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ON THE IWAHORI-SPHERICAL DISCRETE SERIES FOR p-ADIC CHEVALLEY GROUPS; FORMAL DEGREES AND L-PACKETS

BY MARK REEDER

ABSTRACT. – We give a general formula for the formal degrees of those square integrable representations of a p-adic Chevalley group which have both an Iwahori-fixed vector and a Whittaker model. They are worked out in detail for small groups, using a computer. The results are interpreted in terms of L-packets.

1. Introduction

Let F be a non-archimedean local field, and let G denote the F-rational points of a Chevalley group $\mathbf{G}$ of adjoint type, defined over the ring of integers $\mathcal{O}$ in F. We use similar notational distinctions between other algebraic groups over $\mathcal{O}$ and their F-rational points. Let $\mathbf{B}$ be a Borel subgroup containing a maximal F-split torus $\mathbf{A}$, and let $K = \mathbf{G}(\mathcal{O})$ be the integer points of $\mathbf{G}$. $\mathcal{I}$ denotes the Iwahori subgroup whose reduction modulo the prime ideal in $\mathcal{O}$ is $\mathbf{B}(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the residue field of q elements. The main goal of this work is to compute the formal degrees of generic square integrable representations of G which contain a nonzero vector invariant under $\mathcal{I}$. Here, "generic" means the representation has a Whittaker model (see below).

Our approach is straightforward, in principal. We compute the L^2-norm of a matrix coefficient, using a formula for the coefficient due to J.-S. Li. The representations we consider are never supercuspidal, so the matrix coefficents are not compactly supported. The main difficulty however, is writing down the coefficient in a sufficiently explicit way.

Our result says nothing new for GL_n, since the Iwahori spherical discrete series of GL_n consists only of twists of the Steinberg representation. Anyway, there is already the work of Corwin, Moy and Sally [CSM], who computed the formal degrees of all discrete series representatations of GL_n in the tame case. To handle the nonsupercuspidal representations, they use the matching theorem for division algebras. Recently, Bushnell and Kutzko [B-K] have computed all formal degrees for GL_n in the general case.

For more general groups even than those considered here, Borel [B] found the formal degrees of representations whose spaces of $\mathcal{I}$-invariants are one dimensional. This relies on some formulas of Macdonald [M] for generalized Poincaré series of affine Weyl groups. When G has adjoint type, the Steinberg representation and its twists are the only generic representations with with one dimensional $\mathcal{I}$-invariants, so our results are essentially disjoint from those of [B]. This is fortunate, because for small groups (rank≤ 3) our calculations combined with those of [B] give the formal degrees of all $\mathcal{I}$-spherical discrete series representations.

The reader will quickly notice that our formula shares little elegance with the well-known formula for the formal degree of the Steinberg representation. This is partly a fact of life, and partly due to a missing idea. On the one hand, our formula applies to all groups and representations (of the stated class) at once. Moreover, our representations have more complicated Jacquet modules than those considered in [B], hence more complicated matrix coefficients. We are able to integrate one of these coefficients because it transforms nicely under K, but that means it involves the whole Jacquet module, so the value of the integral, hence the formal degree, is a complicated sum. On the other hand, in any specific example (several are given in section 7), the formula seems to reduce to a fairly simple rational function in q, although I needed a machine to see this. (The computations given here for PSP_6, SO_7, SO_9, G_2 and F_4 were done with *Mathematica*, installed on a NeXT workstation.) Perhaps there is a way to rewrite or interpret our formula to make this simplicity apparent.

Motivated by some remarks of Lusztig in [L], we use our computations of degrees to define certain L-packets and suggest a refinement of a conjecture of Lusztig about parametrizing certain square-integrable representations of G in terms of representations of a certain finite group associated to unipotent elements in the dual group of G. See (7.2) for all this. We only remark here that for SO_5, G_2 and F_4, the degrees of some of our $\mathcal{I}$-spherical representations are also the degrees of supercuspidal representations. A. Moy has also noticed this phenomenon. L. Morris has considered these L-packets from a different point of view, and has proven several results on intertwining algebras [Mo] that should be useful in the pursuit of Lusztig's conjecture.

B. Gross and D. Prasad ([GP, (2.6)]) have recently stated some conjectures about generic L-packets which in some respects are much broader than the one made here, and which are corroborated by our results.

I am grateful to P. Sally for his advice, informative discussions and encouragement in these and other mathematical endeavors.

2. Notation and statement of the main result

(2.1) The set of unramified characters of A has the structure of a complex torus T, which is in fact a maximal torus of the Langlands dual group $\hat{G}$ of G. Let W denote the Weyl group of T in $\hat{G}$.

For $\tau \in T$, let $\mathcal{E}_2(\tau)$ be the set of isomorphism classes of square integrable constituents of the principal series representation

$$I(\tau) := \mathrm{Ind}_B^G(\tau) = \{f \in C_c^\infty(G) : f(bg) = \delta^{\frac{1}{2}}(b)\tau(b)f(g) \quad \text{for all} \quad b \in B,\ g \in G\}.$$

Here δ is the modular function of B.

For $w \in W$ and $\tau \in T$, let τw be the unramified character of A given by $\tau w(a) = \tau(waw^{-1})$. From [B], we know that $\mathcal{E}_2(\tau w) = \mathcal{E}_2(\tau)$ for any $w \in W$, and if $\tau' \in T$ is not W-conjugate to τ then $\mathcal{E}_2(\tau') \cap \mathcal{E}_2(\tau) = \varnothing$. Moreover, every square integrable representation of G containing a nonzero $\mathcal{I}$-invariant vector belongs to some $\mathcal{E}_2(\tau)$, and there are only finitely many such representations, up to isomorphism. We will give a formula for the formal degree of a certain distinguished member of $\mathcal{E}_2(\tau)$, namely the unique representation in $\mathcal{E}_2(\tau)$ possessing a Whittaker model.

Recall that a Whittaker model for a representation V is a realization of V as a subrepresentation of $\mathrm{Ind}_N^G \theta$ where N is the unipotent radical of B and θ is a character of N which lies in the open A-orbit on the space of such characters. This orbit is unique, since we are assuming G has adjoint type.

We now review the part of the classification given in [K-L] of Iwahori-spherical representations which is concerned with the discrete series. First, $\mathcal{E}_2(\tau)$ is nonempty if and only if τ is constructed in the following way: Choose $s \in T$ so that its centralizer H in $\hat{G}$ is semisimple (not just reductive). Let n be a distinguished nilpotent element in the Lie algebra of H. This means the centralizer of n in H contains no nontrivial torus. Let $\phi : SL_2(\mathbb{C}) \longrightarrow H$ be the homomorphism arising in the Jacobson-Morozov theorem, such that

$$d\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = n.$$

We can arrange that ϕ maps the diagonal matrices in SL_2 into T. Then set

$$\tau = s\phi\begin{pmatrix} q^{\frac{-1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}.$$

The actual parametrization of $\mathcal{E}_2(\tau)$ involves the geometry of the flag manifold of $\hat{G}$, and goes as follows. Let $X = \hat{G}/\hat{B}$ be the variety of Borel subgroups of $\hat{G}$. Let $X(\tau, n)$ be the subvariety of Borel subgroups containing both τ and $\exp n$. Let $Z(\tau, n)$ be the mutual centralizer in $\hat{G}$ of τ and n, and let $Z_{\hat{G}}$ be the center of $\hat{G}$. Then $Z(\tau, n)/Z_{\hat{G}}$ acts on $X(\tau, n)$, and this induces a linear action of the component group $A(\tau, n)$ of $Z(\tau, n)/Z_{\hat{G}}$ on the homology of $X(\tau, n)$. It is shown in ([K-L]) that there is a bijection between $\mathcal{E}_2(\tau)$ and the set of equivalence classes of representations ρ of $A(\tau, n)$ occuring in the above representation on the homology of $X(\tau, n)$. (Throughout this paper, "homology" means singular homology with complex coefficients.) We let $\mathcal{M}_{\tau,\rho}$ denote the square integrable representation of G corresponding to ρ as above. The dimension of the space of $\mathcal{I}$-invariants in $\mathcal{M}_{\tau,\rho}$ equals the multiplicity of ρ in the $A(\tau, n)$-module $H_\bullet(X(\tau, n))$.

This is the way the classification was envisaged in [L], although its proof in [K-L] requires that it be formulated differently, because one has trouble defining the Hecke algebra action in terms of homology. Let us explain the above interpretation of Kazhdan and Lusztig's result. They actually consider the complexified equivariant K-homology group $K_0^M(X(n))$, where $X(n)$ is the variety of Borel subgroups containing $\exp n$, and M is the algebraic subgroup of $\hat{G} \times \mathbb{C}^\times$ generated by (τ, q). It turns out that $K_0^M(X(n))$ is a module over the complex representation ring R of M, as well as the affine Hecke algebra, and $\mathcal{M}_{\tau,\rho}$ is defined in [K-L, (5.12)] via its $\mathcal{I}$-invariants as

$$\mathcal{M}_{\tau,\rho}^{\mathcal{I}} = \mathrm{Hom}_{A(\tau,n)}\big(\rho, \mathbb{C} \otimes_R K_0^M(X(n))\big),$$

where $\mathbb{C}$ is an R-module via the homomorphism $R \longrightarrow \mathbb{C}$ given by evaluation at (τ, q). Let $\bar{R}$ be the localization of R at the corresponding maximal ideal. The localization theorem [K-L, 1.3(k)] says

$$\bar{R} \otimes_R K_0^M(X(n)) \simeq \bar{R} \otimes_R K_0^M(X(\tau, n)).$$

Since M acts trivially on $X(\tau, n)$, the right side is $\bar{R} \otimes_{\mathbb{C}} K_0(X(\tau, n))$ ([K-L, 1.3(m)]). The ordinary K-group $K_0(X(\tau, n))$ is $A(\tau, n)$-equivariantly isomorphic to the even-degree homology of $X(\tau, n)$ ([K-L, 1.3(m2)]), which is the whole homology of $X(\tau, n)$ by the main result of [CLP]. It follows that

$$\mathbb{C} \otimes_R K_0^M(X(n)) \simeq H_\bullet(X(\tau, n))$$

as $A(\tau, n)$-modules.

Contrary to what is implied by the result announced in [G], not every representation of $A(\tau, n)$ occurs in the homology of $X(\tau, n)$. This happens already for Sp_4. Put another way, the Iwahori-spherical representations do not fill up their L-packets. In [L], Lusztig suggested a way to account for the missing representations, and we shall see that the formal degrees fit in nicely with this, at least in our examples. Indeed we can use the formal degrees to distribute the extra representations into the various L-packets already partially occupied by Iwahori-spherical representations. It is not clear if the representations in a given L-packet have any features in common, other than their formal degrees.

On the other hand, the trivial representation of $A(\tau, n)$ always appears in $H_0(X(\tau, n))$, which is just the permutation representation of $A(\tau, n)$ on the set of connected components of $X(\tau, n)$, and the corresponding Iwahori spherical G-representation $\mathcal{M}_{\tau,1}$ has special properties, which can be seen via another description of $\mathcal{M}_{\tau,1}$, found in [R].

We can choose $\tau \in T$ in its W-orbit so that $I(\tau)$ has an injective Whittaker model (see [R2,(8.1)]). Then the representation $I(\tau)$ has a unique irreducible subrepresentation $\mathcal{U}(\tau)$, appearing in $I(\tau)$ with multiplicity one.

In [R], we showed that

$$\mathcal{M}_{\tau,1} \simeq \mathcal{U}(\tau),$$

hence $\mathcal{M}_{\tau,1}$ is the unique member of $\mathcal{E}_2(\tau)$ having a Whittaker model. Actually, one can prove directly that $\mathcal{U}(\tau)$ is square integrable, using the geometric description of the Jacquet module of $\mathcal{U}(\tau)$ given in [R], along with the elementary argument of Lusztig [L] which gives a geometric interpretation of Casselman's criteria for square integrability.

We are going to compute the formal degree of $\mathcal{U}(\tau)$. Our method rests on the fact that $\mathcal{U}(\tau)$ is also the unique constituent of $I(\tau)$ whose restriction to K contains the Steinberg representation of $\mathbf{G}(\mathbb{F}_\mathbf{q})$, pulled back to K.

(2.2) We need still more notation before we can state our formula for the formal degree of $\mathcal{U}(\tau)$. Recall that the dual group $\hat{G}$ is simply-connected, and we identify the group of unramified characters of A with a fixed maximal torus $T \subseteq \hat{G}$ which is contained in a Borel subgroup $\hat{B} = T\hat{U}$ of $\hat{G}$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of $\hat{G}$ and T, respectively.

Also let Δ, Δ^+, Σ be the roots, positive roots, and simple roots of $\mathfrak{t}$ in $\mathfrak{g}$. These are determined by the pair $(\mathbf{A}, \mathbf{B})$. Let $\langle\ ,\ \rangle$ be a nondegenerate W-invariant inner product on $\mathfrak{t}^*$, where W is the Weyl group.

Let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of $\mathfrak{t}$. For any $\lambda \in \Lambda$, in particular a root, let $e_\lambda : T \longrightarrow \mathbb{C}^\times$ be the corresponding rational character of T. For each $\alpha \in \Delta$ there is a one parameter subgroup

$$h_\alpha : \mathbb{C}^\times \longrightarrow T,$$

defined by the condition $e_\lambda(h_\alpha(t)) = t^{\langle \lambda, \check{\alpha} \rangle}$, where $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. Since $\hat{G}$ is simply connected, every $\tau \in T$ may be written uniquely as

$$\tau = \prod_{\alpha \in \Sigma} h_\alpha(t_\alpha).$$

There is an isomorphism of lattices

$$a \mapsto \lambda_a : A/A \cap K \longrightarrow \Lambda,$$

such that if $\tau = \prod_{\alpha \in \Sigma} h_\alpha(t_\alpha) \in T$, then

$$\tau(a) = \prod_{\alpha \in \Sigma} t_\alpha^{\langle \lambda_a, \check{\alpha} \rangle}.$$

Thus, $e_{\lambda_a}(\tau) = \tau(a)$.

The Weyl group W acts on everything in sight. In particular,

$$w\lambda_a = \lambda_{waw^{-1}}, \quad \text{and} \quad e_{w\lambda}(\tau) = e_\lambda(\tau w).$$

Here we write τw to denote a right action of W on T. If w is represented by an element $\tilde{w}$ in the normalizer in $\hat{G}$ of T then τw is defined as $\tilde{w}^{-1} \tau \tilde{w}$.

For $\tau \in T$, we set

$$\Delta_\tau^+ = \{\alpha \in \Delta^+ : e_\alpha(\tau) = 1\},$$

and let

$$W_\tau = \{w \in W : \ \tau w = \tau\}, \qquad W^1 = \{w \in W : w^{-1} \Delta_\tau^+ \subseteq \Delta^+\}.$$

Since $\hat{G}$ is simply connected, W^1 is a set of representatives for $W_\tau \backslash W$.

Let

$$P(t) = \sum_{w \in W} t^{\ell(w)}$$

be the Poincaré polynomial of W. More generally set, for each subset of the simple roots $J \subseteq \Sigma$,

$$P_J(t) = \sum_{w \in W_J} t^{\ell(w)},$$

where W_J is the parabolic subgroup of W generated by the reflections in J. So $P(t) = P_\Sigma(t)$, and $P_\varnothing = 1$.

We also consider the distinguished representatives W^J for W/W_J, where

$$W^J = \{w \in W : w(J) \subseteq \Delta^+\},$$

and put

$$P^J(t) = \sum_{w \in W^J} t^{\ell(w)}.$$

It is known that

$$P_J(t)P^J(t) = P(t), \quad \text{and} \quad P(t^{-1}) = t^{-\nu}P(t),$$

where

$$\nu = |\Delta^+|.$$

Let $\Lambda^+ \subset \Lambda$ be the set of dominant weights and for each $J \subseteq \Sigma$, put

$$\Lambda_J = \{\lambda \in \Lambda^+ : \langle \lambda, \check{\alpha} \rangle = 0 \iff \alpha \text{ belongs to the span of } J\}.$$

In other words, Λ_J is the set of dominant weights whose stabilizer in W is exactly W_J. In particular, $\Lambda_\varnothing$ is the set of regular dominant weights and $\Lambda_\Sigma = \{0\}$.

For fixed $\tau \in T$, consider the function $M : \Lambda \longrightarrow \mathbb{C}$ first introduced by Macdonald:

$$M(\lambda) = \sum_{w \in W} w \cdot \big(e_\lambda \prod_{\alpha > 0} \frac{1 - q^{-1}e_\alpha}{1 - e_\alpha}\big)(\tau).$$

Here "$w\cdot$" denotes the action of W on the space of rational functions on T. After summing over W, the poles cancel and the whole expression is really a polynomial on T, which is being evaluated at $\tau \in T$. Using a result of [Li], we will express a certain matrix coefficient of $\mathcal{U}(\tau)$ in terms of the function M.

The next two results in tandem give an effective way to compute the formal degree of $\mathcal{U}(\tau)$.

FORMULA A. – *Let $d(\tau)$ be the formal degree of $\mathcal{U}(\tau)$, where the Haar measure on G is such that* vol($\mathcal{I}$)=1. *Then*

$$d(\tau)^{-1} = q^\nu \sum_{J \subseteq \Sigma} \frac{1}{P_J(q)} \sum_{\lambda \in \Lambda_J} |M(\lambda)|^2.$$

This would not be of much use if we did not know how to evaluate $M(\lambda)$, and that is the role of the next result. We define certain polynomials F_w on $\mathfrak{t}^*$, one for each $w \in W^1$, as follows.

$$F_w(\lambda) = \epsilon(w) e_{\rho - w\rho}(\tau) \sum_{J \subseteq \Delta^+_{\tau w}} \big(\nabla_J(Q)\big)(\tau w) \prod_{\beta \in \Delta^+_{\tau w} - J} \langle \lambda - \rho, \beta \rangle,$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha,$$

$$Q = \prod_{\alpha \in \Delta^+} 1 - q^{-1} e_\alpha \in \mathbb{C}[T],$$

and

$$\nabla_J = \prod_{\alpha \in J} \nabla_\alpha,$$

where ∇_β is the derivation of the coordinate ring $\mathbb{C}[T]$ defined by

$$\nabla_\beta e_\mu = \langle \mu, \beta \rangle e_\mu.$$

Explicitly, we have

$$\nabla_J(Q) = \sum_{S \subseteq \Delta^+} (-q^{-|S|}) \Big(\prod_{\alpha \in J} \langle \delta_S, \alpha \rangle \Big) e_{\delta_S},$$

where δ_S is the sum of the roots in S. Finally, $\nabla_\varnothing$ is defined to be the identity operator.

FORMULA B. – *For any dominant weight* λ, *we have*

$$M(\lambda) = C \sum_{w \in W^1} F_w(\lambda) e_\lambda(\tau w),$$

where

$$C = (-1)^\nu \Big(\prod_{\beta \in \Delta_\tau^+} \langle \rho_1, \beta \rangle \prod_{\alpha \in \Delta^+ - \Delta_\tau^+} (e_\alpha(\tau) - 1) \Big)^{-1} \qquad (\text{here } \rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_\tau^+} \beta\ ).$$

It is shown in [R] that the polynomials F_w determine the structure of the Jacquet module of $\mathcal{U}(\tau)$. They also have the following geometric interpretation. Let $\hat{G}_\tau$ be the centralizer of τ in G. This is a connected (because $\hat{G}$ is simply connected) reductive group which contains T, and has Weyl group W_τ. Let $X(\tau)$ be the fixed points of τ in the flag variety X, and for $w \in W$, let $X_w(\tau)$ be the $\hat{G}_\tau$-orbit of $w\hat{B} \in X$. Using the Bruhat decomposition, one sees that $X(\tau)$ is the disjoint union $X(\tau) = \coprod_{w \in W^1} X_w(\tau)$, and each $X_w(\tau)$ is isomorphic to the flag variety $\hat{G}_\tau / \hat{B} \cap \hat{G}_\tau$. It follows that $X(\tau, n)$ is the disjoint union

$$X(\tau, n) = \coprod_{w \in W^1} X_w(\tau, n), \quad X_w(\tau, n) := X(\tau, n) \cap X_w(\tau).$$

It turns out that the term of highest degree in F_w is a harmonic polynomial on $\mathfrak{t}^*$ which represents the fundamental class of $X_w(\tau, n)$ in $H_*(X_w(\tau))$. We define

$$W_0^1 = \{ w \in W^1 :\ X_w(\tau, n) \neq \varnothing \}.$$

It follows that F_w is not the zero polynomial if and only if $w \in W_0^1$, and the sum in Formula B is really a sum over the much smaller set W_0^1.

(2.3) Some special cases are worth considering. If n is regular in H (hence distinguished), then τ is regular ($W_\tau = 1$). Let W_H be the Weyl group of H. Then

$$W_0^1 = \{w \in W^1 : w^{-1}(\Delta_H \cap \Delta^+) \subseteq \Delta^+\} = \{w \in W^1 : Q(\tau w) \neq 0\},$$

for each $w \in W_0^1$ the variety $X_w(\tau, n)$ consists of the single point $\{w\hat{B}\}$, and F_w is the constant polynomial

$$F_w(\lambda) = \epsilon(w) e_{\rho - w\rho}(\tau) Q(\tau w),$$

so we have

$$M(\lambda) = \prod_{\alpha>0} \frac{1 - q^{-1} e_\alpha(\tau)}{1 - e_\alpha(\tau)} \sum_{w \in W_0^1} e_\lambda(\tau w) \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{1 - q e_\alpha(\tau)}{q - e_\alpha(\tau)}.$$

Moreover the group $A(\tau, n)$ is just the center of H modulo the center of $\hat{G}$ (see section 6 below). It acts trivially on $X(\tau, n)$. In this case $\mathcal{U}(\tau)$ is the only discrete series constituent of $I(\tau)$.

If $\Delta_\tau^+ = \{\beta\}$ for some root β then $X_w(\tau, n)$, if nonempty, either is a projective line left invariant by $A(\tau, n)$, or consists of finitely many points permuted transitively by $A(\tau, n)$. The first case occurs if and only if $Q(\tau w) \neq 0$ and then

$$F_w(\lambda) = \epsilon(w) e_{\rho - w\rho}(\tau) Q(\tau w) \left(\langle \lambda - \rho, w^{-1}\beta \rangle - q^{-1} \sum_{\gamma>0} \frac{\langle w\gamma, \beta \rangle e_\gamma(\tau w)}{1 - q^{-1} e_\gamma(\tau w)} \right).$$

The second case occurs if and only if there is exactly one positive root η such that $e_\eta(\tau w) = q$, and for this η we have $\langle w\eta, \beta \rangle \neq 0$. Here $F_w(\lambda)$ is the constant polynomial

$$F_w(\lambda) = -\epsilon(w) e_{\rho - w\rho}(\tau) \langle w\eta, \beta \rangle \prod_{\eta \neq \gamma > 0} 1 - q^{-1} e_\gamma(\tau w).$$

In this situation the F_w's, and their connection with Jacquet modules, were found by Rodier ([Ro]), and his work led me to seek their general form, as they arose from certain Whittaker functions ([R]). Our explicit examples below are covered by this and the regular case.

(2.4) For the record, we show how to express the formal degree in closed form, even though the version above with infinite sums seems better suited to actual computations.

Let $\{\omega_\alpha : \alpha \in \Sigma\}$ be the fundamental dominant weights, where $\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha\beta}$ for α, $\beta \in \Sigma$. If we write a general $\lambda \in \Lambda^+$ as

$$\lambda = \sum n_\alpha \omega_\alpha$$

for nonnegative integers n_α, then

$$\Lambda_J = \{\lambda : \ n_\alpha = 0 \text{ if } \alpha \in J, \quad n_\alpha \geq 1 \text{ if } \alpha \in J_c\},$$

where $J_c = \Sigma - J$.

We now define, for $x, y \in W^1$ and $\mu \in \Lambda^+$, certain complex numbers $C^\mu_{x,y}$ (independent of λ) by the equation

$$F_x(\lambda)\overline{F_y(\lambda)} = \sum_{\mu\in\Lambda^+} C^\mu_{x,y} \prod_{\alpha\in\Sigma} n_\alpha^{\langle\mu,\check{\alpha}\rangle},$$

where "bar" denotes complex conjugation. It is clear that only finitely many $C^\mu_{x,y}$ are nonzero. When τ is regular, $C^\mu_{x,y}$ is nonzero only for $\mu = 0$.

Next we need some combinatorial notation: For $1 \neq z \in \mathbb{C}$ and any integer $k \geq 0$, we set

$$\begin{bmatrix} z \\ k \end{bmatrix} = \left(X\frac{\partial}{\partial X}\right)^k \frac{X}{1-X}\bigg|_{X=z}.$$

For $k \geq 1$ we have

$$\begin{bmatrix} z \\ k \end{bmatrix} = \frac{z(z+1)\cdots(z+k-1)}{(1-z)^{k+1}}.$$

Finally, here is the formula for the formal degree in closed form.

FORMULA C. – *Assume that $\mathcal{E}_2(\tau)$ is nonempty. With Haar measure on G normalized to give $\mathcal{I}$ volume one, the formal degree of $\mathcal{U}(\tau)$ is given by*

$$d(\tau)^{-1} = q^\nu |C|^2 \sum_{\substack{J\subseteq\Sigma \\ (x,y)\in W_0^1\times W_0^1}} \frac{1}{P_J(q)} \sum_{\mu\in\Lambda_J} C^\mu_{x,y} \prod_{\alpha\in\Sigma-J} \begin{bmatrix} e_{\omega_\alpha}(\tau x)\overline{e_{\omega_\alpha}(\tau y)} \\ \langle\mu,\check{\alpha}\rangle \end{bmatrix},$$

where C is as in Formula B.

When τ is regular, the formula becomes

$$d(\tau)^{-1} = q^\nu \left|\prod_{\alpha>0} \frac{1-q^{-1}e_\alpha(\tau)}{1-e_\alpha(\tau)}\right|^2 \sum_{\substack{J\subseteq\Sigma \\ (x,y)\in W_0^1\times W_0^1}} \frac{d_x\overline{d_y}}{P_J(q)} \prod_{\beta\in\Sigma-J}\left(\left[e_{\omega_\beta}(\tau x)\overline{e_{\omega_\beta}(\tau y)}\right]^{-1} - 1\right)^{-1},$$

where

$$d_w = \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \frac{1-qe_\alpha(\tau)}{q-e_\alpha(\tau)}.$$

Note that the sum over Λ_J is in fact a finite sum. Moreover, Casselman's square integrability condition guarantees that $|e_{\omega_\alpha}(\tau x)| < 1$ for all $x \in W_0^1$. Formula C follows directly from Formula A upon substituting the expansion

$$F_x(\lambda)\overline{F_y(\lambda)} = \sum_{\mu\in\Lambda^+} C^\mu_{x,y} \prod_{\alpha\in\Sigma} n_\alpha^{\langle\mu,\check{\alpha}\rangle}$$

into Formula A and remarking that for any complex number z with $|z| < 1$ and nonnegative integer k, we have

$$\sum_{n=1}^{\infty} n^k z^n = \begin{bmatrix} z \\ k \end{bmatrix}.$$

One can hope that Formula C has a topological interpretation, as is the case for the degree of the Steinberg representation.

The proof of Formula A will occupy sections 3 and 4, and Formula B is proved in section 5. In section 6 we make some remarks on computing the Kazhdan-Lusztig parameters, in preparation for our computational examples in section 7, where we also discuss Lusztig's conjecture.

3. The matrix coefficient

Let $\mathcal{H}_0$ be the convolution algebra of $\mathcal{I}$-bi-invariant functions on G which have support in K. This is the Hecke algebra of the finite Weyl group W, and is a subalgebra of the affine Hecke algebra associated to G. For a matrix coefficient of $\mathcal{U}(\tau)$, we use a particular function $\Phi \in \mathcal{U}(\tau)^{\mathcal{I}}$ which can be defined as the unique function in $I(\tau)^{\mathcal{I}}$ transforming by the sign character $\mathcal{H}_0$, taking the value one on $e \in G$. To see Φ explicitly, view the group G as the disjoint union

$$G = \bigcup_{w \in W} Bw\mathcal{I}.$$

For $b \in B, w \in W, k \in \mathcal{I}$, we have

$$\Phi(bwk) = (-q)^{-\ell(w)} \tau\delta^{\frac{1}{2}}(b),$$

where $\ell(w)$ is the length of w, and δ is the modulus of B. In [R], Φ was denoted by Φ_- to distinguish it from the K-spherical function. If T_s is one of the standard generators of $\mathcal{H}_0$ then $T_s\Phi = -\Phi$.

Let $\tilde{\Phi}$ be the analogue of Φ in $I(\tau^{-1})$, which is the contragredient of $I(\tau)$ via the pairing $I(\tau) \times I(\tau^{-1}) \longrightarrow \mathbb{C}$ given by

$$(\phi, \psi) = \int_K \phi(k)\psi(k)dk.$$

Define $\Gamma \in C^\infty(G)$ by

$$\Gamma(g) = (R_g\Phi, \tilde{\Phi}) = \int_K \Phi(kg)\tilde{\Phi}(k)dk,$$

where R_g denotes the action of G on $I(\tau)$ by right translation. Following Casselman [C], who looked at the K-spherical function, J.-S. Li has computed the following formula for Γ.

(3.1) PROPOSITION ([Li]). – *Let $a \in A$ be such that its corresponding weight λ_a is dominant with respect to Δ^+. Then*

$$\Gamma(a) = \delta^{\frac{1}{2}}(a) \sum_{w \in W} w \cdot \left(e_{\lambda_a} \prod_{\alpha > 0} \frac{1 - q^{-1} e_\alpha}{1 - e_\alpha} \right)(\tau) = \delta^{\frac{1}{2}}(a) M(\lambda_a).$$

Our task is to extend this formula to all of G and then make it explicit enough to compute its L^2-norm. Since Γ is clearly $\mathcal{I}$ bi-invariant, and G is the disjoint union

$$G = \bigcup_{W \times A/A \cap K} \mathcal{I} w a \mathcal{I},$$

we need only compute $\Gamma(wa)$ for arbitrary $w \in W$ and $a \in A/A \cap K$.

(3.2) LEMMA. – *Let $g \in G$, $\phi \in I(\tau)^{\mathcal{I}}$, $\psi \in I(\tau^{-1})^{\mathcal{I}}$. Let $T_g \in C_c^\infty(G)$ be the characteristic function of $\mathcal{I} g \mathcal{I}$. Then*

$$(R_g \phi, \psi) = \mathrm{vol}(\mathcal{I} g \mathcal{I})^{-1} (T_g \phi, \psi).$$

(3.3) LEMMA. – *In the previous lemma, if $k \in K$ then*

$$(T_k \phi, \psi) = (\phi, T_{k^{-1}} \psi).$$

These two lemmas are easily proved by interchanging integrals.

Recall that the length function on $W \ltimes A/A \cap K$ is given by the equation

$$\mathrm{vol}(\mathcal{I} w a \mathcal{I}) = q^{\ell(wa)}.$$

The explicit formula is

$$\ell(wa) = \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} |\langle \lambda_a, \check{\alpha} \rangle + 1| + \sum_{\substack{\alpha > 0 \\ w\alpha > 0}} |\langle \lambda_a, \check{\alpha} \rangle|.$$

Using this, it is easy to check that if $s \in W$ is the reflection for a simple root α and $sw > w$ for some $w \in W$, then

$$\ell(swa) = \begin{cases} \ell(wa) + 1, & \text{if} \quad \langle w\lambda_a, \check{\alpha} \rangle \geq 0 \\ \ell(wa) - 1, & \text{if} \quad \langle w\lambda_a, \check{\alpha} \rangle < 0. \end{cases}$$

In the affine Hecke algebra $\mathcal{H}$ of smooth $\mathcal{I}$ bi-invariant functions on G with compact support, we then have

$$T_{swa} = \begin{cases} T_s T_{wa}, & \text{if} \quad \langle w\lambda_a, \check{\alpha} \rangle \geq 0 \\ T_s^{-1} T_{wa}, & \text{if} \quad \langle w\lambda_a, \check{\alpha} \rangle < 0. \end{cases}$$

(3.4) LEMMA. – *For all $a \in A/A \cap K$, and $w \in W$, we have*

$$\Gamma(wa) = \epsilon(w) q^{\ell(a) - \ell(wa)} \Gamma(a),$$

where $\epsilon(w) = (-1)^{\ell(w)}$.

Proof. – Induct on $\ell(w)$. Let s be the simple reflection for $\alpha \in \Sigma$, with $sw > w$ as above. Then

$$\begin{aligned} \Gamma(swa) &= q^{-\ell(swa)}(T_{swa}\Phi, \tilde{\Phi}) \quad \text{by (3.2)} \\ &= q^{-\ell(swa)}(T_s^{\pm 1}T_{wa}\Phi, \tilde{\Phi}) \\ &= q^{-\ell(swa)}(T_{wa}\Phi, T_s^{\pm 1}\tilde{\Phi}) \quad \text{by (3.3)} \\ &= -q^{-\ell(swa)}(T_{wa}\Phi, \tilde{\Phi}) \quad \text{by definition of } \tilde{\Phi} \\ &= -q^{\ell(wa)-\ell(swa)}\Gamma(wa) \\ &= \epsilon(sw)q^{\ell(a)-\ell(swa)}\Gamma(a), \end{aligned}$$

this last by induction.

(3.5) LEMMA. – *For all $w \in W$ and $a \in A/A \cap K$, we have*

$$\ell(waw^{-1}) = \ell(a).$$

Proof. – It is enough to prove this for the simple reflection $s = s_\alpha$. We get

$$\ell(sas) = \sum_{\beta>0} |\langle \lambda_{sas}, \check{\beta}\rangle| = \sum_{\beta>0} |\langle \lambda_a, s\check{\beta}\rangle|.$$

Since s permutes the positive roots other than α among themselves and $|\langle \lambda_a, -\alpha\rangle| = |\langle \lambda_a, \alpha\rangle|$, the sum is unchanged.

(3.6) LEMMA. – *For $s = s_\alpha$ as above and $a \in A/A \cap K$, we have*

$$T_{sas} = \begin{cases} T_sT_aT_s^{-1}, & \text{if} \quad \langle w\lambda_a, \check{\alpha}\rangle \geq 0 \\ T_s^{-1}T_aT_s, & \text{if} \quad \langle w\lambda_a, \check{\alpha}\rangle < 0. \end{cases}$$

Proof. – If $\langle \lambda_a, \check{\alpha}\rangle \geq 0$ then $\ell(sass) = \ell(sa) = \ell(a) + 1 = \ell(sas) + 1$, so $T_{sas}T_s = T_{sass} = T_{sa} = T_sT_a$.

If $\langle \lambda_a, \check{\alpha}\rangle < 0$ then $\ell(ssa) = \ell(a) = \ell(sa) + 1$, so $T_sT_{sa} = T_{ssa} = T_a$. Also, $\ell(sas) = \ell(sa) + 1$, so $T_{sa}T_s = T_{sas}$. Putting these together gives $T_{sas} = T_s^{-1}T_aT_s$, as claimed.

Now we can compute $\Gamma(a)$ for an arbitrary $a \in A/A \cap K$. The next lemma would be clear for the K-spherical zonal function, but does not seem obvious for our Γ, which is only $\mathcal{I}$-bi-invariant

(3.7) LEMMA. – *For all $a \in A/A \cap K$ and $w \in W$, we have*

$$\Gamma(waw^{-1}) = \Gamma(a).$$

Proof. – It is enough to prove this for $w = s = s_\alpha$ as usual. We have

$$\begin{aligned} \Gamma(sas) &= \mathrm{vol}(\mathcal{I}sas\mathcal{I})^{-1}(T_{sas}\Phi, \tilde{\Phi}) \\ &= q^{-\ell(sas)}(T_s^{\pm 1}T_aT_s^{\mp 1}\Phi, \tilde{\Phi}) \\ &= -q^{-\ell(a)}(T_s^{\pm 1}T_a\Phi, \tilde{\Phi}), \end{aligned}$$

where the top row of signs occurs if and only if $\langle \lambda_a, \check{\alpha}\rangle \geq 0$. Lemma (3.3) says that T_s is self-adjoint with respect to (,). Hence T_s^{-1} is also self-adjoint. Moreover, $T_s^{\pm 1}\tilde{\Phi} = -\tilde{\Phi}$, so

$$\Gamma(sas) = -q^{-\ell(a)}(T_a\Phi, T_s^{\pm 1}\tilde{\Phi}) = q^{-\ell(a)}(T_a\Phi, \tilde{\Phi}) = \Gamma(a),$$

by (3.2).

4. The integral of the matrix coefficient

We now modify our matrix coefficient a little, and view it as a function on the weight lattice Λ. Recall that

$$M(\lambda_a) = \delta^{-\frac{1}{2}}(a)\Gamma(a) = \sum_{w\in W} w\cdot \big(e_{\lambda_a}\prod_{\alpha>0}\frac{1-q^{-1}e_\alpha}{1-e_\alpha}\big)(\tau).$$

(4.1) FORMULA A. – *Let $d(\tau)$ be the formal degree of $\mathcal{U}(\tau)$, where the Haar measure on G is such that* vol($\mathcal{I}$)=1. *Then*

$$d(\tau)^{-1} = q^\nu \sum_{J\subseteq\Sigma}\frac{1}{P_J(q)}\sum_{\lambda\in\Lambda_J}|M(\lambda)|^2.$$

Proof. – The formal degree may be computed as

$$d(\tau)^{-1} = \frac{\int_G |\Gamma(g)|^2 dg}{(\Phi,\tilde{\Phi})^2}.$$

Let dk be the restriction of Haar measure on G to K. We first compute

$$\begin{aligned}(\Phi,\tilde{\Phi}) &= \int_K \Phi(k)\tilde{\Phi}(k)dk\\ &= \sum_{w\in W}\int_{(B\cap K)w\mathcal{I}}\Phi(k)\tilde{\Phi}(k)dk\\ &= \sum_{w\in W}\mathrm{vol}[(B\cap K)w\mathcal{I}]q^{-2\ell(w)}\\ &= \sum_{w\in W} q^{-\ell(w)}\\ &= P(q^{-1}).\end{aligned}$$

Next,

$$\begin{aligned}\int_G |\Gamma(g)|^2 dg &= \sum_{W\times A/A\cap K}\mathrm{vol}(\mathcal{I}wa\mathcal{I})|\Gamma(wa)|^2\\ &= \sum_{W\times A/A\cap K} q^{\ell(wa)}\cdot q^{2(\ell(a)-\ell(wa))}|\Gamma(a)|^2\\ &= \sum_{W\times A/A\cap K} q^{2\ell(a)-\ell(wa)}|\Gamma(a)|^2.\end{aligned}$$

Every $a\in A/A\cap K$ is conjugate to a unique element of Λ^+. (We are identifying $A/A\cap K=\Lambda$ here.) For $a\in\Lambda_J\subseteq\Lambda^+$, let $W^a=W^J$, and $W_a=W_J$. Then $\{waw^{-1}: w\in W^J\}$ is the set of distinct W-conjugates of a. Therefore we have

$$\int_G |\Gamma(g)|^2 dg = \sum_{\substack{a\in\Lambda^+\\(w,x)\in W\times W^a}} q^{2\ell(xax^{-1})-\ell(wxax^{-1})}|\Gamma(xax^{-1}|^2$$
$$= \sum_{a\in\Lambda^+} q^{2\ell(a)}\Big[\sum_{(w,x)\in W\times W^a} q^{-\ell(wxax^{-1})}\Big]|\Gamma(a)|^2.$$

We must work on the sum in the square brackets. I claim that in this expression, we have

$$\ell(wxax^{-1}) = \ell(a) + \ell(wx) - \ell(x).$$

For $w = 1$ this is just (3.5), so we can induct on $\ell(w)$. Suppose $sw > w$, where $s = s_\alpha$ is the ubiquitous simple reflection. It is enough to prove that

$$\ell(swxax^{-1}) = \ell(wxax^{-1}) + \ell(swx) - \ell(wx).$$

We know that for any $a' \in A/A\cap K$, we have

$$\ell(swa') = \begin{cases} \ell(wa')+1 & \text{if} \quad \langle w\lambda_{a'}, \check{\alpha}\rangle \geq 0 \\ \ell(wa')-1 & \text{if} \quad \langle w\lambda_{a'}, \check{\alpha}\rangle < 0. \end{cases}$$

Suppose now that $swx > wx$. Then $(wx)^{-1}\alpha > 0$ so $0 \leq \langle\lambda_a, (wx)^{-1}\check{\alpha}\rangle = \langle w\lambda_{xax^{-1}}, \check{\alpha}\rangle$ so the claim is true in this case.

Now suppose $swx < wx$. To get the claim here, we need to show that $\langle\lambda_a, (wx)^{-1}\check{\alpha}\rangle < 0$. Since $(wx)^{-1}\alpha < 0$, we at least have $\langle\lambda_a, (wx)^{-1}\check{\alpha}\rangle \leq 0$. If it actually were zero, we would have $-(wx)^{-1}\alpha$ being a positive root in the span of J, where $a \in \Lambda_J$. But note that $x[-(wx)^{-1}\alpha] = -w^{-1}\alpha < 0$ since $sw > w$. This contradicts the fact that $x \in W^a$, and proves the claim.

Therefore,

$$\sum_{(w,x)\in W\times W^a} q^{-\ell(wxax^{-1})} = q^{-\ell(a)} \sum_{(w,x)\in W\times W^a} q^{-\ell(wx)+\ell(x)}$$
$$= q^{-\ell(a)} \sum_{x\in W^a} q^{\ell(x)} \sum_{w\in W} q^{-\ell(wx)}.$$

We can get rid of x in the sum over W, and the above expression becomes

$$q^{-\ell(a)}P^J(q)P(q^{-1}).$$

But $P^J(q) = P(q)P_J(q)^{-1} = q^\nu P(q^{-1})P_J(q)^{-1}$, where $\nu = |\Delta^+|$ is the number of positive roots, so

$$\int_G |\Gamma(g)|^2 dg = P(q^{-1})^2 q^\nu \sum_{a\in\Lambda^+}\Big[\sum_{x\in W_a} q^{\ell(x)}\Big]^{-1} q^{\ell(a)}|\Gamma(a)|^2.$$

Finally, recall that $\delta(a) = q^{-\ell(a)}$ for $a \in \Lambda^+$, so $q^{\ell(a)}|\Gamma(a)|^2 = |M(\lambda_a)|^2$.

5. Proof of Formula B

This is a simpler version of a computation of certain Whittaker functions in [R, (2.4)]. The idea is to expand the product in $M(\lambda)$, apply the Weyl character formula in an explicit form, and then put the product back together. The notation is that of section two.

We set, for any $\lambda \in \Lambda$,

$$\chi(\lambda) = \frac{e_\rho}{D} \sum_{w \in W} \epsilon(w) e_{w(\lambda+\rho)} \in \mathbb{C}[T],$$

where $D = \prod_{\alpha \in \Delta^+}(e_\alpha - 1)$. If λ is dominant, this is Weyl's formula for the character of the irreducible representation of $\hat{G}$ with highest weight λ. We denote its value at τ by $\chi(\lambda, \tau)$. It is easy to see, using the Weyl dimension formula applied to the centralizer of τ, that

$$\chi(\lambda, \tau) = e_\rho(\tau)(-1)^\nu C \sum_{W_\tau \backslash W} \epsilon(w) e_{w(\lambda+\rho)}(\tau) \prod_{\beta \in \Delta^+_{\tau w}} \langle \lambda + \rho, \beta \rangle.$$

We rewrite $M(\lambda)$ as

$$M(\lambda) = (-1)^\nu \left(\frac{e_\rho}{D} \sum_{w \in W} \epsilon(w) w \cdot (e_{\lambda - \rho} Q) \right)(\tau),$$

and expand

$$Q = \sum_{S \subseteq \Delta^+} (-q)^{-|S|} e_{\delta_S},$$

where δ_S is the sum of the roots in S, to get

$$\begin{aligned} M(\lambda) &= (-1)^\nu \sum_{S \subseteq \Delta^+} (-q)^{-|S|} \chi(\lambda - 2\rho + \delta_S, \tau) \\ &= e_\rho(\tau) C \sum_{S \subseteq \Delta^+} (-q)^{-|S|} \sum_{w \in W_\tau \backslash W} \epsilon(w) e_{w(\lambda - \rho + \delta_S)}(\tau) \prod_{\beta \in \Delta^+_{\tau w}} \langle \lambda - \rho + \delta_S, \beta \rangle \\ &= C \sum_{w \in W_\tau \backslash W} \epsilon(w) e_{\rho - w\rho}(\tau) \left[\sum_{S \subseteq \Delta^+} (-q)^{-|S|} e_{w\delta_S}(\tau) \prod_{\beta \in \Delta^+_{\tau w}} \langle \lambda - \rho + \delta_S, \beta \rangle \right] e_{w\lambda}(\tau). \end{aligned}$$

The term in square brackets is

$$\begin{aligned} &\sum_{\substack{S \subseteq \Delta^+ \\ J \subseteq \Delta^+_{\tau w}}} (-q)^{-|S|} e_{\delta_S}(\tau w) \left(\prod_{\beta \in J} \langle \delta_S, \beta \rangle \right) \left(\prod_{\alpha \in \Delta^+_{\tau w} - J} \langle \lambda - \rho, \alpha \rangle \right) \\ &= \sum_{\substack{S \subseteq \Delta^+ \\ J \subseteq \Delta^+_{\tau w}}} (-q)^{-|S|} \nabla_J(e_{\delta_S})(\tau w) \left(\prod_{\alpha \in \Delta^+_{\tau w} - J} \langle \lambda - \rho, \alpha \rangle \right) \\ &= \sum_{J \subseteq \Delta^+_{\tau w}} \nabla_J(Q)(\tau w) \left(\prod_{\alpha \in \Delta^+_{\tau w} - J} \langle \lambda - \rho, \alpha \rangle \right), \end{aligned}$$

as asserted in Formula B.

6. Remarks on Kazhdan-Lusztig parameters

In general it is a difficult combinatorial problem to determine the varieties $X(\tau, n)$. In principle, one can always resort to the Bruhat decomposition, but this does not seem practical for large groups. However, the number of cases to be considered can be greatly reduced, as we show in this section. Along with an inductive procedure given in [CLP, (3.9)] (which we discuss below in some examples), this will be enough for our purposes.

Recall the notation of (2.1). We have an element $s \in T$ whose centralizer H is semisimple, n is distinguished nilpotent in the Lie algebra of H, with associated homomorphism $\phi : SL_2(\mathbb{C}) \longrightarrow H$, and

$$\tau = s\phi\begin{pmatrix} q^{\frac{-1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}.$$

We show how to reduce the computation of $X(\tau, n)$ and $A(\tau, n)$, along with the homology representation of the latter on the former, to the case $s = 1$, $H = \hat{G}$. Since $s, \tau, \exp n$ all belong to H we have analogous objects $X_H(\tau, n)$, $A_H(\tau, n)$. Let Z_H and $Z_{\hat{G}}$ be the respective centers of H and $\hat{G}$. Let $Z_H(\tau, n)$ be the mutual centralizer in H of τ and $\exp n$. Since $Z_H(\tau, n)$ contains no nontrivial torus yet is reductive by [Car, 5.5.9], it must be finite. Hence we have

$$A_H(\tau, n) = Z_H(\tau, n)/Z_H.$$

Any element of $\hat{G}$ centralizing τ must also centralize its elliptic part s, so we have

$$A(\tau, n) = Z_H(\tau, n)/Z_{\hat{G}},$$

and there is an exact sequence

$$1 \longrightarrow Z_H/Z_{\hat{G}} \longrightarrow A(\tau, n) \longrightarrow A_H(\tau, n) \longrightarrow 1.$$

For example, if n is regular in H, then $Z_H(\tau, n) = Z_H$, so $A(\tau, n) = Z_H/Z_{\hat{G}}$. Let W_H be the Weyl group of T in H, viewed as a subgroup of W.

(6.1) LEMMA. – *The homology group $H_\bullet(X(\tau, n))$ is isomorphic, as an $A(\tau, n)$-module, to the direct sum of $[W : W_H]$ copies of $H_\bullet(X_H(\tau, n))$, where $A(\tau, n)$ acts on the latter homology group via the above exact sequence.*

Proof. – We will actually show that $X(\tau, n)$ is $Z_H(\tau, n)$-equivariantly isomorphic to the disjoint union of $[W : W_H]$-copies of $X_H(\tau, n)$. Moreover, the hypothesis that n be distinguished is not needed here.

Let $X(s)$ be the fixed points of s in X. Using the Bruhat decomposition, one checks that

$$X(s) = \coprod_{w \in W_H \backslash W} Hw\hat{B} \quad \text{disjoint union.}$$

Moreover, the stabilizer in H of $w\hat{B}$ is a Borel subgroup of H, because any positive system of the roots of $\hat{G}$ meets the roots of H in a positive system of the latter. Letting $X_H = H/H \cap \hat{B}$, we therefore have

$$X(s) \simeq \coprod_{w \in W_H \backslash W} X_H$$

as H-varieties. Take the mutual fixed points of $h = \phi\begin{pmatrix} q^{\frac{-1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}$ and $\exp n$ on both sides. Since s acts trivially on X_H, the right side becomes

$$\coprod_{w \in W_H \backslash W} X_H(\tau, n).$$

The left side becomes $X(s) \cap X(h) \cap X(n)$. The proof is completed by showing that $X(s) \cap X(h) = X(\tau)$. One containment is obvious. By uniqueness of expression in the Bruhat decomposition, each point in $X(\tau)$ can be written $uw\hat{B}$, where u belongs to a product of root groups for roots $\alpha \in \Delta_\tau^+$. On the other hand, $e_\alpha(s)$ has complex modulus one, and $e_\alpha(h)$ is positive real, so we must have $e_\alpha(s) = e_\alpha(h) = 1$, showing that $uw\hat{B}$ is also fixed by s and h.

For example, consider those discrete series representations V of G with $\dim V^{\mathcal{I}} = 1$, which are not twists of the Steinberg representation. We call these "Borel representations", and they fit into the Kazhdan-Lusztig classification as follows. The representation $\mathcal{M}_{\tau,\rho}$ is a Borel representation if and only if the irreducible representation ρ of $A(\tau, n)$ appears with multiplicity one in $H_\bullet(X(\tau, n))$ (see (2.1)). By (6.1), we must have $H = \hat{G}$ because the multiplicities are divisible by $|W : W_H|$. Thus n is distinguished in $\mathfrak{g}$, and determined by τ (see [Car, 5.6.2]). One now examines Borel's list [B, 5.8] and finds that for each τ occuring there, with corresponding n, one of two possibilities holds. Either there is a ρ appearing in $H_\bullet(X(\tau, n))$ with multiplicity one, in which case this Borel representation is $\mathcal{M}_{\tau,\rho}$, or there is no such ρ. The latter case arises because Borel is considering p-adic groups which are not necessarily adjoint, and several of his one dimensional representations may be be packaged in a single non-Borel representation of the corresponding adjoint group G.

We describe all of this more explicitly in each case below.

7. Examples

We now give some explicit computations of formal degrees using Formulas A and B. For rank at most three, our computations combined with the results of [B] will give all formal degrees of Iwahori-spherical representations. We will use the formal degrees to define L-packets whose members are either Iwahori-spherical representations, or certain other representations in the discrete series of G.

We first discuss G_2 in some detail. The computations of formal degrees for the other groups are entirely analogous, so we just present the results in those cases.

$\mathbf{G_2}$

Let α and β be the simple roots of G_2, with α the short root. Let $s = s_\alpha$ and $r = s_\beta$ be the corresponding simple reflections in the Weyl group.

There are five Iwahori-spherical discrete series representations of G_2. We list them according to the Kazhdan-Lusztig classification (§2).

$$H_1 = G_2: \quad \tau_1 = h_\alpha(q^{-3})h_\beta(q^{-5}), \quad \mathcal{U}(\tau_1) = \text{the Steinberg representation}, \quad W_0^1 = \{e\}$$
$$H_2 = G_2: \quad \tau_2 = h_\alpha(q^{-1})h_\beta(q^{-2}), \quad n_2 = \text{subregular}, \quad A(\tau_2, n_2) \simeq S_3, W_0^1 = \{e, r\}$$
$$H_3 \simeq (SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/\langle(-1,-1)\rangle: \quad \tau_3 = h_\alpha(-q^{-1})h_\beta(-q^{-2}),$$
$$n_3 = \text{regular in } H, \quad A(\tau_3, n_3) \simeq \mathbb{Z}/2, \quad W_0^1 = \{e, s, r\}$$
$$H_4 \simeq SL_3(\mathbb{C}): \quad \tau_4 = h_\alpha(\zeta q^{-1})h_\beta(q^{-2}),$$
$$n_4 = \text{regular in } H, \quad A(\tau_4, n_4) \simeq \mathbb{Z}/3, \quad W_0^1 = \{e, s\}.$$

In this last, ζ is a nontrivial cube root of unity. The conjugate root gives an isomorphic representation.

There is another square integrable constituent $\mathcal{U}(\tau_2)'$ of $I(\tau_2)$. It is the unique Borel representation, and is placed as follows: There is a chain of subrepresentations

$$0 \subset \mathcal{U}(\tau_2) \subset \mathcal{J}_{\mathcal{O}_3} \subset \mathcal{J}_{\mathcal{O}_2} \subset V \subset I(\tau),$$

with

$$V/\mathcal{J}_{\mathcal{O}_2} \simeq \mathcal{U}(\tau_2)'.$$

The subrepresentations $\mathcal{J}_{\mathcal{O}_d}$ are attached to orbits of the centralizer of τ in the q^{-1}-eigenspace of $Ad(\tau)$ in $\mathfrak{g}$ ([R]). The subscript on $\mathcal{O}$ indicates the dimension. We have $\dim(\mathcal{U}(\tau_2)')^{\mathcal{I}} = 1$, and

$$\mathcal{E}_2(\tau_2) = \{\mathcal{U}(\tau_2), \mathcal{U}(\tau_2)'\}.$$

The variety $X(\tau_2, n_2)$ consists of three points and a projective line. The group $A(\tau_2, n) \simeq S_3$ has three representations: 1=trivial, ϱ=reflection, sgn=signum. This group permutes the three points in $X(\tau_2, n_2)$ in all possible ways, and fixes the line. Thus only the characters 1 and ϱ appear in $H_\bullet(X(\tau_2, n_2))$. In terms of the Kazhdan-Lusztig parametrization,

$$\mathcal{U}(\tau_2)' \simeq \mathcal{M}_{\tau_2, \varrho},$$

Based on formal degrees, we declare a certain supercuspidal representation of G_2 to be the "missing" representation, corresponding to sgn. All other $\mathcal{E}_2(\tau_i)$'s are singletons, with trivial action by $A(\tau_i, n_i)$, and these packets also have missing representations, which we discuss below.

(7.1) PROPOSITION. – *The formal degrees of the Iwahori-spherical representations of G_2 are given as follows (recall $vol(\mathcal{I}) = 1$):*

$$d(\tau_1) = \frac{(q^5-1)(q-1)^2}{(q^6-1)(q+1)}$$
$$d(\tau_2) = \frac{1}{2}d(\tau_2)' = \frac{q(q-1)^3}{6(q^3-1)(q+1)^2}$$
$$d(\tau_3) = \frac{q(q-1)^2}{2(q^3+1)(q+1)}$$
$$d(\tau_4) = \frac{q(q-1)^3(q+1)}{3(q^6-1)}.$$

The formulas for $d(\tau_1)$ and $d(\tau_2)'$ are covered by the formulas in [B]. Before indicating the computations of the others, we discuss the representations of G_2 associated to the representations of the groups $A(\tau, n)$ which do not appear in the homology of $X(\tau, n)$.

It turns out that each formal degree above is also the formal degree of a supercuspidal representation. For the finite group $G_2(\mathbb{F}_q)$ there are four cuspidal unipotent representations. Labelling them as in [Car, p.(460)], they are

$$\begin{aligned} &G_2[1] \quad \text{of dimension} \quad \frac{q(q-1)^2(q^3-1)}{6(q+1)} \\ &G_2[-1] \quad \text{of dimension} \quad \frac{q}{2}(q-1)(q^3-1) \\ &G_2[\zeta] \quad \text{of dimension} \quad \frac{q}{3}(q^2-1)^2 \\ &G_2[\zeta^2] \quad \text{of dimension} \quad \frac{q}{3}(q^2-1)^2. \end{aligned}$$

(Recall that ζ is a cube root of unity.)

Let $d[x]$ be the formal degree of the irreducible supercuspidal representation

$$\mathcal{V}[x] = \mathrm{ind}_K^G G_2[x],$$

with the same normalization $\mathrm{vol}(\mathcal{I}) = 1$ as always. Then

$$d[x] = \frac{\deg G_2[x]}{P(q)},$$

and we find

$$d(\tau_2) = d[1], \quad d(\tau_3) = d[-1], \quad d(\tau_4) = d[\zeta].$$

These results are consistent with the following conjecture, which is a refinement of one due to Lusztig ([L, (1.5), (1.7)]).

(7.2) CONJECTURE. – *Let τ and n be as in section one. Let $\hat{A}$ be the set of irreducible representations of the component group $A(\tau, n)$. Then there is a finite set of square integrable representations of G,*

$$\widetilde{\mathcal{E}}_2(\tau) = \{V_\rho : \rho \in \hat{A}\}$$

such that the following hold.

(1) $\deg(V_\rho) = \dim(\rho)\deg(V_1)$.

(2) $V_1 \simeq \mathcal{U}(\tau)$, *and is the unique member of* $\widetilde{\mathcal{E}}_2(\tau)$ *with a Whittaker model.*

(3) V_ρ *has an Iwahori fixed vector if and only if* ρ *appears in the representation of* $A(\tau, n)$ *on the homology of the variety* $X(\tau, n)$.

(4) *The union of all* $\widetilde{\mathcal{E}}_2(\tau)$*'s is exactly the collection of square integrable representations whose invariants under the "unipotent radical" of a parahoric subgroup contain a unipotent representation of the corresponding Levi subgroup of* $G(\mathbb{F}_q)$.

I have added the first two statements. The others are due to Lusztig. This is of course in accordance with the examples computed here. In the G_2 case, we would have

$$\begin{aligned}
&\widetilde{\mathcal{E}}_2(\tau_1) = \{\text{Steinberg}\},\\
&\widetilde{\mathcal{E}}_2(\tau_2) = \{V_1 = \mathcal{U}(\tau_2),\ V_\varrho = \mathcal{U}(\tau_2)',\ V_\epsilon = \mathcal{V}[1]\} \quad (\text{here } \mathcal{A}(\tau_2, n_2) \simeq \mathcal{S}_3),\\
&\widetilde{\mathcal{E}}_2(\tau_3) = \{V_1 = \mathcal{U}(\tau_3),\ V_{-1} = \mathcal{V}[-1]\}, \quad (\text{here } |\mathcal{A}(\tau_3, n_3)| = 2),\\
&\widetilde{\mathcal{E}}_2(\tau_4) = \{V_1 = \mathcal{U}(\tau_4),\ V_\zeta = \mathcal{V}[\zeta],\ \mathcal{V}_{\zeta^2} = \mathcal{V}[\zeta^2]\}, \quad (\text{here } |\mathcal{A}(\tau_4, n_4)| = 3).
\end{aligned}$$

To verify (2), one must show that none of the non-Iwahori-spherical members of $\widetilde{\mathcal{E}}_2(\tau)$ have Whittaker models. Suppose $\mathcal{V}[x] \subseteq \mathrm{Ind}_N^G \theta$. By the Iwasawa decomposition $G = NAK$, there exists f belonging to the $G_2[x]$-isotypic component in the K-decomposition of $\mathcal{V}[x]$, and $a \in A$ such that $f(a) \neq 0$. Let K_1 be the kernel of the natural map $\pi : K \longrightarrow G_2(\mathbb{F}_q)$, and let $N_1 = N \cap K_1$. The invariance of f under N_1 implies θ is trivial on aN_1a^{-1}. Consider the character $\theta^a(n) = \theta(ana^{-1})$. We have a G-isomorphism $\mathrm{Ind}_N^G\theta \longrightarrow \mathrm{Ind}_N^G\theta^a$ given by $\phi \mapsto \phi^a$, $\phi^a(g) = \phi(ag)$, for $g \in G$. Now $f^a(1) \neq 0$, $\theta^a|_{N_1} \equiv 1$, and f^a is K_1-invariant. It follows that the restriction of f^a to K is a nonzero function of the form $\psi \circ \pi$, where ψ belongs to a Whittaker model of $G_2[x]$. This is a contradiction, because cuspidal unipotent representations of finite Chevalley groups do not have Whittaker models ([Car, p.379]).

We now show how our formula leads to the asserted formal degrees for G_2 which are not contained in [B]. We discuss $d(\tau_2)$ in detail since that is the most tedious one. For the others, we just give the explicit form of $M(\lambda)$ obtained from Formula B, after which it is a long but completely mechanical process to compute the formal degree, using Formula A.

COMPUTATION OF $d(\tau_2)$. – Normalize the inner product $\langle\ ,\ \rangle$ on Λ so that $\langle\beta, \beta\rangle = 3$, so $\langle\alpha, \alpha\rangle = 1$ and $\langle\alpha, \beta\rangle = -\frac{3}{2}$. For an arbitrary weight $\lambda = a\omega_\alpha + b\omega_\beta$, we have

$$\langle\lambda, \alpha\rangle = \frac{1}{2}\langle\lambda, \check{\alpha}\rangle = \frac{a}{2}.$$

Set $\tau = \tau_2$. We have $e_\alpha(\tau) = 1$, $e_\beta(\tau) = q^{-1}$, and $\Delta_\tau^+ = \{\alpha\}$. Also

$$Q(\tau) = \prod_{\gamma > 0} 1 - q^{-1}e_\gamma(\tau) = q^{-12}(q-1)(q^2-1)^4(q^3-1),$$

$$\prod_{\gamma \in \Delta^+ - \Delta_\tau^+} e_\gamma(\tau) - 1 = q^{-6}(1-q)^4(1-q^2), \qquad \prod_{\alpha \in \Delta_\tau^+} \langle\rho_1, \alpha\rangle = \frac{1}{2},$$

and

$$\sum_{\gamma>0} \frac{\langle\alpha,\gamma\rangle e_\gamma(\tau)}{1-q^{-1}e_\gamma(\tau)} = \frac{e_\alpha(\tau)}{1-q^{-1}e_\alpha(\tau)} = \frac{q}{q-1}.$$

In particular, the constant C of Formula B is

$$C = \frac{2q^6}{(1-q)^4(1-q^2)}.$$

Using the special cases in (2.3), one easily checks that only F_e and F_r are nonzero. The corresponding generalized eigenspaces in the Jacquet module are two and one dimensional, respectively. (The rest of the $\tau r\delta^{\frac{1}{2}}$- eigenspace is taken up by $\mathcal{U}(\tau_2)'$.)

For $\lambda = a\omega_\alpha + b\omega_\beta$ as above, we have

$$e_\lambda(\tau) = q^{-a-2b}, \quad e_\lambda(\tau r) = e_{\lambda - b\beta}(\tau) = q^{-a-b}.$$

Also by (2.3) we have

$$F_e(\lambda) = Q(\tau)\left(\frac{a-1}{2} - \frac{1}{q-1}\right),$$
$$F_r(\lambda) = -\epsilon(r)\langle r\beta,\alpha\rangle q^{-1}\frac{Q(\tau)}{1-q^{-1}e_\beta(\tau)} = \frac{3}{2}\frac{q^{-1}Q(\tau)}{(1-q^{-2})}.$$

Using Formula B, we get

$$|M(\lambda)|^2 = \frac{(q^3-1)^2(q^2-1)^4}{q^{12}(q-1)^6}\Big[\big(a^2(q^2-1)^2 - 2a(q^2-1)(q+1)^2 + (q+1)^4\big)q^{-2a-4b} + 6q\big(a(q^2-1)-(q+1)^2\big)q^{-2a-3b} + (9q^2)q^{-2a-2b}\Big].$$

We next compute each sum $\sum_{\Lambda_J} |M(\lambda)|^2$ separately. We abbreviate

$$C' = \frac{(q^3-1)^2(q^2-1)^4}{q^{12}(q-1)^6}.$$

$J = \varnothing$: We sum over all $a, b \geq 1$ and get

$$\sum_{\Lambda_\varnothing} |M(\lambda)|^2 = C'\frac{9q^6 - 3q^5 + 5q^4 + q^2 - q + 1}{(q^2+q+1)(q^2+1)(q^2-1)^2}.$$

$J = \{\beta\}$: Here $b = 0$ and we sum over all $a \geq 1$ to get

$$\sum_{\Lambda_{\{\beta\}}} |M(\lambda)|^2 = C'\frac{(2q^2-2q+1)}{(q^2-1)}.$$

$J = \{\alpha\}$: Here $a = 0$ and we sum over all $b \geq 1$ to get

$$\sum_{\Lambda_{\{\alpha\}}} |M(\lambda)|^2 = C' \frac{(q^2-q+1)(4q^4+q^2+1)}{(q^4-1)(q^2+q+1)}.$$

$J = \{\alpha, \beta\}$: Here $a = b = 0$. We get

$$|M(0)|^2 = C'(q^2-q+1)^2.$$

Using Formula A we find

$$d(\tau_2)^{-1} = q^6 \sum_{J \subseteq \{\alpha,\beta\}} \frac{1}{P_J(q)} \sum_{\lambda \in \Lambda_J} |M(\lambda)|^2 = \frac{6(q+1)^2(q^3-1)}{q(q-1)^3},$$

as claimed.

COMPUTATION OF $d(\tau_3)$. – For $\tau = \tau_3 = h_\alpha(-q^{-1})h_\beta(-q^{-2})$, we have

$$e_\alpha(\tau) = -1, \ e_\beta(\tau) = -q^{-1}, \text{ and } \Delta_\tau^+ = \varnothing.$$

Since τ is regular, $w \in W_0^1$ if and only if $Q(\tau w) \neq 0$, and one checks $W_0^1 = \{e, s, r\}$. Also

$$Q(\tau) = q^{-12}(q+1)(q^4-1)^2(q^3+1), \quad Q(\tau s) = Q(\tau), \quad Q(\tau r) = \frac{2Q(\tau)}{1+q^{-2}},$$

and

$$\prod_{\gamma \in \Delta^+} e_\gamma(\tau) - 1 = 2(1-q^{-2})^2(1+q^{-2}).$$

For $\lambda = a\omega_\alpha + b\omega_b$, we have

$$e_\lambda(\tau) = (-q^{-1})^a(-q^{-2})^b, \quad e_\lambda(\tau s) = (q^{-1})^a(-q^{-2})^b, \quad e_\lambda(\tau r) = (-q^{-1})^a(q^{-1})^b.$$

From (2.3) we get

$$M(\lambda) = \frac{(q+1)(q^2+1)(q^3+1)}{2q^6}\left([1+(-1)^a](q^{-1})^a(-q^{-2})^b + \frac{2q^{-1}(-q^{-1})^a(q^{-1})^b}{1+q^{-2}} \right).$$

Now we compute the sums $\sum_{\Lambda_J} |M(\lambda)|^2$ as before.

COMPUTATION OF $d(\tau_4)$. – For

$$\tau = \tau_4 = h_\alpha(\zeta q^{-1})h_\beta(q^{-2}),$$

we have

$$e_\alpha(\tau) = \bar{\zeta}, \quad e_\beta(\tau) = q^{-1}, \quad e_\lambda(\tau) = (\zeta q^{-1})^a(q^{-2})^b,$$
$$Q(\tau) = q^{-12}(q-\bar{\zeta})(q^6-1)(q^3-1)(q^2-1),$$
$$\prod_{\gamma > o} 1 - e_\gamma(\tau) = q^{-6}(1-\bar{\zeta})(q^3-1)(q^2-1)(q-1).$$

This τ is again regular, and we have $W_0^1 = \{e, s\}$. One checks that $e_\lambda(\tau s) = \overline{e_\lambda(\tau)}$, for every λ, dominant or not. In particular, $Q(\tau s) = \overline{Q(\tau)}$.

This means

$$M(\lambda) = \frac{(q^6-1)}{q^5(q-1)} N_a q^{-(a+2b)}, \quad \text{where} \quad N_a = \begin{cases} 1+q^{-1} & \text{if} \quad a \equiv 0(3) \\ -q^{-1} & \text{if} \quad a \equiv 1(3) \\ -1 & \text{if} \quad a \equiv -1(3). \end{cases}$$

Now we compute as before.

$\mathbf{SO_{2k+1}(F)}$

We now turn to the groups $SO_{2k+1}(F)$. We will use matrix notation. The dual group is $\hat{G} = Sp_{2k}(\mathbb{C})$. If we choose coordinates so that the matrix of the symplectic form is

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix},$$

we can choose T to be the diagonal matrices in $\hat{G}$ and an element $\tau \in T$ can be represented by the k-tuple $\tau = (t_1, \ldots, t_k)$, consisting of the first k diagonal entries in τ. We denote the simple reflections in the Weyl group by $s_1, \ldots, s_k$ where s_k inverts the k^{th} coordinate of τ and s_i swaps the i^{th} and $(i+1)^{\text{st}}$ coordinates, for $1 \le i \le k-1$.

For $0 \le i \le \left[\frac{k}{2}\right]$, let H_i be the centralizer of $(1, \ldots, 1, -1 \ldots, -1)$ (with i negative ones). Then

$$H_i \simeq Sp_{2(k-i)} \times Sp_{2i}.$$

These are the groups H appearing in the Kazhdan-Lusztig classification.

The distinguished nilpotent elements in $\mathfrak{sp}_{2k}$ are parametrized by partitions of k with distinct parts, and can be constructed as follows. Let $\lambda = \lambda_1 > \lambda_2 > \cdots > \lambda_{d(\lambda)}$ be a partition of k. Let $(V(m), \phi_m)$ be the irreducible $SL_2(\mathbb{C})$-representation of dimension $2m$. Then $V(m)$ carries a nondegenerate $SL_2(\mathbb{C})$-invariant symplectic form and the orthogonal direct sum $V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_{d(\lambda)})$ is a symplectic space of dimension $2k$. The homomorphism $\phi : SL_2(\mathbb{C}) \longrightarrow Sp(V)$ is the direct sum of the ϕ_{λ_i}'s. Then we can take

$$\tau = \phi\begin{pmatrix} q^{\frac{-1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}, \quad n = d\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The mutual centralizer $Z(\tau, n)$ of n and τ is the collection of symplectic automorphisms of V acting by $\pm I$ on each $V(\lambda_i)$. Hence $Z(\tau, n) \simeq (\mathbb{Z}/2)^{d(\lambda)}$.

The following observation will suffice to compute $X(\tau, n)$ in our examples. By (6.1) we can assume n is distinguished in $\mathfrak{sp}_{2k}(\mathbb{C})$. The flag variety X is the variety of isotropic flags

$$F = F_1 \subset \cdots \subset F_k$$

with $\dim F_i = i$. Then $F \in X(\tau)$ if and only if each F_i is spanned by τ-eigenvectors. Let $\pi : X \longrightarrow \mathbb{P}(\mathbb{V})$ be the map sending F to F_1. If $F \in X(\tau, n)$ then $F_1 \subset \ker n$. Since n is distinguished, the eigenvalues of τ on $\ker n$ are distinct, so the image of π is finite, and each connected component of $X(\tau, n)$ is contained in an $A(\tau, n)$-stable fiber of π. In turn, each fiber is an $X_L(\tau', n')$ for a proper Levi subgroup L of $Sp_{2k}(\mathbb{C})$ (see [CLP, 3.9]). In our examples, either n' is distinguished, so we can repeat the process, or k is very small and the computation of $X(\tau, n)$ is elementary.

To illustrate, we give the parameters of the Borel representations. Up to twisting by the order two character of $SO_{2k+1}(F)$ corresponding to $-I \in Sp_{2k}(\mathbb{C})$, there is one Borel representation for $k \geq 3$. It occurs when $H = \hat{G}$,

$$\tau = (q^{-\frac{2k-3}{2}}, q^{-\frac{2k-5}{2}}, \ldots, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}),$$

and n is subregular in $\mathfrak{sp}_{2k}$, with partition $(k-1, 1)$. The above inductive procedure reduces us to the case $k = 2$, where $n' \in \mathfrak{sp}_4(\mathbb{C})$ is no longer distinguished, but $X_L(\tau', n') \subset X(\tau, n)$ is easily seen to be a union of a projective line and two points, the latter being interchanged by a nontrivial element of $A(\tau, n)$. We find that the variety $X(\tau, n)$ consists of k points and a projective line. The group $A(\tau, n)$ has order two, and permutes two of the points while fixing the other components. Hence the unique nontrivial character ρ of $A(\tau, n)$ appears with multiplicity one in $H_\bullet(X(\tau, n))$, the unique (up to twist) Borel representation of $SO_{2k+1}(F)$ is $\mathcal{M}_{\tau,\rho}$, and we have $\mathcal{E}_2(\tau) = \{\mathcal{U}(\tau),\ \mathcal{M}_{\tau,\rho}\}$.

We now give some computations of formal degrees and L-packets.

$\mathbf{SO_5(F)}$

The parameters for the Iwahori-spherical discrete series are as follows.

$(i = 0)$ $\tau_1 = (q^{-\frac{3}{2}}, q^{-\frac{1}{2}})$, $n_1 =$ regular, $A(\tau_1, n_1) = 1$, $W_0^1 = \{e\}$

$(i = 1)$ $\tau_2 = (-q^{-\frac{1}{2}}, q^{-\frac{1}{2}})$, $n =$ regular in $\mathfrak{sp}_2 \times \mathfrak{sp}_2$, $A(\tau_2, n_2) \simeq \mathbb{Z}/2$, $W_0^1 = \{e, s_1\}$

In the latter case, the action of $A(\tau_2, n_2)$ on $X(\tau_2, n_2)$ is trivial, because τ_2 is regular. The Iwahori spherical discrete series of $SO_5(F)$ is therefore given by

$$\mathcal{E}(\pm\tau_1) = \{\mathcal{U}(\pm\tau_1)\} \quad \text{(the Steinberg representation and its twist)},$$

$$\mathcal{E}(\pm\tau_2) = \{\mathcal{U}(\pm\tau_2)\}.$$

The formal degrees $d(\pm\tau_1)$ are in [B] and $d(\pm\tau_2)$ is easily computed by hand using Formulas A and B (note that τ_2 is regular). We find

(7.3) PROPOSITION. – *With the Iwahori subgroup having mass one, the formal degrees of the Iwahori-spherical discrete series representations of $SO_5(F)$are given by*

$$d(\pm\tau_1) = \frac{(q^3-1)(q-1)}{2(q^2+1)(q+1)^2}, \qquad d(\pm\tau_2) = \frac{q(q-1)^2}{2(q^2+1)(q+1)^2}.$$

We note ([Car]) that the finite group $SO_5(\mathbb{F}_q)$ has exactly one cuspidal unipotent representation κ, and with the same normalization of Haar measure as above, one finds that the formal degree of $\tilde{\kappa} := ind_K^G \kappa$ equals $d(\tau_2)$. Thus if we set

$$\tilde{\mathcal{E}}(\pm\tau_1) = \mathcal{E}(\pm\tau_1), \qquad \tilde{\mathcal{E}}(\pm\tau_1) = \{\mathcal{U}(\pm\tau_2),\ \pm \otimes \tilde{\kappa}\},$$

where $\pm\otimes$ denotes twisting by the character of $SO_5(F)$ corresponding to $\pm I \in Sp_4(\mathbb{C})$, we see that (7.2) is verified for $SO_5(F)$.

$\mathbf{SO_7(F)}$

Here the parameters are as follows.

$$(i=0)\quad \tau_1 = (q^{-\frac{5}{2}}, q^{-\frac{3}{2}}, q^{-\frac{1}{2}}),\ n_1 = \text{regular},\ A(\tau_1, n_1) = 1,\ W_0^1 = \{e\}$$
$$(i=0)\quad \tau_2 = (q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}),\ n_2 = \text{subregular},\ A(\tau_2, n_2) \simeq \mathbb{Z}/2, W_0^1 = \{e, s_1, s_3\}$$
$$(i=1)\quad \tau_3 = (q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}),\ n = \text{reg. in } \mathfrak{sp}_4 \times \mathfrak{sp}_2,$$
$$A(\tau_3, n_3) \simeq \mathbb{Z}/2,\ W_0^1 = \{e, s_2, s_2 s_3\}.$$

The variety $X(\tau_2, n_2)$ consists of three points and the projective line through $\hat{B}$ and $s_3\hat{B}$ in $X = \hat{G}/\hat{B}$. The group $A(\tau_2, n_2)$ permutes two of the points, leaving the other components fixed. The group $A(\tau_3, n_3)$ acts trivially on $X(\tau_3, n_3)$. Thus we have

$$\mathcal{E}_2(\pm\tau_i) = \{\mathcal{U}(\pm\tau_i)\} \quad \text{for} \quad \text{i} = 1, 3,$$

$$\mathcal{E}_2(\pm\tau_2) = \{\mathcal{U}(\pm\tau_2), \mathcal{U}(\pm\tau_2)'\}.$$

The Steinberg representation and its twist are $\mathcal{U}(\pm\tau_1)$ and the Borel representations are $\mathcal{U}(\pm\tau_2)'$. The formal degrees of these are found in [B]. As before we let $d(\pm\tau_2)'$ denote the formal degree of $\mathcal{U}(\pm\tau_2)'$.

(7.4) PROPOSITION. – *With the volume of the Iwahori subgroup equal to one, the formal degrees of the Iwahori-spherical discrete series representations of $SO_7(F)$ are given as follows.*

$$d(\tau_1) = \frac{(q^5-1)(q-1)^2}{2(q^3+1)(q^2+1)(q+1)}$$
$$d(\tau_2) = d(\tau_2)' = \frac{q(q-1)^3}{4(q^2+1)(q+1)^3}$$
$$d(\tau_3) = \frac{q(q-1)^2(q^3-1)}{4(q^3+1)(q^2+1)(q+1)}.$$

Since the computations are just like those for G_2, we omit the details.

I believe, but have not proven, that (7.2) can be verified in the following way. There is only one representation of an $A(\tau, n)$ not accounted for by Iwahori spherical representations of $SO_7(F)$. This corresponds to the nontrivial representation of $A(\tau_3, n_3)$. Likewise, the only Levi subgroup of $SO_7(\mathbb{F}_q)$ which has a cuspidal unipotent representation is $SO_5(\mathbb{F}_q)$ as mentioned above. Let $\tilde{\kappa}$ be the supercuspidal representation of $SO_5(F)$ already considered. Let L be the unique Levi subgroup of $SO_7(F)$ isomorphic to $F^\times \times SO_5(F)$, and let P be a parabolic subgroup with Levi equal to L. Then the representation

$$\mathrm{Ind}_P^G |\cdot|^s \otimes \tilde{\kappa}$$

is reducible for exactly one positive $s_0 \in \mathbb{R}$ ([S]). According to [C2], this induced representation will then have a unique square integrable subrepresentation V. Note that V is not generic, since $\tilde{\kappa}$ is not generic. We set

$$\tilde{\mathcal{E}}(\pm\tau_3) = \{\mathcal{U}(\pm\tau_3), \pm \otimes V\},$$

along with

$$\tilde{\mathcal{E}}(\pm\tau_i) = \mathcal{E}(\pm\tau_i)$$

for $i = 1, 2$. If the formal degree of V is $\frac{q(q-1)^2(q^3-1)}{4(q^3+1)(q^2+1)(q+1)}$, then conjecture (7.2) will be verified for $SO_7(F)$. According to [S1,(5.5.4.3)] and [S], the formal degree of V is given by the residue at s_0 of a certain rational function in one variable s. Thus, knowing the reducibility point s_0 is equivalent to knowing the formal degree. Unfortunately we cannot use the results of Shahidi ([Sh]) to determine s_0 since the inducing representation is not generic.

$\mathbf{SO_9(F)}$

Here $\hat{G} = Sp_8(\mathbb{C})$, and the parameters for the Iwahori spherical discrete series are as follows.

$$\begin{aligned}
&(i=0) \quad \tau_1 = (q^{-\frac{7}{2}}, q^{-\frac{5}{2}}, q^{-\frac{3}{2}}, q^{-\frac{1}{2}}),\ n_1 = \text{regular},\ A(\tau_1, n_1) = 1,\ W_0^1 = \{e\}\\
&(i=0) \quad \tau_2 = (q^{-\frac{5}{2}}, q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}),\ n_2 = \text{subregular},\\
&\qquad\qquad A(\tau_2, n_2) \simeq \mathbb{Z}/2, \quad W_0^1 = \{e, s_4, s_2, s_2 s_1\}\\
&(i=1) \quad \tau_3 = (q^{-\frac{5}{2}}, q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}),\ n_3 = \text{regular} \ \in \mathfrak{sp}_3 \times \mathfrak{sp}_1,\\
&\qquad\qquad A(\tau_3, n_3) \simeq \mathbb{Z}/2, \quad W_0^1 = \{e, s_3, s_3 s_2, s_3 s_2 s_1\}\\
&(i=1) \quad \tau_4 = (q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}), n_4 = \text{subregular} + \text{regular} \in \mathfrak{sp}_3 \times \mathfrak{sp}_1,\\
&\qquad\qquad A(\tau_4, n_4) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2, \quad |W_0^1| = 12\\
&(i=2) \quad \tau_5 = (q^{-\frac{3}{2}}, -q^{-\frac{3}{2}}, q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}, -q^{-\frac{1}{2}}), n_5 = \text{regular in } \mathfrak{sp}_2 \times \mathfrak{sp}_2,\\
&\qquad\qquad A(\tau_5, n_5) \simeq \mathbb{Z}/2, \quad W_0^1 = \{e, s_1, s_2, s_3, s_1 s_3, s_1 s_3 s_2\}.
\end{aligned}$$

The nonregular characters are τ_2 and τ_4. The variety $X(\tau_2, n_2)$ consists of four points and the line joining $\hat{B}$ and $s_3\hat{B}$. The group $A(\tau_2, n_2)$ permutes two of the points and fixes the remaining components. Using (6.1), we find that the variety $X(\tau_4, n_4)$ consists of twelve points and four lines. The lines join the points $\hat{B}, s_3\hat{B}, s_3 s_2\hat{B}, s_3 s_2 s_1\hat{B}$ to their respective translates by s_2. We can choose generators ϵ and δ for $A(\tau_4, n_4)$ so that ϵ swaps four pairs of points and fixes the remaining eight components, while δ acts trivially on $X(\tau_4, n_4)$.

The Iwahori-spherical discrete series is therefore given by

$$\mathcal{E}_2(\pm\tau_j) = \{\mathcal{U}(\pm\tau_j)\} \quad \text{for} \quad \mathrm{i} = 1, 3, 5,$$

$$\mathcal{E}_2(\pm\tau_j) = \{\mathcal{U}(\pm\tau_j), \mathcal{U}(\pm\tau_j)'\} \quad \text{for} \quad \mathrm{i} = 2, 4.$$

I have only computed the degrees for regular τ's. They are

$$d(\pm\tau_1) = \frac{(q^7-1)(q^5-1)(q-1)^2}{2(q^4+1)(q^3+1)(q^2+1)^2(q+1)^3}$$
$$d(\pm\tau_3) = \frac{q(q^5-1)(q-1)^3}{4(q^4+1)(q^3+1)(q+1)^3}$$
$$d(\pm\tau_5) = \frac{q^2(q^3-1)^2(q-1)^2}{2(q^4+1)(q^2+1)^2(q+1)^4}.$$

The Borel representations $\mathcal{U}(\pm\tau_2)'$ correspond to the nontrivial character of $A(\tau_2, n_2)$ and have degree

$$d(\pm\tau_2)' = \frac{q(q^5-1)(q-1)^3}{4(q^3+1)(q^2+1)^2(q+1)^3}.$$

For τ_3, τ_4 and τ_5 we have one, two and one missing representation(s), respectively.

$\mathbf{PSp_{2k}(F)}$

Next we consider the group $PSp_{2k}(F)$, with dual group $\hat{G} = Spin_{2k+1}(\mathbb{C})$. We return to Chevalley group notation for elements of $\hat{G}$. Denote the simple roots of $\hat{G}$ by $\alpha_1, \alpha_2, \ldots, \alpha_k$ with α_k being short. Let h_i be the one-parameter subgroup h_{α_i}. The semisimple centralizers of semisimple elements are of the form

$$H_i = \big(Spin_{2(k-i)} \times Spin_{2i+1}\big)/Z,$$

where Z is a central subgroup of the direct product of order two, and $0 \le i \le k$, $i \ne k-1$. The center of $Spin_{2(k-i)}$ maps isomorphically onto the center of H_i, which has order four.

The Borel representations arise as follows. For $k \ge 3$ the representation $\mathcal{U}(\tau)$ corresponding to $H_0 = Spin_{2k}$, n regular in H_0, has $\dim \mathcal{U}(\tau)^{\mathcal{I}} = 2$. Here the variety $X(\tau, n)$ consists of two points and the group $A(\tau, n)$ has order two, acting trivially on $X(\tau, n)$.

Let G_0 be the image of $Sp_{2k}(F)$ in $G = PSp_{2k}(F)$ under the natural isogeny. Then G_0 has finite index in G, and the restriction of $\mathcal{U}(\tau)$ to G_0 splits into two representations, each having one-dimensional $\mathcal{I} \cap \mathcal{G}_0$-invariants. These are Borel representations of $Sp_{2k}(F)$, but not of $PSp_{2k}(F)$.

For $k \ge 4$ there is a Borel representation of $PSp_{2k}(F)$ with parameters $H = H_k = \hat{G}$, n regular in H_{k-2}. The latter is a distinguished nilpotent element in $\mathfrak{so}_{2k+1}(\mathbb{C})$ having Jordan blocks of sizes $1, 3, 2k-3$. This is distinquished because $k \ge 4$. The variety $X(\tau, n)$ consists of the following components: $X_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $X_2 \simeq \mathbb{P}^1$, $X_3 \simeq \mathbb{P}^1$, $X_i = \{x_i, x_i'\}$, for $4 \le i \le k+1$, $Y = \{y, y'\}$.

The group $A(\tau, n)$ is a product of two cyclic groups of order two with generators a and b, where a permutes the points y and y', while b permutes x_i and x_i', and both a and b act trivially on the other components. Let ρ be the character of $A(\tau, n)$ which is nontrivial on a and is trivial on b. Then the unique Borel representation of $PSp_{2k}(F)$ is $\mathcal{M}_{\tau,\rho}$.

$\mathbf{PSp_6(F)}$

We now take $k = 3$ and consider $G = PSp_6(F)$ with $\hat{G} = Spin_7(\mathbb{C})$. All τ's with nonempty $\mathcal{E}_2(\tau)$ are regular, and the corresponding nilpotent elements are regular in their H. There are three pairs of them:

$$H_0 = SL_4, \quad \tau_0^{\pm} = h_1(-q^{-2})h_2(q^{-3})h_3(\pm iq^{-\frac{3}{2}}), \ A(\tau_0^{\pm}, n_0) \simeq \mathbb{Z}/2, \ W_0^1 = \{e, s_3\}$$
$$H_1 = \big(SL_2 \times SL_2 \times SL_2\big)/\langle(-I,-I,-I)\rangle, \quad \tau_1^{\pm} = h_1(-q^{-\frac{1}{2}})h_2(-q^{-2})h_3(\pm q^{-1}),$$
$$A(\tau_1^{\pm}, n_1) \simeq \mathbb{Z}/2, \quad W_0^1 = \{e, s_1, s_2, s_3, s_1s_3, s_3s_2\}$$
$$H_3 = \hat{G}, \quad \tau_3^{\pm} = h_1(q^{-3})h_2(q^{-5})h_3(\pm q^{-3}), \quad A(\tau_3^{\pm}, n_3) = 1, \quad W_0^1 = \{e\}.$$

(7.5) PROPOSITION. – *The formal degrees of the Iwahori-spherical discrete series representations of $PSp_6(F)$, with the volume of the Iwahori subgroup equal to one, are given as follows.*

$$d(\tau_0^{\pm}) = \frac{q(q-1)^3}{2(q^3+1)(q+1)^2}$$
$$d(\tau_1^{\pm}) = \frac{q^2(q-1)^3}{2(q^3+1)(q^2+1)(q+1)^2}$$
$$d(\tau_3^{\pm}) = \frac{(q^5-1)(q-1)^2}{2(q^3+1)(q^2+1)(q+1)}.$$

For $\tau_0^{\pm}$ and $\tau_1^{\pm}$ we have one missing representation each.

$\mathbf{F_4}$

We have only one new formal degree to offer for F_4 so we shall be brief. The possible semisimple centralizers H of semisimple elements are of type F_4, B_4, $A_1 \times C_3$, $A_2 \times A_2$, and $A_3 \times A_1$. Up to conjugacy there is a unique distinguished nilpotent element n in F_4 which has $A(\tau, n) = S_4$. Let r be the unique three dimensional irreducible representation of S_4 containing an vector invariant under S_3. Let ϵ be the sign character of S_4. Then the Borel representation has parameters $(\tau, n, r \otimes \epsilon)$. This is verified by the procedure mentioned in section 6, using the description of $X(\tau, n)$ given in [CLP]. The missing representation corresponds to ϵ. If we inflate and induce the cuspidal unipotent representation $F_4^{II}[1]$ ([Car,p.461]) of $\mathbf{F_4}(\mathbb{F}_\mathbf{q})$, we get a supercuspidal representation whose degree is one third that of the Borel representation, in agreement with (7.2). It seems difficult to compute the degree of the generic member of this packet using our formula. Allen Moy has independently noticed the concurrence of the above two formal degrees.

We can implement our formula for the representation with $H = B_4$, n regular in H, $\tau = h_{\alpha_1}(q^{-7})h_{\alpha_2}(q^{-13})h_{\alpha_3}(-q^{-9})h_{\alpha_4}(-q^{-5})$, where α_i are the simple roots numbered sequentially across the Dynkin diagram, and α_1 and α_2 are long. We find, with the volume of $\mathcal{I} = 1$, that

$$\deg \mathcal{U}(\tau) = \frac{q(q^{10}-1)(q^7-1)(q^3-1)(q-1)^3}{2(q^{12}-1)(q^8-1)(q^3+1)(q+1)^2}.$$

This is not the formal degree of the inflation and induction of a cuspidal unipotent representation of $\mathbf{F_4}(\mathbb{F}_\mathbf{q})$.

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