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ON GLOBAL NASH FUNCTIONS

BY JESÚS M. RUIZ (1) AND MASAHIRO SHIOTA

ABSTRACT. — Let $M \subset \mathbb{R}$ be a compact Nash manifold, and $\mathcal{N}(M)$ [resp. $\mathcal{O}(M)$] its ring of global Nash (resp. analytic) functions. A *global Nash* (resp. *analytic*) *set* is the zero set of finitely many global Nash (resp. analytic) functions, and we have the usual notion of *irreducible* set. Then we say that *separation holds for* M if every Nash irreducible set is analytically irreducible. The main result of this paper is that separation holds if and only if *every semialgebraic subset of* M *described by s global analytic inequalities can also be described by s global Nash inequalities*. In passing, we also prove that when separation holds, *every Nash function on a Nash set extends to a global Nash function on* M.

Introduction

Let $M \subset \mathbb{R}^p$ be a Nash manifold, $\mathcal{N}_M$ (resp. $\mathcal{O}_M$) its sheaf of germs of Nash (resp. analytic) functions of M and $\mathcal{N}(M)$ [resp. $\mathcal{O}(M)$] its ring of global Nash (resp. analytic) functions on M. One of the main and oldest open problems on global Nash functions is *separation*. To state it properly, recall that a *Nash set* is a subset $X \subset M$ which is the zero set of a global Nash function $h \in \mathcal{N}(M): X = \{x \in M : h(x) = 0\}$, and X is called *Nash irreducible* if it is not the union of two smaller Nash sets. Of course, this mimics the global analytic notions of Bruhat-Whitney ([BrWh]), and leads to the problem mentioned above, namely:

PROBLEM 1 (Separation). — *Is every Nash irreducible set an irreducible global analytic set?*

In case the answer is yes we say that *separation holds for* M. This problem can be reformulated in another familiar way. Recall that M is a semialgebraic subset of $\mathbb{R}^p$, and that any Nash set $X \subset M$ is semialgebraic too. Then one may ask: *is every semialgebraic global analytic set a Nash set?* This is equivalent to separation by two basic facts: *a*) the irreducible analytic components of a semialgebraic analytic set are also semialgebraic, and *b*) every semialgebraic set is contained in a Nash set of the same dimension.

Actually, the semialgebraic subsets of M are essentially linked to Nash functions: they are exactly the subsets that can be defined with finitely many systems of Nash equalities

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and/or inequalities. In other words, using Nash functions instead of polynomials in the definition of semialgebraic sets does not produce new sets. However, what does change is *complexity*. For instance, to describe the set $S=\{(x, y)\in\mathbb{R}^2 : x^2-y^2>1, x>0\}$ we need two polynomial inequalities, while a single Nash one is enough, namely $S=\{(x, y)\in\mathbb{R}^2 : x>\sqrt{1+y^2}\}$. Again, the open question is the comparison of *Nash* and *analytic* complexity:

PROBLEM 2 (Complexities). — *Is every semialgebraic set* $S\subset M$ *that can be described with s analytic inequalities*:

$$S=\{x\in M : f_1(x)>0, \ldots, f_s(x)>0\} \qquad \textit{with} \quad f_1, \ldots, f_s\in\mathcal{O}(M),$$

also describable with s Nash inequalities:

$$S=\{x\in M : g_1(x)>0, \ldots, g_s(x)>0\} \qquad \textit{with} \quad g_1, \ldots, g_s\in\mathcal{N}(M)?$$

Now if the answer to this is affirmative we will say that M *has equal complexities.*

It is fairly easy to see why complexities are connected to separation. For, suppose separation fails for a Nash set $X\subset M$, that is, X is Nash irreducible but analytically reducible. Then there are two analytic functions f and g neither of which vanishes on X, but their product $h=fg$ does, and we can even suppose that $X=\{h=0\}$. Then the semialgebraic set $S=\{f^2>0\}$ canot be described with one single Nash inequality.

What is more surprising is that the converse is also true if M is compact. This is the main result we will prove in this paper:

THEOREM 1. — *Let* $M\subset\mathbb{R}^p$ *be a compact Nash manifold. Then separation holds for* M *if and only if* M *has equal complexities.*

The proof of this result involves in a crucial way another open problem on global Nash functions. To make this precise we need some more terminology.

Let $X\subset M$ be a Nash set. Then the *ideal of* X is the ideal of all global Nash functions that vanish on $X: I(X)=\{h\in\mathcal{N}(M) : h(x)=0 \ \forall x\in X\}$, and the *sheaf of Nash function germs of* X is the sheaf $\mathcal{N}_X=\mathcal{N}_M/I(X)\mathcal{N}_M$. This seems to be the suitable global notion; in particular, since $\mathcal{N}(M)$ is a noetherian ring, the sheaf $\mathcal{N}_X$ is *globally* finitely presented. A *Nash function on* X is a global section of this sheaf $\mathcal{N}_X$, and we will denote by $\mathcal{N}(X)$ the ring of all Nash functions on X. There is a canonical homomorphism $\mathcal{N}(M)\to\mathcal{N}(X)$ which by obvious reasons we call *restriction*, and we come to another important open question:

PROBLEM 3 (Extension). — *Is every Nash function on a Nash set* $X\subset M$ *the restriction of a Nash function on* M?

If the solution is in the affirmative we will say that *extension holds for* M. Note that every Nash function on a Nash set is always the restriction of a global analytic function, by Cartan's Theorem B, and also the restriction of a global continuous semialgebraic function, by the semialgebraic Tietze theorem, but these two extensions may well not coincide. One key fact we need to prove before Theorem 1 is:

THEOREM 2. — *Let* $M \subset \mathbb{R}^p$ *be a* compact *Nash manifold. If separation holds for* M, *then extension holds too.*

These questions have attracted the interest of many mathematicians since the end of the 70's [BE]. The way they are posed varies from one author to another, and in particular we have chosen geometric versions (later we will be more explicit on this matter). The first relevant result was Efroymson's positive solution to extension in case X is a Nash submanifold of M ([Ef2], [Pk]). Concerning separation, Shiota has proved that a semialgebraic analytic set X which is a closed submanifold is a Nash set ([Sh2]). Quite recently Tancredi and Tognoli have obtained these two results in case X is compact coherent and has only normal singularities [TT]. Also recently, Coste and Diop have come back to some old ideas of Efroymson [Ef1] to solve extension if M is an open semialgebraic subset of $\mathbb{R}^2$ [CtDp]. Furthermore there were many known connections between these problems and other open questions on Nash functions ([BT], [CtDp], [Sh2]), much in the spirit of this work. On the other hand some of our results here can be discussed without compactness assumptions, in a more general sheaf theoretic setting; we will treat this in the forthcoming [RzSh]. Finally, there are several results by Shiota [Sh3] concerning analytic factorization of Nash functions, which settle all these matters for Nash surfaces. For instance, we have:

THEOREM 3. — *Let* $M \subset \mathbb{R}^p$ *be a Nash manifold. Any Nash irreducible set* $X \subset M$ *of dimension* $\leqq 1$ *is an irreducible global analytic set.*

Proof. — We can assume that M is closed in $\mathbb{R}^p$. Then a global analytic set in M is a global analytic set in $\mathbb{R}^p$. Hence it suffices to see that a semialgebraic global analytic set $X \subset \mathbb{R}^p$ of dimension $\leqq 1$ is a Nash set. Let $\pi : \mathbb{R}^p \to \mathbb{R}^2$ be a linear projection such that $\pi | X$ is proper and $\pi | X \setminus X'$ is injective for some finite set X'. Then $\pi(X) \subset \mathbb{R}^2$ is a closed semialgebraic global analytic set of dimension $\leqq 1$ (recall the fact that any closed analytic set in $\mathbb{R}^2$ is globally analytic). Thus we will assume $p=2$; we can suppose also that X is irreducible and, consequently, everywhere of dimension 1.

Let Z denote the Zariski closure of X in $\mathbb{R}^2$, and Y the Nash irreducible component of Z containing X. Clearly Y is everywhere of dimension 1, and X is a global analytic irreducible component of Y. We set $X_1 = X$ and denote by X_2 the union of the other global analytic irreducible components of Y. Thus $Y = X_1 \cup X_2$, and both X_i are semialgebraic global analytic sets everywhere of dimension 1. We see that the ideal I(Y) is principal as follows. Let $\mathscr{P}$ denote the sheaf of ideals of $\mathscr{N}_{\mathbb{R}^2}$ generated by I(Y). Then every stalk $\mathscr{P}_x$, $x \in Y$, is an intersection of height one prime ideals. Hence it is generated by one element: namely, a square root f of $h = \sum_i f_i^2$ for some fixed generators $f_1, \ldots, f_k$ of I(Y). Indeed, locally any of the two square roots generates the sheaf, and we can choose the sign coherently to get a global square root because $\mathbb{R}^2$ is contractible (*see* [Sh1, Lemme 1]). By a similar argument as above, we obtain two analytic functions g_1 and g_2 which generate the ideals of analytic functions vanishing on X_1 and X_2 respectively. The only additional remark in the analytic case is that we do not know that $I(X_i)$ is finitely generated. To settle this difficulty, we consider a Stein open neighborhood Ω of $\mathbb{R}^2$ in $\mathbb{C}^2$ and an extension to Ω of the analytic sheaf $\mathscr{P}_i$

generated by $\mathrm{I}(\mathrm{X}_i)$ (*see* [C]). Then we find countably many complex analytic functions $h_j^{\mathbb{C}}$ on Ω whose restrictions to $\mathbb{R}^2$ generate $\mathscr{P}_i$. Then we can choose small enough positive real numbers c_j such that the series $\sum c_j (h_j^{\mathbb{C}})^2$ is a well defined complex analytic function, and its restriction h to $\mathbb{R}^2$ generates $\mathscr{P}_{i,x}^2$ for all $x \in \mathbb{R}^2$. Now we only have to produce a square root g_i of h as in the Nash case.

Finally we have $f = \phi g_1 g_2$ for some analytic function ϕ, and applying the factorization theorem [Sh3] to $f = (\phi g_i) g_j$ we get two Nash functions h_i, h_j and two positive analytic functions φ_i, φ_j such that

$$\phi g_i = \varphi_i h_i, \qquad g_j = \varphi_j h_j.$$

Thus $\mathrm{X}_j = g_j^{-1}(0) = h_j^{-1}(0)$ is a Nash set. This contradicts the fact that Y is an irreducible Nash set. ■

The paper is organized as follows. In Section 1 we prove Theorem 2. In Section 2 we review what is needed from the theory of the real spectrum and the theory of fans, and show how they work in our geometric setting. The final goal of this section is a reformulation of Theorem 1 adapted to these abstract techniques: the *fan extension theorem*. The proof of this theorem is done in Section 4, after obtaining in Section 3 several preliminary lemmas; in particular, a fan extension lemma for valuation rings and another for henselian excellent rings.

1. Separation and extension

We devote this section to the

Proof of Theorem 2. — Let $\mathrm{M} \subset \mathbb{R}^p$ be a Nash manifold and suppose that separation holds for M. Let φ be a Nash function on a Nash set $\mathrm{X} \subset \mathrm{M}$. Set $\tilde{\mathrm{M}} = \mathrm{M} \times \mathbb{R}$ and denote by $\tilde{\mathrm{X}}$ the graph of φ. First we see that $\tilde{\mathrm{X}} \subset \tilde{\mathrm{M}}$ is a Nash set.

By Cartan's Theorem B, there is an analytic extension $\phi : \mathrm{M} \to \mathbb{R}$ of φ and we get a proper analytic embedding

$$\Phi: \quad \mathrm{M} \to \tilde{\mathrm{M}} : x \mapsto (x, \phi(x)).$$

Let $p : \tilde{\mathrm{M}} \to \mathrm{M}$ denote the canonical projection and $p' : \tilde{\mathrm{M}} \to \mathrm{M}$ a Nash map very close to p in the C^1-Nash topology. For this topology and its properties, *see* [Sh1, II. 1]; as M is compact it is the topology induced by the C^1-Whitney topology. Now since p' is close to p, the composition $\theta = p' \circ \Phi$ is close to $p \circ \Phi = \mathrm{Id}_{\mathrm{M}}$, and consequently it is an analytic diffeomorphism. As $p'(\tilde{\mathrm{X}}) = \theta(\mathrm{X})$ and X is a global analytic set, we conclude that $p'(\tilde{\mathrm{X}})$ is a global analytic set too. Moreover, $p'(\tilde{\mathrm{X}})$ is clearly semialgebraic, and since separation holds for M, $p'(\tilde{\mathrm{X}}) \subset \mathrm{M}$ is a Nash set. This implies that $p'^{-1}(p'(\tilde{\mathrm{X}})) \subset \tilde{\mathrm{M}}$ is a Nash set. Finally, we have

$$\tilde{\mathrm{X}} = \bigcap_{p'} p'^{-1}(p'(\tilde{\mathrm{X}})),$$

where the intersection runs over all the p''s, which shows that $\tilde{\mathrm{X}} \subset \tilde{\mathrm{M}}$ is a Nash set as claimed.

Now let $\mathscr{P}$ and $\tilde{\mathscr{P}}$ denote respectively the sheaves of ideals of $\mathscr{N}_M$ and $\mathscr{N}_{\tilde{M}}$ generated by $I(X)$ and $I(\tilde{X})$. Then for every $x \in M$ the stalk $\mathscr{P}_x$ consists of the Nash function germs whose complexifications vanish on the zero set germ of the complexifications of the germs at x of some fixed generators of $I(X)$. The same statement holds true for $\tilde{\mathscr{P}}(x, y)$, $(x, y) \in \tilde{M}$. This implies that $\tilde{\mathscr{P}}_{(x, y)}$ is generated by $\mathscr{P}_x$ and $y - \Phi_x$, where $\Phi_x \in \mathscr{N}_{M, x}$ is a germ whose class mod $\mathscr{P}_x$ is φ_x.

Now we will extend φ to every member of a finite covering of X by open semialgebraic subsets of M. Let $f_1, \ldots, f_k$ generate $I(\tilde{X})$. We claim that at every $(x, y) \in \tilde{X}$ we can write

$$g_{i, (x, y)} f_{i, (x, y)} = y - \Phi_x \bmod \mathscr{P}_x,$$

for some i and some $g_{i, (x, y)} \in \mathscr{N}_{\tilde{M}, (x, y)}$. Indeed, by the preceding remarks, we have

$$fi, (x, y) = h_{i, (x, y)} (y - \Phi_x) \bmod \mathscr{P}_x, \qquad 1 \leqq i \leqq k,$$

and we want to see that some $h_{i, (x, y)}$ is a unit. But we also have

$$y - \Phi_x = \sum_j \mu_j f_{j, (x, y)},$$

and from this we get a homogeneous system mod $\mathscr{P}_x$

$$0 = (h_{1, (x, y)} \mu_1) f_{1, (x, y)} + \ldots + (h_{i, (x, y)} \mu_i - 1) f_{i, (x, y)} + \ldots + (h_{k, (x, y)} \mu_k) f_{k, (x, y)}, \qquad 1 \leqq i \leqq k,$$

whose determinant has the form

$$(-1)^k + \lambda_1 h_{1, (x, y)} + \ldots + \lambda_k h_{k, (x, y)}.$$

Hence if no $h_{i, (x, y)}$ were a unit, we would conclude

$$f_{i, (x, y)} = 0 \bmod \mathscr{P}_x, \qquad 1 \leqq i \leqq k,$$

and consequently $\tilde{\mathscr{P}}_{(x, y)} = \mathscr{P}_x \mathscr{N}_{\tilde{M}, (x, y)}$. Thus $\tilde{X}$ would have dimension $> \dim(X)$, which is impossible.

Whence the open sets

$$U_i = \{ (x, y) \in \tilde{M} \mid y - \Phi_x \in (f_{i, (x, y)}) + \mathscr{P}_x \}$$

cover $\tilde{X}$. Now, $y - \Phi_x$ is regular with respect to y at every point $(x, y) \in \tilde{X}$, and it follows that f_i is regular with respect to y at every point $(x, y) \in U_i \cap \tilde{X}$. Consequently, shrinking U_i we can assume that f_i is regular with respect to y on U_i. Hence $f_i^{-1}(0) \cap U_i$ is the graph of a Nash function F_i on $V_i = \pi(U_i)$, where $\pi : \tilde{M} \to M$ is the canonical projection. Then $F_i = \varphi$ on $V_i \cap X$.

Next, using a partition of unity we paste the F_i's as follows. Set $V_0 = M \setminus X$. Let $\{ \rho_0, \ldots, \rho_k \}$ be a C^1 Nash ($= C^1$ semialgebraic) partition of unity subordinated to $\{ V_0, \ldots, V_k \}$. Then we denote by F_0 the zero function on V_0, and $\sum_{i=0}^{k} \rho_i F_i$ is a C^1 Nash extension of φ.

Lastly, using the approximation theorem ([Sh2], II.4.1, p. 123) we will smoothen this function. But we need the same trick as in the proof of the extension theorem ([Sh2], II.5.1, pp. 127-129). The graph of $\sum_{i=0}^{k} \rho_i F_i$ is the zero set of the C^1 Nash function $h=\sum_{i=0}^{k} \rho_i (y-F_i)$ on $\tilde{M}$. Note that h is regular with respect to y on $h^{-1}(0)$. Since $y-F_i=\chi_i f_i$ on U_i for some $\chi_i \in \mathcal{N}(U_i)$ $(1 \leqq i \leqq k)$ and $f_0=\sum_{i=1}^{k} f_i^2$ is never zero on $\tilde{M} \setminus \tilde{X}$, we get C^1 Nash functions h_i on $\tilde{M}$ such that $h=\sum_{i=0}^{k} h_i f_i$. Let h_i^* be a Nash approximation of h_i for $0 \leqq i \leqq k$ (for the topology involved in this approximation and its properties *see* [Sh2, II.1]). Then $h^*=\sum_{i=0}^{k} h_i^* f_i$ is a Nash approximation of h, and clearly $h^* \in I(\tilde{X})$. Choose the approximation so that h^* is regular with respect to y. Then $(h^*)^{-1}(0)$ is the graph of a Nash function on M, and that Nash function is what we want. ■

2. Complexities and fan extensions

The contents of this section are more or less known, excpet for some results directly concerning our problem. However we give a quick review with quotations for the convenience of the reader.

(2.1) CONSTRUCTIBLE SETS IN THE REAL SPECTRUM ([BCR, Ch. 4 & 7]). — Let A be a commutative ring with unit. The real spectrum $\mathrm{Spec}_r(A)$ of A is the set of all pairs $\alpha=(\mathfrak{p}_\alpha, >_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $>_\alpha$ is an ordering in the residue field $\kappa(\mathfrak{p}_\alpha)$. The element α is called a *prime cone of* A. It is clear that

$$\mathrm{Spec}_r(A)=\bigcup_{\mathfrak{p}} \mathrm{Spec}_r(\kappa(\mathfrak{p})),$$

where the $\mathfrak{p}$'s run among the prime ideals of A.

We will write $f(\alpha)>0$, $f(\alpha)=0$ to mean $f \bmod \mathfrak{p}_\alpha >_\alpha 0$, $f \bmod \mathfrak{p}_\alpha=0$. In this way we can impose sign conditions on the elements of A and use notations like $\{f_1>0, \ldots, f_s>0\} \subset \mathrm{Spec}_r(A)$ for $\{\alpha \in \mathrm{Spec}_r(A): f_1(\alpha)>0, \ldots, f_s(\alpha)>0\}$. Then we define the *constructible sets of* $\mathrm{Spec}_r(A)$ to be the subsets $C \subset \mathrm{Spec}_r(A)$ of the form

$$C=\bigcup_{i=1}^{p} \{f_{i1}>0, \ldots, f_{ir_i}>0, g_i=0\}.$$

The real spectrum is equiped with the *Harrison topology* generated by the following constructible sets:

$$C=\bigcup_{i=1}^{p} \{f_{i1}>0, \ldots, f_{ir_i}>0\}.$$

We also define the *Zariski topology* by analogy with the Zariski prime spectrum: a subbasis consists of all sets of the form $\{f \neq 0\}$. We distinguish the closure in this topology with an index Z and saying *Zariski closure*. In a somehow mixed way we define the *Zariski boundary* of an <u>open set</u> $C \subset \operatorname{Spec}_r(A)$ to be the Zariski closure of the boundary of C, that is $\partial_Z(C) = \overline{\overline{C} \setminus C}^Z$. It is easy to check that if $C = \{f_1 > 0, \ldots, f_s > 0\}$, then $C \cap \partial_Z(C) = \varnothing$. We will say that C *does not meet its Zariski boundary* instead of writing $C \cap \partial_Z(C) = \varnothing$. With this terminology we can state the main result concerning our problem:

THEOREM 2.1.1 ([Br1, 4.1, p. 76]). — *Let* C *be an open constructible subset of* $\operatorname{Spec}_r(A)$ *which does not meet its Zariski boundary and s an integer. The following assertions are equivalent*:

a) *There are* $f_1, \ldots, f_s \in A$ *such that* $C = \{f_1 > 0, \ldots, f_s > 0\}$.

b) *For every prime ideal* $\mathfrak{p}$ *of* A *there are* $f_1, \ldots, f_s \in A$ *such that*

$$C \cap \operatorname{Spec}_r(\kappa(\mathfrak{p})) = \{f_1 > 0, \ldots, f_s > 0\} \cap \operatorname{Spec}_r(\kappa(\mathfrak{p})).$$

(2.2) FANS AND CONSTRUCTIBLE SETS. — Let K be a field and $\operatorname{Spec}_r(K)$ its real spectrum (usually called its *space of orderings*, because the prime cones of K are exactly the orderings of K). A (*finite*) *fan of* K is a finite set F of orderings of K such that for any three orderings $\alpha_1, \alpha_2, \alpha_3 \in F$, their product $\alpha_4 = \alpha_1 . \alpha_2 . \alpha_3$ is a well-defined ordering and belongs to F (we multiply orderings as signatures). This condition holds trivially if $\#(F) = 1$ or 2, in which case we say that F is *trivial*. A basic fact is that $\#(F)$ is always a power of 2.

The beautiful result that shows the importance of fans is:

THEOREM 2.2.1 ([Br2, 5.4, p. 312]). — *Let* C *be a constructible subset of* $\operatorname{Spec}_r(K)$. *The following assertions are equivalent*:

a) *There are s elements* $f_1, \ldots, f_s \in K$ *such that* $C = \{f_1 > 0, \ldots, f_s > 0\}$.

b) *For every fan* F *of* K *with* $\#(F) = 2^m$ *and* $F \cap C \neq \varnothing$ *we have* $\#(F \cap C) = 2^n$ *with* $0 \leqq m - n \leqq s$.

Putting together the two theorems above we obtain

PROPOSITION 2.3. — *Let* A *be a commutative ring with unit and* C *an open constructible subset of* $\operatorname{Spec}_r(A)$ *which does not meet its Zariski boundary. Let s be an integer. The following assertions are equivalent*:

a) *There are* $f_1, \ldots, f_s \in A$ *such that* $C = \{f_1 > 0, \ldots, f_s > 0\}$.

b) *For every prime ideal* $\mathfrak{p}$ *of* A *and every fan* $F \subset \operatorname{Spec}_r(\kappa(\mathfrak{p}))$ *with* $\#(F) = 2^m$ *and* $F \cap C \neq \varnothing$ *we have* $\#(F \cap C) = 2^n$ *with* $0 \leqq m - n \leqq s$.

(2.4) FANS AND VALUATIONS ([Lm, § 3, 5 & 12]). — Let K be a field and V a valuation ring of K; we will denote by $\mathfrak{m}_V$ the maximal ideal of V and by $k_V = V/\mathfrak{m}_V$ its residue field. We say that an ordering α of K is *compatible with* V if V is convex with respect to α, that is, from $-g < f < g$, $g \in V$, $f \in K$ it follows $f \in V$. In that case, V contains the rationals. If α is compatible with V, then α induces a unique ordering γ in the residue field of V, defined in the obvious way: any element $z \in k$ is the residue class of some unit $u \in V$ whose sign in α is by definition the sign of z in γ. We say that α specializes to γ.

These specializations are ruled by the following theorem:

THEOREM 2.4.1 (Baer-Krull theorem [Lm, 3.10]). — *Let* V *be a valuation ring of the field* K, *and let* Γ *denote the value group of* V. *Fix an ordering* γ *of the residue field* k *of* V. *Then there is a bijection between the set of orderings* α *of* K *compatible with* V *and specializing to* γ *and the set of group homomorphisms* $\phi : \Gamma \to \{+1, -1\}$. *Such a bijection* $\alpha \mapsto \phi$ *can be defined through any fixed ordering* α_0 *specializing to* γ *by*

$$\alpha(f) = \alpha_0(f) . \phi(v(f)),$$

where given $f \in K \setminus \{0\}$, $v(f)$ *stands for its value in* Γ.

Now we say that a fan F of K is *compatible with* V if every $\alpha \in F$ is compatible with V. The main result relating fans and valuations is:

THEOREM 2.4.2 (Bröcker's trivialization theorem [Lm, 5.13]). — *Let* F *be a non-trivial fan of the field* K. *Then* F *is compatible with some non-trivial valuation ring* V *of* K *and the orderings of* F *specialize to at most two distinct orderings in the residue field of* V.

Finally we apply all this abstract stuff in our geometric setting. Let $M \subset \mathbb{R}^p$ be a compact Nash manifold and consider the sheaves $\mathcal{N}_M$, $\mathcal{O}_M$ and rings $\mathcal{N}(M)$, $\mathcal{O}(M)$ as in Section 1. The references for the next two paragraphs are [BCR, Ch. 7 & 8] for the Nash case and [Rz1, 2] for the analytic. For commutative algebra (flatness, completions, excellent rings, etc.) we refer to [Mt].

(2.5) THE NASH AND ANALYTIC TILDA OPERATORS. — Put again $A = \mathcal{N}(M)$ [resp. $\mathcal{O}(M)$]. Then we have the *Nash* (resp. *analytic*) *tilda operator*: $S \mapsto \tilde{S}$, which maps a set $S \subset M$ defined by a system of Nash (resp. analytic) equalities and/or inequalities to the constructible set $\tilde{S} \subset \operatorname{Spec}_r(A)$ given by the same system. The starting fact here is that $S = \varnothing$ if and only if $S = \varnothing$, which is only a sophisticated reformulation of the Artin-Lang homomorphism theorem for global Nash (resp. analytic) functions. From this it follows that the definition above is consistent and gives a bijection that preserves inclusions and Zariski closures. Some further work shows that this bijection preserves closures and interiors, so it preserves Zariski boundaries. Among the consequences of these facts we will use the following form of the real Nullstellensatz: a prime ideal $\mathfrak{p} \subset A$ is real if and only if the Krull dimension of the ring $A/\mathfrak{p}$ coincides with the topological dimension of the zero set $X \subset M$ of $\mathfrak{p}$. Also, the irreducible components of a zero set correspond bijectively to the prime divisors of its zero ideal.

We have defined two different tilda operators. However, we use the same notation for both: the context will always avoid any risk of confusion.

(2.6) THE EXTENSION $\mathcal{N}(M) \subset \mathcal{O}(M)$. — Put $A = \mathcal{N}(M)$ [resp. $\mathcal{O}(M)$]. The ring A is an excellent noetherian ring. Its maximal ideals are exactly the ideals of the points of M: for every $x \in M$ its maximal ideal $\mathfrak{m}_x$ consists of all functions $f \in A$ that vanish at x. Every localization $A_x = A_{\mathfrak{m}_x}$ is a local regular ring whose Krull dimension is the topological dimension of M at x.

Moreover the extension $\mathcal{N}(M) \subset \mathcal{O}(M)$ is faithfully flat and regular, and for every point $x \in M$ we have the following commutative square of regular faithfully flat local homomorphisms

$$\begin{array}{ccc} \mathcal{O}(M)_x & \rightarrow & \mathcal{O}_{M,x} \\ \uparrow & & \uparrow \\ \mathcal{N}(M)_x & \rightarrow & \mathcal{N}_{M,x} \end{array}$$

Furthermore all the four homomorphisms extend to isomorphisms between the respective adic completions. In particular, every prime ideal of height r of any of these rings generates in any other bigger a radical ideal whose prime divisors have all height r.

Now we can prove a result that will be essential later:

PROPOSITION 2.7. – *Every ordering* α *of the residue field* $\kappa(\mathfrak{p})$ *of a prime ideal* $\mathfrak{p} \subset \mathcal{N}(M)$ *extends to an ordering* α' *of the residue field* $\kappa(\mathfrak{q})$ *of some prime divisor* $\mathfrak{q}$ *of the extension* $\mathfrak{p}\,\mathcal{O}(M)$.

Proof. – We first claim that there is a point $x \in M$ with the property that every function positive at x is positive in α. For, consider all the finite intersections of the form

$$S = \bigcap_{f_1(\alpha) \geqq 0, \ldots, f_s(\alpha) \geqq 0} \{x \in M : f_1(x) \geqq 0, \ldots, f_s(x) \geqq 0\}.$$

Clearly $\alpha \in \tilde{S}$, and $\tilde{S} \neq \varnothing$. Since the tilda operator is a bijection, we deduce $S \neq \varnothing$. Then by compactness, $\bigcap S \neq \varnothing$, and any point $x \in \bigcap S$ verifies the statement (actually there is only one).

We will write $\alpha \rightarrow x$. Then we are in the hypotheses needed to apply the going-down theorem in [Rz3], and we obtain a prime ideal $\mathfrak{q} \subset \mathcal{O}(M)$ lying over $\mathfrak{p}$ and an ordering α' in $\kappa(\mathfrak{q})$ that restricts to α in $\kappa(\mathfrak{p})$. Furthermore $\mathfrak{q}$ can be chosen with $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p})$, which means that $\mathfrak{q}$ is a prime divisor of $\mathfrak{p}\,\mathcal{O}(M)$ as required. ∎

The preceding proposition shows that the extension to $\mathcal{O}(M)$ of a *real* prime ideal of $\mathcal{N}(M)$ has some *real* prime divisor. It follows easily from the real Nullstellensatz that in case separation holds, that extension has a *unique* real prime divisor. As a matter of fact we could reformulate separation as follows: *Given a real prime ideal* $\mathfrak{p} \subset \mathcal{N}(M)$, *is the real-radical of* $\mathfrak{p}\,\mathcal{O}(M)$ *a prime ideal*? Now a stronger separation question is whether the ideal $\mathfrak{p}\,\mathcal{O}(M)$ is prime for every prime $\mathfrak{p}$ of $\mathcal{N}(M)$. Concerning extension the difference is similar: we only deal with ideals of the form $I(X)$, *i.e.* real ideals, instead of arbitrary ideals I of $\mathcal{N}(M)$.

After this preparation we are ready to reformulate Theorem 1 in a much more abstract way, but which will be also much more tractable:

THEOREM 2.8 (Fan extension theorem). – *Suppose that separation holds for* M. *Then every fan* F *of the residue field of a prime ideal* $\mathfrak{p}$ *of* $\mathcal{N}(M)$ *extends to a fan* F′ *of the residue field of the unique real prime divisor* $\mathfrak{q}$ *of* $\mathfrak{p}\,\mathcal{O}(M)$, *with* $\#(F) = \#(F')$.

Our claim is that *Theorem* 1 *follows from this fan extension theorem*. Indeed, suppose we are given a semialgebraic set $S \subset M$ which can be described by s analytic inequalities, but cannot by s Nash inequalities. Then using the tilda operators, $\tilde{S} \subset \operatorname{Spec}_r(\mathcal{O}(M))$ can be described with s inequalities, but $\tilde{S} \subset \operatorname{Spec}_r(\mathcal{N}(M))$ cannot. By Proposition 2.3 there are two possibilities:

(i) The constructible set $\tilde{S}$ meets its Zariski boundary in $\operatorname{Spec}_r(\mathcal{N}(M))$.

(ii) There is a prime ideal $\mathfrak{p} \subset \mathcal{N}(M)$ and a fan F of its residue field $\kappa(\mathfrak{p})$ such that $\#(F \cap \tilde{S})$ gives a numerical obstruction.

In the first case, note that the Zariski boundary of $\tilde{S}$ in $\operatorname{Spec}_r(\mathcal{N}(M))$ corresponds by the *Nash* tilde operator to the smallest Nash set T that contains the semialgebraic set $\bar{S} \setminus S$. Consequently $S \cap T \neq \emptyset$. Now by the separation assumption T is also the smallest analytic set that contains $\bar{S} \setminus S$. Indeed, the latter is a union of irreducible analytic components of T, and those irreducible components are semialgebraic. Hence by the equivalent formulation of separation given in the introduction, those irreducible components are Nash sets and their union is a Nash set too. From this we see that T corresponds by the *analytic* tilda operator to the Zariski boundary of $\tilde{S}$ in $\operatorname{Spec}_r(\mathcal{O}(M))$. Since $S \cap T \neq \emptyset$, we conclude that $\tilde{S}$ meets its Zariski boundary in $\operatorname{Spec}_r(\mathcal{O}(M))$. This contradicts the fact that $\tilde{S}$ can be described with s inequalities in $\operatorname{Spec}_r(\mathcal{O}(M))$.

Now suppose (ii). Then by Theorem 2.8 there is a fan F′ of the residue field $\kappa(\mathfrak{q})$ of a prime ideal $\mathfrak{q} \subset \mathcal{O}(M)$ lying over $\mathfrak{p}$, whose restriction to $\kappa(\mathfrak{p})$ is F and $\#(F') = \#(F)$. Clearly, $\#(F' \cap \tilde{S}) = \#(F \cap \tilde{S})$, and we obtain the same numerical obstruction, this time in $\mathcal{O}(M)$. Whence $\tilde{S} \subset \operatorname{Spec}_r(\mathcal{O}(M))$ cannot be described with s inequalities (Proposition 2.3), and applying the analytic tilda operator once more, $S \subset M$ cannot be described with s analytic inequalities, a contradiction.

(2.9) Stability indices. — The method used in the last proof is an example of the systematic approach developed in [AnBrRz2]. It leads to the exact computation of the *stability indices* of the rings $\operatorname{Spec}_r(\mathcal{N}(M))$ and $\operatorname{Spec}_r(\mathcal{O}(M))$. Indeed, by [AnBrRz1, 10.2] any fan in a residue field of $\mathcal{N}(M)$ [resp. $\mathcal{O}(M)$] has $\leqq 2^d$ elements, where d stands for the dimension of M. This, together with Proposition 2.3 and the tilda operator, shows that a semialgebraic set that can be described with s Nash (resp. analytic) inequalities can be described always with no more that d. Note however that this gives no information concerning a fixed semialgebraic set, which is the matter in the equal complexities problem.

3. Preliminaries for fan extensions

In order to prove Theorem 2.7 we will use several results concerning extensions of orderings and fans in various situations. We devote this section to such preliminary lemmas.

First we consider a valuation theory situation:

Proposition 3.1. — *Let* K *be a field,* $V \subset K$ *a valuation ring and* F *a fan of* K *compatible with* V. *Let* $W \subset L$ *be an extension of* $V \subset K$ *which has the same value* *group* Γ *as* V. *Suppose that the orderings of* F *specialize to two orderings* σ_1, σ_2 *of the residue field* k_V *of* V *and these extend to two orderings* τ_1, τ_2 *of the residue field* k_W *of* W. *Then* F *extends to a fan* F′ *of* L *compatible with* W *whose orderings specialize to* τ_1, τ_2 *and such that* $\#(F') = \#(F)$.

Proof. — Let F_i denote the set of orderings of F that specialize to σ_i, and G_i the set of orderings of K compatible with V that specialize to σ_i. Then let G'_i be the set of orderings of L compatible with W that specialize to τ_i. Finally choose $\alpha'_i \in G'_i$ and denote by α_i its restriction to K. By the Baer-Krull theorem (Theorem 2.4.1) we have three bijections

$$\mathrm{Hom}(\Gamma, \{+1, -1\}) \begin{smallmatrix} \nearrow & G'_i \\ & \downarrow \\ \searrow & G_i \end{smallmatrix} \qquad \phi \begin{smallmatrix} \nearrow & \alpha'_i . \phi \\ & \downarrow \\ \searrow & \alpha_i . \phi \end{smallmatrix}$$

the vertical arrow being restriction from L to K. Now for every $\gamma \in F_i$ there exists a unique homomorphism ϕ_γ such that $\gamma = \alpha_i . \phi_\gamma$, and $\gamma' = \alpha'_i . \phi_\gamma \in G'_i$. Then put $F'_i = \{\gamma' : \gamma \in F_i\}$. We claim that $F' = F'_1 \cup F'_2$ is the fan we sought. Indeed, it is clear that F′ is a set of orderings that extend F and $\#(F') = \#(F)$. Hence we only must show that F′ is a fan. But from the general theory of fans we know that $G'_1 \cup G'_2 \supset F'$ is a fan. Consequently, if $\gamma'_1, \gamma'_2, \gamma'_3 \in F'$ their product is a well defined ordering $\gamma' = \gamma'_1 . \gamma'_2 . \gamma'_3 \in G'_i$ for certain i. Then the restriction of γ' to K is an ordering $\gamma \in G_i$. Using the bijections described above we get $\gamma' = \alpha'_i . \phi_\gamma$. On the other hand, $\gamma = \gamma_1 . \gamma_2 . \gamma_3 \in F_i$, because F is a fan, and we conclude that $\gamma' \in F'_i \subset F'$ and we are done. ■

Before proceeding further we need some terminology. Let $\phi : A \to B$ be a local homomorphism of (local) noetherian rings, that is, $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$, where $\mathfrak{m}_A$, $\mathfrak{m}_B$ denote respectively the maximal ideals of A, B. We will say that ϕ *has the approximation property* if given a system of polynomials equations $f_i(x_j) = 0$, $1 \leqq i \leqq p$, $1 \leqq j \leqq q$, with coefficients in A, any solution $x_j = b_j \in B$ can be arbitrarily approximated in the adic topology by solutions $x_j = a_j \in A$. Now the main result is:

THEOREM 3.2. — *Let* $\phi : A \to B$ *be a local homomorphism which has the aproximation property. Suppose that* A *is a domain and let* K *be its quotient field. Then* B *is also a domain and every fan* F *of* K *extends to a fan* F′ *of the quotient field* L *of* B, *with* $\#(F') = \#(F)$.

Proof. — The argument to prove that B is a domain is well-known (*see* for instance [Tg, III.4.4, p. 62]). Thus we turn to the assertion concerning fans. Let $F = \{\alpha_i, 1 \leqq i \leqq 2^m\}$ and set P_i for the positive cone of α_i in A, that is, P_i is the set of elements of A which are positive in α_i. We seek 2^m orderings β_i of L such that $\beta_i | K = \alpha_i$ [or equivalently $g(\beta_i) > 0$ for $g \in P_i$] and $\beta_i . \beta_j . \beta_k = \beta_l$ whenever $\alpha_i . \alpha_j . \alpha_k = \alpha_l$. To that end, we consider the product space Σ of 2^m copies of $\mathrm{Spec}_r(L)$, and the set $E = E_1 \cap E_2 \subset \Sigma$ where

$$E_1 = \{(\beta_i, 1 \leqq i \leqq 2^m) \in \Sigma : g(\beta_i) > 0 \text{ for } g \in P_i \text{ and } 1 \leqq i \leqq 2^m\}$$

and

$$E_2=\{(\beta_i,\ 1\leqq i\leqq 2^m)\in\Sigma: f(\beta_i)f(\beta_j)f(\beta_k)f(\beta_l)>0 \text{ for } f\in L \text{ and } \alpha_i.\alpha_j.\alpha_k=\alpha_l\}.$$

We have to check that $E\neq\varnothing$. Rewriting, we have $E_1=\bigcap_{\mathbf{g}} E_{1\,\mathbf{g}}$ where the intersection runs over all tuples $\mathbf{g}=(g_i,\ 1\leqq i\leqq 2^m)\in P_1\times\ldots\times P_{2^m}$, with

$$E_{1\,\mathbf{g}}=\{\beta_1\in \mathrm{Spec}_r(L): g_1(\beta_1)>0\}\times\ldots\times\{\beta_{2^m}\in \mathrm{Spec}_r(L): g_{2^m}(\beta_{2^m})>0\}$$

Similarly $E_2=\bigcap_f E_{2\,f}$, where the intersection runs over all $f\in L$ and

$$E_{2\,f}=\bigcup_{\varepsilon}\{\beta_1\in \mathrm{Spec}_r(L): \varepsilon_1 f(\beta_1)>0\}\times\ldots\times\{\beta_{2^m}\in \mathrm{Spec}_r(L): \varepsilon_{2^m} f(\beta_{2^m})>0\}$$

with $\varepsilon=(\varepsilon_i,\ 1\leqq i\leqq 2^m)$ verifying $\varepsilon_i=\pm 1$ and $\varepsilon_i\varepsilon_j\varepsilon_k=\varepsilon_l$ whenever $\alpha_i.\alpha_j.\alpha_k=\alpha_l$. We will denote by $\mathscr{E}$ the set of all these tuples ε. Now suppose $E=\varnothing$. Then, since Σ is compact and all sets $E_{1\,\mathbf{g}}$, $E_{2\,f}$ are closed, there exist $\mathbf{g}^1,\ \ldots,\ \mathbf{g}^p, f_1,\ \ldots, f_q$ such that

$$E_{1\,\mathbf{g}^1}\cap\ldots\cap E_{1\,\mathbf{g}^p}\cap E_{2\,f_1}\cap\ldots\cap E_{2\,f_q}=\varnothing.$$

Rewriting this we get

$$\bigcup_{\varepsilon\in\mathscr{E}^q} E_\varepsilon(1)\times\ldots\times E_\varepsilon(2^m)=\varnothing,$$

where $\varepsilon=(\varepsilon_i^j;\ 1\leqq i\leqq 2^m,\ 1\leqq j\leqq q)$ and

$$E_\varepsilon(i)=\{g_i^1>0,\ \ldots,\ g_i^p>0,\ \varepsilon_i^1 f_1>0,\ \ldots,\ \varepsilon_i^q f_q>0\}.$$

Hence, for every $\varepsilon\in\mathscr{E}^q$ there is some i such that $E_\varepsilon(i)=\varnothing$. By the abstract Positivstellensatz, ([BCR, 4.4.1, p. 81]), this equality is equivalent to the fact that the equation

$$\sum_{\nu,\,\mu} y_{i\,\nu\mu}^2 (g_i^1)^{\nu_1}\ldots(g_i^p)^{\nu_p}(\varepsilon_i^1 f_1)^{\mu_1}\ldots(\varepsilon_i^q f_q)^{\mu_q}=-(g_i^1)^{2r_1}\ldots(g_i^p)^{2r_p}(\varepsilon_i^1 f_1)^{2s_1}\ldots(\varepsilon_i^q f_q)^{2s_q},$$

where $\nu_k,\ \mu_l=0,\ 1$, has a solution, say

$$y_{i\,\nu\mu}=t_{i\,\nu\mu}\in B.$$

Collecting these equations for all $\varepsilon\in\mathscr{E}^q$ and replacing the f_j's by indeterminates x_j we get a system

$$\sum_{\nu,\,\mu} y_{i\,\nu\mu}^2 (g_i^1)^{\nu_1}\ldots(g_i^p)^{\nu_p}(\varepsilon_i^1 x_1)^{\mu_1}\ldots(\varepsilon_i^q x_q)^{\mu_q}$$
$$=-(g_i^1)^{2r_1}\ldots(g_i^p)^{2r_p}(\varepsilon_i^1 x_1)^{2s_1}\ldots(\varepsilon_i^q x_q)^{2s_q},\qquad \varepsilon\in\mathscr{E}^q,\quad 1\leqq i\leqq 2^m,$$

which has in B the solution

$$y_{i\,\nu\mu}=t_{i\,\nu\mu},\qquad x_j=f_j.$$

Since the homomorphism ϕ has the approximation property, this solution can be approximated by other in A, say

$$y_{i\,\nu\mu}=z_{i\,\nu\mu}, \qquad x_j=h_j.$$

Going backwards in the Positivstellensatz that means that there are $h_1, \ldots, h_q \in \mathrm{A}$ such that for every $\varepsilon \in \mathscr{E}^q$ the set

$$\mathrm{D}_\varepsilon(i)=\{g_i^1>0, \ldots, g_i^p>0, \varepsilon_i^1 h_1>0, \ldots, \varepsilon_i^q h_q>0\}$$

is empty for some i. Consequently, the product $\mathrm{D}_\varepsilon(1)\times \ldots \times \mathrm{D}_\varepsilon(2^m)$ is empty too. Hence

$$\bigcup_{\varepsilon \in \mathscr{E}^q} \mathrm{D}_\varepsilon(1)\times \ldots \times \mathrm{D}_\varepsilon(2^m)=\varnothing,$$

which is a contradiction, since by construction $(\alpha_i, 1 \leqq i \leqq 2^m)$ belongs to that set. ■

The main example of a homomorphism with the approximation property is provided by the following deep result:

THEOREM 3.3 [Rt, 4.2]. — *Let* A *be a henselian local excellent ring containing the rationals, and* $\mathrm{B}=\hat{\mathrm{A}}$ *its adic completion. Then the canonical homomorphism* $\phi:\mathrm{A}\to\mathrm{B}$ *has the approximation property.*

The first cases to which these results apply are the rings $\mathscr{N}_{\mathrm{M},x}$ and $\mathscr{O}_{\mathrm{M},x}$ of Nash and analytic germs at a point x of a Nash manifold M. As a matter of fact this was the concern of the important paper [Ar]. Furthermore, since the canonical homomorphism $\mathscr{N}_{\mathrm{M},x}\to\mathscr{O}_{\mathrm{M},x}$ induces an isomorphism between the completions, it has the approximation property. We will use this later.

Coming back to our general lemmas we recall ([vdD, II.2.5, p. 75], [Pr, 0.5, p. 131]):

PROPOSITION 3.4 (Amalgamation). — *Let* $\kappa \subset \kappa'$ *be two fields, the smaller algebraically closed in the bigger, and* $k \supset \kappa$ *a third field. Suppose we are given orderings* γ' *and* σ *of* κ' *and* κ *respectively that restrict to the same ordering* γ *of* κ. *Then the ring* $\kappa' \otimes_\kappa k$ *is a domain, and there exists an ordering* τ *of its quotient field* κ^* *that extends both* γ' *and* σ.

The preceding result will be useful in our setting in view of the following fact:

PROPOSITION 3.5. — *Let* $\mathrm{M}\subset\mathbb{R}^p$ *be a compact Nash manifold,* $\mathscr{N}(\mathrm{M})$ *and* $\mathscr{O}(\mathrm{M})$ *its rings of global Nash and global analytic functions. Let* $\mathfrak{p}\subset\mathscr{N}(\mathrm{M})$ *be a* real *prime ideal and* $\mathfrak{q}$ *a prime divisor of* $\mathfrak{p}\mathscr{O}(\mathrm{M})$. *If extension holds, the field* $\kappa(\mathfrak{p})$ *is algebraically closed in* $\kappa(\mathfrak{q})$.

Proof. — We have $\mathrm{A}=\mathscr{N}(\mathrm{M})/\mathfrak{p}\subset\mathscr{O}(\mathrm{M})/\mathfrak{q}$, and correspondingly the field extension $\mathrm{K}=\kappa(\mathfrak{p})\subset\mathrm{L}=\kappa(\mathfrak{q})$. To prove the assertion in the statement, pick an element $h\in\mathrm{L}$ which is algebraic over K: there are $a_0, \ldots, a_m\in\mathrm{A}$ with $a_0 h^m+\ldots+a_m=0$ in K. Then we must see that $h\in\mathrm{K}$.

First, we can suppose that the polynomial $\mathrm{P}(t)=a_0 t^m+\ldots+a_m\in\mathrm{K}[t]$ is irreducible. Now we recall that the total ring of fractions Φ of $\mathrm{B}=\mathscr{O}(\mathrm{M})/\mathfrak{p}\mathscr{O}(\mathrm{M})$ is canonically isomorphic to the product of the residue fields of the prime divisors of

$\mathfrak{p}\,\mathcal{O}(M)$, and so $\Phi \equiv L \times \Phi'$ where Φ' is a product of fields. Thus we can pick $f \in \Phi$ with $f \equiv (h, 0)$, so that f is a root of the polynomial $Q(t) = t\,P(t)$ and for h to be in K it is enough that f is in K. Clearly the discriminant of $Q(t)$, $\delta \in \mathbb{Z}[a_0, \ldots, a_m] \subset A$ is not zero. Second, there is an element $\eta \in A$, $\eta \neq 0$, such that the zero set $X \subset M$ of $\mathfrak{p}$ is a Nash manifold off $\eta = 0$. Indeed, take any generators of $\mathfrak{p}$ and a regular point x of maximal dimension of $X \subset M \subset \mathbb{R}^p$, so that some jacobian of those generators has maximal corank at x (this is possible because $\mathfrak{p}$ being real, $I(X) = \mathfrak{p}$). That jacobian is actually a global Nash function, and does the job. Then we claim that $(\delta\eta)^m f \in B$ for some m.

To prove our claim consider a complexification $X^{\mathbb{C}} \subset M^{\mathbb{C}}$ of the couple $X \subset M$. We can assume that f extends to a meromorphic function $f^{\mathbb{C}}$ of $X^{\mathbb{C}}$, and $a_0, \ldots, a_m, \delta, \eta$ to analytic functions $a_0^{\mathbb{C}}, \ldots, a_m^{\mathbb{C}}, \delta^{\mathbb{C}}, \eta^{\mathbb{C}}$ with $f^{\mathbb{C}}(a_0^{\mathbb{C}}(f^{\mathbb{C}})^m + \ldots + a_m^{\mathbb{C}}) = 0$. Furthermore by the construction of η we also can suppose that $X^{\mathbb{C}}$ is a complex manifold off $\eta^{\mathbb{C}} = 0$. Then at every point $y \in X^{\mathbb{C}} \setminus \{\delta^{\mathbb{C}}\eta^{\mathbb{C}} = 0\}$, the meromorphic function $f^{\mathbb{C}}$ gives a simple root of the polynomial equation $x(a_0^{\mathbb{C}} x^m + \ldots + a_m^{\mathbb{C}}) = 0$ in the complex manifold $X^{\mathbb{C}} \setminus \{\eta^{\mathbb{C}} = 0\}$. Hence $f^{\mathbb{C}}$ is analytic at y. Thus in the diagram

$$\begin{array}{ccc} \mathcal{O}(X^{\mathbb{C}})_y & \rightarrow & \mathcal{O}_{X^{\mathbb{C}}, y} \\ \downarrow & & \downarrow \\ \mathcal{M}(X^{\mathbb{C}})_y & \rightarrow & \mathcal{M}_{X^{\mathbb{C}}, y} \end{array}$$

(where $\mathcal{M}$ means *meromorphic*, or equivalently *total ring of fractions*) the element $f^{\mathbb{C}}$ is in the right upper corner and in the left lower one. By faithful flatness of the first horizontal arrow we conclude that $f^{\mathbb{C}}$ is actually in the left upper corner. In other words, for every $y \in X^{c} \setminus \{\delta^{\mathbb{C}}\eta^{\mathbb{C}} = 0\}$ there is a global representation of the meromorphic function $f^{\mathbb{C}}$ whose denominator does not vanish at y.

Now for every $x \in X$, the ring of germs at x of holomorphic functions of $X^{\mathbb{C}}$ is $\mathcal{O}_{X^{\mathbb{C}}, x} = \mathcal{O}_{M^{\mathbb{C}}, x}/\mathfrak{p}\,\mathcal{O}_{M^{\mathbb{C}}, x}$. Indeed, since $\mathfrak{p}$ is radical, it generates a radical ideal in $\mathcal{O}_{M, x}$, and consequently also in $\mathcal{O}_{M^{\mathbb{C}}, x} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_{M, x}$, so that by Rückert's Nullstellensatz, any holomorphic germ vanishing on $X_x^{\mathbb{C}} =$ zero set germ of the ideal $\mathfrak{p}\,\mathcal{O}_{M^{\mathbb{C}}, x}$ belongs to $\mathfrak{p}\,\mathcal{O}_{M^{\mathbb{C}}, x}$.

Now the germ $f_x^{\mathbb{C}}$ is an element of the total ring of fractions of the ring $\mathcal{O}_{X^{\mathbb{C}}, x}$ and we consider the ideal of denominators of $f_x^{\mathbb{C}}$, that is, the ideal $I_x^{\mathbb{C}}$ of all germs $g_x \in \mathcal{O}_{X^{\mathbb{C}}, x}$ such that $g_x f_x^{\mathbb{C}} \in \mathcal{O}_{X^{\mathbb{C}}, x}$. The zero germ $Y_x^{\mathbb{C}} \subset X_x^{\mathbb{C}}$ of $I_x^{\mathbb{C}}$ is the pole germ of $f^{\mathbb{C}}$. We will see next that the germ $\delta_x^{\mathbb{C}}\eta_x^{\mathbb{C}}$ vanishes on $Y_x^{\mathbb{C}}$. Indeed, otherwise in any representative $Y^{\mathbb{C}}$ of the pole germ there would be points y arbitrarily close to x with $\delta^{\mathbb{C}}(y)\,\eta^{\mathbb{C}}(y) \neq 0$. But we saw before that for such a point y there is a global representation of the meromorphic function f whose denominator does not vanish at $y \in Y^{\mathbb{C}}$. This is impossible, by the definition of the pole germ, and we conclude that $\delta_x^{\mathbb{C}}\eta_x^{\mathbb{C}}$ vanishes on $Y_x^{\mathbb{C}}$. Then by Rückert's Nullstellensatz again, $(\delta_x^{\mathbb{C}}\eta_x^{\mathbb{C}})^{m(x)} \in I_x^{\mathbb{C}}$ for some $m(x)$. Hence $(\delta_x^{\mathbb{C}}\eta_x^{\mathbb{C}})^{m(x)} f_x^{\mathbb{C}} \in \mathcal{O}_{X^{\mathbb{C}}, x}$ and since everything is here a complexification, we get $(\delta_x\eta_x)^{m(x)} f_x \in \mathcal{O}_{M, x}/\mathfrak{p}\,\mathcal{O}_{M, x}$. This implies $(\delta\eta)^{m(x)} f \in B_x$, where B_x stands for the localization of B at the maximal ideal of the point $x \in X$. Thus $(\delta\eta)^{m(x)} f = g/h$, where $g, h \in B$, and $h(x) \neq 0$. Since X is compact, we can pick finitely many fractions g_i/h_i, $1 \leqq i \leqq r$, such that $X=\{h_1\neq 0\}\cup\ldots\cup\{h_r\neq 0\}$. Then putting $m=\max_i(m_i)$ we obtain

$$(\delta\eta)^m f=\frac{(\delta\eta)^{m-m_1}g_1h_1+\ldots+(\delta\eta)^{m-m_r}g_rh_r}{h_1^2+\ldots+h_r^2}.$$

Since the denominator never vanishes on X, a standard application of Cartan's Theorem A for real analytic manifolds shows that the left hand side of the equation above is an element of B, and we have proved our claim.

All this shows that we can assume $f\in B$ without loss of generality, and we will see that $f_x\in\mathcal{N}_{M,x}/\mathfrak{p}\,N_{M,x}$ for every $x\in M$. To that end, we consider the polynomial $Q(t)\in\mathcal{N}_{M,x}[t]$. Since the homomorphism $\mathcal{N}_{M,x}\to\mathcal{O}_{M,x}$ has the approximation property, for every ν we find $f^{(\nu)}\in\mathcal{N}_{M,x}$ such that $Q(f^{(\nu)})\in\mathfrak{p}\,\mathcal{N}_{M,x}$ and the jets of order ν of f and $f^{(\nu)}$ coincide. Then let $\mathfrak{p}'$ be a prime divisor of $\mathfrak{p}\,\mathcal{N}_{M,x}$. Since $\mathfrak{p}'$ lies over $\mathfrak{p}$ and $Q(t)$ is not zero mod $\mathfrak{p}$, we conclude it is not zero mod $\mathfrak{p}'$ either. Consequently, the $f^{(\nu)}$'s are roots of the same non-zero polynomial in the field $\kappa(\mathfrak{p}')$, and hence there is ν_0 with $f^{(\nu)}=f^{(\nu_0)}$ mod $\mathfrak{p}'$ for $\nu\geqq\nu_0$. As there are finitely many $\mathfrak{p}'$'s, we find ν_0 large enough such that $f^{(\nu)}-f^{(\nu_0)}\in\bigcap\mathfrak{p}'=\mathfrak{p}\,\mathcal{N}_{M,x}$ for $\nu\geqq\nu_0$. This implies $f-f^{(\nu_0)}\in\bigcap_{\nu\geqq\nu_0}(\mathfrak{m}_x^\nu+\mathfrak{p}\,\mathcal{N}_{M,x})$, and consequently $f-f^{(\nu_0)}\in\mathfrak{p}\,\mathcal{N}_{M,x}$. In other words, $f\in\mathcal{N}_{M,x}/\mathfrak{p}\,\mathcal{N}_{M,x}$, as wanted.

So we have proved that $\mathbf{f}=\{f_x\}_{x\in M}$ defines a global section of the sheaf $\mathcal{N}_M/\mathfrak{p}\,\mathcal{N}_M$. But we are assuming that extension holds, and consequently there is a Nash function $f'\in\mathcal{N}(M)$ such that $f'_x=f_x$ mod $\mathfrak{p}\,\mathcal{O}_{M,x}$ for every $x\in M$. This shows that $f\in A$, and finishes the proof. ■

Since we have already proved that separation implies extension, from the preceding proposition we get:

COROLLARY 3.6. — *Let* $M\subset\mathbb{R}^p$ *be a compact Nash manifold and suppose that separation holds for* M. *Let* $\mathfrak{p}\subset\mathcal{N}(M)$ *be a* real *prime ideal and* $\mathfrak{q}$ *the unique real prime divisor of* $\mathfrak{p}\,\mathcal{O}(M)$. *Then the field* $\kappa(\mathfrak{p})$ *is algebraically closed in* $\kappa(\mathfrak{q})$.

4. Proof of the fan extension theorem

Let $M\subset\mathbb{R}^p$ be a compact Nash manifold and assume that separation holds. Let $\mathcal{N}(M)$ and $\mathcal{O}(M)$ be the rings of global Nash and global analytic functions on M. Finally consider a prime ideal $\mathfrak{p}_0\subset\mathcal{N}(M)$ and a fan F in the residue field $K=\kappa(\mathfrak{p}_0)$. By separation, the ideal $\mathfrak{p}_0\,\mathcal{O}(M)$ has a unique real prime divisor $\mathfrak{q}_0\subset\mathcal{O}(M)$. Then we seek a fan F′ in the residue field $\kappa(\mathfrak{q}_0)\supset\kappa(\mathfrak{p}_0)=K$ whose restriction to K is F. We separate the argument in nine steps.

STEP I. — By Theorem 2.4.2 the fan F is compatible with a non-trivial valuation V of K and specializes to at most two distinct orderings α_1, α_2 of the residue field k_V of V. Put $A=\mathcal{N}(M)/\mathfrak{p}_0$. We claim that V contains the ring A.

Indeed, let $f \in \mathcal{N}(M)$ and denote by $\bar{f}$ the residue class of f in A. To see that $\bar{f} \in V$, pick any ordering of K, $\gamma_0 \in F$. Then γ_0 is compatible with V, which means that if we find an integer m such that $-m <_{\gamma_0} \bar{f} <_{\gamma_0} m$, then $\bar{f} \in V$. But M is compact, so that there is an integer m such that $-m < f(x) < m$ for all $x \in M$. Using the tilda operator we deduce $-m < f(\gamma) < m$ for all $\gamma \in \mathrm{Spec}_r(\mathcal{N}(M))$. In particular, for $\gamma = \gamma_0$ we get $-m <_{\gamma_0} \bar{f} <_{\gamma_0} m$, and we are done. ■

STEP II. — Let $\mathfrak{p}$ be the center of V in A: $\mathfrak{p} = \mathfrak{m}_V \cap A$, where $\mathfrak{m}_V$ is the maximal ideal of V. Then V dominates the localization $A_{\mathfrak{p}}$. The ideal $\mathfrak{p}$ is the residue class mod $\mathfrak{p}_0$ of some prime ideal of $\mathcal{N}(M)$, which we still denote by $\mathfrak{p}$, and we get a local homomorphism $\rho : \mathcal{N}(M)_{\mathfrak{p}} \to V$. Now notice that $\mathfrak{p}$ is real and since separation holds, the extension $\mathfrak{p}\mathcal{O}(M)$ has a unique real prime divisor, which we denote by $\mathfrak{q}$. Now we produce the following diagram of local homomorphisms

$$\begin{array}{ccccccccc}
\mathcal{O}(M) & \subset & \mathcal{O}(M)_{\mathfrak{q}} & \to & \mathcal{O}(M)^h_{\mathfrak{q}} & \longrightarrow & \mathcal{O}(M)_{\mathfrak{q}}^{\wedge} \equiv \kappa(\mathfrak{q})[[x]] & \equiv & \hat{D} \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathcal{N}(M) & \subset & \mathcal{N}(M)_{\mathfrak{p}} & \to & \mathcal{N}(M)^h_{\mathfrak{p}} & \to & (\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(M)^h_{\mathfrak{p}})_{\mathfrak{n}} = D & \to & D^h \\
& & \downarrow \rho & & & & & & \\
& & (V, F) & & & & & &
\end{array}$$

where:

(i) The index ^{h} denotes henselization, and the index $\hat{}$ adic completion;

(ii) the integer r is the common height of the ideals $\mathfrak{p}$ and $\mathfrak{q}$, or equivalently the common Krull dimension of the regular rings $\mathcal{N}(M)_{\mathfrak{p}}$, $\mathcal{O}(M)_{\mathfrak{q}}$;

(iii) $x = (x_1, \ldots, x_r)$, the x_i's being a regular system of parameters of $\mathcal{N}(M)_{\mathfrak{p}}$ and since $\mathfrak{p}$ generates $\mathfrak{q}\mathcal{O}(M)_{\mathfrak{q}}$ also one of $\mathcal{O}(M)_{\mathfrak{q}}$;

(iv) $\mathfrak{n}$ is the ideal generated by $x_1, \ldots, x_r$ in $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(M)^h_{\mathfrak{p}}$, $D = (\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(M)^h_{\mathfrak{p}})_{\mathfrak{n}}$ is a local regular ring of dimension r and the x_i's are a system of regular parameters of D;

(v) the arrows denoted by $\equiv$ are the canonical isomorphisms obtained by considering the x_i's as indeterminates over $\kappa(\mathfrak{q})$.

Of all of this, only the assertions (iv) concerning the ring D require an explanation. The key fact is that the canonical homomorphism $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(M)^h_{\mathfrak{p}} \to \kappa(\mathfrak{q})[[x]]$ is flat. To see that, after factorizing through $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]]$, it suffices to show that $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] \to \kappa(\mathfrak{q})[[x]]$ is flat. Then, by the characterization of flatness in terms of linear equations ([Mt, Th. 1, p. 17-18]), it is enough to see that $\kappa \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] \to \kappa(\mathfrak{q})[[x]]$ is flat for every finitely generated subextension κ of $\kappa(\mathfrak{q}) \supset \kappa(\mathfrak{p})$. Now, factorizing through $\kappa[[x]]$ and since $\kappa[[x]] \to \kappa(\mathfrak{q})[[x]]$ is flat [Bk, Exercise III.3.17, p. 250], we are reduced to show that $\kappa \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] \to \kappa[[x]]$ is flat. But we can write $\kappa = \kappa(\mathfrak{p})(y)[\theta]$, where $y = (y_1, \ldots, y_s)$ are indeterminates and θ is algebraic over $\kappa(\mathfrak{p})(y)$. Thus our homomorphism comes by the base change $\kappa \otimes_{\kappa(\mathfrak{p})(y)} -$ from $\kappa(\mathfrak{p})(y) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] \to \kappa(\mathfrak{p})(y)[[x]]$. Hence we will see that the latter homomorphism is flat. To that end it suffices to check that

$$\kappa(\mathfrak{p})[y] \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] \to \kappa(\mathfrak{p})(y)[[x]]$$

is flat. Finally $\kappa(\mathfrak{p})[y] \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})[[x]] = \kappa(\mathfrak{p})[[x]][y]$ and the homomorphism under consideration factorizes in the form

$$\kappa(\mathfrak{p})[[x]][y] \to \kappa(\mathfrak{p})[[x]][y]_{(x)} \to \kappa(\mathfrak{p})(y)[[x]],$$

where the two arrows are flat: localization and completion.

One we know that $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(\mathrm{M})_{\mathfrak{p}}^h \to \kappa(\mathfrak{q})[[x]]$ is flat, let $\mathfrak{n}$ be the contraction of the maximal ideal of $\kappa(\mathfrak{q})[[x]]$. Then the homomorphism

$$(\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(\mathrm{M})_{\mathfrak{p}}^h)_{\mathfrak{n}} \to \kappa(\mathfrak{q})[[x]]$$

is faithfully flat, and since $x_1, \ldots, x_r$ generate the maximal ideal of $\kappa(\mathfrak{q})[[x]]$, they generate $\mathfrak{n}$. We conclude at once that $\kappa(\mathfrak{q})[[x]]$ is the adic completion of D, that D is a local regular ring of dimension r and that $x_1, \ldots, x_r$ are a regular system of parameters of D. ■

STEP III. — The ring D and its henselization D^h are excellent rings.

By the jacobian criterion [Mt, Th. 102, p. 291], it is enough to see that the derivations $\partial/\partial x_1, \ldots, \partial/\partial x_r$ of $\kappa(\mathfrak{q})[[x]]$ leave D and D^h invariant. This in turn will follow if $\mathcal{N}(\mathrm{M})_{\mathfrak{p}}$ is invariant, using the general properties of derivations and the definition of the homomorphisms

$$\mathcal{N}(\mathrm{M})_{\mathfrak{p}} \to \mathcal{N}(\mathrm{M})_{\mathfrak{p}}^h \to (\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(\mathrm{M})_{\mathfrak{p}}^h)_{\mathfrak{n}} = \mathrm{D} \to \hat{\mathrm{D}} = \kappa(\mathfrak{q})[[x]].$$

On the other hand $\mathcal{N}(\mathrm{M})_{\mathfrak{p}}$ will be invariant if there is a field $\kappa \subset \mathcal{N}(\mathrm{M})_{\mathfrak{p}}$ such that the canonical extension $\kappa \to \kappa(\mathfrak{p})$ is algebraic and the rank of the module $\mathrm{Der}_{\kappa}(\mathcal{N}(\mathrm{M})_{\mathfrak{p}})$ of derivations of $\mathcal{N}(\mathrm{M})_{\mathfrak{p}}$ over is exactly $\mathrm{ht}(\mathfrak{p})$ ([Mt, Th. 99, p. 288]). Now by the extension property for closed Nash submanifolds ([Sh2, II.5.5, p. 131]), we can write $\mathcal{N}(\mathrm{M}) = \mathcal{N}(\mathbb{R}^p)/\mathrm{I}(\mathrm{M})$, where $\mathrm{I}(\mathrm{M})$ corresponds to any fixed closed embedding $\mathrm{M} \subset \mathbb{R}^p$, and there is a prime ideal $\mathfrak{P} \supset \mathrm{I}(\mathrm{M})$ such that $\mathfrak{p} = \mathfrak{P}/\mathrm{I}(\mathrm{M})$. Then $\mathcal{N}(\mathrm{M})_{\mathfrak{p}} = \mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}}/\mathrm{I}(\mathrm{M})$, and by [Mt, Th. 100, p. 290] we are reduced to find $\kappa \subset \mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}}$ such that $\mathrm{Der}_{\kappa}(\mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}})$ has rank $= \mathrm{ht}(\mathfrak{P})$. Finally put $\mathfrak{O} = \mathfrak{P} \cap \mathbb{R}[\mathrm{X}_1, \ldots, \mathrm{X}_p]$. As is well known, the extension $\mathbb{R}[\mathrm{X}_1, \ldots, \mathrm{X}_p]/\mathfrak{O} \subset \mathcal{N}(\mathbb{R}^p)/\mathfrak{P}$ is algebraic, and the transcendence degree of $\mathcal{N}(\mathbb{R}^p)/\mathfrak{P}$ over $\mathbb{R}$ is $d = p - \mathrm{ht}(\mathfrak{P})$. Now we can make a linear change of coordinates and assume that the canonical homomorphism $\mathbb{R}[\mathrm{X}_1, \ldots, \mathrm{X}_d] \to \mathbb{R}[\mathrm{X}_1, \ldots, \mathrm{X}_p]/\mathfrak{O}$ is finite and injective. This implies that $\mathfrak{P} \cap \mathbb{R}[\mathrm{X}_1, \ldots, \mathrm{X}_d] = \{0\}$ and so $\kappa = \mathbb{R}(\mathrm{X}_1, \ldots, \mathrm{X}_d) \subset \mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}}$ induces and algebraic extension $\kappa \to \kappa(\mathfrak{p})$ and we claim this is the κ we sought. Indeed,

$$\partial_{d+1} = \partial/\partial \mathrm{X}_{d+1}, \ldots, \partial_p = \partial/\partial \mathrm{X}_p$$

are clearly derivations of $\mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}}$ over κ and $\mathrm{X}_{d+1}, \ldots, \mathrm{X}_p \in \mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}}$ verify $\partial_i(\mathrm{X}_j) = \delta_{ij}$. As $p - d = \mathrm{ht}(\mathfrak{P})$, we apply [Mt, Th. 99, p. 288] to conclude that the rank of $\mathrm{Der}_{\kappa}(\mathcal{N}(\mathbb{R}^p)_{\mathfrak{P}})$ is $\mathrm{ht}(\mathfrak{P})$, as wanted. ■

Now the proof becomes a fan chasing through the diagram above. This will be done by successively completing a third row of valuation rings and fans compatible with them.

STEP IV. — Consider the henselization V^h of V. By the universal property of henselizations [Ng, 43.5, p. 181] we have a commutative square of local homomorphisms

$$\begin{array}{ccc} \mathcal{N}(M)_{\mathfrak{p}} & \to & \mathcal{N}(M)^h_{\mathfrak{p}} \\ \downarrow \rho & & \downarrow \rho^h \\ V & \to & V^h \end{array}$$

The henselization of a valuation ring is a valuation ring of a field $K^h \supset K$ and with the same residue field and value group as V. Thus, by Proposition 3.1 the fan F (which is compatible with V) extends to a fan F^h of V^h. Whence, we have added an entry to the diagram of Step II:

$$\begin{array}{ccccccccc} \mathcal{O}(M) \subset & \mathcal{O}(M)_{\mathfrak{q}} & \to & \mathcal{O}(M)^h_{\mathfrak{q}} & \longrightarrow & \mathcal{O}(M)_{\mathfrak{q}}^{\wedge} \equiv \kappa(\mathfrak{q})[[x]] & \equiv & \hat{D} \\ \uparrow & \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{N}(M) \subset & \mathcal{N}(M)_{\mathfrak{p}} & \to & \mathcal{N}(M)^h_{\mathfrak{p}} & \to & (\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \mathcal{N}(M)^h_{\mathfrak{p}})_{\mathfrak{n}} = D & \to & D^h \\ & \downarrow \rho & & \downarrow \rho^h & & & & \\ & (V, F) & \to & (V^h, F^h) & & & & \end{array}$$

STEP V. — We have two orderings σ_1, ρ_2 in the residue field k^h of V^h that restrict to α_1, α_2 in $\kappa(\mathfrak{p}) \subset k^h$. On the other hand, by Proposition 2.7, each α_i lifts to an ordering α'_i in the residue field of some prime divisor of the extension of $\mathfrak{p}$ to $\mathcal{O}(M)$. But $\mathfrak{q}$ is the unique real prime divisor of that extension and so α'_i is an ordering of $\kappa(\mathfrak{q})$. Now we consider the commutative square

$$\begin{array}{ccc} \kappa(\mathfrak{q}) & \to & \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} k^h \\ \uparrow & & \uparrow \\ \kappa(\mathfrak{p}) & \to & k^h \end{array}$$

By separation the field $\kappa(\mathfrak{p})$ is algebraically chlosed in $\kappa(\mathfrak{q})$ (Corollary 3.6), and by amalgamation (Proposition 3.5) the ring in the right upper corner is a domain, and we find two orderings τ_1, τ_2 in its quotient field k^* that extend simultaneously the orderings α'_1, α'_2 of $\kappa(\mathfrak{q})$ and σ_1, σ_2 of $\kappa(\mathfrak{p})$.

STEP VI. — The field k^* is an extension of the residue field k^h of the valuation ring V^h. Then we can construct a valuation V^* of the quotient field K^* of the domain $\kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} K^h$, which extends V^h, has the same value group as V^h and whose residue field is k^*.

Indeed, let k^h, $\mathfrak{m}^h$ and Γ^h denote respectively the residue field, the maximal ideal and the value group of V^h. First consider any subextension $\kappa(\mathfrak{p}) \subset \kappa \subset \kappa(\mathfrak{q})$ and the domain $\kappa \otimes_{\kappa(\mathfrak{p})} k^h$. Then the canonical homomorphism $V^h \to \kappa \otimes_{\kappa(\mathfrak{p})} V^h$ is faithfully flat, and it follows that $\mathfrak{m}^h$ generates in $E = \kappa \otimes_{\kappa(\mathfrak{p})} V^h$ the prime ideal $\mathfrak{m} = \kappa \otimes_{\kappa(\mathfrak{p})} \mathfrak{m}^h$. Thus we get a faithfully flat local embedding $V^h \to E_{\mathfrak{m}}$, and $\mathfrak{m}^h$ generates the maximal ideal of $E_{\mathfrak{m}}$. Furthermore, E_m contains the quotient field k of $\kappa \otimes_{\kappa(\mathfrak{p})} k_V$, and k is a coefficient field of $E_{\mathfrak{m}}$. Finally the quotient field L of E is the quotient field of the domain $\kappa \otimes_{\kappa(\mathfrak{p})} K^h$, and consequently $K^h \subset L \subset K^*$ where K^* denotes the quotient field of the domain $\kappa(\mathfrak{q}) \otimes_{\varkappa(\mathfrak{p})} K^h$. We summarize this construction in the following diagram:

$$\begin{array}{ccccccc}
\kappa(\mathfrak{q}) & \subsetneq & k^* = qf(\kappa(\mathfrak{q}) \otimes_{\varkappa(\mathfrak{p})} k^h) & \subset & E^*_{m^*} & \subset & K^* qf(\kappa(\mathfrak{q}) \otimes_{\varkappa(\mathfrak{p})} K^h) \\
\cup & & \cup & & \uparrow & & \cup \\
\kappa & \subsetneq & k = qf(\kappa \otimes_{\varkappa(\mathfrak{p})} k^h) & \subset & E_m & \subset & L = qf(\kappa \otimes_{\varkappa(\mathfrak{p})} K^h) \\
\cup & & \cup & & \uparrow & & \cup \\
\kappa(\mathfrak{p}) & \subsetneq & k^h = \kappa(\mathfrak{p}) \otimes_{\varkappa(\mathfrak{p})} k^h & \subset & V^h & \subset & K^h = \kappa(\mathfrak{p}) \otimes_{\varkappa(\mathfrak{p})} K^h
\end{array}$$

(with the obvious notations in the first row). Now we consider the collection $\mathscr{W}$ of all pairs (κ, W) where κ is as above and W is a valuation ring of L such that:

(i) W contains the local ring E_m and the inclusion $E_m \subset W$ is a local homomorphism,

(ii) the induced map $E_m/\mathfrak{m} \to W/\mathfrak{m}_W$ is a bijection, and

(iii) W is an extension of V^h with the same value group Γ^h.

It is clear that $\mathscr{W}$ is inductive and contains $(\kappa(\mathfrak{p}), V^h)$. Consequently by Zorn's lemma it has a maximal element (κ, W). We will see that this W is the valuation V^* we seek. For this it is enough to prove that $\kappa = \kappa(\mathfrak{q})$, so we argue by way of contradiction supposing there is an element $t \in \kappa(\mathfrak{q}) \setminus \kappa$. We need the following fact:

Claim. — *Let Φ denote either k^h or K^h. Then the element $t \in \kappa(\mathfrak{q})$ is algebraic over κ if and only if the element $t \equiv t \otimes 1 \in \kappa(\mathfrak{q}) \otimes_{\varkappa(\mathfrak{p})} \Phi$ is algebraic over $\kappa \otimes_{\varkappa(\mathfrak{p})} \Phi$. If that is the case, the irreducible polynomial of t over κ coincides with the irreducible polynomial of t over $\kappa \otimes_{\varkappa(\mathfrak{p})} \Phi$.*

This claim follows from the fact that applying the faithfully flat base change $- \otimes_{\varkappa(\mathfrak{p})} \Phi$ to the homomorphism $\kappa[T] \to \kappa(\mathfrak{q}) : T \mapsto t$ we get $(\kappa \otimes_{\varkappa(\mathfrak{p})} \Phi)[T] \to \kappa(\mathfrak{q}) \otimes_{\varkappa(\mathfrak{p})} \Phi : T \mapsto t \otimes 1$ (T stands for an indeterminate).

Now we distinguish two cases:

Case 1, *t is algebraic over κ*. — Then the claim above for $\Phi = K^h$ shows that t is also algebraic over L, and there is some valuation W′ of $L[t]$ extending W. We put $\kappa' = \kappa[t]$ and will see that the pair (κ', W') is an element of $\mathscr{W}$. As it is strictly bigger than (κ, W), this will be the wanted contradiction.

Since t and $1/t$ are algebraic over κ, they are integral over W and since W′ is integrally closed, we conclude that t is a unit of W′. Hence, $k[t] \subset k_{W'}$. Now by the claim above $[\kappa[t]:\kappa] = [k[t]:k] = [L[t]:L]$, say $= n$. Thus from the fundamental inequality [En, 17.5, p. 128] we deduce

$$n = [L[t]:L] \geqq e.f = [\Gamma_{W'}:\Gamma_W].[k_{W'}:k] \geqq [\Gamma_{W'}:\Gamma_W].n$$

which implies $\Gamma_{W'} = \Gamma_W$ and $k_{W'} = k[t]$. In this situation the construction associated to κ' gives

$$E' = \kappa[t] \otimes_{\varkappa(\mathfrak{p})} V^h = \kappa[t] \otimes_{\varkappa} E = E[t],$$

$$L' = \kappa[t] \otimes_{\varkappa(\mathfrak{p})} K^h = \kappa[t] \otimes_{\varkappa} L = L[t],$$

(using again the claim above). From this it follows easily that W′ dominates $E'_{\mathfrak{m}'}$, and so $(\kappa', W') \in \mathscr{W}$.

Case 2, *t is not algebraic over* κ. — Then the same happens over k and over L and we can treat t as an indeterminate. Hence putting $\kappa'=\kappa(t)$, we get $E'_{m'}=E[t]_{m[t]}$ and $L'=L(t)$. Then $W'=W[t]_{m_W[t]}$ is a valuation ring of L′, and (κ', W') is an element of $\mathscr{W}$ strictly bigger than (k, W). We are done. ■

STEP VII. — Now by Proposition 3.1 there is a fan F^* compatible with V^* that extends F^h. Substituting V^* by its henselization we can assume that V^* is henselian. Note that it has a coefficient field k^* and contains $\kappa(\mathfrak{q})\subset k^*$. Thus we have a canonical homomorphism $\eta:\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}}\to V^*$ that makes commutative the square

$$\begin{array}{ccc} \mathcal{N}(M)^h_{\mathfrak{p}} & \to & \kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}} \\ \downarrow \rho^h & & \downarrow \eta \\ V^h & \longrightarrow & V^* \end{array}$$

Hence, the prime ideal $\eta^{-1}(\mathfrak{m}_{V^*})$ lies over the maximal ideal of $\mathcal{N}(M)^h_{\mathfrak{p}}$, and consequently contains the x_i's. Thus we obtain the ideal $\mathfrak{n}$ defined in Step II and a local homomorphism $D=(\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}})_{\mathfrak{n}}\to V^*$ that, since V^* is henselian, extends to another one $\rho^*:D^h\to V^*$. This adds one more entry to our diagram which looks as follows

$$\begin{array}{ccccccccc} \mathcal{O}(M) & \subset & \mathcal{O}(M)_{\mathfrak{q}} & \to & \mathcal{O}(M)^h_{\mathfrak{q}} & \longrightarrow & \mathcal{O}(M)^\wedge_{\mathfrak{q}}\equiv\kappa(\mathfrak{q})[[x]] & \equiv & \hat{D} \\ \uparrow & & \uparrow & & & & & & \uparrow \\ \mathcal{N}(M) & \subset & \mathcal{N}(M)_{\mathfrak{p}} & \to & \mathcal{N}(M)^h_{\mathfrak{p}} & \to & (\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}})_{\mathfrak{n}}=D & & \to D^h \\ & & \downarrow \rho & & \downarrow \rho^h & & \downarrow & & \swarrow \\ & & (V, F) & \to & (V^h, F^h) & \longrightarrow & (V^*, F^*) & & \rho^* \end{array}$$

and explains our somehow sophisticated choice of the ring D.

STEP VIII. — After all this preparation, the fan F^* restricts to a fan in the residue field of the ideal $\mathfrak{p}_0^*=\ker(\rho^*)$. Since D^h is henselian excellent, Theorem 3.2 applies here: $\mathfrak{p}_0^*$ generates a prime ideal $\mathfrak{q}_0^*$ in the completion $\hat{D}\equiv\mathcal{O}(M)^\wedge_{\mathfrak{q}}$, and F^* extends to a fan $\hat{F}$ of the residue field of $\kappa(\mathfrak{q}_0^*)$. To finish we restrict $\hat{F}$ to $\mathcal{O}(M)$ via the homomorphism $\mathcal{O}(M)\subset\mathcal{O}(M)^\wedge_{\mathfrak{q}}$ as follows: the ideal $\mathfrak{q}_0^*$ lies over a prime ideal $\mathfrak{q}_0\subset\mathcal{O}(M)$ and $\hat{F}$ restricts to a fan F' in the residue field $\kappa(\mathfrak{q}_0)$. It is clear from the construction of the diagram that $\mathfrak{q}_0$ lies over $\mathfrak{p}_0$ and F' restricts to F in $\kappa(\mathfrak{p}_0)$.

STEP IX. — It remains to see that $\mathfrak{q}_0$ is a prime divisor of $\mathfrak{p}_0\mathcal{O}(M)$, since then it will be the *unique real* one, as wanted. For that we consider the ideal $I=\ker(\rho^h)$. Since the base change $\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}-$ is faithfully flat, the kernel of

$$\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\rho^h:\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}}\to\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}V^h$$

is the extension $I(\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}\mathcal{N}(M)^h_{\mathfrak{p}})$. Thus, since V^* is a valuation ring of a field containing the domain $\kappa(\mathfrak{q})\otimes_{\kappa(\mathfrak{p})}V^h$, we deduce that $I'=I.D$ is the kernel of the homomorphism $D\to V^*$. Finally put $J=\mathfrak{q}_0^*\cap\mathcal{O}(M)^h_{\mathfrak{q}}$, which by construction is a prime divisor of the extension $I\mathcal{O}(M)^h_{\mathfrak{q}}$. The following diagram depicts the various ideals involved:

$$\begin{array}{ccccccc} \mathfrak{q}_0 & \to & \mathfrak{q}_0\,\mathcal{O}(M)_\mathfrak{q} & \to & J & \longrightarrow & \mathfrak{q}_0^* \\ & & & & \uparrow & & \uparrow \\ \mathfrak{p}_0 & \to & \mathfrak{p}_0\,\mathcal{N}(M)_\mathfrak{q} & \to & I & \to I' \to & \mathfrak{p}_0^* \end{array}$$

Here an arrow between two ideals means that the target is either the extension or a prime divisor of the source. Note also that these arrows correspond to faithfully flat regular homomorphisms. Then counting heights we obtain

$$\mathrm{ht}(\mathfrak{q}_0)=\mathrm{ht}(J)=\mathrm{ht}(I)=\mathrm{ht}(\mathfrak{p}_0)$$

and since $\mathfrak{q}_0$ lies over $\mathfrak{p}_0$ we conclude that $\mathfrak{q}_0$ is a prime divisor of $\mathfrak{p}_0\,\mathcal{O}(M)$. We are done. ■

This ends the proof of the fan extension theorem.

Remark. — The complicated argument above is not needed if $\mathrm{ht}(\mathfrak{p})=1$. Indeed, in that case $\mathcal{N}(M)_\mathfrak{p}$ is a discrete valuation ring and the fan is determined by the two orderings induced in $\kappa(\mathfrak{p})$ (every one of these two orderings lifts to exactly two orderings of the quotient field of $\mathcal{N}(M)_\mathfrak{p}$ by prescribing the sign of any fixed uniformizer $t\in\mathfrak{p}$ of $\mathcal{N}(M)_\mathfrak{p}$). Since $\mathcal{O}(M)_\mathfrak{q}$ is another discrete valuation ring and t is also a uniformizer of this bigger ring, the fan extends as soon as the orderings in $\kappa(\mathfrak{p})$ extend to $\kappa(\mathfrak{q})$. Thus we bypass henselizations, extensions of residue field and completions.

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