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BIFURCATION DIAGRAMS AND FOMENKO'S SURGERY ON LIOUVILLE TORI OF THE KOLOSSOFF POTENTIAL $U=\rho+(1/\rho)-k\cos\varphi$

By Ljubomir GAVRILOV, Mohammed OUAZZANI-JAMIL and Regis CABOZ

Abstract. — By making use of the rich algebraic structure of the problem and Fomenko's theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential $U=\rho+(1/\rho)-k\cos\varphi$.

I. Introduction

Consider the motion of a particle of unit mass on the plane (x, y) in a potential field

$$U=a\rho+\frac{b}{\rho}+c\cos\varphi+d\sin\varphi,\qquad a, b, c, d\in\mathbf{R}$$

where $x=\rho\cos\varphi$, $y=\rho\sin\varphi$. Without loss of generality one may suppose (after a rotation and $\mathbf{R}$-linear change of ρ and U) that

$$U(x, y)=\pm\rho\pm\frac{1}{\rho}-k\cos\varphi, k\in\mathbf{R}$$

The corresponding Hamiltonian function is:

$$H=\frac{1}{2}(p_x^2+p_y^2)+U(x, y)$$

and the energy level sets $\{H=h\}\subset\mathbf{R}^4$ are compact if $U=\rho+(1/\rho)-k\cos\varphi$. The Hamiltonian system

$$(1)\qquad \begin{cases} x'=\dfrac{dH}{dp_x}, & p'_x=-\dfrac{dH}{dx} \\ y'=\dfrac{dH}{dp_y}, & p'_y=-\dfrac{dH}{dy}\end{cases}\qquad (')=\frac{d}{dt}$$

where

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \rho + \frac{1}{\rho} - k\cos\varphi$$

is integrable and the second integral of motion reads:

$$F = -(k^2 + y^2)p_x^2 + 2y(x-k)p_x p_y - p_y^2(x-k)^2 - \frac{2k(x-k)(kx-1)}{\sqrt{x^2+y^2}}$$

The integrability of the system (1) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets

$$\mathbf{A}_{\mathbf{R}} = \{(x, y, p_x, p_y) \in \mathbf{R}^4 : H = h,\ F = f\} \subset \mathbf{R}^4.$$

For doing that we find first the bifurcation diagram **B** of the problem (1), *i.e.* the set of critical values of the energy-momentum mapping

$$(x, y, p_x, p_y) \rightarrow (F, H).$$

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that **B** is exactly the discriminant locus of a certain polynomial whose coefficients are functions in f, h, k. The latter is closely related to the algebraic structure of the complexified system (1). This structure is studied in section 2 where we prove that the complexified generic level set $\{H=h,\ F=f\}$ is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to H and F do not linearize on this Abelian variety. Thus the system (1) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1]. For non-critical values of F and H the level set $\mathbf{A}_{\mathbf{R}}$ is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (*see* Theorem 2 of section 3).At last, in section 4, we describe the structure of singular level sets $\mathbf{A}_{\mathbf{R}}$. According to Fomenko's theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To "guess" exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as f and h pass through the bifurcation diagram **B** (Theorem 3 and Theorem 4 of section 4).

II. Algebraic structure

Denote by $\mathbf{A_C}$ the complex affine algebraic variety:

$$\mathbf{A_C}=\{(x, y, p_x, p_y, z)\in \mathbf{C}^5 : \mathrm{H}=h,\ \mathrm{F}=f,\ x^2+y^2=z^2,\ z\neq 0\}\subset \mathbf{C}^5,$$

where

$$\mathrm{H}(x, y, p_x, p_y, z)=\frac{1}{2}(p_x^2+p_y^2)+z+\frac{1}{z}-k\frac{x}{z},$$

$$\mathrm{F}(x, y, p_x, p_y, z)=-(k^2+y^2)p_x^2+2y(x-k)p_xp_y-p_y^2(x-k)^2-\frac{2k(x-k)(kx-1)}{z}$$

The variety $\mathbf{A_C}$ is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$\varphi(u)=-2(u^3-hu^2+(1-k^2)u-f/2) \tag{2}$$

and the corresponding hyperelliptic curve

$$\mathrm{K}:\ \{w^2=(u^2-k^2)\varphi(u)\}. \tag{3}$$

Remark. — K is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

THEOREM 1. — *If the polynomial* $(u^2-k^2)\varphi(u)$ *has no double roots then the affine algebraic variety* $\mathbf{A_C}$ *is a smooth complex manifold which is biholomorphically equivalent to the complex manifold* $\tilde{\mathbf{A}}_\mathbf{C}\setminus\mathcal{D}$, *where* $\tilde{\mathbf{A}}_\mathbf{C}$ *is a complex algebraic torus (Abelian variety) and* $\mathcal{D}$ *is a divisor.* $\tilde{\mathbf{A}}_\mathbf{C}$ *is a two-sheeted unramified covering of the Jacobi variety* Jac(K) *of the genus algebraic two curve* K. *The trajectories of the Hamiltonian flow generated by* H *on* $\mathbf{A_C}$ *are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by* $\mathrm{H}+s\mathrm{F}$, $s\neq 0$ *on* $\mathbf{A_C}$ *are not straight lines.*

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following (λ, μ) coordinates (*see* [8] for details):

$$\begin{cases} x=\dfrac{\lambda\mu}{k}+k \\ y=\dfrac{1}{k}\sqrt{(\lambda^2-k^2)(k^2-\mu^2)} \end{cases} \tag{4}$$

The canonical variables $(p_\lambda, p_\mu, \lambda, \mu)$ on $\mathbf{T^* R}^2$ are given by

$$(5)\qquad \left\{\begin{aligned} p_x &= \frac{(\lambda^2-k^2)\,\mu p_\lambda-(\mu^2-k^2)\,\lambda p_\mu}{k(\lambda^2-\mu^2)}\\ p_y &= \frac{\sqrt{(\lambda^2-k^2)(k^2-\mu^2)}\,(\lambda p_\lambda-\mu p_\mu)}{k(\lambda^2-\mu^2)}\end{aligned}\right.$$

In these new variables the integrals of motion take the form

$$\mathrm{H}=\frac{(\lambda^2-k^2)\,p_\lambda^2-(\mu^2-k^2)\,p_\mu^2+2(1-k^2)(\lambda-\mu)+2(\lambda^3-\mu^3)}{2(\lambda^2-\mu^2)},$$

$$\mathrm{F}=\frac{-\mu^2(\lambda^2-k^2)\,p_\lambda^2+\lambda^2(\mu^2-k^2)\,p_\mu^2-2\lambda\mu(\lambda\mu+k^2+k^2-1)(\lambda-\mu)}{(\lambda^2-\mu^2)}$$

and hence on each level set $\mathbf{A_C}$ holds

$$(6)\qquad p_\lambda=\sqrt{\frac{\varphi(\lambda)}{\lambda^2-k^2}},\qquad p_\mu=\sqrt{\frac{\varphi(\mu)}{\mu^2-k^2}}.$$

For a further use we note also the relation

$$(7)\qquad \mathrm{F}=p_\mu^2(\mu^2-k^2)+2\mu^3-2\mu^2\,\mathrm{H}+2\mu(1-k^2).$$

Denote by d/dt_s the time derivative along the Hamltonian flow of the function $\mathrm{H}_s=\mathrm{H}+s\mathrm{F}$. By making use of the equations

$$\frac{d\lambda}{dt_s}=\frac{\partial \mathrm{H}}{\partial p_\lambda},\qquad \frac{d\mu}{dt_s}=\frac{\partial \mathrm{H}_s}{\partial p_\mu}$$

and (6) one obtains

$$(8)\qquad \left\{\begin{aligned} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=-2s\,dt_s\\ \frac{\lambda^2\,d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{\mu^2\,d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=dt_s\end{aligned}\right.$$

The system (8) can be also written in the following equivalent form

$$(9)\qquad \left\{\begin{aligned} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=-2s\,dt_s\\ \frac{\lambda\,d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2-k^2)}}+\frac{\mu\,d\mu}{\sqrt{\varphi(\mu)(\mu^2-k^2)}}&=\frac{1-2s\lambda\mu}{\lambda+\mu}\,dt_s\end{aligned}\right.$$

The flow of Kolossoff system (1) corresponds to $s=0$, and obviously $t_s|_{s=0}=t$. The system (9) implies, roughly speaking, that our initial system linearizes on an Jacobian variety after using a "new time"

$$d\tau = \frac{dt}{\lambda + \mu}. \tag{10}$$

The time τ will play an important role and it is exactly the "Kovalevskaya time" (*see* [8] for details).

Define now the Abel-Jacobi map

$$\zeta : S^2 K \to \operatorname{Jac}(K) : (P_1, P_2) \to \left(\int_{P_\infty}^{P_1} \omega_1 + \int_{P_\infty}^{P_2} \omega_1, \int_{P_\infty}^{P_1} \omega_2 + \int_{P_\infty}^{P_1} \omega_2 \right)$$

where

$$\omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2 - k^2)}}, \qquad \omega_2 = \frac{u\,du}{\sqrt{\varphi(u)(u^2 - k^2)}}$$

P_1, P_2, $\in K$ P_∞ is the "infinite" point on K and S^2 K is the second symetric product of K.

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus x, y, p_x, p_y, $z = \sqrt{x^2 + y^2}$ can be expressed in terms of genus two theta functions living on the Jacobi variety Jac(K). These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symetric product S^2 K of the curve K (which is birational to Jac(K) according to Jacobi theorem) correspond two values of (x, y, p_x, p_y). On the other hand these functions do not have branch points on Jac(K) and hence they are root functions (Wurzelfunktionen [14]) on Jac(K).

Consider the Abelian variety $\tilde{\mathbf{A}}_{\mathbf{C}} = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, 2e_4\}$ where

$$\operatorname{Jac}(K) = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, e_4\}.$$

If the basis (e_1, e_2, e_3, e_4) of the period lattice is chosen in a proper way then the function x, y, p_x, p_y, z become single-valued on $\tilde{\mathbf{A}}_{\mathbf{C}}$. Let us fix such a basis. The natural projection

$$\pi : \tilde{\mathbf{A}}_{\mathbf{C}} \to \operatorname{Jac}(K) \tag{11}$$

corresponds to the involution

$$(x, y, p_x, p_y, z) \to (x, -y, p_x, -p_y, z) \tag{12}$$

on $\tilde{\mathbf{A}}_{\mathbf{C}}$. Consider the mapping

$$i : \mathbf{C}^5 \to \mathbf{CP}^7 : (x, y, z, p_x, p_y) \to [f_0, f_1, \ldots, f_7]$$

where

$$(13)\qquad \begin{cases} f_0=1 \\ f_1=x \\ f_2=y \\ f_3=z \\ f_4=xp_y-yp_x \\ f_5=f_4^2 \\ f_6=f_3(f_4-kp_y) \\ f_7=(p_x^2-p_y^2)y-2p_xp_yx-2f_2f_3. \end{cases}$$

LEMMA 1. — *The functions f_i, $i=0, 1 \ldots, 7$ considered as single-valued meromorphic functions on $\tilde{\mathbf{A}}_{\mathbf{C}}$ provide a smooth embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$.*

Proof of theorem 1 *assuming the above lemma.* — As the functions $f_0, f_1, \ldots, f_7$ provide an embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$ (Lemma 1) then the closure $\overline{i(\mathbf{A_C})}$ of $i(\mathbf{A_C})$ in $\mathbf{CP}^7$ is biholormophically equivalent to $\tilde{\mathbf{A}}_{\mathbf{C}}$. Consider the divisors $\mathscr{D}_\infty$ and $\mathscr{D}'_{2\infty}$ defined by

$$(\lambda\mu)_\infty = 2(\zeta(P_\infty)+\zeta(K)) = 2\mathscr{D}_\infty$$

and

$$(z)_0=(\lambda+\mu)_0=\mathscr{D}'_{2\infty}$$

Obviously $\mathscr{D}'_{2\infty} \sim 2\mathscr{D}_\infty$. It is easily seen that $\mathbf{A_C}$ is biholormopyically equivalent to $\overline{i(\mathbf{A_C})} \setminus \{\mathscr{D}_\infty \cup \mathscr{D}'_{2\infty}\}$. Indeed i is a biholomorphic mapping between some neighbourhood $\mathrm{V}_{\mathbf{A_C}}$ of $\mathbf{A_C}$ in $\mathbf{C}^5 \setminus \{z \neq 0\}$ and $i(\mathrm{V}_{\mathbf{A_C}}) \subset \mathbf{CP}^7$. To check that it suffice to note that if $(x, y, p_x, p_y, z) \in \mathbf{A_C}$ then

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_4, f_6)}{\partial(x, y, p_x, p_y, z)}\right)=kyz$$

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_5, f_7)}{\partial(x, y, p_x, p_y, z)}\right)=-4p_y(p_xx^2y+p_xy^3-p_yx^3-p_yxy^2)$$

and hence $\operatorname{rank}(i)=5$ (otherwise the equality $y=p_y=0$ implies $\operatorname{disc}((k^2-u^2)\varphi(u))=0$). As $i(\mathbf{A_C})=\tilde{\mathbf{A}}_{\mathbf{C}} \setminus \mathscr{D}_\infty$ is a smooth complex manifold, it is concluded that $\mathbf{A_C}$ is also a smooth complex manifold. △

Proof of Lemma 1. — For an arbitrary divisor $\mathscr{D} \subset \tilde{\mathbf{A}}_{\mathbf{C}}$ we denote

$$\mathscr{L}(\mathscr{D}) = \{f \text{ meromorphic on } \tilde{\mathbf{A}}_{\mathbf{C}},\ (f) \geqq -\mathscr{D}\}$$

As $\zeta(\mathrm{K})$ defines (1, 1) polarization on Jac (K) then $\mathscr{D}_\infty = \pi^{-1} \circ \zeta(\mathrm{K})$ defines (1, 2) polarization on $\mathbf{A_C}$. Thus $2\mathscr{D}_\infty$ defines (2, 4) polarization on $\tilde{\mathbf{A}}_{\mathbf{C}}$ and $\dim \mathscr{L}(2\mathscr{D}_\infty) = 2 \times 4 = 8$, [7]. To prove lemma 1, it is enough to check that the functions $f_0, f_1, \ldots, f_7$ provide a basis of $\mathscr{L}(2\mathscr{D}_\infty)$. First of all let us note that f_i blow up only along $\mathscr{D}_\infty$. Indeed in λ, μ

coordinates we have

$$f_1=1$$

$$f_1=\frac{\lambda\mu}{k}+k$$

$$f_2=\frac{1}{k}\sqrt{(\lambda^2-k^2)(k^2-\mu^2)}$$

$$f_3=\lambda+\mu$$

$$f_4=\frac{1}{(\lambda-\mu)}\{\sqrt{(k^2-\mu^2)}\sqrt{\varphi(\lambda)}-\sqrt{(\lambda^2-k^2)}\sqrt{-\varphi(\mu)}\}$$

$$f_5=f_4^2$$

$$f_6=\frac{1}{(\lambda-\mu)}\{\mu\sqrt{(k^2-\mu^2)}\sqrt{\varphi(\lambda)}-\lambda\sqrt{\varphi(\lambda)}-\sqrt{(\lambda^2-k^2)}\sqrt{-\varphi(\mu)}\}$$

$$f_7=\frac{1}{k(\lambda-\mu)}\{2(\lambda\mu-k^2)\sqrt{\varphi(\lambda)}\sqrt{-\varphi(\mu)}-\sqrt{(\lambda^2-k^2)(k^2-\mu^2)}(\varphi(\lambda)+\varphi(\mu))\}-2f_2f_3.$$

To prove that $f_i\in\mathcal{L}(2\mathcal{D}_\infty)$ we shall find, following [1], the asymptotic expansions of x, y, z as functions of the time τ (10) in a neighbourhood of a generic point $\tau^0\in\mathcal{D}_\infty$. Formulae (4) imply that $\lambda+\mu=\sqrt{x^2+y^2}$ and hence the changing of time in the system (1) is equivalent to multiplying each equation by z. According to (9) and (4) the variables x, y, z are meromorphic in τ and the corresponding Laurent series are:

$$(14)\qquad \begin{cases} x=\sum_{j=0}^{\infty} x_j\tau^{j-2}, & p_x=\sum_{j=0}^{\infty} p_{x_j}\tau^{j-1} \\ y=\sum_{j=0}^{\infty} y_j\tau^{j-2}, & p_y=\sum_{j=0}^{\infty} p_{y_i}\tau^{j-1} \\ z=\sum_{j=0}^{\infty} z_j\tau^{j-2} \end{cases}$$

(here τ stays for $\tau-\tau_0$). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients x_j, y_j, z_j. The general solution (14) depends effectively upon three free parameters α, γ, δ:

$$(15)\qquad \begin{cases} x=\dfrac{\alpha}{\tau^2}+\dfrac{(k\beta^3-4\gamma\alpha)}{4\beta}+\delta\tau+\dots \\ y=\dfrac{\beta}{\tau^2}-\dfrac{(k\alpha\beta+4\gamma)}{4}-\dfrac{\alpha\delta}{\beta}\tau+\dots \\ z=\dfrac{-2}{\tau^2}+\dfrac{2\gamma}{\beta}+\dots \end{cases}$$

where $\alpha^2+\beta^2=4$ (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

$$
(16)\qquad \left\{\begin{aligned}
f_0&=1\\
f_1&=\frac{\alpha}{\tau^2}+\dots\\
f_2&=\frac{\beta}{\tau^2}+\dots\\
f_3&=-\frac{2}{\tau^2}+\dots\\
f_4&=\frac{k\beta}{\tau}+\dots\\
f_5&=\frac{k^2\beta^2}{\tau^2}+\dots\\
f_6&=\frac{12\delta}{\beta\tau^2}+\dots\\
f_7&=-2\frac{(k\alpha\beta+6\gamma)}{\tau^2}+\dots
\end{aligned}\right.
$$

The complex constants α (or β such that $\alpha^2+\beta^2=4$), γ, δ parametrize the pole divisor $\mathscr{D}_\infty$. Indeed substituting (15) in $\{\mathrm{H}=h,\ \mathrm{F}=f,\ z^2=x^2+y^2\}$ we obtain the genus three curve

$$
(17)\qquad \left\{\begin{aligned}
\gamma&=\frac{2h\beta-k\alpha\beta}{16},\\
\delta^2&=\frac{\beta}{72}(k^3\alpha\beta^3+8k^2\gamma\beta^2-2k(1+k^2)\alpha\beta-32k^2\gamma-2f\beta),\\
\alpha^2&+\beta^2=4
\end{aligned}\right.
$$

$\mathscr{D}_\infty$ is a double unramified covering of the genus two curve

$$
(18)\qquad \delta^2=\frac{(\alpha^2-4)}{144}(k^3\alpha^3+2hk^2\alpha^2+4k(1-k^2)\alpha+4f)
$$

and obviously this curve (18) coincides with (3) after making the substitution

$$
\alpha\to\frac{2u}{k},\qquad \delta\to\frac{w}{3k}.
$$

Equations (16) and (18) imply that $f_0, f_1, \dots, f_7$ are linearly independent on $\tilde{\mathbf{A}}_{\mathbf{C}}$ which completes the proof of lemma 1. △

III. Topology of Regular Level Sets

In this section we shall describe the topological type of $A_{\mathbf{R}}$ for all generic constants f, h, $k \in \mathbf{R}$. The system (1) will be considered as a real system of differential equations.

According to Theorem 1 $A_{\mathbf{R}}$ is a smooth real manifold if the polynomial $(k^2-u^2)\varphi(u)$ has no double roots. Define the bifurcation set

(19) $$\mathbf{B}=\{(f, h, k)\in \mathbf{R}^3 : \operatorname{disc}((u^2-k^2)\varphi(u))=0\}.$$

It is clear that the topological type of $A_{\mathbf{R}}$ may change only as (f, h, k) passes through $\mathbf{B}$. Thus in each connected component of the set $\mathbf{R}^3 \setminus \mathbf{B}$ the level set $A_{\mathbf{R}}$ has the same topological type. Note that the bifurcation set $\mathbf{B}\subset \mathbf{R}^3\{f, h, k\}$ is invariant under the involution

$$(f, h, k) \to (f, h, -k)$$

and the topological type of the level set $A_{\mathbf{R}}$ is one and the same at the points (f, h, k) and $(f, h, -k)$. Thus it is enough to consider $k \geqq 0$.

THEOREM 2. — *The set* $\{\mathbf{R}^3 \setminus \mathbf{B}\} \cap \{k \geqq 0\}$ *consists of* 12 *connected components. The sections of these components with the plane* $\{k=\text{const.}\}$ *are shown on* figure 1. *If* $(f, h, k)\in \mathbf{R}^3\setminus \mathbf{B}$ *the level set* $A_{\mathbf{R}}$ *is* (*diffeomorphic to*) *a torus, to a disjoint union of two tori, or it is the empty set as it is shown in* table I.

Remark. — The notation 2T in table I means a disjoint union of 2 two-dimensional tori.

Proof of Theorem 2. — The complex conjugation

(20) $$(x, y, z, p_x, p_y) \to (\bar{x}, \bar{y}, \bar{z}, \bar{p}_x, \bar{p}_y)$$

acts as an antiholomorphic involution on $A_{\mathbf{C}}$. The set of its fixed points is the real part $\Re(A_{\mathbf{C}})$ of $A_{\mathbf{C}}$ and $A_{\mathbf{R}}=\Re(A_{\mathbf{C}})\cap\{z>0\}$. Consider also the natural antiholomorphic involution τ of the Kovalevskaya curve (3) given in (w, u) coordinates by:

$$\tau: \quad (w, u) \to (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symetric product $S^2 K$ and hence on Jac(K) and $\tilde{A}_{\mathbf{C}}$. Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on $A_{\mathbf{C}}$. The upshot is that in order to describe $A_{\mathbf{R}}$ it is enough to study the projection

$$\pi: \quad A_{\mathbf{C}} \to \operatorname{Jac}(K)$$

and the pair (K, τ).

Remark. — The pair (K, τ) where K is a Riemann surface and τ is an antiholomorphic involution on K is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

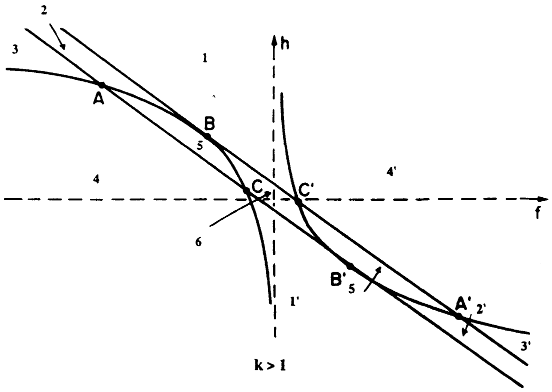

Fig. 1.1

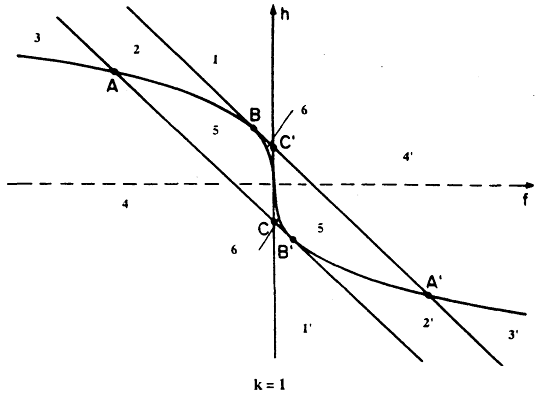

Fig. 1.2

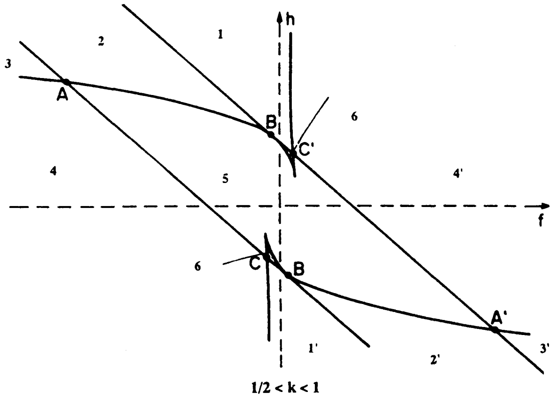

Fig. 1.3

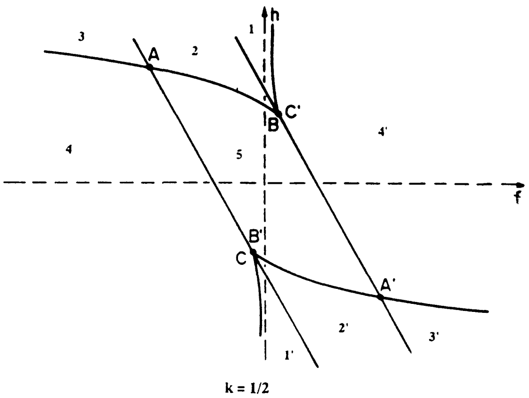

Fig. 1.4

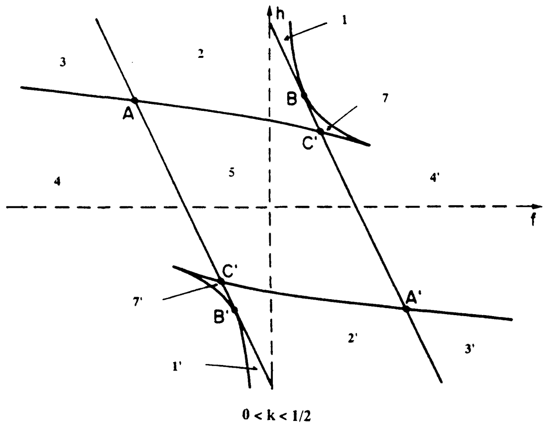

Fig. 1.5

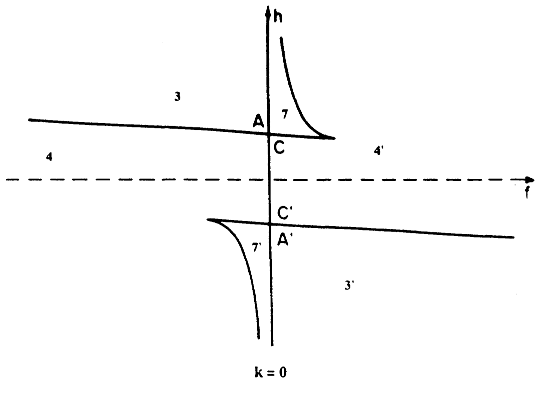

Fig. 1.6

Fig. 1. – The set $\mathbf{B} \cap \{k = \text{const.}\}$ for $k \geqq 0$.

TABLE I

Topological type of $\mathbf{A_R}$ *and real roots of the polynomial* $(u^2-k^2)\varphi(u)$ *for* $(f, h, k) \in \mathbf{R}^3 \setminus \mathbf{B}$ (see fig. 1).

Domain	Roots	Topological type
1	$-k<u_1<u_2<k<u_3$	T
1′	$u_1<-k<u_2<u_3<k$	∅
2	$-k<u_1<k<u_2<u_3$	T
2′	$u_1<u_2<-k<u_3<k$	∅
3	$u_1<-k<k<u_2<u_3$	2 T
3′	$u_1<u_2<-k<k<u_3$	∅
4	$u_1<-k<k$	∅
4′	$-k<k<u_1$	∅
5	$-k<u_1<k$	∅
6	$-k<u_1<u_2<u_3<k$	∅
7	$-k<k<u_1<u_2<u_3$	∅
7′	$u_1<u_2<u_3<-k<k$	∅

DEFINITION. — A connected component of the set of fixed points of τ on K is called an oval.

To determine the ovals of K it suffices to study the real roots of the polynomial $(u^2-k^2)\varphi(u)$ for different values of f, h and k. These roots are shown on table I. Using the formulae (4), (5), (6) and the condition $(x, y, z, p_x, p_y) \in \mathbf{R}^5$ we obtain that $\mathbf{A_R} \neq \varnothing$ only if (f, h, k) belongs to domain 1, 2 or 3. There we find exactly two "admissible" ovals whose projections on the z-plane are given by the intervals Δ_1 and Δ_2 (*see* table II). The product of the "admissible" ovals in $S^2 K$ [and hence in Jac(K)] gives a Liouville torus. Thus we proved that $\pi(\mathbf{A_R})$ consists of a torus T. There are two possibilities for $\mathbf{A_R} = \pi^{-1}(T)$ (recall that $\mathbf{A_C}$ is a double covering of $\mathrm{Jac}(K) \setminus \mathcal{D}$ and the projection is given by the map (11)):

TABLE II

Domain	1	2	3
Projection of the "admissible" ovals on z-plane	$\Delta_1=[u_1, u_2]$ $\Delta_2=[k, u_3]$	$\Delta_1=[u_1, k]$ $\Delta_2=[u_2, u_3]$	$\Delta_1=[-k, k]$ $\Delta_2=[u_2, u_3]$

— $\mathbf{A_R}$ is a disjoint union of two copies of T;

— $\mathbf{A_R}$ is homeomorphic to a torus two times "longer" than T.

To determine which case arises it suffices to note that when λ (respectively μ) makes one turn around the interval Δ_1 (respectively Δ_2) in a complex domain then the function y does not change in the first case, whereas in the second case it changes the sign [we recall that the projection π corresponds to the involution (20)]. Thus we find that in domain 1 and 2 $\mathbf{A_R}$ is a torus and in domain 3 it is a disjoint union of two tori. △

At last we shall find the topological type of the regular energy-level surface $\{H=h\}$

LEMMA 2. — *The bifurcation set* Σ *of the family of surfaces*

$$Q_{h,k}=\left\{\frac{p_x^2+p_y^2}{2}+\rho+\frac{1}{\rho}-k\cos\varphi=h\right\}$$

is given by the union of two lines $\Sigma=\{h=2+k\}\cup\{h=2-k\}\subset\mathbf{R}^2\{h, k\}$. *The set* $\mathbf{R}^2\setminus\Sigma$ *consists of* 4 *components shown on* figure 2. *The topological type of* $Q_{h,k}$ *in each of these domains is given in table* 3.

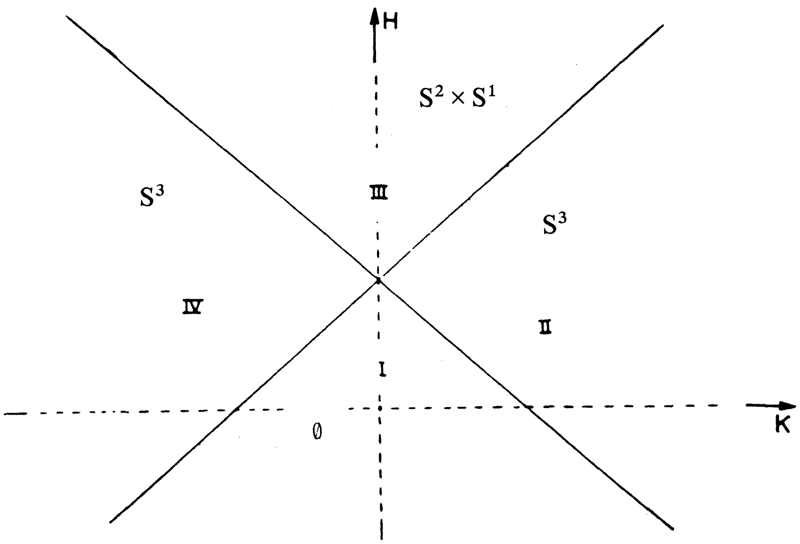

Fig. 2. — The set Σ.

Remarks. — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are: S^3 (the sphere), $\mathbf{R}P^3$ (the projective space), T^3 (the torus) and $S^2\times S^1$ (the direct product), *see* [13] for details.

TABLE III

Topological type of the energy level set $Q_{h,k}=\{H=h\}$ *for* $(h, k)\in\mathbf{R}^2\setminus\Sigma$ (see fig. 2).

Domain	1	2	3	4
Topological type	$\varnothing$	S^3	$S^2\times S^1$	S^3

Proof of Lemma 2. — The function H has exactly two critical points $p_x=p_y=0$, $y=0$, $x=\pm 1$, for $k\neq 0$ and a critical variety $\{p_x=p_y=0, x^2+y^2=1\}$ for $k=0$ with corresponding critical values $h=2\pm k\,(k\neq 0)$ and $h=2\,(k=0)$. Let us compute the topological type of $Q_{h,k}$. If $k=0$ then

$$H=\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{\rho}+2\geqq 2$$

and hence for $h<2$ we have $Q_{h,k}=\varnothing$. This implies that in domain 1 $Q_{h,k}=\varnothing$. Suppose now that $k=0$. On the surface $H=2+\varepsilon$ where ε is small and positive, $\rho-1$ is small together with ε. As.

$$\left\{\frac{p_x^2+p_y^2}{2}+\rho+\frac{1}{\rho}=2+\varepsilon\right\}$$

can be written as

$$\left\{\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{(\rho-1+1)}=\varepsilon\right\}\Leftrightarrow\left\{\frac{p_x^2+p_y^2}{2}+(\rho-1)^2-(\rho-1)^3+\ldots=\varepsilon\right\}\sim S^2$$

Then $Q_{2+\varepsilon,0}$ is topologically equivalent to $S^2\times S^1$ and hence in domain 3 the topological type of $Q_{h,k}$ is $S^2\times S^1$. Consider at last $Q_{2,\varepsilon}$, for ε small and positive

$$Q_{2,\varepsilon}=\left\{\frac{p_x^2+p_y^2}{2}+\frac{(\rho-1)^2}{\rho}=\varepsilon\cos\varphi\right\}.$$

The set $Q_{2,\varepsilon}\cap\{\varphi=\text{const.}\}$ is topologically equivalent to S^2 for $\varphi\in((-\pi/2),(\pi/2))$ and to a point for $\varphi=\pm(\pi/2)$. Hence $Q_{2,\varepsilon}$ is topologically equivalent to S^3. This implies that in domain 2 (and 4 by a symmetry) the topological type of $Q_{h,k}$ is S^3. △

IV. Topology of Singular Level Sets and Surgery on Liouville Tori

In this section we shall find the topological type of the level set $\mathbf{A_R}$ for generic values $(f, h, k)\in\mathbf{B}$ and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set $\mathbf{A_R}$ far from the bifurcation diagram. Suppose now that the constants f, h, k are changed in such a way, that (f, h, k) passes through the bifurcation diagram $\mathbf{B}$. Then the topological type of $\mathbf{A_R}$ may change and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see *fig.* 3).

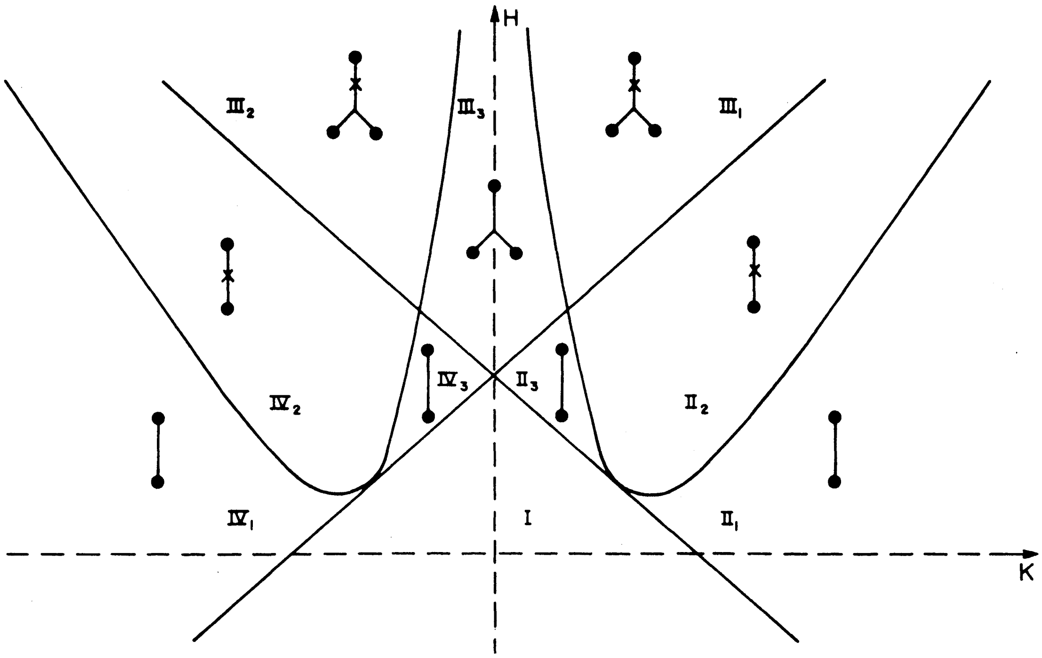

Fig. 3. — Bifurcations of two-dimensional invariant Liouville tori and the corresponding graphs.

1) A (two-dimensional) torus T^2 is contracted to the axial circle S^1 and then vanishes. Denote this surgery as $T \to S^1 \to \varnothing$.

2) A torus T splits into two tori by passing through the complex $S^1 \times \{S^1 \wedge S^1\}$ where $S^1 \wedge S^1$ is a union of two circles having exactly one common point. Denote this bifurcation as $T \to 2T$.

3) A torus T becomes twice "shorter" as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$, and the corresponding bifurcation will be denoted as $T \to T$.

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a "branching" point (see *fig.* 3) stands for $\{S^1 \wedge S^1\} \times S^1$. At last asterisk denotes a set homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \to S^1$.

For fixed constants h and k let us consider the energy level surface $Q_{h,k} = \{H = h\}$. As f varies the Liouville tori contained in the level set $\{F = f\}|_{Q_{h,k}}$ may change its topological type. Denote by $\Gamma(Q_{h,k}, F)$ the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following

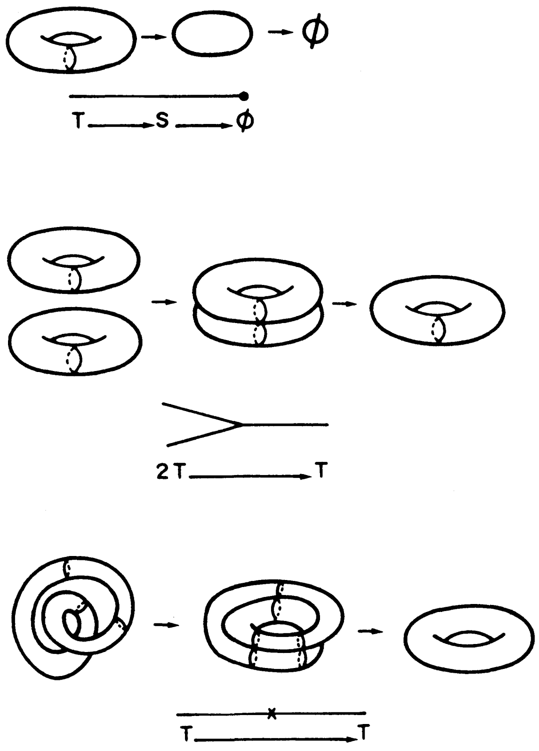

Fig. 4. — The set $\mathscr{D}$ and the graphs $\Gamma(Q_{h,k}, F)$.

THEOREM 3. — *If* (h, k) *belongs to one and the same connected component of the set*

$$\mathscr{D}=\{h\neq 2\pm k\}\cap\left\{h\neq\pm\left(k+\frac{1}{2k}\right)\right\}\subset\mathbf{R}^2\{h, k\}$$

then the graph $\Gamma(Q_{h,k}, F)$ *is the same and it is shown on figure* 4.

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s): s\to(f(s), h(s), k(s))\subset\mathbf{R}^3\{f, h, k\}$$

intersecting the bifurcation diagram $\mathbf{B}$ at $s=s_0$.

DEFINITION. — A bifurcation of Liouville tori contained in the level set

$$A_{\mathbf{R}}\equiv Q_{h(s), k(s)}\cap\{F=f(s)\}$$

as s passes through s_0 is called generic, provided that $\mathbf{B}$ is smooth in a neighbourhood of $(f(s_0), h(s_0), k(s_0))$ and $\gamma(s)$ intersects $\mathbf{B}$ transversally.

THEOREM 4. — *All generic bifurcations of Liouville tori of the system* (1) *are given in table IV.*

TABLE IV

Generic bifurcations of the level set $\mathbf{A_R}$.

1 → 2	1 → 6	1 → 4′	2 → 3	2 → 5	3 → 4
T → T	T → ∅	T → ∅	T → 2 T	T → ∅	T → ∅ T → ∅

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko's theorem [3] (adapted to our case).

DEFINITION. — A smooth function F on a manifold Q is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function F on a manifold Q is called nondegenerated, provided that the Hessian matrix d^2 F is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko's classification theorem of bifurcations of two-dimentional Liouville tori.

THEOREM (Fomenko [3]. — *Let* F *be a Bott integral on a non-singular constant energy surface* Q^3 *of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of* F *on* Q^3 *is a union of circles. Then each bifurcation of Liouville tori contained in the level set* $\{F=f\}$, *as f varies, is a composition of the three bifurcations* $T \to S^1 \to \varnothing$, $T \to 2T$, *and* $T \to T$ *described above.*

Remark. — The condition that each critical manifold of F is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko's theorem we need to check that F is a Bott function when restricted on an energy level surface $Q_{h,k}$.

LEMMA 3. — *The second integral* F *is a Bott function on the non-singular energy level surface* $Q_{h,k}=\{H=h\}$ *provided that* $h \neq \pm(k+(1/2k)$.

Proof of Lemma 3. — Suppose that $Q_{h,k}$ is a non-singular compact manifold, *i.e.* $h \neq 2 \pm k$ (Lemma 2). If F has a critical value f on $Q_{h,k}$ then the corresponding level surface $\mathbf{A_R}=\{H=h, F=f\}$ is degenerated and hence the polynomial $(u^2-k^2)\varphi(u)$ has multiple zeros. The condition $h \neq \pm(k+(1/2k))$ means that $(u^2-k^2)\varphi(u)$ has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the h-coordinates of the points A, B, C′ are $2+k$, $k+(1/2k)$, $2-k$ for $k>0$ and $2-k$, $-k(1/2k)$, $2+k$ for $k<0$. So let us suppose that the level set $\mathbf{A_R}$ is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by (λ, μ), formulae (5), (6) and (7), at least for $\lambda \neq \mu$. If in addition λ and μ are far from a double root of $(u^2-k^2)\varphi(u)$ then the equations (8) imply that the Hamiltonian flows of H and $H+sF$ are linearly independent and hence dH and dF are linearly independent at such point.

Thus critical points of $F|_{Q_{h,k}}$ correspond only to (λ, μ) such that λ (or μ) is a double root of $(u^2-k^2)\varphi(u)$. This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of H on $Q_{h,k}$ has no stationary points and the critical set of F on $Q_{h,k}$ is invariant under the action of this flow.

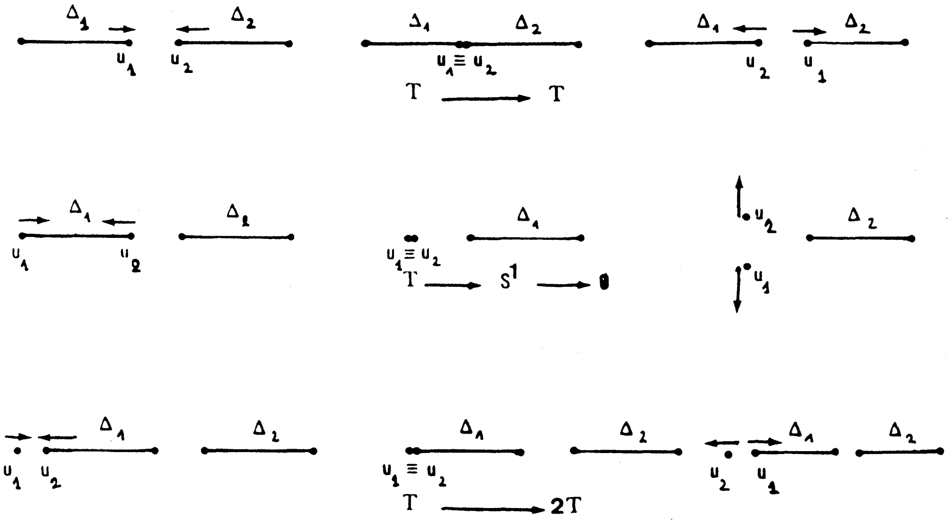

Fig. 5. – Correspondence between bifurcation of roots of the polynomial $(u^2-k^2)(u^3-hu^2+(1-k^2)u-f/2)$ and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of $F|_{Q_{h,k}}$ is non-degenerated of the normal planes to these circles. Let $\mu=\mu_0$ be a double root of $(u^2-k^2)\varphi(u)$. According to (7) we have

$$F|_{Q_{h,k}}=(\mu^2-k^2)p_\mu^2+2\mu^3-2\mu^2 h+2\mu(1-k^2)$$

and a critical circle of the level set $\{F|_{Q_{h,k}}\}=f$ is given by $\mu=\mu_0$, $p_\mu=0$. The normal directions to this circle are given by derivations with respect to μ and p_μ. We have

$$\frac{\partial^2 F}{\partial\mu\,\partial p_\mu}=\begin{pmatrix} 2(\mu_0^2-k^2) & 0 \\ 0 & -\varphi''(\mu_0)\end{pmatrix}$$

and as $\mu_0\neq\pm k$ then $\operatorname{rank}(d^2(F|_{Q_{h,k}}))\geqq 2$. On the other hand the Hessian $d^2(F|_{Q_{h,k}})$ is degenerated on tangent lines to the critical circle and hence $\operatorname{rank}(d^2(F|_{Q_{h,k}})=2$ which completes the proof of lemma 3. △

Proof of Theorem 3. – Let us fix a regular energy level set $Q_{h,k}$ with a Bott integral F on it, and let us consider the corresponding line $h=$const. on figure 1 (plane $k=$const., $h=$const. in the space $\mathbf{R}^3\{f, h, k\}$). As f vary the topological type of $A_{\mathbf{R}}=\{Q_{h,k}\}\cap\{F=f\}$ may change. Using Theorem 2 and the Fomenko's classification theorem we identify several possible bifurcations. For example passing from domain 3 (where $A_{\mathbf{R}} \sim 2T$) to domain 2 where ($A_{\mathbf{R}} \sim T$) on figure 1 we may have the following surgeries: $2T \to T$, or composition of $T \to T$ and $T \to \varnothing$. To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial $(u^2 - k^2)\varphi(u)$, and more specifically the four ends of the "admissible" ovals Δ_1 and Δ_2. The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial $(u^2 - k^2)\varphi(u)$ are easely decribed on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if (h, k) is fixed and belongs to one and the same connected component of the set

$$\mathscr{D} = \{ h \neq 2 \pm k \} \cap \left\{ h \neq \pm \left(k + \frac{1}{2k} \right) \right\},$$

then changing f the same bifurcations of roots of the polynomial $(u^2 - k^2)\varphi(u)$ take place. This implies that if (h, k) belongs to one and the same connected component of the set $\mathscr{D}$ the corresponding Fomenko's graph $\Gamma(Q_{h,k}, F)$ is the same and it is shown on figure 4. This completes the proof of theorem 3. △

DEFINITION. — The straight line $l \subset \mathbf{R}^3 \{f, h, k\}$ is generic provided that it intersects $\mathbf{B}$ transversally.

To prove Theorem 4 we note that instead of a generic smooth curve $l \subset \mathbf{R}^3 \{f, h, k\}$ it suffice to consider a generic straight line

$$\{ c_1 h + c_2 f + c_3 = 0,\ k = \text{const.} \} \subset \mathbf{R}^3 \{f, h, k\}.$$

Then Theorem 4 follows from the following

LEMMA 4. — *Let* $\{ c_1 h + c_2 f + c_3 = 0,\ k = \text{const.} \}$ *be a generic straight line in* $\mathbf{R}^3 \{f, h, k\}$. *Then* $\{ c_1 H + c_2 F + c_3 = 0 \} \subset \mathbf{R}^4 \{x, y, p_x, p_y\}$ *is a smooth surface, and* F *is a Bott integral on it.*

Indeed, instead of H we may take for a Hamiltonian of (1) the function $c_1 H + c_2 F$. The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (wich generalizes Lemma 2 and Lemma 3).

Let $k = k_0$ be fixed, $(f_0, h_0, k_0) \in \mathbf{B}$ be a generic point (*i.e.* in a neighbourhood of it $\mathbf{B}$ is a smooth manifold), and let $q = (x^0, y^0, p_x^0, p_y^0)$ be a point on the level set $\{ H = h_0, F = f_0 \}$. We shall prove that if

$$c_1 \operatorname{grad}(H)|_q + c_2 \operatorname{grad}(F)|_q = 0 \qquad (21)$$

then the straight line $\{ c_1 h + c_2 f + c_3 = 0 \}$ is tangent to $\mathbf{B}$ (and hence it is not generic). As the equation of a straight line tangent to $\mathbf{B}$ at the point (f_0, h_0, k_0) is given by

$$\{ u_0^3 - h u_0^2 + (1 - k^2) u_0 - f/2 = 0 \} \subset \mathbf{R}^2 \{f, h\}$$

where u_0 is the double root of the polynomial $P(u) = (u^2 - k^2)\varphi(u)$ then it is enough to prove that $c_1/c_2 = 2u_0^2$. In $(\lambda, \mu, p_\lambda, p_\mu)$ coordinates defined by (4), (5) we have the identity

(7)

$$F=p_\mu^2(\mu^2-k^2)+2\mu^3-2\mu^2 H+2\mu(1-k^2).$$

Then, at least far from the locus we have

(22) $$\{\lambda=\mu\}\cup\{(\lambda^2-k^2)(\mu^2-k^2)=0\}$$

(23) $$\left\{\begin{aligned}\frac{\partial F}{\partial \mu}&=2\mu p_\mu^2-\varphi'(\mu)-2\mu^2\frac{\partial H}{\partial \mu}\\ \frac{\partial F}{\partial p_\mu}&=2(\mu^2-k^2)p_\mu-2\mu^2\frac{\partial H}{\partial p_\mu}\\ \frac{\partial F}{\partial \lambda}&=-2\mu^2\frac{\partial H}{\partial \lambda}\\ \frac{\partial F}{\partial p_\lambda}&=-2\mu^2\frac{\partial H}{\partial p_\lambda}\end{aligned}\right.$$

As grad (H) and grad (F) are colinear according to (21), then

(24) $$\left\{\begin{aligned}&2(\mu^2-k^2)p_\mu=0\\ &2\mu p_\mu^2-\varphi'(\mu)=0\end{aligned}\right.$$

and hence $p_\mu=0$, $\varphi'(\mu)=0$. Now (6) implies that $\varphi(\mu)=\varphi'(\mu)=0$ and hence μ is a double root of the polynomial $(u^2-k^2)\varphi(\mu)$. Suppose now that λ^0, μ^0, p_λ^0, p_μ^0) belongs to the locus (22) and let $(\lambda, \mu, p_\lambda, p_\mu)$ tends to $(\lambda^0, \mu^0, p_\lambda^0. p_\mu^0)$. The vectors grad (H) and grad (F) tend to some vectors grad $(H)^0$ and grad $(F)^0$ and let us suppose that these vectors are colinear. Using (6), (23) and (24) we conclude that

$$(\mu^2-k^2)\varphi(\mu)\to 0 \qquad \text{and} \quad 2\mu\frac{\varphi(\mu)}{\mu^2-k^2}-\varphi'(\mu)\to 0$$

and hence μ^0 is a double root of the polynomial $(\mu^2-k^2)\varphi(\mu)$. The upshot is that if c_1 grad (H) $+c_2$ grad (F) $=0$ then $c_1/c_2=2\mu_0^2$, where μ_0 is the double root of the polynomial $(\mu^2-k^2)\varphi(\mu)$, and hence the straight line

$$\{c_1 h+c_2 f+c_3=0,\ k=\text{const.}\}$$

is tangent to **B**. This completes the proof of Lemma 4. △

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