Annales scientifiques de l'É.N.S.

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*Annales scientifiques de l'É.N.S. 4e série*, tome 26, nº 2 (1993), p. 175-188
<http://www.numdam.org/item?id=ASENS_1993_4_26_2_175_0>

*Ann. scient. Éc. Norm. Sup.*,
4e série, t. 26, 1993, p. 175 à 188.

# AN EXCESS SPHERE THEOREM

By Peter PETERSEN V (*) and Shun-hui ZHU (**)

Abstract. — We investigate manifolds with bounded curvature, bounded diameter and small excess. In particular, we prove that any manifold with bounded curvature, Ricci curvature $\geqq n-1$ and diameter sufficiently close to $\pi$ is a twisted sphere.

## I. Introduction

Throughout this paper, let M denote a closed connected $n$ dimensional Riemannian manifold with $n \geqq 2$.

If the Ricci curvature $\operatorname{ric}(M) \geqq n-1$, then Myers' theorem (*see* [M]) implies that the diameter $\operatorname{diam}(M) \leqq \pi$ and $\pi_1(M)$ is finite. Furthermore, Cheng's maximal diameter theorem (*see* [C]) says that if $\operatorname{diam}(M) = \pi$, then M is a sphere of constant sectional curvature $\sec(M) = 1$. This has led to many investigations into what manifolds with $\operatorname{ric}(M) \geqq n-1$ and $\operatorname{diam}(M) \sim \pi$ should look like.

Without further conditions, Anderson and Otsu showed that one can not hope for such manifolds to be spheres.

*Example* 1. — ([A1], [O2]) When $n \geqq 4$, there are Riemannian manifolds with $H_2(M) \neq 0$, $\operatorname{ric}(M) \geqq n-1$, $\operatorname{vol}(M) \geqq v$, and $\operatorname{diam}(M) \to \pi$.

Thus one must add some more conditions on M before one can show that M is a sphere at least.

Before we proceed we will sort out the various ways a differentiable $n$ manifold M can be a sphere. Of course, M can be either homotopy equivalent, homeomorphic or diffeomorphic to the standard sphere. The generalized Poincaré conjecture (*see* [Sm], [Fr]) asserts that if $n \geqq 4$, then M is homeomorphic to the standard sphere provided it is homotopy equivalent to the standard sphere. So when $n \neq 3$, there is no difference between these two concepts. In addition, M can also be what we call a twisted

1991 Mathematics Subject Classification 53C20.

(*) Partial supported by NSF and A. P. Sloan Foundation.

(**) Partially supported by NSF.

sphere. This means that M is the union of two discs glued together along their boundaries by a diffeomorphism. A twisted sphere is naturally homeomorphic to a standard sphere. One can, however, say more. Namely, as long as $n \leqq 6$, M has to be diffeomorphic to the standard sphere, while when $n \geqq 7$, all one can say is that M is a differentiable manifold homeomorphic to the standard sphere (*see* [Mu] for $n \leqq 3$, [Ce] for $n=4$ and [KM] for $n \geqq 5$).

The most general sphere theorem results so far are:

THEOREM 2. — ([GrS]) *If* $\sec(\mathrm{M}) \geqq 1$, $\operatorname{diam}(\mathrm{M}) > \pi/2$, *then* M *is a twisted sphere.*

THEOREM 3. — ([GP1]) *Given real numbers* $k$, $v>0$, *there is an* $\varepsilon(n, k, v)$ *such that if*

$$\sec(\mathrm{M}) \geqq -k, \qquad \operatorname{vol}(\mathrm{M}) \geqq v, \qquad \operatorname{ric}(\mathrm{M}) \geqq n-1, \qquad \textit{and} \quad \operatorname{diam}(\mathrm{M}) \geqq \pi-\varepsilon,$$

*then* M *is a twisted sphere.*

THEOREM 4. — ([P]) *Given* $i_0>0$, *there is an* $\varepsilon(n, i_0)>0$ *such that if*

$$\operatorname{ric}(\mathrm{M}) \geqq n-1, \qquad \operatorname{inj}(\mathrm{M}) \geqq i_o, \qquad \textit{and} \quad \operatorname{diam}(\mathrm{M}) \geqq \pi-\varepsilon,$$

*then* M *is a twisted sphere.*

*Remark.* — Bessa ([Be]) recently proved that M is in fact diffeomorphic to a sphere under the conditions in theorem 4. Thus we have an optimal sphere theorem in this case.

In this paper, we prove among other things the following,

THEOREM 5. — *Given* $\mathrm{K}>0$, *there is an* $\varepsilon(n, \mathrm{K})>0$ *such that any* M *satisfying*

$$\sec(\mathrm{M}) \leqq \mathrm{K}, \qquad \operatorname{ric}(\mathrm{M}) \geqq n-1, \qquad \textit{and} \quad \operatorname{diam}(\mathrm{M}) \geqq \pi-\varepsilon,$$

*is a twisted sphere.*

In the case when $n=4$, theorem 5 was proved by Shen ([S]).

*Remark.* — There is some room for improvements both in theorem 3 and 5. Example 1 still shows that we cannot make away with the sectional curvature condition completely. However, theorem 3 could still be true if we eliminate the lower volume bound, and in theorem 5, the upper sectional curvature bound could perhaps be substituted by an upper Ricci curvature bound.

Theorem 3, 4, 5 are in fact all corollaries of more general theorems, which do not use the assumption: $\operatorname{ric}(\mathrm{M}) \geqq n-1$, $\operatorname{diam}(\mathrm{M}) \geqq \pi-\varepsilon$. Rather they use the only things we know about manifolds of positive Ricci curvature and almost maximal diameter, namely, such manifolds have finite fundamental group, bounded diameter and small excess, where the excess ([GP1]) is defined as $e(\mathrm{M}) = \min_{p, q} \max_x (d(p, x) + d(x, q) - d(p, q))$.

The idea of using excess to solve the diameter sphere problem was initiated in [GP1]. It is easy to see that manifolds with $e(\mathrm{M})=0$ are twisted spheres. Conversely, Weinstein has proved that any twisted sphere admits Riemannian metrics with $e(\mathrm{M})=0$ (appendix C in [B]). As with the almost maximal diameter question discussed above, it is therefore natural to believe that manifolds with small excess are related to spheres. Note, however, that any manifold can be scaled to have arbitrarily small diameter and hence also arbitarily small excess. An even more dramatic example by Anderson is the following.

*Example* 6. – ([A2]) There are four dimensional manifolds M satisfying:

$$H_2(M) \neq 0, \qquad |\mathrm{ric}(M)| \leqq K, \qquad \mathrm{vol}(M) \geqq v, \qquad \mathrm{diam}(M) \leqq D, \qquad \text{and} \quad e(M) \to 0.$$

Therefore, to get any kind of sphere theorem for manifolds with small excess, we must use conditions such as injectivity radius or sectional curvature. The two results generalizing theorem 3 and 4 are,

THEOREM 7. – ([GP1]). *Given $k$, $v$, $D > 0$, there is an $\varepsilon(n, k, v, D) > 0$ such that any* M *with*

$$\sec(M) \geqq -k, \qquad \mathrm{vol}(M) \geqq v, \qquad \mathrm{diam}(M) \leqq D, \qquad \textit{and} \quad e(M) \leqq \varepsilon,$$

*is a homotopy sphere.*

THEOREM 8. – ([P]). *Given $k$, $i_0$, $D > 0$, there is an $\varepsilon(n, k, i_0, D) > 0$ such that any* M *with,*

$$\mathrm{ric}(M) \geqq -k, \qquad \mathrm{inj}(M) \geqq i_0, \qquad \mathrm{diam}(M) \leqq D, \qquad \textit{and} \quad e(M) \leqq \varepsilon,$$

*is a twisted sphere.*

Likewise, in this paper, we prove the following,

THEOREM 9. – *Given* K, $D > 0$, *there is an $\varepsilon(n, K, D) > 0$ such that any* M *with,*

$$|\sec(M)| \leqq K \qquad \mathrm{diam}(M) \leqq D, \qquad \pi_1(M) \textit{ finite}, \qquad \textit{and} \quad e(M) \leqq \varepsilon,$$

*is a twisted sphere when $n \neq 3$, and* M *or a double cover of* M *is a lens space when $n = 3$.*

In view of Weinstein's result and example 6, we cannot hope to improve theorem 7 and 8 in any way. The following simple examples show that also theorem 9 is optimal.

*Example* 10. – Let $(M . g_\varepsilon)$ be $S^{n-k} \times S^k$ with $n \geqq 4$ and $k \geqq 2$, where $S^{n-k}$ has $\sec = 1$ and $S^k$ has $\sec = \varepsilon^{-2}$. As $\varepsilon \to 0$, $(M, g_\varepsilon) \to S^{n-k}$. Thus $e(M, g_\varepsilon) \to 0$. Hence the upper curvature bound is necessary.

*Example* 11. – Let $(M, g_\varepsilon)$ be a flat torus with diameter $\varepsilon$. As $\varepsilon \to 0$, $(M, g_\varepsilon) \to$ a point, so $e(M, g_\varepsilon) \to 0$. Thus the finiteness of $\pi_1(M)$ is necessary.

*Example* 12. – (Example 1.4 on P326 of [CG]) Any 3 dimensional lens space L(p, $q$) collapses with bounded curvature and diameter to a rotationally symmetric compact surface (football shaped). Such spaces obviously have $e(M) = 0$. Thus theorem 9 is also optimal in dimension 3.

*Remark*. – We do not know yet whether all 3-manifolds which are double covered by a lense space, admit metrics with small excess.

Note that when $n=3$, theorem 9 does not imply theorem 5. In this case, we need a special 3 dimensional noncollapsing result proved in [BT]. It is still possible that 3-manifolds with $\mathrm{ric} \geqq 2$ and $\mathrm{diam}(M) > \pi/2$ are spheres. But not much is known about this except Hamilton's classification of 3-manifolds with positive Ricci curvature. See, however, [W] for a discussion on 3-manifolds with almost maximal diameter and a lower volume bound.

We would like to point out a difference between theorem 9 and previous sphere theorems. Without the condition on the excess, the classes of manifolds considered in theorem 7 and 8 are well behaved, by which we mean they are either compact or contain only finitely many topological types. Thus they are all proved by showing that the limit space is a sphere. The class in theorem 9, however, can collapse. In this case, it is not possible to show the limit is a sphere. We therefore need to use the full force of the collapsing result of Fukaya. The class in theorem 2 can also collapse, but it is not relevant for the proof. Our results in this paper are the first sphere theorems where one has to deal with the collapsing phenomenon. (*See*, however, [FY1] and [FY2] for other pinching results which use collapsing techniques.)

The proof of theorem 9 is divided into a geometric part (§ 2) and a purely topological part (§ 3). In section 2 we use the results in [F] to exhibit M as the union of two disc bundles over infranil manifolds. Finiteness for the fundamental group is not used here. This structure is then used in Section 3 with some fundamental group considerations to prove theorem 9. In Section 4, we will prove theorem 5 for the case $n=3$, which is the only case not covered by theorem 9. Finally, we conclude in Section 5 with some remarks and questions about manifolds with small excess.

For related results about excess and similar invariants, see [GP1], [P], [S], [O1] and [PSZ].

We would like to thank Professor Karsten Grove for promoting the idea that some positive curvature questions can be attacked by proving a double soul type theorem as we do in Section 2.

## II. A double soul theorem

In this section we prove the following result,

THEOREM 2.1. — *For any positive integer* $n \geqq 2$ *and positive numbers* D, K, *there exists a positive constant* $\varepsilon(n, D, K)$ *such that if* $M^n$ *is a connected Riemannian manifold satisfying*

$$|\sec(M)| \leqq K, \qquad \mathrm{diam}(M) \leqq D, \qquad e(M) \leqq \varepsilon,$$

*then either* M *is infranil or there exist two embedded connected infranilmanifolds* $F_1$ *and* $F_2$ *in* M, *and* M *is diffeomorphic to the union of the normal bundles of* $F_1$ *and* $F_2$. *Furthermore, there is a gradient-like vector field* $\xi$ *on* $M \setminus \{F_1, F_2\}$ *along whose integral curves the distance to* $F_1$ ($F_2$) *is decreasing (increasing.)*

*Proof.* – We prove this by contradiction. Suppose there is a sequence of manifolds $\{M_i\}$ such that

$$|\sec(M_i)| \leqq K, \qquad \operatorname{diam}(M) \leqq D, \qquad e(M_i) \to 0,$$

but $\{M_i\}$ does not satisfy the conclusions of the theorem.

By the fibration theorem of K. Fukaya (theorem 0.12 in [F]), the frame bundles $FM_i$ with the canonical metric (sub)converges in $C^{0,\alpha}$ topology to a $C^{0,\alpha}$ Riemannian manifold Y, and the $O(n)$ action on $FM_i$ induces an $O(n)$ action on Y, the quotient $Y/O(n)=X$ is the Hausdorff limit of $\{M_i\}$. Furthermore, there is a diagram

$$\begin{array}{ccc} FM_i & \xrightarrow{g_i} & Y \\ \downarrow \psi & & \downarrow \varphi \\ M_i & \xrightarrow{f_i} & X \end{array} \tag{1}$$

that commutes. Here $g_i$ is an $O(n)$-invariant almost Riemannian submersion with infranil fibers.

In the trivial case when X is a point, M is diffeomorphic to an infranil manifold. In what follows, we assume X is not a point. Since $e(M_i) \to 0$ when $i \to \infty$, $e(X)=0$. Hence there exist two points $p, q \in X$, such that for any point $x \in X$,

$$d(p, x)+d(x, q)=d(p, q).$$

We claim for any point $x \in X \setminus \{p, q\}$, there exists a unique geodesic $\gamma$ from $p$ to $q$ passing through $x$. In fact, let $\gamma_p : [0, l_p] \to X$, $\gamma_q : [0, l_q] \to X$ be two geodesics such that

$$\gamma_p(0)=p, \qquad \gamma(l_p)=x, \qquad \gamma_q(0)=x, \qquad \gamma_q(l_q)=q.$$

Then $\gamma_p \cup \gamma_q$ must be smooth at $x$. Otherwise, let $\bar{\gamma}_p$ be a geodesic from $p$ to $\gamma_q(t_0)$ for some small $t_0$. Since

$$L(\gamma_p)+L(\gamma_q)=d(p, q)$$
$$L(\bar{\gamma}_p)+L(\gamma_q|_{[t_0, l_q]})=d(p, q),$$

$\gamma_p \cup \gamma_q$ and $\bar{\gamma}_p \cup \gamma_q|_{[t_0, l_q]}$ are both (minimal) geodesics having the segment $\gamma_q|_{[t_0, l_q]}$ in common. This contradicts the fact that geodesics in the limiting space of manifolds with lower curvature bounds do not bifurcate ([GP2]). Thus $\gamma_p \cup \gamma_q$ is the unique smooth geodesic from $p$ to $q$ passing through $x$. Hence X is a suspension over a space S.

From now on, we will drop the subscript $i$ in our discussion. Let $F_1=f^{-1}(p)$, $F_2=f^{-1}(q)$. The above quoted theorem of Fukaya implies $F_1$ and $F_2$ are embedded connected infranil manifolds of M.

To show that M is the union of the normal bundles of $F_1$ and $F_2$, we only need to construct a gradient-like smooth vector field on $M \setminus \{F_1, F_2\}$. Note the tangent vectors $\eta$ to the geodesics from $p$ to $q$ constructed above in X is a gradient field. The idea is to try to lift $\eta$ to a vector field in M. The difficulty is that $f_*$ is not well-defined. (By a more elaborate result of Fukaya, $f_*$ is well-defined along each strata of X. This, however, will not really help us.) We note that in the diagram (1) the map $f$ comes from $g$, and $g$ is a well-defined submersion (since Y is a manifold). Thus to circumvent the difficulty, we first lift $\eta$ to Y, then use $g$ to lift to FM, then pushdown to M to get the desired vector field.

To lift $\eta$ to Y, we work with geodesics. For any point $y \in \mathrm{Y} \diagdown \varphi^{-1}(\{p, q\})$, denote $x = \varphi(y)$. Let $\gamma$ be the unique geodesic of X from $p$ to $q$ passing through $x$. We claim there exists a unique lift of $\gamma$ to a geodesic $\bar{\gamma}$ in $\mathrm{Y} \diagdown \varphi^{-1}(\{\mathrm{p}, q\})$ passing through $y$. In fact, denote the two segments of $\gamma$ divided by $x$ as $\gamma_1$ and $\gamma_2$. By the proposition in the appendix (*see* the remark there), $\gamma_1$ and $\gamma_2$ can be lifted to $\bar{\gamma}_1$ and $\bar{\gamma}_2$ passing through $y$. Since $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are "horizontal", *i. e.*, they realize the distance between their end points, therefore $\mathrm{L}(\bar{\gamma}_1) + \mathrm{L}(\bar{\gamma}_2) = \mathrm{L}(\gamma_1) + \mathrm{L}(\gamma_2) = d(p, q)$. If $\bar{\gamma}_1 \cup \bar{\gamma}_2$ is not smooth at $y$, there is a geodesic from $\varphi^{-1}(p)$ to $\varphi^{-1}(q)$ with length shorter than $\mathrm{L}(\bar{\gamma}_1) + \mathrm{L}(\bar{\gamma}_2) = d(p, q)$. After projecting down to X, we get a geodesic shorter than $d(p, q)$, which is impossible. Thus $\bar{\gamma}_1 \cup \bar{\gamma}_2$ is smooth at $y$. This implies the existence and uniqueness of the lift through $y$. Let $\zeta =$ the tangent vector to $\bar{\gamma}$.

We now show the vector field $\zeta$ is continuous. This follows easily from the uniqueness of lifts. In fact, we only need to show the collection of geodesics $\bar{\gamma}$ is continuous (this is why we work with geodesics instead of vector fields.) Let $y_i \to y$, and corresponding $\bar{\gamma}_i$, $\bar{\gamma}$ such that $\bar{\gamma}_i \to \bar{\sigma}$ which are defined on $[0, l]$ with $l = d(p, q)$. Since geodesics converge to geodesics, $\bar{\sigma}$ is a geodesic through $y$. The continuity of the collection of geodesics $\{\gamma\}$ in X implies that $\bar{\gamma}$ and $\bar{\sigma}$ are the lifts of the same geodesic in X, the unique one passing through $x = \varphi(y)$. Now the uniqueness of lifts we just proved in the previous paragraph implies that $\bar{\sigma} = \bar{\gamma}$. Thus $\zeta$ is continuous.

The lifting of $\zeta$ from Y to a vector field $\chi$ in FM is easy, since Y is a $\mathrm{C}^{0, \alpha}$ Riemannian manifold and $g$ is an almost Riemannian submersion. To make a canonical lifting, we proceed as follows. For $z \in \mathrm{FM} \diagdown (\varphi \circ g)^{-1}(\{\mathrm{p}, q\})$, let $y = g(z)$. Consider the subset of vectors at $z$ defined by $\mathrm{A}_z = \{\vec{v} \mid g_*(\vec{v}) = \zeta(y)\}$. Since $g$ is an almost Riemannian submersion, there are vectors in $\mathrm{A}_z$ of length less than, say, 2. Thus $\min_{\mathrm{A}_z} \|\vec{v}\|$ is achieved by a vector $\chi(z)$. Obviously such minimum is unique since if $\vec{v}_1$ and $\vec{v}_2$ are two such minimums, then $1/2(\vec{v}_1 + \vec{v}_2) \in \mathrm{A}_z$ and its length is strictly shorter than $\|\vec{v}_1\| (\|\vec{v}_2\|)$ unless $\vec{v}_1$ and $\vec{v}_2$ are parallel. We thus obtain a vector field $\chi$ in $\mathrm{FM} \diagdown (\varphi \circ g)^{-1}(\{p, q\})$.

To see $\chi$ is continuous, let $z_i \to z$ and $\chi(z_i) \to \vec{u}$. Since $g_*$ is continuous, we have

$$g_*(\vec{u}) = g_*(\lim \chi(z_i)) = \lim g_*(\chi(z_i)) = \lim \zeta(g(z_i)) = \zeta(g(z)).$$

Thus $\vec{u} \in \mathrm{A}_z$. If $\|\vec{u}\|$ is not the minimum in $\mathrm{A}_z$, there is a vector $\vec{u}'$ in $\mathrm{A}_z$ with $\|\vec{u}'\| < \|\vec{u}\|$. Thus for $i$ big, $\|\vec{u}'\| < \|\chi(z_i)\|$. Now since $g$ is a fiber bundle, there are vectors $\vec{u}_i' \in \mathrm{A}_{z_i}$ such that $\lim \vec{u}_i' = \vec{u}'$. In particular, when $i$ is big, $\|\vec{u}_i'\| < \|\chi(z_i)\|$. This contradicts the definition of $\chi$. Thus $\vec{u}$ has minimum length in $\mathrm{A}_z$, hence $\vec{u} = \chi(z)$. This proves that $\chi$ is continuous.

We note that $\zeta$ is $O(n)$ invariant and perpendicular to the $O(n)$ orbits. Also $\zeta$ is a gradient vector field. Since $g$ is an almost Riemannian submersion, we conclude that $\chi$ is almost perpendicular to the $O(n)$ orbits and is almost a gradient field. The construction also implies that $\chi$ is $O(n)$-invariant (since $g$ is). We now take the projection of $\chi$ in the directions which are perpendicular to the $O(n)$ orbits and, by abuse of notation, denote the resulting vector field again by $\chi$. It follows that $\chi$ is continuous, $O(n)$-invariant and gradient like, *i.e.*, if $\delta$ is a integral curve of $\chi$, then $\mathrm{dist}((\varphi \circ g)^{-1}(p), \delta(t))$ is strictly increasing, and $\mathrm{dist}((\varphi \circ g)^{-1}(q), \delta(t))$ is strictly decreasing.

We now let $\xi = \psi_*(\chi)$ (since $\chi$ is $O(n)$-invariant). Note that $\|\xi\| = \|\chi\|$ since $\chi$ is perpendicular to the $O(n)$ orbits (*i.e.*, $\chi$ is the horizontal lift of $\xi$ with respect to $\psi$) Let $\sigma$ be a integral curve of $\xi$, then $\mathrm{dist}(F_1, \sigma(t)) = \mathrm{dist}((\varphi \circ g)^{-1}(p), \delta(t))$ is strictly increasing. Similarly, $\mathrm{dist}(F_2, \sigma(t))$ is strictly decreasing. Thus $\xi$ is a gradient-like vector field. Obviously $\xi$ is continuous. We can now use the standard procedure to approximate $\xi$ to get a smooth vector field (*see* [GS]), still denoted by $\xi$. This is the desired smooth vector field. This contradicts the assumption at the beginning of the proof.

Q.E.D.

## III. Van Kampen's theorem for double soul manifolds

We are going to use the double soul theorem 2.1 to prove theorem 9. Since the manifolds $F_1$ and $F_2$ are compact $K(\pi, 1)$ spaces, the fundamental groups yield enough information about the manifolds. So our argument will use heavily van Kampen's theorem. The technical difficulty is caused by the fact that, after lifting to the universal cover, the sets involved are not necessarily connected in general, while van Kampen's theorem requires connectedness.

Let $\pi : \tilde{M} \to M$ be the universal covering map. Let $\tilde{F}_1$ and $\tilde{F}_2$ be the preimages of $F_1$ and $F_2$ under $\pi$. Denote by $\tilde{\xi}$ the lifting of the vector field $\xi$.

Since $\pi_1(M)$ is finite, $\tilde{M}$, $\tilde{F}_1$ and $\tilde{F}_2$ are all compact. Note that $\tilde{F}_i$ does not have to be connected, and that when it is not, all connected components are diffeomorphic and have identical normal bundles (identified by the deck transformation.) We will show that when the dimension of M is not three, M is homeomorphic to a sphere, and in dimension three, M or a double cover of M is a lens space. We will divide the proof of theorem 9 into three cases depending on the codimension of $\tilde{F}_i$.

**1. codim** $(\tilde{F}_1) \geqq 2$, **codim** $(\tilde{F}_2) \geqq 2$.

We first show that both $\tilde{F}_1$ and $\tilde{F}_2$ are connected. If $\tilde{F}_1$ is not connected, let $C_1$, $C_2$ be two of its connected components. Take $x_i \in C_i$ $(i = 1, 2)$. Since $\tilde{M}$ is connected, there is a curve $\gamma$ from $x_1$ to $x_2$ in $\tilde{M}$. If $\gamma \cap \tilde{F}_2 = \varnothing$, then deform $\gamma$ along the vector field $\tilde{\xi}$ will give a curve $\bar{\gamma}$ lying entirely in $\tilde{F}_1$. This is not possible since $C_1$ and $C_2$ are different components of $\tilde{F}_1$. Thus $\gamma \cap \tilde{F}_2 \neq \varnothing$, say, $\gamma \cap D \neq \varnothing$ with D a connected components of $\tilde{F}_2$. Consider the sphere bundle $S(D)$ of the normal bundle of D. Since D is connected and $\mathrm{codim}(D) = \mathrm{codim}(\tilde{F}_2) \geqq 2$, $S(D)$ is connected. Hence we can replace the portion of $\gamma$ near D by a curve lying in S(D), the resulting curve $\bar{\gamma}$ does not intersect D. Do the same for all components of $\tilde{F}_2$ with which $\gamma$ intersects (a finite number of such), we will get a curve from $x_1$ to $x_2$, not intersecting $\tilde{F}_2$. Using the flow $\xi$, we will again get a contradiction. Thus $\tilde{F}_1$ is connected. The same argument shows that $\tilde{F}_2$ is also connected. This argument shows that if the sphere bundle of the normal bundle of each connected components of $F_1\,(F_2)$ is connected, then $F_2\,(F_1)$ is connected.

Let $\tilde{U}_1$, $\tilde{U}_2$ be the normal bundles of $\tilde{F}_1$ and $\tilde{F}_2$, then $\tilde{M}=\tilde{U}_1 \cup \tilde{U}_2$. $\tilde{I}=\tilde{U}_1 \cap \tilde{U}_2$ is the sphere bundle over $\tilde{F}_1$ (and $\tilde{F}_2$). Since $\operatorname{codim}(\tilde{F}_1) \geqq 2$, $\tilde{I}$ is connected. A version of the Van Kampen theorem says that in the following diagram

$$\begin{tikzcd} & \pi_1(\tilde{U}_1) \arrow[dr, "j_1"] \arrow[drr, "k_1"] & & \\ \pi_1(\tilde{I}) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(\tilde{M}) \arrow[r, "h"] & H \\ & \pi_1(\tilde{U}_2) \arrow[ur, "j_2"'] \arrow[urr, "k_2"'] & & \end{tikzcd} \tag{2}$$

for any group H, if there are maps $k_1$, $k_2$ making the diagram commute, then there exists a map $h$ making the diagram commute.

1.*a*. $\operatorname{codim}(\tilde{F}_1) \geqq 3$ (or $\operatorname{codim}(\tilde{F}_2) \geqq 3$).

Since $\tilde{I}$ is the sphere bundle over $\tilde{F}_1$, the homotopy sequence for sphere bundles is reduced to,

$$0 \to \pi_1(\tilde{I}) \xrightarrow{i_1} \pi_1(\tilde{F}_1) \to 0,$$

where we have used the map $i_1$ in the above sequence since $\tilde{F}_1$ is homotopic to $\tilde{U}_1$. Thus $i_1$ is an isomorphism. If we choose $H=\pi_1(\tilde{U}_2)$ and $k_2=\mathrm{id}$, $k_1=k_2 \circ i_2 \circ i_1^{-1}$, then diagram (2) commutes. Since $\pi_1(\tilde{M})=\{e\}$, $h$ does not exist unless $\pi_1(\tilde{U}_2)=\{e\}$. Thus $\tilde{F}_2=\{\text{a point}\}$ since $\tilde{F}_2$ is a compact $K(\pi, 1)$ space. Thus $\tilde{I}=S^{\dim(M)-1}$, and the homotopy exact sequence for $\tilde{U}_1 \to \tilde{F}_1$ implies $\pi_1(\tilde{F}_1)=\{e\}$. Hence $\tilde{F}_1=\{\text{a point}\}$, and $F_1 \cong F_2=\{\text{a point}\}$. It follows that M is a suspension, and therefore homeomorphic to a sphere, since M is a manifold.

1.*b*. $\operatorname{codim}(\tilde{F}_1)=\operatorname{codim}(\tilde{F}_2)=2$.

In this case the exact homotopy sequence is,

$$0 \to Z \xrightarrow{l_1,\, l_2} \pi_1(\tilde{I}) \xrightarrow{i_1,\, i_2} \pi_1(\tilde{F}_i) \to 0.$$

Let $l_i(Z)=\langle a_i \rangle$, then in diagram (2) take $H=\pi_1(\tilde{I})/\langle a_1, a_2 \rangle$, and let $k_1$, $k_2$ be the natural projections. Then diagram (2) commutes. Again since $\pi_1(\tilde{M})=\{e\}$, the map $h$

does not exist unless $H=\{e\}$. Thus $\pi_1(\tilde{I})$ is generated by $a_1$ and $a_2$. Hence $\pi_1(\tilde{U}_1)=\pi_1(\tilde{U}_2)=\{e\}$ or Z. Therefore $\pi_1(\tilde{F}_1)=\pi_1(\tilde{F}_2)=\{e\}$ or Z. It then follows that $\tilde{F}_1=\tilde{F}_2=\{\text{a point}\}$ or $S^1$. Note that $\operatorname{codim}(\tilde{F}_1)=\operatorname{codim}(\tilde{F}_2)=2$, thus if $\tilde{F}_1\cong\tilde{F}_2=\{\text{a point}\}$, then $F_1=F_2=\{\text{a point}\}$. Thus $M=S^2$. If $\tilde{F}_1\cong\tilde{F}_2=S^1$, also $F_1\cong F_2=S^1$. Therefore M has a genus one Heegaard splitting. This implies that M is a lens space because $\pi_1(M)$ is finite (*see* [He].)

**2. codim $(\tilde{F}_1)=1$, codim $(\tilde{F}_2)\geqq 2$ (or codim $(\tilde{F}_2)=1$, codim $(\tilde{F}_1)\geqq 2$.)**

Since $\operatorname{codim}(\tilde{F}_2)\geqq 2$, exactly the same argument as in 1 (verbatim) shows $\tilde{F}_1$ is connected. Since $\pi_1(\tilde{M})=0$, $\tilde{F}_1$ disconnects $\tilde{M}$ into two components, in particular, $\tilde{U}_1=\tilde{F}_1\times(-1, 1)$. Thus $\tilde{F}_2$ has two connected components (since $\tilde{F}_1$ is connected.) Thus $\tilde{M}=V_1\cup V_2$ with $V_i$ the normal bundle over a connected components of $\tilde{F}_2$. This is exactly the situation we treated in (1) (codim $\geqq 2$ case). Note that in this situation, although the vector field does not give the flow as in (1), the connectedness is already known. Hence if $\operatorname{codim}(\tilde{F}_2)\geqq 3$, then $\tilde{F}_2$ is a point, which implies $\tilde{I}$ is a sphere. Thus $\tilde{F}_1=\{\text{a point}\}$ or $S^1$. Note that $\operatorname{codim}(\tilde{F}_1)=1$, thus $\dim(\tilde{M})=1$ or 2, this contradicts the assumption that $\operatorname{codim}(\tilde{F}_2)\geqq 3$. If $\operatorname{codim}(\tilde{F}_2)=2$, then the same argument as in $(1.b)$ implies M is three dimensional. Denote $U_i=\pi(\tilde{U}_i)$ for $i=1, 2$, and $I=\pi(\tilde{I})$. Since $F_1$ is a two dimensional infranil manifold, $F_1$ is a torus $T^2$ or a Klein bottle K. Note that $F_2=S^1$. Thus $U_1$ is a line bundle over $T^2$ or K, and $U_2$ is a 2-disc bundle over $S^1$. There are only two 2-disc bundles over $S^1$ up to diffeomorphism, the trivial $S^1\times D^2$ and a nonorientable bundle. In the latter case, M then also becomes nonorientable. But then $\pi_1(M)$ will be infinite (since $0=1-b_1+b_2-b_3=1-b_1+b_2$ implies that $b_1\geqq 1$). In the former case, we know that $I=S^1\times S^1$. If $F_1=T^2$, then the cohomology Mayer-Vietoris sequence says,

$$H^1(M)\to H^1(F_1)\oplus H^1(F_2)\to H^1(I)$$

is exact. Taking, say R as coefficients, we then get that,

$$H^1(M, R)\to R\oplus R\oplus R\to R\oplus R$$

is exact. Thus $b_1(M, R)\geqq 1$, which contradicts that $\pi_1(M)$ is finite. If $F_1=K$, we get the following diagram from the van Kampen theorem (note that I is connected),

$$\begin{tikzcd}
& \pi_1(K) \arrow[dr, "j_1"] \arrow[drr, "k_1"] & & \\
\pi_1(I) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(M) \arrow[r, "h"] & Z_2 \\
& \pi_1(S^1) \arrow[ur, "j_2"'] \arrow[urr, "k_2"'] & &
\end{tikzcd}$$

where $k_2=0$ and $k_1$ is the natural projection $\pi_1(K)\to\pi_1(K)/\pi_1(I)$. Let $\bar{M}$ denote the double cover of M corresponding to the kernel H of $h:\pi_1(M)\to Z_2$. Clearly the image of $\pi_1(K)$ in $\pi_1(M)$ does not lie in H while the image of $\pi_1(S^1)$ lies in H. Thus the lift $\bar{K}$ of K is connected and the lift $\bar{S}^1$ of $S^1$ is the union of two disjoint circles. We then have a genus one Heegaard splitting of $\bar{M}$. This implies as before that $\bar{M}$ is a lens space.

**3. codim** $(\tilde{F}_1) =$ **codim** $(\tilde{F}_2) = 1$

Since $\pi_1(\tilde{M})=0$, both sphere bundles of the normal bundles of $\tilde{F}_1$ and $\tilde{F}_2$ are disconnected. Thus $\tilde{U}_i = \tilde{F}_i \times (-1, 1)$ for $i=1, 2$. It then follows that $\tilde{F}_1$ and $\tilde{F}_2$ are diffeomorphic. We now show that $\tilde{M}$ is a fiber bundle over $S_1$ with $\tilde{F}_1(\tilde{F}_2)$ as fibers. The homotopy sequence will then give a contradiction. In fact, if we enumerate the connected components of $\tilde{U}_i$ $(i=1, 2)$ as $C_1, C_2, \ldots, C_m$ ($m$ is finite since $\tilde{M}$ is compact), then all $\{C_j\}$ are diffeomorphic to $\tilde{F} \times (-1, 1)$ with $\tilde{F}$ a connected component of $\tilde{F}_1(\tilde{F}_2)$. $\tilde{M}$ is obtained by identifying the boundaries of $\{C_j\}$ (which is diffeomorphic to $\tilde{F}$). Notice that after identifying one component of $\partial C_1$ to that of, say, $\partial C_2$, the resulting set is still diffeomorphic to $\tilde{F} \times (-1, 1)$. By induction, $\tilde{M}$ is obtained by identifying the two components of the boundary of $\tilde{F} \times (-1, 1)$, and therefore it is a fiber bundle over $S^1$ with $\tilde{F}$ as fibers. But this contradicts that $\pi_1(\tilde{M})=0$.

This completes the proof of Theorem 9 in all cases.

Q.E.D.

## IV. Noncollapsing in dimension three

In this section we prove theorem 5. Note that when $\dim(M) \neq 3$, theorem 5 is an immediate consequence of theorem 9 and Myers' theorem. We will thus concentrate on the three dimensional case. The argument in $(1.b)$ and 2 of the previous section implies that we only need to prove M is simply connected (this of course also follows from Hamilton's result, *see* [Ha]).

In the proof we need to use the following result,

THEOREM 4.1. — *Given* $i_0$, $K>0$. *If a Riemannian manifold* M *satisfies,*

$$\sec(M) \leqq K, \qquad \operatorname{inj}(M) \geqq i_0,$$

*then any subgroup* G *of the group of isometries of* M, *with*

$$d(G(p), p) \leqq \varepsilon = 1/2 \min\{i_0, \pi/4\sqrt{K}\}$$

*for some point* $p \in M$, *has a fixed point.*

*Proof.* — This is basically the Cartan's theorem which says that any compact group action on a simply connected manifold of nonpositive curvature has a fixed point. In our situation, the condition $d(G(p), p) \leqq \varepsilon$ implies the existence of the center of gravity of the set $G(p)$ (see 8.1.3 of [BK], also [GK]). Theorem 4.1 then follows since a group always fixes the center of gravity of an invariant set.

Q.E.D.

We now prove theorem 5 in dimension three by contradiction. Suppose there is a sequence of three manifolds $\{M_i\}$ satisfying,

(3) $$|\sec(M_i)| \leqq K, \qquad \mathrm{ric}(M_i) \geqq 2 \qquad \mathrm{diam}(M_i) \geqq \pi - \varepsilon,$$

with $\varepsilon \to 0$, but $\{M_i\}$ are not simply connected. Let $\pi_i : \tilde{M}_i \to M_i$ be the universal covering maps. It is easy to see that $\tilde{M}_i$ with the pull-back metric satisfies condition (3). Let $p_i, q_i \in M_i$ such that $d(p_i, q_i) = \mathrm{diam}(M_i)$. For any $\alpha_i \in \pi_1(M_i, q_i)$ which $\alpha_i$ a geodesic loop at $q_i$. Fix $\tilde{q}_i \in M_i$ such that $\pi_i(\tilde{q}_i) = q_i$. Let $\tilde{\alpha}_i$ be the liftings of $\alpha_i$ with base point at $\tilde{q}_i$. A simple volume comparison argument (Lemma in § 3 of [W]) implies $d(\tilde{p}_i, \alpha_i(\tilde{q}_i)) \leqq \tau_i(\varepsilon_i)$ where $\tau_i(\varepsilon_i) \to 0$ when $\varepsilon_i \to 0$. This implies $d(\pi_1(M_i)(\tilde{q}_i), \tilde{q}_i) \leqq \tau_i$.

A theorem of Burago and Toponogov ([BT]) shows that the sequence of manifolds $\{\tilde{M}_i\}$ has a uniform lower bound on the injectivity radius. Theorem 4.1 then implies $\pi_1(M_i)$ has a fixed point when $i$ is big. This contradicts the fact that $\alpha_i$ is a deck transformation, hence fixed-point free. This contradiction implies that $\{M_i\}$ are simply connected, thus completing the proof of theorem 5.

Q.E.D.

*Remark.* — In dimension three, because of the relation between sectional curvature and Ricci curvature, what we have actualy proved is that any three manifold with $2 \leqq \mathrm{ric}(M) \leqq K$, $\mathrm{diam}(M) \geqq \pi - \varepsilon$ is a twisted sphere.

## V. Some remarks

The double soul theorem in Section 2 does not depend on the fact that $\pi_1$ is finite. Thus we still get a structure result for manifolds with infinite $\pi_1$ and small excess. In fact one can go through the fundamental group argument in this case as well, just as we did in Section 3. In the case where one of the $F_i$'s have codim $\geqq 3$, we get as before that $\tilde{F}_i \subset \tilde{M}$ is simply connected and therefore diffeomorphic to some Euclidean space $R^k$. Both of the sets $\tilde{F}_i$ will then be $R^k$ and the universal covering $\tilde{M}$ is a disc bundle over a twisted sphere, because the normal bundles to $\tilde{F}_1$, $\tilde{F}_2$ are trivial. In the special case when both $F_i$ have codim $\leqq 2$, it is harder to say what happens. On the other hand, this can only happen if M collapses to a space X with dimension $\leqq 2$. There are basically only four spaces X with $\dim(X) \leqq 2$, $e(X) = 0$, and $\sec(X) > -\infty$. Either $\dim(X) = 1$ in which case it must be an interval or a circle, or $\dim(X) = 2$ in which case X is the suspension over an interval or a circle. It is therefore in general quite restrictive for a manifold to have bounded curvature, bounded diameter and small excess. One might even ask if such manifolds, or at least a finite cover, fiber over a twisted sphere with infranil fibers (*see* [PSZ]). We leave it to the interested reader to classify all 2, 3-dimensional manifolds satisfying the conclusions of the double soul theorem 2.1.

One might also wonder what happens if there is no upper curvature bound. In that case we have the following

DOUBLE SOUL CONJECTURE (Grove & Petersen). — *Let* $n \geqq 2$, K, $D > 0$ *be given, there is a positive constant* $\varepsilon(n, K, D)$ *such that any Riemannian manifold* $M^n$ *with,*

$$\sec(M) \geqq -K, \qquad \operatorname{diam}(M) \leqq D, \qquad \textit{and} \quad e(M) \leqq \varepsilon,$$

*is the union of the normal bundles over two submanifolds, which are almost nonnegatively curved.*

## APPENDIX

## GROUP ACTIONS AND LIFTINGS OF GEODESICS

In this appendix, we give the lifting properties for certain singular spaces. We used such a property in the proof of Theorem 2.1. To be more precise, let $(M^n, g)$ be a $n$-dimensional Riemannian manifold, and G a compact group acting on M by isometries. The quotient space M/G has a natural metric $d$, defined as $d(\bar{x}, \bar{y}) = d^M(Gx, Gy)$. The compactness of G implies that (M/G, $d$) is an inner metric space, *i.e.*, the distance between two points is realized by a continuous curve connecting them. By definition, a geodesic in M/G is a curve realizing the distance between any of its two points. Consider the natural map $\pi : M \to M/G$. We are interested in when a given geodesic in M/G admitts a "horizontal" lift in M with given initial point, *i.e.*, when there is a geodesic $\bar{\gamma}$ in M such that $\pi(\bar{\gamma}) = \gamma$ and $\bar{\gamma}$ realizes distance between the orbits it passes through. The following proposition gives an affirmative answer without any conditions.

PROPOSITION. — *For a fixed manifold* M *and a compact group* G *of isometries,*

1. *geodesics in* M/G *do not bifurcate,*
2. $\pi : M \to M/G$ *is distance nonincreasing,*
3. *any geodesic in* M/G *has a "horizontal" lift, given an initial point.*

*Proof.* — 1. It is well-known that there exists a sequence of metrics $g_k$ on M such that $\sec(g_k) \geqq -\Lambda^2$ for some $\Lambda$ and $(M, g_k)$ converges with respect to the Hausdorff distance to M/G (*see*, for example, Example 1.2$c$ in [Y]). 1) follows since geodesics in the limit space of a sequence with lower curvature bound do not bifurcate ([GP2]).

2. Let $\gamma$ be a minimal geodesic in M from $x$ to $y$. Consider the orbits $Gx$ and $Gy$, if $Gx \cap Gy \neq \varnothing$, we then have $d(\bar{x}, \bar{y}) = 0$, 2) holds trivially in this case. Thus we consider the case when $Gx \cap Gy = \varnothing$. Let $\delta$ be a geodesic realizing the distance between $Gx$ and $Gy$ ($\delta$ exists since G is compact). Using the group action, we can assume that $\delta(0) = x$ and $\delta$ realizes the distance between $x$ and $Gy$. Thus $L(\gamma) \geqq L(\delta) = d(\bar{x}, \bar{y})$, *i.e.*, $\pi$ is distance nonincreasng. It follows that $\pi$ maps minimal geodesic between two orbits to a geodesic.

3. Let $\sigma : [0, b] \to M/G$ be a geodesic. Given $p \in \pi^{-1}(\sigma(0))$, let $t_0 = \sup \{ t \mid \sigma|_{[0, t]}$ has a "horizontal" lift starting at $p \}$.

We claim that $t_0>0$. To see this, we first note that for $0<t<b$, $\sigma$ is the only geodesic from $\sigma(0)$ to $\sigma(t)$. This follows because a different geodesic will give a bifurcation of $\sigma$ at $\sigma(t)$, which is not possible by 1). Consider the orbits $\pi^{-1}(\sigma(0))$ and $\pi^{-1}(\sigma(t))$. Let $\bar{\sigma}$ be a minimal geodesic from $\pi^{-1}(\sigma(0))$to $\pi^{-1}(\sigma(t))$. Using the group action, we can assume $\bar{\sigma}=p$. Then $\pi(\bar{\sigma})$ is a geodesic from $\sigma(0)$ to $\sigma(t)$ (by 2)). The uniqueness of geodesics we just proved implies $\pi(\bar{\sigma})=\sigma$. Thus $\bar{\sigma}$ is a "horizontal" lift of $\sigma$.

If $t_0<b$, we can pick a number $t$ such that $t_0<t<b$ and use the same argument as above to get a continuation of the lift to $[0, t]$, which contradicts the choice of $t_0$. Thus $t_0=b$.

Q.E.D.

*Remark.* – In the proof of (3), we only used the property that geodesics do not bifurcate, this is guaranteed by the presence of a lower curvature bound. It was in this context we used the above proposition in the proof of theorem 2.1.

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(Manuscript received October 22, 1991,
revised January 27, 1992).

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