Annales scientifiques de l'É.N.S.

H. Hida
J. Tilouine

Anti-cyclotomic Katz p-adic L-functions and congruence modules

Annales scientifiques de l'É.N.S. 4e série, tome 26, nº 2 (1993), p. 189-259
<http://www.numdam.org/item?id=ASENS_1993_4_26_2_189_0>

Ann. scient. Éc. Norm. Sup.,
4^e série, t. 26, 1993, p. 189 à 259.

ANTI-CYCLOTOMIC KATZ p-ADIC L-FUNCTIONS AND CONGRUENCE MODULES

BY H. HIDA [1] AND J. TILOUINE

ABSTRACT. — The purpose of this paper is to prove the divisibility of the characteristic power series of the congruence module of a Hida p-adic family of theta series coming from a CM-field (with fixed CM-type) by the anti-cyclotomic specialisation of the Katz p-adic L-function with auxiliary conductor. This requires to construct first this p-adic L-function since in the original paper by Katz the auxiliary conductor was trivial. The divisibility proven here is one of two steps towards one of the two divisibilities predicted by the (anti-cyclotomic) Iwasawa main conjecture for CM-fields. The second step has been carried out by the authors and will be published elsewhere.

0. Introduction

The purpose of this paper is to prove the divisibility of the characteristic power series $H=\chi(C_0)$ of the congruence module C_0 (of a CM-field M and its CM-type Σ) by the anti-cyclotomic projection L^- of the Katz p-adic L-function of arbitrary auxiliary conductor. In another article [HT2], generalizing an idea developed in [MT] by Mazur and one of the present authors, we will prove another divisibility result asserting that $\chi(C_0)$ divides the characteristic power series $\chi(\mathrm{I}\,w^-)$ of an appropriate Iwasawa module constructed out of the "half p-ramified" p-abelian extension of the anti-cyclotomic tower M^- of the CM-field M. Our method of the proof of the first divisibility is a (many variable) generalization of the method employed in [DH] by Doi and one of the present authors for the special values of these L-functions and in [T] in the one variable case. A summary of these two divisibility results can be found in [HT1]. For the sake of completeness, we included a treatment of the construction of the Katz p-adic L-functions with arbitrary conductor $\mathfrak{C}p^\infty$ for each CM-field M following Katz [K4] where such L-functions are constructed in the case of p-power level p^∞.

Throughout the paper, we use the notation introduced in [H1] without detailed explanation, since this paper is in some sense a continuation of the work done in [H1]. Let F be a totally real number field and M be a totally imaginary quadratic extension of F (hereafter such fields will be called CM fields). We write D_F for the discriminant of F. We write $\mathfrak{R}$ (resp. $\mathfrak{r}$) the integer ring of M (resp. F). We fix a

[1] Supported in part by an N.S.F. grant.

prime p, the algebraic closures $\bar{\mathbf{Q}}$ of $\mathbf{Q}$, $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and the following two embeddings throughout the paper:

$$\iota_\infty\colon\ \bar{\mathbf{Q}} \to \mathbf{C} \qquad \text{and} \qquad \iota_p\colon\ \bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p.$$

We suppose throughout the paper the following ordinarity hypothesis for M and p:

(0.1) *Every prime factor of p in* F *splits in* M.

Then, writing c both for complex conjugation on $\mathbf{C}$ and on $\bar{\mathbf{Q}}$ induced under ι_∞, we choose a set of embeddings Σ of M into $\bar{\mathbf{Q}}$ such that

(0.2 *a*) $\Sigma \cap \Sigma c = \varnothing$, *and* $\Sigma \cup \Sigma c$ *is the set of all embeddings of* M *into* $\bar{\mathbf{Q}}$;

(0.2 *b*) *the p-adic place induced by each element of* Σ *composed with* ι_p *is distinct from any of those induced by elements in* Σc.

The set Σ satisfying (0.2 *a*, *b*) is called a p-adic CM-type. Under the ordinarity hypothesis, we can find a p-adic CM-type, and we fix one such Σ. By abusing the symbol, sometimes we understand Σ as a set of places at p (and hence, a set of prime ideals over p) induced by the embeddings in Σ composed with ι_p.

We now describe the power series L^-. Let $\lambda\colon \mathrm{M}_{\mathbf{A}}^\times/\mathrm{M}^\times \to \mathbf{C}$ be a Hecke character such that

$$\lambda(x_\infty) = \prod_{\sigma\in\Sigma} x_\sigma^{(m_0+d_\sigma(1-c))\sigma} \tag{0.3}$$

where m_0 and d_σ are integers and as usual $\mathrm{M}_{\mathbf{A}}^\times$ is the idele group of M and x_∞ denotes the infinity part of $x \in \mathrm{M}_{\mathbf{A}}^\times$. Then λ has values in $\bar{\mathbf{Q}}$ on the finite part $\mathrm{M}_{\mathbf{A}_f}^\times$ of $\mathrm{M}_{\mathbf{A}}^\times$. Moreover, the map $\hat{\lambda}\colon \mathrm{M}_{\mathbf{A}_f}^\times/\mathrm{M}^\times \to \bar{\mathbf{Q}}_p$ defined by $\hat{\lambda}(x) = \lambda(x) \prod_{\sigma\in\Sigma} x_p^{(m_0+d_\sigma(1-c))\sigma}$ is a well defined continuous character, which is called the p-adic avatar of λ. It is a well known theorem of Shimura (*see* (Sh1] and sections 1 and 4,5 in the text) that the special value $L(0, \lambda)$ of the primitive Hecke L-function $L(s, \lambda)$ is algebraic up to a canonical complex period if λ is Σ-critical (*i.e.* (m_0, d) satisfies the condition in Theorem II, (ii) below). Let $\mathrm{C}(\lambda)$ be the conductor of λ, and write $\mathfrak{C}$ for the prime-to-p-part of $\mathrm{C}(\lambda)$. Then by class field theory, we can regard $\hat{\lambda}$ as a character of the Galois group $\mathrm{G}_\infty(\mathfrak{C})$ of the maximal ray class field modulo $\mathfrak{C}p^\infty$ over M. Then we have a Katz measure μ on $\mathrm{G}_\infty(\mathfrak{C})$ satisfying, for an explicit constant $\mathrm{A}(\lambda)$,

$$\frac{\displaystyle\int \hat{\lambda}\, d\mu}{p\text{-adic period}} = \mathrm{A}(\lambda) \frac{L(0, \lambda)}{\text{complex period}}$$

whenever λ is critical and of conductor divisible by $\mathfrak{C}$ (*see* below Theorem II for details). In particular, the p-adic period is contained in the p-adic completion $\mathfrak{O}$ of the integer ring of the maximal unramified extension of $\mathbf{Q}_p$ inside Ω. Then, the measure μ is defined over $\mathfrak{O}$, and thus we may regard μ as an element of the continuous group algebra $\mathfrak{O}[[\mathrm{G}_\infty(\mathfrak{C})]]$. We now write $\mathrm{G}_{\mathrm{tor}}(\mathfrak{C})$ for the maximal torsion subgroup of $\mathrm{G}_\infty(\mathfrak{C})$

and put $\mathbf{W}=G_\infty(\mathfrak{C})/G_{tor}(\mathfrak{C})$, $\Lambda_0=\mathfrak{O}[[\mathbf{W}]]=\varprojlim_\alpha \mathfrak{O}[\mathbf{W}/\mathbf{W}^{p^\alpha}]$. Then $\mathbf{W}$ is determined canonically independent of $\mathfrak{C}$, and hence the complex conjugation c acts naturally on $\mathbf{W}$ via $w^c=cwc^{-1}$. Similarly, regarding λ as a character of $M_{\mathbf{A}}^\times$, we can let c acts on λ, *i.e.*, $\lambda\circ c(x)=\lambda(x^c)$. We need to fix a pair of characters (ψ, $\psi^-=\psi^{-1}(\psi\circ c)$) of $G_{tor}(\mathfrak{C})$ and $G_{tor}(\mathfrak{C}^-)$, where $\mathfrak{C}^-$ denotes the conductor of ψ^- [as a character of $G_\infty(\mathfrak{C})/\mathbf{W}$]. Replacing $\mathfrak{O}$ by its finite extension, we may assume that ψ has values in $\mathfrak{O}$. Then choosing compatible decompositions

$$G_\infty(1)=G_{tor}(1)\times\mathbf{W},\qquad G_\infty(\mathfrak{C})=G_{tor}(\mathfrak{C})\times\mathbf{W}\qquad\text{and}\qquad G_\infty(\mathfrak{C}^-)=G_{tor}(\mathfrak{C}^-)\times\mathbf{W},$$

we define a projection

$$(0.4)\qquad \pi_\psi^-:\ \mathfrak{O}[[G_\infty(\mathfrak{C}^-)]]\to\mathfrak{O}[[\mathbf{W}]]=\Lambda_0\quad\text{by}\quad \pi_\psi^-(\zeta, w)=\psi^-(\zeta)\,w^{-1}w^c$$

for $(\zeta, w)\in G_{tor}(\mathfrak{C}^-)\times\mathbf{W}$. Then we define

$$(0.5)\qquad L^-=\pi_\psi^-(\mu).$$

We assume that ψ is primitive of conductor $\mathfrak{C}$.

Let $N=\mathfrak{N}_{M/F}(\mathfrak{C})D$ for the relative discriminant D of M/F, and write $\mathsf{h}^{n.ord}(N;\mathfrak{O})$ for the p-adic nearly ordinary Hecke algebra of level N defined in [H1]. With the character ψ, we then associate a canonical algebra homomorphism: $\mathsf{h}^{n.ord}(N;\mathfrak{O})\to\Lambda_0$ (*see* § 6) to which we attach the congruence module $C_0=C_0(\psi)$ (*see* [H1, (5.2)] and (6.9*b*) in the text). The congruence module defined in [HT1, (H5)] might be a bit bigger than the one we use here. The possible difference of their characteristic power series is only a fractional power of p, and hence, this change does not affect to [HT1, Theorems 2.1 and 4.1] if $\#(G_{tor}(1))$ is prime to p; otherwise, we need to use the definition given in [H1, (5.2)]. We write H for the characteristic power series of C_0 in Λ_0. Let E be the set of primes $\mathfrak{q}$ of F in $F\cap\mathfrak{C}+\mathfrak{C}^c$, and let $\Delta(s)$ be the product of Euler factors at primes in E for the product of primitive L-functions $L(s,\chi)L(s,\psi^-)$ for the quadratic character χ of $F_{\mathbf{A}}^\times$ corresponding to M/F; *i.e.* writing the Euler product

$$L(s,\chi)\,L(s,\psi^-)=\prod_{\mathfrak{q}} L_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s})^{-1},$$

we have

$$(0.6a)\qquad \Delta(s)=\prod_{\mathfrak{q}\in E} L_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s}).$$

Then $\Delta(1)\neq 0$. We put

$$(0.6b)\qquad \Delta(M/F;\mathfrak{C})=\Delta(1)\,h(M)/h(F),$$

where $h(M)$ [resp. $h(F)$] is the class number of M (resp. F). The excluded Euler factors $\Delta(1)$ is in fact trivial if

$$(0.7a)\qquad \mathfrak{C}+\mathfrak{C}^c=\mathfrak{R}$$

and is a product of $(1 \pm \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-1})$ if the following condition is satisfied:
(0.7 *b*) *All prime factors in* $\mathfrak{C}$ *outside* $\mathfrak{C}^-$ *are inert or ramified over* F.
Then we have

THEOREM I. — *Suppose that* $p>2$. *In* $\mathfrak{O}[[\mathbf{W}]] \otimes_{\mathbf{Z}} \mathbf{Q}$, L^- *divides* H. *Moreover, if there exist a conductor* $\mathfrak{C}'$ *(prime to p) and a character* $\varphi: G_{tor}(\mathfrak{C}') \to \bar{\mathbf{Q}}^\times$ *such that the* μ-*invariant of the branches of the Katz measure corresponding to* $\psi^{-1}\varphi$ *and* $\psi^{-1}(\varphi \circ c)$ *vanishes simultaneously, then* $\Delta(\mathrm{M}/\mathrm{F}; \mathfrak{C}) L^-$ *divides* H *in* $\mathfrak{O}[[\mathbf{W}]]$.

This theorem will be proven section 8. The proof is based on the comparison formula (8.5 *a*, *b*) between the Katz *p*-adic *L*-functions and the *p*-adic Rankin product (constructed in [H1]) of two Λ-adic theta series ($\Lambda = \mathfrak{O}[[\mathbf{W}]]$) with complex multiplication by M. Naturally, we shall make the following

CONJECTURE. — $\mathrm{H} = \Delta(\mathrm{M}/\mathrm{F}; \mathfrak{C}) L^-$ *up to unit factors in* $\mathfrak{O}[[\mathbf{W}]]$ *if* $p>2$.

This conjecture is known to be true when M is an imaginary quadratic field under a certain additional hypothesis (*see* [MT], [T], [HT1 and 2]).

We now explain the interpolation property of the Katz measure more precisely. We associate to λ its dual λ^* given by $\lambda^*(x) = \lambda(x^c)^{-1} |x|_{\mathbf{A}}$. Then the *p*-adic avatar of λ^* is given by $\hat{\lambda}^*(x) = \hat{\lambda}(x^c)^{-1} \mathfrak{N}(x)^{-1}$ for the cyclotomic character $\mathfrak{N}: G_\infty(1) \to \mathbf{Z}_p^\times$. We fix a finite idele d_{M} of M such that the ideal corresponding to d_{M} is the different ϑ_{M} of M/**Q**. We define the local Gauss sum of λ at prime ideals $\mathfrak{Q}$ dividing the conductor of λ:

$$(0.8) \qquad G(d_{\mathrm{M}}, \lambda_{\mathfrak{Q}}) = \lambda(\varpi_{\mathfrak{Q}}^{-e}) \sum_{u \in (\mathfrak{R}_{\mathfrak{Q}}/\mathfrak{Q}^e)^\times} \lambda_{\mathfrak{Q}}(u)\, \mathbf{e}_{\mathrm{M}}(\varpi_{\mathfrak{Q}}^{-e} d_{\mathfrak{Q}}^{-1} u),$$

where ϖ is a prime element of the $\mathfrak{Q}$-adic completion $\mathrm{M}_{\mathfrak{Q}}$, $\mathfrak{R}_{\mathfrak{Q}}$ is the $\mathfrak{Q}$-adic integer ring $\mathfrak{R}$ of $\mathrm{M}_{\mathfrak{Q}}$, $\lambda_{\mathfrak{Q}}$ is the restriction of λ to $\mathrm{M}_{\mathfrak{Q}}^\times$, $e = e(\mathfrak{Q})$ is the exponent of $\mathfrak{Q}$ in the conductor of λ and $\mathbf{e}_{\mathrm{M}}: \mathrm{M}_{\mathbf{A}}/\mathrm{M} \to \mathbf{C}^\times$ is the standard additive character normalized as $\mathbf{e}_{\mathrm{M}}(x_\infty) = \exp(2\pi i \mathrm{Tr}_{\mathrm{M}/\mathfrak{Q}}(x_\infty))$. Outside the conductor of λ, we simply put $G(d_{\mathrm{M}}, \lambda_{\mathfrak{Q}}) = 1$. The canonical complex period Ω_∞ is in fact an element of $\mathrm{F} \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^\Sigma$, and the *p*-adic period is found in $(\mathfrak{O}^\times)^\Sigma$ [*see* (4.4 *a*, *b*]. Actually, these numbers are well defined modulo $\bar{\mathbf{Q}}^\times$ but the ratio "Ω_∞/Ω_p" is uniquely determined (*i.e.* If Ω_∞ is changed by an algebraic factor, Ω_p is also changed by the same factor). We choose an element $\delta \in \mathrm{M}$ such that
(0.9 *a*) $\delta^c = -\delta$ *and* $\iota_\infty(\mathrm{Im}(\delta^\sigma)) > 0$ *for all* $\sigma \in \Sigma$,
(0.9 *b*) *The alternating form* $\langle x, y \rangle = \mathrm{Tr}_{\mathrm{M}/\mathrm{F}}(xy^c/2\delta)$ *induces an isomorphism* $\mathfrak{R} \wedge \mathfrak{R} \cong \vartheta^{-1} \mathfrak{c}^{-1}$ *for an ideal* $\mathfrak{c}$ *prime to* $p\mathfrak{C}\mathfrak{C}^c$,
where ϑ is the different of F/**Q**. By (0.9 *b*), we can take 2δ or $(2\delta)^c$ as the $\mathfrak{Q}$-component of d_{M} if $\mathfrak{Q}$ is prime to $\mathfrak{c}$. Then we define ε-factors:

$$(0.10) \qquad \mathrm{W}_p(\lambda) = \Big\{ \prod_{\mathfrak{p} \in \Sigma} \mathrm{N}(\mathfrak{P}^{-e(\mathfrak{P})}) G(2\delta; \lambda_{\mathfrak{P}}) \Big\},$$

$$\mathrm{W}'(\lambda) = \prod_{\mathfrak{L} \mid \mathfrak{F}} G((2\delta)^c, \lambda_{\mathfrak{L}}^{-1}) \prod_{\mathfrak{L} \mid \mathfrak{F}_c} G(2\delta, \lambda_{\mathfrak{L}^c}^{-1}) \prod_{\mathrm{I} \mid \mathrm{I}} G((2\delta)^c, \lambda_{\mathrm{I}}^{-1}),$$

where $\mathfrak{L}$ denotes prime ideals in M and we decomposed $\mathfrak{C}=\mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ so that $\mathfrak{F}\mathfrak{F}_c$ consists of split primes over F, $\mathfrak{I}$ consists of inert or ramified primes over F, $\mathfrak{F}+\mathfrak{F}_c=\mathfrak{R}$ and $\mathfrak{F}_c^c \supset \mathfrak{F}$. Then we have

THEOREM II. — *There exists a unique measure* μ *on the ray class group* $G_\infty(\mathfrak{C})$ *modulo* $\mathfrak{C}p^\infty$ *of* M *having values in* $\mathfrak{O}$ *satisfying*

$$\frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu}{\Omega_p^{m_0\Sigma+2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times)\, W_p(\lambda) \frac{(-1)^{m_0\Sigma}\pi^d \Gamma_\Sigma(m_0\Sigma+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times \prod_{\mathfrak{L}\mid\mathfrak{C}} (1-\lambda(\mathfrak{L}))\Big\{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda^*(\mathfrak{P}^c))\Big\} L(0,\lambda)$$

for all Hecke characters λ *modulo* $\mathfrak{C}p^\infty$ *such that*

(i) *the conductor of* λ *is divisible by all prime factors of* $\mathfrak{F}$,

(ii) *the infinity type of* λ *is* $m_0\Sigma+d(1-c)$ *for an integer* m_0 *and* $d=\sum_{\sigma\in\Sigma} d_\sigma\sigma$ *with integers* d_σ *satisfying either* $m_0>0$ *and* $d_\sigma \geqq 0$ *or* $m_0 \leqq 1$ *and* $d_\sigma \geqq 1-m_0$.

Moreover denoting the measure μ *for* $G_\infty(\mathfrak{C}^c)$ *by* μ_c, *we have the following functional equation*

$$\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu = \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1}\lambda(\mathfrak{c}^{-1})\,W'(\lambda)\int_{G_\infty(\mathfrak{C}^c)} \hat{\lambda}^*\, d\mu_c$$

as long as the conductor of λ *is divisible by all prime factors of* $\mathfrak{F}\mathfrak{I}$.

In the expression of the theorem, we used the convention for an element ξ of the formal free module generated by $\Sigma \cup \Sigma c$ and for $x \in \mathbf{C}^\Sigma$:

$$x^\xi = \prod_{\sigma\in\Sigma} x_\sigma^{\xi_\sigma} \prod_{\sigma\in\Sigma} x_\sigma^{c\,\xi_{\sigma c}} \quad \text{and} \quad \Gamma_\Sigma(\xi) = \prod_{\sigma\in\Sigma}\Gamma(\xi_\sigma).$$

The set Σ is also identified with the formal sum $\sum_{\sigma\in\Sigma}\sigma$, and $a\in M$ is considered to be an element of $\mathbf{C}^\Sigma$ via diagonal embedding $a \mapsto (a^\sigma)_{\sigma\in\Sigma}$. Abusing this convention, π is considered to be the diagonal element $(\pi)_{\sigma\in\Sigma}$ in $\mathbf{C}^\Sigma$. The L-functions in the theorem is always the primitive one associated with the primitive Hecke character. We also tacitly agree to put $\lambda(\mathfrak{Q})=0$ if $\mathfrak{Q}$ divides the conductor of λ.

Here is a summary of the paper: After giving a brief review in section 1 of all the necessary items from the theory of p-adic modular forms, we start the construction of Eisenstein measure in section 2 and finishes the construction in section 3. The content of sections 1 and 6-8 was actually presented in a series of seminars held at the Université de Paris-Sud in Winter 1989. We will prove Theorem II by mimicking Katz's method [K4] (*i.e.* by specializing the Eisenstein measure at CM-type abelian varieties) in the subsequent sections 4 and 5. The reader who is willing to admit the interpolation property of the Katz measure presented in Theorem II can skip all the sections from section 1 to 5 and go directly to section 6, in which we start the preparation of the proof of Theorem I. Namely, in section 6, we construct an irreducible component attached to the CM field M of the spectrum of the p-adic nearly ordinary Hecke algebra and define its congruence module C_0. In section 7, we give a formula relating the self Petersson inner product of a primitive cusp form with a special value of its symmetric square L-function. In the final section (§ 8), we prove the main theorem by comparing the Katz p-adic L-function with the p-adic Rankin product associated with the irreducible component constructed in section 6. One of the keys in proving this comparison theorem (Theorem 8.1), which generalizes the one for $F=\mathbf{Q}$ in [T], § 7, is the formula obtained in section 7. The origin of the idea of such comparison goes back to an unpublished paper [DH] of Doi and one of the present authors, where the comparison of special values of Hecke L-functions and a Rankin product of theta series of M was carried out to prove a version of the congruence criterion by those L-values in [H4].

The first named author is grateful to R. Gillard for pointing out some mistakes related to the level structures in the definition of Eisenstein series. We are also grateful to the participants of the seminars mentioned above for their patience towards not so well organized presentation of the material in this paper, [K4] and [HT1 and 2].

NOTATION. — We summarize here some notation we will use. For any number field X, we write I_X (resp. D_X) for the set of embeddings of X into $\bar{\mathbf{Q}}$ (resp. the discriminant (in $\mathbf{Z}$) of X/$\mathbf{Q}$). We write $\mathbf{Z}[I_X]$ for the free module generated by I_X. The formal sum $\sum_{\sigma \in I_X} \sigma$ will be written as t_X. Especially, we write I (resp. t) for I_F (resp. t_F). The integer ring of F (resp. M) is denoted by $\mathfrak{r}$ (resp. $\mathfrak{R}$). We denote by $F_{\mathbf{A}}$ (resp. A) the adele ring of F (resp. $\mathbf{Q}$). We write $F_{\mathbf{A}_f}$ (resp. $\mathbf{A}_f$) for the finite part of $F_{\mathbf{A}}$ (resp. $\mathbf{A}$). Similarly F_∞ denotes the infinite part of $F_{\mathbf{A}}$. Any element $x \in F_{\mathbf{A}}$ (resp. $x \in F_{\mathbf{A}}^\times$) is a sum $x_f + x_\infty$ (resp. *a* product $x_f x_\infty$) for $x_f \in F_{\mathbf{A}_f}$ and $x_\infty \in F_\infty$. For any $x \in F_{\mathbf{A}}$ and a prime ideal $\mathfrak{q}$ of $\mathfrak{r}$, $x_\mathfrak{q}$ is the σ-component of x. For infinite place $\sigma \in I$, we write x_σ for the σ-component of $x \in F_{\mathbf{A}}$. Then we denote by $\mathbf{e}_F$: $F_{\mathbf{A}}/F \to \mathbf{C}^\times$ the standard additive character such that $\mathbf{e}_F(x_\infty) = \exp(2\pi i \sum_\sigma x_\sigma)$. Abusing a little this notation, for any element x and any subset X of $F_{\mathbf{A}}$ or $F_{\mathbf{A}}^\times$ and an ideal N (resp. a set N of places) of $\mathfrak{r}$, we write x_N and X_N for the projection of x and X to $\prod_{\mathfrak{q} \mid N} F_\mathfrak{q}$ (resp. $\prod_{\mathfrak{q} \in N} F_\mathfrak{q}$), where $F_\mathfrak{q}$ is the $\mathfrak{q}$-adic completion of F. Especially $\mathfrak{R}_\Sigma = \prod_{\mathfrak{p} \in \Sigma} \mathfrak{R}_\mathfrak{p}$. We also write $\hat{\mathfrak{r}}$ (resp. $\hat{\mathfrak{R}}$ for the product $\prod_\mathfrak{q} \mathfrak{r}_\mathfrak{q}$ (resp. $\prod_\mathfrak{q} \mathfrak{R}_\mathfrak{q}$) of the $\mathfrak{q}$-adic completions $\mathfrak{r}_\mathfrak{q}$ (resp. $\mathfrak{R}_\mathfrak{q}$) over all prime ideals $\mathfrak{q}$. We denote by $F_{\infty+}^\times$ the connected component of F_∞ with identity. We also write $F_{\mathbf{A}+}^\times$ for $F_{\mathbf{A}_f}^\times F_{\infty+}^\times$. We always write $\mathfrak{N}$: $G_\infty(1)$ (or $M_{\mathbf{A}}^\times$) $\to \mathbf{Z}_p^\times$ for the cyclotomic character. On the other hand, for any number field X, $\mathfrak{N}_{X/\mathbf{Q}}(\mathfrak{a}) \in \mathbf{Z}$ denotes the absolute norm of an ideal $\mathfrak{a}$ in X. We sometimes write simply $N(\mathfrak{a})$ for $\mathfrak{N}_{X/\mathbf{Q}}(\mathfrak{a})$ if X is clear from the context. We use the notation introduced in [H1], [H2] and [H3] throughout the paper with only brief explanation.

Content

1. Summary of Katz's theory of p-adic modular forms

1.1. HILBERT-BLUMENTHAL MODULI SPACE. — We start with a description of the moduli space of Hilbert Blumenthal abelian varieties (HBAV). Let F be a totally real number field with the integer ring $\mathfrak{r}$ and with absolute different ϑ. We consider a Hilbert Blumenthal abelian scheme X/S. By definition, X is a proper smooth group scheme (geometrically connected) over a base scheme S with an isomorphism $\theta: \mathfrak{r} \to \mathrm{End}_S(X)$ such that the sheaf $\mathfrak{Lie}(X/S)$ of its Lie algebras on S is free of rank 1 over $\mathfrak{O}_S \otimes_{\mathbf{Z}} \mathfrak{r}$. Let $X^t = \mathrm{Pic}^0(X/S)$ which is naturally a HBAV. We fix a polarization $\lambda: X^t \cong X \otimes_{\mathfrak{r}} \mathfrak{c}$ for a fractional ideal $\mathfrak{c}$ of $\mathfrak{r}$. Then λ induces an isomorphism:

$$\mathrm{Hom}_{\mathfrak{r},\,\mathrm{sym}}(X, X^t)_+ \cong \mathfrak{c}_+.$$

where $\mathfrak{c}_+$ is the set of all totally positive elements in $\mathfrak{c}$ and $\mathrm{Hom}_{\mathfrak{r},\,\mathrm{sym}}(X, X^t)_+$ is the set of symmetric morphisms induced by ample line bundles. Let N be an ideal of $\mathfrak{r}$ prime to p and take an integer N_0 prime to p in N. We consider the level structure which is an $\mathfrak{r}$-linear closed immersion:

$$i:\quad (\vartheta^{-1}/Np^{\alpha}\vartheta^{-1}) \otimes_{\mathbf{Z}} \mu_{N_0 p^{\alpha}} \to X,$$

where for each positive M, μ_M is the kernel of multiplication by the integer M on $\mathbf{G}_m$ as a finite flat group scheme over $\mathbf{Z}$. Such a triple (X, λ, i) is called a test object. We consider the functor $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; Np^{\alpha})_{/S}$: $\mathfrak{Sch}_{/S} \to \mathfrak{Ens}$ which associates for each $T_{/S}$ the set of isomorphism classes over T of the test objects $(X, \lambda, i)_{/T}$. If $p^{\alpha} \geqq 4$ or $M\mathfrak{r} \supset N$ for an integer $M \geqq 4$ (which will be always supposed tacitly), $\mathfrak{M}$ can be represented by an algebraic space (*see* [R, 1.20, 1.22, 6.16]) which is smooth and of relative dimension $[F: \mathbf{Q}]$ over $\mathbf{Z}$. Since $\mathfrak{M}(\mathfrak{c}; Np^{\alpha})$ is a geometric quotient of $\mathfrak{M}(\mathfrak{c}; Np^{\beta})$ $(\beta > \alpha)$ by a free action of a finite group [DR, p. 255], we know (e.g. [Kn, IV.1])

(1.1) *The canonical morphism:* $\mathfrak{M}(\mathfrak{c}; Np^{\beta}) \to \mathfrak{M}(\mathfrak{c}; Np^{\alpha})$ $(\beta \geqq \alpha)$ *is affine and formally etale.*

By (1.1), the limit $\mathfrak{M}(\mathfrak{c};\, Np^\infty) = \varprojlim_\alpha \mathfrak{M}(\mathfrak{c};\, Np^\alpha)$ exists in the category of algebraic spaces. We write $\mathfrak{X}(\mathfrak{c};\, Np^\alpha)/\mathfrak{M}(\mathfrak{c};\, Np^\alpha)$ for the universal HBAV over $\mathfrak{M}(\mathfrak{c};\, Np^\alpha)$.

1.2. THE GAUSS-MANIN CONNECTION. — Let $\alpha: X \to S$ be an abelian scheme over an algebraic space S which is smooth and of relative dimension g over $\mathbf{Z}$. We consider the sheaf H^1_{DR} of the hyper-cohomology $\mathbb{R}^1 \alpha_* \Omega^\cdot_{X/S}$ under the etale topology. Since X/S is an abelian scheme, we have the Hodge filtration which is given in the form of exact sequence as:

$$0 \to \alpha_* \Omega^1_{X/S} \to \mathbb{R}^1 \alpha_* \Omega^\cdot_{X/S} \to R^1 \alpha_* \mathfrak{O}_X \to 0. \tag{1.2}$$

We repeat here briefly the construction of the Gauss-Manin connection ∇: $H^1_{DR} \to \Omega^1_{S/\mathbf{Z}} \otimes_{\mathfrak{O}_S} H^1_{DR}$ done in Katz-Oda [KO] and [K2, 3.2]. Since X/S is smooth, we have an exact sequence:

$$0 \to \alpha^* \Omega^1_{S/\mathbf{Z}} \to \Omega^1_{X/\mathbf{Z}} \to \Omega^1_{X/S} \to 0. \tag{1.3}$$

Then we have a finite filtration: $\Omega^\cdot_{X/\mathbf{Z}} = F^0 \Omega^\cdot_{X/\mathbf{Z}} \supset F^1 \Omega^\cdot_{X/\mathbf{Z}} \supset \ldots \supset \{0\}$ given by $F^i = \operatorname{Im}(\Omega^{\cdot - i}_{X/\mathbf{Z}} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^i_{S/\mathbf{Z}} \to \Omega^\cdot_{X/\mathbf{Z}}) \cong \Omega^{\cdot - i}_{X/\mathbf{Z}} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^i_{S/\mathbf{Z}}$. By (1.3), we know $F^p/F^{p+1} \cong \Omega^{\cdot - p}_{X/S} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^p_{S/\mathbf{Z}}$ and find two exact sequences of complexes:

$$(*) \qquad 0 \to \Omega^{\cdot - 1}_{X/S} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^1_{S/\mathbf{Z}} \to F^0/F^2 \to \Omega^\cdot_{X/S} \to 0,$$

$$(**) \qquad 0 \to \Omega^{\cdot - 2}_{X/S} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^2_{S/\mathbf{Z}} \to F^1/F^3 \to \Omega^{\cdot - 1}_{X/S} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^1_{S/\mathbf{Z}} \to 0.$$

The connecting map of the long exact sequence of cohomology induces

$$\nabla: \quad H^1_{DR} = \mathbb{R}^1 \alpha_* \Omega^\cdot_{X/S} \to \mathbb{R}^2 \alpha_* (\Omega^{\cdot - 1}_{X/S} \otimes_{\mathfrak{O}_X} \alpha^* \Omega^1_{S/\mathbf{Z}}) = H^1_{DR} \otimes_{\mathfrak{O}_S} \Omega^1_{S/\mathbf{Z}}.$$

$$\nabla_1: \quad H^1_{DR} \otimes_{\mathfrak{O}_S} \Omega^1_{S/\mathbf{Z}} \to H^1_{DR} \otimes_{\mathfrak{O}_S} \Omega^2_{S/\mathbf{Z}}.$$

We have the exterior product $F^{i+j} \supset F^i \wedge F^j$, which is compatible with $(*)$ *and* $(**)$. This shows that ∇ and ∇_1 are connections; namely, for $e \in H^1_{DR}$

$$\nabla(fe) = df \otimes e + f\nabla(e) \qquad \text{et} \qquad \nabla_1(\omega \otimes e) = d\omega \otimes e - \omega \wedge \nabla(e).$$

By construction $\nabla_1 \circ \nabla = 0$ and therefore ∇ is integrable. This connection ∇ is called the Gauss-Manin connection. For each derivation $D \in T_{S/\mathbf{Z}}$, we can define $\nabla(D) \in \operatorname{End}(H^1_{DR})$ by

$$\nabla(D)(e) = \mathrm{id} \otimes D(\nabla(e)) \qquad \text{for} \quad D \in T_{S/\mathbf{Z}}.$$

Then we have the Kodaira-Spencer map:

$$K-S: \quad T_{S/\mathbf{Z}} \to \operatorname{Hom}_{\mathfrak{O}_S}(\omega, \mathfrak{Lie}(X^t/S)), \tag{1.4}$$

where $\underline{\omega} = \underline{\omega}\, x/s = \alpha_* H^0(\Omega^1 x/s)$, $\mathfrak{Lie}(X^t/S) = R^1 \alpha^* \mathfrak{O}_{X/S}$ and

$$K-S(D): \quad \underline{\omega} \to H^1_{DR} \xrightarrow{\nabla(D)} H^1_{DR} \to \mathfrak{Lie}(X^t/S) \quad \text{by (1.2)}.$$

If κ is an algebraically closed field of characteristic $p>0$ and if $A_{/\kappa}$ is an ordinary abelian variety over κ, we can calculate explicitly $K-S$ on the formal deformation space $A_{/\kappa}$ over the ring of Witt vectors $W(\kappa)$ with coefficients in κ (*see* [K3]). This shows that
(1.5) $K-S$ *is an isomorphism if* $X/S=\mathfrak{X}(\mathfrak{c}; Np^{\alpha})/\mathfrak{M}(\mathfrak{c}; Np^{\alpha})$.

1.3. Geometric definition of modular forms. – Write $\mathfrak{X}/\mathfrak{M}$ for $\mathfrak{X}(\mathfrak{c}; Np^{\alpha})/\mathfrak{M}(\mathfrak{c}; Np^{\alpha})$. Let I be the set of all embeddings of F into $\bar{\mathbf{Q}}$. Let Φ be the Galois closure in $\bar{\mathbf{Q}}$ of F. We write $\mathfrak{B}$ for the valuation ring in Φ associated to the fixed embedding $\iota_p: \bar{\mathbf{Q}}\to\bar{\mathbf{Q}}_p$. We always suppose that the ideal of polarization $\mathfrak{c}$ is prime to p. Since $\lambda: \mathfrak{X}^t\cong\mathfrak{X}\otimes_{\mathfrak{r}}\mathfrak{c}$, we have an isomorphism

$$(1.6)\qquad \mathfrak{Lie}(\mathfrak{X}^t/\mathfrak{M})_{/\mathfrak{B}}\cong\mathfrak{Lie}(\mathfrak{X}/\mathfrak{M})_{/\mathfrak{B}}.$$

Hereafter, we write $\mathfrak{Lie}=\mathfrak{Lie}(\mathfrak{X}/\mathfrak{M})$. Let $\underline{\omega}=\alpha_*\Omega^1_{\mathfrak{X}/\mathfrak{M}}$. By definition,

$$(1.7a)\qquad \mathfrak{Lie}\cong\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{M}}}(\underline{\omega}, \mathfrak{O}_{\mathfrak{M}}).$$

Since $\mathfrak{Lie}$ and $\underline{\omega}$ are locally free of rank 1 over $\mathfrak{O}_{\mathfrak{M}}\otimes_{\mathbf{Z}}\mathfrak{r}$, we know that for $R=\mathfrak{r}\otimes_{\mathbf{Z}}\mathfrak{O}_{\mathfrak{M}}$

$$\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{M}}}(\underline{\omega}, \mathfrak{O}_{\mathfrak{M}})\cong\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{M}}}(\underline{\omega}\otimes_R R, \mathfrak{O}_{\mathfrak{M}})\cong\mathrm{Hom}_R(\underline{\omega}, \mathfrak{O}_{\mathfrak{M}}\otimes_{\mathbf{Z}}\vartheta^{-1}).$$

Write M^* for $\mathrm{Hom}_R(M, R)$. Then (1.7*a*) is equivalent to, under $\mathrm{Tr}: \vartheta^{-1}\to\mathbf{Z}$,

$$(1.7b)\qquad \mathfrak{Lie}\otimes_R\underline{\omega}\cong\mathfrak{O}_{\mathfrak{M}}\otimes_{\mathbf{Z}}\vartheta^{-1}\ (\Leftrightarrow\underline{\omega}^*\cong\mathfrak{Lie}\otimes_R(\mathfrak{O}_{\mathfrak{M}}\otimes_{\mathbf{Z}}\vartheta^{-1})^*=\mathfrak{Lie}\otimes_{\mathfrak{r}}\vartheta).$$

Then the Kodaira-Spencer map induces

$$T_{\mathfrak{M}/\mathbf{Z}}\overset{(1.4)}{\cong}\mathrm{Hom}_{\mathfrak{O}_{\mathfrak{M}/\mathfrak{B}}\otimes\mathfrak{r}}(\underline{\omega}, \mathfrak{Lie})\cong\underline{\omega}^*\otimes_R\mathfrak{Lie}\overset{(1.7b)}{\cong}\mathfrak{Lie}\otimes_R\mathfrak{Lie}\otimes_{\mathfrak{r}}\vartheta.$$

By the duality over $\mathfrak{O}_{\mathfrak{M}/\mathfrak{B}}$, we have

$$(1.18)\qquad \Omega^1_{\mathfrak{M}/\mathfrak{B}}\cong\underline{\omega}\otimes_R\underline{\omega}.$$

Let $T=\mathrm{Res}_{\mathfrak{r}/\mathbf{Z}}(\mathbf{G}_{m/\mathfrak{r}})$; *i.e.*, for any commutative algebra A, $T(A)=(A\otimes_{\mathbf{Z}}\mathfrak{r})^{\times}$. Then if A is a $\mathfrak{B}$-algebra, each $\sigma: \mathfrak{r}\to\mathfrak{B}$ $(\sigma\in I)$ induces

$$\sigma=\mathrm{id}\otimes\sigma: T(A)=(A\otimes_{\mathbf{Z}}\mathfrak{r})^{\times}\to A^{\times}=\mathbf{G}_m(A).$$

Therefore one may regard σ as a rational character of T. Then, with $k\in\mathbf{Z}[I]$, we associate the character $\chi_k: T\to\mathbf{G}_m$ defined over $\mathfrak{B}$ by $\chi_k(x)=\prod_\sigma x^{\sigma k_\sigma}$. Since $\underline{\omega}$ is an R-module $(R=\mathfrak{O}_{\mathfrak{M}}\otimes_{\mathbf{Z}}\mathfrak{r})$, $T(\mathfrak{O}_{\mathfrak{M}})=R^{\times}$ operates $\mathfrak{O}_{\mathfrak{M}}$-linearly on $\underline{\omega}$. In particular, we embed for each $\mathfrak{B}$-algebra A (A is assumed to be flat if p ramifies in F), $A\otimes_{\mathbf{Z}}\mathfrak{r}\to A^I=\prod_\sigma A(\sigma)$ where $A(\sigma)\cong A$ is considered to be a $\mathfrak{B}$-algebra by σ. Thus we have a decomposition

$$\underline{\omega}\otimes_{\mathbf{Z}}A=\underline{\omega}\otimes_{\mathfrak{r}}(\mathfrak{r}\otimes_{\mathbf{Z}}A)\to\underline{\omega}\otimes_{\mathfrak{r}}A^I=\prod_\sigma\underline{\omega}(\sigma),$$

where $\underline{\omega}(\sigma)=\underline{\omega}\otimes_{\mathfrak{V}}A(\sigma)$ is an invertible sheaf over $\mathfrak{M}\otimes_{\mathbf{Z}}A$ on which $T(A)$ acts by σ. Thus we can construct an invertible sheaf $\underline{\omega}(k)$ over $\mathfrak{M}/\mathfrak{V}$ by $\otimes_\sigma \underline{\omega}(\sigma)^{k_\sigma}$. Then $T(\mathfrak{V})$ acts on $\underline{\omega}(k)$ by χ_k, and we have a canonical morphism for $\underline{\omega}^{\otimes m}=\mathrm{Symm}^m_{\mathfrak{O}_{\mathfrak{M}}}(\underline{\omega})$:

$$\underline{\omega}^{\otimes m}\otimes_{\mathbf{Z}}\mathfrak{V}\to\prod_{\mathrm{Tr}(k)=m,\,k\geq 0}\underline{\omega}(k),$$

which is an isomorphism if p is unramified in F. For each $\mathfrak{V}$-algebra A, we define the space of modular forms integral over A by

$$\mathfrak{M}_k(\Gamma_{00}(Np^\alpha),\,\mathfrak{c};\,A)=H^0(\mathfrak{M}\otimes_{\mathbf{Z}}A,\,\underline{\omega}(k)).$$

Let, for an affine $\mathfrak{V}$-scheme $S_{/\mathfrak{V}}=\mathrm{Spec}(A)_{/\mathfrak{V}}$, $(X,\lambda,\omega,i)_{/S}$ be a quadruple consisting of a HBAV $X_{/S}$, a nowhere vanishing differential form $\omega\in H^0(S,\underline{\omega}_{X/S})$, a $\mathfrak{c}$-polarization λ and an Np^α-level structure i defined over S. Then we can construct $\underline{\omega}_{X/S}(k)=\otimes_\sigma\underline{\omega}_{X/S}(\sigma)^{k_\sigma}$ out of $\underline{\omega}_{X/S}(\sigma)=\underline{\omega}_{X/S}\otimes_{\mathfrak{V}}\mathfrak{V}(\sigma)$. The natural image $\omega(k)=\omega^k=\otimes_\sigma\omega(\sigma)^{k_\sigma}$ is a nowhere vanishing global section of $\underline{\omega}_{X/S}(k)$. Let $f\in\mathfrak{M}_k(\Gamma_{00}(Np^\alpha),\mathfrak{c};A)$. We pull back f by the unique morphism $\varphi: S\to\mathfrak{M}$ which induces $(X,\lambda,i)=\varphi^*(\mathfrak{X},\lambda^{\mathrm{univ}},i^{\mathrm{univ}})$. Then $f\circ\varphi=\varphi^*f$ is a global section of $\underline{\omega}_{X/S}(k)$. Thus we may write $\varphi^*f=f(X,\lambda,\omega,i)\,\omega(k)$ with $f(X,\lambda,\omega,i)\in A$. Therefore, by tautology, one can define $f\in\mathfrak{M}_k(\Gamma_{00}(Np^\alpha),\mathfrak{c};A)$ as a function of test objects (X,λ,ω,i) satisfying:

M1. $f(X,\lambda,\omega,i)\in A$ *if* (X,λ,ω,i) *is a test object over a* $\mathfrak{V}$-*algebra* A,

M2. $f(X,\lambda,\omega,i)$ *only depend on the isomorphism class of* $(X,\lambda,\omega,i)_{/A}$,

M3. $f(X,\lambda,a\omega,i)=a^{-k}f(X,\lambda,\omega,i)$ *for* $a\in T(A)=(A\otimes_{\mathbf{Z}}\mathfrak{V})^\times$ $(a^{-k}=\chi_k(a)^{-1})$,

M4. *If* $\rho: A\to A'$ *is a homomorphism of* $\mathfrak{V}$-*algebra, then*

$$f((X,\lambda,\omega,i)\times_A A')=\rho(f(X,\lambda,\omega,i)).$$

1.4. Differential operators on modular forms. — We have a canonical morphism over $\mathfrak{M}\otimes_{\mathbf{Z}}\mathfrak{V}$

$$\Omega^1_{\mathfrak{M}/\mathfrak{V}}\cong\underline{\omega}\otimes_{R}\underline{\omega}\to\otimes_\sigma\underline{\omega}(2\sigma). \tag{1.9}$$

We then regard the Gauss-Manin connection as a map

$$\nabla:\quad H^1_{DR}\to\Omega^1_{\mathfrak{M}/\mathbf{Z}}\otimes_{\mathfrak{O}_{\mathfrak{M}}}H^1_{DR}\to\otimes_\sigma(\omega(2\sigma)\otimes_{\mathfrak{O}_{\mathfrak{M}}}H^1_{DR}).$$

We also have the Hodge exact sequence

$$0\to\underline{\omega}\to H^1_{DR}\to\mathfrak{Lie}\to 0. \tag{1.10}$$

Let $\mathfrak{A}_{/\mathfrak{M}}$ a sheaf of $\mathfrak{O}_{\mathfrak{M}}$-algebras (or $\mathfrak{O}_{\mathfrak{M}^{an}}$-algebras) over $\mathfrak{M}$ (or the corresponding analytic $\mathfrak{M}^{an}$). We write $H^1_{DR/\mathfrak{A}}$ for $H^1_{DR}\otimes_{\mathfrak{O}_{\mathfrak{M}}}\mathfrak{A}$ (or $H^1_{DR}\otimes_{\mathfrak{O}_{\mathfrak{M}^{an}}}\mathfrak{A}$). Suppose that the Hodge exact sequence splits after having tensored $\mathfrak{A}$ by a projection $\rho: H^1_{DR/\mathfrak{A}}\to\underline{\omega}_{/\mathfrak{A}}$. By (1.9), the connection ∇ induces another connection

$$\nabla^{\otimes m}:\quad H^{1\,\otimes m}_{DR}\to\otimes_\sigma(\underline{\omega}(2\sigma)\otimes H^{1\,\otimes m}_{DR})$$

by the formula of Leibnitz [D1, I.2.7.2]. Define a differential operator by

$$\nabla_{m,\sigma}:\ H_{DR}^{1\ \otimes m} \to \underline{\omega}(2\sigma)\otimes H_{DR}^{1\ \otimes m} \to H_{DR}^{1\ \otimes (m+2)}$$

which is the σ-component of $\nabla^{\otimes m}$ composed with the inclusion

$$\underline{\omega}(2\sigma)\otimes H_{DR}^{1\ \otimes m} \to H_{DR}^{1\ \otimes (m+2)}.$$

To have this inclusion, we need to tensor **Q** to the base ring if p ramifies in F. This loss of integrality of ∇ in the ramified case does not cause any harm, and the integrality of ∇ will be reestablished later by using q-expansion principle [*see* (1.23)]. Define, on the symmetric algebra $\mathrm{Sym}(H_{DR}^1)$ generated by H_{DR}^1, an operator

$$\nabla(\sigma):\ \mathrm{Sym}(H_{DR}^1)\to \mathrm{Sym}(H_{DR}^1) \quad \text{by} \quad \nabla(\sigma)=\otimes_m \nabla_{m,\sigma}.$$

It is known that $\nabla(\sigma)$'s ($\sigma\in I$) commute each other ([K4, (2.1.14)], *see* also (1.21) in the text). Take k, $d\in \mathbf{Z}[I]$ (k, $d\geqq 0$) such that $\mathrm{Tr}(k)=\sum_\sigma k_\sigma=m$. We then have a differential operator

$$\delta(k,\rho)^d:\ \underline{\omega}(k)\to\underline{\omega}^{\otimes m}\to H_{DR}^{1\ \otimes m}\xrightarrow{\prod_\sigma \nabla(\sigma)^{d\sigma}}(H_{DR}^1)^{\otimes (m+2\,\mathrm{Tr}(d))}$$
$$\xrightarrow{\rho^{\otimes (m+2\,\mathrm{Tr}(d))}} \underline{\omega}^{\otimes (m+2\,\mathrm{Tr}(d))}{}_{/\mathfrak{R}}\to\underline{\omega}(k+2d)_{/\mathfrak{R}}.$$

As examples of $\mathfrak{A}$ and ρ, one can offer:

Case C^∞. – Over the differentiable manifold $\mathfrak{M}^{\mathrm{diff}}$ associated to $\mathfrak{M}^{\mathrm{an}}$, one has the Hodge decomposition: $H_{DR}^1=\underline{\omega}\oplus\underline{\varpi}$. We take the sheaf $\mathfrak{A}_\infty$ of the germs of C^∞-class functions over $\mathfrak{M}^{\mathrm{diff}}$. Then we have the projection ρ_∞: $H^1_{DR/\mathfrak{R}_\infty}\to\underline{\omega}_{/\mathfrak{R}_\infty}$ and the differential operators $\delta(k,\rho_\infty)^d$. We will *see* later in (1.21) that this operator coincides with the classical differential operator of Maass, whose arithmetic implication is studied in depth by Shimura in many circumstances.

Case CM. – Let M/F be a CM field and $(X,\lambda,\omega,i)_{/K}$ be a test object in which X is of CM type (M,Σ). We assume that the test object is defined over an algebraic number field K, which may not be finite degree over **Q** if one considers the Np^∞ level structures. Let x: $\mathrm{Spec}(K)\to\mathfrak{M}$ which induces the test object (X,λ,i). We consider $\mathfrak{A}_x=x_*x^*\mathfrak{O}_{\mathfrak{M}}$. Then we can decompose

$$H^1_{DR/\mathfrak{A}_x}=\underline{\omega}(\Sigma)\otimes\underline{\omega}(\Sigma^c)\qquad\text{and}\qquad\underline{\omega}(\Sigma)=\underline{\omega}_{/\mathfrak{A}_x},$$

where M acts on $\underline{\omega}(\Sigma)\cong K^{[F:Q]}$ by the representation $\Sigma=\otimes_{\sigma\in\Sigma}\sigma$ and on $\underline{\omega}(\Sigma^c)$ by its complex conjugate. We know that $\underline{\omega}\otimes\mathfrak{A}_x\cong\underline{\omega}(\Sigma)$ canonically and therefore for $f\in\mathfrak{M}_k(\Gamma_{00}(Np^\alpha),\mathfrak{c};A)$ with a K-algebra A in **C**

$$(\delta(k,\rho_\infty)^d f)(X,\lambda,\omega,i)=\delta(k,\rho_x)(f(X,\lambda,\omega,i))\in A.$$

In particular, if we fix the transcendental isomorphism $\phi: X(\mathbf{C}) \cong \mathbf{C}^\Sigma/\Sigma(\mathfrak{a})$ for a fractional ideal $\mathfrak{a}$, we can define $\omega_{\text{trans}} = \phi^* du$ for the standard coordinate u on $\mathbf{C}^\Sigma$. Then we can write $\Omega_\infty \omega_{\text{trans}} = \omega$ for the complex period $\Omega_\infty = (\Omega(\sigma))_{\sigma \in \Sigma} \in \mathbf{C}^\Sigma$ and we have the algebraicity theorem of Shimura as presented in [K4, (2.4.5)]:

THEOREM 1.1 (Shimura [Sh, § 1]). — *Let the notation be as above. Then we have, for* $f \in \mathfrak{M}_k(\Gamma_{00} Np^\alpha), \mathfrak{c}; A)$,

$$(1.11) \qquad \frac{(\delta(k, \rho_\infty)^d f)(X, \lambda, \omega_{\text{trans}}, i)}{\Omega_\infty^{k+2d}} = (\delta(k, \rho_\infty)^d f)(X, \lambda, \omega, i) \in A$$

p-adic Case. — Let A be a p-adic algebra over $\mathfrak{V}$. Namely, A is a $\mathfrak{V}$-algebra such that $A = \varprojlim_\alpha A/p^\alpha A$. Consider the formal completion of the algebraic space: $\mathfrak{M}^{(p)} = \{\mathfrak{M}(\mathfrak{c}: Np^\infty)_{A/p^\alpha A}\}_\alpha$ [Kn, V.2]. Then for the structure sheaf $\mathfrak{A}_p$ of $\mathfrak{M}^{(p)}$, we have the Dwork-Katz decomposition [K1]:

$$(1.12) \qquad H^1_{DR/\mathfrak{A}_p} \cong \underline{\omega}_{/\mathfrak{A}_p} \otimes U_{/\mathfrak{A}_p},$$

where U is the maximal sub-sheaf on which the Frobenius map, induced under the identification of De Rham cohomology with crystalline cohomology, is everywhere invertible. According to this decomposition, one can define the p-adic differential operator of Katz $\delta(k, \rho_p)^d$. Under the ordinarity hypothesis (0.2 *a*, *b*), this decomposition (1.12) coincides with the decomposition in Case CM at the point x. Namely, let $\mathfrak{O}$ be a complete discrete valuation ring in the p-adic completion Ω of $\bar{\mathbf{Q}}_p$ with residue field $\bar{\mathbf{F}}_p$ (an algebraic closure of $\mathbf{Z}/p\mathbf{Z} = \mathbf{F}_p$). For each $x = (X, \lambda, \omega, i)_{/\mathfrak{O}}$ with complex multiplication of p-adic CM type (M, Σ) and with a Np^∞ level structure i, etc., we can associate an isomorphism $\phi: \hat{X} \cong \hat{\mathbf{G}}_m \otimes \vartheta^{-1}$ such that the p-part of i is induced by ϕ. Then $\mathfrak{Lie}(\hat{\mathbf{G}}_m \otimes \vartheta_p^{-1}) \cong \vartheta_p^{-1}$ and therefore $\underline{\omega}(\hat{\mathbf{G}}_m \otimes \vartheta^{-1}) \cong \mathfrak{r}_p(dT/T)$. Put $\omega_{\text{can}} = \phi^*(dT/T)$. We assume that ω is defined over $B = \iota_p^{-1}(\iota_p(\bar{\mathbf{Q}}) \cap \mathfrak{O})$. Then we can write $\omega = \Omega_p \omega_{\text{can}}$ with the p-adic period $\Omega_p \in (\mathfrak{O}^\times)^\Sigma$ and we have the algebraicity theorem of Katz:

THEOREM 1.2 (Katz [K4, (2.6.7)]). — *Let the notation be as above. Then we have the identity in* B:

$$(1.13) \qquad \frac{(\delta(k, \rho_p)^d f)(X, \lambda, \omega_{\text{can}}, i)}{\Omega_p^{k+2d}} = (\delta(k, \rho_p)^d f)(X, \lambda, \omega, i)$$
$$= (\delta(k, \rho_\infty)^d f)(X, \lambda, \omega, i) \in B.$$

1.5. DEFINITION OF p-ADIC MODULAR FORMS. — Let $A_0 \supset \mathfrak{V}$ be a p-adic algebra. Consider test objects $(X, \lambda, i)_{/A}$ defined over any p-adic A_0-algebra A. A function f of test objects $(X, \lambda, i)_{/A}$ is said to be a p-adic modular form if f satisfies the following conditions (in this definition, the algebra A is also a "variable"):

Mp1. $f(X, \lambda, i) \in A$ only depends on the isomorphism class of $(X, \lambda, i)_{/A}$;

Mp2. If $\rho: A \to A'$ is a homomorphisme of p-adic A_0-algebras, then

$$f((X, \lambda, i) \times_A A') = \rho(f(X, \lambda, i)_{/A})).$$

We write $V(\mathfrak{c}; A_0) = V(\mathfrak{c}, N; A_0)$ for the A_0-algebra of all p-adic modular forms. By definition, we have

$$(1.14) \qquad V(\mathfrak{c}, A_0) = \varprojlim_{\alpha} V(\mathfrak{c}, A_0/p^\alpha A_0)$$

and if p is nilpotent in A_0, the above definition coincides with the definition of modular functions (*i.e.* modular forms of weight 0: M1-4), because any A_0-algebra is automatically a p-adic algebra. Therefore,

$$V(\mathfrak{c}, A_0/p^\alpha A_0) = \Gamma(\mathfrak{M}(\mathfrak{c}; Np^\infty)_{A_0/p^\alpha A_0}, \mathcal{O}_{\mathfrak{M}}) = \mathfrak{M}_0(\Gamma_{00}(Np^\infty), \mathfrak{c}; A_0/p^\alpha A_0).$$

Thus $V(\mathfrak{c}; A_0)$ is the ring of global sections of the sheaf $\mathcal{O}_{\mathfrak{M}(p)}$. We can evaluate any geometric modular form $f \in \mathfrak{M}_k(\Gamma_{00}(Np^\alpha), \mathfrak{c}; A_0)$ at $(X, \lambda, \omega_{can}, i)_{/A}$; thus, a classical modular form gives rise to a p-adic modular form. Then by q-expansion principle, we shall see later that

the natural map: $\mathfrak{M}_k(\Gamma_{00}(Np^\alpha), \mathfrak{c}; A_0) \to V(\mathfrak{c}, N; A_0)$ *is injective.*

1.6. EXPLICIT DETERMINATION OF THE GAUSS MANIN CONNECTION. — We want to compute ∇ over $\mathbf{C}$. The projection: $\mathfrak{M}(\mathfrak{c}; Np^\beta) \to \mathfrak{M}(\mathfrak{c}; Np^\alpha)$ is formally etale (if $p^\alpha \geqq 4$). On the other hand, the construction of the connection ∇ is local under the etale topology. Therefore locally the connections ∇ over $\mathfrak{M}(\mathfrak{c}; Np^\beta)$ and ∇ over $\mathfrak{M}(\mathfrak{c}; Np^\alpha)$ are the same. Therefore it suffices to compute it for finite α. By the comparison theorem of the algebraic De Rham cohomology over $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; Np^\alpha)$ and the analytic one over $\mathfrak{M}^{an}$ ([D1, II.6.2]), the analytic Gauss-Manin connection ∇^{an} constructed for $H^1_{DR/\mathfrak{M}^{an}}$ induces ∇ for $H^1_{DR/\mathfrak{M}}$.

Over $\mathbf{C}$, as a consequence of the analytic theory of abelian varieties, a giving of a test object (X, λ, ω, i) is equivalent to a giving of a triple $(\mathfrak{L}, \lambda, i)$ consisting of an $\mathfrak{r}$-lattice $\mathfrak{L}$ of $F \otimes_{\mathbf{Q}} \mathbf{C} = \mathbf{C}^I$, a positive $\mathfrak{r}$-linear alternating form $\lambda: \mathfrak{L} \wedge \mathfrak{L} \cong \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ and the level structure $i: \mathfrak{d}^{-1}/Np^\alpha\mathfrak{d}^{-1} \to F\mathfrak{L}/\mathfrak{L}$. The positivity of λ means that we can write $\lambda(u, v) = A^{-1}\,\mathrm{Im}(uv^c)$ for a totally positive element $A \in F \otimes_{\mathbf{Q}} \mathbf{R}$. Put

$$\mathfrak{Z} = \{ z = (z_\sigma)_{\sigma \in I} \in \mathbf{C}^I \mid \mathrm{Im}(z_\sigma) > 0 \}.$$

We fix a pair of fractional ideals $(\mathfrak{a}, \mathfrak{b})$ prime to p such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. Then over $\mathbf{C}$, every test object of level Np^α $(X, \lambda, i)_{/\mathbf{C}}$ is isomorphic to a triple (X_z, λ_z, i_z) indexed by $z \in \mathfrak{Z}$ as follows: The abelian variety X_z is given by: $X_z(\mathbf{C}) = \mathbf{C}^I/\mathfrak{L}_z$ where $\mathfrak{L}_z = 2\pi i(\mathfrak{b}z + \mathfrak{a}^*)$, where $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. Then $H_1(X_z, \mathbf{Z}) \cong \mathfrak{L}_z$ by $c \mapsto \left(\int_c du_\sigma\right)_\sigma$ for the standard coordinate $u = (u_\sigma)$ on $\mathbf{C}^I$. The alternating form λ_z is given by:

$$\lambda_z(2\pi i(az+b), 2\pi i(cz+d)) = -(ad-bc) \in \mathfrak{c}^{-1}.$$

Finally i_z: $p^{-\alpha}N^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} = p^{-\alpha}N^{-1}\mathfrak{a}^*/\mathfrak{a}^* \to F\mathfrak{L}_z/\mathfrak{L}_z$ is given by

$$i_z(a \bmod \mathfrak{d}^{-1}) = 2\pi i a \bmod \mathfrak{L}_z.$$

There is an action of the congruence subgroup

$$\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b})=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\begin{pmatrix} \mathfrak{r} & (\mathfrak{ab})^{*} \\ \mathrm{N}p^{\alpha}\mathfrak{ab}\vartheta & \mathfrak{r}\end{pmatrix}\middle|\, d-1\in \mathrm{N}p^{\alpha}\mathfrak{r}\right\}\cap \mathrm{SL}_2(\mathrm{F})$$

on $\mathfrak{b}\oplus\mathfrak{a}^{*}$ given by $(x, y)\mapsto(x, y)\gamma$. Since $\mathfrak{L}_z=(\mathfrak{b}\oplus\mathfrak{a}^{*})\begin{pmatrix} z \\ 1\end{pmatrix}$, $\gamma\in \mathrm{GL}_2(\mathrm{F})$ acts on lattices by $\mathfrak{L}_z\gamma=(\mathfrak{b}\oplus\mathfrak{a}^{*})\gamma\begin{pmatrix} z \\ 1\end{pmatrix}$. One can easily verify that

$$(1.15)\qquad (\mathfrak{L}_z,\,\lambda_z,\,i_z)\cong(\mathfrak{L}_{\gamma(z)},\,\lambda_{\gamma(z)},\,i_{\gamma(z)})\quad\Leftrightarrow\quad \gamma\in\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b}).$$

Therefore $\mathfrak{M}^{an}=\mathfrak{M}(\mathfrak{c};\,\mathrm{N}p^{\alpha})^{an}\cong\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b})\backslash\mathfrak{Z}$ for finite α.

Since the projection $\mathfrak{Z}\to\mathfrak{M}^{an}$ is etale, we may compute ∇^{an} on $\mathfrak{Z}$. Then each fibre of $\mathfrak{X}/\mathfrak{Z}$ at z is X_z and

$$\mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}_z/\mathbf{C})=\mathrm{H}^1(\mathrm{X}_z,\,\mathbf{C})\cong\mathrm{Hom}_{\mathbf{Z}}(\mathfrak{L}_z,\,\mathbf{C}).$$

Therefore $\mathrm{H}^1_{\mathrm{DR}/\mathfrak{Z}}\cong\mathrm{Hom}_{\mathbf{Z}}(2\pi i(\mathfrak{b}\oplus\mathfrak{a}^{*}),\,\mathbf{C})\otimes_{\mathbf{C}}\mathfrak{O}_{\mathbf{Z}}$. We write X_σ, Y_σ for the global sections of $\mathrm{H}^1_{\mathrm{DR}}$ corresponding to

$$\mathrm{X}_\sigma\colon 2\pi i(\mathfrak{b}\oplus\mathfrak{a}^{*})\to 2\pi i\,\mathfrak{a}^{*}\xrightarrow{\sigma}\mathbf{C}\qquad\text{and}\qquad \mathrm{Y}_\sigma\colon 2\pi i(\mathfrak{b}\oplus\mathfrak{a}^{*})\to 2\pi i\,\mathfrak{b}\xrightarrow{-\sigma}\mathbf{C}.$$

On each fibre X_z, $\mathrm{X}_\sigma((2\pi i(az+b))=2\pi i b^{\sigma}$ and $\mathrm{Y}_\sigma(2\pi i(az+b))=-2\pi i a^{\sigma}$. Thus we have

$$(1.16)\qquad \mathrm{Symm}(\mathrm{H}^1_{\mathrm{DR}/\mathfrak{Z}})=\mathfrak{O}_{\mathfrak{Z}}[\mathrm{X},\,\mathrm{Y}](\mathrm{X}=(\mathrm{X}_\sigma),\,\mathrm{Y}=(\mathrm{Y}_\sigma)).$$

Since the inclusion of $\underline{\omega}$ in $\mathrm{H}^1_{\mathrm{DR}/\mathfrak{Z}}(\cong\mathrm{Hom}_{\mathbf{Z}}(2\pi i(\mathfrak{c}^{-1}\oplus\vartheta^{-1}),\,\mathbf{C})\otimes_{\mathbf{C}}\mathfrak{O}_{\mathbf{Z}})$ is given by $\omega\mapsto\left\{c\mapsto\int_c\omega\right\}$ for $c\in\mathrm{H}_1(\mathrm{X}_z,\,\mathbf{C})=\mathfrak{L}_z$, we see that $\mathrm{X}_\sigma-z_\sigma\mathrm{Y}_\sigma$ corresponding to du gives a global section of $\underline{\omega}_{\mathfrak{Z}}$:

$$(1.17)\qquad \underline{\omega}_{\mathfrak{Z}}\cong\oplus_{\sigma\in I}(\mathrm{X}_\sigma-z_\sigma\mathrm{Y}_\sigma)\mathfrak{O}_{\mathfrak{Z}},\qquad(\omega_{\mathrm{trans}}(\sigma)=(\mathrm{X}_\sigma-z_\sigma\mathrm{Y}_\sigma)).$$

For each global section $\omega\in\mathrm{H}^0(\mathfrak{M}(\mathrm{N}p^{\alpha};\,\mathfrak{c}),\,\underline{\omega}(k))$, if we write $\omega=f.(\mathrm{X}-z\mathrm{Y})^k$ for $(\mathrm{X}-z\mathrm{Y})^k=\prod_\sigma(\mathrm{X}_\sigma-z_\sigma\mathrm{Y}_\sigma)^{k_\sigma}$, the association: $\omega\mapsto f$ gives an isomorphism

$$(1.18)\qquad \mathrm{H}^0(\mathfrak{M}(\mathrm{N}p^{\alpha};\,\mathfrak{c}),\,\underline{\omega}(k))=\mathfrak{M}_k(\Gamma_{00}(\mathrm{N}p^{\alpha}),\,\mathfrak{c};\,\mathbf{C})\cong\mathrm{M}_k(\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b})),$$

where $\mathrm{M}_k(\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b}))$ is the space of classical modular forms on $\mathfrak{Z}$; namely, it is the space of holomorphic functions such that $f(\gamma(z))=f(z)(cz+d)^k$ for all $\gamma\in\Gamma=\Gamma_{00}(\mathrm{N}p^{\alpha};\,\mathfrak{a},\,\mathfrak{b})$ (actually, we need to assume the holomorphy condition at cusps if $\mathrm{F}=\mathbf{Q}$).

We compute ∇^{an} by using the sheaf $\Omega_{\mathfrak{X}/\mathfrak{Z}}^{\bullet\,\mathrm{diff}}$ of relative differentials of C^∞ class. There are two merits in the use of $\Omega_{\mathfrak{X}/\mathfrak{Z}}^{\bullet\,\mathrm{diff}}$:

(a) $\mathfrak{X} \cong X_z \times \mathfrak{Z}$ as a differentiable manifold; therefore, the computation of ∇^{diff} is trivial: $\nabla^{\mathrm{diff}} = \mathrm{id} \otimes d_{\mathfrak{Z}}$,

(b) By the original de Rham theory, there is no need to use hyper-cohomology, and we have $H^1_{\mathrm{DR}} \otimes \mathfrak{O}_{\mathfrak{Z}}^{\mathrm{diff}} = H^1(\alpha_* \Omega_{\mathfrak{X}/\mathfrak{Z}}^{\bullet\, \mathrm{diff}})$.

Since we have $\nabla^{\mathrm{diff}} = \mathrm{id} \otimes d_{\mathfrak{Z}}$, we conclude that on

$$H^1_{\mathrm{DR}/\mathfrak{Z}} \cong \mathrm{Hom}_{\mathbf{Z}}(2\pi i(\mathfrak{c}^{-1} \oplus \vartheta^{-1}), \mathbf{C}) \otimes_{\mathbf{C}} \mathfrak{O}_{\mathbf{Z}},$$

$\nabla^{\mathrm{an}} = \mathrm{id} \otimes d$ for the holomorphic exterior differentiation d on $\mathfrak{O}_{\mathfrak{Z}}$ and thus X, Y are horizontal to ∇. Once one gets the expression $\nabla^{\mathrm{an}} = \mathrm{id} \otimes d$, all the differential operators we have introduced can be made explicit automatically according to the definition: Here are the results of computation: The Kodaira Spencer morphism ι: $\Omega_{\mathfrak{Z}/C} \cong \underline{\omega} \otimes_R \underline{\omega}$ is given by

$$(1.19) \qquad \iota(2\pi i\, dz_\sigma) = (X_\sigma - z_\sigma Y_\sigma)^2 \qquad [\text{K4}, (2.1.21)].$$

The differential operator $\nabla(\sigma)$ is given under $\mathrm{Symm}(H^1_{\mathrm{DR}}) = \mathfrak{O}_{\mathfrak{Z}}[X, Y]$

$$(1.20) \qquad \nabla(\sigma) = \frac{1}{2\pi i}(X_\sigma - z_\sigma Y_\sigma)^2 \frac{\partial}{\partial z_\sigma}.$$

By this formula, one can calculate the effect of $\delta(k, \rho_\infty)$: In fact, if we write $\delta(k, \rho_\infty)^d (f(X - zY)^k) = (\delta_k^d f)(X - zY)^{k+2d}$, we have the differential operator of Maass-Shimura [K4, (2.3.27)]:

$$(1.21) \qquad \delta_k^\sigma = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z_\sigma} + \frac{k_\sigma}{2i\,\mathrm{Im}(z_\sigma)}\right) \qquad \text{and} \qquad \delta_k^d = \prod_\sigma \delta^\sigma_{k_\sigma + 2d_\sigma 2} \cdot \ldots \cdot \delta^\sigma_{k_\sigma}.$$

1.7. q-EXPANSION PRINCIPLE. — For each $f \in M_k(\Gamma_{00}(Np^\alpha; \mathfrak{a}, \mathfrak{b}))$, we have the Fourier expansion

$$f(q) = \sum_{\xi \in \mathfrak{ab}} a(\xi, f)\, q^\xi \qquad \text{for} \quad q^\xi = \exp(2\pi i \sum_\sigma \xi^\sigma z_\sigma).$$

To algebraize this expansion, we recall the construction of the Tate HBAV: We consider a set S of g independent linear forms L: $F \to \mathbf{Q}$ ($g = [F: \mathbf{Q}]$) preserving the natural positivity (*i.e.* $L(x) > 0$ if x is totally positive). We say $\xi \in F$ is S-positive if $L(\xi) \geqq 0$ for all $L \in S$. Then we consider the monoid ring

$$\mathbf{Z}[[q; \mathfrak{a}, \mathfrak{b}; S]] = \Big\{ \sum_{\xi: S-\mathrm{pos},\, \xi \in \mathfrak{ab}} a(\xi)\, q^\xi \,\Big|\, a(\xi) \in \mathbf{Z} \Big\}$$

and its localization Λ_S by the multiplicative set $\{ q^\xi \mid \xi$: S-positive$\}$. We consider the morphism

$$\underline{q}: \quad \mathfrak{ab} \to \mathbf{G}_m(\Lambda_S) \text{ given by } \underline{q}(\xi) = q^\xi.$$

The morphism $\underline{q}$ induces $\underline{q}$: $\mathfrak{b} \to \mathbf{G}_m \otimes \mathfrak{a}^{-1} \vartheta^{-1}$, since

$$\mathrm{Hom}_{\mathbf{Z}}(\mathfrak{ab}, \mathbf{G}_m(\Lambda_S)) \cong \mathrm{Hom}_{\mathfrak{r}}(\mathfrak{b}, \mathrm{Hom}_{\mathbf{Z}}(\mathfrak{a}, \mathbf{G}_m(\Lambda_S))) \cong \mathrm{Hom}_{\mathfrak{r}}(\mathfrak{b}, \mathbf{G}_m \otimes \vartheta^{-1}\mathfrak{a}^{-1}(\Lambda_S)).$$

A result of Mumford [M] assures that the rigid analytic quotient $\mathbf{G}_m \otimes \mathfrak{a}^*/\underline{q}(\mathfrak{b})$ ($\mathfrak{a}^* = \vartheta^{-1}\mathfrak{a}^{-1}$) is algebraizables ([R, §§ 2,4]) to a HBAV $\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q)$ over Λ_S. Moreover, for every positive integer M, the canonical morphism $\mu_M \otimes \mathfrak{a}^* \to \mathbf{G}_m \otimes \mathfrak{a}^*$ induces an exact sequence of group schemes over Λ_S:

$$1 \to \mu_M \otimes \mathfrak{a}^* \overset{\iota}{\to} \mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q) \overset{\pi}{\to} M^{-1}\mathfrak{b}/\mathfrak{b} \to 1.$$

In particular $\iota = i_{can}$: $\mu_{p^\infty} \otimes \vartheta^{-1} \oplus \mu_{N_0} \otimes \mathfrak{a}^*/N\mathfrak{a}^* \to \mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q)$ gives the canonical level structure if $\mathfrak{a}$ is prime to N. If $\mathfrak{a}$ is not prime to N, we just take an isomorphism ε: $\mathfrak{r}/N \cong \mathfrak{a}/N\mathfrak{a}$ and define the level structure $i = i(\varepsilon)$ composing with i_{can} above. We have a pairing of group schemes:

$$\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q)[M] \times \mathrm{Tate}_{\mathfrak{b}, \mathfrak{a}}(q)[M] \to \mu_M$$

given as follows: Taking $x \in \mathbf{G}_m \otimes \mathfrak{a}^*$ and $y \in \mathbf{G}_m \otimes \mathfrak{b}^*$ such that $x^M = q^\xi$ for some $\xi \in \mathfrak{b}$ and $y^M = q^\eta$ for some $\eta \in \mathfrak{a}$, we define

$$\langle [x], [y] \rangle_M = x^\eta / y^\xi \in \mu_M \; ((x^\eta/y^\xi)^M = q^{\xi\eta}/q^{\xi\eta} = 1).$$

Then there exists a unique isomorphism

$$\varphi: \quad {}^t\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q) \cong \mathrm{Tate}_{\mathfrak{b}, \mathfrak{a}}(q) \text{ which induce } \langle\ ,\ \rangle_M \text{ for all M}.$$

Then, the natural morphism λ_{can}: $\mathrm{Tate}_{\mathfrak{b}, \mathfrak{a}}(q) \cong \mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q) \otimes \mathfrak{ab}^{-1}$ gives the canonical $\mathfrak{c}$-polarization ($\mathfrak{c} = \mathfrak{ab}^{-1}$) [R, p. 297]. If $\mathfrak{a}$ is prime to p (then $\mathfrak{b}$ is prime to p because $\mathfrak{c} = \mathfrak{ab}^{-1}$ is prime to p), the identity:

$$\omega_{can}: \quad \mathrm{Lie}(\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q)) = \mathrm{Lie}(\mathbf{G}_m \otimes_{\mathbf{Z}} \vartheta^{-1}\mathfrak{a}^{-1}) \cong A \otimes_{\mathbf{Z}} \vartheta^{-1}\mathfrak{a}^{-1} \cong A \otimes_{\mathbf{Z}} \vartheta^{-1}$$

gives a canonical nowhere vanishing differential ω_{can}. By the existence of $(\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can})$ thus defined over $\bigcap_S \Lambda_S$, we can define the q-expansion $f(q)$ for all $f \in \mathfrak{M}_k(\Gamma_{00}(Np^\alpha), \mathfrak{c}; A)$ or $V(\mathfrak{c}; A)$ by

$$f(q) = f(\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) \in A((q))_{\mathfrak{ab}}.$$

where $A((q))_{\mathfrak{ab}} = \{ \sum_{\xi \in \mathfrak{ab}} a(\xi) q^\xi \mid a(\xi) \in A \}$. This expansion coincides with the Fourier expansion over $\mathbf{C}$ and determine the modular form because the algebraic space $\mathfrak{M}(Np^\alpha; \mathfrak{c})$ is geometrically irreducible [DR, § 4]. Namely we have the q-expansion principle [K4, 1.2, 1.9]:

$$(1.22\,a)\quad \begin{aligned} &\mathfrak{M}_k(\Gamma_{00}(Np^\alpha), \mathfrak{c}; A) = M_k(\Gamma_{00}(Np^\alpha; \mathfrak{c})) \cap A((q))_{\mathfrak{ab}} && \text{if } \mathbf{C} \supset A, \\ &\mathfrak{M}_k(\Gamma_{00}(Np^\alpha, \mathfrak{c}; A) = \mathfrak{M}_k(\Gamma_{00}(Np^\alpha), \mathfrak{c}; A')) \cap A((q))_{\mathfrak{ab}} && \text{if } A' \supset A \\ &V(\mathfrak{c}; A) = V(\mathfrak{c}; A') \cap A((q))_{\mathfrak{c}} && \text{if } A' \supset A \end{aligned}$$

$$(1.22\,b)\qquad f = 0 \iff f(\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q), \lambda_{can}, i(\varepsilon)) = 0 \quad \text{for } f \in V(\mathfrak{c}; A).$$

We shall give a p-adic version of (1.21) shown in [K4, (2.6.27)]. Let $\omega = f\,\omega_{\mathrm{can}}(k) \in \mathrm{H}^0(\mathfrak{M}_p \times_{\mathbf{Z}_p} \mathrm{A}, \underline{\omega}(k))$ $(f \in \mathrm{V}(\mathfrak{c};\, \mathrm{A}))$. We then write

$$\delta(k, \rho_p)^d \omega = (\theta^d f)\,\omega_{\mathrm{can}}(k) \in \mathrm{H}^0(\mathfrak{M}_p \times_{\mathbf{Z}_p} \mathrm{A}, \underline{\omega}(k))$$

and also write the q-expansion $f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i_{\mathrm{can}})$ as $f(q) = \sum_{\xi} a(\xi, f)\, q^{\xi}$. Then we have

$$\theta^d f(q) = \sum_{\xi} \xi^d a(\xi, f)\, q^{\xi}. \tag{1.23}$$

2. Fourier expansion of Eisenstein series

We want to calculate Fourier expansion of classical Eisenstein series according to Hecke and Katz [K4, III] for our later construction of p-adic measure. Let $\mathfrak{d}$ be the different of F/**Q**, and for each ideal $\mathfrak{a}$ we write $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. We fix a fractional ideal $\mathfrak{c}$ and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. Let $\phi: \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}')\} \times \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}'')\} \to \mathbf{C}$ be a locally constant function such that $\phi(\varepsilon^{-1}x, \varepsilon y) = N(\varepsilon)^k \phi(x, y)$ for all $\varepsilon \in \mathfrak{r}^{\times}$, where k is a positive integer and $\mathfrak{f}'$ and $\mathfrak{f}''$ are integral ideals prime to p. We put $\mathfrak{f} = \mathfrak{f}' \cap \mathfrak{f}''$ and suppose that all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are prime to $\mathfrak{f}p$. We regard ϕ as a function of $\mathrm{T} = \mathrm{X} \times \mathrm{Y}$ with $\mathrm{X} = \mathrm{Y} = \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})\}$ via the natural projection of $\{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})\} \times \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})\}$ to $\{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}')\} \times \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}'')\}$. We put $\mathrm{X}_\alpha = (\mathfrak{r}/p^\alpha \mathfrak{r}) \times (\mathfrak{r}/\mathfrak{f})$ and define the partial Fourier transform

$$\mathrm{P}\phi: \ \{\mathrm{F}_p/\mathfrak{d}_p^{-1} \times (\mathfrak{f}^*/\mathfrak{d}^{-1})\} \times \mathrm{Y} = \{\bigcup_{\alpha} (p^\alpha \mathfrak{f})^*/\mathfrak{d}^{-1}\} \times \mathrm{Y} \to \mathbf{C}$$

of ϕ by, taking α so that ϕ factors through $\mathrm{X}_\alpha \times \mathrm{Y}$,

$$\mathrm{P}\phi(x, y) = \begin{cases} p^{-\alpha[\mathrm{F}:\mathbf{Q}]} N(\mathfrak{f})^{-1} \sum_{a \in \mathrm{X}_\alpha} \phi(a, y)\, \mathbf{e}_{\mathrm{F}}(ax) & \text{if } \ x \in (p^\alpha \mathfrak{f})^*/\mathfrak{d}^{-1}, \\ 0 \quad \text{if } \ x \notin (p^\alpha \mathfrak{f})^*/\mathfrak{d}^{-1}, \end{cases} \tag{2.1}$$

where $\mathbf{e}_{\mathrm{F}}$ is the standard additive character of the adele ring F_{A} restricted to the local component $\mathrm{F}_{p\mathfrak{f}}$ at $p\mathfrak{f}$. This definition does not depend on the choice of α. Then we see easily by the Fourier inversion formula that

$$\sum_{a \in \mathrm{X}_\alpha} \mathrm{P}\phi(a, y)\, \mathbf{e}_{\mathrm{F}}(-(ax)_{p\mathfrak{f}}) = \phi(x, y). \tag{2.2}$$

We want to define the Eisenstein series $\mathrm{E}_k(\phi;\, \mathfrak{c})$ as a function of isomorphism classes of triples $(\mathfrak{L}, \lambda, i)$ as introduced in 1.6, where i is of level $p^\infty \mathfrak{f}^2$. Thus i is an injective homomorphism:

$$i: \ \mathrm{F}_p/\mathfrak{d}_p^{-1} \times (\mathfrak{f}^2)^*/\mathfrak{d}^{-1} \to p^{-\infty}\mathfrak{L}/\mathfrak{L} \times \mathfrak{f}^{-2}\mathfrak{L}/\mathfrak{L}.$$

We assume that λ induces an isomorphism $\mathfrak{L}\wedge_{\mathfrak{r}}\mathfrak{L}\cong\vartheta^{-1}\mathfrak{c}^{-1}$. By the Pontryagin duality induced from $\mathrm{Tr}\circ\lambda$, we have the adjoint projection of i restricted to $F_p/\vartheta^{-1}\times\mathfrak{f}^*/\vartheta^{-1}$:

$$\pi':\quad (\mathfrak{L}\otimes_{\mathfrak{r}}\mathfrak{r}_p)\times\mathfrak{L}/\mathfrak{f}\,\mathfrak{L}\to\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f}).$$

We introduce two lattices $\mathfrak{L}''\supset\mathfrak{L}'\supset\mathfrak{L}$ so that

$$\mathfrak{L}''/\mathfrak{L}=i((\mathfrak{f}^2)^*/\vartheta^{-1})\qquad\text{and}\qquad\mathfrak{L}'=\mathfrak{f}\,\mathfrak{L}''+\mathfrak{L}.$$

Then $\mathfrak{L}'/\mathfrak{L}=i(\mathfrak{f}^*/\vartheta^{-1})$. By definition,

$$\mathfrak{L}'/\mathfrak{f}\,\mathfrak{L}''\cong(\mathfrak{f}\,\mathfrak{L}''+\mathfrak{L})/\mathfrak{f}\,\mathfrak{L}''\cong\mathfrak{L}/(\mathfrak{L}\cap\mathfrak{f}\,\mathfrak{L}'')\overset{\pi'}{\cong}\mathfrak{r}/\mathfrak{f}.$$

We consider the sub-$\mathfrak{r}_{\mathfrak{f}p}$-module $\mathrm{PV}(\mathfrak{L})$ of $\mathfrak{L}\otimes_{\mathfrak{r}}F_{\mathfrak{f}p}$ such that $\mathrm{PV}(\mathfrak{L})\supset\mathfrak{L}\otimes_{\mathfrak{r}}\mathfrak{r}_{\mathfrak{f}p}$ and the natural inclusion: $\mathfrak{L}\otimes_{\mathfrak{r}}\mathfrak{r}_{\mathfrak{f}p}\to\mathrm{PV}(\mathfrak{L})$ induces an isomorphism

$$\mathrm{PV}(\mathfrak{L})/\mathfrak{L}\otimes_{\mathfrak{r}}\mathfrak{r}_{\mathfrak{f}p}\cong\mathrm{Im}\,(i:F_p/\vartheta^{-1}\times\mathfrak{f}^*/\vartheta^{-1}\to(p^{-\infty}\,\mathfrak{L}/\mathfrak{L})\times\mathfrak{f}^{-1}\,\mathfrak{L}/\mathfrak{L}).$$

Thus $\mathrm{PV}(\mathfrak{L})\otimes_{\mathfrak{r}}\mathfrak{r}_{\mathfrak{f}}=\mathfrak{L}'_{\mathfrak{f}}=\varprojlim_{\alpha}\mathfrak{L}'/\mathfrak{f}^{\alpha}\,\mathfrak{L}'$. Then i induces an injective homomorphism i: $F_p+\mathfrak{f}^*\,\mathfrak{r}_{\mathfrak{f}p}\to\mathrm{PV}(\mathfrak{L})$ and an exact sequence

$$0\to F_p\oplus\mathfrak{f}\,\mathfrak{L}'_{\mathfrak{f}}\to\mathrm{PV}(\mathfrak{L})\overset{\pi'}{\to}\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\to 0.$$

There is another exact sequence:

$$0\to\mathfrak{L}\otimes_{\mathfrak{r}}\mathfrak{r}_{\mathfrak{f}p}\to\mathrm{PV}(\mathfrak{L})\overset{\pi}{\to}F_p/\vartheta_p^{-1}\times\mathfrak{f}^*/\vartheta^{-1}\to 0,$$

where π is induced by i^{-1}. We now put, for $w\in\mathfrak{f}^{-1}p^{-\infty}\,\mathfrak{L}\cap\mathrm{PV}(\mathfrak{L})=\mathfrak{L}^{(Fp)}$ and $P\phi(w)=P\phi(\pi(w),\pi'(w))$,

$$(2.3)\qquad E_k((\mathfrak{L},\lambda,i);\phi,\mathfrak{c})=\frac{(-1)^{kt}\,\Gamma_F(kt+st)}{\sqrt{|D_F|}}\sum_{w\in\mathfrak{L}^{(Fp)}/\mathfrak{r}^{\times}}\frac{P\phi(w)}{N(w)^k\,|N(w)|^{2s}}\bigg|_{s=0},$$

where for each element $\xi=\sum_{\sigma\in I}\xi_\sigma\sigma$ of $\mathbf{C}[I]$, we write $\Gamma_F(\xi)=\prod_{\sigma\in I}\Gamma(\xi_\sigma)$. A priori speaking, the right hand side converges absolutely and locally uniformly, when the real part of s is sufficiently large; one shows however that it has an analytic continuation to the whole complex s-plane. Then we evaluate this function at $s=0$.

We consider the above sum as a function of $z=(z_\sigma)_{\sigma\in I}\in\mathfrak{H}^I=Z$ via the standard triple $(\mathfrak{L}_z,\lambda_z,i_z)$ introduced in 1.6, where $\mathfrak{L}_z=2\pi i(\mathfrak{a}^*z+\mathfrak{b})$ in $F\otimes_{\mathbf{Z}}\mathbf{C}$. Then we see

$$E_k(z;\phi)=E_k((\mathfrak{L}_z,\lambda_z,i_z);\phi,\mathfrak{c})$$
$$=\frac{(-1)^{kt}\,\Gamma_F(kt+st)}{\sqrt{F_F}\,(2\pi i)^{kt}\,(2\pi)^{2st}}\sum_{(a,b)\in\{(p^{\alpha}\mathfrak{a}\mathfrak{f})^*\times\mathfrak{b}\}/\mathfrak{r}^{\times}}\frac{P\phi(a,b)}{N(a+bz)^k\,|N(a+bz)|^{2s}}\bigg|_{s=0}.$$

We write

$$c=c(k, s)=\frac{(-1)^{kt}\,\Gamma_F(kt+st)}{\sqrt{D_F}\,(2\pi i)^{kt}(2\pi)^{2st}}.$$

Then we have

$$E_k(z;\phi)=c\Bigg\{\sum_{a\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{r}^\times}\frac{P\phi(a, 0)}{N(a)^k|N(a)|^{2s}} + \sum_{b\in(\mathfrak{b}-\{0\})/\mathfrak{r}^\times}\ \sum_{a_0\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{a}^*} P\phi(a_0, b)\sum_{a\in\mathfrak{a}^*}\frac{1}{N(a+a_0+bz)^k|N(a+a_0+bz)|^{2s}}\Bigg\}.$$

Putting

$$S_k(z;\mathfrak{a};s)=\sum_{a\in\mathfrak{a}^*}\frac{1}{N(a+z)^k|N(a+z)|^{2s}}$$

for $z\in(F\otimes_{\mathbf{Q}}\mathbf{C})^\times$, we have

$$E_k(z;\phi)=c\Bigg\{\sum_{a\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{r}^\times}\frac{P\phi(a, 0)}{N(a)^k|N(a)|^{2s}}+\sum_{b\in\{\mathfrak{b}-\{0\}\}/\mathfrak{r}^\times}\ \sum_{a_0\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{a}^*} P\phi(a_0, b)\,S_k(a_0+bz;\mathfrak{a};s)\Bigg\}.$$

Then, we have by the Poisson summation formula

$$S_k(z;\mathfrak{a};s)=N(\mathfrak{a})\sqrt{D_F}\sum_{a\in\mathfrak{a}} C_k(1, z, s)$$

for

$$C_k(x, z, s)=\int_{F\otimes_{\mathbf{Q}}\mathbf{R}}\frac{\mathbf{e}_F(-xt)}{N(t+z)^k|N(t+z)|^{2s}}\,dt.$$

Put

$$C=\frac{(-1)^{kt}\,\Gamma_F(kt+st)\,N(\mathfrak{a})}{(2\pi i)^{kt}(2\pi)^{2st}}.$$

From the formula:

$$C_k(x, a_0+bz, s)=\mathbf{e}_F(x, a_0)\,C_k(x, bz, s),$$

we see

$$E_k(z;\phi)=c\Bigg\{\sum_{a\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{r}^\times}\frac{P\phi(a, 0)}{N(a)^k|N(a)|^{2s}}\Bigg\} +C\Big\{\sum_{b\in(\mathfrak{b}-\{0\})/\mathfrak{r}^\times}\ \sum_{a\in\mathfrak{a}\mathfrak{f}}\Big\{\sum_{a_0\in(p^\alpha \mathfrak{a}\mathfrak{f})^*/\mathfrak{a}^*} P\phi(a_0, b)\,\mathbf{e}_F(-(aa_0)_f)\Big\}\,C_k(a, bz, s)\Big\}$$

$$= c \left\{ \sum_{a \in (p^\alpha \mathfrak{a}\mathfrak{f})^* / \mathfrak{r}^\times} \frac{P\phi(a, 0)}{N(a)^k |N(a)|^{2s}} \right\}$$
$$+ C \left\{ \sum_{b \in (\mathfrak{b} - \{0\})/\mathfrak{r}^\times} \sum_{a \in \mathfrak{a}} \phi(a, b) C_k(a, bz, s) \right\}.$$

As calculated in [K4, (3.2.31)] (*see also* [H1, (6.9*b*)]), we have

$$C_k(x, bz, 0) = \begin{cases} i^{-t}(2\pi)^t, & \text{if } k=1 \text{ and } x=0, \\ \dfrac{(2\pi i)^{kt}(-1)^{kt}}{\Gamma_F(kt)} \operatorname{sgn}(N(x)) N(x)^{k-1} \mathbf{e}_F(x, bz), & \text{if } xb \gg 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we now know, supposing $k \geqq 2$ or $\phi(a, 0) = 0$ for all a,

$$E_k(z; \phi; 0) = c \left\{ \sum_{a \in (p^\alpha \mathfrak{a}\mathfrak{f})^* / \mathfrak{r}^\times} \frac{P\phi(a, 0)}{N(a)^k |N(a)|^{2s}} \right\} \Bigg|_{s=0}$$
$$+ N(\mathfrak{a}) \left\{ \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{b}} \sum_{(a, b) \in (\mathfrak{a} \times \mathfrak{b})/\mathfrak{r}^\times,\, ab = \xi} \phi(a, b) \operatorname{sgn}(N(a)) N(a)^{k-1} \mathbf{e}_F(\xi z) \right\}.$$

Note that by the functional equation of the Hecke L-functions of F, the constant term is equal $2^{-t} L(1-k; \phi, \mathfrak{a})$, where

$$L(s; \phi, \mathfrak{a}) = \sum_{\xi \in (\mathfrak{a} - \{0\})/\mathfrak{r}^\times} \phi(\xi, 0) \operatorname{sgn}(N(\xi))^k |N(\xi)|^{-s}.$$

Thus we get, if $k \geqq 2$ or $\phi(a, 0) = 0$ for all a,

$$(2.4)\quad E_k(\phi, \mathfrak{c})(\mathrm{Tate}_{\mathfrak{a}, \mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}, i_{\mathrm{can}}) = N(\mathfrak{a}) \{ 2^{-t} L(1-k; \phi, \mathfrak{a})$$
$$+ \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{b}} \sum_{(a, b) \in (\mathfrak{a} \times \mathfrak{b})/\mathfrak{r}^\times,\, ab = \xi} \phi(a, b) \operatorname{sgn}(N(a)) N(a)^{k-1} q^\xi \}$$

3. Eisenstein measures

We use the Eisenstein series defined in the previous section to construct the Eisenstein-measure having values in $V(\mathfrak{c}, \mathfrak{f}^2; \mathfrak{O})$ for a p-adic algebra $\mathfrak{O}$ given below. Let κ be an algebraic closure of $\mathbf{F}_p$ and $W(\kappa)$ be the ring of Witt vectors with coefficients in κ. We consider $W(\kappa)$ as a subring of the p-adic completion Ω of $\bar{\mathbf{Q}}_p$. Let $\mathfrak{O}$ be a discrete valuation ring finite flat over $W(\kappa)$ inside Ω. Let $\mathfrak{f}$ be an integral ideal in F prime to p. We decompose $\mathfrak{f} = \prod_{\mathfrak{q}} \mathfrak{q}^{\varepsilon(\mathfrak{q})}$ for prime ideals $\mathfrak{q}$. Let $\mathfrak{f}'$ and $\mathfrak{f}''$ be two ideals prime to p such that $\mathfrak{f}' \cap \mathfrak{f}'' = \mathfrak{f}$. Write $\mathfrak{f}' = \prod_{\mathfrak{q}} \mathfrak{q}^{e'(\mathfrak{q})}$. We choose a prime element $\varpi_\mathfrak{q}$ at $\mathfrak{q}$ in F so that $\varpi^\mathfrak{q}$ is prime to p and $\varpi_\mathfrak{q} \equiv 1 \bmod^\times \mathfrak{f}\mathfrak{q}^{-\varepsilon(\mathfrak{q})}$. We put $\varpi^{e(\mathfrak{f}')} = \prod_{\mathfrak{q}} \varpi_\mathfrak{q}^{e'(\mathfrak{q})}$,

$\varpi^{e(\mathfrak{f}'')}=\prod_{\mathfrak{q}}\varpi_{\mathfrak{q}}^{e''(\mathfrak{q})}$ and $\varpi^{\varepsilon}=\prod_{\mathfrak{q}}\varpi_{\mathfrak{q}}^{\varepsilon(\mathfrak{q})}$. We also fix an ideale d of F such that $d\mathfrak{r}=\vartheta$ and d_{∞}. Let $\mathrm{T}=(\mathfrak{r}_p^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})\times(\mathfrak{r}/\mathfrak{f}))/\overline{\mathfrak{r}^{\times}}$ as a topological space. For each function ϕ on T, we define two functions ϕ^0 and ϕ^* on $\mathfrak{r}_p^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})\times(\mathfrak{r}/\mathfrak{f})$, supposing that ϕ is supported on $(\mathfrak{r}/\mathfrak{f})^{\times}$ to define ϕ^*, as follows:

$$(3.1)\quad \phi^0(x, y, a, b)=\mathrm{P}_a^{-1}\phi(x^{-1}, y, a, b)=\sum_{u\in(\mathfrak{r}/\mathfrak{f})}\phi(x^{-1}, y, u, b)\,\mathbf{e}_{\mathrm{F}}(-uad_{\mathfrak{f}}^{-1}\varpi^{-\varepsilon}),$$

$$\begin{aligned}\phi^*(x', y', a', b')&=\{\mathrm{P}_a^{-1}\mathrm{P}_b\phi(x, y, a, b^{-1})\}(x'^{-1}, y, a', b')\\&=N(\mathfrak{f})^{-1}\sum_{u\in(\mathfrak{r}/\mathfrak{f})}\ \sum_{v\in(\mathfrak{r}/\mathfrak{f})^{\times}}\phi(x'^{-1}, y', u, v^{-1})\,\mathbf{e}_{\mathrm{F}}(-ua'd_{\mathfrak{f}}^{-1}\varpi^{-\varepsilon})\,\mathbf{e}_{\mathrm{F}}(vb'd_{\mathfrak{f}}^{-1}\varpi^{-\varepsilon}).\end{aligned}$$

Note that ϕ^0 and ϕ^* satisfy the relation

$$\Phi(u^{-1}x, uy, u^{-1}a, ub)=\Phi(x, y, a, b)\qquad\text{if}\quad u\in\mathfrak{r}^{\times}.$$

Thus, writing N for the map $N:\mathrm{T}\to\mathbf{Q}_p$ given by $N(x, y, a, b)=\prod_{\sigma\in\mathrm{I}}x^{\sigma}$, we can define two Eisenstein measures $\mathbf{E}_{\mathfrak{c}}$ on T and $\mathbf{E}_{\mathfrak{c}}^*$ on

$$\mathrm{T}'=(\mathfrak{r}_p^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})\times(\mathfrak{r}/\mathfrak{f})^{\times})/\overline{\mathfrak{r}^{\times}}$$

with values in $\mathrm{V}(\mathfrak{c}, \mathfrak{f}^2; \mathfrak{O})$ as follows:

$$\mathbf{E}_{\mathfrak{c}}(N^{-k}\phi)=\mathrm{E}_k(\phi^0; \mathfrak{c})\qquad\text{and}\qquad\mathbf{E}_{\mathfrak{c}}^*(N^{-k}\phi)=\mathrm{E}_k(\phi^*; \mathfrak{c}).$$

In the above definition, the Eisenstein measures are defined only on locally polynomial functions but by continuity, they extend to measure having values in $\mathrm{V}(\mathfrak{c}, \mathfrak{f}^2; \mathfrak{O})$ (*see* [K4, IV]). Let S be the set of all prime ideals in F over p. For each $e(p)=(e(\mathfrak{p}))_{\mathfrak{p}\in\mathrm{S}}\in\mathbf{Z}^{\mathrm{S}}$, we write $\mathfrak{p}^{e(p)}$ for $\prod_{\mathfrak{p}\in\mathrm{S}}\mathfrak{p}^{e(\mathfrak{p})}$. Note that if

$$\phi(x, y, a, b)=\phi(x, a)\,\phi'(y, b)$$

is a character of conductor $\mathfrak{p}^{e(p)}\mathfrak{p}^{e'(p)}\mathfrak{f}'\mathfrak{f}''$ of $\mathrm{T}^{\times}=(\mathfrak{r}_p^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times})/\overline{\mathfrak{r}^{\times}}$ extended by 0 outside $\mathrm{T}^{\times}$, then

$$\begin{aligned}(3.2a)\quad \mathrm{P}\phi^0(x, y, a, b)=\phi_{\mathfrak{f}}(d_{\mathfrak{f}}\varpi^{\varepsilon}a)\,\phi'_{p\mathfrak{f}}(y, b)&\prod_{e(\mathfrak{p})>0}\mathrm{G}(\phi_{\mathfrak{p}}^{-1})\,\phi_{\mathfrak{p}}(\varpi^{e(\mathfrak{p})}x_{\mathfrak{p}})\\&\times\prod_{e(\mathfrak{p})=0}\{\chi_{\mathfrak{p}}(x_{\mathfrak{p}})-N(\mathfrak{p})^{-1}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}x_{\mathfrak{p}})\},\end{aligned}$$

where $\chi_{\mathfrak{p}}$ is the characteristic function of $\mathfrak{r}_{\mathfrak{p}}$ on $\mathrm{F}_{\mathfrak{p}}$ and

$$\mathrm{G}(\phi_{\mathfrak{q}})=N(\mathfrak{q}^{-e(\mathfrak{q})})\sum_{u\in(\mathfrak{r}/\mathfrak{q}^{e(\mathfrak{q})})}\phi_{\mathfrak{q}}(u)\,\mathbf{e}_{\mathrm{F}}(u\varpi_{\mathfrak{q}}^{-e(\mathfrak{q})}d_{\mathfrak{q}}^{-1}).$$

We also have

$$(3.2b)\quad \mathrm{P}\phi^*(x, y, a, b)) = N(\mathfrak{f}^{-1}\mathfrak{f}'')\,\phi(d_{\mathfrak{f}}\varpi^{\varepsilon} a)\,\phi'_p(y)$$
$$\times\Big\{\prod_{e(\mathfrak{p})>0} \mathrm{G}(\phi_{\mathfrak{p}}^{-1})\,\phi_{\mathfrak{p}}(\varpi^{e(\mathfrak{p})} x_{\mathfrak{p}}) \prod_{e(\mathfrak{p})=0} \chi_{\mathfrak{p}}(x_{\mathfrak{p}}) - N(\mathfrak{p})^{-1}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}} x_{\mathfrak{p}})\Big\}$$
$$\times\Big\{\prod_{e(\mathfrak{q})>0} \mathrm{G}(\phi_{\mathfrak{q}}'^{-1})\,\phi'_{\mathfrak{q}}(\varpi^{e''(\mathfrak{q})-\varepsilon(\mathfrak{q})}) x_{\mathfrak{q}} \prod_{e(\mathfrak{q})=0} \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}}^{-\varepsilon(\mathfrak{q})} x_{\mathfrak{q}}) - N(\mathfrak{q}^{-1})\,\chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}}^{1-\varepsilon(\mathfrak{q})} x_{\mathfrak{q}})\Big\}$$

where $\chi_{\mathfrak{q}}$ is the characteristic function on $\mathfrak{r}_{\mathfrak{q}}$.

4. The Katz measure in the right half critical region

We now fix a CM field M/F and a p-adic CM-type Σ [satisfying (0.2 a, b)]. We define the Katz measure on the ray class group of M modulo $\mathfrak{C}p^{\infty}$ for an arbitrary ideal $\mathfrak{C}$ prime to p and evaluate in this section the integral of the p-adic avatars of Hecke characters in the right half of the critical region. The other half will be dealt with by a functional equation in the following section. We fix an integral ideal $\mathfrak{C}$ of M prime to p. Let $\lambda : \mathrm{M}_{\mathrm{A}}^{\times}/\mathrm{M}^{\times} \to \mathbf{C}^{\times}$ be a Hecke character of conductor $\mathfrak{C}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{e(\mathfrak{P})}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{ce(\mathfrak{P}^c)}$ such that $\lambda(x_{\infty}) = x_{\infty}^{\eta}$ with

$$(4.1)\qquad \eta = m_0\Sigma + \sum_{\sigma\in\Sigma} d_{\sigma}(\sigma - \sigma c)\qquad \textit{for}\quad m_0>0\ \textit{and}\ \ d_{\sigma}\geqq 0\ \textit{for all}\ \sigma.$$

We write $\kappa = \bar{\mathbf{F}}_p$ for a fixed algebraic closure of $\mathbf{F}_p$ and $\mathrm{W}(\kappa)$ for the ring of Witt vectors with coefficients in κ. We consider that the ring $\mathrm{W}(\kappa)$ sits in Ω. For each fractional ideal $\mathfrak{U}$ prime to $\mathfrak{C}p$ in M, we take a quadruple $(\mathrm{X}(\mathfrak{U}), \lambda(\mathfrak{U}), \omega(\mathfrak{U}), i(\mathfrak{U}))$ defined as follows: Over $\mathbf{C}$, $\mathrm{X}(\mathfrak{U})(\mathbf{C}) = \mathbf{C}^{\Sigma}/\Sigma(\mathfrak{U})$. We pick an element $\delta\in\mathrm{M}$ such that, for complex conjugation c,

(4.2 a) $\delta^c = -\delta$ *and* $\mathrm{Im}(\delta^{\sigma})>0$ *for all* $\sigma\in\Sigma$,

(4.2 b) $\langle u, v\rangle = (u^c v - uv^c)/2\delta$ *on* $\mathfrak{R}$ *induces an isomorphism* $\mathfrak{R}\wedge_{\mathfrak{r}}\mathfrak{R}\cong\vartheta^{-1}\mathfrak{c}^{-1}$ *for the different* ϑ *of* F/**Q** *and an ideal* $\mathfrak{c}$ *prime to p.*

Then $\langle\ ,\ \rangle$ induces a $\mathfrak{c}(\mathfrak{U}\mathfrak{U}^c)^{-1}$-polarization $\lambda(\mathfrak{U})$ on $\mathrm{X}(\mathfrak{U})$. We decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ so that

$$(4.3a)\qquad \mathfrak{F}+\mathfrak{F}_c=\mathfrak{R},\qquad \mathfrak{F}+\mathfrak{F}^c=\mathfrak{R},\qquad \mathfrak{F}_c+\mathfrak{F}_c^c=\mathfrak{R}\qquad\text{and}\quad \mathfrak{F}_c\supset\mathfrak{F}^c,$$

(4.3 b) $\mathfrak{I}$ consists of ideals inert or ramified in M/F.

Put $\mathfrak{f}' = \mathfrak{F}\mathfrak{I}\cap\mathrm{F}$ and $\mathfrak{f}'' = \mathfrak{F}_c\mathfrak{I}\cap\mathrm{F}$. Then $\mathfrak{f}''\supset\mathfrak{f}'=\mathfrak{f}$. We write $\mathfrak{f}' = \prod_{\mathfrak{l}}\mathfrak{l}^{\varepsilon'(\mathfrak{l})}$ and $\mathfrak{f}'' = \prod_{\mathfrak{l}}\mathfrak{l}^{\varepsilon''(\mathfrak{l})}$. We choose a prime element $\varpi_{\mathfrak{l}}$ for each prime $\mathfrak{l}$ dividing $p\mathfrak{f}$ in F so that $\varpi_{\mathfrak{l}}\equiv 1 \bmod \mathfrak{f}\mathfrak{l}^{-\varepsilon'(\mathfrak{l})}$ (in this formula, $\varepsilon'(\mathfrak{p})=0$ if $\mathfrak{p}\,|\,p$) and $\varpi_{\mathfrak{l}}$ is prime to other prime $\mathfrak{l}'$ dividing $p\mathfrak{f}$. We choose a differental idele $d=d_{\mathrm{F}}$ of F such that $d_{\mathrm{F}} = d_{\mathrm{F}_{p\mathfrak{f}\vartheta}}$ and

$d_{F_q}=(2\delta)_{\mathfrak{Q}}$ for prime ideal $\mathfrak{Q} \mid \mathfrak{F}\,(q=\mathfrak{Q}\cap F)$. Then we define

$$i(\mathfrak{U}):\quad F_p/\vartheta_p^{-1}\times(\mathfrak{f}^2)^*/\vartheta^{-1}\to\{\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-\infty}\}(\mathfrak{F}\mathfrak{I})^{-2}\mathfrak{U}/\mathfrak{U}\quad\text{by } x\mapsto d_F x,$$

which induces $i(\mathfrak{U}):F_p/\vartheta_p^{-1}\times(\mathfrak{f}^2)^*/\vartheta^{-1}\to X(\mathfrak{U})$. Then we can find a model $(X(\mathfrak{U}),\lambda(\mathfrak{U}),i(\mathfrak{U})|_{(\mathfrak{f}^2)^*/\vartheta^{-1}})_{/M'}$ for a finite extension $M'/\mathbf{Q}$ which has good reduction over the valuation ring $\mathfrak{W}'$ of M' corresponding to the embedding $\iota_p:M'\to\bar{\mathbf{Q}}_p$, and there exists a nowhere vanishing differential $\omega_{/\mathfrak{W}'}$ on $X(\mathfrak{U})_{/\mathfrak{W}'}$. Moreover, defining $\mathfrak{O}$ as the composite of $W(\kappa)$ and $\iota_p(\mathfrak{W}')$, $(X(\mathfrak{U}),\lambda(\mathfrak{U}),i(\mathfrak{U}))$ is defined over $B=\iota_p^{-1}(\iota_p(\bar{\mathbf{Q}})\cap\mathfrak{O})$ and has good reduction over B. Namely we find a triple $(X(\mathfrak{U}),\ \lambda(\mathfrak{U}),\ i(\mathfrak{U}))_{/B}$ for $B=\iota_p^{-1}(\iota_p(\bar{\mathbf{Q}})\cap\mathfrak{O})$. Since $X(\mathfrak{U})=X(\mathfrak{R})\otimes_{\mathfrak{R}}\mathfrak{U}$ and $\mathfrak{U}$ is prime to p, we have a canonical isomorphism: $\mathfrak{Lie}(X(\mathfrak{U}))_{/\mathfrak{W}'}\cong\mathfrak{Lie}(X(\mathfrak{R}))_{/\mathfrak{W}'}\otimes_{\mathfrak{R}}\mathfrak{U}\cong\mathfrak{Lie}(X(\mathfrak{R}))_{/\mathfrak{W}'}$. Then by duality, we have $\underline{\omega}(X(\mathfrak{U}))_{/\mathfrak{W}'}\cong\underline{\omega}(X(\mathfrak{R}))_{/\mathfrak{W}'}$, canonically. Thus choosing one nowhere vanishing differential $\omega(\mathfrak{R})$ on $X(\mathfrak{R})_{/\mathfrak{W}'}$, we obtain a nowhere vanishing differential $\omega(\mathfrak{U})$ for all $\mathfrak{U}$ defined over $\mathfrak{W}'$. Over $\mathfrak{O}$, i induces an isomorphism $i_*:\hat{\mathbf{G}}_m\otimes_{\mathbf{Z}}\vartheta^{-1}\cong\hat{X}(\mathfrak{U})$ and therefore gives a nowhere vanishing differential $\omega_{\mathrm{can}}(\mathfrak{U})=i_*(dt/t)$, writing $\mathbf{G}_m=\mathrm{Spec}(\mathbf{Z}[t,t^{-1}])$. It is important that $\omega_{\mathrm{can}}(\mathfrak{U})$ corresponds to $\omega_{\mathrm{can}}(\mathfrak{R})$ under the isomorphism: $\underline{\omega}(X(\mathfrak{U}))_{/\mathfrak{O}}\cong\underline{\omega}(X(\mathfrak{R}))_{/\mathfrak{O}}$ [K4, 5.1.47]. Thus the ratio

$$\Omega_p=\omega(\mathfrak{U})/\omega_{\mathrm{can}}(\mathfrak{U})\in\mathfrak{O}\otimes_{\mathbf{Z}}\mathfrak{r}\quad\text{in } (\mathfrak{O}^{\Sigma})^{\times}\tag{4.4a}$$

is independent of $\mathfrak{U}$. Similarly we define $\omega_{\mathrm{trans}}(\mathfrak{U})$ on $X(\mathfrak{U})(\mathbf{C})=\mathbf{C}^{\Sigma}/\Sigma(\mathfrak{U})$ by $\omega_{\mathrm{trans}}(\mathfrak{U})((u_\sigma)_{\sigma\in I})=u_\sigma$ for the coordinate $(u_\sigma)_{\sigma\in I}$ of $\mathbf{C}^{\Sigma}$. Then the ratio

$$\Omega_\infty=\omega(\mathfrak{U})/\omega_{\mathrm{trans}}(\mathfrak{U})\in(\mathbf{C}^{\Sigma})^{\times}=(F\otimes_{\mathbf{Q}}\mathbf{C})^{\times}\tag{4.4b}$$

is also independent of $\mathfrak{U}$. We put $\mathfrak{s}=\mathfrak{F}_c\cap F=\mathfrak{F}_c^c\cap F$ and $\mathfrak{i}=\mathfrak{I}\cap F$. Consider the ray class group $\mathrm{Cl}_M(\mathfrak{C}p^{\alpha})$ modulo $\mathfrak{C}p^{\alpha}$ of M. We agree to write $\mathrm{Cl}_M(\mathfrak{C}p^{\infty})$ for $\varprojlim_{\alpha}\mathrm{Cl}_M(\mathfrak{C}p^{\alpha})$. Then we have homomorphisms

$$\begin{cases}\iota=\iota_c:G(\mathfrak{C})=\{\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{s})^{\times}\}/\overline{\mathfrak{r}^{\times}}\to\mathrm{Cl}_M(\mathfrak{C}p^{\infty})\\ \iota^*=\iota_c^*:G(\mathfrak{C}^c)=\{\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{s})^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\}/\overline{\mathfrak{r}^{\times}}\to\mathrm{Cl}_M(\mathfrak{C}^c p^{\infty})\end{cases}\tag{4.5}$$

induced by the natural inclusion of F into M, where for ι, the first factor $\mathfrak{r}_p^{\times}$ (resp. the second $\mathfrak{r}_p^{\times}$, $(\mathfrak{r}/\mathfrak{f}')^{\times}$, $(\mathfrak{r}/\mathfrak{s})^{\times}$) is identified with $\prod_{\mathfrak{P}\in\Sigma}\mathfrak{R}_{\mathfrak{P}}^{\times}$ (resp. $\prod_{\mathfrak{P}\in\Sigma^c}\mathfrak{R}_{\mathfrak{P}}^{\times}$, $(\mathfrak{R}/\mathfrak{F})^{\times}\times(\mathfrak{r}/\mathfrak{I}\cap\mathfrak{r})^{\times}$, $(\mathfrak{R}/\mathfrak{F}_c)^{\times}$) and for ι^*, the first $\mathfrak{r}_p^{\times}$ (resp. the second $\mathfrak{r}_p^{\times}$, $(\mathfrak{r}/\mathfrak{f})^{\times}$, $(\mathfrak{r}/\mathfrak{s})^{\times}$) is identified with $\prod_{\mathfrak{P}\in\Sigma}\mathfrak{R}_{\mathfrak{P}}^{\times}$ (resp. $\prod_{\mathfrak{P}\in\Sigma^c}\mathfrak{R}_{\mathfrak{P}}^{\times}$, $(\mathfrak{R}/\mathfrak{F}^c)^{\times}\times(\mathfrak{r}/\mathfrak{I}\cap\mathfrak{r})^{\times}$, $(\mathfrak{R}/\mathfrak{F}_c^c)^{\times}$). The morphisms ι and ι^* have finite kernel, and their cokernel is isomorphic to $\mathrm{Cl}^-(\mathfrak{I})$ which is the quotient of $\mathrm{Cl}_M(\mathfrak{I})$ by the natural image of $(\mathfrak{r}/\mathfrak{i})^{\times}$. We now choose a complete representative set $\{\mathfrak{U}_j\}$ for $\mathrm{Cl}^-(\mathfrak{I})$ consisting of fractional ideals prime to $p\mathfrak{C}\mathfrak{C}^c$. Let $[\mathfrak{U}]$ denote the class of $\mathfrak{U}$ in $\mathrm{Cl}_M(\mathfrak{C}p^{\infty})$. Then

$$\mathrm{Cl}_M(\mathfrak{C}p^{\infty})=\bigcup_j\mathrm{Im}(\iota)[\mathfrak{U}_j]^{-1}\qquad\text{and}\qquad\mathrm{Cl}_M(\mathfrak{C}^c p^{\infty})=\bigcup_j\mathrm{Im}(\iota^*)[\mathfrak{U}_j]^{-1}.$$

We identify $\mathrm{Cl}_{\mathrm{M}}(\mathfrak{C}p^{\infty})$ with the Galois group $\mathrm{G}_{\infty}(\mathfrak{C})$ of the ray class field modulo $\mathfrak{C}p^{\infty}$ over M by the Artin Symbol. We write the ideal of the polarization $\lambda(\mathfrak{U}_j)$ *as* $\mathfrak{c}_j=\mathfrak{c}(\mathfrak{U}_j\mathfrak{U}_j^c)^{-1}$. To define measures on $\mathrm{G}_{\infty}(\mathfrak{C})$ and on $\mathrm{G}_{\infty}(\mathfrak{C}^c)$, we explain how to extend functions defined on $\mathrm{G}_{\infty}(\mathfrak{C})$ or $\mathrm{G}_{\infty}(\mathfrak{C}^c)$ to T of T′ which supports the Eisenstein measure:

$$\mathrm{T}=\{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\times\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\}/\overline{\mathfrak{r}^{\times}},\qquad \mathrm{T}'=\{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\times\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})^{\times}\}/\overline{\mathfrak{r}^{\times}},$$

$$\mathrm{T}_0=\{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})\times\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{s})\}/\overline{\mathfrak{r}^{\times}},\qquad \mathrm{T}_0^{\times}=\{\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{s})^{\times}\}/\overline{\mathfrak{r}^{\times}},$$

$${}^t\mathrm{T}_0^{\times}=\{\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{s})^{\times}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\}/\overline{\mathfrak{r}^{\times}}$$

Write the variable of T as $(x_p, x_{\mathfrak{f}}, y_p, y_{\mathfrak{f}})$ in this order. For each function ϕ on $\mathrm{G}_{\infty}(\mathfrak{C})$ (resp. $\mathrm{G}_{\infty}(\mathfrak{C}^c)$) and index $j\in\mathrm{Cl}^-(\mathfrak{J})$, we define two functions ϕ_j (resp. ϕ_j') on T (resp. T′) as follows:

(4.6 *a*) *First we put on* $\mathrm{T}_0^{\times}$, *which naturally surjects onto* $\mathrm{G}_{\infty}(\mathfrak{C})$, $\phi_j(x)=\phi(x[\mathfrak{U}_j]^{-1})$. *We extend this function to* T_0 *by* 0 *outside* $\mathrm{T}_0^{\times}$. *Then we pull back the function defined on* T_0 *to* T *by the natural projection:* $\mathrm{T}\to\mathrm{T}_0$. *We denote this function on* T *by* ϕ_j;

(4.6 *b*) *We put on* ${}^t\mathrm{T}_0^{\times}$, (*which naturally surjects on to* $G_{\infty}(\mathfrak{C}^c)$): $\phi_j'(x)=\phi(x[\mathfrak{U}_j]^{-1})$. *Then identifying* ${}^t\mathrm{T}_0^{\times}$ *with*

$$\{\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{s})^{\times}\times\{0\}\times\mathfrak{r}_p^{\times}\times(\mathfrak{r}/\mathfrak{f})^{\times}\}/\overline{\mathfrak{r}^{\times}}\quad in\ \mathrm{T}_0'=\{\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{s})\times(\mathfrak{r}/\mathfrak{i})\times\mathfrak{r}_p\times(\mathfrak{r}/\mathfrak{f})^{\times}\}/\overline{\mathfrak{r}^{\times}},$$

we extend this function to T_0' *by* 0 *outside* ${}^t\mathrm{T}_0^{\times}$. *We then pull back this function to* T′ *by the natural projection* $\mathrm{T}'\to\mathrm{T}_0'$. *We write this function on* T′ *as* ϕ_j',

In Case (4.6 *b*), the function ϕ_j' is supported on T′ in T. Thus $\mathbf{E}^*(\phi_j')$ is well defined. We write $\mathfrak{c}_j=\mathfrak{c}(\mathfrak{U}_j\mathfrak{U}_j^c)^{-1}$. Then we define measures φ on $\mathrm{G}_{\infty}(\mathfrak{C})$ and φ^* on $\mathrm{G}_{\infty}(\mathfrak{C}^c)$ by

$$(4.7\,a)\qquad \int_{\mathrm{G}_{\infty}(\mathfrak{c})}\phi\,d\varphi=\sum_j\int_{\mathrm{T}}\phi_j\,d\mathbf{E}_j\qquad\text{for}\quad \mathbf{E}_j=x_j^*\,\mathbf{E}_{\mathfrak{c}_j},$$

$$(4.7\,b)\qquad \int_{\mathrm{G}_{\infty}(\mathfrak{c}^c)}\phi\,d\varphi^*=\sum_j\int_{\mathrm{T}'}\phi_j'\,d\mathbf{E}_j^*\qquad\text{for}\quad \mathbf{E}_j^*=x_j^*\,\mathbf{E}_{\mathfrak{c}_j}^*.$$

The measure φ is essentially equal to the measure μ in Theorem II up to units in the measure algebra. We will normalize φ to get μ in the next section. The measure φ^* is introduced to prove the functional equation in section 5. We first state the result for φ:

THEOREM 4.1. — *Let* M *be a* CM *quadratic extension of* F *and* Σ *be a p-adic* CM-*type of* M. *Let* $\mathfrak{C}$ *be an integral ideal prime to p in* M. *We decompose* $\mathfrak{C}=\mathfrak{F}\mathfrak{F}_c\mathfrak{J}$ *as in* (4.3). *We put*

$$w_p(\lambda)=\{\prod_{\mathfrak{P}\in\Sigma}\mathfrak{N}_{\mathrm{M/Q}}(\mathfrak{P}^{-e(\mathfrak{P})})\,\mathrm{G}(2\delta;\lambda_{\mathfrak{P}})\},$$

$$w(\lambda)=w_p(\lambda)\prod_{\mathfrak{L}\,|\,\mathfrak{F}}\lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon})\prod_{\mathfrak{l}\,|\,\mathrm{I}}\lambda_{\mathfrak{l}}^{-1}(\varpi_{\mathfrak{l}}^{\varepsilon}),$$

Then there exists a measure φ *on* $G_\infty(\mathfrak{C})$ *such that*

$$\frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\varphi}{\Omega_p^{m_0\Sigma+2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times)\, w(\lambda) \frac{(-1)^{m_0 t} \pi^d \Gamma_F(m_0 t + d)}{\sqrt{|D_F|}\, \mathrm{Im}(\delta)^d\, \Omega_\infty^{m_0\Sigma+2d}} \times \prod_{\mathfrak{L} \mid \mathfrak{C}} (1-\lambda(\mathfrak{L})) \Big\{ \prod_{\mathfrak{P} \in \Sigma} (1-\lambda(\mathfrak{P}^c)) \prod_{\mathfrak{P} \in \Sigma} (1-\lambda^*(\mathfrak{P}^c)) \Big\} L(0, \lambda)$$

for all Hecke characters λ *such that*

(i) *the conductor of* λ *divides* $\mathfrak{C} p^\infty$,

(ii) $\lambda_\infty(x_\infty) = x_\infty^{m_0\Sigma + d(1-c)}$ *for* $m_0 > 0$ *and* $d = \sum_{\sigma \in \Sigma} d_\sigma \sigma$ *with* $d_\sigma \geqq 0$.

The result for φ^* is as follows:

THEOREM 4.2. — *Under the notation of Theorem* 4.1, *we put*

$$w^*(\lambda) = w_p(\lambda) \prod_{\mathfrak{L} \mid \mathfrak{F}} \mathfrak{N}_{M/Q}(\mathfrak{L}^{-ce(\mathfrak{L}^c)})\, G(2\delta, \lambda_{\mathfrak{L}^c})\, \lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^\varepsilon)\, \lambda_{\mathfrak{L}^c}(\varpi_{\mathfrak{L}^c}^\varepsilon) \times \prod_{I \mid \mathfrak{I}} \mathfrak{N}_{F/Q}(I^{\varepsilon(I)})\, \mathfrak{N}_{M/Q}(\mathfrak{L}^{-e(I)})\, \lambda_I(\varpi_I^\varepsilon)\, \mathfrak{G}(2\delta; \lambda_I),$$

$$w'(\lambda) = \prod_{\mathfrak{L} \mid \mathfrak{F}} \mathfrak{N}_{M/Q}(\mathfrak{L}^{-e(\mathfrak{L})})\, G(2\delta, \lambda_{\mathfrak{L}})\, \lambda_{\mathfrak{L}}(\varpi_{\mathfrak{L}}^{\varepsilon+e(\mathfrak{L})}) \prod_{I \mid \mathfrak{I}} \mathfrak{N}_{F/Q}(I^{\varepsilon(I)})\, \mathfrak{N}_{M/Q}(\mathfrak{L}^{-e(I)})\, \mathfrak{G}(2\delta, \lambda_I)\, \lambda_I(\varpi_I^{2\varepsilon}).$$

Then there exists a measure φ^* *on* $G_\infty(\mathfrak{C}^c)$ *such that*

$$\frac{\int_{G_\infty(\mathfrak{C}^c)} \hat{\lambda}\, d\varphi^*}{\Omega_p^{m_0\Sigma+2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times)\, w^*(\lambda) \frac{(-1)^{m_0 t} \pi^d \Gamma_F(m_0 t + d)}{\sqrt{|D_F|}\, \mathrm{Im}(\delta)^d\, \Omega_\infty^{m_0\Sigma+2d}} \times \Big\{ \prod_{\mathfrak{L} \mid \mathfrak{F}\mathfrak{I}} (1-\lambda^*(\mathfrak{L}^c)) \prod_{\mathfrak{L} \mid \mathfrak{F}_c} (1-\lambda(\mathfrak{L})) \prod_{\mathfrak{P} \in \Sigma} (1-\lambda(\mathfrak{P}^c)) \prod_{\mathfrak{P} \in \Sigma} (1-\lambda^*(\mathfrak{P}^c)) \Big\} L(0, \lambda)$$

for all Hecke characters λ *such that*

(i) *the conductor of* λ *divides* $\mathfrak{C}^c p^\infty$,

(ii) $\lambda_\infty(x_\infty) = x_\infty^{m_0\Sigma c + d(1-c)}$ *for* $m_0 > 0$ *and* $d = \sum_{\sigma \in \Sigma^c} d_\sigma \sigma$ *with* $d_\sigma \geqq 0$. *Moreover, the measures* φ *and* φ^* *are related in the following way:*

$$\prod_{\mathfrak{L} \mid \mathfrak{F}\mathfrak{I}} (1-\lambda(\mathfrak{L}))^{-1} w'(\lambda) \int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\varphi = \prod_{\mathfrak{L} \mid \mathfrak{F}I} (1-\lambda^*(\mathfrak{L} - \lambda^*(\mathfrak{L}))^{-1} \int_{G_\infty(\mathfrak{C}^c)} \hat{\lambda} \circ c\, d\varphi^*.$$

We prove these two theorems at the same time. We compute

$$\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\varphi \left(\text{resp. } \int_{G_\infty(\mathfrak{C}^c)} \hat{\lambda}\, d\varphi^* \right)$$

for algebraic Hecke characters λ of conductor dividing $\mathfrak{C}p^{\infty}$ (resp. $\mathfrak{C}^c p^{\infty}$), where $\hat{\lambda}$ is the p-adic avatar defined on $G_{\infty}(\mathfrak{C})$ [resp. $G_{\infty}(\mathfrak{C}^c)$]. We write

$$\lambda^*(x)=\lambda(x^{-c})|x|_{\mathbf{A}} \quad \text{and} \quad \hat{\lambda}^*(x)=\hat{\lambda}(cxc^{-1})^{-1}\mathfrak{N}(x)^{-1}$$

for the cyclotomic character $\mathfrak{N}$. We assume (4.1) for λ. Then $\phi_j=\lambda(\mathfrak{U}_j)^{-1}\lambda_{p\mathfrak{C}}^{-1}$ on $T_0^{\times}$ and $\phi'_j=\lambda(\mathfrak{U}_j)^{-1}\lambda_{p\mathfrak{C}^c}^{-1}$ on ${}^tT_0^{\times}$. Thus the computation is basically local, and we can compute the general formula only dealing with the case where $\mathfrak{f}$ is a prime power.

We begin with the case where $\mathfrak{C}=\mathfrak{L}^{\alpha}$ with a prime $\mathfrak{L}$ which is inert or ramified in M/F (thus $\alpha=\varepsilon$ for ε given by $\mathfrak{f}=\mathfrak{C}\cap F=\mathfrak{l}^{\varepsilon}$ if $\mathfrak{l}=\mathfrak{L}$, and $\alpha=2\varepsilon$ or $2\varepsilon-1$ if $\mathfrak{l}=\mathfrak{L}^2$). We hereafter use the capital letters $\mathfrak{L}$, $\mathfrak{P}$ and $\mathfrak{Q}$ (resp. the lower case latter $\mathfrak{l}$, $\mathfrak{p}$ and $\mathfrak{q}$) for prime ideals in M (resp. F). This case is technically more difficult than the case of split primes. Writing $\mathfrak{l}=\mathfrak{L}\cap F$, have

$$\phi_j(x, y)=\lambda(\mathfrak{U}_j)^{-1}\lambda_{\Sigma}(x_p)\lambda_{\Sigma^c}^{-1}(y_p)\phi(x_{\mathfrak{l}}, y_{\mathfrak{l}})$$

for

$$\text{(4.8)} \qquad \phi(a, b)=\begin{cases}\lambda_{\mathfrak{l}}^{-1}(a), & \text{if } (a, b)\in \mathfrak{r}_{\mathfrak{l}}^{\times}\times\mathfrak{r}_{\mathfrak{l}},\\ 0, & \text{otherwise.}\end{cases}$$

Note that the $\mathfrak{l}$-part of $PV(\mathfrak{U}_j)_{\mathfrak{l}}$ is given by $\varpi_{\mathfrak{l}}^{-\varepsilon}\mathfrak{r}_{\mathfrak{l}}+\mathfrak{R}_{\mathfrak{l}}$. Thus $\pi(\varpi_{\mathfrak{l}}^{-\varepsilon})=d_{F_{\mathfrak{l}}}^{-1}\varpi_{\mathfrak{l}}^{-\varepsilon}$ and for any $a\in PV(\mathfrak{U}_j)_{\mathfrak{l}}\cap\mathfrak{f}^{-1}\mathfrak{U}_j$, $P_{\mathfrak{l}}P_a^{-1}\phi(a)=\lambda_{\mathfrak{l}}^{-1}(\varpi_{\mathfrak{l}}^{\varepsilon}a_{\mathfrak{l}})$ if $\varpi_{\mathfrak{l}}^{\varepsilon}a_{\mathfrak{l}} \bmod \mathfrak{f}_{\mathfrak{l}}$ is contained in the image of $(\mathfrak{r}_{\mathfrak{l}}/\mathfrak{f}_{\mathfrak{l}})^{\times}$ in $\mathfrak{R}_{\mathfrak{l}}/\mathfrak{f}_{\mathfrak{l}}$ and otherwise $P_{\mathfrak{l}}P_a^{-1}\phi(a)=0$. For each prime ideal $\mathfrak{Q}$ in M and a character $\lambda_{\mathfrak{Q}}$ of $M_{\mathfrak{Q}}^{\times}$ of conductor $\mathfrak{Q}^e$, we put

$$G(d_{M_{\mathfrak{Q}}}, \lambda_{\mathfrak{Q}})=\lambda(\varpi_{\mathfrak{Q}}^{-e})\sum_{u\in(\mathfrak{R}/\mathfrak{Q}^e)^{\times}}\lambda_{\mathfrak{Q}}(u)\mathbf{e}_M(\varpi_{\mathfrak{Q}}^{-e}d_{M_{\mathfrak{Q}}}^{-1}u),$$

where $\varpi_{\mathfrak{Q}}$ is a prime element at $\mathfrak{Q}$. We define $e(\mathfrak{Q})$ for prime ideals $\mathfrak{Q}\in\Sigma\cup\Sigma^c$ or $\mathfrak{Q}=\mathfrak{L}$ when $\mathfrak{Q}^{e(\mathfrak{Q})}$ is the conductor of $\lambda_{\mathfrak{Q}}$. We also define $e'(\mathfrak{P})$ for $\mathfrak{P}\in\Sigma$ by $e'(\mathfrak{P})=\max(e(\mathfrak{P}), 1)$. Let $\Sigma'=\{\mathfrak{P}\in\Sigma \mid e(\mathfrak{P})=0\}$ and $\Sigma''=\Sigma-\Sigma'$. We also put $e'^0(\mathfrak{Q})=e'(\mathfrak{Q})$ [resp. $e^0(\mathfrak{Q})=e(\mathfrak{Q})$] if $\mathfrak{Q}\in\Sigma$ and otherwise $e'^0(\mathfrak{Q})=e^0(\mathfrak{Q})=0$. Let $\chi_{\mathfrak{P}}$ denote the characteristic function of $\mathfrak{R}_{\mathfrak{P}}$ in $M_{\mathfrak{P}}$. Writing $\varpi^{e'^0}$ for $\prod_{\mathfrak{P}\in\Sigma}\varpi_{\mathfrak{P}}^{e'^0(\mathfrak{P})}$ and a_{Σ} for the projection of $a\in M$ to $\prod_{\mathfrak{P}\in\Sigma}M_{\mathfrak{P}}$, we have by the Fourier inversion formula

$$P(\phi_j^0)(a)=\lambda(\mathfrak{U}_j)^{-1}N\Big(\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e'(\mathfrak{P})}\Big)\lambda_{\mathfrak{l}}^{-1}(\varpi_{\mathfrak{l}}^{\varepsilon}a_{\mathfrak{l}})$$

$$\times\sum_{b\in(\mathfrak{r}/\mathfrak{p}^{e'})}\lambda_p^{-1}(b^{-1}, a_{\Sigma c})\mathbf{e}_F((2\delta)^{-1}ba_{\Sigma})$$

$$\overset{u=\varpi^{e'^0}a_{\Sigma}}{=}\lambda(\mathfrak{U}_j)^{-1}N\Big(\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{e'(\mathfrak{P})}\Big)^{-1}\lambda_{\mathfrak{l}}^{-1}(\varpi_{\mathfrak{l}}^{\varepsilon}a_{\mathfrak{l}})$$

$$\times\sum_{b\in(\mathfrak{r}/\mathfrak{p}^{e'})}\lambda_p^{-1}(b^{-1}, a_{\Sigma c})\mathbf{e}_F((2\delta)^{-1}bu\varpi^{-e'^0})$$

$$=\lambda(\mathfrak{U}_j)^{-1}\lambda_{\mathrm{I}}^{-1}(\varpi_{\mathrm{I}}^{\varepsilon}a_{\mathrm{I}})\prod_{\mathfrak{P}\in\Sigma''}N(\mathfrak{P}^{-e(\mathfrak{P})})\mathrm{G}(2\delta;\lambda_{\mathfrak{P}})\lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\lambda_{\mathfrak{P}^c}^{-1}(a_{\mathfrak{P}^c})$$
$$\times\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}^c}^{-1}(a_{\mathfrak{P}}^{c})\lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})\text{-}N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))$$

if $\varpi^e a\in\mathfrak{R}_{\mathfrak{P}}^{\times}$ for all $\mathfrak{P}\in\Sigma''$ and $\varpi_{\mathrm{I}}^{\varepsilon}a_{\mathrm{I}}$ mod $\mathfrak{f}\,\mathfrak{R}_{\mathrm{I}}\in(\mathfrak{r}_{\mathrm{I}}/\mathfrak{f}_{\mathrm{I}})^{\times}$.

Writing Φ for $\lambda_{\Sigma}(x_p)\lambda_{\Sigma^c}^{-1}(y_p)\phi(x_{\mathrm{I}},y_{\mathrm{I}})$, we have by Theorems 1.1 and 1.2

$$(4.9)\quad \frac{\int_{\mathrm{G}_\infty}\hat{\lambda}\,d\varphi}{\Omega_p^{m_0\Sigma+2}}=\Omega_p^{-(m_0\Sigma+2d)}\sum_j\lambda(\mathfrak{U}_j^{-1})\int_{\mathrm{G}}N(x)^{-m_0}(x^{-1}y)^d\Phi(x,y)\,d\mathbf{E}_j(x,y)$$
$$=\sum_j\frac{\lambda(\mathfrak{U}_i^{-1})\theta^d\,\mathrm{E}_{m_0}((\mathrm{X}(\mathfrak{U}_j),\lambda(\mathfrak{U}_j),\omega_{\mathrm{can}}(\mathfrak{U}_j),i(\mathfrak{U}_j));\Phi)}{\Omega_p^{m_0\Sigma+2d}}$$
$$=\sum_j\frac{\lambda(\mathfrak{U}_i^{-1})\,\delta_{m_0}^d\,\mathrm{E}_{m_0}((\mathrm{X}(\mathfrak{U}_j),\lambda(\mathfrak{U}_j),\omega_{\mathrm{trans}}(\mathfrak{U}_j),i(\mathfrak{U}_j));\Phi)}{\Omega_\infty^{m_0\Sigma+2d}}.$$

Writing $\varpi^{e^0+\varepsilon}$ for $\varpi_{\mathrm{I}}^{\varepsilon}\prod_{\mathfrak{P}\in\Sigma}\varpi_{\mathfrak{P}}^{e^0(\mathfrak{P})}$, we put

$$\mathfrak{B}_j=\varpi^{e^0+\varepsilon}\mathfrak{U}_j\{\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e(\mathfrak{P})}\}\mathrm{I}^{-\varepsilon},\qquad w_p(\lambda)=\{\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})\mathrm{G}(2\delta;\lambda_{\mathfrak{P}})\}$$

and

$$w(\lambda)=\{\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})\mathrm{G}(2\delta;\lambda_{\mathfrak{P}})\}\lambda_{\mathfrak{f}}^{-1}(\varpi_{\mathrm{I}}^{\varepsilon}).$$

By the explicit formula of the Maass differential operator (1.21), we know that each term of the series

$$\lambda(\mathfrak{U}_i^{-1})\,\delta_{m_0}^d\,\mathrm{E}_{m_0}((\mathrm{X}(\mathfrak{U}_j),\lambda(\mathfrak{U}_j),\omega_{\mathrm{trans}}(\mathfrak{U}_j),i(\mathfrak{U}_j));\Phi)$$

is given by $\mathrm{P}\phi_j^0(a)\,a^{-m_0\Sigma-2d(1-c)}$ multiplied by the constant:

$$\frac{(-1)^{m_0t}\pi^d\,\Gamma_{\mathrm{F}}(m_0t+d)}{\sqrt{|\mathrm{D}_{\mathrm{F}}|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}.$$

We compute this number if $\varpi^{e^0+\varepsilon}a\mathfrak{B}_j^{-1}$ is prime to $p\mathfrak{f}$ and $\varpi_{\mathrm{I}}^{\varepsilon}a_{\mathrm{I}}$ mod $\mathfrak{f}\mathfrak{R}_{\mathrm{I}}\in(\mathfrak{r}_{\mathrm{I}}/\mathfrak{f}_{\mathrm{I}})^{\times}$ (otherwise it vanishes). We have

$$\mathrm{P}\phi_j^0(a)\,a^{-m_0\Sigma-2d(1-c)}$$
$$=w_p(\lambda)\lambda(\mathfrak{U}_j^{-1})\lambda_{\mathrm{I}}^{-1}(\varpi_{\mathrm{I}}^{\varepsilon}a_{\mathrm{I}})\lambda_{p\infty}^{-1}(a)\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))$$
$$=w_p(\lambda)\lambda(\mathfrak{U}_j^{-1})\lambda_{\mathfrak{f}}(\varpi_{\mathrm{I}}^{\varepsilon})^{-1}\lambda_{p\mathfrak{f}\infty}^{-1}(a)\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))$$

$$= w(\lambda)\lambda(\mathfrak{B}_j^{-1})\lambda((\varpi^{e^0+\varepsilon})^{(p\mathfrak{F}\infty)})\lambda_{p\mathfrak{f}\infty}^{-1}(a)\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))$$

$$= w(\lambda)\lambda(\varpi^{e^0+\varepsilon}a\mathfrak{B}_j^{-1})\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}})),$$

where we regard λ as a character defined modulo $\prod_{\mathfrak{P}\in\Sigma''}\mathfrak{P}^{\infty}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{c\infty}\mathfrak{f}$ (and hence $\lambda(\varpi^{e^0}a^{(\mathfrak{l})}\mathfrak{B}_j^{-1})$ for $a^{(\mathfrak{l})}=aa_{\mathfrak{l}}^{-1}$ is well defined although $\varpi^{e^0}a^{(\mathfrak{l})}\mathfrak{B}_j^{-1}$ may contain prime factors $\mathfrak{P}\in\Sigma'$). Then we have, for

$$c = w(\lambda)\frac{(-1)^{m_0 t}\pi^d\Gamma_F(m_0 t+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}},$$

$$(4.10)\qquad \frac{\int_{G_\infty}\hat{\lambda}\,d\varphi}{\Omega_p^{m_0\Sigma+2d}}$$

$$= c\sum_j\sum_{a\in\mathfrak{r}+\mathfrak{f}\mathfrak{B}_j'/\mathfrak{r}^\times}\lambda(a\mathfrak{B}_j^{-1})\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}).N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))N(a\mathfrak{B}_j^{-1})^{-s}|_{s=0}$$

$$= (\mathfrak{R}^\times:\mathfrak{r}^\times)\,w(\lambda)\frac{(-1)^{m_0 t}\pi^d\Gamma_F(m_0 t+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$

$$\times\Big\{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda^*(\mathfrak{P}^c))\Big\}(1-\lambda(\mathfrak{L}))L(0,\lambda),$$

because $\{a\mathfrak{B}_j^{-1}\}$ for $a\in(\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{R}^\times$ and $j\in\mathrm{Cl}^-(\mathfrak{F})$ gives a representative set of $\mathrm{Cl}_M(\mathfrak{F})$.

We now compute $P\phi_j'^*$. Write the conductor of $\lambda_{\mathfrak{l}}$ as $\mathfrak{L}^{e(\mathfrak{l})}$ for $0\leqq e(\mathfrak{l})\leqq\varepsilon$ if $\mathfrak{l}$ remains prime in M and $0\leqq e(\mathfrak{l})\leqq 2\varepsilon$ when $\mathfrak{l}=\mathfrak{L}^2$. Writing $\phi_j'(x,y)=\lambda(\mathfrak{U}_j)^{-1}\lambda_\Sigma(x_p)\lambda_{\Sigma c}^{-1}(y_p)\phi'$, we have

$$(4.11)\qquad \phi'(a,b)=\begin{cases}\lambda_{\mathfrak{l}}^{-1}(b), & \text{if } (a,b)\in\{0\}\times\mathfrak{r}_{\mathfrak{l}}^\times,\\ 0, & \text{otherwise.}\end{cases}$$

Note that the dual map $i^*:\mathfrak{U}/\mathfrak{f}\mathfrak{U}\to\mathfrak{r}/\mathfrak{f}$ of $i(\mathfrak{U})_{\mathfrak{f}}:\mathfrak{f}^*/\mathfrak{r}^*\to\mathfrak{f}^{-1}\mathfrak{U}/\mathfrak{U}$ is given by $i^*(x)=\mathrm{Tr}_{M/F}(d_{\mathfrak{l}}x(2\delta)^{-1})$ mod $\mathfrak{f}$. Thus

$$\mathbf{e}_F(i^*(a_{\mathfrak{l}})\varpi_{\mathfrak{l}}^{-\varepsilon}bd_{\mathfrak{l}}^{-1})=\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathfrak{l}}^{-\varepsilon}ba_{\mathfrak{l}}).$$

Then we have

$$P\phi_j'^*(a)=\lambda(\mathfrak{U}_j)^{-1}N(\mathfrak{f})^{-1}N\Big(\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e'(\mathfrak{P})}\Big)\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times}\lambda_{\mathfrak{l}}(v)\,\mathbf{e}_M((2\delta)^{-1}va_{\mathfrak{l}}\varpi_{\mathfrak{l}}^{-\varepsilon})$$

$$\times\sum_{b\in(\mathfrak{r}/p^{e'})}\lambda_p^{-1}(b^{-1},a_{\Sigma c})\,\mathbf{e}_F((2\delta)^{-1}ba_\Sigma)$$

$$\overset{u=\varpi^{e'0}a_\Sigma}{=}\lambda(\mathfrak{U}_j)^{-1}N(\mathfrak{f})^{-1}N\Big(\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{e'(\mathfrak{P})}\Big)^{-1}\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times}\lambda_{\mathfrak{l}}(v)\,\mathbf{e}_M((2\delta)^{-1}va_{\mathfrak{l}}\varpi_{\mathfrak{l}}^{-\varepsilon})$$

$$\times \sum_{b \in (\mathfrak{r}/\mathfrak{p}^{e'})} \lambda_p^{-1}(b^{-1}, a_{\Sigma c})\, \mathbf{e}_F((2\delta)^{-1} bu\varpi_\Sigma^{-e'})$$

$$= \lambda(\mathfrak{U}_j)^{-1} N(\mathfrak{f})^{-1} \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times} \lambda_{\mathrm{I}}(v)\, \mathbf{e}_M((2\delta)^{-1} va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})$$

$$\times \prod_{\mathfrak{P} \in \Sigma''} N(\mathfrak{P}^{-e(\mathfrak{P})})\, G(2\delta;\, \lambda_{\mathfrak{P}})\, \lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\, \lambda_{\mathfrak{P}^c}^{-1}(a_{\mathfrak{P}^c})$$

$$\times \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\, \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})\, (\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1} \chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))$$

if $\varpi^e a \in \mathfrak{R}_{\mathfrak{Q}}^\times$ for all $\mathfrak{Q} \in \Sigma''$ and $a_{\mathrm{I}} \in \varpi_{\mathfrak{L}}^{\varepsilon - e(\mathfrak{l})} \mathfrak{R}_{\mathrm{I}}$. Put

$$\mathfrak{B}_j = \varpi^{e^0} \mathfrak{U}_j \{ \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^{-e(\mathfrak{P})} \}, \qquad \mathfrak{B}'_j = \varpi^{e^0} \mathfrak{U}_j \{ \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^{-e'(\mathfrak{P})} \},$$

$$\mathfrak{D}_j = \varpi^{e^0} \varpi_{\mathfrak{L}}^{e(\mathfrak{L}) - \varepsilon} \mathfrak{U}_j \{ \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^{-e(\mathfrak{P})} \} \mathfrak{L}^{\varepsilon - e(\mathfrak{l})},$$

$$\mathfrak{D}'_j = \varpi^{e^0} \varpi_{\mathfrak{L}}^{e(\mathfrak{l}) - \varepsilon} \mathfrak{U}_j \{ \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^{-e'(\mathfrak{P})} \} \mathfrak{L}^{\varepsilon - e(\mathfrak{l})},$$

$$w_p(\lambda) = \{ \prod_{\mathfrak{P} \in \Sigma} N(\mathfrak{P}^{-e(\mathfrak{P})})\, G(2\delta;\, \lambda_{\mathfrak{P}}) \}$$

and

$$w^*(\lambda) = \{ \prod_{\mathfrak{P} \in \Sigma} N(\mathfrak{P}^{-e(\mathfrak{P})})\, G(2\delta;\, \lambda_{\mathfrak{P}}) \}\, \lambda_{\mathfrak{f}}(\varpi_{\mathrm{I}}^{\varepsilon})\, N_{F/\mathbf{Q}}(\mathfrak{f})\, N_{M/\mathbf{Q}}(\mathfrak{L}^{-e(\mathfrak{l})})\, G(2\delta;\, \lambda_{\mathrm{I}}).$$

By the computation similar to (4.10), our value is expressed as the value at 0 of the infinite sum of $\mathrm{P}\phi_j'^*(a)\, a^{-m_0\Sigma - 2d(1-c)} N(a)^{-s}$ over a in $\varpi^{-e^0}\mathfrak{B}_j$. Then if $\varpi_j^{e^0}.a.\mathfrak{B}_j^{-1}$ is prime to Σ'' [otherwise $\mathrm{P}\phi_j'^*(a)$ vanishes],

$$\mathrm{P}\phi_j'^*(a)\, a^{-m_0\Sigma - 2d(1-c)}$$

$$= N(\mathfrak{f})^{-1} w_p(\lambda)\, \lambda(\mathfrak{U}_j^{-1}) \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times} \lambda_{\mathrm{I}}(v)\, \mathbf{e}_M((2\delta)^{-1} va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})$$

$$\times \lambda_{p\infty}^{-1}(a) \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})\, (\chi_{\mathfrak{P}}) - N(\mathfrak{P})^{-1} \chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))$$

$$= N(\mathfrak{f})^{-1} w_p(\lambda)\, \lambda(\mathfrak{U}_j^{-1}) \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times} \lambda_{\mathrm{I}}(a_{\mathrm{I}} v)\, \mathbf{e}_M((2\delta)^{-1} va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})$$

$$\times \lambda_{p\mathrm{I}\infty}^{-1}(a) \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})\, (\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1} \chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))$$

$$= N(\mathfrak{f})^{-1} w_p(\lambda) \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times} \lambda_{\mathrm{I}}(a_{\mathrm{I}} v)\, \mathbf{e}_M((2\delta)^{-1} a_{\mathrm{I}} v \varpi_{\mathrm{I}}^{-\varepsilon})\, \lambda(\mathfrak{B}_j^{-1})\, \lambda((\varpi^{e^0})^{(p\mathfrak{f}\infty)})$$

$$\times \lambda_{p\mathfrak{f}\infty}^{-1}(a) \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})\, (\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1} \chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))$$

$$= N(\mathfrak{f})^{-1} w_p(\lambda) \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times} \lambda_{\mathrm{I}}(a_{\mathrm{I}} v)\, \mathbf{e}_M((2\delta)^{-1} a_{\mathrm{I}} v \varpi_{\mathrm{I}}^{-\varepsilon})$$

$$\times \lambda(\varpi^{e^0} a^{(\mathfrak{l})} \mathfrak{B}_j^{-1}) \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})\, (\chi_{\mathfrak{P}}) - N(\mathfrak{P})^{-1} \chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}})),$$

where we regard λ as a character defined modulo $\prod_{\mathfrak{P}\in\Sigma''}\mathfrak{P}^{\infty}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{c\infty}\mathfrak{f}$ (and hence $\lambda(\varpi^{e^0}a^{(\mathrm{I})}\mathfrak{B}_j^{-1})$ is well defined although $\varpi^{e^0}a^{(\mathrm{I})}\mathfrak{B}_j^{-1}$ may contain prime factors $\mathfrak{P}\in\Sigma'$ for $a^{(\mathrm{I})}=aa_{\mathrm{I}}^{-1}$). We may assume that $\varpi^{e^0}\equiv 1 \bmod \mathfrak{f}\mathfrak{R}_{\mathrm{I}}$ by choosing $\varpi_{\mathfrak{P}}$ so that $\varpi_{\mathfrak{P}}\equiv 1 \bmod \mathfrak{f}\mathfrak{R}_{\mathrm{I}}$. This shows that

$$\lambda_{\mathrm{I}}(a_{\mathrm{I}}v)\,\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})=\lambda_{\mathrm{I}}(\varpi^{e^0}a_{\mathrm{I}}v)\,\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}\varpi^{e^0}va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon}).$$

Thus in principle, we can compute $\int_{\mathrm{G}_\infty}\hat{\lambda}\,d\varphi^*/\Omega_p^{m_0\Sigma+2d}$. We write

$$c=w^*(\lambda)\frac{(-1)^{m_0t}\pi^d\Gamma_{\mathrm{F}}(m_0t+d)}{\sqrt{|\mathrm{D_F}|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}\qquad\text{and}\qquad \mathfrak{r}^\times(\mathfrak{f})=\{\varepsilon\in\mathfrak{r}^\times\,|\,\varepsilon\equiv 1 \bmod \mathfrak{f}\}.$$

Then we have

$$\begin{aligned}
\frac{\int_{\mathrm{G}_\infty}\hat{\lambda}\,d\varphi^*}{\Omega_p^{m_0\Sigma+2d}}&=c\{\mathrm{G}(2\delta,\lambda_{\mathrm{I}})\lambda_{\mathrm{I}}(\varpi_{\mathrm{I}}^{\varepsilon})N_{\mathrm{M/Q}}(\mathfrak{f}\mathfrak{L}^{-e(\mathrm{I})})\}^{-1}\\
&\quad\sum_j\lambda(\mathfrak{B}_j^{-1})\sum_{a\in\mathfrak{B}_j'/\mathfrak{r}^\times}\lambda(a^{(p\mathfrak{f}\infty)})\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times}\lambda_{\mathrm{I}}v)\,\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}va_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})\\
&\quad\times\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))\,N(a\mathfrak{B}_j^{-1})^{-s}|_{s=0}\\
&=c\{\mathrm{G}(2\delta,\lambda_{\mathrm{I}})\lambda_{\mathrm{I}}(\varpi_{\mathrm{I}}^{\varepsilon})N_{\mathrm{M/Q}}(\mathfrak{f}\mathfrak{L}^{-e(\mathrm{I})}))\}^{-1}\\
&\quad\times\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{r}^\times}\lambda_{\mathrm{I}}(v)\sum_j\lambda(\mathfrak{B}_j^{-1})\sum_{a\in\mathfrak{B}_j'/\mathfrak{r}^\times}\lambda(a^{(p\mathfrak{f}\infty)})\lambda_{\mathrm{I}}(a)\\
&\quad\times\sum_{u\in\mathfrak{r}^\times/\mathfrak{r}^\times(\mathfrak{f})}\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}vua_{\mathrm{I}}\varpi_{\mathrm{I}}^{-\varepsilon})\\
&\quad\times\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))\,N(a\mathfrak{B}_j^{-1})^{-s}|_{s=0}\\
&=c\{\mathrm{G}(2\delta,\lambda_{\mathrm{I}})\lambda_{\mathrm{I}}(\varpi_{\mathrm{I}}^{\varepsilon})N_{\mathrm{M/Q}}(\mathfrak{f}\mathfrak{L}^{-e(\mathrm{I})})\}^{-1}\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{r}^\times}\lambda(v^{(\mathrm{I})})^{-1}\sum_j\lambda(\mathfrak{B}_j^{-1})\\
&\quad\times\sum_{a\in\mathfrak{B}_j'/\mathfrak{r}^\times(\mathfrak{f})}\lambda(a^{(p\mathfrak{f}\infty)})\lambda_{\mathrm{I}}(a)\,\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}va_{\mathrm{I}}\omega_{\mathrm{I}}^{-\varepsilon}\\
&\quad\times\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))\,N(a\mathfrak{B}_j^{-1})^{-s}|_{s=0}\\
&\qquad\text{(choosing representatives of } v \text{ prime to } p \text{ in } \mathfrak{r})\\
&=c\{\mathrm{G}(2\delta,\lambda_{\mathrm{I}})\lambda_{\mathrm{I}}(\varpi_{\mathrm{I}}^{\varepsilon})N_{\mathrm{M/Q}}(\mathfrak{f}\mathfrak{L}^{-e(\mathrm{I})})\}^{-1}\sum_{v\in(\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{r}^\times}\lambda_{p\infty}(v)^{-1}\sum_j(v^{-1}\mathfrak{B}_j^{-1})\\
&\quad\times\sum_{a\in\mathfrak{B}_j'/\mathfrak{r}^\times(\mathfrak{f})}\lambda(a^{(p\mathfrak{f}\infty)})\lambda_{\mathrm{I}}(a)\,\mathbf{e}_{\mathrm{M}}((2\delta)^{-1}va_{\mathrm{I}}\omega_{\mathrm{I}}^{-\varepsilon})
\end{aligned}$$

$$\times \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))\, N(a\mathfrak{B}_j^{-1})^{-s}|_{s=0}$$

$$= c\{G(2\delta, \lambda_I)\lambda_I(\varpi_I^{\varepsilon}) N_{M/Q}(\mathfrak{f}\mathfrak{L}^{-e(I)})\}^{-1} \sum_{v \in (/\mathfrak{f})^\times/\mathfrak{r}^\times} \lambda_{p\infty}(v)^{-1} \sum_j \lambda(v^{-1}\mathfrak{B}_j^{-1})$$

$$\times \sum_{a \in v\mathfrak{B}_j'/\mathfrak{r}^\times(\mathfrak{f})} \lambda((v^{-1}a)^{(p\mathfrak{f}\infty)})\lambda_I(v^{-1}a)\, \mathbf{e}_M((2\delta)^{-1} v a_I \varpi_I^{-\varepsilon})$$

$$\times \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1}(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))\, N(va\mathfrak{B}_j^{-1})^{-s}|_{s=0}$$

use the fact: $\chi_{\mathfrak{Q}}(va_{\mathfrak{Q}}) = \chi_{\mathfrak{Q}}(a_{\mathfrak{Q}}))$

$$= c\{G(2\delta, \lambda_I)\lambda_I(\varpi_I^{\varepsilon}) N_{M/Q}(\mathfrak{f}\mathfrak{L}^{-e(I)})\}^{-1}$$

$$\times \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{r}^\times} \sum_j \lambda(v^{-1}\mathfrak{B}_j^{-1}) \sum_{a \in v\mathfrak{B}_j'/\mathfrak{r}^\times(\mathfrak{f})} \lambda(a^{(p\mathfrak{f}\infty)})\lambda_I(a)\, \mathbf{e}_M((2\delta)^{-1} v a_I \varpi_I^{-\varepsilon})$$

$$\times \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N\mathfrak{B}_j^{-1})(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))\, N(va\mathfrak{B}_j^{-1})^{-s}|_{s=0}$$

$$= c\{G(2\delta, \lambda_I)\lambda_I(\varpi_I^{\varepsilon}) N_{M/Q}(\mathfrak{f}\mathfrak{L}^{-e(I)})\}^{-1}$$

$$\times \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times/\mathfrak{r}^\times} \sum_j \sum_{a \in v\mathfrak{B}_j'/\mathfrak{r}^\times(\mathfrak{f})} \lambda(av^{-1}\mathfrak{B}_j^{-1})\lambda_I(a)\, \mathbf{e}_M((2\delta)^{-1} vua_I \varpi_I^{-\varepsilon})$$

$$\times \prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1}(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))\, N(va\mathfrak{B}_j^{-1})^{-s}|_{s=0}$$

(use the fact: $(\mathfrak{R}^\times\ \mathfrak{r}^\times) = (\mathfrak{R}^\times/\mathfrak{R}^\times(\mathfrak{f}) : \mathfrak{r}^\times/\mathfrak{r}^\times(\mathfrak{f}))(\mathfrak{R}^\times(\mathfrak{f}) : \mathfrak{r}^\times(\mathfrak{f})))$

$$= c(\mathfrak{R}^\times : \mathfrak{r}^\times)\{G(2\delta, \lambda_I)\lambda_I(\varpi_I^{\varepsilon}) N_{M/Q}(\mathfrak{f}\mathfrak{L}^{-e(I)})\}^{-1}$$

$$\times \sum_{v \in (\mathfrak{r}/\mathfrak{f})^\times \mathfrak{R}^\times/\mathfrak{R}^\times} \sum_j \sum_{a \in v\mathfrak{B}_j/\mathfrak{R}^\times(\mathfrak{f})} \lambda(av^{-1}\mathfrak{B}_j^{-1}))\lambda_I(a)\, \mathbf{e}_M((2\delta)^{-1} vua_I \varpi_I^{-\varepsilon})$$

$$\prod_{\mathfrak{P} \in \Sigma'} \lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1}(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}} a_{\mathfrak{P}}))\, N(va\mathfrak{B}_j^{-1}))^{-s}|_{s=0}$$

$$= (\mathfrak{R}^\times : \mathfrak{r}^\times)\, c_1\, c\{G(2\delta, \lambda_I)\lambda_I(\varpi_I^{\varepsilon}) N_{M/Q}(\mathfrak{f}\mathfrak{L}^{-e(I)})\}^{-1}$$

$$\times \int_{M_{\mathbf{A}}} \Phi(x)\, \mathbf{e}_M((2\delta)^{-1}\varpi_I^{-\varepsilon} x_f)\lambda(x)|x|_{\mathbf{A}}^{s} d^\times x|_{s=0}$$

where $c_1 \neq 0$ is a suitable constant, $\Phi = \prod_v \Phi_v(x_v)$ is a Schwartz function such that Φ_v is a characteristic function of $\mathfrak{R}_v$ if v is outside p^∞, $\Phi_\sigma(x_\sigma) = x_\sigma^{c(m_0+2d_\sigma)} \exp(-2\pi \mathfrak{R}_{C/R}(x_v))$ at each infinite place σ, at Σ'' it is a characteristic function of $\mathfrak{R}_v^\times$ times λ_v^{-1} and at $\mathfrak{Q} \in \Sigma'$, it is $(\chi_{\mathfrak{Q}}(x_{\mathfrak{Q}}) - N(\mathfrak{Q})^{-1}\chi_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}} x_v))$. We have chosen the multiplicative Haar measure $d^\times x_f$ so that for any open compact subgroup U of $\hat{\mathfrak{R}}^\times$, $\int_U d^\times x_f = (\hat{\mathfrak{R}}^\times : U)^{-1}$ and at each infinite place σ, we take $d^\times x_\sigma = |x_\sigma|^{-2}|dx_\sigma \wedge dx_\sigma^c|$. Let

$$U = U(\mathfrak{f}) = \{x \in \hat{\mathfrak{R}}^\times \mid x \equiv 1 \bmod \mathfrak{f}\hat{\mathfrak{R}}\}.$$

In fact, decomposing $M_{\mathbf{A}}^{\times}=\bigcup_{j,v} M^{\times} v^{-(p\mathrm{I}\infty)} b_j^{-1} \mathrm{U} M_{\infty}^{\times}$, (where v runs over a representative set for $(\mathfrak{r}/\mathfrak{f})^{\times} \mathfrak{R}^{\times}/\mathfrak{R}^{\times}$), we see

$$\int_{M_{\mathbf{A}}} \Phi(x)\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon} x_{\mathrm{I}})\,\lambda(x)\,|x|_{\mathbf{A}}{}^{s} d^{\times} x\big|_{s=0}$$

$$=\mathrm{vol}(\mathrm{U})\int_{M_{\mathbf{A}}^{\times}/\mathrm{U}} \Phi(x)\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon} x_{\mathrm{I}})\,\lambda(x)\,|x|_{\mathbf{A}^s} d^{\times} x\big|_{s=0}$$

$$=\mathrm{vol}(\mathrm{U})\int_{M^{\times}\backslash M_{\mathbf{A}}^{\times}/\mathrm{U}} \sum_{\xi\in M^{\times}} \Phi(\xi x)\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon} x_{\mathrm{I}})\,\lambda(x)\,|x|_{\mathbf{A}^s} d^{\times} x\big|_{s=0}$$

$$=\mathrm{vol}(\mathrm{U})\sum_{v,j}\lambda(v^{-(p\mathrm{I}\infty)} b_j^{-1})\,|v^{-(p\mathrm{I}\infty)} b_j^{-1}|_{\mathbf{A}^s}$$

$$\times\int_{M_{\infty}^{\times}/\mathfrak{R}^{\times}(\mathfrak{f})} \sum_{\xi\in M^{\times}} \Phi(\xi v^{-(p\mathrm{I}\infty)} b_j^{-1} x_{\infty})\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon}\xi_{\mathrm{I}})\,\lambda(x_{\infty})\,|x_{\infty}|_{\mathbf{A}^s} d^{\times} x\big|_{s=0}$$

$$=\mathrm{vol}(\mathrm{U})\,\mathrm{G}(s)\sum_{v\in(\mathfrak{r}/\mathfrak{f})^{\times}\mathfrak{R}^{\times}/\mathfrak{R}^{\times}}\ \sum_{j}\ \sum_{\xi\in v\mathfrak{B}_j'/\mathfrak{R}^{\times}(\mathfrak{f})}$$

$$\times\,\Phi_f(\xi v^{-(p\mathrm{I}\infty)} b_j^{-1})\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon}\xi_{\mathrm{I}})\,\lambda(\xi_{\mathrm{I}})\,\lambda(\xi v^{-1}\mathfrak{B}_j^{-1})\,N(\xi v^{-1}\mathfrak{B}_j^{-1})^{-s}\big|_{s=0}$$

$$=\mathrm{vol}(\mathrm{U})\,\mathrm{G}(s)\sum_{v\in(\mathfrak{r}/\mathfrak{f})^{\times}\mathfrak{R}^{\times}/\mathfrak{R}^{\times}}\ \sum_{j}\ \sum_{\xi\in v\mathfrak{B}_j'/\mathfrak{R}^{\times}(\mathfrak{f})}$$

$$\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(\xi)\,(\chi_{\mathfrak{P}}(\xi_{\mathfrak{P}})-N(\mathfrak{P})^{-1}(\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}\xi_{\mathfrak{P}})$$

$$\times\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon}\xi_{\mathrm{I}})\,\lambda(\xi_{\mathrm{I}})\,\lambda(\xi v^{-1}\mathfrak{B}_j^{-1})\,N(\xi v^{-1}\mathfrak{B}_j^{-1}))^{-s}\big|_{s=0}$$

where $\mathfrak{R}^{\times}(\mathfrak{f})=\{u\in\mathfrak{R}^{\times}\,|\,u\equiv 1 \bmod \mathfrak{f}\}$ and

$$\mathrm{G}(s)=\prod_{\sigma\in\Sigma}\int_{\mathbf{C}^{\times}}\exp(-2\pi|x_{\sigma}|^2)\,|x_{\sigma}|^{2(s+m_0+d_{\sigma}-1)}\,|dx_{\sigma}\wedge dx_{\sigma}^{c}|$$

$$=(2\pi)^{t-(st+m_0t+d)}\,\Gamma_{\mathrm{F}}(st+m_0t+d).$$

Thus $c_1=\mathrm{vol}(\mathrm{U}(\mathfrak{f}))^{-1}\,\mathrm{G}(0)^{-1}=\varphi(\mathfrak{f})\,\{(2\pi)^{t-(m_0t+d)}\,\Gamma_{\Sigma}(m_0t+d)\}^{-1}$ for the Euler function φ of M. On the other hand, we see

$$\int_{M_{\mathbf{A}}} \Phi(x)\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon} x_{\mathrm{I}})\,\lambda(x)\,|x|_{\mathbf{A}^s} d^{\times} x$$

$$=\int_{M_{\mathrm{I}}}\Phi_{\mathrm{I}}(x)\,\mathbf{e}_M((2\delta)^{-1}\varpi_{\mathrm{I}}^{-\varepsilon} x_{\mathrm{I}})\,\lambda_{\mathrm{I}}(x)\,|x|_{\mathrm{I}}^{s} d^{\times} x_{\mathrm{I}}\prod_{v\neq\mathrm{I}}\int_{M_v}\Phi_v(x)\,\lambda_v(x)\,|x|_v^{s} d^{\times} x_v.$$

Note that

$$\int_{M_I} \Phi_I(x)\, \mathbf{e}_M((2\delta)^{-1}\varpi_I^{-\varepsilon} x_I)\lambda_I(x)|x|_I^s d^\times x_I$$

$$= \sum_{0 \leqq \alpha} \lambda_I(\varpi_{\mathfrak{L}}^{\alpha})|\varpi_{\mathfrak{L}}^{\alpha}|_I^s \int_{\mathfrak{R}_I^\times} \mathbf{e}_M((2\delta)^{-1}\varpi_{\mathfrak{L}}^{\alpha}\varpi_I^{-\varepsilon} x_I)\lambda_I(x)\, d^\times x_I$$

and if λ_I is non-trivial on $\mathfrak{R}_I^\times$, writing $\mathfrak{L}^e$ for the conductor of λ_I, we have

$$\int_{\mathfrak{R}_I^\times} \lambda_I(x)\, \mathbf{e}_M(x\varpi^{\alpha}(2\delta)^{-1})\, d^\times x = \begin{cases} 0, & \text{if } \alpha \neq -e, \\ \varphi(\mathfrak{L}^e)^{-1}\lambda_I(\varpi_{\mathfrak{L}}^{e})\, G(2\delta, \lambda_I), & \text{if } \alpha = -e, \end{cases}$$

and if λ_I is trivial,

$$\int_{\mathfrak{R}_I^\times} \lambda_I(x)\, \mathbf{e}_M(x\varpi^{\alpha}(2\delta)^{-1})\, d^\times x = \frac{N(\mathfrak{L})}{N(\mathfrak{L})-1}\{\chi\mathfrak{R}_I(\varpi^{\alpha}) - N(\mathfrak{L})^{-1}\chi\mathfrak{R}_I(\varpi^{\alpha+1})\}.$$

This shows that

$$c_1 \int_{M_{\mathbf{A}}} \Phi(x)\, \mathbf{e}_M((2\delta)^{-1}\varpi_I^{-\varepsilon} x_I)\lambda(x)|x|_{\mathbf{A}}^s d^\times x|_{s=0}$$

$$= \{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\} L(0, \lambda)$$

$$\times \begin{cases} N_{M/\mathbf{Q}}(\mathfrak{L}^{\varepsilon}\mathfrak{L}^{-e})\lambda_I(\varpi_I^{\varepsilon})\, G(2\delta, \lambda_I) & \text{if } e>0, \\ N_{M/\mathbf{Q}}(\mathfrak{L}^{\varepsilon})\lambda_I(\varpi_I^{\varepsilon})(1-\lambda^*(\mathfrak{L})) & \text{if } e=0. \end{cases}$$

Thus we have

$$(4.12\,a) \quad \frac{\int_{G_\infty} \hat{\lambda}\, d\varphi^*}{\Omega_p^{m_0\Sigma+2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times)\, c\,(1-\lambda^*(\mathfrak{L}))\{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\} L(0, \lambda),$$

where

$$c = w^*(\lambda)\frac{(-1)^{m_0 t}\pi^d\,\Gamma_F(m_0 t+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$

with

$$w^*(\lambda) = \{\prod_{\mathfrak{P}\in\Sigma} N(\mathfrak{P}^{-e(\mathfrak{P})})\, G(2\delta;\lambda_{\mathfrak{P}})\}\lambda_{\mathfrak{l}}(\varpi_I^{\varepsilon})\,\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{l}^{\varepsilon})\,\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{L}^{-e(\mathfrak{l})})\, G(2\delta;\lambda_I).$$

This also shows that

$$(4.12b)\quad (1-\lambda(\mathfrak{L}))\int_{G_\infty}\hat{\lambda}\,d\varphi^*$$

$$=\mathfrak{N}_{F/Q}(\mathfrak{l}^{\varepsilon})\,\mathfrak{N}_{M/Q}(\mathfrak{L}^{-e(\mathfrak{l})})\,G(2\delta,\lambda_{\mathfrak{l}})\,\lambda_{\mathfrak{l}}(\varpi_{\mathfrak{l}}^{2\varepsilon})(1-\lambda^*(\mathfrak{L}))\int_{G_\infty}\hat{\lambda}\,d\varphi.$$

We now treat the case of split primes. We fix a split prime $\mathfrak{l}=\mathfrak{L}\mathfrak{L}^c$ prime to p. Let us write $\mathfrak{f}=\mathfrak{l}^{\varepsilon(\mathfrak{l})}$ and put $\varpi^{\varepsilon}=\prod_{\mathfrak{l}}\varpi_{\mathfrak{l}}^{\varepsilon(\mathfrak{l})}$. According to (3.1), we define

$$\phi^0(x,y)=P_{\mathfrak{f}}^{-1}\,\phi(x_p^{-1},x_{\mathfrak{f}}\,y)=\sum_{a\in(\mathfrak{r}/\mathfrak{f})}\phi(a,y)\,\mathbf{e}_F(-ax_{\mathfrak{f}}\,d_{F_{\mathfrak{f}}}^{-1}\,\varpi_{\mathfrak{F}}^{-\varepsilon}).$$

Write the conductor of λ as $\mathfrak{L}^{e'}\mathfrak{L}^{ce''}$ and $\mathfrak{C}=\mathfrak{L}^{\varepsilon'}\mathfrak{L}^{c\varepsilon''}$ $(0\leqq e'\leqq\varepsilon'$ and $0\leqq e''\leqq\varepsilon'')$. Thus $\varepsilon=\varepsilon'$. Let $G_\infty=Cl_M(\mathfrak{C}p^\infty)$ and identify $\mathfrak{R}/\mathfrak{L}^{\varepsilon'}\times\mathfrak{R}/\mathfrak{L}^{\varepsilon''}$ with $(\mathfrak{r}/\mathfrak{f}')\times(\mathfrak{r}/\mathfrak{f}'')$ for $\mathfrak{f}'=\mathfrak{l}^{\varepsilon'}$ and $\mathfrak{f}''=\mathfrak{l}^{\varepsilon''}$. For each function ϕ on $G_\infty(\mathfrak{C})$, we have associated a function ϕ_j on T by

$$\phi_j(x,y)=\phi((x,y)[\mathfrak{U}_j^{-1}])\quad\text{for}\quad(x,y)\in\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_{\mathfrak{L}}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times,$$

$$\phi_j(x,y)=0\quad\text{outside}\quad\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_{\mathfrak{L}}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times,\quad\text{if }\varepsilon'\varepsilon''\neq 0,$$

$$\phi_j(x,y)=\phi((x,y_p)[\mathfrak{U}_j^{-1}])\quad\text{for}\quad(x,y)\in\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_{\mathfrak{L}}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}\quad\text{if}\quad\varepsilon'<\varepsilon''=0.$$

(If $\varepsilon'=0$, then $\varepsilon''=0$ and thus $\mathfrak{C}=1$). As already chosen, we have $d_{F_{\mathfrak{l}}}=(2\delta)_{\mathfrak{L}}$. If ϕ is a character $\hat{\lambda}$, then

$$\phi_j=\lambda(\mathfrak{U}_j)^{-1}\,\lambda_{p\mathfrak{l}}^{-1}\quad\text{on}\quad\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_{\mathfrak{L}}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times$$

and

$$\lambda_{p\mathfrak{l}}^{-1}(x_p,x_{\mathfrak{L}},y)=\lambda_p^{-1}(x_py_p)\,\lambda_{\mathfrak{L}^c}^{-1}(y_{\mathfrak{L}^c})\,\lambda_{\mathfrak{L}}^{-1}(x_{\mathfrak{L}}).$$

Then by (3.2a)

$$P\phi_j^0(a)=\lambda(\mathfrak{U}_j)^{-1}\,\lambda_{\mathfrak{L}}^{-1}(d_{F_{\mathfrak{l}}}\varpi_{\mathfrak{L}}^{\varepsilon}a_{\mathfrak{L}})\,\lambda_{\mathfrak{L}^c}(a_{\mathfrak{L}^c})\,P_p\lambda_p(a_p).$$

We now put $e^0(\mathfrak{P})$ if $\mathfrak{P}\in\Sigma$ and $e^0(\mathfrak{Q})=0$ otherwise. Put

$$\mathfrak{B}_j=\varpi^{e^0+\varepsilon}\,\mathfrak{U}_j\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e(\mathfrak{P})}\prod_{\mathfrak{L}\mid\mathfrak{F}}\mathfrak{L}^{-\varepsilon(\mathfrak{L})},$$

$$w_p(\lambda)=\Big\{\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})\,G(2\delta;\lambda_{\mathfrak{P}})\Big\}$$

and

$$w(\lambda)=\lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon'})\Big\{\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})\,G(2\delta;\lambda_{\mathfrak{P}})\Big\}.$$

Note that for any $\mathfrak{U}=\mathfrak{U}_j$, $\mathrm{PV}(\mathfrak{U})=\{\mathrm{M}_\Sigma\times\mathfrak{L}^{-\varepsilon}\mathfrak{R}_\mathfrak{L}\}\oplus\mathfrak{R}_{\Sigma^c\mathfrak{L}^c}$, where

$$\mathrm{M}_\Sigma=\prod_{\mathfrak{P}\in\Sigma}\mathrm{M}_\mathfrak{P}\qquad\text{and}\qquad \mathfrak{R}_{\Sigma^c\mathfrak{L}^c}=\mathfrak{R}_{\mathfrak{L}^c}\times\prod_{\mathfrak{P}\in\Sigma}\mathfrak{R}_{\mathfrak{P}^c}.$$

This shows that $\pi(a)=(2\delta)^{-1}a_{\Sigma\mathfrak{L}} \bmod \vartheta_{pl}^{-1}$ and $\pi'(a)=a_{\Sigma^c}\times(a_{\mathfrak{L}^c} \bmod \mathfrak{L}^{c\varepsilon})$. First suppose that $\varepsilon''>0$. Then we have, for $\Sigma'=\{\mathfrak{Q}\in\Sigma\,|\,e(\mathfrak{Q})=0\}$

$$\mathrm{P}\,\phi_j^0(a)\,a^{-m_0\Sigma-2d(1-c)}=w(\lambda)\,\lambda(\varpi^{e^0+\varepsilon}a\,\mathfrak{B}_j^{-1})\prod_{\mathfrak{P}\in\Sigma'}\lambda_\mathfrak{P}(a_\mathfrak{P})(\chi_\mathfrak{P}(a_\mathfrak{P})-N(\mathfrak{P})^{-1}(\chi_\mathfrak{P}(\varpi_\mathfrak{P}a_\mathfrak{P})),$$

where we consider λ as a character defined modulo $\mathfrak{C}\prod_{\mathfrak{P}\in\Sigma''}\mathfrak{P}^\infty\prod_{\mathfrak{P}\in\Sigma c}\mathfrak{P}^\infty$ and thus $\lambda(\varpi^{e^0+\varepsilon}a\,\mathfrak{B}_j^{-1})$ is well defined although $\varpi^{e^0+\varepsilon}a\,\mathfrak{B}_j^{0-1}$ may have a factor $\mathfrak{P}^n$ for $\mathfrak{P}\in\Sigma'$ and $n\leqq-1$. Thus we have

$$(4.14\,a)\quad \frac{\int_{G_\infty}\hat{\lambda}\,d\varphi}{\Omega_p^{m_0\Sigma+2d}}=(\mathfrak{R}^\times:\mathfrak{r}^\times)\,w(\lambda)\,\frac{(-1)^{m_0t}\pi^d\,\Gamma_\mathrm{F}(m_0t+d)}{\sqrt{|\mathrm{D_F}|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times\{(1-\lambda(\mathfrak{L}))(1-\lambda(\mathfrak{L}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\}\,L(0,\lambda).$$

Suppose that $\varepsilon''=0$. Then we have, for $\Sigma'=\{\mathfrak{Q}\in\Sigma\,|\,e(\mathfrak{Q})=0\}$

$$\mathrm{P}\,\phi_j^0(a)\,a^{-m_0\Sigma-2d(1-c)}=w(\lambda)\,\lambda(\varpi^{e^0+\varepsilon}a\,\mathfrak{B}_j^{-1})\,\lambda_{\mathfrak{L}^c}(a)$$
$$\times\prod_{\mathfrak{P}\in\Sigma'}\lambda_\mathfrak{P}(a_\mathfrak{p})(\chi_\mathfrak{P}(a_\mathfrak{P})-N(\mathfrak{P})^{-1}\chi_\mathfrak{P}(\varpi_\mathfrak{P}a_\mathfrak{P})),$$

where we consider λ is a character modulo $\mathfrak{C}\prod_{\mathfrak{P}\in\Sigma''}\mathfrak{P}^\infty\prod_{\mathfrak{P}\in\Sigma c}\mathfrak{P}^\infty$. Thus we have

$$(4.14\,b)\quad \frac{\int_{G_\infty}\hat{\lambda}\,d\varphi}{\Omega_p^{m_0\Sigma+2d}}=(\mathfrak{R}^\times:\mathfrak{r}^\times)\,w(\lambda)\,\frac{(-1)^{m_0t}\pi^d\,\Gamma_\mathrm{F}(m_0t+d)}{\sqrt{|\mathrm{D_F}|}\,\mathrm{Im}\,(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times(1-\lambda(\mathfrak{L}))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\}\,L(0,\lambda).$$

We now compute $\int_{G_\infty(\mathfrak{C}^c)}\hat{\lambda}\,d\varphi^*$ for characters λ of conductor dividing $\mathfrak{C}^c p^\infty$. In this case, φ_j' is given as follows:

$$\phi_j'(x,y)=\phi((x,y)[\mathfrak{U}_j^{-1}])\quad\text{for}\quad (x,y)\in\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_\mathfrak{L}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times,$$
$$\varphi_j'(x,y)=0\qquad\text{outside}\quad \mathfrak{R}_\Sigma^\times\times\mathfrak{R}_\mathfrak{L}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times,$$
$$\phi_j'(x,y)=\phi((x_p,y)[\mathfrak{U}_j^{-1}])\qquad\text{for}\quad (x,y)\in\mathfrak{R}_\Sigma^\times\times\mathfrak{R}_\mathfrak{L}\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times\quad\text{if}\quad \varepsilon''=0,$$
$$\phi_j'(x,y)=0\qquad\text{outside}\quad \mathfrak{R}_\Sigma^\times\times\mathfrak{R}_\mathfrak{L}^\times\times\mathfrak{R}_{\Sigma c}^\times\times\mathfrak{R}_{\mathfrak{L}^c}^\times\quad\text{if}\quad \varepsilon''\neq0.$$

First suppose that $\varepsilon''>0$. When $\phi=\hat{\lambda}$, we write the conductor of λ as $\prod_{\mathfrak{P}|p}\mathfrak{P}^{e(\mathfrak{P})}\prod_{\mathfrak{L}|\mathfrak{l}}\mathfrak{L}^{e(\mathfrak{L})}$. We write $\mathfrak{L}$ for the prime factor of $\mathfrak{F}$. Then we put $\Sigma_1=\{\mathfrak{L}^c\,|\,e(\mathfrak{L}^c)>0\}$, $\Sigma_1'=\{\mathfrak{L}^c\,|\,e(\mathfrak{L}^c)=0\}$, $\Sigma'=\{\mathfrak{P}\in\Sigma\,|\,e(\mathfrak{P})=0\}$ and $\Sigma''=\Sigma-\Sigma'$. Then we see from (3.2*b*), writing $\chi_{\mathfrak{Q}}$ for the characteristic function of $\mathfrak{R}_{\mathfrak{Q}}$ for each prime $\mathfrak{Q}$ of $\mathfrak{R}$,

$$\begin{aligned}P\phi_j'^*(a)=\lambda(\mathfrak{U}_j^{-1})\lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon}a_{\mathfrak{L}})\Big\{\prod_{\mathfrak{P}\in\Sigma''}N(\mathfrak{P}^{-e(\mathfrak{P})})G(2\delta,\lambda_{\mathfrak{P}})\lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\\ \times\prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}^{-1}(a_{\mathfrak{P}})\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))\Big\}\\ \times N(\mathfrak{L}^{-ce(\mathfrak{L}^c)})G(2\delta,\lambda_{\mathfrak{L}^c})\lambda_{\Sigma^c}^{-1}(a_{\Sigma^c})\Big\{\sum_{\mathfrak{Q}\in\Sigma_1}\lambda_{\mathfrak{Q}}^{-1}(\varpi_{\mathfrak{L}^c}^{-\varepsilon}a_{\mathfrak{Q}})\\ \times\prod_{\mathfrak{Q}\in\Sigma_1'}\lambda_{\mathfrak{Q}}^{-1}(\varpi_{\mathfrak{Q}}^{-\varepsilon}a_{\mathfrak{Q}})\lambda_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{-\varepsilon}a_{\mathfrak{Q}})(\chi_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{-\varepsilon}a_{\mathfrak{Q}})-N(\mathfrak{Q})^{-1}\chi_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{1-\varepsilon}a_{\mathfrak{Q}}))\Big\}\end{aligned}$$

if $a\in\mathfrak{U}_j\mathfrak{L}^{-\varepsilon}\mathfrak{L}^{c(\varepsilon-e'(\mathfrak{L}^c))}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e'(\mathfrak{P})}$. Outside $\mathfrak{U}_j\mathfrak{L}^{-\varepsilon}\mathfrak{L}^{c(\varepsilon-e'(\mathfrak{L}^c))}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e'(\mathfrak{P})}$, $P\phi_j'^*(a)=0$ and if $a\mathfrak{B}_j^{-1}$ is divisible by $\mathfrak{Q}$ for one of primes $\mathfrak{Q}$ with either $e(\mathfrak{Q})>0$ or $\mathfrak{Q}=\mathfrak{L}$, then $P\phi_j'^*(a)=0$, where

$$\mathfrak{B}_j=\varpi_{\mathfrak{L}}^{\varepsilon}\varpi_{\mathfrak{L}^c}^{-\varepsilon+e(\mathfrak{L}^c)}\varpi^{e^0}\mathfrak{U}_j\mathfrak{L}^{-\varepsilon}\mathfrak{L}^{c(\varepsilon-e(\mathfrak{L}^c))}\prod_{\mathfrak{P}\in\Sigma}\mathfrak{P}^{-e(\mathfrak{P})}$$

and $e'(\mathfrak{Q})=e(\mathfrak{Q})$ if $e(\mathfrak{Q})>0$, and $e'(\mathfrak{Q})=1$ if $e(\mathfrak{Q})=0$. We also put

$$w^*(\lambda)=N(\mathfrak{L}^{-ce(\mathfrak{L}^c)})G(2\delta,\lambda_{\mathfrak{L}^c})\lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon})\lambda_{\mathfrak{L}^c}^{-1}(\varpi_{\mathfrak{L}^c}^{-\varepsilon})\Big\{\times\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})G(2\delta;\lambda_{\mathfrak{P}})\Big\}.$$

Then we see

$$\begin{aligned}P\phi_j'^*(a)a^{-m_0\Sigma-d(1-c)}=w^*(\lambda)\lambda(\varpi_{\mathfrak{L}}^{\varepsilon}\varpi^{e^0}\varpi_{\mathfrak{L}^c}^{e(\mathfrak{L}^c)-\varepsilon}a\mathfrak{B}_j^{-1})\\ \prod_{\mathfrak{P}\in\Sigma'}\lambda_{\mathfrak{P}}(a_{\mathfrak{P}})(\chi_{\mathfrak{P}}(a_{\mathfrak{P}})-N(\mathfrak{P})^{-1}\chi_{\mathfrak{P}}(\varpi_{\mathfrak{P}}a_{\mathfrak{P}}))\\ \times\prod_{\mathfrak{Q}\in\Sigma_1}{}'\lambda_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{-\varepsilon}a_{\mathfrak{Q}})(\chi_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{-\varepsilon}a_{\mathfrak{Q}})-N(\mathfrak{Q})^{-1}\chi_{\mathfrak{Q}}(\varpi_{\mathfrak{Q}}^{1-\varepsilon}a_{\mathfrak{Q}})).\end{aligned}$$

We then have

$$(4.15a)\quad\frac{\int_{G_\infty}\hat{\lambda}\,d\varphi^*}{\Omega_p^{m_0\Sigma+2d}}=(\mathfrak{R}^\times:\mathfrak{r}^\times)\,w(\lambda)\frac{(-1)^{m_0t}\pi^d\Gamma_F(m_0t+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times(1-\lambda(\mathfrak{L}^c))(1-\lambda^*(\mathfrak{L}^c))\Big\{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\Big\}L(0,\lambda).$$

Now suppose that $\varepsilon''=0$. The computation is essentially same as above. We only stat the result. Put

$$w^*(\lambda)=N(\mathfrak{L}^{-ce(\mathfrak{L}^c)})\,G(2\delta,\lambda^{\mathfrak{L}}{}_c)\,\lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon})\,\lambda^{\mathfrak{L}}{}_c(\varpi_{\mathfrak{L}^c}^{\varepsilon})\Big\{\prod_{\mathfrak{P}\in\Sigma}N(\mathfrak{P}^{-e(\mathfrak{P})})\,G(2\delta;\lambda_{\mathfrak{P}})\Big\}.$$

We then have

$$(4.15b)\qquad \frac{\displaystyle\int_{G_\infty(\mathfrak{C}^c)}\hat{\lambda}\,d\varphi^*}{\Omega_p^{m_0\Sigma+2d}}=(\mathfrak{R}^\times:\mathfrak{r}^\times)\,w^*(\lambda)\,\frac{(-1)^{m_0t}\pi^d\,\Gamma_F(m_0t+d)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times(1-\lambda^*(\mathfrak{L}^c))\Big\{\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))(1-\lambda^*(\mathfrak{P}^c))\Big\}L(0,\lambda).$$

Note that if the conductor of λ divides $\mathfrak{C}p^\infty$, then the conductor of $\lambda\circ c$ divides $\mathfrak{C}^cp^\infty$ and if the infinity type of λ is given by $m_0\Sigma^c+\sum_{\sigma\in\Sigma^c}d_\sigma\sigma(1-c)$. Thus we can construct $d\varphi^*$ out of the CM-type Σc. Since $L(0,\lambda)=L(0,\lambda\circ c)$, comparing $(4.14a, b)$ with $(4.15a, c)$ we have

$$(4.16)\qquad (1-\lambda(\mathfrak{L}))^{-1}w'(\lambda)\int_{G_\infty(\mathfrak{C})}\hat{\lambda}\,d\varphi=(1-\lambda^*(\mathfrak{L}^c))^{-1}\int_{G_\infty(\mathfrak{C}^c)}(\hat{\lambda}\circ c)\,d\varphi^*,$$

where $w'(\lambda)=N(\mathfrak{L}^{-e(\mathfrak{L})})\,G(2\delta,\lambda_{\mathfrak{L}})\,\lambda_{\mathfrak{L}}(\varpi_{\mathfrak{L}}^{\varepsilon+e(\mathfrak{L})})$. The above formula is the formula relating φ and φ^* in Theorem 4.2, when $\mathfrak{f}$ is a power of a split prime $\mathfrak{L}$. The case of inert or ramified prime power is proven as $(4.12b)$. The general case follows from these prime power case as already explained. By (4.10) and $(4.14a, b)$, we obtain Theorem 4.1. The first part of Theorem 4.2 follows from $(4.12a)$ and $(4.15a, b)$.

5. Functional equation of the Katz p-adic L-function

In this section, we prove Theorem II of the introduction. Namely we study the functional equation of Eisenstein series and the Katz p-adic L-functions. According to [K 4, 3.3], we construct a dual $(X^t, \lambda^t, i^t)_{/A}$ out of a test object $(X, \lambda, i)_{/A}$ as follows. Here i is a $\Gamma_{00}(Np^\infty)$-level structure for N prime to p, and λ is a $\mathfrak{c}$-polarization for an ideal $\mathfrak{c}$ prime to Np^∞. The $\mathfrak{c}^{-1}$-polarization λ^t is given by the following commutative diagram:

$$\begin{array}{rccl}\lambda^t: & (X^t)^t & \to & X^t\otimes_{\mathfrak{r}}\mathfrak{c}^{-1}\\ & \| & & \downarrow\lambda\otimes\mathrm{id}\\ & X & = & X\otimes_{\mathfrak{r}}\mathfrak{c}\otimes_{\mathfrak{r}}\mathfrak{c}^{-1}\end{array}$$

Take a positive integer N_0 prime to p in N and consider $\mu_{Np^\infty}\otimes\vartheta^{-1}$, which is the maximal subgroup of $\mu_{N_0p^\infty}\otimes\vartheta^{-1}$ killed by Np^∞; *i.e.*

$$\mu_{Np^\infty}\otimes\vartheta^{-1}=\mathrm{Hom}_{gr}((F_p/\vartheta_p^{-1})\times N^*/\mathfrak{r}^*,\ \mu_{N_0p^\infty}\otimes\vartheta^{-1}).$$

Then $i:\mu_{Np^\infty}\otimes\vartheta^{-1}\to X$ is a closed immersion. We define $i^t:\mu_{Np^\infty}\otimes\vartheta^{-1}\to X^t$ by the commutative diagram:

$$\begin{array}{cccc} i^t: & \mu_{Np^\infty}\otimes\vartheta^{-1} & \rightarrow & X^t \\ & \| & & \downarrow\lambda \\ & \mu_{Np^\infty}\otimes_{\mathbf{Z}}\vartheta^{-1}\otimes_{\mathfrak{r}}\mathfrak{c} & \xrightarrow{i\otimes\mathrm{id}} & X\otimes_{\mathfrak{r}}\mathfrak{c}. \end{array}$$

Now suppose that A is an algebra over the localization $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$ at prime ideal p. Then $\mathfrak{c}\otimes_{\mathbf{Z}}A$. Given a nowhere vanishing differential $\omega_{/X}$, we construct $\omega^t_{/X^t}$ by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Lie}(X^t) & \overset{\omega^t}{\cong} & \vartheta^{-1}\otimes_{\mathbf{Z}}A \\ \downarrow\lambda & & \| \\ \mathrm{Lie}(X\otimes_{\mathfrak{r}}\mathfrak{c}) & & \vartheta^{-1}(\mathfrak{r}\otimes_{\mathbf{Z}}A) \\ \| & & \| \\ \mathrm{Lie}(X)\otimes_{\mathfrak{r}}\mathfrak{c} & \overset{\omega\otimes\mathrm{id}}{\cong} & \vartheta^{-1}\mathfrak{c}\otimes_{\mathbf{Z}}A \end{array}$$

We apply this construction to the Tate HBAV $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}, i_{\mathrm{can}})$. Then it is obvious by construction (*see* § 1.7) that

$$(5.1)\qquad (\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}, i_{\mathrm{can}})^t=(\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}, i_{\mathrm{can}}).$$

Since the correspondence $(X, \lambda, i)\mapsto(X^t, \lambda^t, i^t)$ gives an anti-equivalence of the category of test objects, we have an involution

$$f\mapsto f^t:\quad V(\mathfrak{c}^{-1}, N; A)\cong V(\mathfrak{c}, N; A)$$

and

$$\mathfrak{M}_k(\mathfrak{c}^{-1}, \Gamma_{00}(Np^\infty); A)\cong\mathfrak{M}_k(\mathfrak{c}, \Gamma_{00}(Np^\infty); A)$$

given by $f^t(X, \lambda, i)=f(X^t, \lambda^t, i^t)$ and $f^t(X, \lambda, \omega, i)=f(X^t, \lambda^t, \omega^t, i^t)$.

Let $\phi:\{\mathfrak{r}_p^\times\times\mathfrak{r}_p^\times\times(\mathfrak{r}/\mathfrak{f})\times(\mathfrak{r}/\mathfrak{f})\}\to\mathfrak{O}$ be a continuous function with $\phi(\varepsilon x, \varepsilon y)=\phi(x, y)$ for all $\varepsilon\in\mathfrak{r}^\times$ and put as in (3.1)

$$\phi^0(x, y, a, b)=P_a^{-1}\phi(x_p^{-1}, y, a, b)$$

$$\phi^*(x', y', a', b')=\{P_a^{-1}P_b\phi(x, y, a, b^{-1})\}(x'^{-1}, y', a', b')$$

and we write $\mathsf{E}(\phi; \mathfrak{c})$ for the Eisenstein series whose q-expansion at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i_{\mathrm{can}})$ is given as

$$N(\mathfrak{a})\Big\{\sum_{0\ll\xi\in\mathfrak{ab}}\ \sum_{(a,\,b)\in(\mathfrak{a}\times\mathfrak{b})/\mathfrak{r}^\times,\ ab=\xi}\phi^0(a, b)\,sgn(N(a))\,N(a)^{-1}\mathbf{e}_F(\xi z)\Big\}.$$

Then by (5.1), we see

(5.2) $\mathsf{E}(\phi;\, \mathfrak{c}^{-1})^t((\mathrm{Tate}_{\mathfrak{a},\,\mathfrak{b}}(q),\, \lambda_{\mathrm{can}},\, \omega_{\mathrm{can}},\, i_{\mathrm{can}}))$

$$= \mathsf{E}(\phi,\, \mathfrak{c}^{-1})((\mathrm{Tate}_{\mathfrak{b},\,\mathfrak{a}}(q),\, \lambda_{\mathrm{can}},\, \omega_{\mathrm{can}},\, i_{\mathrm{can}}))$$

$$= N(\mathfrak{b})\Big\{\sum_{0 \ll \xi \in \mathfrak{ab}} \;\sum_{(a,\, b) \in (\mathfrak{b} \times \mathfrak{a})/\mathfrak{r}^{\times},\, ab=\xi} \phi^{0}(a,\, b)\,|\,N(a)\,|^{-1}\,\mathbf{e}_{\mathrm{F}}(\xi z)\Big\}$$

$$= N(\mathfrak{c}^{-1}\mathfrak{a})\Big\{\sum_{0 \ll \xi \in \mathfrak{ab}} \mathbf{e}_{\mathrm{F}}(\xi z) \times \sum_{(a,\, b) \in (\mathfrak{a} \times \mathfrak{b})/\mathfrak{r}^{\times},\, ab=\xi} \phi^{0t}(a,\, b)\,\mathfrak{N}(ab^{-1})\,|\,N(a)\,|^{-1}\Big\}$$

$$= N(\mathfrak{c}^{-1})\,\mathsf{E}^{*}(\hat{\phi};\, \mathfrak{c}),$$

where $\phi^{0t}(a,\, b) = \phi^{0}(b,\, a)$ and

(5.3 a) $$\hat{\phi}(x,\, y) = N(\mathfrak{f})\,\mathrm{P}_{a}\,\phi(y^{-1},\, x^{-1},\, -b^{-1},\, a)\,\mathfrak{N}(xy)^{-1}.$$

Now we compute the dual of $(\mathrm{X}(\mathfrak{U}),\, \lambda(\mathfrak{U}),\, i(\mathfrak{U}))_{/\mathrm{B}}$. We see easily that $\mathrm{X}(\mathfrak{U})^t = \mathrm{X}(\mathfrak{U}) \otimes_{\mathfrak{r}} \mathfrak{c}(\mathfrak{U}\mathfrak{U}^{c})^{-1} = \mathrm{X}(\mathfrak{c}\,\mathfrak{U}^{-c})$ because $\lambda(\mathfrak{U})$ is a $\mathfrak{c}(\mathfrak{U}\mathfrak{U}^{c})^{-1}$-polarization. By the commutative diagram:

$$\begin{array}{llcl} \lambda(\mathfrak{U})^t\colon & (\mathrm{X}(\mathfrak{U})^t)^t & \to & \mathrm{X}(\mathfrak{c}\,\mathfrak{U}^{-c}) \otimes_{\mathfrak{r}} \mathfrak{c}^{-1}\,\mathfrak{U}\mathfrak{U}^{c} \\ & \Vert & & \downarrow \lambda(\mathfrak{u}) \otimes \mathrm{id} \\ & \mathrm{X}(\mathfrak{U}) & = & \mathrm{X}(\mathfrak{U}) \otimes_{\mathfrak{r}} (\mathfrak{c}\,\mathfrak{U}\mathfrak{U}^{c})^{-1} \otimes_{\mathfrak{r}} \mathfrak{c}^{-1}\,\mathfrak{U}\mathfrak{U}^{c}, \end{array}$$

we know that $\lambda(\mathfrak{U})^t = \lambda(\mathfrak{c}\,\mathfrak{U}^{-c})$. Write $\mathfrak{f} = \mathfrak{s}\mathfrak{i}$, so that $\mathfrak{s}$ consists of split primes in M and $\mathfrak{i}$ consists of ramified or inert primes in M. To compute $i(\mathfrak{U})^t$, we see the diagram:

$$\begin{array}{llcl} i(\mathfrak{U})^t\colon & \mu_{f^2 p^\infty} \otimes \vartheta^{-1} & \to & \mathrm{X}(\mathfrak{c}\,\mathfrak{U}^{-c})[\mathfrak{f}^2 p^\infty] \\ & \Vert & & \Vert \\ & \mu_{f^2 p^\infty} \otimes_{\mathbf{Z}} \vartheta^{-1} \otimes_{\mathfrak{r}} \mathfrak{c} & \xrightarrow{i \otimes \mathrm{id}} & (\mathrm{X}(\mathfrak{U}) \otimes_{\mathfrak{r}} \mathfrak{c})[\mathfrak{f}^2 p^\infty]. \end{array}$$

Then

$$i(\mathfrak{U})^t\colon\quad \mathrm{F}_p/\vartheta_p^{-1} \times (\mathfrak{s}^2)^*/\mathfrak{r}^* \times (\mathfrak{i}^2)^*/\mathfrak{r}^* \ni (x_p,\, x',\, x'') \mapsto (2\,\delta x_p,\, 2\,\delta x',\, d_{\mathrm{F}_{\mathfrak{i}}} x'') \in \mathrm{X}(\mathfrak{c}\,\mathfrak{U}^{-c}).$$

Thus

$$i(\mathfrak{U})^t = i(\mathfrak{c}\mathfrak{U}^{-c})$$

and

$${}^t(\mathrm{X}(\mathfrak{U}),\, \lambda(\mathfrak{U}),\, i(\mathfrak{U}))_{/\mathrm{B}} = (\mathrm{X}(\mathfrak{c}\,\mathfrak{U}^{-c}),\, \lambda(\mathfrak{c}\,\mathfrak{U}^{-c}),\, i(\mathfrak{c}\,\mathfrak{U}^{-c}))_{/\mathrm{B}}.$$

By using this formula, we now compute

$$\int_{\mathrm{G}_\infty(\mathfrak{c})} \hat{\lambda}\, d\varphi = \sum_j \mathsf{E}(\phi_j;\, \mathfrak{c}_j)^t((\mathrm{X}(\mathfrak{c}\mathfrak{U}_j^{-c}),\, \lambda(\mathfrak{c}\mathfrak{U}_j^{-c}),\, i(\mathfrak{c}\mathfrak{U}_j^{-c}))_{/\mathrm{B}})$$

$$= \sum_j N(\mathfrak{c}_j)\,\mathsf{E}^{*}(\hat{\phi}_j,\, \mathfrak{c}_j^{-1})(\mathrm{X}(\mathfrak{c}\mathfrak{U}_j^{-c}),\, \lambda(\mathfrak{c}\mathfrak{U}_j^{-c}),\, i(\mathfrak{c}\,\mathfrak{U}_j^{-c}))_{/\mathrm{B}}.$$

Let ϕ_j (resp. ϕ'_j) be the function ϕ_j (resp. ϕ'_j) corresponding λ [resp. $\lambda^*=(\lambda\circ c)^{-1}N^{-1}$] relative to $\{\mathfrak{U}_j\}$ (resp. $\{c\,\mathfrak{U}_j^{-c}\}$). We regard sometimes that the prime to p-part of these functions are defined on M_c having value 0 outside $\mathfrak{R}_c$. At each split prime ideal $\mathfrak{l}=\mathfrak{L}\mathfrak{L}^c$ in $\mathfrak{f}$, by definition, the $\mathfrak{L}$-part $\phi_{j,\mathfrak{L}}$ of $\lambda(\mathfrak{U}_j)\phi_j$ is given by

$$\phi_{j,\mathfrak{L}}(x)=\begin{cases}\lambda_{\mathfrak{L}}^{-1}(x) & \text{on } \mathfrak{R}_{\mathfrak{L}}^{\times},\\ 0 & \text{outside } \mathfrak{R}_{\mathfrak{L}}^{\times}.\end{cases}$$

The $\mathfrak{L}^c$-part $\phi_{j,\mathfrak{L}^c}$ of $\lambda(\mathfrak{U}_j)\phi_j$ is given by, when $\varepsilon''>0$,

$$\phi_{j,\mathfrak{L}^c}(x)=\begin{cases}\lambda_{\mathfrak{L}^c}^{-1}(x) & \text{on } \mathfrak{R}_{\mathfrak{L}^c}^{\times},\\ 0 & \text{outside } \mathfrak{R}_{\mathfrak{L}^c}^{\times}.\end{cases}$$

and, when $\varepsilon''=0$, $\phi_{j,\mathfrak{L}^c}(x)=\chi_{\mathfrak{L}^c}(x)$.
On the other hand, we have for the $\mathfrak{L}^c$-part

$$\phi^{*\prime}_{j,\mathfrak{L}^c}(x)=\begin{cases}\lambda_{\mathfrak{L}}(x^c) & \text{on } \mathfrak{R}_{\mathfrak{L}^c}^{\times},\\ 0 & \text{outside } \mathfrak{R}_{\mathfrak{L}^c}^{\times}.\end{cases}$$

The $\mathfrak{L}$-part $\phi^{*\prime}_{j,\mathfrak{L}}$ of $\lambda^*(c\,\mathfrak{U}_j^{-c})\phi^{*\prime}_j$ is given by, when $\varepsilon''>0$.

$$\phi^{*\prime}_{j,\mathfrak{L}}(x)=\begin{cases}\lambda_{\mathfrak{L}^c}(x^c) & \text{on } \mathfrak{R}_{\mathfrak{L}}^{\times},\\ 0 & \text{outside } \mathfrak{R}_{\mathfrak{L}}^{\times},\end{cases}$$

and when $\varepsilon''=0$, $\phi^{*\prime}_{j,\mathfrak{L}}(x)=\chi_{\mathfrak{L}}(x)$.

By the Fourier inversion formula, we see

$$\hat{\phi}_{j,\mathfrak{L}}(x)=N(\mathfrak{L}^{\varepsilon-e(\mathfrak{L}^c)})\,\lambda_{\mathfrak{L}^c}(\varpi_{\mathfrak{L}^c}^{-e(\mathfrak{L}^c)})\,\mathrm{G}(2\delta,\lambda_{\mathfrak{L}^c}^{-1})\,\phi^{*\prime}_{j,\mathfrak{L}}(\varpi_{\mathfrak{L}}^{e(\mathfrak{L}^c)-\varepsilon}a),$$
$$\hat{\phi}_{j,\mathfrak{L}^c}(x)=\lambda_{\mathfrak{L}}(-1)\,\phi^{*\prime}_{j,\mathfrak{L}^c}(x).$$

When $\mathfrak{l}$ is inert or ramified, we see

$$\phi_{j,\mathfrak{l}}(a,b)=\begin{cases}\lambda_{\mathfrak{l}}^{-1}(a) & \text{if } (a,b)\in\mathfrak{r}_{\mathfrak{l}}^{\times}\times\mathfrak{r}_{\mathfrak{l}},\\ 0 & \text{otherwise},\end{cases}$$

and

$$\phi^{*\prime}_{j,\mathfrak{l}}(a,b)=\begin{cases}\lambda_{\mathfrak{l}}(b) & \text{if } (a,b)\in\{0\}\times\mathfrak{r}_{\mathfrak{l}}^{\times}\times\mathfrak{r}_{\mathfrak{l}},\\ 0 & \text{otherwise},\end{cases}$$

and

$$\phi^{*\prime}_{j,\mathfrak{l}}(a,b)=\begin{cases}\lambda_{\mathfrak{l}}(b) & \text{if } (a,b)\in\{0\}\times\mathfrak{r}_{\mathfrak{l}}^{\times},\\ 0 & \text{otherwise}.\end{cases}$$

This shows that $\hat{\phi}_{j,\mathfrak{l}}=\lambda_{\mathfrak{l}}(-1)N(\mathfrak{l}^{\varepsilon})\phi^{*\prime}_{j,\mathfrak{l}}$. We put

$$\phi^{*\prime\prime}_{j,\mathfrak{l}}(x)=\lambda_{\mathfrak{L}}(\varpi^{\varepsilon-e(\mathfrak{L}^c)})\phi^{*\prime}_{j,\mathfrak{L}}(x_{\mathfrak{L}})\phi^{*\prime}_{j,\mathfrak{L}^c}(x_{\mathfrak{L}^c}) \qquad \text{if} \quad \mathfrak{l}=\mathfrak{L}\mathfrak{L}^c\,(\mathfrak{L}\neq\mathfrak{L}^c),$$

$$\phi^{*\prime\prime}_{j,\mathfrak{l}}=\phi^{*\prime}_{j,\mathfrak{l}} \quad \text{if } \mathfrak{l} \text{ is inert or ramified in M},$$

$$\phi^{*\prime\prime}_j(x)=\lambda^*(\mathfrak{c}^{-1}\mathfrak{U}^c_j)\prod_{\mathfrak{l}}\phi^{*\prime\prime}_{j,\mathfrak{l}}(x_{\mathfrak{l}})=N(\mathfrak{c})^2\lambda(\mathfrak{c})N(\mathfrak{U}_j)^{-1}\lambda(\mathfrak{U}_j^{-1})\prod_{\mathfrak{l}}\phi^{*\prime\prime}_{j,\mathfrak{l}}(x_{\mathfrak{l}}).$$

We now have

$$(5.4)\quad \int_{G_\infty(\mathfrak{C})}\hat{\lambda}\,d\varphi=\sum_j \mathsf{E}(\phi_j;\,\mathfrak{c}_j)^t((X(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,\lambda(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,i(\mathfrak{c}\,\mathfrak{U}_j^{-c}))_{/B})$$

$$=\sum_j N(\mathfrak{c}_j)\,\mathsf{E}^*(\hat{\phi}_j,\,\mathfrak{c}_j^{-1})(X(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,\lambda(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,i(\mathfrak{c}\,\mathfrak{U}_j^{-c}))_{/B}).$$

$$=c\,N(\mathfrak{c})^{-1}\lambda(\mathfrak{c}^{-1})\sum_j \mathsf{E}^*(\phi^{*\prime\prime}_{j,}\,\mathfrak{c}_j^{-1})(X(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,\lambda(\mathfrak{c}\,\mathfrak{U}_j^{-c}),\,i(\mathfrak{c}\,\mathfrak{U}_j^{-c}))_{/B})$$

$$=c\,\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1}\lambda(\mathfrak{c}^{-1})\prod_{\mathfrak{L}\mid\mathfrak{F}}\lambda_{\mathfrak{L}^c}(\varpi_{\mathfrak{L}^c}^{e(\mathfrak{L}^c)-\varepsilon})\,\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{L}^{e(\mathfrak{L}^c)-\varepsilon})\int_{G_\infty(\mathfrak{C}^c)}\hat{\lambda}^*\,d\varphi^*,$$

where

$$c=\prod_{\mathfrak{L}\mid\mathfrak{F}}\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{L}^{\varepsilon-e(\mathfrak{L}^c)})\,\lambda_{\mathfrak{L}^c}(\varpi^{-e(\mathfrak{L}^c)})\,G(2\delta,\,\lambda_{\mathfrak{L}^c}^{-1})\,\lambda_{\mathfrak{L}}(-1)\prod_{\mathfrak{l}\mid\mathfrak{i}}\lambda_{\mathfrak{l}}(-1)\,\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{l}^{\varepsilon(\mathfrak{l})}).$$

We define the standard measure μ on $G_\infty(\mathfrak{C})$ by

$$(5.5)\qquad \int_{G_\infty(\mathfrak{C})}\phi\,d\mu=\phi\Big(\prod_{\mathfrak{L}\mid\mathfrak{F}}\varpi_{\mathfrak{L}}^{\varepsilon(\mathfrak{L})}\prod_{\mathfrak{L}\mid\mathfrak{i}}\varpi_{\mathfrak{l}}^{\varepsilon(\mathfrak{l})}\Big)\int_{G_\infty(\mathfrak{C})}\phi\,d\varphi.$$

Then we have

THEOREM 5.1. – *Let* M *be a CM quadratic extension of* F *and* Σ *be a p-adic CM-type of* M. *Let* $\mathfrak{C}$ *be an integral ideal prime to p in* M. *We decompose* $\mathfrak{C}=\mathfrak{F}\mathfrak{F}_c\mathfrak{i}$ *as in* (4.3). *We put*

$$W_p(\lambda)=\Big\{\prod_{\mathfrak{P}\in\Sigma}\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{P}^{-e(\mathfrak{P})})\,G(2\delta;\,\lambda_{\mathfrak{p}})\Big\},$$

$$W'(\lambda)=\prod_{\mathfrak{L}\mid\mathfrak{F}}G((2\delta)^c,\,\lambda_{\mathfrak{L}}^{-1})\prod_{\mathfrak{L}\mid\mathfrak{F}_c}G(2\delta,\,\lambda_{\mathfrak{L}^c}^{-1})\prod_{\mathfrak{l}\mid\mathfrak{i}}G((2\delta)^c,\,\lambda_{\mathfrak{l}}^{-1}).$$

Then there exist a (unique) measure μ *on* $G_\infty(\mathfrak{C})$ *with values in* $\mathfrak{O}$ *such that*

$$(5.6a)\quad \frac{\int_{G_\infty(\mathfrak{C})}\hat{\lambda}\,d\mu}{\Omega_p^{m_0\Sigma+2d}}=(\mathfrak{R}^\times:\mathfrak{r}^\times)\,W_p(\lambda)\,\frac{(-1)^{m_0 t}\pi^d\,\Gamma_F(m_0t+d)}{\sqrt{|D_F|}\,\operatorname{Im}(\delta)^d\,\Omega_\infty^{m_0\Sigma+2d}}$$
$$\times\prod_{\mathfrak{L}\mid\mathfrak{C}}(1-\lambda(\mathfrak{L}))\Big\{\prod_{\mathfrak{p}\in\Sigma}(1-\lambda(\mathfrak{P}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda^*(\mathfrak{P}^c))\Big\}L(0,\lambda)$$

for all Hecke characters λ *modulo* $\mathfrak{C}p^\infty$ *such that*

$$\lambda_\infty(x_\infty) = x_\infty^{m_0\Sigma + d(1-c)} \quad \textit{for} \quad m_0 > 0$$

and

$$d = \sum_{\sigma \in \Sigma} d_\sigma \sigma \quad \textit{with} \quad d_\sigma \geqq 0$$

and

$$(5.6b) \quad \frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu}{\Omega_p^{m_0\Sigma + 2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times) \frac{(-1)^{m_0 t} \pi^d \Gamma_F(m_0 t + d)}{\sqrt{|D_F|}\, \mathrm{Im}(\delta)^d \Omega_\infty^{m_0\Sigma + 2d}} W_p(\lambda^*) \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1} \lambda(\mathfrak{c})^{-1} W'(\lambda) \times \Big\{ \prod_{\mathfrak{L} \mid \mathfrak{F}} (1 - \lambda^*(\mathfrak{L}^c)) \prod_{\mathfrak{L} \mid \mathfrak{F}_c \mathfrak{i}} (1 - \lambda(\mathfrak{L})) \prod_{\mathfrak{P} \in \Sigma} (1 - \lambda(\mathfrak{P}^c)) \prod_{\mathfrak{P} \in \Sigma} (1 - \lambda^*(\mathfrak{P}^c)) \Big\} L(0, \lambda^*)$$

for all Hecke characters λ *modulo* $\mathfrak{C}p^\infty$ *such that*

$$\lambda_\infty(x_\infty) = x_\infty^{(2-m_0)\Sigma + (d + (m_0 - 1)\Sigma)(1-c)} \quad \textit{for} \quad m_0 > 0$$

and

$$d = \sum_{\sigma \in \Sigma} d_\sigma \sigma \quad \textit{with} \quad d_\sigma \geqq 0.$$

Proof. — The formula (5.6*a*) follows from Theorem 4.1 and the definition of μ. We prove (5.6*b*). By Theorem 4.2 and (4.12*a*), (4.15*a*, *b*) and (5.4), we see that

$$\frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu}{\Omega_p^{m_0\Sigma + 2d}} = (\mathfrak{R}^\times : \mathfrak{r}^\times) \frac{(-1)^{m_0 t} \pi^d \Gamma_F(m_0 t + d)}{\sqrt{|D_F|}\, \mathrm{Im}(\delta)^d \Omega_\infty^{m_0\Sigma + 2d}} W \times \Big\{ \prod_{\mathfrak{L} \mid \mathfrak{F}} (1 - \lambda^*(\mathfrak{L}^c)) \prod_{\mathfrak{L} \mid \mathfrak{F}_c \mathfrak{i}} (1 - \lambda(\mathfrak{L})) \times \prod_{\mathfrak{P} \in \Sigma} (1 - \lambda(\mathfrak{P}^c)) \prod_{\mathfrak{P} \in \Sigma} (1 - \lambda^*(\mathfrak{P}^c)) \Big\} L(0, \lambda^*),$$

where, for c in (5.4) and $w^*(\lambda^*)$ in Theorem 4.2,

$$W = c\, \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1} \lambda(\mathfrak{c}^{-1}) w^*(\lambda^*) \prod_{\mathfrak{L} \mid \mathfrak{F}} \lambda_{\mathfrak{L}}(\varpi_{\mathfrak{L}}^{\varepsilon}) \prod_{\mathfrak{l} \mid \mathfrak{i}} \lambda_{\mathfrak{l}}(\varpi_{\mathfrak{l}}^{\varepsilon}) \prod_{\mathfrak{L} \mid \mathfrak{F}} \lambda_{\mathfrak{L}}^*(\varpi_{\mathfrak{L}}^{\varepsilon - e(\mathfrak{L})})$$
$$= c\, \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1} \lambda(\mathfrak{c}^{-1}) \prod_{\mathfrak{L} \mid \mathfrak{F}} \lambda_{\mathfrak{L}^c}^{-1}(\varpi_{\mathfrak{L}^c}^{\varepsilon - e(\mathfrak{L}^c)}) N(\mathfrak{L}^{e(\mathfrak{L}^c) - \varepsilon})$$

$$\times \prod_{\mathfrak{L}|\mathfrak{F}} (\varpi_{\mathfrak{L}}^{\varepsilon}) \prod_{\mathfrak{l}|\mathfrak{i}} \lambda_{\mathfrak{l}}(\varpi_{\mathfrak{l}}^{\varepsilon}) W_p(\lambda^*) \prod_{\mathfrak{L}|\mathfrak{F}} N(\mathfrak{L}^{-ce(\mathfrak{L})}) G(2\delta, \lambda_{\mathfrak{L}^c}^*) \lambda_{\mathfrak{L}}^{*-1}(\varpi_{\mathfrak{L}}^{\varepsilon}) \lambda_{\mathfrak{L}^c}^*(\varpi_{\mathfrak{L}^c}^{\varepsilon})$$

$$\times \prod_{\mathfrak{l}|\mathfrak{i}} \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{l}^{-\varepsilon(\mathfrak{l})}) \lambda_{\mathfrak{l}}^*(\varpi_{\mathfrak{l}}^{\varepsilon}) G(2\delta; \lambda_{\mathfrak{l}})$$

$$= c\, \mathfrak{N}_{F/Q}(\mathfrak{c})^{-1} \lambda(\mathfrak{c}^{-1}) W_p(\lambda^*) \prod_{\mathfrak{L}|\mathfrak{F}} \lambda_{\mathfrak{L}}(\varpi_{\mathfrak{L}}^{\varepsilon})$$

$$\times \prod_{\mathfrak{l}|\mathfrak{i}} \lambda_{\mathfrak{l}}(\varpi_{\mathfrak{l}}^{\varepsilon}) \Big\{ \prod_{\mathfrak{L}|\mathfrak{F}} \lambda_{\mathfrak{L}^c}^{-1}(\varpi_{\mathfrak{L}^c}^{\varepsilon - e(\mathfrak{L}^c)}) N(\mathfrak{L}^{e(\mathfrak{L}^c)-\varepsilon}) \Big\}$$

$$\times \Big\{ \prod_{\mathfrak{L}|\mathfrak{F}} G(2\delta, \lambda_{\mathfrak{L}}^{-1}) \lambda_{\mathfrak{L}}^{-1}(\varpi_{\mathfrak{L}}^{\varepsilon}) \lambda_{\mathfrak{L}^c}(\varpi_{\mathfrak{L}^c}^{\varepsilon}) \Big\}$$

$$\times \prod_{\mathfrak{l}|\mathfrak{i}} \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{l}^{-\varepsilon(\mathfrak{l})}) \lambda_{\mathfrak{l}}^{-1}(\varpi_{\mathfrak{l}}^{\varepsilon}) G(2\delta; \lambda_{\mathfrak{l}}^{-1}) \Big\}$$

$$= W_p(\lambda^*) \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-1} \lambda(\mathfrak{c})^{-1} W'(\lambda).$$

We now deal with complex functional equations to show that the value of the right hand side of (5.6b) is equal to that of (5.6a), which finishes the proof of Theorem II. As a differental idele of M, we take d_M such that $d_{M_\mathfrak{q}} = (-2\delta)_\mathfrak{q}$ if $\mathfrak{q}$ divides $\mathfrak{C}p \cap \vartheta_M$ and $d_{M_\mathfrak{q}} = 1$ outside $\mathfrak{C}p \cap \vartheta_M$ for the absolute different ϑ_M of M. We write D_M for the discriminant of M/**Q**. Then as is well known (e. g. [W]), we have

(5.7a) $G_{\lambda_\infty}(s) L(s, \lambda^u)$

$$= \kappa \lambda^u(b) \lambda_\infty(-1) (|D_M| N(C))^{(1/2)-s} G_{\lambda_\infty^{-1}}(1-s) L(1-s, (\lambda^u)^{-1}),$$

where (i) $= b = \varpi_C^e (d_M)_{C\vartheta M}$ if the conductor of λ is $C = \prod_{\mathfrak{L}} \mathfrak{L}^{e(\mathfrak{L})}$, (ii) λ^u is the unitarization of λ, *i.e.* $\lambda/|\lambda|$, (iii) $\kappa = \prod_v \kappa_v$, κ_v is defined as follows: for each finite place v with $e(v) = 0$, $\kappa_v = 1$, and for each finite place v with $e(v) > 0$

$$\kappa_v = |\varpi_v^{e(v)}|_v^{-1/2} \int_{\mathfrak{R}_v^\times} \lambda^{-1}(x) \mathbf{e}_{M_v}(b_v^{-1} x)\, dx$$

for the additive Haar measure dx with volume 1 on $\mathfrak{R}_v$. For an infinite place v, write $\lambda_v^u(x) = x^{-A} \bar{x}^{-B} |x|^{A+B}$ with $AB = 0$ and $A, B \geqq 0$. Then, we have $\kappa_v = i^{A+B}$ and $G_{\lambda_\infty}(s) = \prod_{v \in \infty} G_{\lambda_v}(s)$ for

$$G_{\lambda_v}(s) = (2\pi)^{1-(s+(A+B)/2)} \Gamma(s + (A+B)/2).$$

Since the choice of the additive character in [W] is $\mathbf{e}_F(-x)$, the formula (5.7a) differs from the one given in [W] by this sign. If the infinity type of λ^* is $m_0 \Sigma + d(1-c)$, then the infinity type of λ is

$$m'_0 \Sigma + d'(1-c) = (2 - m_0) \Sigma + (d + (m_0 - 1)\Sigma)(1-c)$$

and thus the integers A and B of $\lambda^{*u}(x)=N(x)^{m_0/2}\lambda^*$ at $\sigma\in I$ given by $A=0$ and $B=m_0+2d_\sigma$. Then $L(0,\lambda^*)=L(m_0/2,\lambda^{*u})$ and

$$G_{\lambda^{*u}_\infty}\left(\frac{m_0}{2}\right)=(2\pi)^{t-m_0-d}\Gamma_F(m_0t+d).$$

Note that $L(0,\lambda)=L(1-(m_0/2),(\lambda^{*u})^{-1})$. Thus we have

(5.7 *b*) $\pi^d\Gamma_F(m_0t+d)L(0,\lambda^*)$

$$=\kappa\lambda^{*u}(b)\lambda_\infty(-1)(|D_M|N(C))^{(1-m_0)/2}2^{-d}(2\pi)^{-t+m_0+d}\Gamma_F(t+d)L(0,\lambda),$$
$$=\kappa\lambda^{*u}(b)\lambda_\infty(-1)(|D_M|N(C))^{(1-m_0)/2}2^{(m_0-1)t}\pi^{d'}\Gamma_F(m_0't+d')L(0,\lambda),$$

where C is the conductor of λ.

Now we compute $\kappa(\lambda^*\circ c)^u(b)\lambda_\infty(-1)(|D_M|N(C))^{(1-m_0)}/2$, which is equal to $\kappa\lambda^{*u}(b)\lambda_\infty(-1)(|D_M|N(C))^{(1-m_0)/2}$:

$$\begin{aligned}\kappa(\lambda^*\circ c)^{*u}(b)\lambda_\infty(-1)(|D_M|\mathfrak{N}_{M/Q}(C))^{(1-m_0)/2}\\ =\lambda_C(-1)i^{m_0\Sigma+2d}\mathfrak{N}_{M/Q}(C)^{-1/2}N(C\vartheta_M)^{m_0/2}(\lambda^*\circ c)_{C\vartheta_M}(d_M(\varpi^e C))\lambda_C(\varpi^e)\\ \times\{\prod_{\mathfrak{L}\mid\mathfrak{C}}G(d_{M_\mathfrak{L}},\lambda_\mathfrak{L})\}(|D_M|\mathfrak{N}_{M/Q}(C))^{(1-m_0)/2}\\ =\lambda_C(-1)i^{m_0\Sigma+2d}\mathfrak{N}_{M/Q}(C)^{-1}|D_M|^{1/2}(\lambda^*\circ c)_{C\vartheta_M}(d_M)\{\prod_{\mathfrak{L}\mid C}G(d_{M_\mathfrak{L}},\lambda_\mathfrak{L})\}.\end{aligned}$$

Note that $G(d_{M_\mathfrak{M}},\lambda_\mathfrak{L})=\lambda_\mathfrak{L}(-1)N(\mathfrak{L}^{e(\mathfrak{L})})G(d_{M_\mathfrak{L}},\lambda_\mathfrak{L}^{-1})^{-1}$. Thus we know

$$\kappa\lambda^{*u}(b)\lambda_\infty(-1)(|D_M|\mathfrak{N}_{M/Q}(C))^{(1-m_0)/2}$$
$$=i^{m_0\Sigma+2d}|D_M|^{1/2}(\lambda^*\circ c)_{C\vartheta_M}(d_M)\{\prod_{\mathfrak{L}\mid C}G(d_{M_\mathfrak{L}},\lambda_\mathfrak{L}^{-1})^{-1}\}.$$

Note that $2\delta=\vartheta_M\mathfrak{c}$ for the absolute different ϑ_M of M. Thus, the prime-to-C ϑ_M part of the ideal (2δ) is $\mathfrak{c}$ and we see

$$\begin{aligned}(\lambda^*\circ c)_{C\vartheta_M}(-2\delta)=\lambda^*(\mathfrak{c})^{-1}(2\delta)^{-m_0\Sigma-d(1-c)}\\ =\lambda(\mathfrak{c})\mathfrak{N}_{F/Q}(\mathfrak{c})\,\mathrm{Im}(2\delta)^{-m_0\Sigma}i^{-m_0\Sigma}(-1)^d\\ =i^{-m_0\Sigma}\lambda(\mathfrak{c})\mathfrak{N}_{F/Q}(\mathfrak{c})|D_M|^{-1/2}\mathrm{Im}(2\delta)^{(1-m_0)\Sigma}(-1)^d,\end{aligned}$$

since $\mathrm{Im}(2\delta)^\Sigma=\mathfrak{N}_{M/Q}(\vartheta_M\mathfrak{c})^{1/2}=|D_M|^{1/2}\mathfrak{N}_{F/Q}(\mathfrak{c})$. This shows

$$\begin{aligned}\kappa(\lambda^*\circ c)^u(b)\lambda_\infty(-1)(|D_M|\mathfrak{N}_{M/Q}(C))^{(1-m_0)/2}\\ =\lambda(\mathfrak{c})(\lambda^*\circ c)_{C\vartheta_M}((-2\delta)^{-1}d_M)\mathfrak{N}_{F/Q}(\mathfrak{c})\\ \mathrm{Im}(2\delta)^{(1-m_0)\Sigma}\{\prod_{\mathfrak{L}\mid C}G(d_{M_\mathfrak{L}},\lambda_\mathfrak{L}^{-1})^{-1}\}.\end{aligned}$$

Note that $G(d_{M_{\mathfrak{L}}}, \lambda_{\mathfrak{L}}^{-1}) = \lambda_{\mathfrak{L}}((-2\delta) d_{M_{\mathfrak{L}}}^{-1}) G((-2\delta), \lambda_{\mathfrak{L}}^{-1})$ and hence

$$(5.7c)\quad \kappa(\lambda^* \circ c)^u(b)\lambda_\infty(-1)(|D_M| \mathfrak{N}_{M/\mathbf{Q}}(C))^{(1-m_0)/2} = \lambda(\mathfrak{c})\mathfrak{N}(\mathfrak{c})\,\mathrm{Im}\,(2\delta)^{(1-m_0)\Sigma}\{\prod_{\mathfrak{L}|C} G(-2\delta, \lambda_{\mathfrak{L}}^{-1})^{-1}\}.$$

Since $(-1)^{m_0 t} = (-1)^{m'_0 t}$, we know from Theorem 5.1 that

$$\frac{\int_{G_\infty(\mathfrak{c})} \hat{\lambda}\, d\mu}{\Omega_p^{m'_0\Sigma + 2d'}} = (\mathfrak{R}^\times : \mathfrak{r}^\times)\frac{(-1)^{m'_0 t}\pi^{d'}\Gamma_F(m'_0 t + d')}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^{d'}\Omega_\infty^{m'_0\Sigma + 2d'}} W_p(\lambda^*)\{\prod_{\mathfrak{L}|p} G(-2\delta, \lambda_{\mathfrak{L}}^{-1})^{-1}\}$$
$$\times\{\prod_{\mathfrak{L}|\mathfrak{F}_c i}(1-\lambda^*(\mathfrak{L}))\prod_{\mathfrak{L}|\mathfrak{F}}(1-\lambda(\mathfrak{L}))$$
$$\times\prod_{\mathfrak{P}\in\Sigma}(1-\lambda(\mathfrak{P}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda^*(\mathfrak{P}^c))\}\, L(0, \lambda)$$
$$= (\mathfrak{R}^\times : \mathfrak{r}^\times)\frac{(-1)^{m'_0 t}\pi^{d'}\Gamma_F(m'_0 t + d')}{\sqrt{|D_F|}\,\mathrm{Im}\,(\delta)^{d'}\Omega_\infty^{m'_0\Sigma + 2d'}} W_p(\lambda)$$
$$\times\{\prod_{\mathfrak{L}|\mathfrak{F}}(1-\lambda^*(\mathfrak{L}^c))\prod_{\mathfrak{L}|\mathfrak{F}_c i}(1-\lambda(\mathfrak{L}))$$
$$\times\prod_{\mathfrak{p}\in\Sigma}(1-\lambda(\mathfrak{P}^c))\prod_{\mathfrak{P}\in\Sigma}(1-\lambda^*(\mathfrak{P}^c))\}\, L(0, \lambda).$$

This conclude the proof of Theorem II.

6. Theta measures attached to CM-fields

In this section, we shall construct the theta measure of CM-type (M, Σ) having values in the space of nearly ordinary (p-adic) modular forms and then study the congruence module $C_0(\psi)$ attached to ψ. We suppose the ordinarity conditions (0.1) and (0.2a, b) for (p, M, Σ). We shall use the same notation introduced in [H 1, § 1] for the space of classical and p-adic Hilbert modular forms. Let $I_M = \Sigma \cup \Sigma_c$ be the set of all embeddings of M into $\bar{\mathbf{Q}}$. Let λ be a Hecke character of $M_{\mathbf{A}}^\times/M^\times$ with values in $\mathbf{C}^\times$ whose infinity type is given by $\lambda(x_\infty) = x_\infty^{-\eta}$ for $\eta = \sum_\sigma \eta_\sigma \sigma \in \mathbf{Z}[I_M]$ and whose conductor is $C(\lambda) = \mathfrak{C}\mathfrak{P}^e$. Here we use the notation introduced in section 4 and thus $\mathfrak{p}^e$ denotes the product $\prod_{\mathfrak{P}\in\Sigma\cup\Sigma_c}\mathfrak{P}^{e(\mathfrak{P})}$. We suppose that $\eta_\sigma \geq \eta_{\sigma c}$ for all $\sigma\in\Sigma$. Since λ is invariant under $M^\times$, we know that $\eta + \eta c = m_0 t_M$ for an integer m_0 and $t_M = \sum_{\sigma\in I_M}\sigma$. Let $\mathrm{Res}: Z[I_M] \to \mathbf{Z}[I]$ be the restriction map and define $v\in\mathbf{Z}[I]$ and $n\in\mathbf{Z}[I]$ by

$$(6.1)\qquad v = \sum_{\sigma\in\Sigma c}\eta_\sigma\,\mathrm{Res}(\sigma) \quad \textit{and} \quad n = \sum_{\sigma\in\Sigma}(\eta_\sigma - \eta_{\sigma\rho} - 1)\,\mathrm{Res}(\sigma).$$

Then we see easily that $n+2v=(m_0-1)t$ for $t=\sum_{\sigma\in I}\sigma$. Let $\hat{\lambda}: M_{\mathbf{A}}^{\times}/M^{\times}\to\bar{\mathbf{Q}}_p^{\times}$ be the p-adic avatar introduced in section 0. For each idele $y\in F_{\mathbf{A}}^{\times}$, we define a formal q-expansion coefficient $\mathbf{a}_p(y, \theta(\lambda))\in\bar{\mathbf{Q}}_p$ (cf. [H 1, § 1] by

$$(6.2)\qquad \mathbf{a}_p(y, \theta(\lambda))=\begin{cases}\displaystyle\sum_{xx^c=y;x\Sigma=1}\hat{\lambda}(x), & \text{if } y\mathfrak{r} \text{ is an integral ideal,}\\ 0 & \text{otherwise,}\end{cases}$$

and $\mathbf{a}_{0,p}(y, \theta(\lambda))=0$
where x runs over all integral ideles in $M_{\mathbf{A}_f}^{\times}$ modulo $U^{(p\mathfrak{C}\mathfrak{C}^c)}=\{u\in\hat{\mathfrak{R}}^{\times}\,|_{u_{p\mathfrak{C}\mathfrak{C}^c}}=1\}$ such that (i) $x\mathfrak{R}$ is integral, (ii) $\mathfrak{N}_{M/F}(x)=y$ and (iii) $x_{\mathfrak{Q}}=1$ for prime ideal $\mathfrak{Q}$ in Σ and in $\mathfrak{C}$. We define a subset $R(y, \mathfrak{C})$ of $M_{\mathbf{A}}^{\times}$ by the condition (i)-(iii) as above. Note that $R(y, \mathfrak{C})/U^{(p\mathfrak{C}\mathfrak{C}^c)}$ is a finite set. We define a pair of weight (k, w) by $k=n+2t$ and $w=t-v$. We also define a pair of finite order characters $\psi: F_{\mathbf{A}}^{\times}/F^{\times}\to\bar{\mathbf{Q}}^{\times}$ and $\psi': \mathfrak{r}_p^{\times}\to\bar{\mathbf{Q}}^{\times}$ by

$$\psi(x)=\chi(x)\lambda(x)\,|x|_{\mathbf{A}}^{m_0}\qquad \text{for}\quad x\in F_{\mathbf{A}}^{\times}$$

and

$$\psi'(y)\,y^{v}=\hat{\lambda}(y)^{-1}\qquad \text{for}\quad y\in\mathfrak{r}_p^{\times},$$

where we have identified $\mathfrak{r}_p$ with $\mathfrak{R}_{\Sigma c}=\prod_{\mathfrak{P}\in\Sigma c}\mathfrak{R}_{\mathfrak{P}}$ and $\chi(\mathfrak{q})=(M/F/\mathfrak{q})$ is the quadratic character associated with the extension M/F by class fied theory. Then we shall prove

THEOREM 6.1. – *Let* $C(\lambda)$ *be the conductor of* λ *and* D *be the relative discriminant of* M/F. *Suppose* (0.1) *and* (0.2 *a*, *b*). *Then there exists a unique modular form* $\theta(\lambda)$ *in* $M_{k,w}(\mathfrak{N}_{M/F}(C(\lambda))Dp, \psi', \psi; \mathbf{C})$ *whose q-expansion coefficients in the sense of* [H 1, *Th.* 1.1] *are given by* $\mathbf{a}_p(y, \theta(\lambda))$. *Moreover the automorphic representation of* $GL_2(F_{\mathbf{A}})$ *spanned by the right translations of* $\theta(\lambda)$ *has conductor* $\mathfrak{N}_{M/F}(C(\lambda))D$.

Proof. – Let λ_0 be the unitarization of λ, *i.e.*, $\lambda_0(x)=\lambda(x)\,|x|_{\mathbf{A}}^{m_0/2}$. Then it is well known (e.g. [Y, Th. 2], [G]) that we have a primitive modular form $\Theta(\lambda_0)$ in $S_{k,k/2}(\mathfrak{N}_{M/F}(\mathfrak{C}\mathfrak{P}^e)D, \mathrm{id}, \psi; \mathbf{C})$ if λ is not of the form $\mu\circ\mathfrak{N}_{M/F}$ for a Hecke character μ of $F_{\mathbf{A}}^{\times}/F^{\times}$ and otherwise in $M_{k,k/2}(\mathfrak{N}_{M/F}(\mathfrak{C}\mathfrak{P}^e)D, \mathrm{id}, \psi; \mathbf{C})$. The Fourier expansion of $\Theta(\lambda_0)$ is given by

$$\Theta(\lambda_0)\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)=\sum_{0\ll\xi\in F}a(\xi y d_F, \Theta(\lambda_0))(\xi y_\infty)^{k/2}\,\mathbf{e}_F(i\xi y_\infty)\,\mathbf{e}_F(\xi x),$$

$$a(y, \Theta(\lambda_0))=\mathfrak{N}_{F/\mathbf{Q}}(y\mathfrak{r})^{-1/2}\sum_{\mathfrak{B}\mathfrak{B}^{\rho}=y\mathfrak{r}}\lambda_0^{*}(\mathfrak{B}),$$

where $\mathfrak{B}$ runs over all integral ideals such that $\mathfrak{B}\mathfrak{B}^{\rho}=y\mathfrak{r}$, λ_0^{*} is the ideal character corresponding to λ_0 in the following way: $\lambda_0^{*}(\mathfrak{Q})=\lambda_0(\varpi_{\mathfrak{Q}})$ for all prime ideal $\mathfrak{Q}$ outside

$C(\lambda)$ ($\lambda_0^*(\mathfrak{B})=0$ if $\mathfrak{B}$ has a non-trivial common divisor with $C(\lambda)$). First suppose that

(6.3) $C(\lambda)$ *is prime to all* $\mathfrak{P}$ *in* Σ^c.

Then we put, for $m=m_0-1$,

$$\theta'(\lambda)(x)=|d|_{\mathbf{A}}^{-(m/2)-1}\Theta(\lambda_0)(x)|\det(x)|_{\mathbf{A}}^{-m/2}\in\mathfrak{M}_{k,w}(\mathfrak{N}_{M/F}(C(\lambda))D, \mathrm{id}, \psi; \mathbf{C}).$$

In fact, according to the formula for the weight of $\Theta(\lambda)\otimes||_{\mathbf{A}}^{-m/2}$ in [H 1, § 7.E, p. 369], the weight w of $\theta'(\lambda)$ is given by $(k-mt)/2=t-v$. Thus we see

$$\theta'(\lambda)\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)=|d|_{\mathbf{A}}^{-(m/2)-1}|y|_{\mathbf{A}}^{-m/2}\Theta(\lambda_0)\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)$$

and hence the coefficient of $\theta'(\lambda)$ of $\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x)$ is given by

$$\begin{aligned}
&|d_F|_{\mathbf{A}}^{-(m/2)-1}|y|_{\mathbf{A}}^{-m/2}a(\xi y d_F, \Theta(\lambda_0))(\xi y_\infty)^{k/2}\\
&\qquad=|y|_{\mathbf{A}}|\xi y d_F|_{\mathbf{A}}^{-(m/2)-1}\mathfrak{N}_{F/\mathbf{Q}}(\xi y\vartheta)^{-1/2}\sum_{\mathfrak{B}\mathfrak{B}^\rho=\xi y\vartheta}\lambda_0^*(\mathfrak{B})(\xi y_\infty)^{k/2}\\
&\qquad=|y|_{\mathbf{A}}\mathfrak{N}_{F/\mathbf{Q}}(\xi y\vartheta)^{m_0/2}\sum_{\mathfrak{B}\mathfrak{B}^\rho=\xi y\vartheta}\lambda_0^*(\mathfrak{B})(\xi y_\infty)^{k/2-mt/2-t}\\
&\qquad=|y|_{\mathbf{A}}\sum_{\mathfrak{B}\mathfrak{B}^\rho=\xi y\vartheta}\lambda^*(\mathfrak{B})(\xi y_\infty)^{-v}\\
&\qquad=|y|_{\mathbf{A}}\{(\xi y d_F)^v\}(\xi y_\infty)^{-v}\sum_{\mathfrak{B}\mathfrak{B}^\rho=\xi y\vartheta}\lambda^*(\mathfrak{B})\{(\xi y d_F)^{-v}\}.
\end{aligned}$$

Therefore we know from [H 2, Prop. 4.1] (with the notation in [H 1]) that $\mathbf{a}(y, \theta'(\lambda))=\sum_{\mathfrak{B}\mathfrak{B}^\rho=y\mathfrak{r}}\lambda^*(\mathfrak{B})\{y^{-v}\}$ and hence by [H 1, (1.3 b)] which tells us: $\mathbf{a}_p(y, \mathbf{f})=\mathbf{a}(y, \mathbf{f})\{y^v\}(y_p)^{-v}$, we have $\mathbf{a}_p(y, \theta'(\lambda))=\sum_{\mathfrak{B}\mathfrak{B}^\rho=y\mathfrak{r}}\lambda^*(\mathfrak{B})y_p^{-v}$. Modifying $\theta'(\lambda)$ as in [H 1, § 5, Lemma 5.3, (iii)] to exclude the coefficients at $\Sigma\mathfrak{C}$ and writing the result as $\theta(\lambda)$, we have

$$\mathbf{a}_p(y, \theta(\lambda))=\sum_{xx^\rho=y}\lambda^*(x^{(\Sigma\mathfrak{C})}\mathfrak{R})y_p^{-v},$$

where $x^{(\Sigma\mathfrak{C})}=xx_{\Sigma\mathfrak{C}}^{-1}$, and x runs over a representave set of the set of ideles satisfying (i)-(iii) of (6.2) modulo $U^{(p\mathfrak{C}\mathfrak{C}^c)}$). Note that $x_{\Sigma c}=y_p$ under the identification: $\mathfrak{r}_p=\mathfrak{R}_{\Sigma c}$ and that $\hat{\lambda}(x)=\lambda(x_f)x_p^{-\eta}$. Especially, if $x_{\Sigma\mathfrak{C}}=1$ and $xx^c=y$, then $\hat{\lambda}(x)=\lambda^*(x\mathfrak{R})y_p^{-v}$ since $v=\sum_{\sigma\in\Sigma c}\eta_\sigma\mathrm{Res}(\sigma)$. Therefore we know that

$$\mathbf{a}_p(y, \theta(\lambda))=\sum_{xx^c=y}\lambda^*(x^{(\Sigma\mathfrak{C})}\mathfrak{R})y_p^{-v}=\sum_{x\in R(y,\ \mathfrak{C})/U(p\mathfrak{C}\mathfrak{C}^c)}\lambda_p(x),$$

which shows the desired assertion when (6.3) is satisfied. We prove the general case of the theorem (*i.e.* the case where $C(\lambda)$ has common factor with Σ^c) after proving the following p-adic version of Theorem 6.1.

THEOREM 6.2. – *Let $\mathfrak{O}$ be as in section 4. Then, there is a unique $\mathfrak{O}$-linear measure* $\theta : \mathfrak{C}(G_\infty(\mathfrak{C});\mathfrak{O}) \to S^{n\cdot \mathrm{ord}}(\mathfrak{N}_{M/F}(\mathfrak{C})D;\mathfrak{O})$ *which is given by*

$$\mathbf{a}_p(y, \theta(\phi)) = \begin{cases} \sum_{x \in R(y, \mathfrak{C})/U(p\mathfrak{C}\mathfrak{C}^c)} \phi(x), & \text{if } y \text{ is integral}, \\ 0, & \text{otherwise}, \end{cases}$$

for each function $\phi \in S(G_\infty(\mathfrak{C});\mathfrak{O})$, where ϕ is regarded as a function on $M_{\mathbf{A}}^\times$ via the natural projection map which sends $\varpi_{\mathfrak{q}}$ for each prime ideal $\mathfrak{q}$ outside $\mathfrak{C}_p$ to the class of the ideal $\mathfrak{q}$.

Proof. – Keeping the assumption that λ is unramified at Σ^c, we now treat imprimitive λ. Write the conductor of λ as $C(\lambda') = \mathfrak{C}' \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^{e(\mathfrak{P})}$, where $\mathfrak{C}'$ is a divisor of $\mathfrak{C}$. Then, again modifying as in [H 1, Lemma 5.3, (iii)] $\theta'(\lambda)$ in the proof of Theorem 6.1 of lever $D\mathfrak{N}_{M/F}(C(\lambda'))$, we still have $\theta(\lambda)$ of level Np^∞ given as in Theorem 6.1 for $N = D\mathfrak{N}_{M/F}(\mathfrak{C})$. We have a natural exact sequence

$$1 \to \mathfrak{R}_{\Sigma^c}^\times (= \mathfrak{r}_p^\times) \to G_\infty(\mathfrak{C}) \to Z(\mathfrak{C}) \to 1,$$

where $Z(\mathfrak{C}) = \varprojlim_\alpha \mathrm{Cl}_M(\mathfrak{C} \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^\alpha)$. Let Ω be the completion of $\bar{\mathbf{Q}}_p$ under $|\ |_p$ and consider the subspace A of $S(G_\infty(\mathfrak{C});\Omega)$ spanned by characters $\hat{\lambda}$ for all (primitive or imprimitive) Hecke characters λ unramified at Σ^c. We claim that

(6.4) A *is dense in* $S(G_\infty(\mathfrak{C});\Omega)$.

This is obvious because A contains any finite order character factoring through $Z(\mathfrak{C})$ and any character of the form $x \mapsto x^{-\eta}$ for $\eta \in \mathbf{Z}[I_M]$ with $\eta + \eta c \in \mathbf{Z} t_M$ (*cf.* [H 3, Lemma 3.9]). Since the Ω-linear map θ is well defined on A by Theorem 1 and is bounded with measure norm 1, θ naturally extends to $\mathfrak{C}(G_\infty(\mathfrak{C});\Omega)$ by continuity, and the extended measure still has the given q-expansion. By [H 1, (2.2 b)], we see $\theta(\lambda) \mid T(\varpi_{\mathfrak{p}}) = \hat{\lambda}(\varpi_{\mathfrak{P}})\, \varpi_{\mathfrak{p}}^v \theta(\lambda)$ for $\varpi_{\mathfrak{P}} \in \mathfrak{r}_p \cong \mathfrak{R}_{\Sigma^c}$. This shows that θ has values in the nearly ordinary space. This finishes the proof.

Proof of Theorem 6.1 *in the general case.* – We now prove the complex case of Theorem 6.2. Namely we show $\theta(\lambda)$ exists as a complex modular form even if $C(\lambda)$ is not prime to Σ^c. We already know that $\theta(\lambda)$ is meaningful as a p-adic nearly ordinary form even when (6.3) is not satisfied. If $n \geqq 0$, it is well known that any nearly ordinary common eigenform of $T(\varpi_{\mathfrak{q}})$ for all prime $\mathfrak{q}$ is classical ([H 3, Cor. 2.5], [H 1, Cor. 3.3]). This shows the desired assertion in the case of $n \geqq 0$. There is a more general argument which is valid without assuming $n \geqq 0$ and without using p-adic theory. Let $\pi = \otimes_{\mathfrak{q}} \pi_{\mathfrak{q}}$ be the automorphic representation generated by $\Theta(\lambda_0) \otimes |\ |_{\mathbf{A}}^{-m_0/2}$, which always exists. Let $V(\pi_{\mathfrak{q}})$ denote the representation space of

$\pi_{\mathfrak{q}}$. For each prime factor $\mathfrak{p}$ of p in F, by the ordinary assumption (0.1), $\mathfrak{p}$ splits into a product of two primes: $\mathfrak{p}=\mathfrak{P}\mathfrak{P}^c$ in M, *i.e.* $\mathfrak{P}\in\Sigma$ and $\mathfrak{P}^c\in\Sigma^c$. Then the local representation $\pi_{\mathfrak{p}}=\pi(\lambda_{\mathfrak{P}}, \lambda_{\mathfrak{P}^c})$ is a principal representation, which can be realized on the function space $B=B(\lambda_{\mathfrak{P}}, \lambda_{\mathfrak{P}^c})$ consisting of smooth functions ϕ on $GL_2(F_{\mathfrak{p}})$ such that $\phi\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} x\right)=|a|_{\mathfrak{p}}\lambda_{\mathfrak{P}}(d)\lambda_{\mathfrak{P}^c}(a)\phi(x)$ for all upper triangular matrices (here $GL_2(F_{\mathfrak{p}})$ acts on B by right translation). Writing $C(\lambda)=\mathfrak{C}\prod_{\mathfrak{P}\mid p}\mathfrak{P}^{e(\mathfrak{P})}$, let $\delta=\max(e(\mathfrak{P}), e(\mathfrak{P}^c))$ and define, for $\mathfrak{p}=F\cap\mathfrak{P}$,

$$U=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathfrak{r}_{\mathfrak{p}}) \,\middle|\, c\in\mathfrak{p}^{\delta} \text{ and } a\equiv d\equiv 1 \bmod \mathfrak{p}^{\delta}\right\}.$$

Then we define a function Φ on $GL_2(F_{\mathfrak{p}})$ by

$$\Phi(x)=\begin{cases} |a|_{\mathfrak{p}}\lambda_{\mathfrak{p}}(d)\lambda_{\mathfrak{p}^c}(a), & \text{if } x=\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} u \text{ for } u\in U, \\ 0, & \text{otherwise.}\end{cases}$$

Then $\Phi\in B$ and $\Phi\mid T(\varpi_{\mathfrak{p}})(x)=\sum_{r\in\mathfrak{r}_{\mathfrak{p}}/\mathfrak{p}}\Phi\left(x\begin{pmatrix}\varpi_{\mathfrak{p}} & r \\ 0 & 1\end{pmatrix}\right)=\lambda_{\mathfrak{P}^c}(\varpi_{\mathfrak{p}})\Phi(x)$. Thus in $V(\pi_{\mathfrak{p}})$, there is a non-zero vector $v_{\mathfrak{p}}$ which is an eigen vector of $T(\varpi_{\mathfrak{p}})$ with eigenvalue $\lambda_{\mathfrak{P}^c}(\varpi_{\mathfrak{p}})$. Outside p, it is well known that there is a unique (up to constant factors) new vector $v_{\mathfrak{p}}$ in $V(\pi_{\mathfrak{q}})$ satisfying $v_{\mathfrak{q}}\mid T(\mathfrak{q})=\mathbf{a}_p(\varpi_{\mathfrak{q}}, \theta(\lambda))v_{\mathfrak{q}}$. Especially, if $\pi_{\mathfrak{q}}$ is spherical, $v_{\mathfrak{q}}$ is the unique spherical vector. Thus the automorphic form corresponding to $\otimes_{\mathfrak{q}}v_{\mathfrak{q}}$ must be a non-zero constant multiple of $\theta(\lambda)$ because of the duality theorem [H 1, Th. 2.2], which finishes the proof.

Let $N=\mathfrak{N}_{M/F}(\mathfrak{C})D$ and $S^{n.\text{ord}}(N; \mathfrak{O})$ be the space of $\mathfrak{O}$-integral nearly ordinary p-adic modular forms introduced in [H 1, § 3, after Cor. 3.], where it is written as $\bar{S}^{n.\text{ord}}(N)$. By Theorem 6.2, we have an 0-linear map $\theta: S(G_\infty(\mathfrak{C}); \mathfrak{O})\to S^{n.\text{ord}}(N; \mathfrak{O})$, which induces

(6.5) θ: $S(G_\infty(\mathfrak{C}); K/\mathfrak{O})\to S^{n.\text{ord}}(N; K/\mathfrak{O})=S^{n.\text{ord}}(N; \mathfrak{O})\otimes_0 K/\mathfrak{O}$.

We now want to dermine the kernel of the above map. Write $M^*=\text{Hom}_0(M, \mathfrak{O})$ for any $\mathfrak{O}$-module M, and let $\mathbf{h}^{n.\text{ord}}(N; \mathfrak{O})$ be the p-adic nearly ordinary (cuspidal) Hecke algebra introduced in [H 3] and [H 1, § 3]. Then it is known that

(6.6 *a*) $S^{n.\text{ord}}(N; \mathfrak{O})^*\cong\mathbf{h}^{n.\text{ord}}(N; \mathfrak{O})$ canonically ([H 1, Th. 3.1]).

It is taulogical that

(6.6 *b*) $S(G_\infty(\mathfrak{C}); \mathfrak{O})^*\cong\mathfrak{O}[[G_\infty(\mathfrak{C})]]$ as topological algebras,

where on the left-hand side, the ring structure is given by the convolution product of p-adic measures. Thus θ induces a morphism by duality

$$\theta^*: \mathbf{h}^{n.\text{ord}}(N; \mathfrak{O})\to\mathfrak{O}[[G_\infty(\mathfrak{C})]].$$

Note that $S(G_\infty(\mathfrak{C}); \mathfrak{O})$ is naturally an $\mathfrak{O}[[G_\infty(\mathfrak{C})]]$-module via the convolution product of functions and measures. Then we see easily from the construction of (6.6 *a*, *b*) and θ that

$$(6.7a)\qquad \theta(\phi\,|\,\theta^*(h)) = \theta(\phi)\,|\,h \quad \text{for} \quad h \in \mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O}).$$

Thus especially θ^* is an algebra homomorphism. We have a natural group morphism $\iota : M_\Sigma^\times \times M_{\Sigma c}^\times \to G_\infty(\mathfrak{C})$ because $G_\infty(\mathfrak{C})$ is a quotient of $M_{\mathbf{A}}^\times$. Let $\mathbf{T}(y)$ for $y \in \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times$ be the (normalized) Hecke operator introduced in [H 1, § 3] and $\langle \mathfrak{q} \rangle$ be the operator given by the right translation $\mathbf{f}\,|\,\langle \mathfrak{q} \rangle(x) = \mathbf{f}(x\varpi_{\mathfrak{q}})$ for each modular form $\mathbf{f}$ in [H 1, § 1] and prime $\mathfrak{q}$ outside $\mathfrak{C}p$. Then by Theorem 6.2, identifying $\mathfrak{r}_p^\times$ with $\mathfrak{R}_{\Sigma c}^\times$, we conclude

$$(6.7b)\qquad \begin{cases} \theta^*(\mathbf{T}(u)) = \iota(1, u) \quad \text{for} \quad u \in \mathfrak{r}_p^\times, \\ \theta^*(\mathbf{T}(\varpi_{\mathfrak{p}})) = \iota(1, \varpi_{\mathfrak{p}}) \textit{ for each prime } \mathfrak{p} \textit{ dividing } p, \\ \theta^*(\mathbf{T}(\mathfrak{q})) = \begin{cases} [\mathfrak{Q}] + [\mathfrak{Q}^c] & \textit{if} \quad \mathfrak{q} = \mathfrak{Q}\mathfrak{Q}^c\ (\mathfrak{Q} \neq \mathfrak{Q}^c), \\ [\mathfrak{Q}] & \textit{if} \quad \mathfrak{Q} \textit{ ramifies in } M/F, \\ 0 & \textit{if} \quad \mathfrak{q} \textit{ remains prime in } M, \end{cases} \\ \theta^*(\langle \mathfrak{q} \rangle) = \chi(\mathfrak{q})\,\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-1}[\mathfrak{q}], \end{cases}$$

where $[\mathfrak{Q}]$ is the image of the prime ideal $\mathfrak{Q}$ under the Artin symbol and we agree to put $[\mathfrak{Q}] = 0$ in $\mathfrak{O}[[G_\infty(\mathfrak{C})]]$ if $\mathfrak{Q}$ divides $\mathfrak{C}$.

Now suppose that $\mathfrak{C}^c = \mathfrak{C}$. Then complex conjugation c acts naturally on $G_\infty(\mathfrak{C})$. Let $U^{(\Sigma\mathfrak{C})} = \{ u \in \hat{\mathfrak{R}}^\times \,|\, u_{\Sigma\mathfrak{C}} = 1 \}$.

PROPOSITION 6.3. – *Suppose that* $\mathfrak{C}^0 = \mathfrak{C}$ *and let* H *be a subgroup of* $G = G_\infty(\mathfrak{C})$ *generated by* uu^{-c} *for all* $u \in U^{(\Sigma\mathfrak{C})} \cup (\mathfrak{R}_{\mathfrak{C}\Sigma c} \cap M_{\mathfrak{C}\Sigma c}^\times)$. *Then we have*

$$\mathrm{Ker}(\theta) = \{ \phi \in \mathfrak{C}(G/H; K/\mathfrak{O}) \,|\, \phi \circ c = -\phi \}.$$

Proof. – For any split prime $\mathfrak{q}$ outside $\mathfrak{C}$ and $u \in U^{(\Sigma\mathfrak{C})} \cup (\mathfrak{R}_{\mathfrak{C}\Sigma c} \cap M_{\mathfrak{C}\Sigma c}^\times)$,

$$\mathbf{a}_p(\varpi_{\mathfrak{q}}\varpi_{\mathfrak{q}}^c uu^c, \theta(\phi)) = \phi(\varpi_{\mathfrak{q}} u) + \phi(\varpi_{\mathfrak{q}}^c u).$$

Thus if $\phi \in \mathrm{Ker}(\theta)$, then by Čebotarev density theorem, we know that $\phi \circ c = -\phi$. Thus $\phi(\varpi_{\mathfrak{q}} u) = -\phi(\varpi_{\mathfrak{q}c} u) = \phi(\varpi_{\mathfrak{q}} u^c)$. Again by Čebotarev density theorem, we see $\phi\,|\,u = \phi\,|\,u^c$ for all $u \in U^{(\Sigma\mathfrak{C})} \cup (\mathfrak{R}_{\mathfrak{C}\Sigma c} \cap M_{\mathfrak{C}\Sigma c}^\times)$, where $\phi\,|\,z(z') = \phi(zz')$ for $z, z' \in G$. Thus $\phi\,|\,uu^{-c} = \phi$ for all $u \in U^{(\Sigma\mathfrak{C})} \cup (\mathfrak{R}_{\mathfrak{C}\Sigma c} \cap M_{\mathfrak{C}\Sigma c}^\times)$. Then ϕ factors through G/H. The converse assertion is obvious.

Let $N = D\mathfrak{N}_{M/F}(\mathfrak{C})$ and define a compact group

$$\mathbf{G} = \mathbf{G}(N) = Z(N) \times \mathfrak{r}_p^\times$$

for

$$Z(N) = F_{\mathbf{A}}^\times / \overline{F^\times U_F(Np^\infty) F_{\infty+}^\times} = \varprojlim_\alpha \mathrm{Cl}_F(Np^\alpha).$$

The inclusion F_A into M_A induces a group homomorphism of $Z(N)$ into $G_\infty(\mathfrak{C})$ with finite kernel. The identification r_p with $\mathfrak{R}_{\Sigma c}$ combined with the natural morphism: $\mathfrak{R}_{\Sigma c}^\times \to M_A^\times \to G_\infty(\mathfrak{C})$ induces another group homomorphism of $r_p^\times$ into $G_\infty(\mathfrak{C})$. Thus we have a canonical morphism $i:\mathbf{G}(N) \to G_\infty(\mathfrak{C})$ with finite kernel and cokernel. In particular, we can choose the free part $\mathbf{W}$ of $G_\infty(\mathfrak{C})$ and W of $\mathbf{G}$ so that i induces an isomorphism of W into $\mathbf{W}$ with finite cokernel. We have two characters

$$r_p^\times \ni u \mapsto \mathbf{T}(u^{-1}) \in \mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O})$$

and

$$Z(N) \ni [q] \mapsto \langle q \rangle \in \mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O}).$$

These characters induces a $\mathfrak{O}[[\mathbf{G}]]$-algebra structure on $\mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O})$. The use of the character: $u \mapsto \mathbf{T}(u^{-1})$ [instead of $u \mapsto \mathbf{T}(u)$] looks artificial but it is in fact natural because $[\xi\,\mathfrak{R}]$ for $\xi \equiv 1 \bmod p\mathfrak{C}$ ($\xi \in M^\times$) is equal to $i(\xi_\Sigma^{-1}, \xi_{\Sigma c}^{-1})$ in $G_\infty(\mathfrak{C})$. Let $\mathbf{A} = \mathfrak{O}[[W]]$ and $\Lambda_0 = \mathfrak{O}[[\mathbf{W}]]$. Then by (6.7), θ^* is almost a $\mathbf{A}$-algebra homomorphism (*i.e.* θ^* becomes an $\mathbf{A}$-algebra homomorphism if we twist the $\mathbf{A}$-algebra structure of $\mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O})$ by the cyclotomic character). It is known [H 3, Th. 2.4] that

(6.8) $\mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O})$ *is torsion free and of finite type as* $\mathbf{A}$-*module.*

With the notation of Proposition 6.3, the $\mathbf{Q}_p$-dimension of $(G/H)\otimes_Z \mathbf{Q}$ is $d = [F:\mathbf{Q}]$, less than that of $G\otimes\mathbf{Q}$ because H is of finite index in $G^- = \{x \in G \mid x^c = x^{-1}\}$. Therefore the Pontryagin dual module $\mathrm{Ker}(\theta)^*$ of $\mathrm{Ker}(\theta)$ is pseudo-null as $\mathfrak{O}[[\mathbf{G}]]$-module if $F \neq \mathbf{Q}$ and is always torsion $\mathfrak{O}[[\mathbf{W}]]$-module. In the following corollary, we do not assume that $\mathfrak{C} = \mathfrak{C}^c$.

COROLLARY 6.4. — *Let* $G_{\mathrm{tor}}(\mathfrak{C})$ *be the torsion part of* $G_\infty(\mathfrak{C})$ *and* $\psi: G_{\mathrm{tor}}(\mathfrak{C}) \to \mathfrak{O}^\times$ *be a character. Let* $\psi_*: \mathfrak{O}[[G_\infty(\mathfrak{C})]] \to \mathfrak{O}[[\mathbf{W}]]$ *be the projection induced by* ψ, i.e. $\psi_*(\zeta, w) = \psi(\zeta)w) \in G_{\mathrm{tor}} \times \mathbf{W})$. *Let* $\theta^*: \mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O}) \to \mathfrak{O}[[G_\infty(\mathfrak{C})]]$ *for* $N = \mathfrak{N}_{M/F}(\mathfrak{C})D$ *be the dual map of* (6.5), *which is an algebra homomorphism. Then* $\lambda = \psi_* \circ \theta^*$ *is surjective if* $\psi \not\equiv \psi \circ c \bmod \mathfrak{m}$ *for the maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{O}$. *Without assuming that* $\psi \not\equiv \psi \circ c \bmod \mathfrak{m}$, λ *becomes surjective after localizing at any height one prime* P *of* $\mathfrak{O}[[W]]$ *if* $F \neq \mathbf{Q}$. *When* $F = \mathbf{Q}$, *possibly for all but one height one prime* P_0, *the morphism* λ *induces a surjection after localization.*

Proof. — Applying Proposition 6.3 to $\mathfrak{C}' = \mathfrak{C} \cap \mathfrak{C}^c$, we have a morphism $\lambda' = \psi^* \circ \theta^*: \mathbf{h}^{n.\mathrm{ord}}(\mathfrak{N}_{M/F}(\mathfrak{C}')D; \mathfrak{O}) \to \mathfrak{O}[[\mathbf{W}]]$. The cokernel of this morphism is given by the Pontryagin dual $\mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] \cap \mathrm{Ker}(\theta)$, where

$$\mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] = \{\phi \in \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})\phi \mid x = \psi(x)\phi \text{ for } x \in G_{\mathrm{tor}}(\mathfrak{C}')\}.$$

We see easily that $(\phi \circ c) \mid x = (\phi \mid x^c) \circ c$. Thus if $\phi \in \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] \cap \mathrm{Ker}(\theta)$, then $\phi \circ c = -\phi$ and $\phi \circ c \in \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi \circ c]$. Namely

$$\mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] \cap \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi \circ c] \supset \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] \cap \mathrm{Ker}(\theta).$$

We know that

$$\mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi] \cap \mathscr{C}(G_\infty(\mathfrak{C}'); K/\mathfrak{O})[\psi \circ c] = 0 \qquad \text{if} \quad \psi \not\equiv \psi \circ c \bmod \mathfrak{m}.$$

This shows the assertion because $\mathrm{Im}(\lambda) \supset \mathrm{Im}(\lambda')$ by (6.7 *b*). For any height one prime P of $\mathfrak{O}[[\mathbf{W}]]$, as already seen, $\lambda_P = \lambda \bmod P$ satisfies $\lambda_P \neq \lambda_P \circ c$ if $F \neq \mathbf{Q}$. When $F = \mathbf{Q}$, there is at most one height one prime P_0 modulo which $\lambda_P = \lambda_P \circ c$ holds. In fact, this prime corresponds to the cyclotomic projection $\mathfrak{O}[[\mathbf{W}]] \to \mathfrak{O}[[\Gamma]]$ for $\Gamma = 1 + p\mathbf{Z}_p$ given by $w \mapsto ww^c = \mathfrak{N}_{M/\mathbf{Q}}(w)$. As long as $\lambda_P \neq \lambda_P \circ c$ holds, the same proof works well, and we have the surjectivity of λ after localization at P.

We now have an algebra homomorphism

$$\lambda: \quad \mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O}) \to \Lambda_0 = \mathfrak{O}[[\mathbf{W}]] \qquad (N = \mathfrak{N}_{M/F}(\mathfrak{C})\, D),$$

which is generically surjective. Let $\mathbf{G}_{\mathrm{tor}}$ be the torsion part of $\mathbf{G}(N)$. Let ψ_0 be the restriction of λ to $\mathbf{G}_{\mathrm{tor}}$. Since $\mathbf{G} = \mathfrak{r}_p^\times \times Z(N)$, we can write $\mathbf{G}_{\mathrm{tor}} = \mu \times Z_{\mathrm{tor}}$ for the torsion parts μ of $\mathfrak{r}_p^\times$ and Z_{tor} of $Z(N)$. Thus we have two characters ψ' and ψ^+ such that $\psi_0(\zeta, z) = \psi'(\zeta)\psi^+(z)$ for $(\zeta, z) \in \mu \times Z_{\mathrm{tor}}$. Then $\psi' = \psi^{-1} \circ i|_\mu$ and $\psi^+ = \psi\omega^{-1}\chi|_{Z_{\mathrm{tor}}}$, where $\chi(\mathfrak{q}) = ((M/F)/\mathfrak{q})$. There (ψ', ψ^+) is the character of λ in the sense of [H 1, § 5]. As seen in [H 1, § 5], we can decompose $\mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O}) \otimes_{\mathbf{Z}} \mathbf{Q} = H(\psi^+, \psi^+) \oplus B$ as an algebra direct sum so that $H(\psi^+, \psi')$ is the maximal quotient on which $\mathbf{G}_{\mathrm{tor}}$ acts via the character (ψ^+, ψ'). Let $\mathbf{h}(\psi^+, \psi')$ be the image of $\mathbf{h}^{n.\mathrm{ord}}(N; \mathfrak{O})$ in $H(\psi^+, \psi')$. Suppose that ψ as a character of $G_\infty(\mathfrak{C})/\mathbf{W} = G_{\mathrm{tor}}$ has conductor divisible by $\mathfrak{C}$. Then λ is primitive in the sense of [H 1, Th. 3.4] by theorem 6.1. Let $\mathbf{K}$ be the quotient field of Λ_0. Then λ induces a $\mathbf{K}$-algebra decomposition

$$\mathbf{h}(\psi^+, \psi') \otimes_{\mathbf{A}} \mathbf{K} = \mathbf{K} \oplus \mathbf{B}$$

so that the first projection coincides with λ on $\mathbf{h}(\psi^+, \psi')$. Writing R for $\mathbf{h}(\psi^+, \psi') \otimes_{\mathbf{A}} \Lambda_0$, we consider the images $R(\mathbf{K}) \cong \Lambda_0$ and $R(\mathbf{B})$ in $\mathbf{K}$ and $\mathbf{B}$, respectively. Then the congruence module $C_0(\lambda)$ is defined by

$$(6.9) \qquad C_0(\lambda) = R(\mathbf{K}) \otimes_R R(\mathbf{B}) \cong (R(\mathbf{K}) \oplus R(\mathbf{B}))/R \cong R(\mathbf{K})/\{R \cap (R(\mathbf{K}) \oplus 0)\}$$

which is a torsion Λ_0-module of finite type ([H 6, § 6]). Let H be a generator of the smallest principal ideal in Λ_0 containing the ideal $R \cap (R(\mathbf{K}) \oplus 0)$. Thus H is a characteristic power series of $C_0(\lambda)$.

7. Petersson inner product and symmetric square *L*-functions

In this section, we generalize the formula in [H 4, Th. 5.1] relating the self Petersson inner product (**f**, **f**) (of a primitive Hilbert modular form **f**) with a special value of the symmetric square *L*-function of **f**. We use the same notation as in [H 1] for complex and *p*-adic Hilbert modular forms. Let **f** in $S_{k,w}(N, \psi', \psi; \mathbf{C})$ be a common eigenform of Hecke operators $T(\varpi_\mathfrak{q})$ for all primes $\mathfrak{q}$. The ideal N is assumed to be the smallest possible level of **f**. Let $\mathbf{f}^0$ be the primitive form in $S_{k,w}(\mathrm{C}, \mathrm{id}, \psi; \mathbf{C})$ in the sense of [H 1, § 5] associated with **f**, where $\mathrm{C}=\mathrm{C}(\mathbf{f})$ is the conductor of **f**. Then $k-2w=mt$ with an integer $m\geqq 0$ and $t=\sum_\sigma \sigma$. Write $\mathbf{f}^0|\mathrm{T}(\mathfrak{q})=\mathbf{a}(\mathfrak{q})\,\mathbf{f}^0$ for each prime ideal $\mathfrak{q}$ of F and $\pi=\otimes_\mathfrak{q}\pi_\mathfrak{q}$ for the automorphic representation of $\mathrm{GL}_2(\mathrm{F_A})$ generated by the right translations of **f**. We call $\pi_\mathfrak{q}$ is minimal if $\mathrm{C}(\pi_\mathfrak{q})\supset\mathrm{C}(\pi_\mathfrak{q}\otimes\xi)$ as ideal in $\mathfrak{r}_\mathfrak{q}$ for all quasi characters $\xi:\mathrm{F}_\mathfrak{q}^\times\to\mathbf{C}^\times$. Here for local representation $\pi_\mathfrak{q}$, $\mathrm{C}(\pi_\mathfrak{q})$ denotes its conductor. When $\pi_\mathfrak{q}$ is a minimal principal series representation or minimal special representation, we may write $\pi_\mathfrak{q}=\pi(\eta,\eta')$ or $\sigma(\eta,\eta')$ so that η is unramified and $\mathbf{a}(\mathfrak{q})=\eta(\varpi_\mathfrak{q})+\eta(\varpi_\mathfrak{q})$ according as $\pi_\mathfrak{q}$ is spherical or not. When $\pi_\mathfrak{q}$ is principal, we define $\alpha_\mathfrak{q}=\eta(\varpi_\mathfrak{q})|\varpi_\mathfrak{q}|_\mathfrak{q}^{(m+1)/2}$ and $\beta_\mathfrak{q}=\eta'(\varpi_\mathfrak{q})|\varpi_\mathfrak{q}|_\mathfrak{q}^{(m+1)/2}$. We first define an imprimitive adjoint lift L-function $\mathfrak{L}(s,\mathrm{Ad}(\mathbf{f}))$ by the following Euler product:

$$\mathfrak{L}(s,\mathrm{Ad}(\mathbf{f}))=\prod_\mathfrak{q}\mathfrak{L}_\mathfrak{q}(N(\mathfrak{q})^{-s})^{-1} \tag{7.1}$$

where

$$\mathfrak{L}_\mathfrak{q}(\mathrm{X})=\begin{cases}(1-\bar\alpha_\mathfrak{q}\beta_\mathfrak{q}\mathrm{X})(1-\bar\beta_\mathfrak{q}\alpha_\mathfrak{q}\mathrm{X}), & \text{if}\quad \pi_\mathfrak{q} \text{ is spherical,}\\ (1-|\varpi_\mathfrak{q}|_\mathfrak{q}\mathrm{X}), & \text{if}\quad \pi_\mathfrak{q} \text{ is special,}\\ 1, & \text{otherwise.}\end{cases}$$

Next we denote by $L(s,\mathrm{Ad}(\mathbf{f}))$ the primitive L-function attached to the adjoint lift of π to GL(3) by Gelbart-Jacquet [GJ]. Note that $L(s,\mathbf{f})$ is independent of the twist of **f**. Let S be the set of primes $\mathfrak{q}$ dividing C such that

(7.2) $\pi_\mathfrak{q}$ *is supercuspidal and* $\pi_\mathfrak{q}\otimes\chi_\mathfrak{q}\cong\pi_\mathfrak{q}$ *for the unique unramified quadratic character* $\chi_\mathfrak{q}$ *of* $\mathrm{F}_\mathfrak{q}$.

When $\pi_\mathfrak{q}$ is principal or special, we write $\pi_\mathfrak{q}=\pi(\eta_\mathfrak{q},\eta'_\mathfrak{q})$ or $\sigma(\eta_\mathfrak{q},\eta'_\mathfrak{q})$ and define

$$\mathrm{S}'=\{\mathfrak{q}\,|\,\mathrm{C}(\mathbf{f})\,|\,\pi_\mathfrak{q}=\pi(\eta_\mathfrak{q},\eta'_\mathfrak{q}) \text{ and } \eta_\mathfrak{q}=\eta'_\mathfrak{q} \text{ on } \mathfrak{r}_\mathfrak{q}^\times\},$$

$$\Xi_p=\{\mathfrak{q}\,|\,\mathrm{C}(\mathbf{f})\,|\,\pi_\mathfrak{q}=\pi(\eta_\mathfrak{q},\eta'_\mathfrak{q}) \text{ and } \pi_\mathfrak{q} \text{ is minimal}\},$$

$$\Xi_s=\{\mathfrak{q}\,|\,\mathrm{C}(\mathbf{f})\,|\,\pi_\mathfrak{q}=\sigma(\eta_\mathfrak{q},\eta'_\mathfrak{q}) \text{ and } \pi_\mathfrak{q} \text{ is minimal}\},$$

$$\Xi'=\{\mathfrak{q}\,|\,\mathrm{C}(\mathbf{f})\,|\,\mathfrak{q}\notin\Xi,\ \pi_\mathfrak{q}=\pi(\eta_\mathfrak{q},\eta'_\mathfrak{q}) \text{ and } \eta_\mathfrak{q}\neq\eta'_\mathfrak{q} \text{ on } \mathfrak{r}_\mathfrak{q}^\times\},$$

$$\Xi''=\{\mathfrak{q}\,|\,\mathrm{C}(\mathbf{f})\,|\,\mathfrak{q}\notin\Xi,\ \pi_\mathfrak{q}=\sigma(\eta_\mathfrak{q},\eta'_\mathfrak{q}) \text{ and } \eta_\mathfrak{q} \text{ is ramified}\}$$

We assume that

(7.3 *a*) **f** *is primitive in the sense of* [H 1, § 5],

which is equivalent to the following two conditions:

(i) For $\mathfrak{q}\in\Xi$, $\eta_\mathfrak{q}$ is unramified, and $\mathbf{f}|\mathrm{T}(\varpi_\mathfrak{q})=\eta(\varpi_\mathfrak{q})\,\mathbf{f}$.

(ii) $\mathbf{f}|\mathrm{T}(\varpi_\mathfrak{q})=0$ *if* $\mathfrak{q}\in\Xi'\cup\Xi''\cup\mathrm{S}'\cup\mathrm{S}$.

Then we define for each prime ideal $\mathfrak{q}$ dividing $C(\mathbf{f})$ (*cf.* [H5, § 6])

$$(7.3\,c)\qquad \begin{cases} L_{\mathfrak{q}}(X)=\mathfrak{L}_{\mathfrak{q}}(X) \text{ if } \pi_{\mathfrak{q}} \textit{ is either special and } \mathfrak{q}\in\Xi \textit{ or spherical},\\ L_{\mathfrak{q}}(X)=(1-X) \text{ if } \mathfrak{q}\in\Xi_p\cup\Xi',\\ L_{\mathfrak{q}}(X)=(1-|\varpi_{\mathfrak{q}}|_{\mathfrak{q}}X) \quad \textit{if} \quad \mathfrak{q}\in\Xi''\cup\Xi_s,\\ L_{\mathfrak{q}}(X)=(1+X) \quad \textit{if} \quad \mathfrak{q}\in S,\\ L_{\mathfrak{q}}(X)=(1-\gamma_{\mathfrak{q}}X)(1-X)(1-\bar{\gamma}_{\mathfrak{q}}X)\\ \textit{for} \quad \gamma_{\mathfrak{q}}=\eta'_{\mathfrak{q}}(\varpi_{\mathfrak{q}})/\eta_{\mathfrak{q}}(\varpi_{\mathfrak{q}}) \quad \textit{for} \quad \mathfrak{q}\in S',\\ L_{\mathfrak{q}}(X)=1 \qquad \textit{otherwise}. \end{cases}$$

Here we note that $L_{\mathfrak{q}}(X)=\mathfrak{L}_{\mathfrak{q}}(X)$ if either $\mathfrak{q}\in\Xi_p\cup\Xi_s$ or $\pi_{\mathfrak{q}}$ is supercuspidal but $\mathfrak{q}\notin S$, and otherwise $\mathfrak{L}_{\mathfrak{q}}(X)=1$ for $\mathfrak{q}\,|\,C$ because of (ii). Thus $\mathfrak{L}_{\mathfrak{q}}(X)$ is different from $L_{\mathfrak{q}}(X)$ only for $\mathfrak{q}$ in $E=S\cup S'\cup\Xi'\cup\Xi''$. Namely

$$L_E(s, \mathrm{Ad}(\mathbf{f}))=\prod_{\mathfrak{q}\in E} L_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s})\, L(s, \mathrm{Ad}(\mathbf{f}))=\mathfrak{L}(s, \mathrm{Ad}(\mathbf{f})).$$

We then define

$$(7.3\,c)\qquad L(s, \mathrm{Ad}(\mathbf{f}))=\prod_{\mathfrak{q}} L_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}.$$

Let ρ be the 2-dimensional Galois representation into $GL_2(\bar{\mathbf{Q}}_l)$ attached to $\mathbf{f}$. Namely ρ is unramified outside Cl and for every prime $\mathfrak{q}$ outside Cl,

$$\det(1_2-\rho(\mathrm{Frob}_{\mathfrak{q}})X)=(X-\alpha_{\mathfrak{q}})(X-\beta_{\mathfrak{q}}).$$

Then writing Ad for the adjoint representation of the algebraic group $GL(2)$, the L-function $L(s, \mathrm{Ad}(\mathbf{f}))$ is the L-function of the Galois representation $\mathrm{Ad}\circ\rho$. It is obvious that the Euler factor at $\mathfrak{q}$ for $\mathfrak{q}$ outside Cl as above is a correct factor for the Galois representation $\mathrm{Ad}\circ\rho$ by definition. We can vary l and therefore, the prime l does not pose any problem. The determination of the Euler factors at primes dividing C according to the classification of the automorphic representation is a subtle question, which is basically solved by [GJ]. A good and clear summary of the result in [GJ] can be found in [Sch, § 1] (there is a minor misprint in the formula (1.7) in [Sch], and we need to replace $\mathrm{Sym}^2(\sigma_\infty)$ there by $\mathrm{Sym}^2(\sigma_\infty)\otimes\mathrm{sgn}^k$; *see* also [H5, § 6]).

THEOREM 7.1. – *Let* $\mathbf{f}$ *be a primitive form in* $\mathbf{S}_{k,w}(C, \psi', \psi; \mathbf{C})$ *satisfying* (7.3 *a*, *b*). *Let* $E=\Xi'\cup\Xi''\cup S\cup S'$. *Then we have*

$$(\mathbf{f}^u, \mathbf{f}^u)_C=|D_F|\,\Gamma_F(k)\,\mathfrak{N}_{F/\mathbf{Q}}(C)\,2^{-2\{k\}+1}\pi^{-d-\{k\}}L_E(1, \mathrm{Ad}(\mathbf{f})),$$

where

$$L_E(s, \mathrm{Ad}(\mathbf{f}))=\prod_{\mathfrak{q}\in E} L_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s})\, L(s, \mathrm{Ad}(\mathbf{f}))=\mathfrak{L}(s, \mathrm{Ad}(\mathbf{f})), \qquad \Gamma_F(k)=\prod_{\sigma\in I}\Gamma(k_\sigma),$$

$\{k\} = \sum_{\sigma \in I} k_\sigma$, $\mathbf{f}^u \in \mathbf{S}_{k, k/2}(\mathbf{C}; \psi', \psi; \mathbf{C})$ *is the unitarization of* $\mathbf{f}$ *defined in* [*H*1, (4.2 *a*)] *and* $(\mathbf{f}^u, \mathbf{f}^u)_\mathbf{C}$ *is the self Petersson inner product of* $\mathbf{f}^u$ *defined in* [*H*1, (7.1)].

Before starting the proof of the theorem, we prepare a lemma concerning the Rankin product L-functions: let $\mathbf{f}$ (resp. $\mathbf{g}$) be primitive forms in $\mathbf{S}_{k, w}(\mathrm{N}, \mathrm{id}, \psi; \mathbf{C})$ (resp. $\mathbf{S}_{\varkappa, \omega}(\mathrm{M}, \mathrm{id}, \chi; \mathbf{C})$). Write

$$k\text{-}2w = mt \qquad \text{and} \qquad \kappa\text{-}2\omega = \mu t \, (m, \mu \in \mathbf{Z}).$$

Let $\pi = \otimes_q \pi_q$ (resp. $\pi' = \otimes_q \pi'_q$) be the automorphic representation spanned by $\mathbf{f}$ and $\mathbf{g}$. When π_q (resp. π'_q) is principal or special, we write $\pi_q = \pi(\eta_q, \eta'_q)$ or $\sigma(\eta_q, \eta'_q)$ (resp. $\pi'_q = \pi(\xi_q, \xi'_q)$ or $\sigma(\xi_q, \xi'_q)$) and assume that $\mathbf{f} \mid \mathrm{T}(\varpi_q) = \eta(\varpi_q)\mathbf{f}$ and η is unramified (resp. $\mathbf{g} \mid \mathrm{T}(\varpi_q) = \xi(\varpi_q)\mathbf{g}$ and ξ is unramified) if π_q (resp. π'_q) is minimal but neither supercuspidal nor spherical. Then we define whenever π_q (resp. π'_q) is minimal but not supercuspidal,

$$(7.4) \quad \begin{cases} \alpha_q = \eta(\varpi_q) \, |\varpi_q|_q^{(m+1)/2} & \text{and} \quad \beta_q = \eta'(\varpi_q) \, |\varpi_q|_q^{(m+1)/2} \\ a_q = \eta(\varpi_q) \, |\varpi_q|_q^{(m/2)+1} & \text{and} \quad b_q = \eta'(\varpi_q) \, |\varpi_q|_q^{(m/2)+1} \\ (\text{resp. } \alpha'_q = \xi(\varpi_q) \, |\varpi_q|_q^{(\mu+1)/2} & \text{and} \quad \beta'_q = \xi'(\varpi_q) \, |\varpi_q|_q^{(\mu+1)/2}, \\ a'_q = \xi'(\varpi_q) \, |\varpi_q|_q^{(\mu/2)+1} & \text{and} \quad b'_q = \xi'(\varpi_q) \, |\varpi_q|_q^{(\mu/2)+1}) \end{cases}$$

We now write the Fourier expansion of $\mathbf{f}^u$ as

$$(7.5) \quad \mathbf{f}^u\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \xi \in \mathrm{F}} a(\xi y d, \mathbf{f}) (\xi y_\infty)^{k/2} \mathbf{e}_\mathrm{F}(i \xi y_\infty) \mathbf{e}_\mathrm{F} \xi x) \qquad \text{for} \quad d = d_\mathrm{F},$$

where the function $y \mapsto a(y, \mathbf{f})$ is defined on finite ideles and vanishes outside integral ideles. We then know (*see* [H1, (4.3 *b*). (2.2)] that

$$(7.6) \qquad a(\varpi)_q^n, \mathbf{f}) = |\varpi_q^n|_q^{1/2} \frac{\alpha_q^{n+1} - \beta_q^{n+1}}{\alpha_q - \beta_q} = \frac{a_q^{n+1} - b_q^{n+1}}{a_q - b_q}$$

and $\alpha_q \beta_q = \psi(\varpi_q)$ *if* π_q *is spherical,*

$a(\varpi_q^n, \mathbf{f}) = |\varpi_q^n|_q^{1/2} \alpha_q^n = a_q^n$ *is spherical,*

$a(\varpi_q^n, \mathbf{f}) = |\varpi_q^n|_q^{1/2} \alpha_q^n = a_q^n$ *if* π_q *is minimal but neither spherical nor supercuspidal,*

$a(\varpi_q^n, \mathbf{f}) = 0$ *if* $n > 0$ *and* π_q *is non-minimal or supercuspidal.*

Similar description of $a(\varpi_q^n, \mathbf{g})$ holds using α'_q, β'_q and b'_q. We divide our argument into the following 5 cases:

Case A: π_q and π'_q are both spherical,

Case B: π_q is minimal but neither spherical nor supercuspidal and π'_q is spherical,

Case C : π'_q is minimal but neither spherical nor supercuspidal and π_q is spherical,

Case D: π_q and π'_q and both minimal but neither spherical nor supercuspidal

Case E: one of π_q and π'_q is non-minimal or supercuspidal.

Then we define Euler factors

$$(7.7)\quad \begin{cases} \mathfrak{D}_{\mathfrak{q}}(X)=(1-\alpha_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X)(1-\alpha_{\mathfrak{q}}\beta'_{\mathfrak{q}}X)(1-\beta_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X)(1-\beta_{\mathfrak{q}}\beta'_{\mathfrak{q}}X) \text{ in Case A,} \\ \mathfrak{D}_{\mathfrak{q}}(X)=(1-\alpha_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X)(1-\alpha_{\mathfrak{q}}\beta'_{\mathfrak{q}}X) \text{ in Case B,} \\ \mathfrak{D}_{\mathfrak{q}}(X)=(1-\alpha_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X)(1-\beta_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X) \text{ in Case C,} \\ \mathfrak{D}_{\mathfrak{q}}(X)=(1-\alpha_{\mathfrak{q}}\alpha'_{\mathfrak{q}}X) \text{ in Case D, } \mathfrak{D}_{\mathfrak{q}}(X)=1 \text{ Case E.} \end{cases}$$

We define

$$(7.8)\quad \begin{cases} \mathfrak{D}(s, \mathbf{f}, \mathbf{g})=\prod_{\mathfrak{q}} \mathfrak{D}_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s})^{-1} \\ \text{and} \\ D(s, \mathbf{f}, \mathbf{g})=\sum_{\mathfrak{a}} a(\mathfrak{a}, \mathbf{f})\, a(\mathfrak{a}, \mathbf{g})\, \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{a})^{-s}, \end{cases}$$

where $\mathfrak{a}$ runs over all integral ideal of F. It is easy to see that $\mathfrak{D}(s, \mathbf{f}, \mathbf{g})$ is equal to the standard zeta function $L(s, \pi^u \times \pi'^u)$ of $GL(2) \times GL(2)_{/F}$ up to finitely many Euler factors (*see* [H1, § 0 and § 5]).

LEMMA 7.2. – *Let* L *be the product of primes* $\mathfrak{q}$ *for which one of* $\pi_{\mathfrak{q}}$ *and* $\pi'_{\mathfrak{q}}$ *are not spherical. Then we have*

$$\mathfrak{D}(s+1, \mathbf{f}, \mathbf{g})=L_L(2s+2, \chi\psi)\, D(s, \mathbf{f}, \mathbf{g}),$$

where $L_L(s, \chi\psi)=\{\prod_{\mathfrak{q}|L}(1-\chi\psi(\mathfrak{q})N(\mathfrak{q})^{-s})\}L(s, \chi\psi)$ *for the primitive Hecke L-function with character* $\chi\psi$.

Proof. – By definition, we need to compute

$$P(X)=\sum_{n=0}^{\infty} a(\varpi_{\mathfrak{q}}^n, \mathbf{g})\, X^n.$$

We start the computation in Case A. Then we see

$$P(X)=\sum_{n=0}^{\infty} |\varpi_{\mathfrak{q}}^n|_{\mathfrak{q}} \frac{(\alpha_{\mathfrak{q}}^{n+1}-\beta_{\mathfrak{q}}^{n+1})(\alpha_{\mathfrak{q}}'^{n+1}-\beta_{\mathfrak{q}}'^{n+1})}{(\alpha_{\mathfrak{q}}-\beta_{\mathfrak{q}})(\alpha'_{\mathfrak{q}}-\beta'_{\mathfrak{q}})} X_n.$$

Thus we see

$$\begin{aligned} P(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})X) &= \frac{1}{(\alpha_{\mathfrak{q}}-\beta_{\mathfrak{q}})(\alpha'_{\mathfrak{q}}-\beta'_{\mathfrak{q}})} \sum_{n=0}^{\infty} (\alpha_{\mathfrak{q}}^{n+1}-\beta_{\mathfrak{q}}^{n+1})(\alpha_{\mathfrak{q}}'^{n+1}-\beta_{\mathfrak{q}}'^{n+1}) X^n \\ &= \frac{1}{(\alpha_{\mathfrak{q}}-\beta_{\mathfrak{q}})(\alpha'_{\mathfrak{q}}-\beta'_{\mathfrak{q}})} \sum_{n=0}^{\infty} \{(\alpha_{\mathfrak{q}}\alpha'_{\mathfrak{q}})^{n+1}-(\alpha_{\mathfrak{q}}\beta'_{\mathfrak{q}})^{n+1}-(\beta_{\mathfrak{q}}\alpha'_{\mathfrak{q}})^{n+1}+(\beta_{\mathfrak{q}}\beta'_{\mathfrak{q}})^{n+1}\} X^n \\ &= (1-\alpha_{\mathfrak{q}}\beta_{\mathfrak{q}}\alpha'_{\mathfrak{q}}\beta'_{\mathfrak{q}}X^2)\, \mathfrak{D}_{\mathfrak{q}}(X)^{-1}=(1-\chi\psi(\varpi_{\mathfrak{q}})X^2)\, \mathfrak{D}_{\mathfrak{q}}(X)^{-1}. \end{aligned}$$

This shows the assertion in Case A. Now we go into Case B. We have

$$P(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})X)=\sum_{n=0}^{\infty}\frac{\alpha_{\mathfrak{q}}^{n+1}(\alpha_{\mathfrak{q}}'^{n+1}-\beta_{\mathfrak{q}}'^{n+1})}{\alpha_{\mathfrak{q}}(\alpha_{\mathfrak{q}}'-\beta_{\mathfrak{q}}')}X^n=\mathfrak{D}_{\mathfrak{q}}(X)^{-1},$$

which shows the desired assertion. The similar computation yields in Case C that

$$P(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})X)=\mathfrak{D}_{\mathfrak{q}}(X)^{-1}.$$

In Case D and E, the computation is much simpler and yield the same result, which finishes the proof of the lemma.

Proof of Theorem 7.1. — We now take a primitive from $\mathbf{f}\in\mathbf{S}_{k,w}(C, \mathrm{id}, \psi; \mathbf{C})$. Let $\mathbf{f}^c$ be the complex conjugate of $\mathbf{f}$; *i.e.* $\mathbf{a}(y, \mathbf{f}^c)=\mathbf{a}(y, \mathbf{f})^c$ for all $y\in F_{\mathbf{A}_f}^\times$. Then we consider $\mathfrak{D}(s, \mathbf{f}, \mathbf{f}^c)$. Here Cases A, D and E only can occur. We divides $\Xi=\Xi_p\cup\Xi_s$ so that

$$\Xi_p\,(\text{resp. } \Xi_s)=\{\mathfrak{q}\in\Xi\,|\,\pi_{\mathfrak{q}} \text{ is principal (resp. special)}\}.$$

Then we have

$$\mathfrak{D}_{\mathfrak{q}}(X)=(1-X)L_{\mathfrak{q}}(X)\quad\text{in Case A,}\qquad \mathfrak{D}_{\mathfrak{q}}(X)=L_{\mathfrak{q}}(X)\quad\text{when } \mathfrak{q}\in\Xi_s,$$
$$\mathfrak{D}_{\mathfrak{q}}(X)=(1-X)\quad\text{when } \mathfrak{q}\in\Xi_p\qquad\text{and}\qquad \mathfrak{D}_{\mathfrak{q}}(X)=1\quad\text{when } \mathfrak{q}\in E.$$

Thus we see for the Dedekind zeta function ζ of F

$$\begin{aligned}\zeta_C(2s+2)D(s, \mathbf{f}, \mathbf{f}^c)&=\mathfrak{D}(s+1, \mathbf{f}, \mathbf{f}^c)\\ &=\prod_{\mathfrak{q}\,|\,C}(1-\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s-1})\zeta(s+1)\prod_{\mathfrak{q}\notin\Xi,\,\mathfrak{q}\,|\,C}L_{\mathfrak{q}}(\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s})L(s+1, \mathrm{Ad}(\mathbf{f}))\\ &=\prod_{\mathfrak{q}\,|\,C}(1-\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-s-1})\zeta(s+1)L_E(s+1, \mathrm{Ad}(\mathbf{f})),\end{aligned}$$

since for $\mathfrak{q}\,|\,C$ but $\mathfrak{q}\notin E$, $L_{\mathfrak{q}}(X)=1$. Thus we have

$$\mathrm{Res}_{s=0}\,\zeta_C(2s+2)D(s, \mathbf{f}, \mathbf{f}^c)=\{\mathrm{Res}_{s=1}\,\zeta(s)\}\prod_{\mathfrak{q}\,|\,C}(1-\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-1})L_E(1, \mathrm{Ad}(\mathbf{f})).$$

On the other hand, with the notation of [H1, Th. 6.1], we see

$$\begin{aligned}\mathrm{Res}_{s=1}\,G_0(x, \mathrm{id}, \mathrm{id}; s)&=\mathrm{Res}_{s=1}\,E_0(x, \mathrm{id}, \mathrm{id}; s)=\pi^d\,\mathrm{Res}_{s=1}\,\zeta_C(2s-1)\\ &=2^{-1}\pi^d\,\mathrm{Res}_{s=1}\,\zeta_C(s)=2^{-1}\pi^d\sum_{\mathfrak{q}\,|\,C}(1-\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})^{-1})\,\mathrm{Res}_{s=1}\,\zeta(s).\end{aligned}$$

The first equality of the above formula follows from the definition:

$$G_0(x, \mathrm{id}, \mathrm{id}; s)=\mathfrak{N}_{F/\mathbf{Q}}(C)^{s-1}E_0(x\tau, \mathrm{id}, \mathrm{id}; s)$$

for

$$\tau=\begin{pmatrix}0 & -1\\ m & 0\end{pmatrix}\in GL_2(F_{\mathbf{A}_f}),$$

where m is a finite idele with $m\mathfrak{r}=\mathrm{C}$. Then we can see the second equality follows looking into the Fourier expansion of $\mathrm{E}_0(x, \mathrm{id}, \mathrm{id}; s)$ given in [H1, Th. 6.1]. From the formula [H1, (4.7), (4.2 a)], we see

$$\mathrm{Z}(s.\ \mathbf{f}^c, \mathbf{f}, \mathrm{id})=\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathrm{C})^{-1}\,|\mathrm{D}_\mathrm{F}|^{-(1/2)}\int_{\mathrm{X}_0(\mathrm{C})}\overline{\mathbf{f}^u(x)}\,\mathbf{f}^u(x)\,\mathrm{E}_0^*(x, \mathrm{id}, \mathrm{id}; s+1)\,d\mu_\mathrm{C}(x).$$

Therefore we have

$$\zeta_\mathrm{C}(2s+2)\,\mathrm{Z}(s.\ \mathbf{f}^c, \mathbf{f}, \mathrm{id})=\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathrm{C})^{-1}\,|\mathrm{D}_\mathrm{F}|^{-(1/2)}\int_{\mathrm{X}_0(\mathrm{C})}\overline{\mathbf{f}^u(x)}\,\mathbf{f}^u(x)\,\mathrm{E}_0(x, \mathrm{id}, \mathrm{id}; s+1)\,d\mu_\mathrm{C}.$$

Here the Eisentein series E_0 and E_0^* have the relation:

$$\mathrm{E}_0(x, \mathrm{id}, \mathrm{id}; s)=\zeta_\mathrm{C}(2s)\,\mathrm{E}_0^*(x, \mathrm{id}, \mathrm{id}; s) \qquad [\mathrm{H1}, (4.8\ c)],$$

from which the above formula follows. Thus we have

$$\begin{aligned}&\mathrm{Res}_{s=0}\,\zeta_\mathrm{C}(2s+2)\,\mathrm{Z}(s, \mathbf{f}^c, \mathbf{f}, \mathrm{id})\\&\quad=\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(C)^{-1}\,|\mathrm{D}_\mathrm{F}|^{-(1/2)}\,2^{-1}\,\pi^d\prod_{\mathfrak{q}\mid\mathrm{C}}(1-\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathfrak{q})^{-1})\,\mathrm{Res}_{s=1}\,\zeta(s)\,(\mathbf{f}^u, \mathbf{f}^u)_\mathrm{C}.\end{aligned}$$

Moreover we know from [H1, (4.6)]

$$\mathrm{Z}(s, \mathbf{f}^c, \mathbf{f}, \mathrm{id})=|\mathrm{D}_\mathrm{F}|^{(1/2)+s}\,(4\pi)^{-ds-\{k\}}\,\Gamma_\mathrm{F}(s+k)\,\mathrm{D}(s, \mathbf{f}^c, \mathbf{f}),$$

where $\Gamma_\mathrm{F}(s+k)=\prod_{\sigma\in\mathrm{I}}\Gamma(s+k_\sigma)$ and $\{k\}=\sum_{\sigma\in\mathrm{I}}k_\sigma\in\mathbf{Z}$. Therefore we have

$$\begin{aligned}&\mathrm{Res}_{s=0}\,\zeta_\mathrm{C}(2s+2)\,\mathrm{D}(s, \mathbf{f}, \mathbf{f}^c)\\&\quad=|\mathrm{D}_\mathrm{F}|^{-1/2}\,(4\pi)^k\prod_{\sigma\in\mathrm{I}}\Gamma(k)^{-1}\,\mathrm{Res}_{s=0}\,\zeta_\mathrm{C}(2s+2)\,\mathrm{Z}(s, \mathbf{f}_c, \mathbf{f}, \mathrm{id})\\&\quad=|\mathrm{D}_\mathrm{F}|^{-1}\,\Gamma_\mathrm{F}(k)^{-1}\,\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathfrak{C})^{-1}\,2^{2\{k\}-1}\,\pi^{d+\{k\}}\\&\qquad\qquad\prod_{\mathfrak{q}\mid\mathrm{C}}(1-\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathfrak{q})^{-1})\,\{\mathrm{Res}_{s=1}\,\zeta(s)\}\,(\mathbf{f}^u, \mathbf{f}^u)_\mathrm{C},\end{aligned}$$

which is in turn equal to

$$\{\mathrm{Res}_{s=1}\zeta(s)\}\prod_{\mathfrak{q}\mid\mathrm{C}}(1-\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathfrak{q})^{-1})\,L_\mathrm{E}(1, \mathrm{Ad}(\mathbf{f})).$$

This shows that

$$(\mathbf{f}^u, \mathbf{f}^u)_\mathrm{C}=|\mathrm{D}_\mathrm{F}|\,\Gamma_\mathrm{F}(k)\,\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathrm{C})\,2^{-2\{k\}+1}\,\pi^{-d-\{k\}}\,L_\mathrm{E}(1, \mathrm{Ad}\,(\mathbf{f})).$$

8. Proof of Theorem I, Comparison of p-adic L-functions

The idea of the proof of Theorem I is the comparison of two p-adic L-functions (8.5 a, b): one is the Katz p-adic L-function given in [K4] and Theorem II and the other is the p-adic Rankin product L-function constructed in [H1, Th. 5.2]. We use the notation introduced in Theorem II and in the previous sections. We fix a conductor $\mathfrak{C}$ prime to p and a character $\psi : G_{tor}(\mathfrak{C}) \to \bar{\mathbf{Q}}^{\times}$ of conductor $\mathfrak{C}$. Here we regard ψ as a character of $G_{\infty}(\mathfrak{C})/W$ and in this sense its conductor is defined. Thus we have the associated projection $\psi_* : \mathfrak{O}[[G]] \to \Lambda_0 = \mathfrak{O}[[W]]$ for $G = G_{\infty}(\mathfrak{C})$. Let

$$\mathfrak{X} = \mathrm{Hom}_{\mathfrak{O}\text{-alg}}(\Lambda_0, \bar{\mathbf{Q}}_p) = \mathrm{Spec}(\Lambda_0)(\bar{\mathbf{Q}}_p).$$

Then for each $P \in \mathfrak{X}$, we have a continuous character $\lambda_P = P \circ \psi_* : G \to \bar{\mathbf{Q}}_p$. We say λ_P^{-1} (or P) is critical (of type Σ) if, regarding $G_{\infty}(\mathfrak{C})$ as a quotient of $M_{\mathbf{A}}^{\times}$, we have $\lambda_P(x_p) = x_p^{-\eta}$ for $\eta \in \mathbf{Z}[I_M]$ such that

(8.1) *the p-type η of λ_P^{-1} is $m_0\Sigma + d(1-c)$ for an integer m_0 and $d = \sum_{\sigma \in \Sigma} d_\sigma \sigma$ with integers d_σ satisfying either $m_0 > 0$ and $d_\sigma \geqq 0$ or $m_0 \leqq 1$ and $d_\sigma \geqq 1 - m_0$.*

Then, $m_0 = (\eta_\sigma + \eta_{\sigma c})$ for all $\sigma \in \Sigma$, and there exists a complex avatar $\lambda_{P,\infty} : M_{\mathbf{A}}^{\times} \to \mathbf{C}^{\times}$ such that $\iota_\infty^{-1}(\lambda_{P,\infty}(\mathfrak{a})) = \iota_p^{-1}(\lambda_P(\mathfrak{a}))$ for all ideals $\mathfrak{a}$ prime to $p\mathfrak{C}$. Let

$$L(s, \lambda_{P,\infty}) = \sum_{\mathfrak{a}} \lambda_{P,\infty}(\mathfrak{a}) \mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{a})^{-s}$$

be the complex L-function. Then, writing $L_p(\lambda_P)$ [resp. $L_\infty(\lambda_P)$] for $\int \lambda \, d\mu$ for the Katz measure μ in Theorem II [resp. $L(0, \lambda_{P,\infty})$], we have

$$(8.2)\quad \frac{L_p(\lambda_P^{-1})}{\Omega_p^{m_0\Sigma+2d}} = (\mathfrak{R}^{\times} : \mathfrak{r}^{\times}) W_p(\lambda_P^{-1}) \prod_{\mathfrak{L} \mid \mathfrak{C}} (1 - \lambda_P^{-1}(\mathfrak{L})) \prod_{\mathfrak{P} \in \Sigma} (1 - \lambda_{P,\infty}^{-1}{}^*(\mathfrak{P}^c))(1 - \lambda_{P,\infty}^{-1}(\mathfrak{P}^c)) \times \frac{(-1)^{m_0[F:\mathbf{Q}]} \pi^d \Gamma_F(m_0 t + d)}{|D_F|^{1/2} \mathrm{Im}(\delta)^d \Omega_\infty^{m_0\Sigma+2d}} L_\infty(\lambda_P^{-1}).$$

In [H1, Th. I, Th. 5.2], a measure related to μ has been constructed. We now want to compare these two measures. As in section 4, we assume $\mathfrak{O}$ to be a complete discrete valuation ring in Ω with residue field $\kappa = \bar{\mathbf{F}}_p$. Then we put $\mathbf{A} = \mathfrak{O}[[\mathbf{W}]]$ and $\Lambda_0 = \mathfrak{O}[[W]]$. Since W is naturally a subgroup of finite index of $\mathbf{W}$, Λ_0 is a finite flat algebra extension of $\mathbf{A}$. Let $\mathbf{h}^{n.ord}(N; \mathfrak{O})$ be the nearly ordinary Hecke algebra of (prime to p) level $N = D\mathfrak{N}_{M/F}(\mathfrak{C})$. Let $\iota : \mathbf{G} = Z(N) \times \mathfrak{r}_p^{\times} \to G = G_\infty(\mathfrak{C})$ be the natural morphism defined in section 6, which sends $(1, u)$ with $u \in \mathfrak{r}_p^{\times}$ to $u \in \mathfrak{R}_{\Sigma c}^{\times} (= \mathfrak{r}_p^{\times})$. Then the θ-measure constructed in Theorem 6.2 induces an algebra homomorphism

$$\theta_N^* : \mathbf{h}^{n.ord}(N; \mathfrak{O}) \to \mathfrak{O}[[G]]$$

satisfying (6.7 *b*). Let $\mathfrak{C}'$ be another conductor prime to p. We fix another primitive character $\varphi : G_{tor}(\mathfrak{C}') \to \mathfrak{O}^\times$. Then ψ (resp. φ) induces a projection morphism $\psi_* : \mathfrak{O}[[G_\infty(\mathfrak{C})]] \to \Lambda_0$ (resp. $\varphi_* : \mathfrak{O}[[G_\infty(\mathfrak{C}')]] \to \Lambda_0$). We put

$$\lambda = \psi_* \circ \theta_N^* : \mathbf{h}^{n.ord}(N; \mathfrak{O}) \to \Lambda_0 \qquad \text{and} \qquad \nu = \varphi_* \circ \theta_{N'}^* : \mathbf{h}^{n.ord}(N'; \mathfrak{O}) \to \Lambda_0$$

for $N' = \mathfrak{N}_{M/F}(\mathfrak{C}')D$. Then λ and ν are primitive in the sense of [H1, Th. 3.4]. Let P, $Q \in \mathfrak{A}(\Lambda_0)$, then $\lambda_P = P \circ \lambda : G_\infty(\mathfrak{C}) \to \bar{\mathbf{Q}}_p^\times$ and $\nu_Q = Q \circ \nu : G_\infty(\mathfrak{C}') \to \bar{\mathbf{Q}}_p^\times$ have their complex avatard for which we use the same symbol. Let η and ξ be the infinity type of λ_P^{-1} and ν_Q^{-1}, respectively. We define m_0 and $m'_0 \in \mathbf{Z}$ by $\eta + \eta c = m_0 t_M$ and $\xi + \xi c = m'_0 t_M$. Then we have, with the notation introduced in [H1, § 0],

$$(8.3\ a) \qquad \begin{cases} m_0 - 1 = m(P), & v(P) = \sum_{\sigma \in \Sigma} \eta_{\sigma c}\, \sigma|_F, \\ m'_0 - 1 = m(Q), & v(Q) = \sum_{\sigma \in \Sigma} \xi_{\sigma c}\, \sigma|_F. \end{cases}$$

Then $\theta(\lambda_P)$ (resp. $\theta(\nu_Q)$) is the modular form belonging to λ at P in $\mathbf{S}_{k,w}(N p^\alpha; \psi'_P, \psi_P^+; \mathfrak{O})$ [resp. ν at Q in $\mathbf{S}_{k',w'}(N' p^\alpha; \varphi'_Q, \varphi_Q^+; \mathfrak{O})$] for $N' = \mathfrak{N}_{M/F}(\mathfrak{C}')D$, $k = n(P) + 2t$ and $w = t - v(P)$ (resp. $k' = n(Q) + 2t$ and $w' = t - v(Q)$) for suitable $\alpha = (\alpha(\mathfrak{p})) \in \mathbf{Z}^\Sigma$, where

$$(8.3\ b) \qquad \begin{cases} \psi'_P(x) x^{v(P)} = \lambda_P^{-1}(\iota(1, x)), & \varphi'_Q(x) x^{v(Q)} = \nu_Q^{-1}(\iota(1, x)) \\ \psi_P^+(x) = \chi(x) \lambda_P(x) |x|_{F_\mathbf{A}}^{m_0}, & \varphi_Q^+(x) = \chi(x) \nu_Q(x) |x|_{F_\mathbf{A}}^{m'_0} \quad \text{for } x \in F_\mathbf{A}^\times. \end{cases}$$

Here χ is the quadratic character of G corresponding to M/F. Let $\pi(P) = \otimes_\mathfrak{q} \pi_\mathfrak{q}$ (resp. $\pi'(Q) = \otimes \mathfrak{q}\, \pi'_\mathfrak{q}$) be the automorphic representation generated by $\theta(\lambda_P)$ (resp. $\theta(\nu_Q)$). Then we can write

$$\pi_\mathfrak{q} = \begin{cases} \pi((\lambda_P)_\mathfrak{Q}, (\lambda_P)_{\mathfrak{Q}^c}) & \text{if } \mathfrak{q} = \mathfrak{Q}\mathfrak{Q}^c, \\ \pi(\delta_\mathfrak{q}, \delta_\mathfrak{q} \chi_\mathfrak{q}) & \text{if } \mathfrak{q} = \mathfrak{Q} \quad \text{or} \quad \mathfrak{q} = \mathfrak{Q}^2 \text{ and } (\lambda_P)_\mathfrak{q} = \delta_\mathfrak{q} \circ \mathfrak{N}_{M/F}, \end{cases}$$

$$\pi'_\mathfrak{q} = \begin{cases} \pi((\nu_Q)_\mathfrak{Q}, (\nu_Q)_{\mathfrak{Q}^c} & \text{if } \mathfrak{q} = \mathfrak{q}^c, \\ \pi(\delta'_\mathfrak{q}, \delta'_\mathfrak{q} \chi_\mathfrak{q}) & \text{if } \mathfrak{q} = \mathfrak{Q} \quad \text{or} \quad \mathfrak{q} = \mathfrak{Q}^2 \text{ and } (\nu_Q)_\mathfrak{q} = \delta'_\mathfrak{q} \circ \mathfrak{N}_{M/F}, \end{cases}$$

where $\mathfrak{Q}$ is a prime ideal of M dividing $\mathfrak{q}$, and $\delta_\mathfrak{q}$ and $\delta'_\mathfrak{q}$ are the characters of $F_\mathfrak{q}^\times$. If $\mathfrak{q}$ is inert or ramified in M/F and there are no characters $\delta_\mathfrak{q}$ (resp. $\delta'_\mathfrak{q}$) such that $(\lambda_P)_\mathfrak{q} = \delta_\mathfrak{q} \circ \mathfrak{N}_{M/F}$ (resp. $(\nu_Q)_\mathfrak{q} = \delta'_\mathfrak{q} \circ \mathfrak{N}M/F$), then $\pi_\mathfrak{q}$ (resp. $\pi'_\mathfrak{q}$) is supercuspidal.

We fix a group decomposition $\mathfrak{r}_p^\times = W^- \times \mu(\mathfrak{r}_p)$ of topological group such that $\mu(\mathfrak{r}_p)$ is the subgroup of roots of unity in $\mathfrak{r}_p$. Thus for each given character $\eta : \mu(\mathfrak{r}_p) \to \bar{\mathbf{Q}}^\times$, we can find $v \in \mathbf{Z}[I]$ such that $\eta(\zeta) = \zeta^v$ for all $\zeta \in \mu(\mathfrak{r}_p)$. Let $J = \{ w \in \mathbf{Z}[I] \mid \zeta^w = 1 \text{ for all } \zeta \in \mu(\mathfrak{r}_p) \}$. Then $\mathbf{Z}[I]/J \cong \mathrm{Hom}(\mu(\mathfrak{r}_p), \bar{\mathbf{Q}}^\times)$, which is a finite module. We apply this argument to the characters

$$\psi : \mu(\mathfrak{R}_{\Sigma_c}) = \mu(\mathfrak{r}_p) \to \bar{\mathbf{Q}}^\times \qquad \text{and} \qquad \varphi : \mu(\mathfrak{R}_{\Sigma_c}) = \mu(\mathfrak{r}_p) \to \bar{\mathbf{Q}}^\times,$$

and write $v(\psi)$ and $v(\varphi)$ for the corresponding elements in $\mathbf{Z}[I]$. Then we consider the subset $\mathfrak{Y}$ of $\mathfrak{X}\times\mathfrak{X}$ ($\mathfrak{X}=\mathrm{Spec}(\Lambda_0)(\Omega)$) consisting of (P, Q) such that

(8.4 *a*) $t\leqq n(\mathrm{P})-n(\mathrm{Q}), \quad n(\mathrm{Q})-n(\mathrm{P})+2t\leqq(m(\mathrm{P})-m(\mathrm{Q}))t \quad$ *and* $\quad v(\mathrm{Q})\geqq v(\mathrm{P})$,

(8.4 *b*) ψ'_P *and* φ'_Q *are both induced by finite order Hecke characters of* $\mathrm{F}^\times_{\mathbf{A}}/\mathrm{F}^\times$ *unramified outside* p.

Since we know from (8.3 *b*) that

$$\psi'_{\mathrm{P}}\zeta)=\psi_{\Sigma c}^{-1}(\zeta)\zeta^{-v(\mathrm{P})}=\zeta^{-v(\psi)-v(\mathrm{P})}$$

and

$$\psi'_P(\zeta)=\psi_{\Sigma c}^{-1}(\zeta)\zeta^{-v(\mathrm{P})}=\zeta^{-v(\psi)-v(\mathrm{P})},$$

the set $\mathfrak{Y}_0$ consisting of $(\mathrm{P},\mathrm{Q})\in\mathfrak{X}\times\mathfrak{X}$ such that (P, Q) satisfies (8.4 *a*), $v(\mathrm{Q})+v(\varphi)\equiv v(\mathrm{P})+v(\psi)\equiv 0 \bmod \mathrm{J}$ and $(\psi'_{\mathrm{P}})|_{\mathrm{W}_-}=(\varphi'_{\mathrm{Q}})|_{\mathrm{W}_-}=\mathrm{id}$ is a subset of $\mathfrak{Y}$. The subset $\mathfrak{Y}_0$ is obviously Zariski dense in $\mathfrak{X}\times\mathfrak{X}$, and hence $\mathfrak{Y}$ is dense in $\mathfrak{X}\times\mathfrak{X}$.

Let $(\mathrm{P},\mathrm{Q})\in\mathfrak{Y}$ and $\theta(\lambda_{\mathrm{P}})^0$ (resp. $\theta(\nu_{\mathrm{Q}})^{c0}$) be the primitive form associated with $\theta(\lambda_{\mathrm{P}})$ (resp. the complexe conjugate $\theta(v_{\mathrm{Q}})^c$ of $\theta(\nu_{\mathrm{Q}})$). Then we define $\mathfrak{D}(s,\theta(\lambda_{\mathrm{P}}),\theta(\nu_{\mathrm{Q}})^c)=\mathfrak{D}'(s,\theta(\lambda_{\mathrm{P}})^0,\theta(\nu_{\mathrm{Q}})^{c0},\mathrm{id})$ as in (7.8). Write $\mathfrak{D}_p(s,\theta(\lambda_{\mathrm{P}}),\theta(\nu_{\mathrm{Q}})^c)$ for the L-function obtained from $\mathfrak{D}(s,\theta(\lambda_{\mathrm{P}}),\theta(\nu_{\mathrm{Q}})^c)$ excluding Euler p-factors. Let Ξ_0 (resp. Ξ'_0) be the set consisting of prime ideals $\mathbf{Q}$ outside p (in M) such that one of the conductors of ψ and φ^{-1} (resp. ψ and $\varphi^{[c]}$) is divisible by $\mathbf{Q}$ but the conductor of $\psi\varphi^{-1}$ (resp. $\psi\varphi^{[c]}$) is prime to $\mathbf{Q}$, where $\varphi^{[c]}(x)=\varphi(x^c)^c=(\varphi\circ c)^{-1}(x)$ as idele character having values in $\bar{\mathbf{Q}}$. For any ideal Hecke character η, we write η^0 for the primitive character associated with η. Applying the Euler product expansion of $\mathfrak{D}(s,\theta(\lambda_{\mathrm{P}}),\theta(\nu_{\mathrm{Q}})^c)$ given in (7.7), we see easily that

(8.5 *a*) $$\mathfrak{D}_p(s,\theta(\lambda_{\mathrm{P}}),\theta(\nu_{\mathrm{Q}})^c)=\mathrm{E}'(\mathrm{P},\mathrm{Q};s)\,L_p(s,(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{cu})^0)\,L_p(s,(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{[c]u})^0),$$

where $\nu_{\mathrm{Q}}^{[c]}(x)=\nu_{\mathrm{Q}}(x^c)^c$ for $x\in\mathrm{M}^\times_{\mathbf{A}}$ with $x_p=1$. Here $\mathrm{E}'(\mathrm{P},\mathrm{Q};s)$ is a product of Euler factors of the form $(1-(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{cu})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-s})$ or $(1-(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{[c]u})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-s})$. In fact, there exists certain subset Ξ (resp. Ξ') in the set Ξ_0 (resp, Ξ'_0) such that

$$\mathrm{E}'(\mathrm{P},\mathrm{Q};s)=\prod_{\mathfrak{Q}\in\Xi}(1-(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{cu})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-s})\prod_{\mathfrak{Q}\in\Xi'}(1-(\lambda_{\mathrm{P}}^u\nu_{\mathrm{Q}}^{[c]u})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-s}).$$

The difference of $\Xi_0-\Xi$ and $\Xi'_0-\Xi'$ is contained in the set of ramified primes in M/F and depends only on ψ and φ. Then writing

$$\Psi_1(\varepsilon^*_{\mathrm{P},\mathrm{Q}})=\prod_{\mathfrak{Q}\in\Xi}(1-(\lambda_{\mathrm{P}}\nu_{\mathrm{Q}}^{-1})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-1})$$

$$\Psi_2(\varepsilon'^*_{\mathrm{P},\mathrm{Q}})=\prod_{\mathfrak{Q}\in\Xi'}(1-(\lambda_{\mathrm{P}}(\nu_{\mathrm{Q}}^0c)^{-1})^0(\mathfrak{Q})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q})^{-1})$$

for $\varepsilon^*_{P,Q}=((\lambda_P \nu_Q^{-1})\circ c)^0 \mathfrak{N}^{-1}(\varepsilon_{P,Q}(\lambda_P^{-1}\nu_Q)^0)$ and $\varepsilon'^*_{P,Q}=(\nu_Q^{-1}(\lambda_P\circ c))^0 \mathfrak{N}^{-1}(\varepsilon'_{P,Q}=(\lambda_P^{-1}(\nu_Q\circ c))^0)$, we see

$$(8.5\,b)\quad \mathfrak{D}_p\left(1+\frac{m(Q)-m(P)}{2},\ \theta(\lambda_P),\ \theta(\nu_Q)^c\right)$$
$$=\Psi_1(\varepsilon^*_{P,Q})\,\Psi_2(\varepsilon^*_{P,Q})\,L_p(0,\varepsilon^*_{P,Q})\,L_p(0,\varepsilon'^*_{P,Q}).$$

Let $C_0(\lambda)$ be the congruence module of λ defined in (6.9) and H be its characteristic power series. Then by [H1, Th. 5.1-2],

(8.6) *there exists an element* $\mathfrak{D}$ *of the quotient field of* $\Lambda_0\hat{\otimes}_0\Lambda_0$ *such that:*

(i) $$(H\otimes 1)\,\mathfrak{D}\in\Lambda_0\hat{\otimes}_0\Lambda_0$$

(ii) *Suppose that* $(P, Q)\in\mathfrak{Y}$. *Then we have*

$$\mathfrak{D}(P,Q)=C(P,Q)\,W(P,Q)\,W(P,Q)\,S(P)^{-1}\,E(P,Q)$$
$$\times\frac{\mathfrak{D}_p(1+((m(Q)-m(P))/2),\ \theta(\lambda_P),\ \theta(\nu_Q)^c)}{(\theta(\chi_P)\otimes\psi'^0_P,\ \theta(\lambda_P)\otimes\psi'^0_P)}.$$

Here $\theta(\lambda_P)\otimes\psi'^0_P$ *is the primitive form associated with* $\pi(P)\otimes\psi'_P$, *and* E(P, Q), S(P), W(P, Q) *and* C(P, Q) *are given as follows:*

$$C(P,Q)=\frac{i^{n(Q)-n(P)}\,|D_F|^{1+m(Q)-m(P)}\,\pi^{\{2v(P)-2v(Q)-n(Q)-3t\}}}{2^{\{n(Q)+v(Q)-v(P)+2t\}+\{v(Q)-v(P)+t\}+\{n(P)+t\}}}$$
$$\times\Gamma_F(n(Q)+v(Q)-v(P)+2t)\,\Gamma_F(v(Q)-v(P)+t);$$

To write down the ε-factor W(P, Q), we put for simplicity

$$G(\varepsilon)=\sum_{u \bmod C(\varepsilon_\Sigma)}\varepsilon_\Sigma(u)\,\mathbf{2}_F(u/\varpi^e d)=\prod_{\mathfrak{p}\in\Sigma}\lambda(\varpi_{\mathfrak{p}}^{e(\mathfrak{p})})\,G(d,\varepsilon_{\mathfrak{p}}),$$

writing $C(\varepsilon_\Sigma)=\mathfrak{p}^e=\prod_{\mathfrak{p}\in\Sigma}\mathfrak{p}^{e(\mathfrak{p})}$ and $\varpi^e=\prod_{\mathfrak{p}\mid p}\varpi_{\mathfrak{p}}^{e(\mathfrak{p})}$. Then $W(P,Q)=W'(P,Q)\,W_p(P,Q)$ and

$$W_p(P,Q)=\frac{(\nu_{Q,p}(d_p)\,|\,\lambda_{P,p}(d_p)\,|\,G(\nu_{Q,\Sigma^c}^{-1}\lambda_{P,\Sigma^c})\,G(\nu_{Q,\Sigma}^{-1}\lambda_{P,\Sigma^c}))}{(\lambda_{P,p}(d_p)\,|\,\nu_{Q,p}(d_p)\,|\,G(\lambda_{P,\Sigma}^{-1}\lambda_{P,\Sigma^c}))},$$

$$W'(P,Q)=(\varphi_Q^+\varphi'_Q\psi_P^+\psi'_P)_\infty(-1)\frac{\mathfrak{N}_{M/Q}(\mathfrak{C}')^{(m(Q)/2)+1}\,\mathfrak{N}_{F/Q}(D)^{1+((m(Q)-m(P))/2)}\,W'(\theta(\nu_Q))}{\mathfrak{N}_{M/Q}(\mathfrak{C})^{m(P)/2}(\theta(\lambda_P))}$$

$$E(P,Q)=E_1(P,Q)\,E_2(P,Q)$$

and

$$E_1(P,Q)=\prod_{\mathfrak{P}\in\Sigma}\frac{(1-(\varepsilon_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'_{P,Q})^0(\mathfrak{P}^c))}{(1-(\varepsilon^*_{P,Q})^0(\mathfrak{P}))(1-(\varepsilon'^*_{P,Q})^0(\mathfrak{P}))}$$

$$= \prod_{\mathfrak{P} \in \Sigma} \frac{(1-(\varepsilon_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon^*_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'_{P,Q})^0(\mathfrak{P}^c))}{(1-(\varepsilon^*_{P,Q})^0(\mathfrak{P}))(1-(\varepsilon^*_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'^*_{P,Q})^0(\mathfrak{P}))(1-(\varepsilon'^*_{P,Q})^0(\mathfrak{P}^c))},$$

$$E_2(P, Q) = \prod_{\mathfrak{P} \in \Sigma} \lambda^{-1}_{P,\Sigma^c} \nu_{Q\Sigma}(\varpi_{\mathfrak{p}}^{\gamma'}) \lambda^{-1}_{P,\Sigma^c} \nu_{Q,\Sigma^c}(\varpi_{\mathfrak{p}}^{\gamma}) = \prod_{\mathfrak{P} \in \Sigma} \frac{((\varepsilon^*_{P,Q})_{\Sigma}^{-1}(\varpi_{\mathfrak{p}}^{\gamma})(\varepsilon'^*_{P,Q})_{\Sigma}^{-1}(\varpi_{\mathfrak{p}}^{\gamma'}))}{\mathfrak{N}_{M/Q}(\mathfrak{P}^{\gamma}\mathfrak{P}^{\gamma'})};$$

$$S(P) = S_1(P) S_2(P),\ S_1(P) = \prod_{\mathfrak{P} \in S} (1-(\varepsilon_P)^0(\mathfrak{P}^c))(1-((\varepsilon^*_P)^0(\mathfrak{P}^c)),$$

$$S_2(P) = ((\varepsilon^*_P)_{\Sigma}^{-1} \mathfrak{N}^{-2})(\varpi_{\mathfrak{p}}^{\delta})$$

where

$$\varepsilon^*_{P,Q} = ((\lambda_P \nu_Q^{-1}) \circ c) \mathfrak{N}^{-1} \quad (\varepsilon_{P,Q} = \lambda_P^{-1} \nu_Q)$$

and

$$\varepsilon'^*_{P,Q} = \nu_Q^{-1} (\lambda_P \circ c) \mathfrak{N}^{-1} \quad (\varepsilon'_{P,Q} = \lambda_P^{-1} (\nu_Q \circ c)),$$

$$C(\varepsilon^*_{P,Q}) = \prod_{\mathfrak{P} \in \Sigma} (\mathfrak{P}^{\gamma} \mathfrak{P}^{c\alpha}), \qquad C(\varepsilon'^*_{P,Q}) = \prod_{\mathfrak{P} \in \Sigma} (\mathfrak{P}^{\gamma'} \mathfrak{P}^{c\alpha'}),$$

$$\varepsilon^*_P = \lambda_P^{-1} (\lambda_P^0 c) \mathfrak{N}^{-1} = (\psi_0^{-1} \mathfrak{N}_{M/F})(\lambda_P^0 c)^2 \mathfrak{N}^{-m(P)-2} \quad (\varepsilon_P = \lambda_P^{-1} (\lambda_P^0 c))$$

and

$$C(\varepsilon^*_P) = \prod_{\mathfrak{P} \in \Sigma} (\mathfrak{P}^{\delta} \mathfrak{P}^{c\delta}) \qquad \text{for} \quad \alpha, \alpha', \gamma, \gamma', \delta \in \mathbf{Z}[\Sigma].$$

By (8.6) (i), $(H \otimes 1)\mathfrak{D}$ generates a pseudo-null module in the quotient field of $\Lambda_0 \hat{\otimes}_0 \Lambda_0$ modulo $\Lambda_0 \hat{\otimes}_0 \Lambda_0$, which is in fact null. Thus

(8.7) $\Phi = (H \otimes 1)\mathfrak{D}$ *belongs to*

$$\Lambda_0 \hat{\otimes}_0 \Lambda_0 \cong \mathfrak{O}[[X_0, \ldots, X_{d+s}, Y_0, \ldots, Y_{d+s}]] \qquad \textit{and} \qquad \mathfrak{D} = \Phi/H.$$

Since the conductor of $\theta(\lambda_P) \otimes \psi_P'^0$ is $N = \mathfrak{N}_{M/F}(\mathfrak{C}) D$ times the conductor of $\otimes_{\mathfrak{p} \in \Sigma} \pi(\lambda_{\mathfrak{p}} \psi'_P, \lambda^c_{\mathfrak{p}} \psi'_P)$ and on $\mathfrak{R}_{\Sigma}^{\times}$, $\lambda_{\Sigma} \psi'_P$ coincides with $\lambda_{P,\Sigma}(\lambda^0_{P,\Sigma^c} c)^{-1} = \varepsilon_P |_{\mathfrak{R}_{\Sigma}^{\times}}$, we know that $C(\theta(\lambda_P) \otimes \psi_P'^0) = N \mathfrak{p}^{\delta}$. Let S be the set of primes $\mathfrak{q}$ dividing N such that

(8.8 *a*)
- (i) $\mathfrak{q}$ *is inert or ramified in* M/F;
- (ii) $\lambda_{\mathfrak{q}} \neq \lambda_{\mathfrak{q}} \circ c$ *when* $\mathfrak{q}$ *is inert* M/F;
- (iii) $\lambda_{\mathfrak{q}} \eta_{\mathfrak{q}} = \lambda_{\mathfrak{q}} \circ c$ *for the unique unramified quadratic character* $\eta_{\mathfrak{q}}$ *of* $M_{\mathfrak{q}}$ *if* $M_{\mathfrak{q}}/F_{\mathfrak{q}}$ *is ramified.*

Then it is well known that S is the set given by (7.2) for $\theta(\lambda_P) \otimes \psi_P'^0$. Thus we also know from Theorem 7.1 that

(8.8 *b*) $(\theta(\lambda_P) \otimes \psi_P'', \theta(\lambda_P) \otimes \psi_P'')$

$$= |D_F| \Gamma_F(n(P) + 2t) N_{F/\mathbf{Q}}(N \mathfrak{p}^{\delta}) 2^{-2\{n(P)+2t\}+1} \pi^{-\{n(P)+3t\}} \Delta(1) L(0, \varepsilon^*_P) L(1, \chi),$$

where $\Delta(1)$ is an in (0.6 *a*) and

$$L(1, \chi) = 2^{[F:\mathbf{Q}]-1} (\mathfrak{N}_{F/\mathbf{Q}}(D) |D_F|)^{-1/2} \pi^{[F:\mathbf{Q}]} (\mathfrak{R}^{\times} : \mathfrak{r}^{\times})^{-1} h(M)/h(F)$$

for the class number $h(M)$ [resp. $h(F)$] of M (resp. F).

Combining (8.5), (8.6) and (8.8), we can now express the value $\mathfrak{D}(P, Q)=\Phi(P, Q)/H(P)$ in terms of $L_\infty(\varepsilon_{P,Q})L_\infty(\varepsilon^*_{P,Q})/L_\infty(\varepsilon_P)$ up to a specific constant. We now interpret this expression in terms of power series in Λ_0 interpolating $L_p(\varepsilon_{P,Q})L_p(\varepsilon^*_{P,Q})/L_p(\varepsilon_P)$.

Let L_1, L_2, L_1^*, $L_2^*\in\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]=\Lambda_0\hat{\otimes}_0\Lambda_0$ and L^-, $L^{-*}\in\mathfrak{O}[[\mathbf{W}]]=\Lambda_0$ be such that $L_1(P, Q)=L_p(\varepsilon_{P,Q})$, $L_1^*(P, Q)=L_p(\varepsilon^*_{P,Q})$, $L_2(P, Q)=L_p(\varepsilon'_{P,Q})$, $L_2^*(P, Q)=L_p(\varepsilon'^*_{P,Q})$ and $L^-(P)=L_p(\varepsilon_P)$, $L^{-*}(P)=L_p(\varepsilon_P^*)$ for all arithmetic points $(P, Q)\in\mathfrak{Y}$. By the result of Katz [K4], Theorem II and (8.2), to show the uniqueness of these L^-, L_1, L_2, L^{-*}, L_1^* and L_2^*, it is sufficient to show that the points $(P, Q)\in\mathfrak{Y}$ are dense in $\mathrm{Spec}(\Lambda_0\hat{\otimes}_0\Lambda_0)$, which is already seen. We will show the existence of these elements in $\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$ later. By the functional equation obtained in [K4, (5.3.7)] and Theorem 5.2 in the text, L^- (resp. L_1 and L_2) is a unit multiple of L^{-*} (resp. L_1^* and L_2^*). We now determine the infinity type of $\varepsilon^*_{P,Q}$, $\varepsilon'^*_{P,Q}$ and ε^*_P. Write $\infty(\varepsilon)$ for the infinity type of each Hecke character ε. We write

$$\infty(\varepsilon_P^*)=-m\Sigma-\sum_{\sigma\in\Sigma}d_\sigma(\sigma-\sigma c),\qquad \infty(\varepsilon'^*_{P,Q})=-m'\Sigma-\sum_{\sigma\in\Sigma}d'_\sigma(\sigma-\sigma c),$$

and

$$\infty(\varepsilon_P^*)=-m''\Sigma-\sum_{\sigma\in\Sigma}d''_\sigma(\sigma-\sigma c).$$

We also write $\infty(\lambda_P)=\eta$, $\infty(\nu_Q)=\xi$, $m_0 t_M=\eta+\eta c$ and $m'_0 t_M=\xi+\xi c$. Then

$$(8.9\,a)\qquad \begin{cases} m(P)=m_0-1, \quad m(Q)=m'_0-1, \quad n(P)=\sum_{\sigma\in\Sigma}(\eta_\sigma-\eta_{\sigma\rho}-1)\sigma|_F\\ n(Q)=\sum_{\sigma\in\Sigma}(\xi_\sigma-\xi_{\sigma\rho}-1)\sigma|_F, \quad v(P)=\sum_{\sigma\in\Sigma}\eta_{\sigma\rho}\sigma|_F, \quad v(Q)=\sum_{\sigma\in\Sigma}\xi_{\sigma\rho}\sigma|_F. \end{cases}$$

From this, we know

$$(8.9\,b)\qquad \begin{cases} d|_F=\sum_{\sigma\in\Sigma}(\eta_\sigma-\xi_\sigma-1)\sigma|_F=n(P)+v(P)-n(Q)-v(Q)-t,\\ m=2-m_0+m'_0=m(Q)-m(P)+2,\\ (d+m\Sigma)|_F=\sum_{\sigma\in\Sigma}(\xi_{\sigma\rho}-\eta_{\sigma\rho}+1)\sigma|_F=v(Q)-v(P)+t\\ (2d+m\Sigma)|_F=\sum_{\sigma\in\Sigma}(\eta_\sigma-\eta_{\sigma\rho}-1)\sigma|_F-\sum_{\sigma\in\Sigma}(\xi_\sigma-\xi_{\sigma\rho}-1)\sigma|_F=n(P)-n(Q) \end{cases}$$

$$(8.9\,c)\qquad \begin{cases} d'|_F=\sum_{\sigma\in\Sigma}(\eta_\varepsilon-\xi_{\sigma\rho}-1)\sigma|_F=n(P)+v(P)-v(Q),\\ m'=2-m_0+m'_0=m(Q)-m(P)+2,\\ (d'+m'\Sigma)|_F=\sum_{\sigma\in\Sigma}(\xi_\sigma-\eta_{\sigma\rho}+1)\sigma|_F=n(Q)+v(Q)-v(P)+2t\\ (2d'+m'\Sigma)|_F=\sum_{\sigma\in\Sigma}(\eta_\sigma-\xi_{\sigma\rho}-1)\sigma|_F+\sum_{\sigma\in\Sigma}(\xi_\sigma-\eta_{\sigma\rho}+1)\sigma|_F=n(P)+n(Q)+2t \end{cases}$$

(8.9 d)
$$\begin{cases} d''|_F=\sum_{\sigma\in\Sigma}(\eta_\sigma-\eta_{\sigma\rho}-1)\sigma|_F=n(P), \qquad m''=2, \\ (d''+m''\Sigma)|_F=n(P)+2t, \qquad (2d''+m''\Sigma)|_F=2n(P)+2t. \end{cases}$$

We may consider $\Psi=\Psi_1\Psi_2$ in (8.5 b) as an element in $\Lambda_0\hat{\otimes}_0\Lambda_0$ because λ and ν have values in Λ_0. We put

$$U=\frac{\mathfrak{D}L^{-*}}{\Psi L_1^* L_2^*}=\frac{\Phi L^{-*}}{\Psi L_1^* L_2^* H}.$$

We compute U(P, Q) and show that U is a unit in $\Lambda_0\hat{\otimes}_0\Lambda_0\otimes_{\mathbf{Z}}\mathbf{Q}$. For that purpose, we shall compare the values of Φ/H and $L_1^* L_2^*/L^{-*}$ at each $(P, Q)\in\mathfrak{Y}$ factor by factor. We start from the extra Euler factors:

Extra Euler factors. — The extra Euler factors coming from Φ/H are:

$$\prod_{\mathfrak{p}\in\Sigma}\frac{(1-(\varepsilon_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon_P^*)^0(\mathfrak{P}^c))(1-(\varepsilon'_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'^*_{P,Q})^0(\mathfrak{P}^c))}{(1-(\varepsilon_P)^0(\mathfrak{P}^c))(1-(\varepsilon_P^*)^0(\mathfrak{P}^c))}$$

The extra Euler factors coming from $L_1^* L_2^*/L^{-*}$ are:

$$\prod_{\mathfrak{P}\in\Sigma}\frac{(1-(\varepsilon_P^*)^0(\mathfrak{P}^c))(1-(\varepsilon_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'^*_{P,Q})^0(\mathfrak{P}^c))(1-(\varepsilon'_{P,Q})^0(\mathfrak{P}^c))}{(1-(\varepsilon_P)^0(\mathfrak{P}^c))(1-(\varepsilon_P^*)^0(\mathfrak{P}^c))}.$$

Thus they will be cancelled out in U.

ε-factors at p: W_p. — The Gauss sum factor at p coming from Φ/H is:

(8.10 a)
$$\begin{aligned}&\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{P}^\delta)^{-1}E_2(P,Q)S_2(P)^{-1}\\ &\qquad\times G(\nu_{Q,\Sigma^c}^{-1}\lambda_{P,\Sigma^c})G(\nu_{Q,\Sigma}^{-1}\lambda_{P,\Sigma^c})/G(\lambda_{P,\Sigma}^{-1}\lambda_{P,\Sigma^c})\\ &=\frac{((\varepsilon^*_{P,Q})_\Sigma^{-1}(\varpi_p^\gamma)(\varepsilon'^*_{P,Q})_\Sigma^{-1}(\varpi_p^{\gamma'})G((\varepsilon'^*_{P,Q})_\Sigma)G((\varepsilon^*_{P,Q})_\Sigma))\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{P}^\delta)}{\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{P}^\gamma\mathfrak{P}^{\gamma'})((\varepsilon_P^*)_\Sigma^{-1}(\varpi_p^\delta))G((\varepsilon_P^*)_\Sigma)}\\ &=\frac{W_p(\varepsilon'^*_{P,Q})W_p(\varepsilon^*_{P,Q})}{W_p(\varepsilon_P^*)},\end{aligned}$$

where the first $\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{P}^\delta)^{-1}$ comes from the same factor in (8.8 b) and $W_p(\varepsilon)$ is as in (0.10). Thus we have

(8.10 b)
$$\begin{aligned}&\mathfrak{N}_{M/\mathbf{Q}}(\mathfrak{P}^\delta)^{-1}E_1(P,Q)S_2(P)^{-1}W_p(P,Q)\\ &\qquad=\frac{(\nu_{Q,p}(d_p)|\lambda_{P,p}(d_p)|)W_p(\varepsilon'^*_{P,Q})W_p(\varepsilon^*_{P,Q})}{(\lambda_{P,p}(d_p)|\nu_{Q,p}(d_p)|)W_p(\varepsilon_P^*)}.\end{aligned}$$

We define an idele C by $2\delta=d_M C$. Then

$$\begin{aligned}\lambda_P(d_M)|\lambda_P(d_M)|^{-1}&=\lambda_P(C)^{-1}|\lambda_P(C)|=\lambda_P(C_f)^{-1}|\lambda_P(C_f)|\,i^{\{n(P)+t\}}\\ &=\psi_*(C_f)^{-1}(P)(-1)^{v(P)}\mathfrak{N}_{F/\mathbf{Q}}(c)^{m(P)+1}\,i^{\{n(P)+t\}},\end{aligned}$$

because the value of the complex character $\lambda_P(C_p)$ is equal to

$$\hat{\lambda}_P(C_p)\, C_p^{-v(P)} = \hat{\lambda}_P(C_p)(-1)^{v(P)}$$

for its p-adic avatar $\hat{\lambda}_P$. Here we regard $\psi_*(C_f)\in\Lambda_0$ as a function on $\mathfrak{X}$ by $\psi_*(C_f)(P)=\psi_*(C_f) \bmod P$. Thus the quotient of the ε-factors of $\Psi L_1^* L_2^*/L^{-*}$ and Φ/H is given by

$$(8.10\,c)\quad \frac{(\nu_{Q,p}(d_p)\,|\,\lambda_{P,p}(d_p)\,|)}{(\lambda_{P,p}(d_p)\,|\,\nu_{Q,p}(d_p)\,|)} = \frac{(\lambda_P(d_M^{(p)})\,|\,\nu_Q(d_M^{(p)})\,|)}{(\nu_Q(d_M^{(p)})\,|\,\lambda_P(d_M^{(p)})\,|)}\,\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{m(Q)-m(P)}\,\varphi_*(C_f)(Q)^{-1}\,\psi_*(C_f)(P)\,i^{\{m(Q)-m(P)\}}.$$

The factor $\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{c})^{m(Q)-m(P)}$ will be cancelled out modulo units in Λ_0 by the same factor in (8.12 *b*) below.

Complex and p-adic periods. — Since the period factor coming from $L_p(\varepsilon)$ with $\infty(\varepsilon)=m_0\Sigma+d(1-c)$ is given by $(\Omega_\infty^{m_0\Sigma+2d}/\Omega_p^{m_0\Sigma+2d})$, we see that the factor corresponding to $\Psi L_1^* L_2^*(P, Q)/L^{-*}(P)$ is

$$(8.11)\qquad \left(\frac{\Omega_\infty^{m\Sigma+2d}}{\Omega_p^{m\Sigma+2d}}\right)\left(\frac{\Omega_\infty^{m'\Sigma+2d'}}{\Omega_p^{m'\Sigma+2d'}}\right)\left(\frac{\Omega_p^{m''\Sigma+2d''}}{\Omega_\infty^{m''\Sigma+2d''}}\right).$$

By (8.9 *b*, *c*, *d*), we get

$$m\Sigma+2d+m'\Sigma+2d'-m''\Sigma-2d''=n(P)-n(Q)+n(P)+n(Q)+2t-2n(P)-2t=0.$$

Thus (8.11) is reduced to 1. This is consistent with the fact that $\mathfrak{D}=\Phi/H$ does not contain any period factor.

2-power factor. — The 2-power appearing in Φ-part is

$$2^{-\{n(Q)+v(Q)-v(P)+2t\}-\{v(Q)-v(P)+t\}-\{n(P)+t\}+2\{n(P)+2t\}-1}=2^{(m(P)-m(Q))[F:\mathbf{Q}]-1}$$

On the other hand, we see from (8.9 *b*, *c*, *d*) that

$$(8.12\,a)\qquad d+d'-d''=(m(P)-m(Q))\,t-t$$

and we see also from $\{u^c v-uv^c/2\delta\,|\,u, v\in\mathfrak{R}\}=\vartheta^{-1}\mathfrak{c}$ with the notation of section 4 that $\delta\mathfrak{R}=2^{-1}\vartheta_{M/F}\vartheta\mathfrak{c}$, where $\vartheta_{M/F}$ is the relative different of M/F. Thus Im(δ)-part in the evaluation formula in Theorem II gives

$$(8.12\,b)\quad \operatorname{Im}(\delta)^{-d-d'+d''}=\operatorname{Im}(\delta)^{-(m(P)-m(Q))t+1} = 2^{(m(P)-m(Q)-1)[F:\mathbf{Q}]}\,\mathfrak{N}_{F/\mathbf{Q}}(D\,\mathfrak{c}^2)^{-(m(P)-m(Q)-1)/2}\,|\,D_F\,|^{m(Q)-m(P)+1},$$

Thus, from the factor $2^{[F:\mathbf{Q}]-1}$ coming from (8.8 *b*), we see
(8.12 *c*) *the 2-part of* H/Φ *and* $L_1^* L_2^*/L^{-*}$ *are the same.*

π-factor. — On the side of Φ/H, we have

$$\pi^{\{2v(\mathrm{P})-2v(\mathrm{Q})-n(\mathrm{Q})-3t\}+\{n(\mathrm{P})+2t\}}=\pi^{(m(\mathrm{P})-m(\mathrm{Q})-1)[\mathrm{F}:\mathbf{Q}]}$$

On the other hand, on the side of $\Psi L_1^* L_2^*/L^{-*}$, we have

$$\pi^{\{d+d'-d''\}}=\pi^{(m(\mathrm{P})-m(\mathrm{Q})-1)[\mathrm{F}:\mathbf{Q}]} \quad \text{by } (8.12\ a).$$

Γ-factor. — On the side of Φ/H, we have

$$\frac{\Gamma_{\mathrm{F}}(n(\mathrm{Q})+v(\mathrm{Q})-v(\mathrm{P})+2t)\,\Gamma_{\mathrm{P}}(v(\mathrm{Q})-v(\mathrm{P})+t)}{\Gamma_{\mathrm{F}}(n(\mathrm{P})+2t)}$$

On the side of $\Psi L_1^* L_2^*/L^{-*}$, we have by (8.9 *b*, *c*, *d*)

$$\frac{\Gamma_{\mathrm{F}}(mt+d)\,\Gamma_{\mathrm{F}}(m't+d')}{\Gamma_{\mathrm{F}}(m''t+d'')}=\frac{\Gamma_{\mathrm{F}}(n(\mathrm{Q})+v(\mathrm{Q})-v(\mathrm{P})+2t)\,\Gamma_{\mathrm{F}}(v(\mathrm{Q})-v(\mathrm{P})+t)}{\Gamma_{\mathrm{F}}(n(\mathrm{P})+2t)}.$$

D_{F}-power. — From $\mathrm{C}(\mathrm{P},\mathrm{Q})$ and (8.8), we have $|\mathrm{D}_{\mathrm{F}}|^{(1/2)+(m(\mathrm{Q})-m(\mathrm{P}))}$ on the Φ/H-side and by (8.12 *b*) and Theorem II, we have $|\mathrm{D}_{\mathrm{F}}|^{(1/2)+(m(\mathrm{Q})-m(\mathrm{P}))}$ on the $\Psi L_1^* L_2^*/L^{-*}$-side. Thus

(8.13) *the* D_{F}*-part of* Φ/H *and* $\Psi L_1^* L_2^*/L^{-*}$ *are equal.*

Unit index factor. — We have $(\mathfrak{R}^\times : \mathfrak{r}^\times)^{-1}$ in $(\theta(\lambda_{\mathrm{P}})\otimes\psi_{\mathrm{P}}'', \theta(\lambda_{\mathrm{P}})\otimes\psi_{\mathrm{P}}')$ by (8.8), which will be cancelled out by the same factor on the side of $\Psi L_1^* L_2^*/L^{-*}$.

W′-part. — We first compute $\mathrm{W}'(\theta(\lambda_{\mathrm{P}}))$. As seen in [H1, (4.10 *a*)], $\mathrm{W}(\theta(\lambda_{\mathrm{P}}))\,i^{\{-n(\mathrm{P})-2t\}}$ gives the root number of the functional equation of $L(s,\lambda_{\mathrm{P}})$. With the notation of (5.7 *a*, *c*) applied to $\lambda=\lambda_{\mathrm{P}}$, we see

$$\mathrm{W}'(\theta(\lambda_{\mathrm{P}}))\,i^{\{-n(\mathrm{P})-2t\}}\prod_{\mathfrak{P}\mid p}\lambda_{\mathrm{P}}^u(\varpi_{\mathrm{P}}^{e(\mathfrak{P})}d_{\mathrm{M}_{\mathfrak{P}}})\,\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{P}^{e(\mathfrak{P})})^{-1/2}\,\mathrm{G}(\lambda_{\mathrm{P},\mathfrak{P}}^{-1})$$

$$=\mathrm{W}(\theta(\lambda_{\mathrm{P}}))\,i^{\{-n(\mathrm{P})-2t\}}=\kappa\lambda_{\mathrm{P}}^u(b)\,\lambda_{\mathrm{P},\infty}(-1)$$

$$=\lambda_{\mathrm{P},\infty}(-1)\,i^{\{m_0\Sigma+2d\}}\prod_{\mathfrak{Q}\mid \mathrm{C}\vartheta_{\mathrm{M}}}\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q}^{e(\mathfrak{Q})})^{-1/2}\,\lambda_{\mathrm{P}}^u(\varpi_{\mathfrak{Q}}^{e(\mathrm{Q})}d_{\mathrm{M}_{\mathfrak{Q}}})\,\mathrm{G}(\lambda_{\mathfrak{Q}}^{-1})$$

Since $\{m_0\Sigma+2d\}=\{n(\mathrm{P})+t\}$ by (6.1), we see

$$\mathrm{W}'(\theta(\lambda_{\mathrm{P}}))=\lambda_{\mathrm{P},\infty}(-1)\,i^{\{2n(\mathrm{P})+3t\}}$$

$$\times\prod_{\mathfrak{Q}\mid \mathrm{C}\vartheta_{\mathrm{M}},\ \mathfrak{Q}+p\mathfrak{R}=\mathfrak{R}}\mathfrak{N}_{\mathrm{M}/\mathbf{Q}}(\mathfrak{Q}^{e(\mathfrak{Q})})^{-1/2}\,\lambda_{\mathrm{P}}^u(\varpi_{\mathfrak{Q}}^{e(\mathfrak{Q})}d_{\mathrm{M}_{\mathfrak{Q}}})\,\mathrm{G}(\lambda_{\mathfrak{Q}}^{-1})$$

$$=\lambda_{\mathrm{P},\infty}(-1)\,\lambda_{\mathrm{P}}(d_{\mathrm{M}}^{(p)})\,|\lambda_{\mathrm{P}}(d_{\mathrm{M}}^{(p)})|^{-1}\,i^{\{2n(\mathrm{P})+3t\}}\,\mathfrak{N}_{\mathrm{F}/\mathbf{Q}}(\mathfrak{C})^{-(m(\mathrm{P})/2)-1}\,\mathrm{W}'(\lambda_{\mathrm{P}}),$$

where d_{M} is a differential idele of $\mathrm{M}/\mathbf{Q}$ for which we assumed as in section 4 that $d_{\mathrm{M}_{\mathfrak{Q}}}=(2\delta)^c$ for $\mathfrak{Q}\mid d_{\mathrm{M}}p\mathfrak{C}$. We see from the functional equation (Theorem 5.2) that there exists $\mathrm{W}'(\lambda)$ in the quotient field of Λ_0 such that $\mathrm{W}'(\lambda)(\mathrm{P})=\mathrm{W}'(\lambda_{\mathrm{P}})$ for all $\mathrm{P}\in\mathfrak{X}$. We know that for all critical P, $\mathrm{W}'(\lambda_{\mathrm{P}})$ is a p-adic unit. This shows that $\mathrm{W}'(\lambda)\in\Lambda_0^\times$.

Similarly we can define $W'(\nu)\in\Lambda_0^\times$. Thus we see

$$(8.14\,a)\quad W'(P, Q)=(\psi_P^+\psi'_P\varphi_Q^+\varphi'_Q\lambda_P\nu_Q)_\infty (-1)\, i^{\{2n(Q)-2n(P)\}}\mathfrak{N}_{F/Q}(D)^{(m(Q)-m(P))/2+1} \times\frac{\nu_Q(d_M^{(p)})\,|\,\lambda_P(d_M^{(p)})\,|\,\mathfrak{N}_{M/Q}(\mathfrak{C})\,W'(\nu)(Q)}{\lambda_P(d_M^{(p)})\,|\,\nu_Q(d_M^{(p)})\,|\,W'(\lambda)(P)}.$$

Since $\lambda_{P,\mathfrak{C}}(-1)$ [resp. $\nu_{Q,\mathfrak{C}'}(-1)$] is independent of P (resp. Q) and

$$\lambda_{P,p\mathfrak{C}}(-1)=(-1)^{n(P)+t}\qquad\text{and}\qquad \nu_{Q,p\mathfrak{C}'}(-1)=(-1)^{n(Q)+t},$$

we see easily from definition that

$$(8.14\,b)\quad (\psi_P^+\psi'_P\varphi_Q^+\varphi'_Q\lambda_P\nu_Q)_\infty (-1)\, i^{\{2n(Q)-2n(P)\}}(-1)^{v(P)+v(Q)+n(P)+n(Q)}\in\{\pm 1\}$$

is a constant independent of P *and* Q.

Thus we see from (8.10 *c*) that

$$(8.14\,c)\quad \mathfrak{N}_{M/Q}(\mathfrak{P}^\delta)^{-1}E_2(P, Q)\,S_2(P)^{-1}W(P, Q)$$
$$=\mathfrak{N}_{M/Q}(\mathfrak{P}^\delta)^{-1}E_2(P, Q)\,S_2(P)^{-1}W_p(P, Q)\,W'(P, Q)$$
$$=\frac{W_p(\varepsilon'^*_{P,Q})\,W_p(\varepsilon^*_{P,Q})}{W_p(\varepsilon^*_P)}\mathfrak{N}_{F/Q}(D)^{(m(Q)-m(P))/2+1}\,\mathfrak{N}_{F/Q}(c)^{m(Q)-m(P)} \times\frac{\mathfrak{N}_{M/Q}(\mathfrak{C})\,W'(\nu)(Q)}{W'(\lambda)(P)}\varphi_*(C_f)(Q)^{-1}\,\psi^*(C_f)(P)\, i^{\{n(Q)-n(P)\}}.$$

Note that $\mathfrak{N}(d)\,\mathfrak{N}_{M/Q}(\mathfrak{C})\,W'(\nu)\,W'(\lambda)^{-1}\varphi_*(C_f)^{-1}\psi_*(C_f)$ is a unit in Λ_0. The factor $i^{\{n(Q)-n(P)\}}$ is cancelled by the same factor in C(P, Q) by (8.14 *b*). The other factor $\mathfrak{N}_{F/Q}(D)^{(m(Q)-m(P))/2+1}\,\mathfrak{N}_{F/Q}(\mathfrak{C})^{m(Q)-m(P)}$ is cancelled out by (8.12 *b*).

Thus we have finished comparing all major factors of Φ/H and $\Psi L_1^* L_2^*/L^{-*}$, and by (8.8) the remaining difference is

$$(8.15)\qquad \Delta(M/F;\mathfrak{C})=\Delta(1)\,h(M)/h(F).$$

Since L^{-*}, L_1^* and L_2^* are unit multiple of L^-, L_1 and L_2 by the functional equation in [K4, (5.3.7)] in the text, we have

THEOREM 8.1. — *We have*

$$\frac{\Phi}{H}=\frac{U_1\Psi L_1^* L_2^*}{\Delta(M/F;\mathfrak{C})\,L^{-*}}=\frac{U_2\Psi L_1 L_2}{\Delta(M/F;\mathfrak{C})\,L^-}$$

for units U_1 *and* U_2 *in* $\Lambda_0\hat{\otimes}_0\Lambda_0$.

We now explain how to construct L_1. Complex conjugation c induces an automorphism of $G_\infty(1)$ which hence preserves $G_{tor}(1)$ and induces an automorphism of

$\mathbf{W}=G_\infty(1)/G_{tor}(1)$. Let $\mathfrak{m}$ be the prime-to-p-part of the conductor of $\psi^{-1}\varphi$ and consider the character $\varepsilon_1 : G_\infty(\mathfrak{m}) \to \mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$ defined by

$$\varepsilon_1(\zeta, w)=\psi^{-1}\varphi(\zeta)(w^{-1}, w)\in\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]] \qquad \text{for} \quad (\zeta, w)\in G_{tor}(\mathfrak{m})\times\mathbf{W},$$

which induces an algebra homomorphism $\varepsilon_1 : \mathfrak{O}[[G_\infty(\mathfrak{m})]]\to\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$. Let L_1 be the image of the Katz measure μ in $\mathfrak{O}[[G_\infty(\mathfrak{m})]]$ under ε_1. Then by definition, we see

$$L_1(P, Q)=\int_{G_\infty(\mu)}\varepsilon_{P, Q}\, d\mu=L_p(\varepsilon_{P, Q}).$$

Similarly, we can construct L_2, L_1^* and L_2^*. For example, to construct L_2, we use the morphism

$$\varepsilon_2 : \quad G_\infty(C((\psi^0 c)^{-1}\varphi))\to\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$$

given by $\varepsilon_2(\zeta, w)=(\psi^0 c)^{-1}\varphi)(\zeta)(w^{-c}, w)$, and then we see $L_2=\varepsilon_2(\mu)$ for μ on $G_\infty(C((\psi^0 c)^{-1}\varphi))$. To construct L^-, we consider the morphism for $\mathfrak{C}^-=C(\psi^-)$:

$$\varepsilon:\quad G_\infty(\mathfrak{C}^-)\to\mathfrak{O}[[\mathbf{W}]] \quad \text{given by } \varepsilon(\zeta, w)=\psi^{-1}(\psi^0 c)(\zeta)\, w^{-1} w^c.$$

Then $L^-=\varepsilon(\mu)$ for μ on $G_\infty(C(\psi^{-1}(\psi^0 c))$. Similarly, we can define L^{-*}.

We now finish the proof of Theorem I. For a given character $P : \mathbf{W}\to\mathfrak{O}^\times$, the map $\iota_P : \mathfrak{O}[[\mathbf{W}]]\to\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]\to\mathfrak{O}[[\mathbf{W}]]$ given by $\iota_P(h)=P\otimes\mathrm{id}(1\otimes h)$ is an automorphism of $\mathfrak{O}[[\mathbf{W}]]$, where $P\otimes\mathrm{id}$ is an algebra homomorphism induced by the character:

$$\mathbf{W}\times\mathbf{W}\to\mathfrak{O}[[\mathbf{W}]] \text{ which takes } (w, w') \text{ to } P(w)\, w'.$$

Let L be the image of μ under $((\psi\circ c)^{-1}\varphi)_*$. Then we see easily that $L_{2, P}=P\otimes\mathrm{id}(L_2)=\iota_P(L)$. Thus the μ-invariant of L_2 and the specialization $L_{2, P}$ are equal to that of L and are independant of P. Here the μ-invariant of $X\in\mathfrak{O}[[\mathbf{W}]]$ or $\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$ is by definition the maximal exponent of the prime element $\varpi\in\mathfrak{O}$ which divides X. Similarly, $L_{1, P}=P\otimes\mathrm{id}(L_1)$ has μ-invariant equal to that of $(\psi^{-1}\varphi)_*(\mu)$. It is easy to see that the Euler factor Ψ has trivial μ-invariant in $\mathfrak{O}[[\mathbf{W}\times\mathbf{W}]]$. Suppose that one of the prime factors $\mathbf{P}$ of L^- in $\mathfrak{O}[[\mathbf{W}]]$ outside p divides L_i (for one of $i=1, 2$). Then $|\mathbf{P}(P)|_p\to 0$ as P approaches a zero of $\mathbf{P}$. Write $G_i=W\,\mathbf{P}^m$ so that X is prime to $\mathbf{P}$. Then we can let P approach to a zero of $\mathbf{P}$ without letting $X(P, Q)$ be identically zero as a function of Q. Since $Q\mapsto L_i(P, Q)$ is nothing but $L_{i, P}$, writing $L_{i, P}=\mathbf{P}(P)^m X_P$ for $X_P(Q)=X(P, Q)$, the μ-invariant of $L_{i, P}$ becomes arbitrary large if P approach to a zero of $\mathbf{P}$ outside the zero of X. This is a contradiction because the μ-invariant of $L_{i, P}$ is independent of P. Thus inside $\mathfrak{O}[[\mathbf{W}(M)\times\mathbf{W}(M)]]\otimes_{\mathbf{Z}}\mathbf{Q}$, L is prime to $L_1 L_2$. Since each factor of Ψ is of the form $1\text{-}\mathfrak{N}^{-1}(Q)\psi_*\varphi_*^{-1}(Q)$ or $1\text{-}\mathfrak{N}^{-1}(Q)(\psi_*(\varphi_*^{-1}\circ c))(Q)$, the same argument shows that L^- is also prime to Ψ. Namely we have proven Theorem I:

THEOREM 8.2. — *L^- divides* H *in* $\mathfrak{O}[[\mathbf{W}]]\otimes_{\mathbf{Z}}\mathbf{Q}=\Lambda_0\otimes_{\mathbf{Z}}\mathbf{Q}$. *Moreover if one can choose the character* φ *so that the* μ*-invariant of* $L_1 L_2$ *is equal to* 0, *then* $\Delta(M/F; \mathfrak{C})\, L^-$ *divides* H *in* Λ_0 *for* $\Delta(M/F; \mathfrak{C})$ *in* (8.15).

REFERENCES

[D1] P. DELIGNE, *Equations différentielles à points singuliers réguliers* (*Lec. Notes in Math.*, No. 163, Springer, 1970).

[DH] K. DOI and H. HIDA, *On a Certain Congruence of Cusp Forms and the Special Values of their Dirichlet Series*, unpublished, 1979.

[DR] P. DELIGNE and K. A. RIBET, *Values of Abelian L-functions at Negative Integers Over Totally Real Field* (*Invent. Math.*, Vol. 59, 1980, pp. 227-286).

[G] P. B. GARRETT, *Holomorphic Hilbert Modular Forms*, Wadsworth & Brooks/Cole, 1990.

[GJ] S. GELBART and H. JACQUET, *A Relation Between Automorphic Representations of* GL(2) *and* GL(3) (*Ann. Scient. Éc. Norm. Sup.*, 4[e] série, T. 11, 1978, pp. 471-542).

[H1] H. HIDA, *On* p*-adic* L*-functions of* GL(2) × GL(2) *Over Totally Real Fields* (*Ann. Institut Fourier*, Vol. 41, 1991, pp. 311-391).

[H2] H. HIDA, *On* p*-adic Hecke Algebras for* GL_2 *Over Totally Real Fields* (*Ann. of Math.*, Vol. 128, 1988, pp. 295-384).

[H3] H. HIDA, *On Nearly Ordinary Hecke Algebras for* GL(2) *Over Totally Real Fields* (*Advanced Study in Pure Math.*, Vol. 17, 1989, pp. 139-169).

[H4] H. HIDA, *Congruences of Cusp Forms and Special Values of their Zeta Functions* (*Inventiones Math.*, Vol. 63, 1981, pp. 225-261).

[H5] H. HIDA, *p-adic L-Functions for Base Change Lifts of* GL_2 *to* GL_3, *to appear in Proc. Ann Arbor Conference on Automorphic forms, Shimura Varieties, and L-functions*, 1988, vol. II, pp. 93-142, *Perspectives in Math.*, 11, 1990.

[H6] H. HIDA, *Modules of Congruence of Hecke Algebras and L-functions Associated with Cusp Forms* (*Amer. J. Math.*, Vol. 110, 1988, pp. 323-382).

[HT1] H. HIDA and J. TILOUINE, *Katz p-adic L-functions, Congruence Modules and Deformation of Galois Representations* (*Proceedings of Durham Symposium on Arithmetic of L-functions*, 1989).

[HT2] H. HIDA and J. TILOUINE, *On the Anti-cyclotomic Main Conjecture for* CM *Fields*, preprint.

[K1] N. M. KATZ, *Travaux de Dwork* (*Séminaire Bourbaki*, 409, 1972).

[K2] N. M. KATZ, *Nilpotent Connections and the Monodromy Theorem. Applications of a Result of Turrittin* (*Publ. Math. I.H.E.S.*, Vol. 39, 1970, pp. 175-232).

[K3] N. M. KATZ, *Serre-Tate Local Moduli*, in *Surfaces algébriques* (*Lecture Notes in Math.*, No. 868, Springer, 1978, pp. 138-202.

[K4] N. M. KATZ, p*-adic L-functions for* CM *Fields* (*Invent. Math.*, Vol. 49, 1878, pp. 199-297).

[KO] N. M. KATZ, and T. ODA, *On the Differentiation of the De Rham Cohomology Classes with Respect to Parameters* (*J. Math. Kyoto Univ.*, Vol. 8, 1968, pp. 199-213).

[Kn] D. KNUTSON, *Algebraic Spaces* (*Springer Lecture Notes in Math.*, No. 203).

[M] D. MUMFORD, *An Analytic Construction of Degenerating Abelian Varieties Over Complete Rings* (*Comp. Math.*, Vol. 24, 1972, pp. 239-272).

[MT] B. MAZUR and J. TILOUINE, *Représentations galoisiennes, différentielles de Kähler et conjectures principales* (*Publ. I.H.E.S.*, No. 71, 1990, pp. 65-103).

[R] M. RAPOPORT, *Compactifications de l'espace de modules de Hilbert-Blumenthal* (*Composition Math.*, Vol. 36, 1978, pp. 255-335).

[Sch] C.-G.SCHMIDT, P*-adic Measures Attached to Automorphic Representations of* GL(3) [*Inventiones Math.*, Vol. 92, 1988, pp. 597-631].

[Sh] G. SHIMURA, *On Some Arithmetic Properties of Modular Forms of One and Several Variables* (*Ann. of Math.*, Vol. 102, 1975, pp. 491-515).

[T] J. TILOUINE, *Sur la conjecture principale anticyclotomique* (*Duke Math. J.*, Vol. 59, 1989, pp. 629-673).

[W] A. WEIL, *Basic Number Theory*, Springer, 1974.

[Y] H. YOSHIDA, *On the Representation of the Galois Groups Obtained from Hilbert Modular Forms* (*Thesis*, Princeton University, 1973).

(Manuscrit reçu le 22 mars 1991,
révisé le 10 avril 1992).

H. HIDA,
Dept. of Math.,
U.C.L.A.,
Los Angeles, Ca 90024,
U.S.A.

J. TILOUINE,
Université de Paris-Sud,
Mathématique, Bât. 425,
Centre d'Orsay,
91405 Orsay Cedex,
France.