SO}_{2n} \times \mathrm{GL}_\ell$. The proof that it has a meromorphic continuation in (ζ, s) which is continuous on $V_{\rho_{\tau,s}} \times V_{\pi_{\sigma,\zeta}}$ is exactly the same as for $A(W, \xi_{\tau,s})$ in case $\ell \geq n$. The identity takes place in the domain of convergence of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ (where the r.h.s. is the analytic continuation of the written integral). The proportionality of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ and the r.h.s. follows from the Uniqueness Theorem. The factor of proportionality is obtained by a calculation. Theorem C for the case $\ell \geq n$ is proved using an analogous identity (Section 7). Theorem C (in both cases) shows that $\gamma(\tau, \Lambda^2, 2s-1, \widetilde{\psi})\Gamma(\pi \times \tau, s, \psi)$ is (up to a sign which depends on $\omega_\tau(-1), \ell, n$) the Shahidi local coefficient for $\pi \otimes \tau$ [Sh1], which, by [Sh2], equals the Artin gamma factor of $\pi \otimes \tau$ (up to a power of i, which depends on n, ℓ and ψ). Thus we get, setting $\gamma(\pi \times \tau, s, \psi) = \gamma(\tau, \Lambda^2, 2s-1, \widetilde{\psi})\Gamma(\pi\times, \tau, s, \psi)$,

COROLLARY 1. – *Up to a constant of absolute value one, which depends on $\omega_\tau(-1), \ell, n$ and $\psi, \gamma(\pi \times \tau, s, \psi)$ is equal to the Artin gamma factor associated with $\pi \otimes \tau$ and ψ.*

In [S] we proved the general multiplicativity property of $\gamma(\pi \times \tau, s, \psi)$ in the variable π, when $\ell < n$ and the field is nonarchimedean. We can prove that $\gamma(\pi \times \tau, s, \psi)$ is multiplicative in both π and τ for all ℓ, n (over a nonarchimedean field). This will appear in a forthcoming publication. This together with Corollary 1 and the general observation in [G.S.], p.114, show

COROLLARY 2. – *Over any local field and up to a constant of absolute value one, which depends on $\omega_\tau(-1), \ell, n$ and ψ, $\gamma(\pi \times \tau, s, \psi)$ is equal to the local coefficient of Shahidi, associated to $\pi \otimes \tau$.*

3. Proof of the Uniqueness Theorems

Notation and assumptions are as in section 2.

PROOF OF THE UNIQUENESS THEOREM IN CASE $\ell < n$. – Put, (*see* (2.1))

$$R = Y^{(\ell,n)} i_{\ell,n}(G_\ell) .$$

A continuous bilinear form on $V_\pi \times V_{\rho_{\tau,s}}$, which satisfies the equivariance property (2.1) is an element of $Bil_R\left(V_{\pi\cdot\psi_\alpha^{-1}}, V_{\rho_{\tau,s}}\right)$. Here $V_{\pi\cdot\psi_\alpha^{-1}}$ is V_π as a space, on which $Y^{(\ell,n)}$ acts through ψ_α^{-1}. We want to show that

$$\dim Bil_R\left(V_{\pi\cdot\psi_\alpha^{-1}}, V_{\rho_{\tau,s}}\right) \leq 1 ,$$

except possibly for a discrete set of values of s. By Frobenius reciprocity [Wr. Theorem 5.3.3.1], there is an isomorphism of vector spaces

$$Bil_R\left(V_{\pi\cdot\psi_\alpha^{-1}}, V_{\rho_{\tau,s}}\right) \simeq Bil_{H_n}\left(\operatorname{Ind}^c{}_R^{H_n} \pi \cdot \psi_\alpha^{-1}, V_{\rho_{\tau,s}}\right) .$$

$\operatorname{Ind}^c{}_R^{H_n} \pi \cdot \psi_{\alpha^{-1}}$ is the differentiably induced representation, where the (vector) functions on H_n have compact support modulo R. *See* [Wr. 5.3.1]. By Bruhat Theory (Theorem 5.3.2.3 in [Wr.]), there is an embedding

$$Bil_{H_n}\left(\operatorname{Ind}^c{}_R^{H_n} \pi \cdot \pi_\alpha^{-1}, V_{\rho_{\tau,s}}\right) \hookrightarrow \bigoplus_{\gamma \in Q_n \backslash H_n / R} T(\gamma) .$$

$T(\gamma)$ is a certain space of $V_\pi \widehat{\otimes} V_\tau$ – distributions on an open $Q_n \times R$-invariant subset Ω_γ of H_n. The orbit (double coset) γ is contained in Ω_γ as a closed subset. Note that $Q_n \backslash H_n / R$ is finite. It is described in [S], Section 0. We have ([Wr.], 5.3.2.3),

$$\dim T(\gamma) \leq \sum_{k=0}^{\infty} \dim\left(Bil_{Q_n \cap \gamma R \gamma^{-1}}(\tau_s \otimes (\pi\psi_\alpha^{-1})^\gamma, \Lambda_k)\right) .$$

Λ_k are finite dimensional algebraic representations coming from derivatives. We identify, to our convenience, a representation with a space on which it is realized. An element of $Bil_{Q_n\cap\gamma R\gamma^{-1}}(\tau_s\bigotimes(\pi\psi_\alpha^{-1})^\gamma,\Lambda_k)$, when considered as a $U_n\cap\gamma Y^{(\ell,n)}\gamma^{-1}$ equivariant bilinear form, embeds $(\psi_\alpha)^\gamma|_{U_n\cap\gamma Y^{(\ell,n)}\gamma^{-1}}$ in Λ_k. Λ_k as an algebraic representation of a subgroup of U_n cannot have nontrivial eigenvalues. Thus we have $\psi_\alpha \mid \gamma^{-1}U_n\gamma\cap Y^{(\ell,n)}\equiv 1$. This happens only on the open orbit $Q_n\beta_{\ell,n}R$ (where there are no derivatives). *See* [S], Section 0. Thus

$$\dim\left(Bil_R\left(V_{\pi\cdot\psi_\alpha^{-1}},V_{\rho_{\tau,s}}\right)\right)\le\dim\left(Bil_{Q_n\cap\beta_{\ell,n}R\beta_{\ell,n}^{-1}}\left(\tau_s,(\pi\psi_\alpha^{-1})^{\beta_{\ell,n}}\right)\right).$$

The last space is the space of continuous bilinear forms T on $V_\pi\times V_\tau$, such that

$$(3.1)\quad T\left(\pi\begin{pmatrix}g&x&c\\&1&\tilde{x}\\&&g^*\end{pmatrix}w,\ \tau\left(\begin{array}{cc|c}g&x&\\0&1&Y\\\hline&&z\end{array}\right)v\right)=\psi\left(\begin{array}{c|c}I_{\ell+1}&Y\\\hline&z\end{array}\right)|\det g|^{s'}T(w,v).$$

Here $\begin{pmatrix}g&x&c\\&1&\tilde{x}\\&&g^*\end{pmatrix}\in P_\ell, z\in Z_r, Y\in M_{(\ell+1)\times r}(F)$. s' is a certain shift of $-s$. Now write π as a quotient of an induced representation (in the differentiable sense) from the Borel subgroup B_ℓ of G_ℓ and a quasicharacter η. Thus we may assume that $\pi=\operatorname{Ind}_{B_\ell}^{G_\ell}\eta$. Consider

$$L: C_c^\infty(G_\ell)\to\operatorname{Ind}_{B_\ell}^{G_\ell}\eta,$$

given by

$$(3.2)\qquad L_\phi(g)=\int_{B_\ell}\delta_{B_\ell}^{-1/2}(b)\eta^{-1}(b)\phi(bg)db,$$

where db is a right invariant measure on B_ℓ. By a theorem of Bruhat (Lemma 5.1.1.4 in [Wr.]), L is continuous, open and surjective. Put, for $\phi\in C_c^\infty(G_\ell)$, $v\in V_\tau$,

$$S(\phi,v)=T(L_\phi,v).$$

S is a separately continuous bilinear map on $C_c^\infty(G_\ell)\times V_\tau$, and so extends to a continuous liner map on $C_c^\infty(G_\ell)\overline{\bigotimes}V_\tau$ (inductive tensor product) and this space, since V_τ is a Fréchet space, is isomorphic to $C_c^\infty(G_\ell;V_\tau)$, by a well known theorem of Grothendieck ([G], Chap.2, p.84). Thus S defines a V_τ-distribution of G_ℓ (in the sense of Appendix 2.3 in [Wr.]). Let λ,ρ denote left and right translations respectively on G_ℓ. From (3.2), we have for $b\in B_\ell$, $g\in G_\ell$ and $\phi\in C_c^\infty(G_\ell)$

$$(3.3)\qquad L_{\lambda(b)\phi}=\delta_{B_\ell}^{1/2}(b)\eta^{-1}(b)L_\phi,$$

$$(3.4)\qquad L_{\rho(g)\phi}=\pi(g)L_\phi.$$

Thus, from (3.1), it follows that

$$
\begin{aligned}
&S\left(\lambda(b)\rho\begin{pmatrix} g & x & c \\ & 1 & \widetilde{x} \\ & & g^* \end{pmatrix}\phi,\ \tau\left(\begin{array}{cc|c} g & x & \\ & 1 & Y \\ \hline & & z \end{array}\right)v\right) \\
(3.5)\qquad &=\delta_{B_\ell}^{1/2}(b)\eta^{-1}(b)|\det g|^{s'}\psi\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z \end{array}\right)S(\phi, v)\ .
\end{aligned}
$$

We are now at the situation of [Wr.] 5.2.4. Indeed consider the left action of the group $G = B_\ell \times P_\ell$ on (the manifold) $M = G_\ell$, given by

$$
(b,p)\circ m = bmp^{-1}\ .
$$

Put, for $h=(b,p)\in G$ and $\varphi \in C_c^\infty(M;V_\tau)$,

$$
\begin{aligned}
&\varphi^h(m) = \varphi(h^{-1}\cdot m)\ , \\
&S^h(\varphi) = S\left(\varphi^{h^{-1}}\right) = S\big(\lambda(b^{-1})\rho(p^{-1})\varphi\big)\ , \\
&\big({}^tU_{\eta,s}(h)S\big)(\varphi) = \delta_{B_\ell}^{-1/2}(b)\eta(b)|\det g|^{-s'}S\left(\tau\begin{pmatrix} g & x & \\ & 1 & \\ & & I_r \end{pmatrix}\varphi\right)\ .
\end{aligned}
$$

Here $p = \begin{pmatrix} g & x & c \\ & 1 & \widetilde{x} \\ & & g^* \end{pmatrix}$. We have for $\varphi = \phi\otimes v$, $\phi \in C_c^\infty(G_\ell)$, $v\in V_\tau$, and $h=(b,p)$ as above,

$$
\begin{aligned}
S^h(\varphi) &= S\big(\lambda(b^{-1})\rho(p^{-1})\phi, v\big) = \delta^{-1/2}(b)\eta(b)|\det g|^{-s'}S\left(\phi, \tau\begin{pmatrix} g & x & \\ & 1 & \\ & & I_r \end{pmatrix}v\right) \\
&= \big({}^tU_{\eta,s}(h)S\big)(\varphi)\ .
\end{aligned}
$$

Thus, for all $\varphi\in C_c^\infty(M;V_\tau)$, $h\in G$,

$$
(3.6)\qquad S^h(\varphi) = \big({}^tU_{\eta,s}(h)S\big)(\varphi)\ ,
$$

and we can use Theorem 5.2.4.5 in [Wr.]. Note that S satisfies the additional property

$$
(3.7)\qquad S\left(\tau\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z \end{array}\right)\circ\varphi\right) = \psi\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z \end{array}\right)S(\varphi)\quad ,\quad z\in Z_r\ .
$$

We have to first analyze the orbit space $G\backslash M = B_\ell\backslash G_\ell/P_\ell$. The orbits are those of the action of P_ℓ on the flag variety $B_\ell\backslash G_\ell$. Let $F^{2\ell+1}$ (columns) be equipped with the symmetric bilinear form defined by $J_{2\ell+1}$, and let $\{e_1, \dots, e_{2\ell+1}\}$ be the standard basis of $F^{2\ell+1}$. Let for $i \le \ell$, $W_i = \mathrm{Span}\{e_1, \dots, e_i\}$ and denote $e_{\ell+2} = e_{-\ell}, e_{\ell+3} = e_{-\ell+1}, \dots, e_{2\ell+1} = e_{-1}$. G_ℓ acts on the variety of maximal isotropic flags $D = \{0 = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_\ell\}$ by $D \cdot g = \{0 = g^{-1}D_0 \subset g^{-1}D_1 \subset \cdots \subset g^{-1}D_\ell\}$. B_ℓ is the stabilizer of $\{0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_\ell\}$. The orbit under P_ℓ of a flag D is determined by the vector

$$(\dim(D_1 \cap W_\ell), \dim(D_2 \cap W_\ell), \dots, \dim(D_\ell \cap W_\ell)) \;. \tag{3.8}$$

If the vector (3.8) is

$$(\underbrace{0, \dots, 0}_{j_1}, \underbrace{1, \dots, 1}_{j_2}, \underbrace{2, \dots, 2}_{j_3}, \dots, \underbrace{k-1, \dots, k-1}_{j_k}) \;,$$

then a representative for this orbit is obtained according to the following basis of a maximal isotropic subspace

$$\begin{aligned} &\underbrace{e_{-\ell},\; e_{-\ell+1}, \dots, e_{-\ell+j_1-1}}_{j_1};\; \underbrace{e_1, e_{-\ell+j_1},\; e_{-\ell+j_1+1}, \dots, e_{-\ell+j_1+j_2-1}}_{j_2}; \\ &\underbrace{e_2, e_{-\ell+j_1+j_2}, \dots, e_{-\ell+j_1+j_2+j_3-1}}_{j_3};\; \underbrace{e_3, e_{-\ell+j_1+j_2+j_3}, \dots}_{j_4}, \dots \end{aligned} \tag{3.9}$$

That is $D_1 = \mathrm{Span}\{e_{-\ell}\}, D_2 = \mathrm{Span}\{e_{-\ell}, e_{-\ell+1}\}$, etc. If $k = 1$, then (3.9) is the basis $\{e_{-\ell}, e_{-\ell+1}, \dots, e_{-1}\}$ and (3.8) is the zero vector. The stabilizer of the corresponding flag in P_ℓ is

$$\widehat{B}_{\mathrm{GL}_\ell(F)} = \left\{ \widehat{b} = \begin{pmatrix} b & & \\ & 1 & \\ & & b^* \end{pmatrix} \middle| b \in B_{\mathrm{GL}_\ell(F)} \right\} , \tag{3.10}$$

($B_{\mathrm{GL}_\ell(F)}$ is the standard Bore subgroup of $\mathrm{GL}_\ell(F)$.) If $k > 1$, the projection of the stabilizer in P_ℓ of the flag determined by (3.9) to the upper left $(\ell+1) \times (\ell+1)$ block is

$$\left\{ \begin{pmatrix} b & * & * \\ & b' & 0 \\ & & 1 \end{pmatrix} \middle| \begin{array}{l} b \in B_{\mathrm{GL}_{k-1}(F)} \\ b' \in B_{\mathrm{GL}_{\ell-k+1}(F)} \end{array} \right\} . \tag{3.11}$$

By Theorem 5.2.4.5 in [Wr.] (and the proofs of Theorems 5.2.2.1, 5.2.3.1), the dimension of the space of V_τ-distributions on H_n, which satisfy (3.6), (3.7) is majorized by the sum of dimensions of the spaces of continuous bilinear forms E on $V_\tau \times F^N$ which satisfy an equivariance property of the form

$$\begin{aligned} &E\left(\tau\left(\begin{array}{ccc|c} b & x & c & \\ & b' & 0 & Y \\ & & 1 & \\ \hline & & & z \end{array}\right)v\, ,\;\; \Lambda_N\begin{pmatrix} b & x & c \\ & b' & 0 \\ & & 1 \end{pmatrix}\xi\right) \\ &= \widetilde{\eta}(b, b')|(\det b)(\det b')\,|^{-s'}\; \psi\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z \end{array}\right)E(v, \xi) \;. \end{aligned} \tag{3.12}$$

Here $b \in B_{\mathrm{GL}_{k-1}}(F)$, $b' \in B_{\mathrm{GL}_{\ell-k+1}}(F)$, $\widetilde{\eta}$ is a certain quasicharacter (obtained from η), N is a nonnegative integer (if $k = 1$, $N = 0$), Λ_N is a certain *algebraic* representation in F^N and k varies with all the possible orbits, defined by (3.9). Since Λ_N is algebraic, then by passage to a subquotient of Λ_N, there are certain algebraic characters $\{\alpha_i\}_{i \in I_N}$ of $(F^*)^\ell$, (the diagonal of $\left\{\begin{pmatrix} b & & \\ & b' & \\ & & 1\end{pmatrix}\right\}$), such that E gives rise to a continuous linear form E' on V_τ, which satisfies

(3.13)

$$E'\left(\tau\left(\begin{array}{ccc|c} b & x & c & \\ & b' & 0 & Y \\ & & 1 & \\ \hline & & & z\end{array}\right)v\right) = \widetilde{\eta}\alpha^{-1}(b,b')|(\det b)(\det b')|^{-s'}\psi\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z\end{array}\right)E'(v)\ ;$$

α belongs to $\{\alpha_i\}_{i \in I_N}$. Note that when N varies, then α belongs to a countable discrete set of (algebraic) characters of $(F^*)^\ell$. Now we can use Bruhat Theory again to study the space of functionals (3.13). Write τ as a quotient of an induced representation from $B_{\mathrm{GL}_n}(F)$. As before E' is determined by a distribution $\mathcal{E}$ on $\mathrm{GL}_n(F)$, which satisfies

$$\mathcal{E}\left(\lambda(\beta)\rho\left(\begin{array}{ccc|c} b & x & c & \\ & b' & 0 & Y \\ & & 1 & \\ \hline & & & z\end{array}\right)\phi\right)$$

$$=\chi_\tau(\beta)\eta_{\pi,k}\alpha^{-1}(b,b')(|\det b||\det b'|)^{-s'}\psi\left(\begin{array}{c|c} I_{\ell+1} & Y \\ \hline & z\end{array}\right)\mathcal{E}(\phi)\ , \tag{3.14}$$

for $\phi \in C_c^\infty(\mathrm{GL}_n(F))$, $\beta \in B_{\mathrm{GL}_n(F)}$; χ_τ is a quasi character which depends on τ, $\eta_{\pi,k} = \widetilde{\eta}$. The rest of the notation in (3.14) is as in (3.13) and (3.12). The orbit space relevant to (3.14) is $B_{\mathrm{GL}_n(F)}\backslash \mathrm{GL}_n(F)/B_{\ell,k}$, where $B_{\ell,k}$ is the group of all $\left(\begin{array}{ccc|c} b & x & c & \\ & b' & 0 & Y \\ & & 1 & \\ \hline & & & z\end{array}\right)$ as above. The number of orbits is finite and we take the following representatives

$$g_{w,e}^{(k)} = w\begin{pmatrix} I_{k-1} & 0 & 0 & \\ & I_{\ell-k+1} & e & \\ & & 1 & \\ & & & I_r\end{pmatrix},$$

where w is in the Weyl group of $\mathrm{GL}_n(F)$, e has coordinates which are in $\{0, 1\}$, and the positions where e has 1, depend on w. Let $H_{w,e}^{(k)}$ be the stabilizer in $B_{\mathrm{GL}_n(F)} \times B_{\ell,k}$ of $g_{w,e}^{(k)}$. By Bruhat Theory, the orbit of $g_{w,e}^{(k)}$ "contributes" continuous linear functions $\mathcal{E}'$ on finite dimensional spaces $F^{N'}$, which $H_{w,e}^{(k)}$-intertwine an algebraic representation $\lambda_{N'}$ (of $H_{w,e}^{(k)}$) on $F^{N'}$ and $\chi_\tau(\beta)\eta_{\pi,k}\alpha^{-1}(bb')(|\det b||\det b'|)^{-s'}\psi\begin{pmatrix} I_{\ell+1} & Y \\ & z\end{pmatrix}$, for

$\left(\beta, \left(\begin{array}{ccc|c} b & x & c & \\ & b' & 0 & Y \\ & & 1 & \\ \hline & & & z \end{array}\right)\right) \in H^{(k)}_{w,e}$. Assume that $k > 1$. Then for $d = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{k-1} \end{pmatrix}$,

the element $\left(\begin{pmatrix} d & \\ & I_{n-k+1} \end{pmatrix}^w, \begin{pmatrix} d & \\ & I_{n-k+1} \end{pmatrix}\right)$ lies in $H^{(k)}_{w,e}$. Again, the functionals $\mathcal{E}'$ "live" on irreducible subquotients of $\lambda_{N'}$, and these act on $\begin{pmatrix} d & \\ & I_{n-k} \end{pmatrix}^w$ according to algebraic characters $\omega(d)$; ω varies (with N') in a discrete (at most) countable set. So we must have an equality

$$\omega(d) = \chi_\tau\left(\begin{pmatrix} d & \\ & I_{n-k+1} \end{pmatrix}^w\right)\eta_{\pi,k}\alpha^{-1}(d, I)|\det d|^{-s'}\ . \tag{3.15}$$

Thus, if s lies outside a certain discrete set, (3.15) will not be satisfied and $\mathcal{E}'$ above must be zero for all possible $k > 1$. It remains to treat case $k = 1$. In this case, the orbit space is $B_{\mathrm{GL}_n(F)}\backslash \mathrm{GL}_n(F)/B_{\ell,1}$, where $B_{\ell,1}$ is the group of all $\left(\begin{array}{cc|c} b & 0 & \\ & 1 & Y \\ \hline & & z \end{array}\right)$. (Recall that $b \in B_{\mathrm{GL}_\ell(F)}$, $z \in Z_r$, $Y \in M_{(\ell+1)\times r}(F)$.) Consider a representative $g^{(1)}_{w,e}$. If w transforms one of the simple root subgroups

$$\begin{pmatrix} I_\ell & & & & \\ & 1 & * & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} I_{\ell+1} & & & & \\ & 1 & * & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \dots, \begin{pmatrix} I_{n-2} & & \\ & 1 & * \\ & & 1 \end{pmatrix} \tag{3.16}$$

to a root subgroup which corresponds to a positive root, then either $\left(\begin{pmatrix} I_\ell & 0 & 0 \\ & 1 & \overline{x} \\ & & I_r \end{pmatrix}^w, \begin{pmatrix} I_\ell & 0 & -e\overline{x} \\ & 1 & \overline{x} \\ & & I_r \end{pmatrix}\right) \in H^{(1)}_{w,e}$, for $\overline{x} = (x, 0, \dots, 0) \in F^r$, or there is a simple root subgroup Z_γ in Z_r, such that $\left(\begin{pmatrix} I_{\ell+1} & \\ & z \end{pmatrix}^w, \begin{pmatrix} I_{\ell+1} & \\ & z \end{pmatrix}\right) \in H^{(1)}_{w,e}$, for $z \in Z_\gamma$. In each of these cases $\mathcal{E}'$ gives rise to an intertwining map between $\psi(t), t \in F$, and a finite dimensional algebraic representation of F. This is impossible and hence $\mathcal{E}'$ is zero in this case. So assume that w takes every root subgroup in (3.16) into a "negative" one. This means that w has the form

$$w = \begin{array}{c} \\ \left(\begin{array}{c|ccccc} & \scriptstyle \ell+1 & \scriptstyle \ell+2 & \scriptstyle \ell+3 & \scriptstyle \cdots & \scriptstyle n \\ & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \end{array}\right) \end{array},$$

that is, the line $e_{\ell+1}$ in w is below the line $e_{\ell+2}$, and this is below the line $e_{\ell+3}$ etc. ($\{e_1, \ldots, e_n\}$ is the standard basis of row vectors in F^n.) Now note that if e (in $g^{(1)}_{w,e}$) has a zero in some coordinate, then there is a diagonal subgroup of $B_{\ell,1}$, which commutes with $\begin{pmatrix} I_\ell & e & \\ & 1 & \\ & & I_r \end{pmatrix}$, and so, as in case $k > 1$, we can deduce an analog of (3.15), which is impossible for s outside a (larger) discrete set. Thus we may assume that $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, and then, it is easy to see that we may assume that w has the form $\begin{pmatrix} & w_{n-\ell} \\ w' & \end{pmatrix}$, where $w_{n-\ell} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ $((n-\ell) \times (n-\ell))$ and w' is in the Weyl group of $\mathrm{GL}_\ell(F)$. Thus the representative $g^{(1)}_{w,e}$ looks like

$$\begin{pmatrix} & w_{n-\ell} \\ w' & \end{pmatrix} \left(\begin{array}{c|c|c} I_\ell & \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} & \\ \hline & 1 & \\ \hline & & I_r \end{array}\right).$$

Note that for $m = \begin{pmatrix} a_1 & & & & 1-a_1 \\ & a_2 & & & 1-a_2 \\ & & \ddots & & \vdots \\ & & & a_{\ell-1} & 1-a_{\ell-1} \\ & & & & 1 \end{pmatrix}$, $\begin{pmatrix} m & \\ & I_{n-\ell} \end{pmatrix}$ commutes with

$\begin{pmatrix} I_\ell & \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} & \\ \hline & 1 & \\ \hline & & I_r \end{pmatrix}$. Thus, if w' takes one of the root subgroups

$$\begin{pmatrix}1 & & & *\\ & \ddots & & \\ & & & 1\end{pmatrix}, \begin{pmatrix}1 & & & \\ & 1 & & *\\ & & \ddots & \\ & & & 1\end{pmatrix}, \dots, \begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & *\\ & & & \ddots & \\ & & & & 1\end{pmatrix}, \dots, \begin{pmatrix}1 & & & \\ & \ddots & & \\ & & 1 & *\\ & & & 1\end{pmatrix},$$

to a "positive" root subgroup, then there is $j \leq \ell - 1$, such that

$$\begin{pmatrix}1 & & & & & 0\\ & \ddots & & & & \\ & & 1 & & & 0\\ & & & a_j & & 1-a_j\\ & & & & 1 & 0\\ & & & & \ddots & \vdots\\ & & & & & 1\end{pmatrix}^{w'} \in B_{\mathrm{GL}_\ell}(F) ,$$

and we get a condition of type (3.15), which cannot be satisfied for s outside a discrete set. If w' does not satisfy the above property, then it has the form $w' = \begin{pmatrix} & 1\\ w'' & \end{pmatrix}$, where w'' is in the Weyl group of $\mathrm{GL}_{\ell-1}(F)$. Continuing in the same manner, possibly enlarging the discrete set of values that s should avoid, we see that all orbits in $B_{\mathrm{GL}_n(F)}\backslash \mathrm{GL}_n(F)/B_{\ell,1}$ contribute the zero space of linear forms $\mathcal{E}'$ except the open orbit, with representative $g^{(1)}_{w_{n,e}}$, where $w_n = \begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, $e = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$; the stabilizer in $B_{\mathrm{GL}_n(F)} \times B_{\ell+1}$, in this case, is trivial (and $N' = 0$ since the orbit is open). Thus the open orbit contributes a space of dimension 1 for all s. This proves the uniqueness theorem for $\ell < n$.

THE CASE $\ell > n$. – Let

$$R = Y_{(n,\ell)} j_{n,\ell}(H_n) .$$

See (2.2). We have to show that

$$\dim Bil_R\Big(V_\pi, V_{\rho_{\tau,s}\cdot\psi_\alpha^{-1}}\Big) \leq 1$$

except, possibly, for a discrete set of values of s. By Frobenius reciprocity, we have

$$Bil_R\Big(V_\pi, V_{\rho_{\tau,s}\cdot\psi_\alpha^{-1}}\Big) \simeq Bil_{G_\ell}\Big(V_\pi, \mathrm{Ind}^c{}_R^{G_\ell}(\rho_{\tau,s}\cdot\psi_\alpha^{-1})\Big) .$$

Write π as a quotient of $\operatorname{Ind}_{P_\ell}^{G_\ell} \sigma$, where σ is a smooth, finitely generated representation, with a unique Whittaker model. Again, we may assume that $\pi = \operatorname{Ind}_{P_\ell}^{G_\ell} \sigma$. We are now in the framework of Bruhat Theory, and we leave the rest of this standard proof (similar to the previous case) to the reader. ∎

4. Asymptotic Expansions of Whittaker Functions

Our aim in this section is to obtain an "asymptotic" expression for Whittaker functions along the center of a Levi subgroup, in such a way that the terms of this expansion are products of polynomials, exponentials (which depend only on the representation) and of certain continuous functions, satisfying growth estimates which can be made better and better, according to the "degree" of the expansion. Before we give the precise statements, let us illustrate this with the low rank example of $\mathrm{GL}_3(\mathbb{R})$. Let $v \mapsto W_v(g)$ be the Whittaker model for a nice representation π of $\mathrm{GL}_3(\mathbb{R})$, which acts on the space V. We consider

$$W_v\begin{pmatrix} ab & & \\ & b & \\ & & 1 \end{pmatrix} \qquad \text{as} \qquad (a,b) \to (0,0) \ .$$

We will present this function, for $a = e^{-x}, b = e^{-t}$, as a sum of terms of the form

$$p(x,t)e^{cx+c't}f_0(v),\ p(x,t)e^{cx}f_1(t,v),\ p(x,t)e^{c't}f_1(x,v),\ p(x,t)f_2(x,t,v)\ ,$$

($i.e.$
$\widetilde{p}(\log a, \log b)a^{-c}b^{-c'}\widetilde{f}_0(v)$, $\widetilde{p}(\log a, \log b)a^{-c}\widetilde{f}_1(\log b, v)$, $\widetilde{p}(\log a, \log b)b^{-c'}\widetilde{f}_1(\log a, v)$, $\widetilde{p}(\log a, \log b)\widetilde{f}_2(\log a, \log b, v)$), where $p(x,t)$ are polynomials which belong to a certain finite set, independent of v, c and c' vary in the set of exponents of π, up to a given level k in the Jacquet module filtration. The functions f_i are linear in v and satisfy nice estimates of the form

$$\begin{aligned}
&|f_0(v)| \le q(v)\ ,\\
&|f_1(y,v)| \le \delta(y)e^{dy}q(v)\ ,\\
&|f_2(x,t,v)| \le \ell(x,t)e^{dx+d't}q(v)\ ,
\end{aligned}$$

where $\delta(y)$ and $\ell(x,t)$ are polynomials (independent of v), d and d' can be chosen "very negative" (k is chosen according to d and d') and $q(v)$ is a continuous seminorm on V. The advantage of such an expression is that it shows us the meromorphic continuation of an integral of the form

$$\int_0^1\int_0^1 W_v\begin{pmatrix} ab & & \\ & b & \\ & & 1 \end{pmatrix}|a|^{s_1}|b|^{s_2}\,da\,db$$

in larger and larger right half planes (in s_1 and in s_2), and moreover, by the estimates above, this continuation is continuous in v, with respect to the Fréchet topology. Let us begin with the preparation towards the precise formulation of our theorems.

Let g_ℓ be the Lie algebra of G_ℓ. Let π be a representation of G_ℓ on a Fréchet space $V = V_\pi$ which is smooth and of moderate growth. We assume that the subspace V_0 of K_{G_ℓ}-finite vectors of V is a Harish-Chandra (g_ℓ, K_{G_ℓ})-module. Assume that λ is a Whittaker functional on V, *i.e.* λ is a continuous linear functional which satisfies $\lambda(\pi(u)v) = \psi(u)\lambda(v)$, for $v \in V, u \in N_\ell$, and ψ is a nondegenerate character of N_ℓ. (We assume that ψ is the standard nondegenerate character of N_ℓ.)

Let

$$W_v(g) = \lambda(\pi(g)v) \ .$$

Our aim is to describe the asymptotic behaviour of $W_v(a)$, for $a = \widehat{b}_1 \cdot \widehat{b}_2 \cdot \cdots \cdot \widehat{b}_\nu$, where $b_j = \begin{pmatrix} a_j I_{m_j} & \\ & I_{\ell-m_j} \end{pmatrix}$, $1 \leq m_1 < m_2 < \cdots < m_\nu \leq \ell$, and $(a_1, \ldots, a_\nu) \to 0$. As we mentioned in Section 2, the methods and ideas of proof are in [W3, 15.2], where the case $\nu = 1$ is treated. What we will do is to keep a careful track of Wallach's derivation, so that we are able to do an inductive process in case $\nu > 1$.

Let P_m be the standard parabolic subgroup of G_ℓ, which preserves an m-dimensional isotropic subspace. Its Levi decomposition is $P_m = M_m \ltimes Y_m$,

$$M_m = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} \middle| \begin{matrix} g \in \mathrm{GL}_m(F) \\ h \in G_{\ell-m} \end{matrix} \right\}, \quad Y_m = \left\{ \begin{pmatrix} I_m & x & z \\ & I & x' \\ & & I_m \end{pmatrix} \in G_\ell \right\} .$$

Let $\mathcal{Y}_m$ denote the Lie algebra of Y_m in g_ℓ. Put

$$H_m = \begin{pmatrix} -I_m & & \\ & 0 & \\ & & I_m \end{pmatrix} \in g_\ell \ .$$

H_m is in the (one dimensional) center of Lie (M_m).

For an integer $k \geq 1$, consider $V/\mathcal{Y}_m^k V$, the $k+1$-th term in the Jacquet module filtration with respect to $\mathcal{Y}_m$. The space of K_{M_m}-finite vectors of $V/\mathcal{Y}_m^k V$ is an admissible, finitely generated (Lie (M_n), K_{M_m})-module, and as H_m is central, $V/\mathcal{Y}_m^k V$ is a direct sum of finitely many generalized eigenspaces for $\pi(H_m)$. *See* [W2, 4.3, 4.4], and note that the replacement of V_0 by V causes no harm. Let $E_k^{(m)}(V)$ be the finite set of eigenvalues of $\pi(H_m)$ on $V/\mathcal{Y}_m^k V$. For $\xi \in E_k^{(m)}(V)$, $\pi(H_m) - \xi I$ has a bounded degree of nilpotence on $V/\mathcal{Y}_m^k V$. As in [W2, 4.4.3],

$$\bigcup_{k=1}^{\infty} E_k^{(m)}(V) \subset \{\xi - n \mid \xi \in E_1^{(m)}(V),\ n = 0, 1, 2, \ldots\} \ .$$

Arrange the elements of $\bigcup_{k=1}^{\infty} E_k^{(m)}(V) = \{\xi_i^{(m)}\}_{i=1}^{\infty}$, so that $\mathrm{Re}\,\xi_1^{(m)} \geq \mathrm{Re}\,\xi_2^{(m)} \geq \cdots$. Let $1 \leq N_1^{(m)} < N_2^{(m)} < \ldots$ be the "jumps", *i.e.* $N_1^{(m)} + 1$ is the first index, such that $\mathrm{Re}\,\xi_{N_1^{(m)}+1}^{(m)} < \mathrm{Re}\,\xi_1^{(m)}$, $N_2^{(m)} + 1$ is the first such that $\mathrm{Re}\,\xi_{N_2^{(m)}+1}^{(m)} < \mathrm{Re}\,\xi_{N_1^{(m)}+1}^{(m)}$ etc.

Fix a norm $\| \quad \|$ on G_ℓ as in [W2, p.71]. In particular, we have $\|g\| \geq 1$, $\|hg\| \leq \|h\| \cdot \|g\|$ for $g, h \in G_\ell$, and for $g = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_\ell \end{pmatrix}^\wedge$, we may assume that $\|g\| = \max\{1, |x_i|, |x_i|^{-1}\}$, $1 \leq i \leq \ell$. By the continuity of the functional λ, there exist a continuous seminorm q on V and a positive constant μ, such that

$$|W_v(g)| \leq \|g\|^\mu q(v) \quad , \quad g \in G_\ell \ . \tag{4.1}$$

Put, for $t \in \mathbb{R}$,

$$a_t^{(m)} = \exp tH_m = \begin{pmatrix} e^{-t}I_m & & \\ & I_{2(\ell-m)+1} & \\ & & e^t I_m \end{pmatrix} .$$

From (4.1), it follows that for $t \geq 0$,

$$|W_v(ga_t^{(m)})| \leq \|g\|^\mu e^{\mu t} q(v) \ . \tag{4.2}$$

Assume that $1 \leq m_1 < m_2 < \cdots < m_\nu \leq \ell$. Let

$$d_j = \xi^{(m_j)}_{N^{(m_j)}_{i_j}}, \ j = 1, \ldots, \nu, \quad \text{and} \quad i_1, \ldots, i_\nu \geq 1 \ .$$

Choose k_j, such that

$$-k_j + \mu < \mathrm{Re}(d_j) - 1 \ . \tag{4.3}$$

Given a subset $J = \{\ell_1, \ldots, \ell_u\}$ of $\{1, \ldots, \nu\}$, such that $\ell_1 < \ell_2 < \cdots < \ell_u$, and a row vector $(y_1, \ldots, y_\nu)$, let $y_J = (y_{\ell_1}, \ldots, y_{\ell_u})$ and $\mathrm{Re}(y_J) = (\mathrm{Re}(y_{\ell_1}), \ldots, \mathrm{Re}(y_{\ell_u}))$. Given another vector $(z_1, \ldots, z_\nu)$, let $y_J \cdot z_J = \sum_{i=1}^n y_{\ell_i} z_{\ell_i}$. Also write $J' = \{1, 2, \ldots, \nu\} \backslash J$ (and order the elements of J').

For $x = (x_1, \ldots, x_\nu)$, $x_i \geq 0$, put

$$a_x = a_{x_1}^{(m_1)} a_{x_2}^{(m_2)} \cdot \cdots \cdot a_{x_\nu}^{(m_\nu)} \ .$$

If $m_\nu < \ell$, let b have the form $a_{z_1}^{(m_\nu+1)} a_{z_2}^{(m_\nu+2)} \cdot \cdots \cdot a_{z_{\ell-m_\nu}}^{(\ell)}$, with $z_i \in \mathbb{R}$. If $m_\nu = \ell$, set $b = I$. Now we are ready to state the main theorem of this section. We fix d_j and k_j, $j = 1, \ldots, \nu$, as above. Some of the objects mentioned in Theorem 1 depend on $(k_1, \ldots, k_\nu)$, but to lighten the notation, we do not mention this dependence.

THEOREM 1. – *There are finite subsets $C_j \subset \bigcup_{r=1}^{k_j} E_r^{(m_j)}(V)$, $j = 1, \ldots, \nu$, a finite set of polynomials $\mathcal{P} \subset \mathbb{C}[x_1, \ldots, x_\nu]$, and for each subset J of $\{1, \ldots, \nu\}$, there is a finite set Ω_J of functions $f(x_{J'}, b, v)$ on $\mathbb{R}_+^{J'} \times \mathbb{R}^{\ell-m_\nu} \times V$, continuous in $(x_{J'}, b)$ and linear on V, such that for all $v \in V$, $W_v(ba_x)$ has the form*

$$W_v(ba_x) = \sum p(x_1, \ldots, x_\nu) e^{c_J \cdot x_J} f(x_{J'}, b, v) \ , \tag{4.4}$$

where the $p(x_1,\ldots,x_\nu)$ *are taken from* $\mathcal{P}$, J *varies over subsets of* $\{1,\ldots,\nu\}$ *as above,* $c_J \in \prod_{j\in J} C_j$ *and for* $f(x_{J'},b,v)$, *there exist a polynomial* $\delta(x_{J'})$ *(independent of* (b,v)*) and a continuous seminorm* $\widetilde{q}$ *on* V, *such that*

$$|f(x_{J'},b,v)| \le \delta(x_{J'})e^{x_{J'}\cdot \mathrm{Re}(d_{J'})}\|b\|^{\mu}\widetilde{q}(v)\ . \tag{4.5}$$

We prove Theorem 1 by induction on ν. Case $\nu = 1$ and the induction step are provided by the following theorem. Put (just for Theorem 2), $m = m_\nu,\ t = x_\nu,\ a_t = a_{x_\nu}^{(m_\nu)},\ d = d_\nu,\ k = k_\nu,\ a' = a_{x_1}^{(m_1)}\cdot\cdots\cdot a_{x_{\nu-1}}^{(m_{\nu-1})}$.

THEOREM 2. – *There exist a finite set* $\mathcal{E} \subset \mathcal{U}((g_\ell)_{\mathbb{C}})$, *a finite subset* $C \subset \bigcup_{r=1}^{k} E_r^{(m)}(V)$, *a finite set* S *of polynomials in* $\mathbb{C}[t]$ *and a finite set* L *of nonnegative integers, such that for all* v, $W_v(ba'a_t)$ *has the form*

$$\begin{aligned} W_v(ba'a_t) &= \sum e^{ct}p(t)W_{\pi(E)v}(ba') + \sum_{\mathrm{Re}(c)\ge \mathrm{Re}(d)} e^{ct}h(t)\varphi_{c,\alpha,D}(b,a',v) \\ &\quad + \sum e^{ct}r(t)\phi_{c,\alpha,D}(t,b,a',v)\ . \end{aligned} \tag{4.6}$$

Here $c \in C$, $p(t),\ h(t),\ r(t)$ *are taken from* S; $E, D \in \mathcal{E}$, $\alpha \in L$. *We have*

$$\begin{aligned} \varphi_{c,\alpha,D}(b,a',v) &:= \int_0^\infty e^{-\tau(k+c)}\tau^\alpha W_{\pi(D)v}(ba_\tau a')d\tau \qquad (\mathrm{Re}(c)\ge \mathrm{Re}(d)) \\ \phi_{c,\alpha,D}(t,b,a',v) &= \int_t^\infty e^{-\tau(k+c)}\tau^\alpha W_{\pi(D)v}(ba_\tau a')d\tau \qquad (\mathrm{Re}(c)\ge \mathrm{Re}(d)) \\ \phi_{c,\alpha,D}(t,b,a',v) &= \int_0^t e^{-\tau(k+c)}\tau^\alpha W_{\pi(D)v}(ba_\tau a')d\tau \qquad (Re(c) < Re(d))\ . \end{aligned} \tag{4.7}$$

These three functions are continuous in (t,b,a'), *linear in* v *and satisfy*

$$\begin{aligned} &|\varphi_{c,\alpha,D}(b,a'v)| \le c_\alpha \|ba'\|^{\mu} q(\pi(D)v)\ , \\ &|\phi_{c,\alpha,D}(t,b,a',v)| \le e^{-\mathrm{Re}(c)t+dt}\|ba'\|^{\mu}\delta_{c,\alpha}(t)q(\pi(D)v)\ , \end{aligned} \tag{4.8}$$

where c_α *is a constant,* $\delta_{c,\alpha}(t)$ *is a polynomial and* q *is the continuous seminorm in (4.2).*

The importance of Theorem 2 is that in (4.6), we achieve a separation of t from ba' in the first two terms, while for the third term, we have the estimate (4.8).

Let us go back to the example of $\mathrm{GL}_3(\mathbb{R})$ and explain how to use Theorem 2 in order to obtain the required expression for $W_v\begin{pmatrix} e^{-(x+t)} & & \\ & e^{-t} & \\ & & 1\end{pmatrix}$, for $x,t \ge 0$. By (4.6), we have

$$\begin{aligned} W_v\begin{pmatrix} e^{-(x+t)} & & \\ & e^{-t} & \\ & & 1\end{pmatrix} &= \sum e^{ct}p(t)W_{\pi(E)v}\begin{pmatrix} e^{-x} & & \\ & 1 & \\ & & 1\end{pmatrix} \\ &\quad + \sum_{Re(c)\ge Re(d)} e^{ct}h(t)\varphi_{c,\alpha,D}(x,v) + \sum e^{ct}r(t)\phi_{c,\alpha,D}(t,x,v)\ . \end{aligned} \tag{4.6$'$}$$

We took $b = I$ and $a' = \begin{pmatrix} e^{-x} & & \\ & 1 & \\ & & 1 \end{pmatrix}$ in (4.6). Now we use "induction", that is we apply Theorem 2 again to $W_{\pi(E)v}\begin{pmatrix} e^{-x} & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $\varphi_{c,\alpha,D}(x,v)$ and $\phi_{c,\alpha,D}(t,x,v)$. The application of (4.6) to $W_{\pi(E)v}\begin{pmatrix} e^{-x} & & \\ & 1 & \\ & & 1 \end{pmatrix}$ is direct and clear. In order to apply Theorem 2 to $\varphi_{c,\alpha,D}(x,v)$ and $\phi_{c,\alpha,D}(t,x,v)$, we have to use their explicit form, given in (4.7), and the estimates (4.8). For example,

$$(4.7)' \qquad \varphi_{c,\alpha,D}(x,v) = \int_0^\infty e^{-\tau(k+c)}\tau^\alpha W_{\pi(D)v}\begin{pmatrix} e^{-(\tau+x)} & & \\ & e^{-\tau} & \\ & & 1 \end{pmatrix} d\tau \ .$$

Here we use Theorem 2, with $b = \begin{pmatrix} e^{-\tau} & & \\ & e^{-\tau} & \\ & & 1 \end{pmatrix}$ and $a' = I$, to express $W_{\pi(D)v}\begin{pmatrix} e^{-(\tau+x)} & & \\ & e^{-\tau} & \\ & & 1 \end{pmatrix}$ as a sum of terms of the form

$$e^{c'x}p'(x)W_{\pi(E')v}\begin{pmatrix} e^{-\tau} & & \\ & e^{-\tau} & \\ & & 1 \end{pmatrix}, \quad e^{c'x}h'(x)\varphi_{c',\alpha',D'}(\tau,v), \quad e^{c'x}r'(x)\phi_{c',\alpha',D'}(x,\tau,v).$$

When we substitute these elements in (4.7)' and then examine their contribution to (4.6)' (the second term), we obtain sums of elements of the following three types

$$(i) \qquad h(t)p'(x)e^{ct+c'x}\int_0^\infty e^{-\tau(k+c)}\tau^\alpha W_{\pi(E')v}\begin{pmatrix} e^{-\tau} & & \\ & e^{-\tau} & \\ & & 1 \end{pmatrix} d\tau \ ,$$

$$(ii) \qquad h(t)h'(x)e^{ct+c'x}\int_0^\infty e^{-\tau(k+c)}\tau^\alpha \varphi_{c',\alpha',D'}(\tau,v)d\tau \ ,$$

$$(iii) \qquad h(t)r'(x)e^{ct+c'x}\int_0^\infty e^{-\tau(k+c)}\tau^\alpha \phi_{c',\alpha',D'}(x,\tau,v)d\tau \ .$$

Note again that in (i) and (ii), we have elements of the form

$$p(x,t)e^{ct+c'x}f_0(v) \ ,$$

where the variables (t, x) are separated from v, and $f_0(v)$ satisfies the estimate (4.8). In the element of type (iii), t is separated from (x, v), so we view (iii) as follows,

$$p(x,t)e^{ct}f_1(x,v) \ ,$$

where

$$f_1(x,v) = e^{c'x}\int_0^\infty e^{-\tau(k+c)}\tau^\alpha \phi_{c',\alpha',D'}(x,\tau,v)d\tau \ .$$

The estimate (4.8) implies

$$|f_1(x,v)| \le e^{Re(c')x}\int_0^\infty e^{-\tau(k+Re(c))}\tau^\alpha e^{(-Re(c')+Re(d'))x}e^{\mu\tau}\delta_{c'\alpha'}(x)q(\pi(D')v)d\tau \ .$$

Here d' is the "d" in (4.8), which corresponds to c' and $\begin{pmatrix} e^{-x} & & \\ & 1 & \\ & & 1 \end{pmatrix}$. *See* (4.3). We get

$$|f_1(x,v)| \le \delta_{c'\alpha'}(x)e^{Re(d')x}\left(\int_0^\infty e^{-\tau(k+Re(c)-\mu)}\tau^\alpha d\tau\right)\widetilde{q}(v) \ .$$

The last integral is bounded by $c_\alpha = \int_0^\infty e^{-\tau}\tau^\alpha d\tau$, since $Re(c) \ge Re(d) > -k+\mu+1$. Similar arguments, though a little more complicated, apply for the third term of (4.6)'. Thus the rough idea is to "keep separating" the variables $x_\nu, x_{\nu-1}, \dots$ in $W_v(a_x)$, "as much as possible", so that when we reach a place where we cannot separate variables "any more", then we at least have a nice estimate of type (4.5). Let us now proceed to the proof of Theorem 2 (the induction step).

Proof. – The proof is essentially a repetition of [W3, 15.2.4], and keeps a careful track of the form of the coefficients and of the difference in the asymptotic expansion. Also *see* [W1,72.].

In [W3, 15.2.4], Wallach constructs a set $\{E_i\}_{i=1}^T \subset \mathcal{U}((g_\ell)_{\mathbb{C}})$, $E_1 = I$, and for each $1 \le i \le T$, a set $\{D_{r,i}\} \subset \mathcal{U}((g_\ell)_{\mathbb{C}})$, where r indexes the basis of monomials $\{\overline{Y}_r\}$ of $\mathcal{Y}_m^k$, in the standard basis elements of $\mathcal{Y}_m$. For these sets, there is a $T \times T$ matrix $B = B^{(m)} = (b_{ij})$, such that for all $v \in V$, $1 \le i \le T$,

$$\pi(H_m)\pi(E_i)v = \sum_{j=1}^T b_{ij}\pi(E_j)v + \sum_r \pi(\overline{Y}_r)\pi(D_{r,i})v \ . \tag{4.9}$$

Fortunately, we do not have to know more about (4.9) than what we have already said. This equation is the only nontrivial fact needed from [W3] for the proof. The rest (follows Wallach) is given in full detail and is completely elementary. We see from (4.9) that the projection of $\mathrm{Span}\{\pi(E_i)v\}_{i=1}^T$ in $V/\mathcal{Y}_m^k V$ is a finite dimensional space, stable by $\pi(H_m)$, which acts on the spanning set according to the matrix B (modulo $\mathcal{Y}_m^k$). Put

$$F(t,b,a',v) = \begin{pmatrix} W_{\pi(E_1)v}(ba'a_t) \\ \vdots \\ W_{\pi(E_T)v}(ba'a_t) \end{pmatrix} \quad , \quad G(t,b,a',v) = \sum_r \begin{pmatrix} W_{\pi(\overline{Y}_r)\pi(D_{r,1})v}(ba'a_t) \\ \vdots \\ W_{\pi(\overline{Y}_r)\pi(D_{r,T})v}(ba'a_t) \end{pmatrix} .$$

Then, by (4.9),

$$\frac{d}{dt}F(t,b,a',v)=B\cdot F(t,b,a',v)+G(t,b,a',v)\ , \tag{4.10}$$

and the solution of (4.10) is (see the appendix to Section 8 in [C])

$$F(t,b,a',v)=e^{tB}F(0,b,a',v)+e^{tB}\int_0^t e^{-\tau B}G(\tau,b,a',v)d\tau\ . \tag{4.11}$$

Let C be the set of eigenvalues of B on $\mathbb{C}^T$. For $c\in C$, let P_c be the projection of $\mathbb{C}^T$ on the c-generalized eigenspace. Put $Q=\sum_{\mathrm{Re}(c)<\mathrm{Re}(d)}P_c \quad , \quad R=\sum_{\mathrm{Re}(c)\geq\mathrm{Re}(d)}P_c$, $(Q+R=I)$. Now rewrite (4.11) in the form

(4.12)
$$\begin{aligned}F(t,b,a',v)=e^{tB}F(0,b,a',v)&+R(e^{tB}\int_0^\infty e^{-\tau B}G(\tau,b,a',v)d\tau)\\ &-R(e^{tB}\int_t^\infty e^{-\tau B}G(\tau,b,a',v)d\tau)+Q(e^{tB}\int_0^t e^{-\tau B}G(\tau,b,a',v)d\tau)\ .\end{aligned}$$

The integrals in the middle two terms of (4.12) are absolutely convergent, as we shall now show. First, note that for a monomial $\overline{Y}_r$,

$$W_{\pi(\overline{Y}_r)\pi(D_{r,i})v}(ba'a_\tau)=\begin{cases}e^{-k\tau}W_{\pi(D_{r,i})v}(ba'a_\tau)\\ 0\end{cases}\ .$$

The first case occurs if and only if $\overline{Y}_r=X_m^k$, where $X_m=\left(\begin{array}{c|c|c}0_m & 1 & \\ \hline & 0 & -1\\ \hline & & 0_m\end{array}\right)$, the standard basis vector of the root subspace of $\mathcal{Y}_m$, which corresponds to the simple root defining P_m. Put $D'_{r,i}=D_{r,i}$ in the first case and $D'_{r,i}=0$ in the second case. Thus

$$G(\tau,b,a',v)=e^{-k\tau}L(\tau,b,a',v)\ ,$$

where

$$L(\tau,b,a',v)=\sum_r\begin{pmatrix}W_{\pi(D'_{r,1})v}(ba'a_\tau)\\ \vdots\\ W_{\pi(D'_{r,T})v}(ba'a_\tau)\end{pmatrix}\ .$$

To show the absolute convergence mentioned above, it is sufficient to consider $P_c\big(e^{tB}\int_0^\infty e^{-\tau b}G(\tau,b,a',v)d\tau\big)$, for $\mathrm{Re}(c)\geq\mathrm{Re}(d)$. We have

$$\begin{aligned}||P_c\Big(e^{(t-\tau)B}G(\tau,b,a',v)\Big)||&=e^{-k\tau}||P_c\Big(e^{(t-\tau)B}L(\tau,b,a',v)\Big)||\\ &\leq e^{-k\tau}f_c(t-\tau)e^{(t-\tau)\,\mathrm{Re}(c)}||L(\tau,b,a',v)||\ .\end{aligned}$$

Here f_c is some polynomial which depends on c and on B. By (4.2), there is a continuous seminorm q_1 on V, such that

$$\|L(\tau, b, a', v)\| \leq e^{\mu\tau}\|ba'\|^{\mu} q_1(v) .$$

Thus

$$\begin{aligned}\|P_c\Big(e^{(t-\tau)B}G(\tau, b, a', v)\Big)\| &\leq f_c(t-\tau)e^{t\,\mathrm{Re}(c)} \cdot e^{(-k-\mathrm{Re}(c)+\mu)\tau}\|ba'\|^{\mu} q_1(v) \\ &\leq f_c(t-\tau)e^{t\,\mathrm{Re}(c)}e^{-\tau}\|ba'\|^{\mu} q_1(v) .\end{aligned} \tag{4.13}$$

We used (4.3) and the fact that $\mathrm{Re}(c) \geq \mathrm{Re}(d)$. This implies the required absolute convergence.

We will obtain the expansion (4.6) by equating the first coordinates of both sides of (4.12). Recall that the first coordinate of $F(t, b, a', v)$ is $W_v(ba'a_t)$. The first coordinate of the first term in the r.h.s. of (4.12) has the form

$$\sum_{i=1}^{T} \varphi_i(t) W_{\pi(E_i)v}(ba') ,$$

where

$$\varphi_i(t) = \sum_{c \in C} e^{ct} P_{i,c}(t)$$

and $P_{i,c}(t)$ are certain polynomials. The first coordinate of the second term in the r.h.s. of (4.12) has the form

$$\sum_{i=1}^{T}\sum_{r}\int_0^{\infty} \eta_i(t-\tau)e^{-k\tau}W_{\pi(D'_{r,i})v}(ba'a_\tau)d\tau , \tag{4.14}$$

where

$$\eta_i(x) = \sum_{\mathrm{Re}(c) \geq \mathrm{Re}(d)} e^{cx} h_{i,c}(x)$$

and $h_{i,c}(x)$ are certain polynomials. Each term in (4.14) is a sum of terms of the form

$$e^{ct}h(t)\int_0^{\infty} e^{-\tau(k+c)}\tau^{\alpha}W_{\pi(D)v}(ba'a\tau)d\tau \equiv e^{ct}h(t)\varphi_{c,\alpha D}(b, a', v) ,$$

where $\mathrm{Re}(c) \geq \mathrm{Re}(d)$, $h(t)$ belongs to a certain finite set of polynomials, α belongs to a certain finite set of nonnegative integers, and D is of the form $D'_{r,i}$. As in (4.13), we have

$$|\varphi_{c,\alpha,D}(b, a', v)| \leq \|ba'\|^{\mu}\left(\int_0^{\infty} e^{-\tau}\tau^{\alpha}d\tau\right)q(\pi(D)v) = c_{\alpha}\|ba'\|^{\mu}q(\pi(D)v) .$$

In a similar way, the first coordinate of the third term in the r.h.s. of (4.12) is a sum of terms of the form

$$e^{ct}r(t)\int_t^{\infty} e^{-\tau(k+c)}\tau^{\alpha}W_{\pi(D)v}(ba'a_\tau)d\tau \equiv e^{ct}r(t)\phi_{c,\alpha,D}(t, b, a', v) ,$$

where $\mathrm{Re}(c) \geq \mathrm{Re}(d)$, $r(t)$ belongs to a certain finite set of polynomials, α belongs to a certain finite set of nonnegative integers and D is of the form $D'_{r,i}$. We have

$$|\phi_{c,\alpha,D}(t,b,a',v)| \leq e^{-\mathrm{Re}(c)t+t(-k+\mu)}\|ba'\|^{\mu}\delta_{c,\alpha}(t)q(\pi(D)v) ,$$

where $\delta_{c,\alpha}(t)$ is a certain polynomial. Finally, the first coordinate of the fourth term in the r.h.s. of (4.12) is a sum of terms of the form

$$e^{ct}r(t)\int_0^t e^{-\tau(k+c)}\tau^{\alpha}W_{\pi(D)v}(ba'a_{\tau})d\tau \equiv e^{ct}r(t)\phi_{c,\alpha,D}(t,b,a',v) ,$$

where $\mathrm{Re}(c) < \mathrm{Re}(d)$, $r(t)$ belongs to a certain finite set of polynomials, α belongs to a certain finite set of nonnegative integers and D is of the form $D'_{r,i}$. We have, in this case,

$$|\phi_{c,\alpha,D}(t,b,a',v)| \leq e^{-\mathrm{Re}(c)t+dt}\|ba'\|^{\mu}\delta_{c,\alpha}(t)q(\pi(D)v) ,$$

where $\delta_{c,\alpha}(t)$ is the polynomial $\frac{t^{\alpha+1}}{\alpha+1}$. The continuity in (t,b,a') of the functions in (4.7) is clear. This completes the proof of Theorem 2. ■

Proof of Theorem 1. – Case $\nu = 1$ is already proved in Theorem 2. In this case (4.6) reads

$$(4.15)\quad \begin{aligned} W_v(ba_t) &= \sum e^{ct}p(t)W_{\pi(E)v}(b) + \sum_{\mathrm{Re}(c)\geq\mathrm{Re}(d)} e^{ct}h(t)\varphi_{c,\alpha,D}(b,I,v) \\ &\quad + \sum e^{ct}r(t)\phi_{c,\alpha,D}(t,b,I,v) . \end{aligned}$$

The first two terms of (4.15) have summands which correspond to the subset $J = \{1\}$ of $\{1\}$ (and $J' = \phi$) in (4.4), so that $f(x_{J'},b,v)$ is either of the form $W_{\pi(E)v}(b)$ or $\varphi_{c,\alpha,D}(b,I,v)$, both of which define continuous linear functions on V, which satisfy (4.5) (as follows from (4.2) and (4.8)). The third term of (4.15) has summands which correspond to the subset $J = \phi$ of $\{1\}$ (and $J' = \{1\}$), taking in this case $f(x_{J'},b,v) = e^{ct}\phi_{c,\alpha D}(t,b,I,v)$ in (4.4). By (4.8), we have

$$\begin{aligned} |f(x_{J'},b,v)| &\leq e^{t(-k+\mu)}\|b\|^{\mu}\delta_{c,\alpha}(t)q(\pi(D)v) \\ &\leq e^{t\,\mathrm{Re}(d)}\|b\|^{\mu}\delta_{c,\alpha}(t)q(\pi(D)v) . \end{aligned}$$

This shows (4.5).

Now use induction on ν. We consider each term of (4.6). In the first term of (4.6), using induction on ν, we see that $W_{\pi(E)v}(ba')$ is of the form (4.4),

$$W_{\pi(E)v}(ba') = \sum \widetilde{p}(x_1,\ldots,x_{\nu-1})e^{c_J\cdot x_J}\widetilde{f}(x_{J'},b,\pi(E)v) ,$$

where J varies over subsets of $\{1,\ldots,\nu-1\}$. Substituting this in the first term of (4.6) (putting back $t = x_\nu$, $c = c_\nu$ etc.), gives a sum of the form

$$\sum p(x_1,\ldots,x_\nu)e^{c_J\cdot x_J+c_\nu x_\nu}f(x_{J'},b,v) ,$$

which corresponds to subsets of $\{1,\ldots,\nu\}$, which contain ν. Here $f(x_{J'},b,v) = \widetilde{f}(x_{J'},b,\pi(E)v)$.

The properties (4.5) are satisfied by the induction hypothesis. For the second term of (4.6), using (4.7) and the induction hypothesis (now replacing b by ba_τ), we have

$$\varphi_{c_\nu,\alpha,D}(b,a',v) = \sum \int_0^\infty e^{-\tau(k_\nu+c_\nu)}\tau^\alpha \widetilde{p}(x_1,\ldots,x_{\nu-1})e^{c_J\cdot x_J}\widetilde{f}(x_{J'},ba_\tau,\pi(D)v)d\tau \ ,$$

where J varies over subsets of $\{1,\ldots,\nu-1\}$. Thus the second term of (4.6) has the form

$$\sum p(x_1,\ldots,x_\nu)e^{c_J\cdot x_J+c_\nu x_\nu} f(x_{J'},b,v)$$

which corresponds to subsets of $\{1,\ldots,\nu\}$, which contain ν. Here

$$f(x_{J'},b,v) = \int_0^\infty e^{-\tau(k_\nu+c_\nu)}\tau^\alpha \widetilde{f}(x_{J'},ba_\tau,\pi(D)v)d\tau \ .$$

We have, by the induction hypothesis (4.5) and by (4.3) (recall that in this case $\mathrm{Re}(c_\nu) \geq \mathrm{Re}(d_\nu)$),

$$\begin{aligned} |f(x_{J'},b,v)| &\leq \left(\int_0^\infty e^{\tau(-k_\nu-\mathrm{Re}(c_\nu)+\mu)}\tau^\alpha d\tau\right)\delta(x_{J'})e^{x_{J'}\cdot\mathrm{Re}(d_{J'})}\|b\|^\mu\widetilde{q}(\pi(D)v) \\ &\leq \left(\int_0^\infty e^{-\tau}\tau^\alpha d\tau\right)\delta(x_{J'})e^{x_{J'}\cdot\mathrm{Re}(d_{J'})}\|b\|^\mu\widetilde{q}(\pi(D)v) \\ &= c_\alpha\delta(x_{J'})e^{x_{J'}\cdot\mathrm{Re}(d_{J'})}\|b\|^\mu\widetilde{q}(\pi(D)v) \ . \end{aligned}$$

This shows that (4.5) is satisfied for $f(x_{J'},b,v)$. (We use the notation of Theorem 1.) This also implies the continuity of f in $(x_{J'},b)$. Now consider the third term of (4.6). Again, using (4.7) and the induction hypothesis (replacing b by ba_τ), we have for $\mathrm{Re}(c_\nu) \geq \mathrm{Re}(d_\nu)$,

$$\phi_{c_\nu,\alpha,D}(x_\nu,b,a',v) = \sum \int_{x_\nu}^\infty e^{-\tau(k_\nu+c_\nu)}\tau^\alpha \widetilde{p}(x_1,\ldots,x_{\nu-1})e^{c_J\cdot x_J}\widetilde{f}(x_{J'},ba_\tau,\ \pi(D)v)d\tau \ ,$$

where J varies over subsets of $\{1,\ldots,\nu-1\}$. The corresponding sum in (4.6) has the form

$$\sum_{\mathrm{Re}(c_\nu)\geq\mathrm{Re}(d_\nu)} p(x_1,\ldots,x_\nu)e^{c_J\cdot x_J} f((x_{J'},x_\nu)b,v) \ ,$$

which corresponds to subsets J of $\{1,\ldots,\nu\}$, which do not contain ν. (J' denotes the complement of J in $\{1,\ldots,\nu-1\}$. $J'\cup\{\nu\}$ is the complement of J in $\{1,\ldots,\nu\}$.) Here

$$f((x_{J'},x_\nu),b,v) = e^{c_\nu x_\nu}\int_{x_\nu}^\infty e^{-\tau(k_\nu+c_\nu)}\tau^\alpha \widetilde{f}(x_{J'},ba_\tau,\pi(D)v)d\tau \ .$$

We have, by the induction hypothesis (4.5), (4.3) (and $\mathrm{Re}(c_\nu) \geq \mathrm{Re}(d_\nu)$),

$$
\begin{aligned}
|f((x_{J'}, x_\nu), b, v)| &\leq \left(\int_{x_\nu}^{\infty} e^{\tau(-k_\nu+\mu)} \cdot e^{-\mathrm{Re}(c_\nu)(\tau - x_\nu)} \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'} \cdot \mathrm{Re}(d_{J'})} \|b\|^\mu \widetilde{q}(\pi(D)v) \\
&\leq \left(\int_{x_\nu}^{\infty} e^{\tau(-k_\nu - \mathrm{Re}(d_\nu)+\mu)} \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'}\,\mathrm{Re}(d_{J'}) + x_\nu\,\mathrm{Re}(d_\nu)} \|b\|^\mu \widetilde{q}(\pi(D)v) \\
&\leq \left(\int_{x_\nu}^{\infty} e^{-\tau} \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'} \cdot \mathrm{Re}(d_{J'}) + x_\nu\,\mathrm{Re}(d_\nu)} \|b\|^\mu \widetilde{q}(\pi(D)v) \\
&= \delta(x_{J'}, x_\nu) e^{x_{J'} \cdot \mathrm{Re}(d_{J'}) + x_\nu\,\mathrm{Re}(d_\nu)} \|b\|^\mu \widetilde{q}(\pi(D)v) \ .
\end{aligned}
$$

$\delta(x_{J'}, x_\nu)$ is a polynomial. This shows (4.5) in this case and also the continuity of f in $(x_{J'}, x_\nu, b)$. Finally, using (4.7) and the induction hypothesis (replacing b by ba_τ), we have, for $\mathrm{Re}(c_\nu) < \mathrm{Re}(d_\nu)$,

$$
\phi_{c_\nu \alpha, D}(x_\nu, b, a', v) = \sum \int_0^{x_\nu} e^{-\tau(k_\nu + c_\nu)} \tau^\alpha \widetilde{p}(x_1, \ldots, x_{\nu-1}) e^{c_J \cdot x_J} \widetilde{f}(x_{J'}, ba_\tau, \pi(D)v) d\tau \ ,
$$

where J varies over subsets of $\{1, \ldots, \nu-1\}$. The corresponding sum in (4.6) has the form

$$
\sum_{\mathrm{Re}(c_\nu) < \mathrm{Re}(d_\nu)} p(x_1, \ldots, x_\nu) e^{c_J \cdot x_J} f((x_{J'}, x_\nu), b, v) \ ,
$$

which corresponds to subsets J of $\{1, \ldots, \nu\}$, which do not contain ν. Here

$$
f((x_{J'}, x_\nu), b, v) = e^{c_\nu x_\nu} \int_0^{x_\nu} e^{-\tau(k_\nu + c_\nu)} \tau^\alpha \widetilde{f}(x_{J'}, ba_\tau, \pi(D)v) d\tau \ .
$$

We have, by the induction hypothesis (4.5), (4.3) (and $\mathrm{Re}(c_\nu) < \mathrm{Re}(d_\nu)$),

$$
\begin{aligned}
|f(x_{J'}, x_\nu), b, v)| &\leq \left(\int_0^{x_\nu} e^{\tau(-k_\nu+\mu)} \cdot e^{\mathrm{Re}(c_\nu)(x_\nu - \tau)} \cdot \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'} \cdot \mathrm{Re}(d_{J'})} \|b\|^\mu \widetilde{q}(\pi(D)v) \\
&\leq \left(\int_0^{x_\nu} e^{\tau(-k_\nu - \mathrm{Re}(d_\nu)+\mu)} \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'} \cdot \mathrm{Re}(d_{J'}) + x_\nu\,\mathrm{Re}(d_\nu)} \|b\|^\mu \widetilde{q}(\pi(D)v) \\
&\leq \left(\int_0^{x_\nu} e^{-\tau} \tau^\alpha d\tau \right) \delta(x_{J'}) e^{x_{J'} \cdot \mathrm{Re}(d_{J'}) + x_\nu\,\mathrm{Re}(d_\nu)} \|b\|^\mu \widetilde{q}(\pi(D)v) \ .
\end{aligned}
$$

As in the last case, this proves (4.5) and the continuity in $(x_{J'}, x_\nu, b)$. The proof of Theorem 1 is complete. ■

DEPENDENCE ON PARAMETERS. – We review and modify the arguments of Theorems 1,2 in order to see the dependence of the terms in (4.4) and (4.6) on parameters. For example, in the case of GL(3), we want to deduce that the meromorphic continuation in (s_1, s_2) of the integral

$$\int_0^1 \int_0^1 W_v \begin{pmatrix} ab & & \\ & b & \\ & & 1 \end{pmatrix} |a|^{s_1} |b|^{s_2} da db$$

is also analytic in the parameters of the representation π. Our treatment here is just to penetrate deeper into proofs of the previous two theorems.

We assume that our representation is induced from a lower parabolic subgroup P of G_ℓ which has Levi part $\mathrm{GL}_{t_1}(F) \times \cdots \times \mathrm{GL}_{t_e}(F) \times G_{\ell-(t_1+\cdots+t_e)}$ and a generic representation $\sigma_1 \cdot |\det \cdot|^{z_1} \otimes \cdots \otimes$
$\sigma_e |\det \cdot|^{z_e} \otimes \sigma'$. Put $z = (z_1, \ldots, z_e)$ and denote the representation by π_z. Denote by V_z the space of the induced representation and by V the space of restrictions of the functions in V_z to K_{G_ℓ}. We realize π_z in the space V. Let $\lambda_1, \ldots, \lambda_e, \lambda'$ be Whittaker functionals for $\sigma_1, \ldots, \sigma_e, \sigma'$. For $v \in V$, there is a unique $v_z \in V_z$ such that $v_z\big|_{K_{G_\ell}} = v$. Put

$$W_z(v) = \int_U (\lambda_1 \otimes \cdots \otimes \lambda_e \otimes \lambda')(v_z(u))\Theta^{-1}(u) du$$

Here U is the radical of the opposite to P and Θ is the restriction of a nondegenerate character of N_ℓ to U. The integral converges absolutely if $(\mathrm{Re}(z_1 - z_2), \ldots, \mathrm{Re}(z_{e-1} - z_e), \mathrm{Re}(z_e))$ is in an appropriate translation of the negative quadrant of $\mathbb{R}^e$ and it has a holomorphic extension which defines a Whittaker functional for π_z. We assume, for simplicity, that $W_z(v)$ is a Whittaker functional with respect to the standard nondegenerate character of G_ℓ. We shall denote for $g \in G_\ell$, $v \in V$ and $E \in \mathcal{U}((g_\ell)_{\mathbb{C}})$,

$$W_{v,z}(g) = W_z(\pi_z(g)v) \ ,$$
$$W_{\pi_z(E)v}(g) = W_z(\pi_z(g)\pi_z(E)v) \ .$$

Given a compact subset Ω in $\mathbb{C}^e$, there are a continuous seminorm q on V and a positive number μ, such that

$$|W_{v,z}(g)| \leq \|g\|^\mu q(v) \ , \tag{4.16}$$

for all $g \in G_\ell$, $v \in V$, $z \in \Omega$. In particular, we have

$$|W_{v,z}(g a_t^{(m)})| \leq \|g\|^\mu e^{\mu t} q(v) \ , \tag{4.17}$$

for all $g \in G_\ell$, $t \geq 0$, $m \leq \ell$, $v \in V$, $z \in \Omega$. Inequalities (4.16), (4.17) replace (4.1) and (4.2). We shall assume that Ω is connected with a nonempty interior. Let $1 \leq m \leq \ell$ and $k \geq 1$ be integers. Denote by $E_{k,z}^{(m)}(V)$ the finite set of eigenvalues of $\pi_z(H_m)$ on the space $V/\mathcal{Y}_m^k V$. Recall that an element of $E_{k,z}^{(m)}(V)$ is of the form $c(z) - n$ where

$c(z) \in E^{(m)}_{1,z}(V)$ and n is a nonnegative integer. By [W3, 12.4.7], $E^{(m)}_{1,z}(V)$ consists of linear functions in $z_1, \ldots, z_e$ (this includes a translation). We now modify the argument of Theorem 2. Choose d, a negative number, as large in absolute value as we may want. Choose k, a positive integer, such that

$$-k + \mu < d - 1 \ . \tag{4.18}$$

With these μ, k and d, let us use the same notation as in the proof of Theorem 2. We note that the sets of elements $\{E_i\}_{i=1}^T$ and $\{D_{r,i}\}_r$ in $\mathcal{U}((\mathfrak{g}_\ell)_{\mathbb{C}})$ do not depend on the representation. This is clear from [W3, 15.2.4]. Let

$$F_z(t,b,a',v) = \begin{pmatrix} W_{\pi_z(E_1)v}(ba'a_t) \\ \vdots \\ W_{\pi_z(E_T)v}(ba'a_t) \end{pmatrix}, \quad G_z(t,b,a',v) = \sum_r \begin{pmatrix} W_{\pi_z(\overline{Y}_r)\pi_z(D_{r,1})v}(ba'a_t) \\ \vdots \\ W_{\pi_z(\overline{Y}_r)\pi_z(D_{r,T})v}(ba'a_t) \end{pmatrix}.$$

We have

$$\pi_z(H_m)\pi_z(E_i)v = \sum_{j=1}^{T} b_{ij}(z)\pi_z(E_j)v + \sum_r \pi_z(\overline{Y}_r)\pi_z(D_{r,i})v \ . \tag{4.19}$$

By [W3, 12.4.7, 15.2.4], $b_{ij}(z)$ are rational in z. The eigenvalues of $B(z) = (b_{ij}(z))$ lie of course in $E^{(m)}_{k,z}(V)$. It follows that the matrix $e^{tB(z)}$ has coordinates which are of the form

$$\sum p_z(t)e^{tc(z)} \ , \tag{4.20}$$

where $p_z(t) \in \mathbb{C}(z)[t]$ and $c(z) \in E^{(m)}_{k,z}(V)$. The elements $p_z(t)$ lie in a finite subset of $\mathbb{C}(z)[t]$ (which depends on k). Let C be the set of eigenvalues of $B(z)$ on $\mathbb{C}^T$, considered as a set of linear functions in z. As in (4.10), (4.11), equation (4.19) gives rise to a differential equation satisfied by F_z, whose solution is

$$F_z(t,b,a',v) = e^{tB(z)}F_z(0,b,a',v) + e^{tB(z)} \int_0^t e^{-\tau B(z)}G_z(\tau,b,a',v)d\tau \ . \tag{4.21}$$

Again, we equate the first coordinates of both sides of (4.21). The first coordinate of the l.h.s. is $W_{v,z}(ba'a_t)$. By (4.20), the first coordinate of the first term in the r.h.s. of (4.21) has the form

$$\sum p_z(t)e^{tc(z)}W_{\pi_z(E)v}(ba') \ . \tag{4.22}$$

$p_z(t)$ lies in a certain finite set of $\mathbb{C}(z)[t]$, E lies in a certain finite set of $\mathcal{U}((\mathfrak{g}_\ell)_{\mathbb{C}})$ and $c \in C$. The first coordinate of the second term in the r.h.s. of (4.21) has the form

$$\sum h_z(t)e^{tc(z)} \int_0^t e^{-\tau(k+c(z))}\tau^\alpha W_{\pi_z(D)v}(ba'a_\tau)d\tau \ . \tag{4.23}$$

Again $h_z(t)$ lies in a certain finite set of $\mathbb{C}(z)[t]$, D lies in a certain finite set of $\mathcal{U}((g_\ell)_{\mathbb{C}})$, α lies in a certain finite set of nonnegative integers and $c \in C$. Put

$$\phi_{c,\alpha,D,z}(t,b,a',v) = \int_0^t e^{-\tau(k+c(z))}\tau^\alpha W_{\pi_z(D)v}(ba'a_\tau)d\tau \ , \tag{4.24}$$

and let

$$\Omega_{c,1} = \left\{z \in \Omega | -k - \operatorname{Re} c(z) + \mu \geq -\frac{1}{2}\right\} ,$$
$$\Omega_{c,2} = \left\{z \in \Omega | -k - \operatorname{Re} c(z) + \mu \leq -\frac{1}{3}\right\} .$$

$\Omega = \Omega_{c,1} \cup \Omega_{c,2}$. If the union is disjoint, then one of the subsets $\Omega_{c,1},\ \Omega_{c,2}$ is empty. First, note that $\phi_{c,\alpha,D,z}(t,b,a',v)$ is holomorphic in z in Ω, and continuous in (t,b,a'), uniformly in Ω. Indeed, since Ω is compact, there is A such that

$$-\operatorname{Re} c(z) \leq A \quad , \quad \forall z \in \Omega \ .$$

Thus, using (4.17), we have for $\tau \geq 0$ (bounding $q(\pi_z(D)v)$, on Ω, by another continuous seminorm which we denote again by q),

$$|e^{-\tau(k+c(z))}\tau^\alpha W_{\pi_z(D)v}(ba'a_\tau)| \leq e^{\tau(-k+A+\mu)}\tau^\alpha \|ba'\|^\mu q(v) \ .$$

This implies the holomorphicity in Ω, and also

$$|\phi_{c,\alpha,D,z}(t,b,a',v)| \leq \|ba'\|^\mu q(v) \int_0^t e^{\tau(-k+A+\mu)}\tau^\alpha d\tau \ .$$

From this the continuity in (t,b,a') (uniform in Ω) is easy to obtain. Now let us estimate $\phi_{c,\alpha,D,z}(t,b,a',v)$ on $\Omega_{c,1}$ and on $\Omega_{c,2}$. Assume that $z \in \Omega_{c,1}$. Then

$$\begin{aligned} |\phi_{c,\alpha,D,z}(t,b,a',v)| &\leq \|ba'\|^\mu q(v) \int_0^t e^{\tau(-k-\operatorname{Re} c(z)+\mu+1/2)}e^{-\tau/2}\tau^\alpha d\tau \\ &\leq e^{t(-k-\operatorname{Re} c(z)+\mu+1/2)}\|ba'\|^\mu q(v)\delta_\alpha(t) \ , \end{aligned}$$

where $\delta_\alpha(t) = \frac{t^{\alpha+1}}{\alpha+1}$. It follows from (4.18) that on $\Omega_{c,1}$

$$|e^{tc(z)}\phi_{c,\alpha,D,z}(t,b,a',v)| \leq e^{dt}\delta_\alpha(t)\|ba'\|^\mu q(v) \ .$$

Assume that $z \in \Omega_{c,2}$. Then write

$$\begin{aligned} \phi_{c,\alpha,D,z}(t,b,a',v) &= \int_0^\infty e^{-\tau(k+c(z))}\tau^\alpha W_{\pi_z(D)v}(ba'a_\tau)d\tau \\ &- \int_t^\infty e^{-\tau(k+c(z))}\tau^\alpha W_{\pi_z(D)v}(ba'a_\tau)d\tau \end{aligned} \ . \tag{4.25}$$

The absolute convergence of the integrals in (4.25) follows from

$$|e^{-\tau(k+c(z))}\tau^{\alpha}W_{\pi_z(D)v}(ba'a_\tau)| \leq \|ba'\|^{\mu}q(v)e^{\tau(-k-\mathrm{Re}\,c(z)+\mu)}\tau^{\alpha} \leq \|ba'\|^{\mu}q(v)e^{-\frac{\tau}{3}}\tau^{\alpha}$$

In particular, this convergence is uniform on $\Omega_{c,2}$, and hence each summand in (4.25) is holomorphic on $\Omega_{c,2}$. Each summand is continuous in (t,b,a'), uniformly in $z \in \Omega_{c,2}$ as will be clear from the following. Let

$$\varphi_{c,\alpha,D,z}(b,a',v) = \int_0^{\infty} e^{-\tau(k+c(z))}\tau^{\alpha}W_{\pi_z(D)v}(ba'a_\tau)d\tau \ , \tag{4.26}$$

$$\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v) = \int_t^{\infty} e^{-\tau(k+c(z))}\tau^{\alpha}W_{\pi_z(D)v}(ba'a_\tau)d\tau \ . \tag{4.27}$$

We have

$$|\varphi_{c,\alpha,D,z}(b,a'v)| \leq \delta_\alpha\|ba'\|^{\mu}q(v) \ ,$$

for $z \in \Omega_{c,2}$. $\delta_\alpha = \int_0^{\infty} e^{-\tau/3}\tau^{\alpha}d\tau$.

The continuous function $z \mapsto -k - \mathrm{Re}\,c(z) + \mu$ is nonzero on the compact set $\Omega_{c,2}$ and so there is $A_0 > 0$ such that

$$|-k-\mathrm{Re}\,c(z)+\mu| \geq A_0 > 0 \ , \qquad \forall z \in \Omega_{c,2} \ . \tag{4.28}$$

We have

$$\begin{aligned}|\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)| &\leq \left(\int_t^{\infty} e^{\tau(-k-\mathrm{Re}\,c(z)+\mu)}\tau^{\alpha}d\tau\right)\|ba'\|^{\mu}q(v)\\ &= e^{t(-k-\mathrm{Re}\,c(z)+\mu)}\left(-\frac{t^{\alpha}}{e_z}+\frac{\alpha}{e_z^2}t^{\alpha-1}-\cdots+\frac{(-1)^{\alpha+1}\alpha!}{e_z^{\alpha+1}}\right)\|ba'\|^{\mu}q(v) \ .\end{aligned}$$

Here $e_z = -k - \mathrm{Re}\,c(z) + \mu$. By (4.28), $|e_z| \geq A_0 > 0$ for all z in $\Omega_{c,2}$. Thus

$$|\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)| \leq e^{t(-k-\mathrm{Re}\,c(z)+\mu)}\delta'_{c,\alpha}(t)\|ba'\|^{\mu}q(v) \ ,$$

where $\delta'_{c,\alpha}(t) = \frac{t^{\alpha}}{A_0} + \frac{\alpha t^{\alpha-1}}{A_0^2} + \cdots + \frac{\alpha!}{A_0^{\alpha+1}}$. This implies that on $\Omega_{c,2}$.

$$|e^{tc(z)}\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)| \leq e^{dt}\delta'_{c,\alpha}(t)\|ba'\|^{\mu}q(v) \ .$$

Summing up, we proved the following analog of Theorem 2. (We keep the notation of Theorem 2. We fix Ω, μ, d and k as above.)

THEOREM 3. – *There exist a finite set $\mathcal{E} \subset \mathcal{U}((g_\ell)_{\mathbb{C}})$, a finite subset $C \subset \bigcup_{r=1}^{k} E_{r,z}^{(m)}(V)$, a finite set of polynomials S in $\mathbb{C}(z)[t]$, and a finite set L of nonnegative integers, such that for all $v \in V$ and z in Ω, $W_{v,z}(ba'a_t)$ has the form*

$$W_v(ba'a_t) = \sum p_z(t)e^{tc(z)}W_{\pi_z(E)v}(ba') + \sum h_z(t)e^{tc(z)}\phi_{\alpha,c,D,z}(t,b,a',v) \ , \tag{4.29}$$

where $p_z(t),\ h_z(t) \in S,\ c \in C,\ E, D \in \mathcal{E},\ \alpha \in L.$ $\phi_{\alpha,c,D,z}(t,b,a',v)$ *is given by (4.24); it is holomorphic in* $z \in \Omega$, *uniformly continuous in* (t,b,a') *(as* z *varies in* Ω*). For* $z \in \Omega_{c,1}$, *we have*

$$|e^{tc(z)}\phi_{c,\alpha,D,z}(t,b,a',v)| \le e^{dt}\delta_\alpha(t)\|ba'\|^\mu q(v)\ , \tag{4.30}$$

where $\delta_\alpha(t) = \frac{t^{\alpha+1}}{\alpha+1}$ *and* q *is a continuous seminorm on* V *which is independent of* z *and of* D. *For* $z \in \Omega_{c,2}$, *we have*

$$\phi_{c,\alpha,D,z}(t,b,a',v) = \varphi_{c,\alpha,D,z}(b,a',v) - \widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)\ , \tag{4.31}$$

where $\varphi_{c,\alpha,D,z}(b,a',v)$, $\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)$ *are given by (4.25), (4.26). They are both holomorphic in* $z \in \Omega_{c,2}$, *uniformly continuous in* (t,b,a') *as* z *varies in* $\Omega_{c,2}$ *and*

$$|\varphi_{c,\alpha,D,z}(b,a',v)| \le \|ba'\|^\mu \delta_\alpha q(v)\ , \tag{4.32}$$

$$|\widetilde{\varphi}_{c,\alpha,D,z}(t,b,a',v)| \le e^{dt}\delta'_{c,\alpha}(t)\|ba'\|^\mu q(v)\ . \tag{4.33}$$

Here δ_α *is a constant and* $\delta'_{c,\alpha}(t)$ *is a polynomial (independent of* z *or* D*).* ■

Remark. – C in Theorem 3 is thought of as a finite set of linear functions in z.

We now prove the analog of Theorem 1. We choose (very) negative numbers $d_1, \dots, d_\nu$ and positive integers $k_1, \dots, k_\nu$, such that

$$-k_j + \mu < d_j - 1\ , \qquad j = 1, \dots, \nu\ .$$

We use the notation of Theorem 1. By (4.29), applied to $t = x_\nu$, $m = m_\nu$, we have that $W_{v,z}(ba_x)$ is a sum of terms of the following two types.

$$p_z(x_\nu)e^{x_\nu c_\nu(z)}W_{\pi_z(D)v}\left(ba_{x_1}^{(m_1)}\cdot \dots \cdot a_{x_{\nu-1}}^{(m_{\nu-1})}\right)\ , \tag{4.34}$$

$$p_z(x_\nu)e^{x_\nu c_\nu(z)}\int_0^{x_\nu} e^{-\tau_\nu(k_\nu + c_\nu(z))}\tau_\nu^{\alpha_\nu}W_{\pi_z(D)v}\left(ba_{x_1}^{(m_1)}\cdot \dots \cdot a_{x_{\nu-1}}^{(m_{\nu-1})}a_{\tau_\nu}^{(m_\nu)}\right)d\tau_\nu\ . \tag{4.35}$$

Here $p_z(x_\nu)$ is taken from a finite set of $\mathbb{C}(z)[x_\nu]$, $c_\nu \in \bigcup_{r=1}^{k_\nu} E_{r,z}^{(m_\nu)}(V)$, α_ν- from a finite set of nonnegative integers, and D is taken from a finite set of $\mathcal{U}((g_\ell)_{\mathbb{C}})$. Now apply (4.29) again to (4.34) and to (4.35). A term of type (4.34) is then a sum of terms of the following two types

$$p_z(x_{\nu-1}, x_\nu)e^{x_{\nu-1}c_{\nu-1}(z) + x_\nu c_\nu(z)}W_{\pi_z(E)v}\left(ba_{x_1}^{(m_1)}\cdot \dots \cdot a_{x_{\nu-2}}^{(m_{\nu-2})}\right)$$

and

$$p_z(x_{\nu-1}, x_\nu)e^{x_{\nu-1}c_{\nu-1}(z) + x_\nu c_\nu(z)}\int_0^{x_{\nu-1}} e^{-\tau_{\nu-1}(k_{\nu_1} + c_{\nu-1}(z))}\tau_{\nu-1}^{\alpha_{\nu-1}}$$
$$W_{\pi_z(D)v}\left(ba_{x_1}^{(m_1}\cdot \dots \cdot a_{x_{\nu-2}}^{(m_{\nu-2})}a_{\tau_{\nu-1}}^{(m_{\nu-1})}\right)d\tau_{\nu-1}\ .$$

A term of type (4.35) is a sum of terms of the following two types

$$p_z(x_{\nu-1},x_\nu)e^{x_{\nu-1}c_{\nu-1}(z)+x_\nu c_\nu(z)}\int_0^{x_\nu} e^{-\tau_\nu(k_\nu+c_\nu(z))}\tau_\nu^{\alpha_\nu} W_{\pi_z(D)v}\left(ba_{x_1}^{(m_1)}\cdot\ldots\cdot a_{x_{\nu-2}}^{(m_{\nu-2})}a_{\tau_\nu}^{(m_\nu)}\right)d\tau_\nu$$

and

$$p_z(x_{\nu-1},x_\nu)e^{x_{\nu-1}c_{\nu-1}(z)+x_\nu c_\nu(z)}\int_0^{x_\nu}\int_0^{x_{\nu-1}} e^{-\tau_{\nu-1}(k_{\nu-1}+c_{\nu-1}(z))-\tau_\nu(k_\nu+c_\nu(z))}\tau_{\nu-1}^{\alpha_{\nu-1}}\tau_\nu^{\alpha_\nu} W_{\pi_z(D)}\left(ba_{x_1}^{(m_1)}\cdot\ldots\cdot a_{x_{\nu-2}}^{(m_{\nu-2})}a_{\tau_{\nu-1}}^{(m_{\nu-1})}a_{\tau_\nu}^{(m_\nu)}\right)d(\tau_{\nu-1},\tau_\nu)\ .$$

Again $p_z(x_{\nu-1},x_\nu)$ are taken from a finite set of $\mathbb{C}(z)[x_{\nu-1},x_\nu]$, etc. We apply again (4.24) to each element of the last four types and so on. Finally, we get a sum of elements of the following types

$$p_z(x_1,\ldots,x_\nu)\prod_{i=1}^{\nu} e^{x_ic_i(z)}\int_0^{x_{i_1}}\cdots\int_0^{x_{i_u}}\prod_{j=1}^{u}\left(e^{-\tau_{i_j}(k_{i_j}+c_{i_j}(z))}\tau_{i_j}^{\alpha_{i_j}}\right) W_{\pi_z(D)v}\left(ba_{\tau_{i_1}}^{(m_{i_1})}\cdot\ldots\cdot a_{\tau_{i_u}}^{(m_{i_u})}\right)d(\tau_{i_1},\ldots,\tau_{i_u})\ . \tag{4.36}$$

Here $\{i_1,\ldots,i_u\}$, $i_1<\cdots<i_u$, is a subset of $\{1,\ldots,\nu\}$ (it may be empty), $p_z(x_1,\ldots,x_\nu)$ is taken from a finite set of $\mathbb{C}(z)[x_1,\ldots,x_\nu]$, $c_i\in\bigcup_{r=1}^{k_i}E_{r,z}^{(m_i)}(V)$, $\alpha_1,\ldots,\alpha_\nu$ are taken from a finite set of nonnegative integers and D from a finite set in $\mathcal{U}(g_\ell)_{\mathbb{C}}$). As in the case of one variable, the integral in (4.36) is holomorphic in $z\in\Omega$ and uniformly continuous in $(x_{i_1},\ldots,x_{i_u},b)$ as z varies in Ω. Put

$$\Omega_{c_{i_1},\ldots,c_{i_u};j_1,\ldots,j_k}=\Omega_{c_{i_1},j_1}\cap\Omega_{c_{i_1},j_2}\cap\ldots\cap\Omega_{c_{i_u},j_u},\quad j_1,\ldots,j_u=1,2\ .$$

Assume that $z\in\Omega_{c_{i_1},\ldots,i_u;j_1,\ldots,j_u}$. Now for such z, rewrite (4.36) as follows. If $j_r=1$, leave the integration $\int_0^{x_{i_r}}$ as it is, and if $j_r=2$, write the integration $\int_0^{x_{i_r}}$ as $\int_0^\infty-\int_{x_{i_r}}^\infty$; $r=1,\ldots,u$. Thus the element (4.36) becomes a sum of elements which are obtained from (4.36) by replacing some of the integrals $\int_0^{x_{i_r}}$ by either $\int_0^\infty$ or $\int_{x_{i_r}}^\infty$. The analysis in the proof of Theorem 3, shows that each such term has the form

$$p_z(x_1,\ldots,x_\nu)\prod_{i=1}^{m} e^{x_{\gamma_i}c_{\gamma_i}(z)}f_z(x_{\gamma'_1},\ldots,x_{\gamma'_{\nu-m}},b,v)\ ,$$

where $\{1,\ldots,\nu\}$ is the disjoint union of $\{\gamma_1,\ldots,\gamma_m\}$ and $\{\gamma'_1,\ldots,\gamma'_{\nu-m}\}$, $\{\gamma'_1,\ldots,\gamma'_{\nu-m}\}\subset\{i_1,\ldots,i_u\}$. The functions $f_z(x_{\gamma'_1},\ldots,x_{\gamma'_{\nu-m}},b,v)$ are linear in v, holomorphic in $z\in\Omega_{c_{i_1},\ldots,c_{i_u};j_1,\ldots,j_u}$, uniformly continuous in $(x_{\gamma'_1},\ldots,x_{\gamma'_{\nu-m}},b)$

as z varies in $\Omega_{c_{i_1},\dots,c_{i_u};j_1,\dots,j_u}$, and there exists a continuous seminorm q on V and a polynomial $\delta(x_{\gamma'_1},\dots,x_{\gamma'_{\nu-m}})$ such that

$$(4.37)\qquad \begin{aligned}&|f_z(x_{\gamma'_1},\dots,x_{\gamma'_{\nu-m}},b,v)|\\&\leq e^{x_{\gamma'_1}d_{\gamma'_1}+\dots+x_{\gamma'_{\nu-m}}d_{\gamma'_{\nu-m}}}\delta(x_{\gamma'_1},\dots,x_{\gamma'_{\nu-m}})\|b\|^{\mu}q(v)\ .\end{aligned}$$

Let us denote this space of functions $f_z(x_{\gamma'_1},\dots,x_{\gamma'_{\nu-m}},b,v)$ on $\mathbb{R}_+^{\nu-m}\times\mathbb{R}_+^{\ell-m_\nu}\times V$ by $F_{J,I,\bar{j}}$, where $J=\{i_1,\dots,i_u\}$, $I=\{\gamma'_1,\dots,\gamma'_{\nu-m}\}$, $\bar{j}=(j_1,\dots,j_u)$. Note that if $j_r=1$ then I contains i_r. Put also

$$\Omega_{c_J,\bar{j}}=\Omega_{c_{i_1},\dots,c_{i_u};j_1,\dots,j_u}$$

$$(4.38)\qquad \begin{aligned}\phi_{c_J,\alpha_J,D,z}(x_J,b,v)=&\int_0^{x_{i_1}}\cdots\int_0^{x_{i_u}}\prod_{j=1}^{u}\left(e^{-\tau_{i_j}(k_{i_j}+c_{i_j}(z))}\tau_{i_j}^{\alpha_{i_j}}\right)\\&W_{\pi_z(D)v}\left(ba_{\tau_{i_1}}^{(m_{i_1})}\cdot\dots\cdot a_{\tau_{i_u}}^{(m_{i_u})}\right)d(\tau_{i_1},\dots,\tau_{i_u})\ ,\end{aligned}$$

for $c_J=(c_{i_1},\dots,c_{i_u}),\alpha_J=(a_{i_1},\dots,\alpha_{i_u}),x_J=(x_{i_1},\dots,x_{i_u})$. Now we can state the analog of Theorem 1, which we just proved. Fix Ω, $d_1,\dots,d_\nu,k_1,\dots,k_\nu$ as before.

THEOREM 4. – *There are finite subsets $C_j\subset\bigcup_{r=1}^{k_j}E_{r,z}^{(m_j)}(V)$, $j=1,\dots,\nu$, (viewed as subsets of linear functions in z), a finite subset $\mathcal{E}\subset\mathcal{U}((g_\ell)_{\mathbb{C}})$, a finite subset $\mathcal{P}$ in $\mathbb{C}(z)[x_1,\dots,x_\nu]$ and a finite subset L of nonnegative integers, such that for all $v\in V$, $z\in\Omega$,*

$$W_v(ba_x)=\sum p_z(x_1,\dots,x_\nu)\prod_{i=1}^{\nu}e^{x_ic_i(z)}\phi_{c_J,\alpha_J,D,z}(x_J,b,v)\ .$$

Here $p_z(x_1,\dots,x_\nu)\in\mathcal{P}$, $c_i\in C_i$, J varies over subsets of $\{1,\dots,\nu\}$, $\alpha_J\in L^{|J|}$ and $D\in\mathcal{E}$. $\phi_{c_J,\alpha_J,D,z}(x_J,b,v)$ is given by (4.38). It is linear in v, holomorphic in $z\in\Omega$, uniformly continuous in x_J, as z varies in Ω. Let $\bar{j}\in\{1,2\}^{|J|}$. Then on $\Omega_{c_{J,\bar{j}}}$, we can write

$$\prod_{i=1}^{\nu}e^{x_ic_i(z)}\phi_{c_J,\alpha_J,D,z}(x_J,b,v)=\sum e^{x_{I'}\cdot c_{I'}(z)}f_z(x_I,b,v)$$

where I varies over certain subsets of J, $I'=\{1,\dots,\nu\}\backslash I$, and the functions $f_z(x_I,b,v)$ belong to a certain finite subset of $F_{J,I,\bar{j}}$. ■

Finally we remark that the four theorems of this section hold in the case of general real (split) reductive groups, in particular for GL_n and the simple algebraic groups. The proofs are the same word for word, noting that for $(x_1,\dots,x_\nu)\in\mathbb{R}^v$, a_x represents an element of the (connected) center A of the Levi part of an arbitrary standard parabolic subgroup and for $x_1,\dots,x_\nu\geq 0$, $a_x\in\overline{A}^+$. In this paper, we will apply these results to $\mathrm{GL}_n(F)$, $G_\ell=\mathrm{SO}_{2\ell+1}(F)$, $H_n=\mathrm{SO}_{2n}(F)$.

5. Proof of Theorem A in case $\ell \geq n$

We will use several reductions, so that it will suffice to prove Theorem A only for integrals of the form

$$R(W_v, \xi_{\tau,s}) = \int\limits_{A_n^{(+)}} W_v\left(\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right)\xi_{\tau,s}(I,a)\delta^{-1}(a)|\det a|^{s+\frac{3n-2\ell-2}{2}}da\ , \tag{5.1}$$

for $v \in V_\pi$, $\xi_{\tau,s} \in V_{\rho_{\tau,s}}$. $A_n^{(+)} = \left\{a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} \middle| a_i > 0,\ i - 1, \dots, n\right\}$. $\delta(a) = \delta(m(a))$ is the modular function for H_n, with respect to the standard Borel subgroup. Put, for short, $s' = s + \frac{3n-2\ell-2}{2}$. For (5.1), we apply the results of section 4. In the sequel, whenever we write that $B(W_v, \xi_{\tau,s})$ satisfies the properties of Theorem A, we mean that $B(W_v, \xi_{\tau,s})$, initially defined as an absolutely convergent integral in some half plane, has a meromorphic continuation to the whole s-plane, which is continuous on $V_\pi \times V_{\rho_{\tau,s}}$ and meromorphic in $(s_1, \dots, s_k)$ if $W_v = W_{v;s_1,\dots,s_k}$. Put

$$B(W_v, \xi_{\tau,s}) = \int\limits_{A_n^{(+)}}\int\limits_{\overline{X}_{(n,\ell)}} W_v\left(\overline{x}\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right)\xi_{\tau,s}(I,a)\delta^{-1}(a)|\det a|^{s+\frac{n-2}{2}}d\overline{x}da\ . \tag{5.2}$$

Recall that this integral converges absolutely for $\mathrm{Re}(s)$ large enough ([S], 5.3).

LEMMA 1. – *If $B(W_v, \xi_{\tau,x})$ satisfies the properties of Theorem A, then so does $A(W_v, \xi_{\tau,s})$.*

Proof. – By the Iwasawa decomposition, we have (first for $\mathrm{Re}(s)$ large enough),

$$A(W_v, \xi_{\tau,s}) = \int\limits_{K_{H_n}} B(\pi(j_{n,\ell}(k))W_v, \rho_{\tau,s}(k)\xi_{\tau,s})dk\ . \tag{5.3}$$

Since $B(W_v, \xi_{\tau,s})$ is a meromorphic function of s, which is continuous on $V_\pi \times V_{\rho_{\tau,s}}$, it follows that for s which is not a pole, the function $k \mapsto B(\pi(j_{n,\ell}(k)W_v, \rho_{\tau,s}(k)\xi_{\tau,s})$ is continuous on K_{H_n}, and hence bounded, when v and $\xi_{\tau,s}$ are fixed. In particular, the integral (5.3) converges absolutely, if s is not a pole. By the Banach-Steinhaus theorem, since $(v, \xi_{\tau,s}) \mapsto B(\pi(j_{n,\ell}(k))W_v, \rho_{\tau,s}(k)\xi_{\tau,s})$ is, as k varies over K_{H_n}, a pointwise bounded family of continuous bilinear forms on $V_\pi \times V_{\rho_{\tau,s}}$, it is equicontinuous, and hence $A(W_v, \xi_{\tau,s})$ is continuous on $V_\pi \times V_{\rho_{\tau,s}}$. (Note, that since $V_\pi, V_{\rho_{\tau,s}}$ are Fréchet spaces, the notions of continuity and separate continuity on $V_\pi \times V_{\rho_{\tau,s}}$ coincide.) The convergence of the integral (5.3) is uniform in s, when it varies in a compact set which does not contain a pole (and, similarly, it is uniform in $(s_1, \dots, s_k)$, when it varies in a compact set, when we consider $W_v = W_{v;s_1,\dots,s_k}$). Thus $A(W_{v;s_1,\dots,s_k}, \xi_{\tau,s})$ is meromorphic in $s, s_1, \dots, s_k$. ■

Our aim is to "get rid" of the $d\overline{x}$-integration in (5.2). Put

$$B_j(W_v,\xi_{\tau,s})$$

(5.4)
$$= \int\limits_{A_n^{(+)}} \int\limits_{M_{j\times n}(F)} W_v\left(\begin{pmatrix} a & & \\ x & I_j & \\ 0 & 0 & I_{\ell-n-j}\end{pmatrix}^{\wedge}\right)\xi_{\tau,s}(I,a)\delta^{-1}(a)|\det a|^{s'}dxda\ .$$

Note that $B(W_v,\xi_{\tau,s}) = B_{\ell-n}(W_v,\xi_{\tau,s})$.

LEMMA 2. – *Assume that $\ell > n$, and let $0 \leq j < \ell - n$. If $B_j(W_v,\xi_{\tau,s})$ satisfies the properties of Theorem A, then so does $B_{j+1}(W_v,\xi_{\tau,s})$.*

Proof. – The proof is similar to ([S], 5.4). First assume that $j = \ell - n - 1$. Consider the subgroup C_n of elements in G_ℓ, of the form

$$\eta(u,z) = \begin{pmatrix} I_n & 0 & u & 0 & z \\ & I_{\ell-n} & 0 & 0 & 0 \\ & & 1 & 0 & u' \\ & & & I_{\ell-n} & 0 \\ & & & & I_n \end{pmatrix}.$$

Let $\phi \in S(C_n)$, the space of Schwartz functions on C_n, and put for $v \in V_\pi$,

$$\pi^{(\ell-n-1)}(\phi)v = \int\limits_{C_n} \phi(\eta(u,z))\pi(\eta(u,z))v d\eta(u,z)\ .$$

Let $\overline{x} = \begin{pmatrix} I_n & \\ x & I_{\ell_n}\end{pmatrix}^{\wedge}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_{\ell-n}\end{pmatrix} \in M_{(\ell-n)\times n}(F)$. We have, as in ([S], 5.4),

$$W_{\pi^{(\ell-n-1)}(\phi)v}\left(\overline{x}\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right) = \widehat{\phi}(x_{\ell-n}a)W_v\left(\overline{x}\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right),$$

where

$$\widehat{\phi}(y) = \int\limits_{C_n} \phi(\eta(u,z))\psi(yu)d\eta(u,z)\ ,$$

for $y \in F^n$ (row vectors). $\widehat{\phi}$ lies in $S(F^n)$ and the map $\phi \mapsto \widehat{\phi}$ is continuous from $S(C_n)$ to $S(F^n)$. Put for $\varphi \in S(F^n)$ and $v \in V_\pi$,

$$\pi_{(1)}(\varphi) = \int\limits_{F^n} \varphi(y)\pi\left(\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ y & 0 & 1\end{pmatrix}^{\wedge}\right)v dy\ .$$

Then (first for $\mathrm{Re}(s) \gg 0$),

$$(5.5)\quad B\left(W_{\pi^{(\ell-n-1)}(\phi)v}, \xi_{\tau,s}\right) = B_{\ell-n}\left(W_{\pi^{(\ell-n-1)}(\phi)v}, \xi_{\tau,s}\right) = B_{\ell-n-1}\left(W_{\pi_{(1)}(\widehat{\phi})v}, \xi_{\tau,s}\right) .$$

Assume that $B_{\ell-n-1}(W_v, \xi_{\tau,s})$ satisfies the properties of Theorem A. Then it follows that for s which is not a pole, the trilinear map $T_{\ell-n-1} : S(C_n) \times V_\pi \times V_{\rho_{\tau,s}} \to \mathbb{C}$, given by $T_{\ell-n-1}(\phi, v, \xi_{\tau,s}) = B_{\ell-n-1}\left(W_{\pi_{(1)}(\widehat{\phi})v}, \xi_{\tau,s}\right)$ is continuous, and so the bilinear map $T'_{\ell-n-1} : \left(S(C_n)\widehat{\otimes}V_\pi\right) \times V_{\rho_{\tau,s}} \to \mathbb{C}$ defined by $T'_{\ell-n-1}(\phi \otimes v, \xi_{\tau,s}) = B_{\ell-n}\left(W_{\pi^{(\ell-n-1}((\phi)v}, \xi_{\tau,s}\right)$ is continuous. The map $S(C_n)\widehat{\otimes}V_\pi \to V_\pi$, given by $\phi \otimes v \mapsto \pi^{(\ell-n-1)}(\phi)v$ is continuous. It is also surjective (by the Dixmier-Malliavin Lemma [DM]) and it is open (by the open mapping theorem). Therefore, the bilinear form $(v, \xi_{\tau,s}) \mapsto B_{\ell-n}(W_v, \xi_{\tau,s})$ is (separately continuous and hence) continuous. The assumptions about the meromorphic properties of $B_{\ell-n-1}(W_{v;s_1,\ldots,s_k}, \xi_{\tau,s})$ clearly imply the same properties for $B_{\ell-n}(W_{v;s_1,\ldots,s_k}, \xi_{\tau,s})$. Next, we repeat the argument for $j < \ell - n - 1$. Assume that $B_j(W_v, \xi_{\tau,s})$ satisfies the properties of Theorem A. Let L_j be the subgroup of elements in G_ℓ of the form

$$\eta(u) = \begin{pmatrix} I_n & 0 & u \\ & I_{j+1} & 0 \\ & & I_{\ell-n-j-1} \end{pmatrix}^{\wedge}, \quad u = (u_1, u_2, \ldots, u_{\ell-n-j-1}) \in M_{n\times(\ell-n-j-1)}(F) .$$

Put, for $\phi \in S(L_j)$, $v \in V_\pi$,

$$\pi^{(j)}(\phi)v = \int_{L_j} \phi(\eta(u))\pi(\eta(u))v d\eta(u) .$$

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_{j+1} \end{pmatrix} \in M_{(j+1)\times n}(F)$. We have

$$W_{\pi^{(j)}(\phi)v}\left(\begin{pmatrix} a & & \\ x & I_{j+1} & \\ 0 & 0 & I_{\ell-n-j-1} \end{pmatrix}^{\wedge}\right) = \widehat{\phi}(x_{j+1})W_v\left(\begin{pmatrix} a & & \\ x & I_{j+1} & \\ 0 & 0 & I_{\ell-n-j-1} \end{pmatrix}^{\wedge}\right),$$

where, for a row vector $y \in F^n$,

$$\widehat{\phi}(y) = \int_{L_j} \phi(\eta(u))\psi(yu_1)d\eta(u) .$$

$\widehat{\phi}$ lies in $S(F^n)$. Put for $\varphi \in S(F^n)$, $v \in V_\pi$,

$$\pi_{(\ell-n-j)}(\varphi)v = \int_{F^n} \varphi(y)\pi\left(\begin{pmatrix} I_n & & & \\ 0 & I_j & & \\ y & 0 & 1 & \\ 0 & 0 & 0 & I_{\ell-n-j-1} \end{pmatrix}^{\wedge}\right)vdy .$$

Then

$$B_{j+1}\left(W_{\pi^{(j)}(\phi)v},\xi_{\tau,s}\right)=B_j\left(W_{\pi_{(\ell-n-j)}(\widehat{\phi})v},\xi_{\tau,s}\right).$$

Now we are at the same situation as in (5.5), and we proceed as in case $j=\ell-n-1$. ■

Proof of Theorem A (in case $\ell \geq n$). – By the previous two lemmas, it is enough to prove the theorem for $R(W_v,\xi_{\tau,s})$ instead of $A(W_v,\xi_{\tau,s})$. Write in (5.1)

$$\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}=a_1^{(1)}a_2^{(2)}\cdots a_n^{(n)}\ ,\ a_i>0, i=1,2,\ldots,n\ ,$$

where $a^{(j)}=\begin{pmatrix} aI_j & \\ & I_{\ell-j}\end{pmatrix}^{\wedge}$. Put $\check{a}^{(j)}=\begin{pmatrix} aI_j & \\ & I_{n-j}\end{pmatrix}$ (in $\mathrm{GL}_n(F)$). If $\ell=n$, define for $\phi\in S(N_\ell)$ and $v\in V_\pi$,

$$\pi(\phi)v=\int\limits_{N_\ell}\phi(u)\pi(u)v\,du\ .$$

If $\ell>n$, define for $\phi\in S(Z_{n+1})$ and $v\in V_\pi$,

$$\pi(\phi)v=\int\limits_{Z_{n+1}}\phi(u)\pi\left(\begin{pmatrix} u & \\ & I_{\ell-n-1}\end{pmatrix}^{\wedge}\right)v\,du\ .$$

We have

$$W_{\pi(\phi)v}\left(\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right)=\widehat{\phi}(a_1,\ldots,a_n)W_v\left(\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right),$$

where

$$\widehat{\phi}(a_1,\ldots,a_n)=\int\limits_{Z_{n+1}}\phi(u)\psi(aua^{-1})du\quad,\quad\text{if}\quad \ell>n\ ,$$

$$\widehat{\phi}(a_1,\ldots,a_n)=\int\limits_{N_\ell}\phi(u)\psi(\widehat{a}u\widehat{a}^{-1})du\quad,\quad\text{if}\quad \ell=n\ .$$

We first consider

$$\text{(5.6)}\qquad\begin{aligned}&R(W_{\pi(\phi)v},\xi_{\tau,s})\\&=\int\limits_{A_n^{(+)}}W_v\left(\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right)\xi_{\tau,s}(I,a)\widehat{\phi}(a_1,\ldots,a_n)^{-1}\delta^{-1}(a)|\det a|^{s'}da\ .\end{aligned}$$

The integral (5.6) is a sum of the following integrals $R_\eta(\phi,v,\xi_{\tau,s})$. Let $\eta=\{m_1,m_2,\ldots,m_\nu\}$,

$m_1 < m_2 < \cdots < m_\nu$, be a subset of $\{1,2,\ldots,n\}$, and let $\eta' = \{m'_1, m'_2, \ldots, m'_{n-\nu}\}$, $m'_1 < m'_2 < \cdots < m'_{n-\nu}$, be its complement in $\{1,2,\ldots,n\}$. Then

$$R_\eta(\phi, v, \xi_{\tau,s})$$
$$(5.7)\qquad = \int_{\substack{0<a_i\leq 1,\ i\in\eta \\ 1<a_i,\ i\in\eta'}} W_v\left(\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge}\right)\xi_{\tau,s}(I,a)\widehat{\phi}(a_1,\ldots,a_n)\delta^{-1}(a)|\det a|^{s'}da\ .$$

We have to distinguish two cases according to $n \in \eta$ or $n \in \eta'$. Both are treated in the same way, so let us assume that $n \in \eta$. (This is the less convenient case....) Denote, for a subset $\gamma = \{i_1, \ldots, i_u\}$ of $\{1,\ldots,n\}$, $i_1 < \cdots < i_u$,

$$a(\gamma) = \prod_{i\in\gamma} a_i^{(i)}\ ,\ \check{a}(\gamma) = \begin{cases} \prod_{i\in\gamma} \check{a}_i^{(i)} & , \text{ if } \ n \notin \gamma \\ \prod_{\substack{i\in\gamma\\ i\neq n}} \check{a}_i^{(i)} & , \text{ if } \ n \in \gamma \end{cases}\ ,\ a_\gamma = (a_{i_1}, \ldots, a_{i_u})\ .$$

Also put $\check{\eta} = \eta\backslash\{n\}$.

Then (5.7) can be written as

$$R_\eta(\phi, v, \xi_{\tau,s}) = \int_{\substack{0<a_i\leq 1,\ i\in\eta \\ 1\leq a_i,\ i\in\eta'}} W_{\pi(a(\eta'))v}(a(\eta))\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))(\check{a}(\eta))$$
$$\cdot\, \omega_\tau(a_n)\widehat{\phi}(a_1,\ldots,a_n)\cdot\delta^{-1}(a)|\det a|^{s'}da\ .$$

Here we denote $\xi_{\tau,s}(h)$ to be the function on H_n, which takes values in $W(\tau,\psi^{-1})$, so that $\xi_{\tau,s}(h)(m) = \xi_{\tau,s}(h,m)$ for $h \in H_n$ and $m \in \mathrm{GL}_n(F)$. ω_τ is the central character of τ. Now let us use Theorem 1 in section 4 (which is valid as remarked there for any real reductive quasi split group). Thus, fix $\mu > 0$ and continuous seminorms q – on V_π, and h – on V_τ, such that

$$|W_v(g)| \leq \|g\|^\mu q(v) \quad , \quad v \in V\pi,\ g \in G_\ell\ ,$$
$$|W'_v(m)| \leq \|m\|^\mu h(v) \quad , \quad v \in V_\tau, m \in \mathrm{GL}_n(F)\ .$$

Here $v \mapsto W'_v(I)$ is the Whittaker functional on V_τ, with respect to ψ^{-1}. (The norm $\|m\|$ on $\mathrm{GL}_n(F)$ is assumed to be such that $\|c\cdot m\| = \|m\|$, for $c \in F^*$. We may assume that for $m = \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & \\ & & & 1\end{pmatrix}$, $\|m\| = \max\{1, |x_i|,\ |x_i|^{-1}\}_{i\leq n}$.) Consider $\bigcup_{k=1}^\infty E_k^{(m)}(V_\pi) = \{\xi_i^{(m)}\}_{i=1}^\infty$, so that $\mathrm{Re}\,\xi_1^{(m)} \geq \mathrm{Re}\,\xi_2^{(m)} \geq \cdots$ and $\bigcup_{k=1}^\infty E_k^{(m)}(V_\tau) = \{\check{\xi}_i^{(m)}\}_{i=1}^\infty$, so that $\mathrm{Re}\,\check{\xi}_1^{(m)} \geq \mathrm{Re}\,\check{\xi}_2^{(m)} \geq \cdots$ (In the second case, the superscript (m), means that we consider the maximal parabolic subgroup of $\mathrm{GL}_n(F)$, which is of type $(m, n-m)$, and consider the generalized eigenspaces in V_τ for the action of $H_m = \begin{pmatrix} -I_m & \\ & 0 \end{pmatrix}$ in $V_\tau/\mathcal{Y}_m^k V_\tau$; $\mathcal{Y}_m$ is the Lie algebra of the unipotent radical.) Denote the "jump" indices in the first case by $N_1^{(m)} < N_2^{(m)} < \cdots$ and in the second case by $\check{N}_1^{(m)} < \check{N}_2^{(m)} < \cdots$. Let $d_j = \xi^{(m_j)}_{N^{(m_j)}_{i_j}}$, $j = 1, \ldots, \nu$, and $\check{d}_j = \check{\xi}^{(m_j)}_{\check{N}^{(m_j)}_{i_j}}$, $j = 1, \ldots, \nu - 1$. The indices i_j are chosen to be large (as large as we want). Recall that $m_\nu = n$, since $n \in \eta$. Choose positive integers k_j, $j = 1, \ldots, \nu$, such that

$$-k_j + \mu < \mathrm{Re}(d_j),\ \mathrm{Re}(\check{d}_j) \quad , \quad \text{for} \quad j = 1, \ldots, \nu - 1$$

and

$$-k_\nu + \mu < \mathrm{Re}(d_\nu) \ .$$

By Theorem 1, section 4, there are finite subsets $C_j \subset \bigcup_{r=1}^{k_j} E_r^{(m_j)}(V_\pi)$, $j = 1, \ldots, \nu$ $\check{C}_j \subset \bigcup_{r=1}^{k_j} E_r^{(m_j)}(V_\tau)$, $j = 1, \ldots, \nu - 1$, such that for $0 < a_i \le 1$, $i \in \eta$

$$W_v(a(\eta)) = \sum r(\log a_\eta) a_J^{-c_J} \varphi(a_{J'_\eta}, v) \ . \tag{5.9}$$

Here $\log a_\eta = (\log a_{m_1}, \ldots, \log a_{m_\nu})$, r is taken from a fixed finite set in $\mathbb{C}[x_1, \ldots, x_\nu]$, (it depends on $(k_1, \ldots, k_\nu)$, but not on v). J varies over subsets of η and J'_η denotes its complement in η. Let $J = \{i_1, i_2, \ldots, i_u\}$, $i_1 < i_2 < \cdots < i_u$. Then c_J has the form $(c_{i_1}, c_{i_2}, \ldots, c_{i_u})$, with $c_j \in C_j$, and $a_J^{-c_J} = a_{i_1}^{-c_{i_1}} \cdot \cdots \cdot a_{i_u}^{-c_{i_n}}$. φ is taken from a fixed finite set of functions, continuous in $a_{J'_n}$, linear in v, for which there exist polynomials β, in $\nu - u$ variables (independent of v), and continuous seminorms $\widetilde{q}$, such that

$$|\varphi(a_{J'_\eta}, v)| \le \beta(\log a_{J'_\eta}) a_{J'_\eta}^{-\mathrm{Re}(d_{J'_\eta})} \quad \widetilde{q}(v) \ , \ v \in V_\pi \ . \tag{5.10}$$

Similarly, for $\omega \in V_\tau$

$$W'_\omega(\check{a}(\eta)) = \sum b(\log a_{\check{\eta}}) a_Q^{-\check{c}_Q} e(a_{Q'_{\check{\eta}}}, \omega) \ . \tag{5.11}$$

Here Q varies over subsets of $\check{\eta}$. $Q'_{\check{\eta}}$ is the complement of Q in $\check{\eta}$. If $Q = \{j_1, \ldots, j_t\}$, then $\check{c}_Q$ has the form $(\check{c}_{j_1}, \ldots, \check{c}_{j_t})$, with $\check{c}_i \in \check{C}_i$. b is taken from a certain set of polynomial in $\nu - 1$ variables (independent of ω) and e – from a certain set of functions, continuous in $a_{Q'_{\check{\eta}}}$, linear in $\omega \in V_\tau$, for which there exist polynomials ε in $\nu - 1 - t$ variables (independent of ω) and continuous seminorms $\widetilde{h}$, such that

$$|e(a_{Q'_{\check{\eta}}}, \omega)| \le \varepsilon(\log a_{Q'_{\check{\eta}}})^{-\mathrm{Re}(\check{d}_{Q'_{\check{\eta}}})} \widetilde{h}(\omega) \ . \tag{5.12}$$

(Recall again, that if $Q' = \{j_1, \ldots, j_{\nu-1-t}\}$ is the complement of Q in $\{1, \ldots, \nu - 1\}$, then $a_{Q'_{\check{\eta}}}^{\mathrm{Re}(\check{d}_{Q'_{\check{\eta}}})} = a_{j_1}^{\mathrm{Re}\,\check{d}_{j_1}} \cdot \cdots \cdot a_{j_{\nu-1-t}}^{\mathrm{Re}\,\check{d}_{\nu-1-t}}$.) Substitute (5.9) (with $\pi(a(\eta')v)$ instead of v,) and (5.11) (with $\tau(\check{a}(\eta')\xi_{\tau,s}(I)$ instead of ω), in (5.8). We get that $R(W_{\pi(\phi)v},\xi_{\tau,s})$ is a sum of integrals of the form

$$\int\limits_{\substack{0<a_i\le 1,\ i\in\eta\\ 1<a_i,\ i\in\eta'}} p(\log a_\eta)a_J^{-c_J}\cdot a_Q^{-\check{c}_Q}\cdot\varphi(a_{J'_\eta},\pi(a(\eta'))v)e\Big(a_{Q'_{\check{\eta}}},\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\Big)\tag{5.13}$$
$$\omega_\tau(a_n)\widehat{\phi}(a_1,\ldots,a_n)\delta^{-1}(a)|\det a|^{s'}da\ ,$$

where $p(x_1,\ldots,x_\nu)$ is a polynomial of the form $r(x_1,\ldots,x_\nu)b(x_1,\ldots,x_{\nu-1})$, J and Q vary over subsets of $\{1,\ldots,\nu\}$ and $\{1,\ldots,\nu-1\}$ respectively. Let Ω be the complement in η of $J'_\eta\cup Q'_{\check{\eta}}$. Let $\overline{\Omega}=(J\cup Q)\backslash\Omega$. Assume first that $n\notin\Omega$. Then we can write

$$\begin{aligned}&a_J^{-c_J}\cdot a_Q^{-\check{c}_Q}\varphi(a_{J'_\eta},\pi(a(\eta'))v)e\Big(a_{Q'_{\check{\eta}}},\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\Big)\\&=a_\Omega^{-(c_\Omega+\check{c}_\Omega)}f\Big(a_{\Omega'_\eta},\pi(a(\eta'))v,\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\Big)\ ,\end{aligned}\tag{5.14}$$

where f is continuous in $a_{\Omega'_\eta}$, linear in the remaining two variables, for which there is a polynomial δ, a nonnegative number M_0, which depends only on π,τ and continuous seminorms, $\widetilde{q}$ – on V_π, $\widetilde{h}$ – on V_τ, such that

$$\begin{aligned}&|f\Big(a_{\Omega'_\eta},\pi(a(\eta'))v,\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\Big)|\\&\le\delta(\log a_{\Omega'_\eta})a_{\Omega'_\eta}^{-\operatorname{Re}(d_{\Omega'_\eta}+\check{d}_{\Omega'_\eta})}\cdot a_{\overline{\Omega}}^{M_0}\cdot\widetilde{q}(\pi(a(\eta'))v)\widetilde{h}(\tau(\check{a}(\eta'))(\xi_{\tau,s}(I)))\ .\end{aligned}\tag{5.15}$$

This follows from (5.10), (5.12) and the fact that for $c_j\in C_j$, we know that c_j has the form ξ_j-n, for $\xi_j\in E_1^{(m_j)}(V_\pi)$, and n is a nonnegative integer, and thus if $0<a\le 1$, $a^{-c_j}\le a^{-\xi_j}$ and so $a^{-c_j}\le a^{M'_0}$ for $M'_0=\max\{-\xi_j|\xi_j\in E_1^{(m_j)}(V_\pi)\}$. This holds, similarly, for $c_i\in\check{C}_i$. Thus the product of terms in $a_J^{-c_j}\cdot a_Q^{-\check{c}_Q}$ (in (5.14)) which belongs to $\overline{\Omega}$ is bounded by a fixed power $a_{\overline{\Omega}}^{M_0}$. (If $\overline{\Omega}=\{\ell_1,\ldots,\ell_u\}$, then $a_{\overline{\Omega}}^{M_0}=(a_{\ell_1}\cdot\cdots\cdot a_{\ell_u})^{M_0}$.) Let $D_j=-d_j+M_0$, $\check{D}_j=-\check{d}_j+M_0$. From (5.15), it follows, using that π and τ are of moderate growth, that there are a positive number M_1 (fixed) and continuous seminorms q_1 – on V_π and h_1 – on V_τ, such that

$$\begin{aligned}&|f\Big(a_{\Omega'_\eta},\pi(a(\eta')v,\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\Big|\\&\le\delta(\log a_{\Omega'_\eta})a_\Omega^{\operatorname{Re}(D_j+\check{D}_j)}\|a(\eta')\|^{M_1}\widetilde{q}_1(v)\widetilde{h}_1(\xi_{\tau,s}(I))\ .\end{aligned}\tag{5.16}$$

The case $n\in\Omega$ is treated similarly. Thus (5.13) is a sum of integrals of the form

$$\int\limits_{\substack{0<a_i\le 1,\ i\in\eta\\ 1<a_i,\ i\in\eta'}} p(\log a_\eta)a_\Omega^{-(c_\Omega+\check{c}_\Omega)}f\left(a_{\Omega'_\eta},\pi(a(\eta'))v,\tau(\check{a}(\eta'))(\xi_{\tau,s}(I))\right)\tag{5.17}$$
$$\widehat{\phi}(a_1,\ldots,a_n)\omega_\tau(a_n)\delta^{-1}(a)|\det a|^{s'}da\ .$$

If $n \in \Omega$ in (5.17), we modify $\check{c}_\Omega$ by adding to it one more coordinate, set to be zero. Assume now that ϕ is of the following form if $\ell > n$,

$$\phi\begin{pmatrix} 1 & u_{12} & & & & u_{ij} \\ & 1 & u_{23} & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & u_{n,n+1} \\ & & & & & 1 \end{pmatrix}$$

$$= \phi_1(u_{12})\phi_2(u_{23})\cdot\cdots\cdot\phi_n(u_{n,n+1})\Phi(u_{13},\dots,u_{n-1,n+1})$$

and if $\ell = n$,

$$\phi\begin{pmatrix} 1 & u_{12} & & & & & & u_{ij} & \\ & 1 & u_{23} & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & u_{n,n+1} & & & & \\ & & & & 1 & -u_{n,n+1} & & & \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & -u_{12} \\ & & & & & & & & 1 \end{pmatrix}$$

$$= \phi_1(u_{12})\cdot\cdots\cdot\phi_n(u_{n,n+1})\Phi(u_{13},\dots).$$

Then

$$\widehat{\phi}(a_1,\dots,a_n) = \widehat{\phi}_1(a_1)\cdot\cdots\cdot\widehat{\phi}_n(a_n)\alpha(\Phi)\ ,$$

where

$$\alpha(\Phi) = \int \Phi(u_{13},\dots)d(u_{13},\dots)\ .$$

Now consider (5.17), where instead of p, we take just one monomial of p. Let $\Omega'_\eta = \{j_1,\dots,j_\zeta\}$. We get an integral which is a product of terms of the following types

$$\alpha(\Phi)\int_0^1 (\log a_{m_i})^{p_i} a_{m_i}^{-(c_i+\check{c}_i)+m_i s'+\delta_i}\widehat{\phi}_{m_i}(a_{m_i})da_{m_i} \quad , \quad n \neq i \in \Omega\ , \tag{5.18}$$

$$\int_0^1 (\log a_n)^{p_\nu} a_n^{-c_\nu+ns'+\delta_n}\omega_\tau(a_n)\widehat{\phi}_n(a_n)da_n \quad , \quad \text{if} \quad n \in \Omega\ , \tag{5.19}$$

(5.20)

$$\int\limits_{\substack{0<a_{j_i}\le 1,\ i\le\zeta \\ 1<a_{m'_i},i\le n-\nu}} \prod_{i=1}^{\zeta}\Big[(\log a_{j_i})^{p_{j_i}} a_{j_i}^{j_i s'+\delta_{j_i}}\widehat{\phi}_{j_i}(a_{j_i})\Big] f\left(a_{\Omega'_\eta},\pi(a(\eta'))v,\tau(\check{a}(\eta'))\right)(\xi_{\tau,s}(I))\cdot$$

$$\cdot\prod_{i=1}^{n-\nu} a_{m'_i}^{m'_i s+\delta_{m'_i}}\widehat{\phi}_{m'_i}(a_{m'_i})d\Big(a_{j_1},\dots,a_{j_\zeta},a_{m'_1},\dots,a_{m'_{n-\nu}}\Big)\ .$$

(Recall that $\eta' = \{m'_1, \dots, m'_{n-\nu}\}$.) If $n \notin \Omega$, we have to modify the last term by multiplying the $j_\zeta = n$ factor by $\omega_\tau(a_n)$. δ_i is the exponent which $\delta^{-1}(a)$ contributes at the i-th coordinate. We see that (5.18) and (5.19) do not depend on v or $\xi_{\tau,s}$, and they represent meromorphic functions which are continuous in ϕ_{m_i} (and in ϕ_n in (5.19)). By (5.16), the integral (5.20) converges absolutely in a half plane $\mathrm{Re}(s) \geq s_0$, and since $D_j, \check{D}_j$ can be taken as large as we want, s_o can be taken negative and large in absolute value, as much as we want. This establishes the meromorphic continuation to such a half plane, and the estimate (5.16) implies that this meromorphic continuation is continuous on $V_\pi \times V_{\rho_{\tau,s}}$. Clearly (5.20) is continuous in the variables ϕ_{j_i} and $\phi_{m'_i}$. We thus have proved that $R(W_\pi(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n \otimes \Phi)v, \xi_{\tau,s})$ has a meromorphic continuation which is continuous in $(\phi_1, \dots, \phi_n, \Phi, v, \xi_{\tau,s})$. The same reasoning as in Lemma 2 shows that $R(W_v, \xi_{\tau,s})$ has a meromorphic continuation which is continuous in $(v, \xi_{\tau,s})$ (whenever s is not a pole). Finally, it is easy to repeat the above arguments, for $W_{v;s_1,\dots,s_k}$, this time using Theorem 4 in Section 4, to see that $R(W_{v;s_1,\dots,s_k}, \xi_{\tau,s})$ is also meromorphic in $(s_1, \dots, s_k)$. Of course the same proofs, apply to $\widetilde{A}$ in exactly the same way. This concludes the proof of Theorem A for case $\ell \geq n$. Recall that now Theorem B (for $\ell \geq n$) also follows (from the uniqueness theorem of Section 3 and from Theorem A). ∎

6. Proof of Theorems A,B,C in case $\ell < n$

The proofs of Theorems A,B,C that we give here (in case $\ell < n$), are tied together. In particular, Theorems B and C are obtained as a result of certain identities (the theorem below). However, the uniqueness theorem of Section 3 plays an important role in the proof of these identities, and so our proofs weave Theorems A,B,C together. The proof is in the spirit of the proof of multiplicativity of the gamma factor in the p-adic case ([S], Sec.11).

Assume that π is a quotient of $\mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma$, where σ is a generic representation of $\mathrm{GL}_\ell(F)$. Since the Whittaker model of π is identical as a space to that of $\mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma$, and since the quotient map from $\mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma$ on π is open, we may assume from now on that $\pi = \mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma$. We realize σ in its Whittaker model $W(\sigma, \psi)$. The space of π consists of smooth functions ϕ on G_ℓ, which take values in $W(\sigma, \psi)$, so that

$$\phi(\overline{u}\widehat{m}g; h) = |\det m|^{-\ell/2}\phi(g; hm), \quad g \in G_\ell, \overline{u} \in \overline{Y}_\ell, m, h \in \mathrm{GL}_\ell(F) .$$

Define

$$W_\phi(g) = \int_{Y_\ell} \phi(ug, I_\ell)\psi^{-1}(u)du . \tag{6.1}$$

Formally $\phi \mapsto W_\phi(I)$ defines a Whittaker functional on the space of π and we want to relate $A(W_\phi, \xi_{\tau,s})$ with the $\mathrm{GL}_n \times \mathrm{GL}_\ell$-gamma factors of $\tau \times \sigma$ and $\tau \times \widehat{\sigma}$. The integral (6.1) may diverge, and so for $\zeta \in \mathbb{C}$, we consider

$$\pi_{\sigma,\zeta} = \mathrm{Ind}_{\overline{P}_\ell}^{G_\ell} \sigma_{-\zeta} ,$$

where

$$\sigma_{-\zeta} = \sigma \cdot |\det\cdot|^{-\zeta}$$

Then for a holomorphic section $\phi_{\sigma,\zeta}$ which now satisfies

$$\phi_{\sigma,\zeta}(\overline{u}\widehat{m}g; h) = |\det m|^{-\zeta-\ell/2}\phi_{\sigma,\zeta}(g; hm) ,$$

the integral

$$W_{\phi_{\sigma,\zeta}}(g) = \int\limits_{Y_\ell} \phi_{\sigma,\zeta}(ug, I_\ell)\psi^{-1}(u)du \tag{6.2}$$

converges absolutely for $\mathrm{Re}(\zeta) \gg 0$ and has an analytic continuation for all ζ. $W_{\phi_{\sigma,\zeta}}(I)$ defines a continuous Whittaker functional on $V_{\phi_{\sigma,\zeta}}$. If $\phi_{\sigma,\zeta_0} \neq 0$, then there is g such that $W_{\phi_{\sigma,\zeta_0}}(g) \neq 0$. Given $W \in W(\pi,\psi)$, there is a standard section $\phi_{\sigma,\zeta}$, such that $W_{\phi_{\sigma,0}} = W$. We will prove that the following *identity* holds.

$$\omega_\tau(-1)^\ell \frac{\gamma(\sigma\times\tau, s-\zeta,\psi)\gamma(\widehat{\sigma}\times\tau, s+\zeta,\psi)}{\gamma(\tau,\Lambda^2, 2s-1,\psi)} A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) = \widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \ . \tag{6.3}$$

Together with this proof the meromorphic continuation in (s,ζ) of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ and $\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ will be established. Theorem C then follows by letting ζ tend to zero. Define (first for $\mathrm{Re}(s) \gg 0$) the following Whittaker model for $\rho_{\tau,s}$.
If n is even,

$$W_{\xi_{\tau,s}}(h) = \int \xi_{\tau,s}(w_n^{-1}u(x)h, I)\psi(x_{n-1,1})dx \ . \tag{6.4}$$

If n is odd,

$$W_{\xi_{\tau,s}}(h) = \int \xi_{\tau,s}\left(w_n^{-1}\begin{pmatrix} I_{\ell+1} & v & 0 & y \\ 0 & 1 & 0 & 0 \\ & & 1 & v' \\ & & & I_{n-1}\end{pmatrix} h, I\right)\psi(v_{n-1})dvdy \ . \tag{6.5}$$

(The Weyl element w_n is defined in Section 2). These integrals have holomorphic continuations to the whole plane and define the Whittaker model of $\rho_{\tau,s}$. Note that $W_{\xi_{\tau,s}}(h)$ is a Whittaker function with respect to the standard maximal unipotent subgroup and the character

$$\begin{pmatrix} z & x \\ & z^* \end{pmatrix} \mapsto \begin{cases} \psi(z_{12}+z_{23}+\cdots+z_{n-1,n}-x_{n-1,1}) & n \text{ even} \\ \psi(z_{12}+z_{23}+\cdots+z_{n-2,n-1}-z_{n-1,n}+x_{n-1,1}) & n \text{ odd} \ . \end{cases} \tag{6.6}$$

Put

$$\widetilde{\xi}_{\tau^*,1-s} = M(w_n, \xi_{\tau,s})$$

(*see* Section 2). This is a section in the induced representation from $\tau^* \cdot |\det\cdot|^{1-s}$ to H_n. The induction is from the parabolic subgroup Q_n, if n is even, and from the parabolic subgroup $w_n\overline{Q}_n w_n^{-1}$, if n is odd. $\tau^*(m) = \tau\left(\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} {}^t m^{-1} \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}\right)$. For this representation define (first for $\mathrm{Re}(s) \ll 0$) the following Whittaker model.

If n is even,

$$W_{\widetilde{\xi}_{\tau^*,1-s}}(h) = \int \widetilde{\xi}_{\tau^*,1-s}(w_n u(x)h, b_n^*)\psi(x_{n-1,1})dx \ . \tag{6.7}$$

If n is odd,

$$W_{\widetilde{\xi}_{\tau^*,1-s}}(h) = \int \widetilde{\xi}_{\tau^*,1-s}\big(w_n u(x)h, b_{0,n}^*\big)\psi^{-1}(x_{n-1,1})dx \tag{6.8}$$

(b_n, $b_{0,n}$ are given in Section 2). These integrals have a holomorphic extension to the whole plane (provided $\widetilde{\xi}_{\tau^*,1-s}$ is holomorphic; otherwise it will be meromorphic with possible poles coming from $\widetilde{\xi}_{\tau^*,1-s}$ only). They define Whittaker models (for $\widetilde{\rho}_{\tau^*1-s}$) with respect the standard maximal unipotent subgroup and the character (6.6). The Shahidi local coefficient $\gamma(\tau, \Lambda^2, 2s-1, \psi)$ in (6.3) is defined such that

$$\gamma(\tau, \Lambda^2, 2s-1, \psi)W_{\widetilde{\xi}_{\tau^*,1-s}}(I) = W_{\xi_{\tau,s}}(I) \ . \tag{6.9}$$

Put

$$\beta'_{\ell,n} = w_n m\begin{pmatrix} & I_{n-\ell} \\ I_\ell & \end{pmatrix}\beta_{\ell,n} \ .$$

Recall that $r = n - \ell - 1$. The proof of (6.3) together with the proof of Theorem A center around the following

THEOREM. – *We have the following identities (whose interpretation is given in the proof).*

$$\begin{aligned} &\omega_\sigma(-1)^{n-1}\gamma(\sigma\times\tau, s-\zeta, \psi^{-1})A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \\ &= \int\limits_{\overline{Y}_\ell\widehat{Z}_\ell\backslash G_\ell}\ \int\limits_{M_{r\times\ell}(F)} W_{\xi_{\tau,s}}\left(m\begin{pmatrix} I_\ell & & \\ y & I_r & \\ 0 & 0 & 1\end{pmatrix}\beta'_{\ell,n}i_{\ell,n}(g)\right)\phi_{\sigma,\zeta}(g,I)dydg \ \cdot \end{aligned} \tag{6.10}$$

$$\begin{aligned} &\omega_\sigma(-1)\gamma\big(\sigma\times\widehat{\tau}, 1-s-\zeta, \psi^{-1}\big)\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \\ &= \int\limits_{\overline{Y}_\ell\widehat{Z}_\ell\backslash G_\ell}\ \int\limits_{M_{r\times\ell}(F)} W_{\widetilde{\xi}_{\tau^*,1-s}}\left(m\begin{pmatrix} I_\ell & & \\ y & I_r & \\ 0 & 0 & 1\end{pmatrix}\beta'_{\ell,n}i_{\ell,n}(g)\right)\phi_{\sigma,\zeta}(g,I)dydg \ . \end{aligned} \tag{6.11}$$

($W_{\xi_{\tau,s}}$ and $W_{\widetilde{\xi}_{\tau^*,1-s}}$ are defined by (6.4)-(6.8)).

Proof. – These identities are obtained formally exactly as in ([S] 11.4, 11.14) for case $k = \ell$. See also the first part of the proof of the theorem in [S] Section 12. This includes among other manipulations an application of the functional equation for $\sigma \times \tau$ (or $\sigma \times \widehat{\tau}$) to an inner integral of a multiple integral which converges absolutely in a vertical strip in s (which depends on ζ). The justification of the formal calculations which lead to the identities (6.10), (6.11) is as follows. Denote the right hand side of (6.10) by $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ (the right hand side of (6.11) is then $B(W_{\widetilde{\xi}_{\tau^*,1-s}}, \phi_{\sigma,\zeta})$). Note how $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ resembles the local integrals for case $\ell \geq n$, only the roles of $\operatorname{Ind}_{Q_n}^{H_n} \tau_s$ and $\operatorname{Ind}_{P_\ell}^{G_\ell} \sigma \cdot |\det \cdot|^{-\zeta}$ are interchanged, and it has the form of an integral of Rankin-Selberg type for $SO_{2n} \times \mathrm{GL}_\ell$. The identity (6.10) (still formal) brings out a duality between $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ and $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ and relates these two local integrals, one for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$ and one for $SO_{2n} \times \mathrm{GL}_\ell$ by a functional equation. Since the justifications for both (6.10) and (6.11) are the same, we explain just the first case. As in [S], Section 5, it is easy to see that for fixed s and $\mathrm{Re}(\zeta)$ large enough (depending on s), the integral $B(W_{\xi_{\tau,s}} \phi_{\tau,\zeta})$ converges absolutely, and moreover, exactly as in the proof of Theorem A for case $\ell \geq n$ (Section 5), we see that $B(W_{\xi_{\tau,s}} \phi_{\sigma,\zeta})$ has a meromorphic continuation in (ζ, s) to $\mathbb{C}^2$ which is (when defined) continuous on $V_{\rho_{\tau,s}} \times V_{\pi_{\sigma,\zeta}}$. Since $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ is obtained by formal manipulations on $A(W_{\phi_{\sigma,\xi}}, \xi_{\tau,s})$, it is easy to see that $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ satisfies the equivariance property (2.1), and so by the uniqueness theorem of Section 3, $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ is proportional to $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$, in the domain of convergence of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$. The continuity of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ in its domain of convergence is explained in Lemma 1 below. To find the proportionality factor, we calculate $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ and $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ for a special substitution. This is done in Lemma 2 below, and the theorem is proved. ■

Remark. – The proof above follows the same line of argument that we used in the p-adic case in [S], Section 11, when we proved the multiplicativity of the gamma factor. There we already knew in advance that $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ is a rational function in q^{-s}, and the uniqueness theorem ([S], Section 8) had no continuity requirements. Here we overcome this crucial issue of continuity, by first establishing it in the case of integrals of type $B(W, \phi_{\sigma,\zeta})$ for $\mathrm{SO}_{2n} \times \mathrm{GL}_\ell$, $n > \ell$ (exactly as in Section 5), and then we compare $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ and $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ in the region of convergence of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$.

Proof of Theorem A (in case $\ell < n$). – The identities (6.10) and (6.11) which initially hold in the domain of convergence of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ (resp. $\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$) already provide the meromorphic continuation in (s, ζ) of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ and $\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ as well as their continuity on $V_{\pi_{\sigma,\zeta}} \times V_{\rho_{\tau,s}}$. At this point we note, continuing the above remark, that this is analogous to the way we obtained the unramified computation of $A(W, \xi_{\tau,s})$ in [S] Section 12. We computed $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ for unramified data through the identity (6.10), the computation for $B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$ being known. If π is a quotient of $I(\sigma_1, \ldots, \sigma_n; \sigma', s_1, \ldots, s_k)$ as in Theorem A, we write σ' as a quotient of $\operatorname{Ind}_{P_{\ell-(t_1+\cdots+t_k)}}^{G_{\ell-(t_1+\cdots+t+k)}} \sigma''$, where σ'' is a generic representation of $\mathrm{GL}_{\ell-(t_1+\cdots+t_k)}(F)$, and then replace σ by the representation $\sigma_{s_1,\ldots,s_k} = \sigma_1 |\det \cdot|^{-s_1} \otimes \cdots \otimes \sigma_k |\det \cdot|^{-s_k} \otimes \sigma''$. The results of Section 4 imply that $B(W_{\xi_{\tau,s}}, W_{\phi_{s_1,\ldots,s_k;\zeta}})$ is meromorphic in $(s_1, \ldots, s_k, \zeta, s)$. (Again, the proof is as in Section 5.) Thus the same is true for $A(W_{\phi_{\sigma_{s_1,\ldots,s_k;\zeta}}}, \xi_{\tau,s})$

by (6.10), and similarly for $\widetilde{A}(W_{\phi_{\sigma_{s_1,\ldots,s_k;\zeta}}}, \xi_{\tau,s})$ by (6.11). This completes the proof of Theorem A for case $\ell < n$.

Proof of Theorem C (in case $\ell < n$). – This proof follows exactly as in ([S] 11.4, 11.14). We bring it for completeness sake. Consider identity (6.11) (of meromorphic functions in (s,ζ))

$$\omega_\sigma(-1)\gamma(\sigma \times \widehat{\tau}, 1-s-\zeta, \psi^{-1})\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) = B(W_{\widetilde{\xi}_{\tau^*,1-s}}, \phi_{\sigma,\zeta}) \; .$$

By (6.9), we have

$$\gamma(\tau, \Lambda^2, 2s-1, \psi)B(W_{\widetilde{\xi}_{\tau^*,1-s}}, \phi_{\sigma,\zeta}) = B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta})$$

and by (6.10), we have

$$B(W_{\xi_{\tau,s}}, \phi_{\sigma,\zeta}) = \omega_\sigma(-1)^{n-1}\gamma(\sigma \times \tau, s-\zeta, \psi^{-1})A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \; .$$

It follows that

$$\begin{aligned}&\omega_\sigma(-1)^n\gamma(\sigma \times \tau, s-\zeta, \psi^{-1})A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})\\ &= \gamma(\tau, \Lambda^2, 2s-1, \psi)\gamma(\sigma \times \widehat{\tau}, 1-s-\zeta, \psi^{-1})\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}).\end{aligned}$$

Thus we get

$$\omega_\sigma(-1)^n \frac{\gamma(\sigma \times \tau, s-\zeta, \psi^{-1})\gamma(\widehat{\sigma} \times \tau, s+\zeta, \psi)}{\gamma(\tau, \Lambda^2, 2s-1, \psi)} A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) = \widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \; .$$

Finally, note that

$$\omega_\sigma(-1)^n\gamma(\sigma \times \tau, s-\zeta, \psi^{-1}) = \omega_\tau(-1)^\ell\gamma(\sigma \times \tau, s-\zeta, \psi) \; .$$

This proves Theorem C (for $\ell < n$). ■

It remains to prove the following two lemmas, which were needed in the proof of the theorem of this section. These lemmas concern the identity (6.10). The two analogous lemmas for (6.11) are obtained similarly.

LEMMA 1. – *For $\ell < n$, $A(W_v, \xi_{\tau,s})$ is continuous on $V_\pi \times V_{\rho_{\tau,s}}$, if $\mathrm{Re}(s)$ is large enough.*

Proof. – As in Section 5, it is enough to prove the continuity of

$$R(v, \phi, \xi_{\tau,s}) = \int\limits_{A_\ell^{(+)}} W_v(\widehat{b})\phi(b_1, \ldots, b_\ell)\delta^{-1}(b) \int\limits_{\overline{X}^{(\ell,n)}} f_{\xi_{\tau,s}}\left(\overline{x}m\begin{pmatrix} b & \\ & I_{n-\ell}\end{pmatrix}\right)\psi_a(\overline{x})d\overline{x}db$$

in $\phi \in S(F^\ell), v \in V_\pi, \xi_{\tau,s} \in V_{\rho_{\tau,s}}$, for $\mathrm{Re}(s) \gg 0$. b has the form $\mathrm{diag}(b_1 b_2 \cdot \cdots \cdot b_\ell, b_2 \cdot \cdots \cdot b_\ell, \ldots, b_{\ell-1}b_\ell, b_\ell), b_i > 0$, and $\delta(b) = \delta(\widehat{b})$ is the modular function of G_ℓ with respect to the standard Borel subgroup. There are a positive number μ and continuous seminorms q on V_π and p on $V_{\rho_{\tau,s}}$ such that

$$(6.12)\qquad \begin{aligned} &|W_v(g)| \le \|g\|^\mu q(v)\ , && v\in V_\pi,\ g\in G_\ell\ ,\\ &|\xi_{\tau,s}(k,m)| \le \|m\|^\mu p(\xi_{\tau,s})\ , && \xi_{\tau,s}\in V_{\rho_{\tau,s}},\ k\in K_{H_n},\ m\in \mathrm{GL}_n(F)\ .\end{aligned}$$

(The norm on $\mathrm{GL}_n(F)$ is one which comes from $P\mathrm{GL}_n(F)$). Thus

$$(6.13)\qquad \begin{aligned} |R(v,\phi,\xi_{\tau,s})| \le \int\limits_{A_\ell^{(+)}} &\|\widehat{b}\|^\mu \delta^{-1}(b)|\det b|^{-r}\phi(b_1,\dots,b_\ell)\\ &\int\limits_{\overline{X}^{(\ell,n)}} \left|f_{\xi_{\tau,s}}\left(m\begin{pmatrix} b & \\ & I_{n-\ell}\end{pmatrix}\overline{x}\right)\right| d\overline{x}db\cdot q(v)\ .\end{aligned}$$

Now write the Iwasawa decompostion of $\overline{x}=\overline{u}(x)=v_x m(t_x)k_x$, with $v_x\in V_n,\ t_x\in A_N^{(+)},\ k_x\in K_{H_n}$. By [S], 7.3 Lemma 3, we have

$$(6.14)\qquad (1+\|x\|^2)^{-\frac{n}{2}} \le \det(t_x) \le (1+\|x\|^2)^{-1/2}$$

and there is a positive number M such that

$$(6.15)\qquad \|t_x\| \le (1+\|x\|^2)^M$$

($\|x\|$ is the Euclidean norm of the matrix x). It follows from (6.12)-(6.15) that for $\mathrm{Re}(s)\gg 0$,

$$\begin{aligned} |R(v,\phi,\xi_{\tau,s})| \le \int\limits_{A_\ell^{(+)}} &\|\widehat{b}\|^\mu \left\|\begin{pmatrix} b & \\ & I_{n-\ell}\end{pmatrix}\right\|^\mu \delta^{-1}(b)|\det b|^{\mathrm{Re}(s)+\frac{n-2}{2}-r}\phi(b_1,\dots,b_\ell)db\cdot\\ &\int\limits_{\overline{X}^{(\ell,n)}} (1+\|x\|^2)^{-\frac{1}{2}(\mathrm{Re}(s)+\frac{n-2}{2})+M\mu}d\overline{x}\cdot q(v)p(\xi_{\tau,s})\ .\end{aligned}$$

The db-integral converges for $\mathrm{Re}(s)$ large enough and is continuous in ϕ. The dx-integral converges for $\mathrm{Re}(s)$ large enough, and the Lemma is proved. ■

Lemma 2. – *Let $\ell < n$. Given $W_\sigma\in W(\sigma,\psi)$ and $W_\tau\in W(\tau,\psi^{-1})$, there are $\phi^{(j)}_{\sigma,\zeta}\in V_{\pi_{\sigma,\zeta}}$ and $\xi^{(j)}_{\tau,s}\in V_{\rho_{\tau,s}}$, $1\le j\le N$, such that*

$$(6.16)\qquad \sum_{j=1}^N A(W_{\phi^{(j)}_{\sigma,\zeta}},\xi^{(j)}_{\tau,s}) = \int\limits_{Z_\ell|\mathrm{GL}_\ell(F)} W_\tau\begin{pmatrix} m & \\ & I_{n-\ell}\end{pmatrix} W_\sigma(m)|\det m|^{s-\zeta-\frac{n-\ell}{2}}dm$$

and

(6.17)

$$\sum_{j=1}^{N} B(W_{\xi^{(j)}_{\tau,s}}\phi^{(j)}_{\sigma,\zeta}) = \int\limits_{Z_\ell\backslash \mathrm{GL}_\ell(F)}\ \int\limits_{M_{\ell\times r}(F)} W_\tau\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_r \\ m & 0 & y\end{pmatrix} W_\sigma(m)|\det m|^{s-\zeta-\frac{n-\ell}{2}}dydm\ .$$

Proof. – The equalities (6.16), (6.17) should be interpreted as equalities of meromorphic functions. Note that the right hand sides of (6.16), (6.17) are the typical local integrals, which appear in the local functional equation for $\mathrm{GL}_n(F)\times \mathrm{GL}_\ell(F)$ and $\tau\times\sigma$. *See* [J.S.]. The substitutions which give (6.16) are described in [S], Proposition 7.3. In exactly the same way, it is easy to see that those same substitutions give (6.17) as well. ∎

7. Proof of Theorem C in case $\ell \geq n$

We use the notation of Section 6. We will prove that the following *identity* holds

$$\omega_\tau(-1)^{\ell+n}\frac{\gamma(\sigma\times\tau, s-\zeta,\psi)\gamma(\widehat{\sigma}\times\tau, s+\zeta,\psi)}{\gamma(\tau,\Lambda^2, 2s-1,\psi')}A(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) = \widetilde{A}(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) \tag{7.1}$$

The precise definition of $\gamma(\tau,\Lambda^2,2s-1,\psi')$ is as follows. Define (first for $\mathrm{Re}(s)\gg 0$) the following Whittaker model for $\rho_{\tau,s}$.
If n is even,

$$W'_{\xi_{\tau,s}}(h) = \int \xi_{\tau,s}(w_n^{-1}u(x)h, I)\psi(\tfrac{1}{2}x_{n-1,1})dx \tag{7.2}$$

If n is odd,

$$W'_{\xi_{\tau,s}}(h) = \int \xi_{\tau,s}\left(w_n^{-1}\left(\begin{array}{cc|cc} I_{n-1} & v & 0 & y \\ 0 & 1 & 0 & 0 \\ \hline & & 1 & v' \\ & & & I_{n-1}\end{array}\right)h, I\right)\psi(\tfrac{1}{2}v_{n-1})dvdy\ . \tag{7.3}$$

These integrals have holomorphic continuation to the whole plane and define the Whittaker model of $\rho_{\tau,s}$, $W'_{\xi_{\tau,s}}(h)$ is a Whittaker function with respect to the standard maximal unipotent subgroup and the character

$$\begin{pmatrix} z & x \\ 0 & z^*\end{pmatrix} \mapsto \begin{cases} \psi(z_{12}+\cdots+z_{n-1,n}-\tfrac{1}{2}x_{n-1,1})\ , & n \text{ even} \\[2ex] \psi(z_{12}+\cdot+z_{n-2,n-1}-\tfrac{1}{2}z_{n-1,n}+x_{n-1,1})\ , & n \text{ odd}\ .\end{cases} \tag{7.4}$$

Also define (as in Section 6), first for $\mathrm{Re}\, s \ll 0$, the following Whittaker models for $\widetilde{\xi}_{\tau^*,1-s}$.
If n is even,

$$W'_{\widetilde{\xi}_{\tau^*,1-s}}(h) = \int \widetilde{\xi}_{\tau^*,1-s}(w_n u(x)h, b_n^*)\psi(\tfrac{1}{2}x_{n-1,1})dx \tag{7.5}$$

If n is odd,

$$W'_{\widetilde{\xi}_{\tau^*,1-s}}(h) = \int \widetilde{\xi}_{\tau^*,1-s}\left(w_n u(x) m\begin{pmatrix} I_{n-1} & \\ & -2 \end{pmatrix} h, b_n^*\right)\psi(\tfrac{1}{2}x_{n-1,1})dx \ . \tag{7.6}$$

These define Whittaker models for $\widetilde{\xi}_{\tau^*,1-s}$, with respect to the standard maximal unipotent subgroup and the character (7.4). The local coefficient $\gamma(\tau, \Lambda^2, 2s-1, \psi')$ in (7.1) is defined such that

$$\gamma(\tau, \Lambda^2, 2s-1, \psi') W'_{\widetilde{\xi}_{\tau^*,1-s}}(I) = W'_{\xi_{\tau,s}}(I) \ . \tag{7.7}$$

As in Section 6, the proof of (7.1) is based on the identities in the following theorem.

THEOREM. – *We have the following identities in case $\ell > n$.*

$$\begin{aligned} &\omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\tau, s-\zeta, \psi) A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \\ &= \int\limits_{V_n\backslash H_n} W'_{\xi_{\tau,s}}(g) \int \phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix} \mu_{n,\ell} j_{n,\ell}(g); I_\ell\right) \\ &\psi^{-1}(v_{\ell-n}) d(v,c,y) dg \ , \end{aligned} \tag{7.8}$$

$$\begin{aligned} &\omega_\sigma(-1)^n \omega_\tau(-1)^{\ell+n-1}\gamma(\sigma\times\widehat{\tau}, 1-s-\zeta, \psi)\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \\ &= \int\limits_{V_n\backslash H_n} W'_{\widetilde{\xi}_{\tau^*,1-s}}(g) \int \phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix} \mu_{n,\ell} j_{n,\ell}(g); I_\ell\right) \\ &\psi^{-1}(v_{\ell-n}) d(v,c,y) dg \ . \end{aligned} \tag{7.9}$$

Here

$$\begin{aligned} \mu_{n,\ell} &= \begin{pmatrix} I_{\ell-n} & & & & \\ & I_n & e_1 & A_1 & \\ & & 1 & e_1' & \\ & & & I_n & \\ & & & & I_{\ell-n} \end{pmatrix} m\begin{pmatrix} & I_{\ell-n} \\ I_n & \end{pmatrix} j_{n,\ell}(w_n^{-1}); \\ e_1 &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A_1 = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \ . \end{aligned} \tag{7.10}$$

If $\ell = n$, we have

$$\begin{aligned}\omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\tau, s-\zeta,\psi)A(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) \\ = \int_{V_n\backslash H_n} W'_{\xi_{\tau,s}}(g)\phi_{\sigma,\zeta}(\mu_n j_n(g), I_n)dg\ ,\end{aligned} \tag{7.11}$$

$$\begin{aligned}\omega_\sigma(-1)^n\omega_\tau(-1)^{\ell+n-1}\gamma(\sigma\times\widehat{\tau}, 1-s-\zeta,\psi)\widetilde{A}(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) \\ = \int_{V_n\backslash H_n} W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)\phi_{\sigma,\zeta}(\mu_n j_n(g); I_n)dg\ .\end{aligned} \tag{7.12}$$

Here $j_n(g) = j_{n,n}(g)$ and

$$\mu_n = \begin{pmatrix} I_n & e_1 & A_1 \\ & 1 & e_1' \\ & & I_n \end{pmatrix} j_n(w_n^{-1})\ . \tag{7.13}$$

Proof. – Since Theorem A is proved in this case (Section 5), then the formal manipulations that we are going to use are interpreted in exactly the same way as in Section 6, using Theorem A and the uniqueness theorem of Section 3. We therefore explain just the formal proof of the identities above. Note again that these identities reflect a duality of local Rankin Selberg integrals, one for $\mathrm{SO}_{2\ell+1}\times\mathrm{GL}_n$ and one for $\mathrm{SO}_{2n}\times\mathrm{GL}_\ell$. Assume first that $\ell > n$. Substitute $W_{\phi_{\sigma,\zeta}}$ (from (6.2)) in $A(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s})$. We get

$$\int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}}\int_{Y_\ell}\phi_{\sigma,\zeta}(u\overline{x}j_{n,\ell}(g); I_\ell)\psi^{-1}(u)f_{\xi_{\tau,s}}(g)dud\overline{x}dg\ . \tag{7.14}$$

Factor the integration on $V_n\backslash H_n$ through L_n. We get

$$\int_{Q_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}}\int_{Y_\ell}\int_{Z_n\backslash\mathrm{GL}_n(F)}\phi_{\sigma,\zeta}\left(u\overline{x}\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge} j_{n,\ell}(g); I_\ell\right)\ .$$

$$\psi^{-1}(u)f_{\xi_{\tau,s}}(m(a)g)|\det a|^{1-n}dadud\overline{x}dg$$

The precise meaning of $\int\limits_{Q_n\backslash H_n}$ is $\int\limits_{K_{H_n}}$. Write

$$\overline{x}\begin{pmatrix} a & \\ & I_{\ell-n}\end{pmatrix}^{\wedge} = \begin{pmatrix} a & \\ xa & I_{\ell-n}\end{pmatrix}^{\wedge}, \quad x\in M_{(\ell-n)\times n}(F),\ a\in\mathrm{GL}_n(F)\ .$$

Change variable $x\mapsto xa^{-1}$, and write

$$\begin{pmatrix} a & \\ x & I_{\ell-n}\end{pmatrix}^{\wedge} = \begin{pmatrix} a & & \\ r & I_{\ell-n-1} & \\ 0 & 0 & 1\end{pmatrix}^{\wedge}\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1\end{pmatrix}^{\wedge}\ .$$

Conjugate u by $\begin{pmatrix} a & & \\ r & I_{\ell-n-1} & \\ 0 & 0 & 1 \end{pmatrix}^{\wedge}$ ($\psi(u)$ is not affected). We get

$$(7.16)\quad \int\limits_{Q_n\backslash H_n}\int\limits_{t\in F^n}\int\limits_{Y_\ell}\psi^{-1}(u)\Bigg(\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int\limits_{M_{(\ell-n-1)\times n}(F)}\phi_{\sigma,\zeta}\Bigg(u\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1 \end{pmatrix}^{\wedge} j_{n,\ell}(g);\begin{pmatrix} a & & \\ r & I_{\ell-n-1} & \\ 0 & 0 & 1 \end{pmatrix}\Bigg)\xi_{\tau,s}(g,a)\cdot|\det a|^{s-\zeta-\frac{\ell-n}{2}}drda\Bigg)dudtdg\ .$$

The inner $drda$ integration in (7.16) is (for t,u,g fixed) a local integral for the theory of $\mathrm{GL}_\ell\times\mathrm{GL}_n$. Thus, using the local functional equation in this case, we have

$$(7.17)\quad \omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\tau,s-\zeta,\psi)A(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s})=\int\limits_{Q_n\backslash H_n}\int\limits_{t\in F^n}\int\limits_{Y_\ell}\psi^{-1}(u)\Bigg(\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\phi_{\sigma,\zeta}\Bigg(u\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1 \end{pmatrix}^{\wedge} j_{n,\ell}(g);\begin{pmatrix} & I_{\ell-n} \\ a & \end{pmatrix}\Bigg)\xi_{\tau,s}(g,a)\cdot|\det a|^{s-\zeta-\frac{\ell-n}{2}-1}da\Bigg)dudtdg\ .$$

Conjugate $\begin{pmatrix} a & \\ & I_{\ell-n} \end{pmatrix}^{\wedge}$ across in (7.17) and let

$$w_{n,\ell}=\begin{pmatrix} & I_{\ell-n} \\ I_n & \end{pmatrix}\ .$$

We get

(7.18)

$$\int\limits_{Q_n\backslash H_n}\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int\limits_{Y_\ell}\psi^{-1}(u)\phi_{\sigma,\zeta}\Bigg(\widehat{w}_{n,\ell}u\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1 \end{pmatrix}^{\wedge} j_{n,\ell}(m(a)g);I_\ell\Bigg) f_{\xi_{\tau,s}}(m(a)g)|\mathrm{deta}|^{1-n}dudtdadg$$

$$=\int\limits_{V_n\backslash H_n}\int\limits_{t\in F^n}\int\limits_{Y_\ell}\psi^{-1}(u)\phi_{\sigma,\zeta}\Bigg(\widehat{w}_{n,\ell}u\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1 \end{pmatrix}^{\wedge} j_{n,\ell}(g);I_\ell\Bigg) f_{\xi_{\tau,s}}(g)dudtdg\ .$$

Write

$$u=\begin{pmatrix} I_n & 0 & \widetilde{v} & * & * \\ 0 & I_{\ell-n} & v & * & * \\ & & 1 & v' & \widetilde{v}' \\ & & & I_{\ell-n} & 0 \\ & & & & I_n \end{pmatrix}\ .$$

Then

$$\phi_{\sigma,\zeta}\left(\widehat{w}_{n,\ell}u\begin{pmatrix} I_n & & \\ 0 & I_{\ell-n-1} & \\ t & 0 & 1\end{pmatrix}j_{n,\ell}(g);I_\ell\right)=$$

$$=\phi_{\sigma,\zeta}\left(\widehat{w}_{n,\ell}\begin{pmatrix} I_n & 0 & \widetilde{v} & * & * \\ 0 & I_{\ell-n} & v-\begin{pmatrix}0\\t\end{pmatrix}\widetilde{v} & * & * \\ & & 1 & * & \widetilde{v}' \\ & & & I_{\ell-n} & 0 \\ & & & & I_n\end{pmatrix}j_{n,\ell}(g);\begin{pmatrix} I_{\ell-n-1} & 0 & 0 \\ & 1 & t \\ & & I_n\end{pmatrix}\right).$$

Changing variable $v\mapsto v+\begin{pmatrix}0\\t\end{pmatrix}\widetilde{v}$ in (7.18), we get that the (formal...)dt integration in (7.18) forces $\widetilde{v}=e_1=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$. Write

$$u=\begin{pmatrix} I_n & 0 & e_1 & 0 & A_1 \\ & I_{\ell-n} & 0 & 0 & 0 \\ & & 1 & 0 & e_1 \\ & & & I_{\ell-n} & 0 \\ & & & & I_n\end{pmatrix}\cdot\begin{pmatrix} I_n & 0 & 0 & c' & 0 \\ & I_{\ell-n} & 0 & 0 & c \\ & & 1 & 0 & 0 \\ & & & I_{\ell-n} & c \\ & & & & I_n\end{pmatrix}\cdot$$
$$\begin{pmatrix} I_n & 0 & 0 & 0 & z \\ 0 & I_{\ell-n} & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & I_{\ell-n} & 0 \\ & & & & I_n\end{pmatrix}\cdot\begin{pmatrix} I_n & & & & \\ & I_{\ell-n} & v & y & \\ & & 1 & v' & \\ & & & I_{\ell-n} & \\ & & & & I_n\end{pmatrix}.$$

Denote the second matrix in this product by $\alpha(c)$ and the last by $\nu(v,y)$. Note that the third matrix is $j_{n,\ell}(u(z))$ and that $\nu(v,y)$ commutes with $j_{n,\ell}(g)$. The integral (7.18) becomes (7.19)

$$\int\limits_{V_n\backslash H_n}\int\limits_{U_n}\left(\int\phi_{\sigma,u}\left(\gamma_{n,\ell}\widehat{w}_{n,\ell}\alpha(c)j_{n,\ell}(u(z)g)\nu(v,y);I_\ell\right)\right.$$
$$\left.\psi^{-1}(v_{\ell-n})f_{\xi_{\tau,s}}(u(z)g)d\nu(v,y)dc\right)du(z)dg$$

$$=\int\limits_{m(Z_n)\backslash H_n}\int\phi_{\sigma,u}\left(\gamma_{n,\ell}\widehat{w}_{n,\ell}\alpha(c)j_{n,\ell}(g)\nu(v,y);I_\ell\right)\psi^{-1}(v_{\ell-n})f_{\xi_{\tau,s}}(g)d\nu(v,y)dcdg\,.$$

Here

$$\gamma_{n,\ell}=\begin{pmatrix} I_{\ell-n} &&&& \\ & I_n & e_1 & A_1 & \\ && 1 & e_1' & \\ &&& I_n & \\ &&&& I_{\ell-n}\end{pmatrix}.$$

Now factor the dg integration in (7.19) through $\overline{U}_n$. We have
(7.20)

$$\gamma_{n,\ell}\widehat{w}_{n,\ell}\alpha(c)j_{n,\ell}(\overline{u}(x))$$
$$=\begin{pmatrix} I_{\ell-n} &&&& \\ & I_n &&& \\ & * & 1 && \\ & * & * & I_n & \\ &&&& I_{\ell-n}\end{pmatrix}\begin{pmatrix} I_{\ell-n} & \\ & I_n+A_1x\end{pmatrix}\gamma_{n,\ell}\begin{pmatrix} I_{\ell-n} & c'x & 0 & c & 0 \\ & I_n & 0 & 0 & c' \\ && 1 & 0 & 0 \\ &&& I_n & -xc \\ &&&& I_{\ell-n}\end{pmatrix}\widehat{w}_{n,\ell}.$$

Thus

$$\phi_{\sigma,\zeta}\left(\gamma_{n,\ell}\widehat{w}_{n,\ell}\alpha(c)j_{n,\ell}(\overline{u}(x)g)\nu(v,y),I_\ell\right)$$
$$=\psi(\tfrac{1}{2}x_{n-1,1})\phi_{\sigma,\zeta}\left(\gamma_{n,\ell}\begin{pmatrix} I_{\ell-n} & cx & 0 & c & 0 \\ & I_n & 0 & 0 & c' \\ && 1 & 0 & 0 \\ &&& I_n & -xc' \\ &&&& I_{\ell-n}\end{pmatrix}\widehat{w}_{n,\ell}j_{n,\ell}(g)\nu(v,y);I_\ell\right).$$

The integral (7.19) becomes

$$\int\limits_{m(Z_n)\overline{U}_n\backslash H_n}\int\phi_{\sigma,\zeta}\left(\gamma_{n,\ell}\begin{pmatrix} I_{\ell-n} & cx & v & c & y \\ & I_n & 0 & 0 & c' \\ && 1 & 0 & v' \\ &&& I_n & -xc' \\ &&&& I_{\ell-n}\end{pmatrix}\widehat{w}_{n,\ell}j_{n,\ell}(g);I_\ell\right)$$
$$\psi(\tfrac{1}{2}x_{n-1,1})\psi^{-1}(v_{\ell-n})\cdot f_{\xi_{\tau,s}}(\overline{u}(x)g)d(x,v,y,c)dg\ .$$

We have

$$\gamma_{n,\ell}\begin{pmatrix} I_{\ell-n} & cx & v & c & y \\ & I_n & 0 & 0 & c' \\ && 1 & 0 & v' \\ &&& I_n & -xc' \\ &&&& I_{\ell-n}\end{pmatrix}\gamma_{n,\ell}^{-1}$$
$$=\begin{pmatrix} I_{\ell-n} & cx & v-cxe_1 & c(I+xA_1)-ve_1' & y \\ & I_n & 0 & 0 & (I-A_1x)c'+e_1v' \\ && 1 & 0 & v'-e_1'xc' \\ &&& I_n & -xc' \\ &&&& I_{\ell-n}\end{pmatrix}.$$

Change variable $v \mapsto v + cxe_1$. Note that $\psi((cx)_{\ell-n,1}) = \psi((cxe_1)_{\ell-n})$ and that $c(I + xA_1) - v'e_1 \mapsto c(I - xA_1) - ve_1'$. Change variable again $c \mapsto c(I - xA_1)^{-1}$ and then $c \mapsto c + ve_1'$. We get

(7.21)

$$\int\limits_{m(Z_n)\overline{U}_n\backslash H_n} \Bigg(\int \phi_{\sigma,\zeta}\Bigg(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n}\end{pmatrix}\gamma_{n,\ell}\widehat{w}_{n,\ell}j_{n,\ell}(g); I_\ell\Bigg)\psi^{-1}(v_{\ell-n})d(v,c,y)\Bigg)$$

$$\cdot\Bigg(\int \psi(\tfrac{1}{2}x_{n-1,1})f_{\xi_{\tau,s}}(\overline{u}(x)g)dx\Bigg)dg\ .$$

The inner dx integral in (7.21) is a Whittaker integral for $\rho_{\tau,s}$, and it equals $W'_{\xi_{\tau,s}}(w_n g)$ (*see* (7.2), (7.3)). Thus (7.21) becomes

(7.22)

$$\int\limits_{V_n\backslash H_n} W'_{\xi_{\tau,s}}(g)\int \phi_{\sigma,\zeta}\Bigg(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n}\end{pmatrix}\mu_{n,\ell}j_{n,\ell}(g); I_\ell\Bigg)\psi^{-1}(v_{n-\ell})d(v,c,y)dg\ ,$$

where $\mu_{n,\ell} = \gamma_{n,\ell}\widehat{w}_{n,\ell}j_{n,\ell}(w_n^{-1})$. This proves (7.8).

Assume now that $\ell = n$. The (formal) proof is similar and should be modified as follows. Note that there is no unipotent integration in $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$. As in (7.15), $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ equals

$$(7.23)\qquad \int\limits_{Q_n\backslash H_n}\int\limits_{Y_\ell}\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} \phi_{\sigma,\zeta}\Big(u\widehat{a}j_n(g); I_n\Big)\psi^{-1}(u)f_{\xi_{\tau,s}}(m(a)g)|\det a|^{1-n}dadudg\ .$$

Here $j_n(g) = j_{n,n}(g)$. Note that $\psi(\widehat{a}u\widehat{a}^{-1}) = \psi(\varepsilon_n av)$ where $\varepsilon_n = (0\cdots 01)$ (n coordinates), $u = y(v,z) = \begin{pmatrix} I_n & v & z \\ & 1 & v' \\ & & I_n\end{pmatrix}$. Again "$\int\limits_{Q_n\backslash H_n}$" means "$\int\limits_{K_{H_n}}$". The integral (7.23) becomes

$$(7.24)\qquad \begin{aligned}&\int\limits_{Q_n\backslash H_n}\int\limits_{Y_\ell}\Bigg(\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} \phi_{\sigma,\zeta}\Big(y(v,z)j_n(g); a\Big)\xi_{\tau,s}(g,a)\\ &\psi^{-1}(\varepsilon_n av)|\det a|^{s-\zeta}da\Bigg)dy(v,z)dg\ .\end{aligned}$$

Formally, the inner da integral, is (for $y(v,z), g$ fixed) a local integral for $\mathrm{GL}_n(F)\times \mathrm{GL}_n(F)$ and (σ,τ). The function $\varphi_v(x_1,\ldots,x_n) = \psi^{-1}(\sum_{i=1}^n x_iv_i)$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}$ plays the role of the "Schwartz function". Now applying the functional equation for (σ,τ), we get

$$\begin{aligned}
&\omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\tau, s-\zeta,\psi)A(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) \\
&= \int\limits_{Q_n\backslash H_n}\int\limits_{Y_\ell}\Bigg(\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} \phi_{\sigma,\zeta}(y(v,z)j_n(g);a)\xi_{\tau,s}(g,a) \\
&\qquad \widehat{\varphi}_v(\varepsilon_1\cdot {}^ta^{-1})|\det a|^{1+s-\zeta}da\Bigg)dy(v,z)dg\ .
\end{aligned}\tag{7.25}$$

Here $\varepsilon_1 = (10\cdots 0)$.

We have $\widehat{\varphi}_v(\varepsilon_1{}^ta^{-1}) = \delta_{v,a^{-1}e_1}$, and (7.25) becomes

$$\int\limits_{Q_n\backslash H_n}\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int \phi_{\sigma,\zeta}(y(a^{-1}e_1,z)j_n(g);a)\xi_{\tau,s}(g,a)|\det a|^{1+s-\xi}dadzdg \tag{7.26}$$

$$= \int\limits_{Q_n\backslash H_n}\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int \phi_{\sigma,\zeta}\Big(\gamma_nj_n(u(z)m(a)g);I_n\Big)\xi_{\tau,s}(u(z)m(a)g;I_n)|\det a|^{1-n}dadzdg$$

$$= \int\limits_{m(Z_n)\backslash H_n} \phi_{\sigma,\zeta}(\gamma_nj_n(g);I_n)f_{\xi_{\tau,s}}(g)dg\ . \tag{7.27}$$

Here $\gamma_n = \begin{pmatrix} I_n & e_1 & A_1\\ & 1 & e_1'\\ & & I_n\end{pmatrix}$. Now factor the integration in (7.27) through $\overline{U}_n$. As in (7.20)

$$\gamma_nj_n(\overline{u}(x)) = \begin{pmatrix} I_n & & \\ * & 1 & \\ * & * & I_n\end{pmatrix}(I_n + A_1x)^\wedge\gamma_n\ .$$

Thus the integral (7.27) becomes

$$\begin{aligned}
&\int\limits_{m(Z_n)\overline{U}_n\backslash H_n} \phi_{\sigma,\zeta}(\gamma_nj_n(g);I_n)\int\psi(\tfrac{1}{2}x_{n-1,1})f_{\xi_{\tau,s}}(\overline{u}(x)g)dxdg \\
&= \int\limits_{V_n\backslash H_n} W'_{\xi_{\tau,s}}(g)\phi_{\sigma,\zeta}(\mu_nj_n(g);I_n)dg\ ,
\end{aligned}\tag{7.28}$$

where $\mu_n = \gamma_nj_n(w_n^{-1})$. This proves (7.11).

[Again the interpretation and justification of the above formal proof are as in case $\ell<n$. If some readers still find some of the above steps too outrageous, here is an example. Consider the passage from (7.24) to (7.26). The convergence of these integrals in their appropriate domains can be established in a way similar to case $\ell<n$. Note

that in Section 5, we proved that the meromorphic continuation of $A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$ exists and is continuous on $V_{\pi_{\sigma,\zeta}} \times V_{\rho_{\tau,s}}$ as well as meromorphic in parameters. Clearly both integrals satisfy the equivariance property (2.2), and so they differ by a proportionality factor (Section 3). To find the factor, we use the following substitutions. Let $\xi_{\tau,s}$ have support in $Q_n\overline{U}_n$, such that

$$\xi_{\tau,s}(m(a)\overline{u}(x); I_n) = |\det a|^{s+\frac{n-2}{2}}\varphi(x)W_\tau(a) ,$$

where φ is a Schwartz function and $W_\tau \in W(\tau, \psi^{-1})$. In (7.24), " $\int\limits_{Q_n\backslash H_n}$ " may also be interpreted as $\int\limits_{\overline{U}_n}$. Thus (7.24) becomes

$$\int\limits_{Y_\ell}\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} \pi_{\sigma,\zeta}(\varphi)\phi_{\sigma,\zeta}(y(v,z);a)W_\tau(a)\psi^{-1}(\varepsilon_n av)|\det a|^{s-\zeta}dady(v,z) ,$$

where

$$\pi_{\sigma,\zeta}(\varphi)\phi_{\sigma,\zeta} = \int \varphi(x)\pi_{\sigma,\zeta}(j_n(\overline{u}(x)\phi_{\sigma,\zeta}dx .$$

By [D.M.], every $\phi_{\sigma,\zeta}$ (smooth) can be written as a linear combination $\sum \pi_{\sigma,\zeta}(\varphi_i)\phi^{(i)}_{\sigma,\zeta}$. Thus for a given $\phi_{\sigma,\zeta}$, there is a linear combination of integrals of type (7.24), which equals

$$\int\limits_{Y_\ell}\int\limits_{Z_n\mathrm{GL}_n(F)} \phi_{\sigma,\zeta}(y(v,z);a)W_\tau(a)\psi^{-1}(\varepsilon_n av)|\det a|^{s-\zeta}dady(v,z) . \tag{7.29}$$

Now take $\phi_{\sigma,\zeta}$ to have support in $P_n\overline{Y}_n$ and such that

$$\phi_{\sigma,\zeta}(y(v,z);a) = \phi_1(v)\phi_2(z)W_\sigma(a) ,$$

where ϕ_1 and ϕ_2 are Schwartz functions and $W_\sigma \in W(\sigma,\psi)$. The integral (7.29) then equals

$$c\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} W_\sigma(a)W_\tau(a)\widehat{\phi}_1(-\varepsilon_n a)|\det a|^{s-\zeta}da , \tag{7.30}$$

where $c = \int \phi_2(z)dz$. When we multiply (7.30) by $\omega_\tau(-1)^{n-1}\gamma(\sigma\times\tau, s-\zeta,\psi)$ and apply the local functional equation for $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$, we get

$$c\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} W_\sigma(a)W_\tau(a)\phi_1(\varepsilon_1{}^t a^{-1})|\det a|^{1+s-\zeta}da . \tag{7.31}$$

Now apply the same substitutions and linear combinations to (7.26). We get

$$\begin{aligned}
&\int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int \phi_{\sigma,\zeta}(y(a^{-1}e_1,z);a)W_\tau(a)|\det a|^{1+s-\zeta}dzda\\
&= \int\limits_{Z_n\backslash \mathrm{GL}_n(F)}\int \phi_1(\varepsilon_1{}^t a^{-1})\phi_2(z)W_\sigma(a)W_\tau(a)|\det a|^{1+s-\zeta}dzda\\
&= c\int\limits_{Z_n\backslash \mathrm{GL}_n(F)} W_\sigma(a)W_\tau(a)\phi_1(\varepsilon_1{}^t a^{-1})|\det a|^{1+s-\zeta}da .
\end{aligned}$$

This proves that the proportionality factor in (7.25) is $\omega_\tau(1)^{n-1}\gamma(\sigma\times\tau, s-\zeta,\psi)$.] Let us proceed to the proof of (7.9) and (7.12). It is almost the same as that of (7.8), (7.11), but we have to be careful about the modifications. Assume that $\ell > n$. If n is even, then $\widetilde{A}(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s}) = A(W_{\phi_{\sigma,\zeta}},\widetilde{\xi}_{\tau^*,1-s})$ and $\widetilde{\xi}_{\tau^*,1-s}(h,I_n) = M(w_n,\xi_{\tau,s})(h,b_n^*)$ lies in $\rho_{\tau^*,1-s}$ and (7.9) in this case is a special case of (7.8) (with $\widetilde{\xi}_{\tau^*,1-s}$ replacing $\xi_{\tau,s}$). Assume that n is odd. Then

$$\widetilde{A}(W_{\phi_{\sigma,\xi}},\xi_{\tau,s}) = A(W_{\pi_{\sigma,\zeta}(d_{n,\ell})\phi_{\sigma,\zeta}},\xi^*_{\tau^*,1-s})$$

where

$$d_{n,\ell} = \begin{pmatrix} I_n & & \\ & I_{2(\ell-n)+1} & \\ & & I_n \end{pmatrix} j_{n,\ell}(\omega)$$

and

$$\xi^*_{\tau^*,1-s}(h,c) = M(w_n,\xi_{\tau,s})(h^\omega, b_n^* c^*)$$

(*see* Section 2). $\xi^*_{\tau^*,1-s}$ lies in the space of $\rho_{\tau^*,1-s}$. By (7.21) we conclude that

$$\omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\widehat{\tau},1-s-\xi,\psi)\widetilde{A}(W_{\phi_{\sigma,\zeta}},\xi_{\tau,s})$$

$$(7.32)\qquad = \int_{m(Z_n)\overline{U}_n\backslash H_n}\left(\int \phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n}\end{pmatrix}\gamma_{n,\ell}\widehat{w}_{n,\ell}j_{n,\ell}(g)d_{n,\ell}; I_\ell\right)\right.$$

$$\left.\psi^{-1}(v_{n-\ell})d(v,c,y)\right)\cdot\left(\int \psi(\tfrac{1}{2}x_{n-1,1})f_{\xi^*_{\tau^*,1-s}}(\overline{u}(x)g)dx\right)dg\ .$$

The inner (Whittaker) dx integral in (7.32) equals

$$(7.33)\qquad \begin{aligned}&\int \psi(\tfrac{1}{2}x_{n-1,1})M(w_n,\xi_{\tau,s})((\overline{u}(x)g)^\omega, b_n^*)dx\\ &= \int \psi(\tfrac{1}{2}x_{n-1,1})M(w_n,\xi_{\tau,s})(w_n u(x) w_n^{-1} g^\omega, b_n^*)dx\\ &= W'_{\widetilde{\xi}_{\tau^*,1-s}}\left(m\begin{pmatrix} I_{n-1} & \\ & -1/2\end{pmatrix} w_n^{-1} g^\omega\right)\end{aligned}$$

(*see* (7.6)). Thus change variable in (7.32)

$g \mapsto \left(w_n m\begin{pmatrix} I_{n-1} & \\ & -2\end{pmatrix} g\right)^\omega = w_n^{-1} m\begin{pmatrix} I_{n-1} & \\ & -1/2\end{pmatrix} g^\omega$. Note that

$$j_{n,\ell}\left(w_n^{-1} m\begin{pmatrix} I_{n-1} & \\ & -1/2\end{pmatrix} g^\omega\right) d_{n,\ell}$$

$$= \left(\begin{array}{c|ccc|c} & & & & -1/2 \\ \hline & I_{n-1} & & & \\ & & -I_{2(\ell-n)+1} & & \\ & & & I_{n-1} & \\ \hline -2 & & & & \end{array}\right) j_{n,\ell}(w_n^{-1} g)\ .$$

We have

$$\gamma_{n,\ell}\widehat{w}_{n,\ell}\left(\begin{array}{c|ccc|c} & & & & -\frac{1}{2} \\ \hline & I_{n-1} & & & \\ & & -I_{2(\ell-n)+1} & & \\ & & & I_{n-1} & \\ \hline -2 & & & & \end{array}\right)$$

$$=\left(\begin{array}{c|ccccc|c} I_{n-1} & & & & & & \\ \hline & 1 & 0 & -2 & 0 & -2 & \\ & 0 & I_{n-1} & 0 & 0 & 0 & \\ & 2 & 0 & -3 & 0 & -2 & \\ & 0 & 0 & 0 & I_{n-1} & 0 & \\ & -2 & 0 & 2 & 0 & 1 & \\ \hline & & & & & & I_{\ell-n} \end{array}\right)\gamma_{n,\ell}\widehat{w}_{n,\ell}\,.$$

Write

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & -3 & -2 \\ -2 & 2 & 1 \end{pmatrix}=\begin{pmatrix} 1 & & \\ 2 & 1 & \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We get

$$\phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix}\gamma_{n,\ell}\widehat{w}_{n,\ell}j_{n,\ell}\left(w_n^{-1}m\begin{pmatrix} I_{n-1} & \\ & -1/2 \end{pmatrix}g^{\omega}\right)d_{n,\ell};I_\ell\right)$$

$$=\phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 2v-2c_2 & 0 & v-2c_2 & c_1 & c_2 & y \\ & 1 & 0 & 0 & 0 & 0 & c_2' \\ & & I_{n-1} & 0 & 0 & 0 & c_1' \\ & & & 1 & 0 & 0 & v'-2c_2' \\ & & & & I_{n-1} & 0 & 0 \\ & & & & & 1 & 2v'-2c_2' \\ & & & & & & I_{\ell-n} \end{pmatrix}\right.\cdot$$

$$\left.\left(\begin{array}{c|ccccc|c} -I_{\ell-n} & & & & & & \\ \hline & 1 & 0 & -2 & 0 & -2 & \\ & 0 & I_{n-1} & 0 & 0 & 0 & \\ & 0 & 0 & 1 & 0 & 2 & \\ & 0 & 0 & 0 & I_{n-1} & 0 & \\ & 0 & 0 & 0 & 0 & 1 & \\ \hline & & & & & & -I_{\ell-n} \end{array}\right)\gamma_{n,\ell}\widehat{w}_{n,\ell}j_{n,\ell}(w_n^{-1}g);I_\ell\right)$$

(we wrote $c = (c_1, c_2)$ where $c_1 \in M_{(\ell-n)\times(n-1)}(F)$)

$$=\psi\Big(2(v_{\ell-n} - (c_2)_{\ell-n})\Big)\phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & -v+2c_2 & -c & y \\ & I_n & 0 & 0 & -c' \\ & & 1 & 0 & -v'+2c_2' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix}\cdot\right.$$

$$\left.\gamma_{n,\ell}^{-1}\widehat{w}_{n,\ell}j_{n,\ell}(w_n^{-1}g); \begin{pmatrix} -I_{\ell-n} & \\ & I_n \end{pmatrix}\right).$$

Change variable (in (7.32)) $v \mapsto v + 2c_2$ (note the cancellation of $\psi(2(c_2)_{\ell-n})$ and then $v \mapsto -v, c \mapsto -c$, we get

$$\int\limits_{V_n\backslash H_n} W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)\int \phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix}\cdot\right.$$

$$\left.\gamma_{n,\ell}^{-1}\widehat{w}_{n,\ell}j_{n,\ell}(w_n^{-1}g); \begin{pmatrix} -I_{\ell-n} & \\ & I_n \end{pmatrix}\right)\psi^{-1}(v_{\ell-n})\cdot d(v,c,y)dg\ .$$

Now change $g \mapsto -g$. Note that $W'_{\widetilde{\xi}_{\tau^*,1-s}}(-g) = \omega_\tau(-1)W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)$. We have

$$\gamma_{n,\ell}^{-1}\widehat{w}_{n,\ell}j_{n,\ell}(-I_{2n}) = \begin{pmatrix} I_{\ell-n} & \\ & -I_n \end{pmatrix}^{\wedge}\gamma_{n,\ell}\widehat{w}_{n,\ell}\ .$$

Conjugating further to the left with $\begin{pmatrix} I_{\ell-n} & \\ & -I_n \end{pmatrix}^{\wedge}$, we get

$$\omega_\tau\omega_\sigma(-1)\int\limits_{V_n\backslash H_n} W'_{\widetilde{\xi}_{\tau*,1-s}}(g)$$

$$\int \phi_{\sigma,\zeta}\left(\begin{pmatrix} I_{\ell-n} & 0 & v & c & y \\ & I_n & 0 & 0 & c' \\ & & 1 & 0 & v' \\ & & & I_n & 0 \\ & & & & I_{\ell-n} \end{pmatrix}\mu_{n,\ell}j_{n,\ell}(g); I_\ell\right)d(v,c,y)dg\ .$$

This proves (7.9).

Finally consider case $\ell = n$. If n is even, then (7.12) follows from (7.11). Assume that n is odd. Again, using (7.28), we get

$$\begin{aligned} &\omega_\tau(-1)^{n-1}\gamma(\sigma\times\widehat{\tau}, 1-s-\zeta,\psi)\widetilde{A}(W)_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) \\ (7.34)\qquad &= \int\limits_{m(Z_n)\overline{U}_n\backslash H_n} \phi_{\sigma,\zeta}(\gamma_n j_n(g)d_n; I_n)\int \psi(\tfrac{1}{2}x_{n-1,1})f_{\xi^*_{\tau^*,1-s}}(\overline{u}(x)g)dxdg\ . \end{aligned}$$

Here $d_n = \begin{pmatrix} I_n & & \\ & -1 & \\ & & I_n \end{pmatrix} j_n(\omega)$. As in (7.33), the (Whittaker) dx integral in (7.34) equals $W'_{\widetilde{\xi}_{\tau^*,1-s}}\left(m\begin{pmatrix} I_{n-1} & \\ & -1/2 \end{pmatrix} w_n^{-1} g^{\omega}\right)$, and the l.h.s. of (7.34) becomes

$$\int\limits_{V_n\backslash H_n} \phi_{\sigma,\zeta}\left(\gamma_n j_n\left(\left(w_n m\begin{pmatrix} I_{n-1} & \\ & -2 \end{pmatrix} g\right)^{\omega}\right) d_n; I_n\right) W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)dg\ .$$

Exactly as in the last case, this integral equals

$$\int\limits_{V_n\backslash H_n} \phi_{\sigma,\zeta}\left(\gamma_n^{-1} j(w_n^{-1} g); I_n\right) W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)dg$$
$$= \omega_\sigma\omega_\tau(-1) \int\limits_{V_n\backslash H_n} \phi_{\sigma,\zeta}\left(\mu_n j_n(g); I_n\right) W'_{\widetilde{\xi}_{\tau^*,1-s}}(g)dg\ .$$

This proves (7.12) and the theorem. ∎

Proof of Theorem C (case $\ell \geq n$). – As in case $\ell < n$, the proof follows immediately from the identities (7.8),(7.9),(7.11),(7.12) and the functional equation which defines the local coefficient (7.7). For example, if $\ell > n$, then (7.8),(7.9) and (7.7) imply that

$$\omega_\tau(-1)^{\ell-1}\gamma(\sigma\times\tau, s-\zeta, \psi)A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})$$
$$=\gamma(\tau,\Lambda^2, 2s-1,\psi')\omega_\sigma(-1)^n\omega_\tau(-1)^{\ell+n-1}\gamma(\sigma\times\widehat{\tau}, 1-s-\zeta,\psi)\widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})\ ,$$

and so

$$\omega_\sigma(-1)^n\omega_\tau(-1)^n\frac{\gamma(\sigma\times\tau, s-\zeta,\psi)\gamma(\widehat{\sigma}\times\tau, s+\zeta,\psi^{-1})}{\gamma(\tau,\Lambda^2,2s-1,\psi')}A(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s}) = \widetilde{A}(W_{\phi_{\sigma,\zeta}}, \xi_{\tau,s})\ .$$

Finally, (7.1) is established using $\omega_\sigma(-1)^n\gamma(\widehat{\sigma}\times\tau, s+\zeta,\psi^{-1}) = \omega_\tau(-1)^\ell\gamma(\widehat{\sigma}\times\tau, s+\zeta,\psi)$. The proof in case $\ell = n$, is similar. ∎

REFERENCES

[C] W. CASSELMAN, *Canonical extensions of Harish-Chandra modules* (*Can. J. Math.*, Vol. 41, 1989, pp. 315-438).

[DM] J. DIXMIER and P. MALLIAVIN, *Factorisations des fonctions et de vecteurs indéfiniment différentiables* (*Bull. Sci. Math.* II, Sér. 102, 1978, pp. 307-330).

[GS] S. GELBART and F. SHAHIDI, *Analytic properties of automorphic L-functions* (*Perspectives in Math.*, Vol. 6, Academic Press, 1988).

[G] A. GROTHENDIECK, *Produits tensoriels topologiques et espaces nucléaires* (*Memoirs AMS*, N°. 16, 1955).

[J.S.] H. JACQUET and J. SHALIKA, *Rankin-Selberg convolutions: archimedean theory* (*Israel Mathematical Conference Proc., Festschrift in honor of I. Piatetski-Shapiro*, S. Gelbart, R. Howe, P. Sarnak, Ed., Part I, 1990, pp. 125-207).

[Sh1] F. SHAHIDI, *On certain L-functions* (*American J. Math.*, Vol. 103, 1981, pp. 297-355).

[Sh2] F. SHAHIDI, *Local coefficients as Artin factors for real groups* (*Duke Math. J.*, Vol. 52, 1985, pp. 973-1007).

[S] D. SOUDRY, *Rankin-Selberg convolutions for $SO_{2\ell+1} \times GL_n$: local theory* (*Memoirs of AMS*, No. 500, 1993, pp. 1-100).

[W1] N. WALLACH, *Asymptotic expansions of generalized matrix entries of representations of real reductive groups, in Lie Group Representations* I (Proceedings, University of Maryland 1982-1983, *Springer Lecture Notes in Mathematics*, Vol. 1024, pp. 287-369).

[W2] N. WALLACH, *Real reductive groups* I, Academic Press, 1988.

[W3] N. WALLACH, *Real reductive groups* II, Academic Press, 1992.

[Wr] G. WARNER, *Harmonic analysis on semi-simple Lie groups* I (*Grund. Math. Wiss.*, Vol. 188, Springer-Verlag, 1972).

(Manuscript received January 4, 1993;
revised September 9, 1993).

D. SOUDRY
Tel Aviv University,
School of Mathematical Sciences,
Ramat Aviv, 69978 Tel Aviv, Israel.