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TRANSGRESSED EULER CLASSES OF SL $(2n, \mathbb{Z})$ VECTOR BUNDLES, ADIABATIC LIMITS OF ETA INVARIANTS AND SPECIAL VALUES OF L-FUNCTIONS

BY JEAN-MICHEL BISMUT AND JEFF CHEEGER

ABSTRACT. — In this paper, we refine a result of Sullivan, who showed that the Euler class of a SL $(2n, \mathbb{Z})$ vector bundle vanishes rationally, by explicitly transgressing the Chern-Weil differential form representing the Euler Class. This transgression is obtained by using techniques which are inspired by the local families index theorem. In particular, we prove Poisson formulas with Grassmann variables. We show that the transgression, used in combination with some previous work of ours can be employed to calculate the adiabatic limit of eta invariant of torus bundles. As an application, we obtain a new proof of the result of Atiyah-Donnelly-Singer and Müller on the signature of Hilbert modular varieties. Finally, using the transgression and the theory of differential characters, we define a characteristic cohomology class $\hat{\chi} \in H^{2n-1}(B, \mathbb{R}/\mathbb{Z})$ for SL $(2n, \mathbb{Z})$ vector bundles on B.

Keywords: Zeta and L-functions, characteristic classes and numbers, Index theory and fixed point theory.

RÉSUMÉ. — Dans cet article, on donne un raffinement d'un résultat de Sullivan relatif à l'annulation de la classe d'Euler rationnelle d'un SL $(2n, \mathbb{Z})$ fibré vectoriel, par transgression explicite des formes de Chern-Weil représentant cette classe d'Euler. La transgression de ces formes de Chern-Weil est réalisée par des techniques inspirées par le théorème d'indice local des familles. On démontre en particulier des formules de Poisson avec des variables grassmanniennes. On utilise ensuite cette transgression en combinaison avec des résultats obtenus antérieurement par nous pour calculer la limite adiabatique d'invariants êta sur des variétés fibrées en tores. On en déduit une nouvelle démonstration du calcul par Atiyah-Donnelly-Singer et Müller de la signature des variétés modulaires de Hilbert. Enfin, en utilisant la transgression et la théorie des caractères différentiels, on définit une classe de cohomologie $\hat{\chi} \in H^{2n-1}(B, \mathbb{R}/\mathbb{Z})$ associée à un SL $(2n, \mathbb{Z})$ fibré vectoriel sur B.

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The initial purpose of this paper was to give a simple direct proof of the Hirzebruch conjecture [H] on the signature of the Hilbert modular varieties, by using our previous results [BC] on adiabatic limits of eta invariants. The Hirzebruch conjecture was first proved by Atiyah-Donnelly-Singer [ADS] and Müller [Mü1,2] (compare also Stern [St]).

In [BC], we considered the case of a fibration $\pi: M \to B$ with even-dimensional fibres Z, where M is an odd dimensional oriented spin manifold. Let g^{TM} be a metric on the tangent bundle TM, and let g^{TB} be a metric on TB. For $\varepsilon > 0$, we equip TM with the metric $g_\varepsilon^{TM} = g^{TM} + (1/\varepsilon)\pi^* g^{TB}$. Let D_ε be the Dirac operator on M acting on twisted

TM spinors associated to the metric g_ε^{TM} and to a unitary connection on the twisting bundle ξ. Let $\bar{\eta}_\varepsilon(0)$ be the reduced eta invariant of the operator D_ε in the sense of Atiyah-Patodi-Singer [APS1]. We showed in [BC] that, when the Dirac operators acting on the fibres Z are invertible, then as $\varepsilon \to 0$, $\bar{\eta}_\varepsilon(0)$ has a limit which can be calculated in terms of a differential form $\tilde{\eta}(0)$, which is local on the base B and global in the fibres Z. It is obtained by using the Levi-Civita superconnection of the fibration, which was introduced by Bismut [B1] in his proof of the local families Index Theorem. The exterior derivative $d\tilde{\eta}(0)$ is given by the integral along the fibre of a form which is local on M. As pointed out in [BC], the form $\tilde{\eta}(0)$ can still be defined when the kernel of the Dirac operators along the fibres Z has constant dimension. In this case, $d\tilde{\eta}(0)$ can be expressed in terms of the difference of the integral along the fibre of a form which is local on M and of the Chern character form in Chern-Weil theory for the index bundle of the given family of Dirac operators acting on the fibres Z.

The process of blowing up the metric on B is called passing to the adiabatic limit.

As explained in Atiyah-Donnelly-Singer [ADS], by using the index theorem of Atiyah-Patodi-Singer [APS1] for manifolds with boundary, the calculation of the signature of the Hilbert modular varieties can be reduced to the evaluation of the adiabatic limit of the eta invariants of the signature operators of certain fibrations, where the fibres are tori which are quotients of $SL(2n, \mathbb{Z})$ vector bundles. These bundles admit natural connections relevant for our problem, which in this specific situation, are unitarily flat.

It turns out that for fibrations by tori, which come from $SL(2n, \mathbb{Z})$ vector bundles E (which are not necessarily unitarily flat), the form $\tilde{\eta}(0)$ can be explicitly calculated. Essentially this is because the spectral theory of the Dirac operators on tori can be reduced to easy computations with Fourier series. As we shall see in Section 2 of the present paper, the form $\tilde{\eta}(0)$ can actually be evaluated in terms of non trivial sums over the lattice $\Lambda \subset E$. These sums can themselves be handled by classical techniques which are used in the analysis on tori, in particular the Poisson summation formula.

What has been described so far led us to quite unexpected results. Recall that in [Su1], Sullivan proved that the Euler class of an oriented $SL(2n, \mathbb{Z})$ vector bundle E with lattice Λ over the base B vanishes rationally. His proof is quite elementary and purely geometrical.

A more sophisticated proof of Sullivan's result can be given by applying the Atiyah-Singer families index Theorem [AS] to the torus fibration E/Λ. Indeed let Z be the fibre of the fibration $E/\Lambda \to B$, and let TZ be the relative tangent bundle to the fibres Z. By applying the families index theorem of Atiyah-Singer [AS], we find that if sign (Z) is the virtual signature bundle over B associated with the fibres Z, and $\mathscr{L}$ is the multiplicative Hirzebruch genus associated with the power series $(x/2)/\tanh(x/2)$, then

$$\mathrm{ch}(\mathrm{sign}(Z)) = \int_Z 2^n \mathscr{L}(TZ) \quad \text{in } H^*(B, \mathbb{Q}) \tag{0.1}$$

Now since TZ is the pullback to E/Λ of the vector bundle E over B, the integral along the fibre in the right-hand side of (0.1) vanishes. Also a direct computation, which is made in equation (2.60), shows that ch (sign (Z)) is proportional to the Euler class e (E) of E, with an invertible factor. From this, one easily finds that the Euler class of E vanishes rationally.

It is then not unnatural that the local families index theorem of Bismut [B1], which is a Chern-Weil theoretic version of the families index Theorem of Atiyah-Singer [AS], permits us to explicitly transgress the Chern character forms of the signature bundle in Chern-Weil theory. This is exactly the role played by the form $\tilde{\eta}(0)$.

By employing Poisson summation like formulas on tori, it turns out one can forget entirely about the families index Theorem, and construct directly a form $\gamma(0)$ which transgresses the Euler form of E in Chern-Weil theory. Moreover, in this specific case, we obtain much more precise results than by using the local families index theorem of [B1]. Essentially, this is because small time asymptotic expansions of heat kernels on flat tori can be completely calculated.

Thus, in retrospect, it becomes apparent that the results of Sullivan [Su1], and Atiyah-Donnelly-Singer [ADS] and Müller [Mü1,2] are deeply related.

This paper is divided into five Sections. In Section 1, we consider a $SL(2n, \mathbb{Z})$ vector bundle E on a manifold B, equipped with a scalar product g^E and an Euclidean connection ∇^E. If R^E is the curvature of ∇^E, we construct a form $\gamma(0)$ such that $d\gamma(0) = \mathrm{Pf}[R^E/2\pi]$, *i.e.* $\gamma(0)$ transgresses the Chern-Weil representative $\mathrm{Pf}[R^E/2\pi]$ of the rational Euler class of E. To construct $\gamma(0)$, we make use of results of Mathai and Quillen [MQ], who calculated explicit representatives of the Thom class of a vector bundle. We also need the Poisson summation formula and a non trivial result on Berezinians [Ma]. Of course in this way, we reobtain the result of Sullivan [Su1].

In Section 2, we interpret the results of Section 1 in the context of the local families index Theorem of [B1]. Also, the form $\tilde{\eta}(0)$ of Bismut-Cheeger [BC] is completely calculated, for the spin complex of E, and also for the signature complex of E.

In Section 3, we calculate the adiabatic limit of the eta invariant of the signature operator of the manifold $M = E/\Lambda$. We cannot directly apply the results of [BC], since the fibres of the fibration $M : E/\Lambda \to B$ have non zero cohomology. Still the proof is an easy adaptation of the methods of [BC]. The adiabatic limit is expressed in terms of the form $\gamma(0)$ and of the eta invariant of a non trivial generalized signature operator $\tilde{D}^E$ on the base B with coefficients in the flat bundle ΛE^*. Our main result is the prototype of a very general result of Dai [D], for general fibrations.

In Section 4, we apply the results of Section 3 to torus fibrations associated to homogeneous vector bundles, and more specifically, to the solvmanifolds which appear as cross sections of Hilbert modular varieties. We thus give a new proof of the conjecture of Hirzebruch [H] on the signature of Hilbert modular varieties.

In Section 5, we put the results of Section 3 in context by making some general observations concerning eta invariants of torus fibrations. Then, using the form $\gamma(0)$ and the differential characters of [CS], we define a secondary characteristic cohomology class $\hat{\chi} \in H^{2n-1}(B, \mathbb{R}/\mathbb{Z})$, for $SL(2n, \mathbb{Z})$ vector bundles, whose Bockstein is the negative of the Euler class. We relate $\hat{\chi}$ to the eta invariant and use this to prove that $\hat{\chi}$ is a torsion class.

I. A canonical transgression of the Euler class of a $\mathrm{SL}(2n, \mathbb{Z})$ vector bundle

Let E be a $\mathrm{SL}(2n, \mathbb{Z})$ oriented vector bundle on a manifold B. A result by Sullivan [Su1] asserts that the Euler class of E vanishes rationally.

Let now g^{E} be an Euclidean product on E. Let ∇^{E} be a connection on E preserving g^{E}, let R^{E} be the curvature of ∇^{E}. Then $\mathrm{Pf}[\mathrm{R}^{\mathrm{E}}/2\pi]$ is the classical Chern-Weil representative of the Euler class of E. The purpose of this Section is to construct a differential form $\gamma(0)$ on B such that $d\gamma(0) = \mathrm{Pf}[\mathrm{R}^{\mathrm{E}}/2\pi]$.

We will use the Mathai-Quillen construction [MQ] of the Thom class of a vector bundle E equipped with a scalar product and a Euclidean connection. The Mathai-Quillen formalism involves some intriguing algebra, which we briefly describe. Combining this algebra with the Poisson summation formula on tori permits us to construct a family of differential forms $\gamma(s)$ $(s \in \mathbb{C})$ depending meromorphically on $s \in \mathbb{C}$. $\gamma(s)$ has a simple pole at $s = \dim \mathrm{E}/2$, with a residue $\tilde{\omega}/(\sqrt{2\pi})^{\dim \mathrm{E}}$. Also $d\gamma(0) = \mathrm{Pf}[\mathrm{R}^{\mathrm{E}}/2\pi]$. The form $\tilde{\omega}$ vanishes for the natural connection to which the local families index Theorem of Bismut [B1] applies. This fact, and more generally, the deep connection of this Section with the local families index Theorem, will be explained in Section 2.

This Section is organized as follows.

In (*a*), we briefly recall the construction by Mathai-Quillen [MQ] of the Thom form of an Euclidean vector bundle with connection.

In (*b*), we introduce a $\mathrm{SL}(2n, \mathbb{Z})$ vector bundle E with metric g^{E}, and we construct a canonical connection ${}^{0}\nabla^{\mathrm{E}}$ on E preserving g^{E}.

In (*c*), by a summation procedure over the dual lattice Λ^* in E^*, we construct forms β_t $(t > 0)$ on B which are closed, and whose cohomology class does not vary with t. By using the Poisson summation formula, we express β_t as a sum over the lattice $\Lambda \subset \mathrm{E}$. In this way, we can study the asymptotics at $t \to 0$ and $t \to +\infty$ of β_t. As $t \to 0$, β_t has an expansion of the type

$$\beta_t = \frac{\omega}{(\sqrt{2\pi t})^{\dim \mathrm{E}}} + O(e^{-c'/t}); \qquad c' > 0.$$

The form ω turns out to be a Berezinian [Ma, p. 166, 167]. By using a fundamental identity on Berezinians, we show that if $\nabla^{\mathrm{E}} = {}^{0}\nabla^{\mathrm{E}}$, then $\omega = 0$. As we shall see in Section 2, this fact is connected with the local families index Theorem of Bismut [B1].

In (*d*) we construct the forms $\gamma(s)$, and we show that $d\gamma(0) = \mathrm{Pf}[\mathrm{R}^{\mathrm{E}}/2\pi]$. We thus obtain a refinement of a result of Sullivan [Su1].

In (*e*), we specialize the results of (*d*) to the case where the connection ∇^{E} is flat.

Finally in (*f*), we express the form $\gamma(s)$ as a sum over the orbits of the lattice Λ^* under the action of $\pi_1(\mathrm{B})$.

(*a*) The Mathai-Quillen conventions. — Let V be an Euclidean oriented vector space of even dimension $2n$. If $\mathrm{A} \in \mathrm{End}(\mathrm{V})$ is antisymmetric, we identify A with the 2-form $\mathrm{X}, \mathrm{Y} \in \mathrm{V} \to \langle \mathrm{X}, \mathrm{AY} \rangle$. In particular $\mathrm{A}^{\wedge 2}, \ldots, \mathrm{A}^{\wedge n}$ denote the exterior powers of

A. Let dx be the oriented volume form on V. By definition, the Pfaffian Pf(A) of A is defined by the relation

$$\frac{A^{\wedge n}}{n!} = \mathrm{Pf}(A)\, dx. \tag{1.1}$$

Assume temporarily that A is invertible. Then $A^{-1} \in \mathrm{End}(V)$ is also antisymmetric. $(A^{-1})^{\wedge 2}, \ldots, (A^{-1})^{\wedge n}$ denote the corresponding exterior powers. In [MQ], Mathai and Quillen observed that the forms $\mathrm{Pf}[A/2\pi]A^{-1}$, $\mathrm{Pf}(A)\ (A^{-1})^{\wedge 2}, \ldots, \mathrm{Pf}[A/2\pi](A^{-1})^{\wedge n}$ are in fact polynomial functions of A, and so they can be extended by continuity to arbitrary (*i.e.* non necessarily invertible) A.

Let now f be a formal power series $f(x) = \sum_{k=0}^{+\infty} a_k x^k$. Let $A \in \mathrm{End}(V)$ be antisymmetric. Then

$$\mathrm{Pf}\left[\frac{A}{2\pi}\right] f(A^{-1}) = \sum_{k=0}^{n} a_k \mathrm{Pf}\left[\frac{A}{2\pi}\right] (A^{-1})^{\wedge k} \tag{1.2}$$

is a well-defined even form on V.

Let * be the usual Hodge operator acting on $\Lambda(V^*)$. Then, one easily verifies that

$$\mathrm{Pf}(A)\, e^{-A^{-1}} = {*e^{A}} \tag{1.3}$$

From (1.3), one can derive useful formulas for the form $\mathrm{Pf}[A/2\pi](A^{-1})^{\wedge k}$ $(0 \leqq k \leqq \dim E)$.

Similarly if $Y \in V$, we identify $A^{-1}Y \in V$ with the 1-form

$$Z \in V \to \langle A^{-1}Y, Z \rangle = -i_Y A^{-1}.$$

Then the form

$$\mathrm{Pf}\left[\frac{A}{2\pi}\right] A^{-1} Y f(A^{-1}) \tag{1.4}$$

is also well-defined for arbitrary antisymmetric $A \in \mathrm{End}(V)$.

Let B be a smooth manifold. Let $\pi : E \to B$ be a Euclidean oriented vector bundle of even dimension $2n$ equipped with a Euclidean connection ∇^E. The connection ∇^E induces a splitting of TE

$$TE = \pi^* E \oplus T^H E \tag{1.5}$$

If $Y \in TE$, let Y^V be the projection of Y on E with respect to the splitting (1.5) of TE.

If $A \in \mathrm{End}(E)$ is antisymmetric, we identify A with the 2-form on the total space of $E : Y, Z \in TE \to \langle Y^V, AZ^V \rangle$. $A^{\wedge 2}, \ldots, A^{\wedge n}$ denote the exterior powers of A in $\Lambda(T^*E)$. If f is a formal power series, if $Y \in E$, then $\mathrm{Pf}[A/2\pi] f(A^{-1})$, $\mathrm{Pf}[A/2\pi] A^{-1} Y f(A^{-1})$ are forms in $\Lambda(T^*E)$.

Let now R^E be the curvature of the connection ∇^E. We identify R^E with $\pi^* R^E$, *i.e.* we consider R^E as a 2-form on the total space of E which takes values in antisymmetric elements of End(E). By antisymmetrization in all indices, $\mathrm{Pf}[R^E/2\pi] f((R^E)^{-1})$, $\mathrm{Pf}[R^E/2\pi] (R^E)^{-1} Y f((R^E)^{-1})$ are differential forms on the total space of E.

$(R^E)^{-1}$ should be considered as a formal form on E of total degree zero. In fact the curvature tensor R^E is a 2-form on B, which lifts to E. $(R^E)^{-1}$ is formally a form of degree -2 on E with values in 2-forms on E, and so it has total degree zero. Similarly $(R^E)^{-1} Y$ must be considered as a form of degree -1. Also, note that if $Y \in E$

$$i_Y \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] f((R^E)^{-1}) = -\mathrm{Pf}\left[\frac{R^E}{2\pi}\right] (R^E)^{-1} Y f'((R^E)^{-1}) \tag{1.6}$$

DEFINITION 1.1. — For $t>0$, let α_t, $\tilde{\alpha}_t$ be the forms of degree $2n$, $2n-1$ respectively on E

$$\left\{ \begin{aligned} \alpha_t &= \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] \exp\left\{-t\left(\frac{|Y|^2}{2} + (R^E)^{-1}\right)\right\} \\ \tilde{\alpha}_t &= \frac{1}{2} \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] (R^E)^{-1} Y \exp\left\{-t\left(\frac{|Y|^2}{2} + (R^E)^{-1}\right)\right\}. \end{aligned} \right. \tag{1.7}$$

By Mathai-Quillen [MQ, Theorem 4.10], we know that for any $t>0$, the form α_t is closed and represents the Thom class of E. In particular, the cohomology class of α_t does not vary with $t>0$. By the transgression formula of Mathai-Quillen [MQ, Section 7], we also know that

$$\frac{\partial}{\partial t} \alpha_t = d\tilde{\alpha}_t. \tag{1.8}$$

Let L be the orientation bundle of B. Since E is oriented, L lifts to the orientation bundle of the total space of E.

Let K be a compact set in E. Let $\| \ \|_{C^1(K)}$ be a natural norm on the set of differential forms on the total space of E with support in K which are continuous and have continuous first derivatives.

The following result is proved in Bismut-Gillet-Soulé [BGS2, Theorem 3.12].

PROPOSITION 1.2. — *Let* K *be a compact set in* B. *There exists a constant* $C>0$ *such that if* μ *is a smooth differential form on* E *with values in* L, *whose support lies in* K, *then for* $t \geqq 1$

$$\left| \int_E \mu \tilde{\alpha}_t \right| \leqq \frac{C}{t^{3/2}} \| \mu \|_{C^1(K)}. \tag{1.9}$$

Following [BGS2, Section 3*f*)], we now set the following definition.

DEFINITION 1.3. — Let ψ be the current on E

$$\psi=-\int_0^{+\infty}\tilde{\alpha}_t\,dt. \tag{1.10}$$

We identify B with the zero section of E. Let δ_B the current of integration on B.

The following result is proved in [BGS2, Theorems 3.14 and 3.15].

THEOREM 1.4. — *The current ψ is locally integrable on* E. *It verifies the equation of currents on* E

$$d\psi=\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]-\delta_B \tag{1.11}$$

Also, we have the identify of locally integrable currents on the total space of E

$$\psi=-\frac{1}{2}\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]\frac{(R^E)^{-1}Y}{(|Y|^2/2)+(R^E)^{-1}}. \tag{1.12}$$

Remark 1.5. — The restriction of ψ of $E\setminus B$ was also considered by Mathai-Quillen [MQ, Section 7]. This restriction is a smooth form which transgresses the closed form $\mathrm{Pf}[R^E/2\pi]$ on $E\setminus B$. Remarkably enough, the form ψ on $E\setminus B$ is exactly the one which was constructed by Chern [Ch] to prove the Chern-Gauss-Bonnet formula. This can be seen easily by using in particular formula (1.3).

Let ε be the oriented volume form along the fibres of E. Using the splitting (1.5) of TE, we can consider ε as a $\dim E$ form on the total space of E, such that if $U\in T^HE$, $i_U\varepsilon=0$.

THEOREM 1.6. — *Assume that the connection ∇^E is flat*, i.e. *that* $R^E=0$. *Then for any* $t>0$, *the following identities hold*

$$\begin{aligned}\alpha_t&=t^{\dim E/2}\exp\left(-t\frac{|Y|^2}{2}\right)\frac{\varepsilon}{(2\pi)^{\dim E/2}}\\ \tilde{\alpha}_t&=\frac{1}{2}t^{(\dim E/2)-1}\exp\left(-t\frac{|Y|^2}{2}\right)\frac{i_Y\varepsilon}{(2\pi)^{\dim E/2}}\end{aligned} \tag{1.13}$$

Also

$$\psi=-\frac{1}{2}\frac{((\dim E/2)-1)!}{(2\pi)^{\dim E/2}}\frac{i_Y\varepsilon}{(|Y|^2/2)^{\dim E/2}}. \tag{1.14}$$

Proof. — Equations (1.13), (1.14) follow from (1.7), (1.12) together with some easy algebra which is left to the reader. □

Remark 1.7. — Using (1.14), one easily verifies that in full generality, the restriction of $-\psi$ to a fibre of E is exactly the solid angle form with respect to the metric g^E.

(*b*) $\mathrm{SL}(2n, \mathbb{Z})$ VECTOR BUNDLES AND EUCLIDEAN CONNECTIONS. — Let B be a connected manifold, and let $\tilde{B}$ be its universal covering. If $\pi_1(B)$ is the fundamental group of B, then $\tilde{\pi}: \tilde{B} \to B$ is a principal $\pi_1(B)$ bundle.

Let $n \in \mathbb{N}$. Let ρ be a group homomorphism from $\pi_1(B)$ into $\mathrm{SL}(2n, \mathbb{Z})$. Let E be the real vector bundle on B

$$E = \tilde{B} \times_\rho \mathbb{R}^{2n} \tag{1.15}$$

Then E is a flat oriented vector bundle on B of dimension $2n$. Let $\Lambda \subset E$ be the lattice

$$\Lambda = \tilde{B} \times_\rho \mathbb{Z}^{2n} \tag{1.16}$$

Let E^* be the dual vector bundle of E, and let $\Lambda^* \subset E^*$ be the dual lattice

$$\Lambda^* = \{ \mu \in E^*; \forall \lambda \in \Lambda; \langle \mu, \lambda \rangle \in 2\pi\mathbb{Z} \}.$$

Let ∇ be the canonical flat connection on E. Then ∇ preserves Λ. By duality, ∇ induces on E^* the corresponding flat connection which preserves Λ^*, and which we still denote by ∇.

Let g^E be a smooth Euclidean product on E. The metric g^E induces an identification $i: E \to E^*$. Note that except when ∇ preserves g^E, the flat connections on E and E^* do not correspond under this identification. Thus $i^{-1}(\Lambda^*)$ is a lattice in E, which is in general not preserved by ∇.

DEFINITION 1.8. — Let ${}^0\theta$ be the 1-form on B with values in self-adjoint endomorphisms of E given by

$${}^0\theta = \frac{1}{2} i^{-1} (\nabla i) \tag{1.17}$$

Let ${}^0\nabla^E$ be the connection on E

$${}^0\nabla^E = \nabla + {}^0\theta \tag{1.18}$$

One easily verifies that ${}^0\nabla^E$ preserves the scalar product g^E. Equivalently, ${}^0\nabla^E$ is a Euclidean connection on E naturally associated to the connection ∇.

Let ${}^0\nabla^{E^*}$ be the connection on E^*

$${}^0\nabla^{E^*} = i({}^0\nabla^E) \tag{1.19}$$

Then ${}^0\nabla^{E^*}$ can be also be obtained from the flat connection on E^* by a formula similar to (1.18).

PROPOSITION 1.9. — *The curvature* ${}^0R^E$ *of the connection* ${}^0\nabla^E$ *is given by*

$${}^0R^E = -({}^0\theta)^2 \tag{1.20}$$

Proof. — From (1.18), we deduce that

$$ {}^0R^E = [\nabla, {}^0\theta] + ({}^0\theta)^2 \tag{1.21}$$

Now

$$ [\nabla, {}^0\theta] = -2\,({}^0\theta)^2 \tag{1.22}$$

Then (1.20) follows from (1.21), (1.22). □

(*c*) MATHAI-QUILLEN'S FORMS AND POISSON SUMMATION FORMULAS. — We make the same assumptions as in Section (1 *b*).

Let ∇^E be a Euclidean connection on E, with curvature R^E. Set

$$ \begin{cases} A = \nabla^E - \nabla \\ \theta = 2\,{}^0\theta - A \end{cases} \tag{1.23}$$

Observe that if $\nabla^E = {}^0\nabla^E$, then $\theta = {}^0\theta$.

PROPOSITION 1.10. — *If* e^* *is a smooth section of* E^*, *then*

$$ \nabla^E i^{-1}(e^*) = i^{-1}(\nabla e^*) - \theta\, i^{-1}(e^*). \tag{1.24}$$

Proof. — Clearly

$$ \nabla^E i^{-1}(e^*) = \nabla i^{-1}(e^*) + A\, i^{-1}(e^*) = i^{-1}(\nabla e^*) + (A - 2\,{}^0\theta)\, i^{-1}(e^*) \tag{1.25}$$

which gives (1.24). □

Note that locally on B, $\mu \in \Lambda^*$ is a well-defined section of E^*, which is tautologically parallel with respect to the flat connection ∇.

We now use the notation of Section (1 *a*). In particular the forms α_t, $\tilde{\alpha}_t$ of Definition 1.1 are calculated using the connection ∇^E and its curvature R^E.

DEFINITION 1.11. — For $t > 0$, let β_t, $\tilde{\beta}_t$ be the forms of degree $2n$, $2n-1$ respectively on B

$$ \begin{cases} \beta_t = \sum_{\mu \in \Lambda^*} (i^{-1}\mu)^* \alpha_t \\ \tilde{\beta}_t = \sum_{\mu \in \Lambda^*} (i^{-1}\mu)^* \tilde{\alpha}_t \end{cases} \tag{1.26}$$

Let θ^* be the 1-form on B with values in End(E) which is the adjoint of θ.

If $A \in \mathrm{End}(E)$ is antisymmetric and invertible, $\theta^* A^{-1} \theta$ is a 2-form on B taking values in self-adjoint endomorphisms of E.

We denote by $1 + \theta^* A^{-1} \theta$ the 2-form on B taking values in symmetric forms on E^* given by

$$ e^*, e'^* \in E^* \to \langle i^{-1}(e^*), (1 + \theta^* A^{-1} \theta)\, i^{-1}(e'^*) \rangle. \tag{1.27}$$

Let $\Delta^{1+\theta^* A^{-1}\theta}$ denote the scalar Laplacian on the fibres of E/Λ associated with the symmetric form $1+\theta^* A^{-1}\theta$ on E^*. If $e_1, \ldots, e_{2n}$ is a base of E for some fibre of E, and if $e^1, \ldots, e^{2n}$ is the corresponding dual base of E^*, then

$$\Delta^{1+\theta^* A^{-1}\theta}=\sum \langle i^{-1}(e^j), (1+\theta^* A^{-1}\theta)\, i^{-1}(e^k)\rangle \frac{\partial^2}{\partial e_j \partial e_k}. \tag{1.28}$$

From now on, we use Mathai-Quillen's conventions described in Section (1 *a*) in expressions like

$$\operatorname{Pf}\left[\frac{A}{2\pi}\right]\operatorname{Tr}\left[\exp\left(\frac{t}{2}\Delta^{1+\theta^* A^{-1}\theta}\right)\right] \tag{1.29}$$

i.e. we replace in (1.29) A^{-1} by $(R^E)^{-1}$.

Let $\operatorname{Vol}(E/\Lambda)$ be the volume of the fibres of E/Λ with respect to the metric g^E.

THEOREM 1.12. — *The forms β_t are closed, and their cohomology class does not depend on $t>0$. Moreover for any $t>0$, the following identities hold*

$$\left\{\begin{aligned} \beta_t &= \operatorname{Pf}\left[\frac{R^E}{2\pi}\right]\sum_{\mu\in\Lambda^*}\exp\left\{\frac{-t}{2}\langle i^{-1}(\mu), (1+\theta^*(R^E)^{-1}\theta)\, i^{-1}(\mu)\rangle\right\},\\ \beta_t &= \operatorname{Pf}\left[\frac{R^E}{2\pi}\right]\operatorname{Tr}\left[\exp\left(\frac{t}{2}\Delta^{1+\theta^*(R^E)^{-1}\theta}\right)\right],\\ \beta_t &= \sum_{\lambda\in\Lambda}\frac{\operatorname{Vol}(E/\Lambda)}{(\sqrt{2\pi t})^{\dim E}}\,\frac{\operatorname{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)}\\ &\qquad\times\exp\left\{\frac{-1}{2t}\langle \lambda, (1+\theta^*(R^E)^{-1}\theta)^{-1}\lambda\rangle\right\}.\end{aligned}\right. \tag{1.30}$$

Proof. — Locally on B, if $\mu\in\Lambda^*$, the form $(i^{-1}\mu)^*\alpha_t$ is closed. Therefore the form β_t is closed. Also, by (1.8), $\frac{\partial\alpha_t}{\partial t}$ is an exact form. Therefore, the form $\frac{\partial\beta}{\partial t}$ is also exact, *i.e.* the cohomology class of β_t does not depend on $t>0$.

By Proposition 1.10, if μ is locally constant in Λ^*, then

$$\nabla^E i^{-1}(\mu) = -\theta\, i^{-1}(\mu) \tag{1.31}$$

Therefore

$$(i^{-1}\mu)^*(R^E)^{-1} = \frac{1}{2}\langle i^{-1}(\mu), \theta^*(R^E)^{-1}\theta\, i^{-1}(\mu)\rangle \tag{1.32}$$

From (1.32), we get the first identity in (1.30). The second identity in (1.30) is a trivial reformulation of the first identity. The third identity in (1.30) follows by Poisson's sumation formula. □

We rewrite the third expression in (1.30) for β_t in the following form

$$(1.33)\quad \beta_t = \frac{\mathrm{Vol}(E/\Lambda)}{(\sqrt{2\pi t})^{\dim E}} \frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)} + \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{\mathrm{Vol}(E/\Lambda)}{(\sqrt{2\pi t})^{\dim E}} \frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)} \exp\left\{\frac{-1}{2t}\langle\lambda,(1+\theta^*(R^E)^{-1}\theta)^{-1}\lambda\rangle\right\}.$$

We emphasize that terms like

$$\frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)}$$

or

$$\frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)} \exp\left\{\frac{-1}{2t}\langle\lambda,(1+\theta^*(R^E)^{-1}\theta)^{-1}\lambda\rangle\right\}$$

cannot be interpreted as the "products" of two or three pieces. They must be calculated in Mathai-Quillen's formalism. First R^E has to be replaced by $A\in\mathrm{End}(E)$ antisymmetric and invertible. Since the obtained expression is polynomial in A, we can finally replace A by R^E.

In the sequel, we will show that under certain circumstances, the first term

$$\frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)},$$

which appears in the right-hand side of (1.33), vanishes. However this does not imply that β_t vanishes. This is very similar to the fact that $x\times(1/x)=1$, even though at 0, $x=0$.

THEOREM 1.13. — *Let ω be the form of degree* 2*n on* B

$$(1.34)\qquad \omega = \mathrm{Vol}(E/\Lambda)\frac{\mathrm{Pf}[R^E/2\pi]}{\det^{1/2}(1+\theta^*(R^E)^{-1}\theta)}$$

Then

$$(1.35)\qquad \omega = \mathrm{Vol}(E/\Lambda)\,\mathrm{Pf}\left[\frac{R^E+\theta\theta^*}{2\pi}\right]$$

Moreover the form ω is exact. In particular if $\nabla^E = {}^0\nabla^E$, *and if* ${}^0\omega=\omega$ *denotes the corresponding differential form, then*

$$(1.36)\qquad {}^0\omega = 0.$$

Proof. — Let A be an antisymmetric invertible matrix in End(E). Note that $E\oplus E$ is a $\mathbb{Z}_2$-graded vector bundle. Then $\mathrm{End}(E\oplus E)$ is a $\mathbb{Z}_2$-graded bundle of algebras, the even (resp. odd) part of $\mathrm{End}(E\oplus E)$ preserves (resp. exchanges) the two copies of E.

Let C be the even element of $\Lambda(T^*B)\hat{\otimes}\operatorname{End}(E\oplus E)$

$$C=\begin{bmatrix} A & \theta \\ -\theta^* & 1 \end{bmatrix}$$

By a formula in [Ma, p. 167], we know that the Berezinian Ber(C) of C is given by

$$\operatorname{Ber}(C)=\frac{\det A}{\det(1+\theta^* A^{-1}\theta)}. \tag{1.37}$$

By another formula in [Ma, p. 166], we also have the formula for Ber(C)

$$\operatorname{Ber}(C)=\det(A+\theta\theta^*). \tag{1.38}$$

From (1.37), (1.38), we deduce that

$$\frac{\det A}{\det(1+\theta^* A^{-1}\theta)}=\det(A+\theta\theta^*) \tag{1.39}$$

The form $\theta\theta^*$ is a 2-form on B taking values in antisymmetric elements of End(E). Therefore the Pfaffian $\operatorname{Pf}(A+\theta\theta^*)$ is well-defined. We claim that by taking square roots in (1.39), we get

$$\frac{\operatorname{Pf}(A)}{\det^{1/2}(1+\theta^* A^{-1}\theta)}=\operatorname{Pf}(A+\theta\theta^*). \tag{1.40}$$

In fact we can multiply θ by $z\in\mathbb{C}$, while leaving θ^* unchanged. Instead of (1.39), we now get

$$\frac{\det A}{\det(1+z\theta^* A^{-1}\theta)}=\det(A+z\theta\theta^*). \tag{1.41}$$

By taking square roots in (1.41) and using analyticity in the variable z, we find that near $z=0$

$$\frac{\operatorname{Pf}(A)}{\det^{1/2}(1+z\theta^* A^{-1}\theta)}=\operatorname{Pf}(A+z\theta\theta^*). \tag{1.42}$$

Since both sides of (1.42) are analytic in $z\in\mathbb{C}$, (1.40) holds.

By replacing A by R^E in (1.40), we get (1.35). If $\nabla^E={}^0\nabla^E$, then $\theta={}^0\theta$. Also by Proposition 1.9, if ${}^0R^E$ is the curvature of ${}^0\nabla^E$, then

$${}^0R^E+({}^0\theta)^2=0 \tag{1.43}$$

Using (1.35), (1.43), we find that the form ${}^0\omega$ associated to ${}^0\nabla^E$ vanishes.

By formula (1.30), as $t\to 0$

$$\beta_t=\frac{1}{(\sqrt{2\pi t})^{\dim E}}\omega+O(t^\infty). \tag{1.44}$$

Since the forms β_t are closed, ω is closed as well. Since ω vanishes for $\nabla^E = {}^0\nabla^E$, a simple deformation argument shows that ω is exact. To prove that ω is exact, we can also use the fact that the cohomology class of β_t does not vary with $t>0$. □

Remark 1.14. — As we shall see in Proposition 1.24, the form ω can be written canonically as a coboundary. Moreover by Remark 2.18, the fact that ${}^0\omega = 0$ can also be viewed as the consequence of the local families index Theorem of Bismut [B1, Theorems 4.12 and 4.16] applied to the fibration $E/\Lambda \to B$.

Incidently, note that by Proposition 1.10, if $\theta = 0$, then $R^E = 0$. Therefore, if $\theta = 0$, $\omega = 0$ as well.

We now study the asymptotics of the forms β_t as $t \to 0$ and $t \to +\infty$.

THEOREM 1.15. — *For any compact* K *in* B, *there exist* $c>0$, $c'>0$ *such that on* K, *as* $t \to +\infty$

$$\beta_t = \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] + O(e^{-ct}). \tag{1.45}$$

and as $t \to 0$

$$\beta_t = \frac{\omega}{(\sqrt{2\pi t})^{\dim E}} + O(e^{-c'/t}). \tag{1.46}$$

Proof. — Equation (1.45) follows from the first line in (1.30) and (1.46) from the third line in (1.30). □

In the theorem which follows, we reobtain a result of Sullivan [Su1], who proved that the Euler class of E vanishes rationally.

THEOREM 1.16. — *The form* $\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]$ *is exact*, i.e. *the Euler class of* E *vanishes in* $H^*(B, \mathbb{Q})$. *For any* $t>0$, *the forms* β_t *are exact.*

Proof. — Let M be a smooth chain without boundary in B. Since the forms β_t are closed and lie in the same cohomology class, $\int_M \beta_t$ does not depend on $t>0$. By Theorem 1.15, it is clear that

$$\left\{\begin{aligned} &\lim_{t\to+\infty} \int_M \beta_t = \int_M \mathrm{Pf}\left[\frac{R^E}{2\pi}\right], \\ &\lim_{t\to 0} \int_M \beta_t = 0. \end{aligned}\right. \tag{1.47}$$

We thus find that $\int_M \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] = 0$. Therefore the forms β_t and $\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]$ are exact. □

Remark 1.17. — The machine which we have just built will permit us to considerably strengthen Theorem 1.16. In fact, we will make the differential form $\mathrm{Pf}[\mathrm{R}^{\mathrm{E}}/2\pi]$ exact in a canonical way.

(*d*) A CANONICAL TRANSGRESSION OF THE EULER FORM OF E.

PROPOSITION 1.18. — *For any $t>0$, the following identity holds*

$$\frac{\partial}{\partial t}\beta_t = d\tilde{\beta}_t. \tag{1.48}$$

Proof. — Equation (1.48) immediately follows from (1.8) and (1.26). □

We now briefly show how to apply the results of Section (1 *c*) to the form $\tilde{\beta}_t$.

We lift the vector bundle E on B to $\mathrm{B}\times\mathrm{R}_+^*$. We continue to denote this bundle by E. For every $s>0$, we equip the vector bundle E over $\mathrm{B}\times\{s\}$ with the metric g^{E}/s. Over $\mathrm{E}\times\mathrm{R}_+^*$, E is now equipped with a metric $g^{\mathrm{E,\,tot}}$.

The flat connection ∇ on E lifts to a connection ∇^{tot}. Let $\nabla^{\mathrm{E,\,tot}}$ be the connection on E over $\mathrm{B}\times\mathrm{R}_+^*$ which has the following two properties:

- For $s>0$, $\nabla^{\mathrm{E,\,tot}}$ restricts to ∇^{E} on $\mathrm{B}\times\{s\}$.
- If X is a smooth section of E over B, then

$$\nabla^{\mathrm{E,\,tot}}_{\partial/\partial s}\mathrm{X} = -\frac{1}{2}\frac{\mathrm{X}}{s}. \tag{1.49}$$

Then one easily verifies that $\nabla^{\mathrm{E,\,tot}}$ preserves the metric $g^{\mathrm{E,\,tot}}$. Also the curvature $\mathrm{R}^{\mathrm{E,\,tot}}$ of $\nabla^{\mathrm{E,\,tot}}$ is the lift of R^{E} to $\mathrm{B}\times\mathrm{R}_+^*$. Finally, as is obvious by (1.17)

$${}^{0}(\nabla^{\mathrm{tot}})^{\mathrm{E}} = {}^{0}\nabla^{\mathrm{E,\,tot}}. \tag{1.50}$$

Let α_t^{tot} be the form α_t on the total space of E over $\mathrm{B}\times\mathrm{R}_+^*$ associated to the connection $\nabla^{\mathrm{E,\,tot}}$.

PROPOSITION 1.19. — *On the total space of* E *over* $\mathrm{B}\times\mathrm{R}_+^*$, *for any $t>0$, the following identity of forms of degree $2n$ holds*

$$\alpha_t^{\mathrm{tot}} = \alpha_{t/s} - \frac{t\,ds}{s^2}\wedge\tilde{\alpha}_{t/s} \tag{1.51}$$

Proof. — The proof of (1.51) is easy and is left to the reader. □

Remark 1.20. — Since the form α_t^{tot} is closed on $\mathrm{B}\times\mathrm{R}_+^*$, we immediately deduce (1.8) from (1.51).

Let now β_t^{tot} be the form β_t on the manifold $\mathrm{B}\times\mathrm{R}_+^*$ associated to the connection $\nabla^{\mathrm{E,\,tot}}$.

PROPOSITION 1.21. — *On* $\mathrm{B}\times\mathrm{R}_+^*$, *for any $t>0$, the following identity of forms of degree $2n$ holds*

$$\beta_t^{\mathrm{tot}} = \beta_{ts} + t\,ds\wedge\tilde{\beta}_{ts}. \tag{1.52}$$

Proof. – Relation (1.52) is an easy consequence of Proposition 1.19. □

Remark 1.22. – Instead of scaling the metric g^E, one can replace the lattice Λ by the scaled lattice $\Lambda/\sqrt{s}$. This also leads directly to Proposition 1.21.

DEFINITION 1.23. – Let $\tilde{\omega}$ be the form of degree $2n-1$ on B

$$\tilde{\omega}=\frac{\mathrm{Vol}(E/\Lambda)}{4}\mathrm{Pf}\left[\frac{R^E+\theta\theta^*}{2\pi}\right]\mathrm{Tr}[(R^E+\theta\theta^*)^{-1}(\theta-\theta^*)] \tag{1.53}$$

Observe that if $\nabla^E={}^0\nabla^E$, since ${}^0\theta={}^0\theta^*$, then $\tilde{\omega}=0$.

PROPOSITION 1.24. – *The following identity holds on* B

$$\omega=-\frac{2}{\dim E}d\tilde{\omega} \tag{1.54}$$

Proof. – Let θ^{tot}, ω^{tot} be the forms θ, ω on $B\times R_+^*$ constructed in (1.23), (1.34). Then one easily verifies that

$$\theta^{tot}=\theta-\frac{ds}{2s} \tag{1.55}$$

Using Theorem 1.12, we find that

$$\omega^{tot}=\frac{\mathrm{Vol}(E/\Lambda)}{s^{\dim E/2}}\mathrm{Pf}\left[\frac{R^E+\theta^{tot}\theta^{tot*}}{2\pi}\right] \tag{1.56}$$

From (1.55), (1.56) we get

$$\omega^{tot}=\frac{\omega}{s^{\dim E/2}}+\frac{ds}{s^{\dim E/2+1}}\wedge\tilde{\omega} \tag{1.57}$$

Since ω^{tot} is closed, we obtain (1.54). □

THEOREM 1.25. – *For any compact set* K *in* B, *there exist* $c>0$, $c'>0$ *such that on* K, *as* $t\to+\infty$

$$\tilde{\beta}_t=O(e^{-ct}) \tag{1.58}$$

and as $t\to 0$

$$\tilde{\beta}_t=\frac{\tilde{\omega}}{(\sqrt{2\pi})^{\dim E}\,t^{\dim E/2+1}}+O(e^{-c'/t}). \tag{1.59}$$

Proof. – We already know that $R^{E,tot}$ is the lift to $B\times R_+^*$ of R^E. We now use Theorem 1.15 applied to the forms β_t^{tot}, ω^{tot} given by formulas (1.52), (1.57), to obtain (1.58), (1.59). □

Let $\Gamma(\Lambda(T^*B))$ be the space of smooth sections of $\Lambda(T^*B)$.

DEFINITION 1.26. — For $s \in \mathbb{C}$, $\operatorname{Re}(s) > \dim E/2$, let $\gamma(s)$ be the smooth form of degree $2n-1$ on B

$$\gamma(s) = \int_0^{+\infty} t^s \tilde{\beta}_t \, dt. \tag{1.60}$$

THEOREM 1.27. — *For* $s \in \mathbb{C}$, $\operatorname{Re}(s) > \dim E/2$, *the following identity holds*

$$\gamma(s) = \Gamma(s+1) \sum_{\substack{\mu \in \Lambda^* \\ \mu \neq 0}} (i^{-1}\mu)^* \left[\frac{(1/2)\operatorname{Pf}[R^E/2\pi](R^E)^{-1} Y}{((|Y|^2/2) + (R^E)^{-1})^{s+1}} \right] \tag{1.61}$$

The map $s \in \mathbb{C}$, $\operatorname{Re}(s) > \dim E/2 \to \gamma(s) \in \Gamma(\Lambda(T^* B))$ *extends to a meromorphic map on* $\mathbb{C}$ *with a simple pole at* $s = \dim E/2$ *and residue* $\tilde{\omega}/(\sqrt{2\pi})^{\dim E}$.

Moreover

$$d\gamma(0) = \operatorname{Pf}\left[\frac{R^E}{2\pi}\right]. \tag{1.62}$$

Proof. — Equation (1.61) follows from (1.7), (1.26) and (1.60). The second statement is a consequence of Theorem 1.25.

Let us now prove (1.62). For $\operatorname{Re}(s) > \dim E/2$, we know that

$$d\gamma(s) = \int_0^{+\infty} t^s d\tilde{\beta}_t \, dt. \tag{1.63}$$

By Proposition 1.18, we find that

$$d\gamma(s) = \int_0^{+\infty} t^s \frac{\partial}{\partial t}\left(\beta_t - \operatorname{Pf}\left[\frac{R^E}{2\pi}\right]\right) dt. \tag{1.64}$$

Using Theorem 1.15 and (1.64), we get

$$d\gamma(s) = -s \int_0^{+\infty} t^{s-1}\left(\beta_t - \operatorname{Pf}\left[\frac{R^E}{2\pi}\right]\right) dt. \tag{1.65}$$

Using Theorem 1.15 again together with (1.65), we obtain (1.62). The proof of Theorem 1.27 is complete. □

Remark 1.28. — From (1.12), (1.61), we find that the form $\gamma(0)$ can be regarded as the pull-back of the form $-\psi$ via the multivalued section $i^{-1}\Lambda^*$. It is also interesting to observe that $\gamma(s)$ can be considered as a generalized Eisenstein series.

(*e*) THE CASE WHERE $R^E = 0$. — We make the same assumptions as in Sections (1 *b*)-(1 *d*). Recall that the form ε on the total space of E associated to the metric g^E and the connection ∇^E was defined in Section (1 *a*).

THEOREM 1.29. — *If* $R^E=0$, *for any* $s\in\mathbb{C}$, $\operatorname{Re}(s)>\dim E/2$, *the following identity holds*

$$\gamma(s)=\sum_{\substack{\mu\in\Lambda^*\\ \mu\neq 0}}(i^{-1}\mu)^*\frac{\Gamma(s+(\dim E/2))}{(2\pi)^{\dim E/2}}\frac{i_Y\varepsilon}{2(|Y|^2/2)^{s+\dim E/2}}. \tag{1.66}$$

Moreover $\gamma(0)$ *is a closed form.*

Proof. — Theorem 1.29 follows immediately from Theorem 1.6 and Definition 1.26. □

Remark 1.30. — Using (1.14), Remark 1.7 and (1.66), we find that if $R^E=0$, the form $\gamma(s)$ is formally the pull-back of a weighted solid angle form by the multivalued section $i^{-1}\Lambda^*$. This fact will play a key role in our proof in Theorem 4.4 of the result of [ADS] and [Mü1,2].

(*f*) THE FORMS $\gamma(s)$ AS SUMS OVER ORBITS. — If $z\in\Lambda^*$, let $[z]$ denote the orbit of z under parallel transport by the flat connection ∇ of z. W can then express the lattice Λ^* as a disjoint union of orbits. The set of orbits will be denoted $\pi_1(B)\diagdown\Lambda^*$. Among the orbits, there is the distinguished orbit 0.

DEFINITION 1.31. — For $[z]\in\pi_1(B)\diagdown\Lambda^*$ a nonzero orbit in Λ^*, and $s\in\mathbb{C}$, $\operatorname{Re}(s)>\dim E/2$, set

$$\gamma_{[z]}(s)=\Gamma(s+1)\sum_{\mu\in[z]}(i^{-1}\mu)^*\frac{1}{2}\frac{\operatorname{Pf}[R^E/2\pi](R^E)^{-1}Y}{((|Y|^2/2)+(R^E)^{-1})^{s+1}} \tag{1.67}$$

PROPOSITION 1.32. — *For* $s\in\mathbb{C}$, $\operatorname{Re}(s)>\dim E/2$, *the following identity of* $\dim E-1$ *forms on* B *holds*

$$\gamma(s)=\sum_{\substack{[z]\in\pi_1(B)\diagdown\Lambda^*\\ [z]\neq 0}}\gamma_{[z]}(s) \tag{1.68}$$

Proof. — Equation (1.68) follows from (1.61) and (1.67). □

Now take $z\in\Lambda^*$.

DEFINITION 1.33. — Let $\pi_1(B)_z$ be the stabilizer of z, *i.e.*

$$\pi_1(B)_z=\{\beta\in\pi_1(B),\tilde{\rho}^{-1}(\beta)z=z\}. \tag{1.69}$$

Set

$$B_z=\pi_1(B)_z\diagdown\tilde{B}. \tag{1.70}$$

Let π_z be the covering map $B_z\to B$. The vector bundle E^* lifts to a vector bundle $\pi_z^*E^*$ on B_z. Tautologically, $\pi_z^*[z]$ has a canonical section μ_z on B_z, which is parallel with respect to the connection $\pi_z^*\nabla$. Thus, we will consider μ_z as a section of $\pi_z^*E^*$ over B_z.

Let $\tilde{\pi}_z$ be the map $\pi_z^*E\to E$.

PROPOSITION 1.34. — *Let* ω *be a smooth form on* B. *Let* $z\in\Lambda^*$, $z\neq 0$. *If* $s\in\mathbb{C}$, $\mathrm{Re}(s)>\dim \mathrm{E}/2$, *then*

$$(1.71)\qquad \int_{\mathrm{B}} \omega\gamma_{[z]}(s)=\int_{\mathrm{B}_z} \pi_z^*\,\omega\,\Gamma(s+1)\,(i^{-1}\mu_z)^*\,\tilde{\pi}_z^*\left\{\frac{1}{2}\,\frac{\mathrm{Pf}[\mathrm{R^E}/2\pi]\,(\mathrm{R^E})^{-1}\mathrm{Y}}{((|\mathrm{Y}|^2/2)+(\mathrm{R^E})^{-1})^{s+1}}\right\}.$$

Proof. — Relation (1.71) is trivial and is left to the reader. □

Remark 1.35. — One easily verifies directly that the right-hand side of (1.71) only depends on the orbit $[z]$.

II. Bundles of tori and the local families index theorem

Again, we consider a $\mathrm{SL}(2n,\mathbb{Z})$ vector bundle E with lattice Λ. Let M be the total space of E/Λ. Let g^{E} be a metric on E. The purpose of this section it to apply the local families index theorem of Bismut [B1] in this situation. Thus, we construct the Levi-Civita superconnection A_t, $t>0$, for a family of Dirac operators acting on the fibres of M, which are associated to the spin complex of the fibres (when E in spin) or to the signature complex of the fibres. The corresponding Chern character forms in the sense of [B1] are expressed very simply in terms of the forms ${}^0\beta_{2t}$ of Section 1 associated to the connection ${}^0\nabla^{\mathrm{E}}$.

By using a construction of Bismut-Cheeger [BC], which relies on a Quillen type of transgression for superconnections [Q], we construct a differential form, noted $\tilde{\eta}(0)$ for the spin complex and $\tilde{\eta}'(0)$ for the signature complex on B. In Section 3, the form $\tilde{\eta}'(0)$ will play a key role in our new proof of the result of Atiyah-Donnelly-Singer [ADS] and Müller [Mü1,2] on the signature of Hilbert modular varieties.

The main result of this Section is to show that both $\tilde{\eta}(0)$ and $\tilde{\eta}'(0)$ are proportional — up to an elementary factor — to the form ${}^0\gamma(0)$, which is the form $\gamma(0)$ associated to the connection ${}^0\nabla^{\mathrm{E}}$ constructed in Section 1. Also, all the remarkable properties of $\tilde{\eta}(0)$ and $\tilde{\eta}'(0)$ which are proved in [BC] by using local families index techniques, here directly follow from Section 1, and in particular from the Poisson summation formula.

This Section is organized as follows. In (*a*), we briefly establish elementary differential geometric results on a torus fibration, in particular in relation with the general setting of Bismut [B1].

In (*b*), we construct the family of Dirac operators D.

In (*c*), we give a very short summary of the superconnection formalism of Quillen [Q].

In (*d*), we construct the Levi-Civita superconnection associated to the family of operators D in the sense of [B1].

In (*e*), we express the form $\tilde{\eta}(0)$ associated to the spin complex of E/Λ in terms of the form ${}^0\gamma(0)$ constructed in Section 1.

Finally in (*f*), we obtain corresponding results for the form $\tilde{\eta}'(0)$.

(*a*) DIFFERENTIAL GEOMETRY OF THE TORUS FIBRATION. — We make the same assumptions and use the same notation as in Sections (1 *b*)-(1 *d*).

Let M be the total space of E/Λ. Then $\pi: M \to B$ is a fibration with fibre $Z = E/\Lambda$. Let TZ be the relative tangent subbundle, *i. e.* the subbundle of TM of vectors tangent to the fibre Z. Clearly, we have a canonical identification

$$TZ = \pi^* E. \tag{2.1}$$

The connection ∇ on E induces the splitting

$$TM = TZ \oplus T^H M. \tag{2.2}$$

In (2.2), $T^H M$ is the horizontal subbundle of TM associated with ∇. Also

$$T^H M = \pi^* TB. \tag{2.3}$$

In view of (2.1)-(2.3), we find that

$$TM = \pi^* E \oplus \pi^* TB. \tag{2.4}$$

If $U \in TB$, let U^H be the horizontal lift of U in $T^H M$. Thus $U^H \in T^H M$, $\pi_* U^H = U$.

The canonical volume for $dx^1 \wedge \ldots \wedge dx^{2n}$ on R^{2n} is invariant under $SL(2n, \mathbb{Z})$. Therefore E carries a natural volume form ε. Clearly

$$\int_Z \varepsilon = 1 \tag{2.5}$$

Let g^E be a smooth Euclidean product on E. g^E induces a metric g^{TZ} on TZ.

We assume that the volume form on E associated to g^E is exactly ε. Therefore the volume of the fibres Z with respect to g^{TZ} is equal 1.

By [B1, Theorem 1.9], the data consisting of g^{TZ}, and the splitting $TM = T^H Z \oplus TM$, uniquely determine an Euclidean connection ∇^{TZ} on $TZ = \pi^* E$.

PROPOSITION 2.1. — *The following identity of connections on* TZ *holds*

$$\nabla^{TZ} = \pi^* {}^0\nabla^E. \tag{2.6}$$

Proof. — Relation (2.6) follows from [BC, eq. (4.8)]. □

Let g^{TB} be a Euclidean metric on TB. We equip $T^H M = \pi^* TB$ with the metric $\pi^* g^{TB}$. Let g^{TM} be the metric on TM which is the orthogonal sum of the metrics g^{TZ} and $\pi^* g^{TB}$ on $TM = TZ \oplus T^H M$. Let $\langle\ ,\ \rangle$ be the corresponding scalar product on TM.

Now, in the special situation considered here, we describe the tensors T and S on M, which were constructed for general fibrations in [B1, Section 1 *c*)].

Note that T is a 2-form on M with values in TZ.

PROPOSITION 2.2. — *If* U *and* V *both lie in* TZ *or in* $T^H M$, *then*

$$T(U, V) = 0. \tag{2.7}$$

If $U \in T^H M$, $X \in TZ$, *then*

$$T(U,X) = {}^0\theta(\pi_* U)X. \tag{2.8}$$

Proof. — By construction [B1, Theorem 1.9], if U, $V \in TZ$, $T(U,V)=0$. If f, f' are smooth sections of TB and if P^{TZ} is the projection operator $TM = TZ \oplus T^H M \to TZ$, one verifies easily that

$$T(f^H, f'^H) = -P^{TZ}[f^H, f'^H] \tag{2.9}$$

Since the connection ∇ on E is flat, $T(f^H, f'^H)=0$. Thus, we have proved (2.7).

Let X be a locally defined parallel section of E with respect to the connection ∇. By [B1, Definition 1.7], we know that

$$T(f^H, X) = {}^0\nabla^E_f X - [f^H, X] \tag{2.10}$$

Clearly $\nabla_f X = [f^H, X]$. Using (1.18), (2.10), we get (2.8). □

By [B1, Definition 1.8 and equation (1.28)], the tensor S is a 1-form on M with values in antisymmetric elements of End(TM) and is characterized by the fact that if X, Y, $Z \in TM$

$$\begin{cases} 2\langle S(X)Y, Z\rangle + \langle T(X,Y), Z\rangle + \langle T(Z,X), Y\rangle - \langle T(Y,Z), X\rangle = 0, \\ S(X)Y - S(Y)X + T(X,Y) = 0. \end{cases} \tag{2.11}$$

PROPOSITION 2.3. — *For any* $X \in TM$, $S(X)$ *interchanges* TZ *and* $T^H M$. *If* X, $Y \in TZ$, U, $V \in T^H M$, *then*

$$\begin{cases} S(X)U = {}^0\theta(\pi_* U)X \\ S(U)X = 0 \\ S(U)V = 0 \\ \langle S(X)Y, U\rangle = -\langle {}^0\theta(\pi_* U)X, Y\rangle = -\langle {}^0\theta(\pi_* U)Y, X\rangle \end{cases} \tag{2.12}$$

Proof. — By [B1, Theorem 1.9], for any $X \in TM$, $S(X)$ maps TZ into $T^H M$. Using Proposition 2.2 and the first identity in (2.11), it is clear that for any $X \in TM$, $S(X)$ maps $T^H M$ into TZ. If X, $Y \in TZ$, U, $V \in T^H M$, then

$$\langle S(U)X, Y\rangle = 0. \tag{2.13}$$

Using the second identity in (2.11) and (2.13), we find that

$$\langle S(X)U, Y\rangle = \langle T(U,X), Y\rangle \tag{2.14}$$

Thus

$$S(X)U = T(U,X) \tag{2.15}$$

The first identity in (2.12) follows from (2.8), (2.15). The second identity in (2.12) is a consequence of the second identity in (2.11) and of (2.15). Moreover

$$\langle S(U)V, X\rangle = -\langle V, S(U)X\rangle. \tag{2.16}$$

Since $S(U)X=0$, from (2.16) we derive the third identity in (2.12). Also

$$\langle S(X)Y, U\rangle = -\langle Y, S(X)U\rangle \tag{2.17}$$

or equivalently

$$\langle S(X)Y, U\rangle = -\langle Y, {}^0\theta(\pi_* U)X\rangle \tag{2.18}$$

Since ${}^0\theta(\pi_* U)$ is self-adjoint, we obtain from (2.18) that

$$\langle S(X)Y, U\rangle = -\langle {}^0\theta(\pi_* U)Y, X\rangle. \tag{2.19}$$

We have thus proved the fourth identity in (2.12). □

Remark 2.4. – The proof of the local families Index Theorem of [B1, Section 4] relies on a remarkable identity verified by the curvature R^{TZ} of the connection ∇^{TZ}. In fact let P^{TZ} be the projection operator from $TM=TZ\oplus T^H M$ on TZ. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}). We denote by ∇' the connection $\nabla^{TZ}\oplus\pi^*\nabla^{TB}$ on $TM=TZ\oplus\pi^* T^H M$. The identity of [B1, Theorem 4.14] or [B2, Theorem 2.3] shows that if $X, Y\in TZ$, $U, V\in TM$ then

$$\begin{aligned}\langle P^{TZ}U, R^{TZ}(X,Y)P^{TZ}V\rangle + \langle U, (\nabla'_X S(Y) - \nabla'_Y S(X))V\rangle \\ -\langle P^{TZ}S(X)U, P^{TZ}S(Y)V\rangle + \langle P^{TZ}S(Y)U, P^{TZ}S(X)V\rangle \\ = \langle X, R^{TZ}(U,V)Y\rangle.\end{aligned} \tag{2.20}$$

By Proposition 2.1, $\nabla^{TZ}=\pi^*\,{}^0\nabla^E$, and so $R^{TZ}=\pi^*\,{}^0R^E$. Using Proposition 2.3, we find that (2.20) is equivalent to

$$\begin{aligned}\langle X, {}^0R^E(\pi_*, U, \pi_* V)Y\rangle \\ = \langle {}^0\theta(\pi_* U)Y, {}^0\theta(\pi_* V)X\rangle - \langle {}^0\theta(\pi_* U)X, {}^0\theta(\pi_* V)Y\rangle,\end{aligned} \tag{2.21}$$

which is exactly Proposition 1.9.

PROPOSITION 2.5. – *Let* λ, μ *be locally constant sections of the lattices* Λ, Λ^* *respectively. Then*

$$\begin{cases} {}^0\nabla^E\lambda = {}^0\theta\lambda \\ {}^0\nabla^E(i^{-1}\mu) = -{}^0\theta\, i^{-1}\mu. \end{cases} \tag{2.22}$$

Proof. – The first identity in (2.22) follows from (1.18). The second identity in (2.22) was proved in Proposition 1.10. □

(*b*) A FAMILY OF DIRAC OPERATORS ASSOCIATED TO THE SPIN COMPLEX OF E/Λ. – We now assume for convenience that E is a spin vector bundle. Let $F^E=F^E_+\oplus F^E_-$ be the

Hermitian $\mathbb{Z}_2$-graded vector bundle of E spinors associated with the metric g^E. Since $TZ = \pi^* E$, TZ is also an oriented spin vector bundle, and $\pi^* F^E = \pi^* F^E_+ \oplus \pi^* F^E_-$ if the corresponding Hermitian $\mathbb{Z}_2$-graded vector bundle of TZ spinors associated with the metric g^{TZ}.

For $x \in B$, $\Gamma_x = \Gamma_{+,x} \oplus \Gamma_{-,x}$ (resp. $\Gamma^0_x = \Gamma^0_{+,x} \oplus \Gamma^0_{-,x}$) denotes the set of smooth (resp. measurable and square integrable) sections of $\pi^* F^E_x = \pi^* F^E_{+,x} \oplus \pi^* F^E_{-,x}$ over the fibre Z_x. We equip Γ^0_x with the obvious L_2 Hermitian product.

Note that for any $x \in B$

$$\left\{\begin{array}{l}\Gamma_x = F^E_x \otimes C^\infty(Z_x; \mathbb{C}) \\ \Gamma^0_x = F^E_x \otimes L_2(Z_x; \mathbb{C}).\end{array}\right. \tag{2.23}$$

In the sequel, we make systematic use of the identification (2.23).

DEFINITION 2.6. — For $x \in B$, $D_x = \begin{bmatrix} 0 & D_{-,x} \\ D_{+,x} & 0 \end{bmatrix}$ denotes the Dirac operator associated with the metric g^E on $TZ_x \cong \pi^* TE_x$ acting on $\Gamma_x = \Gamma_{+,x} \oplus \Gamma_{-,x}$.

If $X \in E$, let $c(X)$ denote the Clifford multiplication operator acting on F^E. If $e_1, \ldots, e_{2n}$ is an orthonormal base of E_x, then

$$D_x = \sum_1^{2n} c(e_i) \nabla_{e_i} \tag{2.24}$$

D_x is an unbounded self-adjoint operator acting on Γ^0_x.

Recall that we identify E^* with E by the metric g^E. Also if $\mu \in \Lambda^*_x$, the function $y \in Z_x \to e^{\sqrt{-1}\langle \mu, y \rangle}$ is well-defined.

THEOREM 2.7. — *For any $x \in B$, we have the following orthogonal splittings of Hilbert spaces*

$$\left\{\begin{array}{l}\Gamma^0_x = \bigoplus_{\mu \in \Lambda^*_x} (F^E_x \otimes \{e^{\sqrt{-1}\langle \mu, y \rangle}\}) \\ \Gamma^0_{\pm,x} = \bigoplus_{\mu \in \Lambda^*_x} (F^E_{\pm,x} \otimes \{e^{\sqrt{-1}\langle \mu, y \rangle}\})\end{array}\right. \tag{2.25}$$

For any $\mu \in \Lambda^*_x$, D_x acts on $F^E_x \otimes \{e^{\sqrt{-1}\langle \mu, y \rangle}\}$ as the Clifford multiplication operator $\sqrt{-1}\, c(i^{-1}\mu) \otimes 1$. Also for any $\mu \in \Lambda^*_x$, $F^E_x \otimes \{e^{\sqrt{-1}\langle \mu, y \rangle}\}$ is an eigenspace of D^2_x, and the corresponding eigenvalue is equal to $|\mu|^2$. In particular

$$\operatorname{Ker} D_{\pm,x} = F^E_{\pm,x} \tag{2.26}$$

Proof. — Relation (2.25) is a standard statement on Fourier series. Using (2.24), it is clear that D_x acts on $F^E_x \otimes \{e^{\sqrt{-1}\langle \mu, y \rangle}\}$ by $\sqrt{-1}\, c(i^{-1}\mu) \otimes 1$. The remaining statements in our Theorem are trivial. □

Remark 2.8. — One can identify the Hilbert space Γ^0_x with a countable direct sum of copies of F^E_x indexed by Λ^*, and if $\mu \in \Lambda^*_x$, D_x acts on the corresponding copy of F^E_x as

$\sqrt{-1}\, c(i^{-1}\mu)$. Note that here, it is essential that the volume of Z with respect to the metric g^{E} is equal to 1.

As in [B1, Section 1*f*)], we now consider Γ, $\Gamma_{\pm}$ as smooth infinite dimensional vector bundles on B.

The Euclidean connection ${}^{0}\nabla^{E}$ on E induces a unitary connection ${}^{0}\nabla^{F^{E}}$ on $F^{E}=F^{E}_{+}\oplus F^{E}_{-}$, which preserves F^{E}_{+} and F^{E}_{-}.

We now define a connection $\tilde{\nabla}$ on Γ as in [B1, Definition 1.10].

DEFINITION 2.9. — If h is a smooth section of Γ, if $f\in TB$, set

$$\tilde{\nabla}_{f} h={}^{0}\nabla^{F^{E}}_{f^{H}} h. \tag{2.27}$$

As in [B1], one verifies easily that $\tilde{\nabla}$ is a connection on Γ.

Let $e_1, \ldots, e_{2n}$ be an orthonormal base of E. By Proposition 2.3, we find that if $U\in TB$

$$\left\langle \sum_{1}^{2n} S(e_i)\, e_i, U^{H} \right\rangle = -\operatorname{Tr}[{}^{0}\theta(U)] \tag{2.28}$$

Since the connection ∇ preserves ε, then $\operatorname{Tr}[{}^{0}\theta]=0$. From (2.28) we get

$$\sum_{1}^{2n} S(e_i)\, e_i=0. \tag{2.29}$$

It follows from [BF, Proposition 1.4] and from (2.29) that $\tilde{\nabla}$ preserves the natural Hermitian product on Γ induced by the Hermitian product of Γ^{0}. Of course, this can be checked directly by using only (2.25), (2.27).

$C^{\infty}(Z,\mathbb{C})$ can also be considered as an infinite dimensional vector bundle on B.

DEFINITION 2.10. — For k a smooth section of $C^{\infty}(Z,\mathbb{C})$ and $f\in TB$, set

$$\nabla^{C^{\infty}}_{f} k=f^{H} k.$$

One easily verifies that $\nabla^{C^{\infty}}$ is a connection on $C^{\infty}(Z,\mathbb{C})$ which preserves the natural L^2 Hermitian product of $C^{\infty}(Z,\mathbb{C})$.

PROPOSITION 2.11. — *Under the identification* $\Gamma=F^{E}\otimes C^{\infty}(Z,\mathbb{C})$, *we have the identities*

$$\left\{\begin{aligned} &\tilde{\nabla}={}^{0}\nabla^{F^{E}}\otimes 1+1\otimes\nabla^{C^{\infty}}\\ &\tilde{\nabla}^{2}=({}^{0}\nabla^{F^{E}})^{2}\otimes 1\end{aligned}\right. \tag{2.30}$$

Proof. — The first line in (2.30) is trivial. Since the connection ∇ is flat, it is clear that $(\nabla^{C^{\infty}})^{2}=0$. The second line in (2.30) follows. □

By Proposition 2.11, it is clear that $\tilde{\nabla}$ preserves the locally defined vector bundles $F^{E}\otimes\{e^{\sqrt{-1}\langle\mu,y\rangle}\}$. Moreover by Theorem 2.7, D also preserves these vector bundles. Therefore for any $f \in TB$, $\tilde{\nabla}_f D$ preserves the vector bundles

$$F^E \otimes \{e^{\sqrt{-1}\langle \mu, y\rangle}\}.$$

PROPOSITION 2.12. – *If* $f \in TB$, $\mu \in \Lambda^*$, *then* $\tilde{\nabla}_f D$ *acts on* $F^E \otimes \{e^{\sqrt{-1}\langle \mu, y\rangle}\}$ *as the operator* $\sqrt{-1}\, c({}^0\nabla^E_f(i^{-1}\mu)) \otimes 1$.

Proof. – By Theorem 2.7, D acts on $F^E \otimes \{e^{\sqrt{-1}\langle \mu, y\rangle}\}$ as $\sqrt{-1}\, c(i^{-1}\mu) \otimes 1$. Using Proposition 2.11, our proposition follows. □

Remark 2.13. – By [B1, Theorem 2.5], we know that if $e_1, \ldots, e_{2n}$ are taken as in (2.24), then

$$\tilde{\nabla}_f D = -\sum_1^{2n} c(e_i) \otimes \nabla_{T(f^H, e_i)}. \tag{2.31}$$

Clearly

$$\nabla_{T(f^H, e_i)} e^{\sqrt{-1}\langle \mu, y\rangle} = \sqrt{-1}\langle \mu, T(f^H, e_i)\rangle e^{\sqrt{-1}\langle \mu, y\rangle}. \tag{2.32}$$

Using Proposition 2.2 and (2.32), we get

$$\nabla_{T(f^H, e_i)} e^{\sqrt{-1}\langle \mu, y\rangle} = \sqrt{-1}\langle e_i, T(f^H, i^{-1}\mu)\rangle e^{\sqrt{-1}\langle \mu, y\rangle}. \tag{2.33}$$

By (2.31), (2.33), we find that $\tilde{\nabla}_f D$ acts on $F^E \otimes \{e^{\sqrt{-1}\langle \mu, y\rangle}\}$ as the operator $\sqrt{-1}\, c(T(i^{-1}\mu, f^H)) \otimes 1$. Now by Propositions 2.2 and 2.5, we know that

$${}^0\nabla^E_f(i^{-1}\mu) = T(i^{-1}\mu, f^H). \tag{2.34}$$

Therefore (2.31), (2.34) fit with Proposition 2.12.

(*c*) THE SUPERCONNECTION FORMALISM OF QUILLEN. – We now briefly describe the superconnection formalism of Quillen [Q].

Let $U = U_+ \oplus U_-$ be a $\mathbb{Z}_2$-graded vector bundle on B. Let $\tau \in \text{End}(U)$ be the involution defining the $\mathbb{Z}_2$-grading, *i.e.* $\tau = \pm 1$ on $U_\pm$. The bundle of algebras End(U) is $\mathbb{Z}_2$-graded, the even (resp. odd) elements of End(U) commuting (resp. anticommuting) with τ. If $C \in \text{End}(U)$, its supertrace $\text{Tr}_s[C]$ is defined by

$$\text{Tr}_s[C] = \text{Tr}[\tau C]$$

We extend $\text{Tr}_s[C]$ to a linear map from the graded tensor product $\Lambda(T^*_{\mathbb{R}} B) \hat{\otimes} \text{End}(U)$ in $\Lambda(T^*_{\mathbb{R}} B)$, with the convention that if $\omega \in \Lambda(T^*_{\mathbb{R}} B)$, $C \in \text{End}(U)$, then

$$\text{Tr}_s[\omega C] = \omega\, \text{Tr}_s[C]$$

Let ∇^U be a connection on U which preserves U_+ and U_-. Let H be a smooth section of $(\Lambda(T^*_{\mathbb{R}} B) \hat{\otimes} \text{End}(U))^{\text{odd}}$. Then $\nabla^U + H$ is a superconnection on the $\mathbb{Z}_2$-graded vector bundle $U = U_+ \oplus U_-$ in the sense of Quillen [Q]. The curvature $(\nabla^U + H)^2$ is a smooth section of $(\Lambda(T^*_{\mathbb{R}} B) \hat{\otimes} \text{End}(U))^{\text{even}}$.

(*d*) THE LEVI-CIVITA SUPERCONNECTION. — We construct the Levi-Civita superconnection in the sense of [B1, Section 3].

DEFINITION 2.14. — For $t>0$, let A_t be the superconnection on the $\mathbb{Z}_2$-graded vector bundle $\Gamma=\Gamma_+\oplus\Gamma_-$

$$A_t=\tilde{\nabla}+\sqrt{t}D. \tag{2.35}$$

By Proposition 2.2, the tensor T vanishes on $T^H M\times T^H M$. Then by comparing with [BF, Proposition 1.18], one verifies that A_t is indeed the Levi-Civita superconnection of the family.

Observe that if μ is a locally constant section of Λ^*, the connection $\tilde{\nabla}$ preserves the locally defined vector bundle $F^E\otimes\{e^{\sqrt{-1}\langle\mu,y\rangle}\}$. If we identify this vector bundle with F^E, then $\tilde{\nabla}$ restricts to ${}^0\nabla^{F^E}$. D also acts on this vector bundle. Therefore the restriction A_t^μ of the superconnection A_t to $F^E\otimes\{e^{\sqrt{-1}\langle\mu,y\rangle}\}\cong F^E$ is (locally) well defined.

Thus from Theorem 2.7 and from the previous considerations, we obtain.

PROPOSITION 2.15. — *If μ is a locally constant section of Δ^*, the restriction A_t^μ of A_t to the vector bundle $F^E\otimes\{e^{\sqrt{-1}\langle\mu,y\rangle}\}\cong F^E$ is given by*

$$A_t^\mu=({}^0\nabla^{F^E}+\sqrt{t}\sqrt{-1}\,c(i^{-1}\mu))\otimes 1. \tag{2.36}$$

The natural Chern-Weil representative of the Chern character of $F_+^E-F_-^E$ associated with the connection ${}^0\nabla^{F^E}$ is the form $\mathrm{Tr}_s[\exp(-({}^0\nabla^{F^E})^2/2i\pi)]$, which we will also denote $\mathrm{ch}(F_+^E-F_-^E)$.

Let $\hat{A}$ be the Hirzebruch polynomial.

PROPOSITION 2.16. — *The following identity holds*

$$\mathrm{ch}(F_+^E-F_-^E)=(-1)^{\dim E/2}\,\mathrm{Pf}\left[\frac{{}^0R^E}{2\pi}\right]\hat{A}^{-1}\left(\frac{{}^0R^E}{2\pi}\right) \tag{2.37}$$

Proof. — Formula (2.37) is an obvious consequence of [ABo, p. 484]. □

Let φ be the map of $\Lambda^{\text{even}}(T^*B)$ into itself which to $\omega\in\Lambda^{2p}(T^*B)$ associates $(2i\pi)^{-p}\omega\in\Lambda^{2p}(T^*B)$.

We will use the notation of Bismut [B1]. By [B1, Theorems 2.6 and 3.6], we know that for any $t>0$, the forms $\varphi\,\mathrm{Tr}_s[\exp(-A_t^2)]$ are closed and represent in cohomology the Chern character of the Atiyah-Singer index bundle $\mathrm{Ker}D_+-\mathrm{Ker}D_-$, which by (2.26) coincides here with $F_+^E-F_-^E$.

As before, we will denote with a ^{0} the objects constructed in Section 1 which are associated to the connection ${}^0\nabla^E$ like ${}^0\alpha_t$, ${}^0\beta_t$, ...

THEOREM 2.17. — *For any $t>0$, the following identity holds*

$$\varphi\,\mathrm{Tr}_s[\exp(-A_t^2)]=(-1)^{\dim E/2}\hat{A}^{-1}\left(\frac{{}^0R^E}{2\pi}\right){}^0\beta_{2t} \tag{2.38}$$

Proof. — Let ${}^0\alpha'_t$ be the form on the total space of E

$$(2.39)\qquad {}^0\alpha'_t=\varphi\,\mathrm{Tr}_s\left[\exp\left(-\left({}^0\nabla^{\mathrm{FE}}+\sqrt{-1}\sqrt{\frac{t}{2}}c(\mathrm{Y})\right)^2\right)\right]$$

By Mathai-Quillen [MQ, Theorem 4.5], we know that

$$(2.40)\qquad {}^0\alpha'_t=(-1)^{\dim \mathrm{E}/2}\,\hat{\mathrm{A}}^{-1}\left(\frac{{}^0\mathrm{R}^{\mathrm{E}}}{2\pi}\right){}^0\alpha_t$$

By Theorem 2.7 and Proposition 2.15, it is clear that

$$(2.41)\qquad \varphi\,\mathrm{Tr}_s[\exp(-\mathrm{A}_t^2)]=\sum_{\mu\in\Lambda^*}(i^{-1}\mu)^*\,{}^0\alpha'_{2t}$$

From (1.26), (2.40), (2.41), we obtain (2.38). □

Remark 2.18. — Let K be a compact subset of B. By Theorems 1.15 and 2.17, we find that there exist $c>0$, $c'>0$ such that on K, as $t\to+\infty$

$$(2.42)\qquad \varphi\,\mathrm{Tr}_s[\exp(-\mathrm{A}_t^2)]=(-1)^{\dim \mathrm{E}/2}\,\mathrm{Pf}\left[\frac{{}^0\mathrm{R}^{\mathrm{E}}}{2\pi}\right]\hat{\mathrm{A}}^{-1}\left(\frac{{}^0\mathrm{R}^{\mathrm{E}}}{2\pi}\right)+O(e^{-ct})$$

and as $t\to 0$

$$(2.43)\qquad \varphi\,\mathrm{Tr}_s[\exp(-\mathrm{A}_t^2)]=O(e^{-c'/t})$$

Because A_t is the Levi-Civita superconnection for parameter value t, the fact that as $t\to 0$, $\varphi\,\mathrm{Tr}_s[\exp(-\mathrm{A}_t^2)]$ has a limit is a special case of the local families index theorem of Bismut [B1, Theorems 4.12 and 4.16]. The proofs of this result given in [B1] and in our paper seem to differ dramatically. Note that in the very special case which is considered here, by Theorem 1.15, any other connection than ${}^0\nabla^{\mathrm{E}}$ is almost as good as ${}^0\nabla^{\mathrm{E}}$ itself for the purpose of obtaining an analogue of (2.43). In particular the asymptotic expansion of the form β_t as $t\to 0$ can be explicitly calculated.

(*e*) A TRANSGRESSION FORMULA. — By proceeding as in [Q], we know that

$$(2.44)\qquad \frac{\partial}{\partial t}\mathrm{Tr}_s[\exp(-\mathrm{A}_t^2)]=-d\,\mathrm{Tr}_s\left[\frac{\mathrm{D}}{2\sqrt{t}}\exp(-\mathrm{A}_t^2)\right]$$

THEOREM 2.19. — *For any $t>0$, the following identity of forms on* B *holds*

$$(2.45)\qquad (2i\pi)^{-1/2}\varphi\,\mathrm{Tr}_s\left[\frac{\mathrm{D}}{2\sqrt{t}}\exp(-\mathrm{A}_t^2)\right]=-2(-1)^{\dim \mathrm{E}/2}\,\hat{\mathrm{A}}^{-1}\left(\frac{{}^0\mathrm{R}^{\mathrm{E}}}{2\pi}\right){}^0\tilde{\beta}_{2t}$$

Proof. — Equation (2.45) follows from a calculation very similar to (2.38). Details are left to the reader. □

Remark 2.20. — By a general result of [BGS1, Theorem 2.11], we know that as $t \to 0$

$$\operatorname{Tr}_s\left[\frac{D}{2\sqrt{t}}\exp(-A_t^2)\right]=O(1). \tag{2.46}$$

Using the notation of Section (1 *d*), we see that $\tilde{\omega}=0$. Let K be a compact subset of B. By Theorems 1.25 and 2.19, we deduce that there exist $c>0$, $c'>0$ such that on K, as $t \to +\infty$

$$\operatorname{Tr}_s\left[\frac{D}{2\sqrt{t}}\exp(-A_t^2)\right]=O(e^{-ct}) \tag{2.47}$$

and that as $t \to 0$

$$\operatorname{Tr}_s\left[\frac{D}{2\sqrt{t}}\exp(-A_t^2)\right]=O(e^{-c'/t}) \tag{2.48}$$

DEFINITION 2.21. — For $s \in \mathbb{C}$, set

$$\tilde{\eta}(s)=\frac{2^s}{(2i\pi)^{1/2}}\int_0^{+\infty} t^s \varphi \operatorname{Tr}_s\left[\frac{D}{2\sqrt{t}}\exp(-A_t^2)\right]dt \tag{2.49}$$

By (2.47), (2.48), $\tilde{\eta}(s)$ is a holomorphic function of $s \in \mathbb{C}$.

Let ${}^0\gamma(s)$ be the form $\gamma(s)$ defined in Definition 1.26, which is associated to the connection ${}^0\nabla^E$.

THEOREM 2.22. — *For any* $s \in \mathbb{C}$, *the following identity holds*

$$\tilde{\eta}(s)=-(-1)^{\dim E/2}\,\hat{A}^{-1}\left(\frac{{}^0R^E}{2\pi}\right){}^0\gamma(s). \tag{2.50}$$

Also

$$d\tilde{\eta}(0)=-(-1)^{\dim E/2}\operatorname{Pf}\left[\frac{{}^0R^E}{2\pi}\right]\hat{A}^{-1}\left(\frac{{}^0R^E}{2\pi}\right). \tag{2.51}$$

Proof. — Equation (2.50) follows immediately from (1.60) and (2.45), while (2.51) is a consequence of (1.62) and (2.50). □

Remark 2.23. — For general fibrations, when D is invertible, the form $\tilde{\eta}(0)$ was first explicitly constructed in Bismut-Cheeger [BC, Definition 4.33 and Remark 4.88], and a local formula over B was given for $d\tilde{\eta}(0)$ in [BC, Theorem 4.35]. More generally, as pointed out in [BC, Section 4 *d*)], for general fibrations, if Ker D_+, Ker D_- are smooth vector bundles over B, the form $\tilde{\eta}(0)$ still makes sense. Equation (2.51) can then be considered as a consequence of [BC].

(*f*) THE SIGNATURE COMPLEX. — We no longer assume that the vector bundle E is spin.

We now briefly state results for signature operators which correspond to those given in Sections (1 *b*), (1 *d*), (1 *e*) for Dirac operators acting on the spin complex.

The Hodge operator * acts naturally on ΛE^*. If $\alpha \in \Lambda^p E^*$, set

$$\tau^E \alpha = i^{n+p(p-1)} *. \tag{2.52}$$

Then $\tau^2 = 1$. Put

$$\Lambda_\pm E^* = \{\alpha \in \Lambda E^*; \tau^E \alpha = \pm \alpha\}. \tag{2.53}$$

Then $\Lambda E^* = \Lambda_+ E^* \oplus \Lambda_- E^*$. In case E is spin, using the notation of Section (2 *b*), we have the identifications

$$\begin{cases} \Lambda E^* = F^E \otimes F^{E^*} \\ \Lambda_\pm E^* = F^E_\pm \otimes F^{E^*} \end{cases} \tag{2.54}$$

To the $\mathbb{Z}_2$-graded vector bundle $\Lambda E^* = \Lambda_+ E^* \oplus \Lambda_- E^*$, we can associate the objects which we associated in Section (2 *b*) to $F = F^E_+ \oplus F^E_-$. To distinguish these objects from the objects of Section (2 *b*), we denote them with a $'$.

For $x \in B$, let d_x be the exterior differentiation operator acting on Γ'_x. Let d_x^* be the adjoint of d_x with respect to the obvious Euclidean product on Γ'^0_x. Set

$$D'_x = d_x + d_x^* \tag{2.55}$$

Then D'_x exchanges $\Gamma'_{+,x}$ and $\Gamma'_{-,x}$. Let $D'_{\pm,x}$ be the restriction of D'_x to $\Gamma'_{\pm,x}$. We write D'_x in matrix form

$$D'_x = \begin{bmatrix} 0 & D'_{-,x} \\ D'_{+,x} & 0 \end{bmatrix} \tag{2.56}$$

By (2.54), ΛE^* is a c(E)-Clifford module, and D'_x is formally given by the right-hand side of (2.24).

The obvious analogue of Theorem 2.7 holds for D'_x. In particular

$$\operatorname{Ker} D'_{\pm,x} = \Lambda_\pm E^*_x. \tag{2.57}$$

The Euclidean connection ${}^0\nabla^E$ induces a Euclidean connection ${}^0\nabla^{\Lambda E^*}$ on ΛE^*, which preserves the splitting $\Lambda E^* = \Lambda_+ E^* \oplus \Lambda_- E^*$. For h a smooth section of Γ', and $f \in TB$, set

$$\tilde{\nabla}'_f h = {}^0\nabla^{\Lambda E^*}_{f^H} h. \tag{2.58}$$

Then the analogues of Propositions 2.11 and 2.12 still hold.

For $t > 0$, set

$$A'_t = \tilde{\nabla}' + \sqrt{t} D'. \tag{2.59}$$

Then A'_t is a superconnection on $\Gamma' = \Gamma'_+ \oplus \Gamma'_-$. The natural Chern-Weil representative of the Chern character of $\Lambda_+ E^* - \Lambda_- E^*$ associated to the connection ${}^0\nabla^{\Lambda E^*}$ is denoted $\mathrm{ch}(\Lambda_+ E^* - \Lambda_- E^*)$. An easy computation shows that

$$\mathrm{ch}(\Lambda_+ E^* - \Lambda_- E^*) = (-2)^{\dim E/2}\, \mathrm{Pf}\left[\frac{{}^0R^E}{2\pi}\right] \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right) \tag{2.60}$$

The analogues of formulas (2.38), (2.45) are now

$$\left\{\begin{aligned} &\varphi\,\mathrm{Tr}_s[\exp(-A_t'^2)] = (-2)^{\dim E/2}\, \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right)\beta_{2t},\\ &(2i\pi)^{-1/2}\varphi\,\mathrm{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right] = -2(-2)^{\dim E/2}\, \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right)\tilde{\beta}_{2t}. \end{aligned}\right. \tag{2.61}$$

Moreover the analogues of (2.47), (2.48) hold as well.

DEFINITION 2.24. — For $s \in \mathbb{C}$, set

$$\tilde{\eta}'(s) = \frac{2^s}{(2i\pi)^{1/2}} \int_0^{+\infty} t^s \varphi\,\mathrm{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right] dt. \tag{2.62}$$

Then $\tilde{\eta}'(s)$ depends holomorphically on $s \in \mathbb{C}$. Of course $\tilde{\eta}'(s)$ should not be confused with the derivative $\partial\tilde{\eta}(s)/\partial s$ of the form $\tilde{\eta}(s)$ described in Section (2 e).

THEOREM 2.25. — *For any $s \in \mathbb{C}$, the following identity holds*

$$\left\{\begin{aligned} &\tilde{\eta}'(s) = -(-2)^{\dim E/2}\, \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right){}^0\gamma(s)\\ &d\tilde{\eta}'(0) = -(-2)^{\dim E/2}\, \mathrm{Pf}\left[\frac{{}^0R^E}{2\pi}\right]\hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right) \end{aligned}\right. \tag{2.63}$$

Proof. — Using (2.61), the proof of Theorem 2.25 is the same as the proof of Theorem 2.22. □

III. Adiabatic limits of eta invariants of signature operators of torus bundles

We make the same assumptions and we use the same notation as in Sections 1 and 2. In addition, we assume that B is compact, oriented and odd dimensional. The purpose of this Section is to calculate the adiabatic limit of the eta invariant of the signature operator on the manifold M in the sense of [BC]. The final answer is expressed in terms of the eta invariant of a certain signature operator on B with coefficients in ΛE^*, and the form $\gamma(0)$ associated to the connection ${}^0\nabla^E$, constructed in Section (1 d).

The formalism of the local families index is still used in this Section. The form $\gamma(0)$ appears through the form $\tilde{\eta}'(0)$ considered in Section (2 f).

This Section is organized as follows. In (*a*), we state some known results on the action of the Clifford algebra of an odd dimensional vector space on its exterior algebra. Such results are implicit in Atiyah-Patodi-Singer [APS1].

In (*b*), we recall the definition in [APS1] of the signature operator of an odd dimensional oriented Riemannian manifold.

In (*c*), we specialize the definition of the signature operator to our torus bundle M. In particular we establish useful formulas for the signature operator of M in terms of various Clifford multiplication operators.

In (*d*), we define a signature operator of B with coefficients in ΛE^*. One peculiar feature of this operator is that it does not preserve the grading in ΛE^*.

In (*e*), we prove that the restriction of the signature operator of M to sections of $\Lambda^{\text{even}}(T^* M)$ which are fibrewise constant is essentially the signature operator of B with coefficients in ΛE^*.

Finally in (*f*), we calculate the adiabatic limit of the eta invariant of the signature operator of M, by an easy adaptation of the techniques of Bismut-Cheeger [BC].

(*a*) Exterior algebras and Clifford algebras: the odd dimensional case. — We here state a few results on Clifford algebras of odd dimensional vector spaces, which are implicit in [APS1], and which we need in a more explicit form.

Let V be an oriented Euclidean vector space of odd dimension $2l-1$. Let $c(V)$ be the Clifford algebra of V.

Let S be the Hermitian vector space of V-spinors. Then S is a $c(V)$ Clifford module of dimension 2^{l-1}. If $e \in V$, with some abuse of notation, we identify $c(e)$ with the corresponding element in End (S). Let $\hat{c}(e)$ denote the negative of the transpose of $c(e)$, which now acts on the dual S^* of S.

We know that

$$\Lambda^{\text{even}}(V^*) = S \otimes S^* \tag{3.1}$$

is a left and right Clifford module. In particular if $e \in E$, $c(e)$ and $\hat{c}(e)$ both act on $\Lambda^{\text{even}}(V^*)$.

Let $*^V$ be the Hodge star operator acting on $\Lambda^{\text{even}}(V^*)$. It depends on the Euclidean metric of V, and on the orientation of V. If $e \in V$, let $e^* \in V^*$ correspond to e under the isomorphism induced by the metric of V.

Then by some tedious calculations, which involve in particular the identification of S with the positive spinors in $V \oplus R$, we find that the actions of $c(e)$, $\hat{c}(e)$ on $\Lambda^{2q}(V^*)$ are given by

$$\begin{cases} c(e) = i^l(-1)^q (e^* \wedge - i_e) *^V \\ \hat{c}(e) = i^l(-1)^q (e^* \wedge + i_e) *^V \end{cases} \tag{3.2}$$

Equivalently

$$\begin{cases} c(e) = i^l(-1)^q (e^* \wedge *^V - *^V e^* \wedge) \\ \hat{c}(e) = i^l(-1)^q (e^* \wedge *^V + *^V e^* \wedge). \end{cases} \tag{3.3}$$

More generally $c(e)$, $\hat{c}(e)$ act on $\Lambda(V^*)$ so that if $\alpha\in\Lambda^p(V^*)$

$$\left\{\begin{array}{l} c(e)\alpha=i^{l+p(p-1)}((-1)^p e^*\wedge *^V - *^V e^*\wedge). \\ \hat{c}(e)\alpha=i^{l+p(p-1)}((-1)^p e^*\wedge *^V + *^V e^*\wedge). \end{array}\right. \tag{3.4}$$

Of course, $c(e)$, $\hat{c}(e)$ preserve $\Lambda^{\text{even}}(V^*)$ and $\Lambda^{\text{odd}}(V^*)$. Let ρ, $\hat{\rho}$, be the one to one linear maps from $\Lambda(V^*)$ to itself, such that if $\alpha\in\Lambda^p(V^*)$, then

$$\left\{\begin{array}{l} \rho(\alpha)=i^{l+p(p-1)} *^V \alpha\in\Lambda^{2l-1-p}(V^*) \\ \hat{\rho}(\alpha)=i^{l+p(p+1)} *^V \alpha\in\Lambda^{2l-1-p}(V^*). \end{array}\right. \tag{3.5}$$

One easily verifies that

$$\left\{\begin{array}{l} c(e)\rho=\rho c(e) \\ \hat{c}(e)\hat{\rho}=\hat{\rho}\hat{c}(e). \end{array}\right. \tag{3.6}$$

Therefore, the representations c (resp. $\hat{c}$) of the Clifford algebra $c(V)$ on $\Lambda^{\text{even}}(V^*)$ and $\Lambda^{\text{odd}}(V^*)$ are equivalent.

(*b*) THE SIGNATURE OPERATOR OF AN ODD DIMENSIONAL ORIENTED MANIFOLD. — Let X be a compact oriented manifold of odd dimension $2l-1$. Let g^{TX} be a smooth Euclidean metric on TX. Let $*^{TX}$ be the Hodge operator acting on $\Lambda(T^*X)$ associated with the metric g^{TX}.

When X is spin, let F be the Hermitian vector bundle of TX spinors associated with the metric g^{TX}. By (3.1), we know that

$$\Lambda^{\text{even}}(T^*X)=F\otimes F^*. \tag{3.7}$$

Let $\tilde{D}$ be the operator acting on $\Gamma(\Lambda^{\text{even}}(T^*X))$, introduced by Atiyah-Patodi-Singer [APS1, p. 63] such that if $\alpha\in\Gamma(\Lambda^{2q}(T^*X))$, then

$$\tilde{D}=i^l(-1)^q(d*^{TX}-*^{TX}d). \tag{3.8}$$

Then $\tilde{D}$ is formally self-adjoint with respect to the natural L_2 Hermitian product on $\Gamma(\Lambda^{\text{even}}(T^*X))$.

Let δ be the adjoint of d with respect to this Hermitian product on $\Gamma(\Lambda(T^*X))$. One easily verifies that

$$\tilde{D}^2=d\delta+\delta d. \tag{3.9}$$

Therefore, the kernel of $\tilde{D}$ consists of the even harmonic froms on X, and so

$$\operatorname{Ker}\tilde{D}\cong H^{\text{even}}(X,\mathbb{C}) \tag{3.10}$$

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}). The connection ∇^{TX} induces a connection $\nabla^{\Lambda^{\text{even}}(T^*X)}$ on $\Lambda^{\text{even}}(T^*X)$, and connections ∇^F, ∇^{F^*} on F, F*. Under the identification (3.7), the connections $\nabla^{\Lambda^{\text{even}}(T^*X)}$ and $\nabla^F\otimes 1+1\otimes\nabla^{F^*}$ on $F\otimes F^*$ can be identified.

Let $e_1, \ldots, e_{2l-1}$ be an orthonormal base of TX. Then as explained in Section (3 *a*), $c(e_1), \ldots, c(e_{2l-1})$ act on $\Lambda^{\text{even}}(T^* X)$. Using (3.3), (3.8), we find easily that

$$\tilde{D} = \sum_{1}^{2l-1} c(e_i) \nabla_{e_i}^{\Lambda^{\text{even}}(T^* X)} \tag{3.11}$$

Therefore $\tilde{D}$ is a classical Dirac operator in the sense of Lichnerowicz [L], Atiyah-Bott-Patodi [ABoP]. It will be called the signature operator associated with the metric g^{TX}.

More generally, let $\tilde{D}^{\text{tot}}$ be the operator acting on $\Gamma(\Lambda(T^* X))$, such that if $\alpha \in \Gamma(\Lambda^p(T^* X))$, then

$$\tilde{D}^{\text{tot}} \alpha = i^{l+p(p-1)}((-1)^p d *^{TX} - *^{TX} d) \alpha \tag{3.12}$$

Then $\tilde{D}^{\text{tot}}$ preserves $\Gamma(\Lambda^{\text{even}}(T^* X))$ and $\Gamma(\Lambda^{\text{odd}}(T^* X))$. Also, the restriction of $\tilde{D}^{\text{tot}}$ to $\Gamma(\Lambda^{\text{even}}(T^* X))$ is exactly $\tilde{D}$. Finally if ρ is the one to one map from $\Lambda(T^* X)$ into $\Lambda(T^* X)$ defined as in (3.5), then by (3.4), (3.6), (3.12)

$$\tilde{D}^{\text{tot}} \rho = \rho \tilde{D}^{\text{tot}}. \tag{3.13}$$

Equivalently, $\tilde{D}^{\text{tot}}$ is the direct sum of two operators equivalent to $\tilde{D}$.

(*c*) THE SIGNATURE OPERATOR OF M. — We now use the notation of Section (2 *a*). We will assume that B is compact, oriented, and has odd dimension $2k-1$. Then M is a compact oriented manifold, of odd dimension $2n+2k-1$.

Let g^{TB} be an Euclidean metric on TB. For $\varepsilon > 0$, let g_ε^{TM} be the metric on $TM = T^H M \oplus \pi^* E$

$$g_\varepsilon^{TM} = \frac{1}{\varepsilon} \pi^* g^{TB} \oplus \pi^* g^E \tag{3.14}$$

Using the identification (2.4), we also have the identification

$$\Lambda(T^* M) = \pi^*(\Lambda(E^*) \hat{\otimes} \Lambda(T^* B)) \tag{3.15}$$

To make our arguments simpler we will for convenience assume that TB and E are spin. Since in essence, all our constructions are purely local, this assumption can actually be dropped everywhere.

Let F^{TB}, $F^E = F_+^E \oplus F_-^E$ be the vector bundles of TB, E spinors associated with the metrics g^{TB}, g^E. Then the vector bundle F^{TM} of TM spinors associated with the metric g_1^{TM} is given by

$$F^{TM} = \pi^*(F^E \otimes F^{TB}). \tag{3.16}$$

The Clifford algebra $c(E)$ acts naturally on $F^E \otimes F^{TB}$. If $U \in TB$, $f \in F^E$, $g \in F^{TB}$, the action of the Clifford multiplication operator $c(U)$ on $f \otimes g$ is given by

$$c(U)(f \otimes g) = (-1)^{\deg(f)} f \otimes c(U) g \tag{3.17}$$

We then obtain an action of the Clifford algebra $c(\mathrm{TM})=c(\mathrm{E})\hat{\otimes} c(\mathrm{TB})$ on F^{TM} which is exactly the natural action of $c(\mathrm{TM})$ on F^{TM}. Also

$$\mathrm{F}^{\mathrm{TM}^*}=\pi^*(\mathrm{F}^{\mathrm{E}^*}\otimes \mathrm{F}^{\mathrm{TB}^*}). \tag{3.18}$$

The analogue of (3.17) holds for the negative of the adjoint action $\hat{c}$ of the Clifford algebra of TM on both sides of (3.18).

By (3.7), we know that.

$$\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{M})=\mathrm{F}^{\mathrm{TM}}\otimes \mathrm{F}^{\mathrm{TM}^*} \tag{3.19}$$

Using (3.16), (3.18), (3.19), we get

$$\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{M})=\pi^*((\mathrm{F}^{\mathrm{E}}\otimes \mathrm{F}^{\mathrm{TB}})\otimes(\mathrm{F}^{\mathrm{E}^*}\otimes \mathrm{F}^{\mathrm{TB}^*})). \tag{3.20}$$

On the other hand

$$\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{M})=\pi^*((\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{B})\hat{\otimes}\Lambda^{\mathrm{even}}(\mathrm{E}^*))\oplus(\Lambda^{\mathrm{odd}}(\mathrm{T}^*\mathrm{B})\hat{\otimes}\Lambda^{\mathrm{odd}}(\mathrm{E}^*))) \tag{3.21}$$

Let $*^{\mathrm{TB}}$ be the Hodge operator acting on $\Lambda(\mathrm{T}^*\mathrm{B})$ associated to the metric g^{TB}. By (3.5) the linear map

$$\alpha\in\Lambda^{2q}(\mathrm{T}^*\mathrm{B})\to i^k(-1)^q *^{\mathrm{TB}}\omega\in\Lambda^{2k-1-2q}(\mathrm{T}^*\mathrm{B}) \tag{3.22}$$

provides a canonical identification

$$\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{B})\cong\Lambda^{\mathrm{odd}}(\mathrm{T}^*\mathrm{B}) \tag{3.23}$$

Relation (3.23) reconciles (2.54), (3.20), (3.21) and (3.22).

Definition 3.1. – For $\varepsilon>0$, let $\tilde{\mathrm{D}}_\varepsilon$ denote the signature operator acting on $\Gamma(\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{M}))$ associated to the metric $g_\varepsilon^{\mathrm{TM}}$.

Let ∇^{TB} be the Levi-Civita connection on TB associated to the metric g^{TB}. Let $\nabla^{u,\mathrm{TM}}$ be the connection on $\mathrm{TM}=\pi^*(\mathrm{E}\oplus\mathrm{TB})$

$$\nabla^{u,\mathrm{TM}}=\pi^*({}^0\nabla^{\mathrm{E}}\oplus\nabla^{\mathrm{TB}}). \tag{3.24}$$

Then $\nabla^{u,\mathrm{TM}}$ preserves the metrics $g_\varepsilon^{\mathrm{TM}}$ on TM.

The connection $\nabla^{u,\mathrm{TM}}$ induces a connection on $\Lambda(\mathrm{T}^*\mathrm{M})$ which we denote by ∇^u.

Let $e_1,\ldots,e_{2n}$ be an orthonormal base of E, and let $f_1,\ldots,f_{2k-1}$ be an orthonormal base of TB. Let $e^1,\ldots,e^{2n}$ and $f^1,\ldots,f^{2k-1}$ be the corresponding dual bases of E^* and $\mathrm{T}^*\mathrm{B}$.

Definition 3.2. – Let D', D^{H}, D'^{H} be the operators acting on

$$\Gamma(\Lambda^{\mathrm{even}}(\mathrm{T}^*\mathrm{M}))=\Gamma(\pi^*((\mathrm{F}^{\mathrm{E}}\otimes\mathrm{F}^{\mathrm{E}^*})\otimes(\mathrm{F}^{\mathrm{TB}}\otimes\mathrm{F}^{\mathrm{TB}^*})))$$

given by

$$(3.25)\quad \begin{cases} D' = \sum_1^{2n} c(e_i) \nabla^u_{e_i} \\ D^H = \sum_1^{2k-1} c(f_\beta) \nabla^u_{f^H_\beta} \\ D'^H = \sum_1^{2k-1} c(f_\beta) \nabla^u_{f^H_\beta} + \frac{1}{2} \sum_{\substack{1 \leq i, j \leq 2n \\ 1 \leq \beta \leq 2k-1}} \langle {}^0\theta(f_\beta) e_i, e_j \rangle c(e_i) \hat{c}(f_\beta) \hat{c}(e_j) \end{cases}$$

PROPOSITION 3.3. – *The following identity of operators acting on* $\Gamma(\Lambda^{\text{even}}(T^* M))$ *holds*

$$(3.26)\quad \tilde{D}_\varepsilon = \sqrt{\varepsilon}\, D'^H + D'.$$

Proof. – Let ∇^{TM}_ε be the Levi-Civita connection on TM associated to the metric g^{TM}_ε on TM. Let $\nabla^{F^{TM^*}}_\varepsilon$ be the connection on F^{TM^*} induced by the connection ∇^{TM}_ε. Let $\nabla^{u, F^{TM}}$, $\nabla^{u, F^{TM^*}}$ be the connections on F^{TM}, F^{TM^*} induced by the connection $\nabla^{u, TM}$. Let ∇^u_ε be the connection on $F^{TM} \otimes F^{TM^*}$

$$(3.27)\quad \nabla^u_\varepsilon = \nabla^{u, F^{TM}} \otimes 1 + 1 \otimes \nabla^{F^{TM^*}}_\varepsilon.$$

Note that $\tilde{D}_\varepsilon$ is a standard Dirac operator acting on smooth sections of $F^{TM} \otimes F^{TM^*}$ associated with the Levi-Civita connection ∇^{TM}_ε. Recall that by Proposition 2.2, T vanishes on $T^H M \times T^H M$.

Using a formula in Bismut-Cheeger [BC, eq. (4.26)] – in which the twisting bundle with connection (ξ, ∇^ξ) is in the present case $(F^{TM^*}, \nabla^{F^{TM^*}}_\varepsilon)$ – we find that

$$(3.28)\quad \tilde{D}_\varepsilon = \sum_1^{2n} c(e_i) \nabla^u_{\varepsilon, e_i} + \sqrt{\varepsilon} \sum_1^{2k-1} c(f_\beta) \nabla^u_{\varepsilon, f^H_\beta}$$

Recall that the tensor S was introduced in [B1, Definition 1.8] and calculated on M in Proposition 2.3. In particular by [B1, Definition 1.8], we know that $\nabla^{TM}_1 = \nabla^{u, TM} + S$. Then we know that

$$(3.29)\quad \nabla^{F^{TM^*}}_\varepsilon = \nabla^{u, F^{TM^*}} + \frac{1}{4} \langle S(\cdot) e_i, e_j \rangle \hat{c}(e_i) \hat{c}(e_j) + \frac{\varepsilon}{4} \langle S(\cdot) f_\alpha, f_\beta \rangle \hat{c}(f_\alpha) \hat{c}(f_\beta) + \frac{\sqrt{\varepsilon}}{2} \langle S(\cdot) f_\beta, e_j \rangle \hat{c}(f_\beta) \hat{c}(e_j)$$

Using Proposition 2.3 and (3.28), (3.29), we get (3.26). □

(*d*) THE SIGNATURE OPERATOR ON $\Lambda(E^*)$. – Assume temporarily that E is a spin vector bundle. Let $\tau^E = \pm 1$ on $F^E_\pm$. Then τ^E acts naturally on $\Lambda(E^*) = F^E \otimes F^{E^*}$.

Let $*^E$ be the Hodge operator acting on $\Lambda(E^*)$. By (2.52)-(2.54), if $\alpha \in \Lambda^p(E^*)$, then

$$(3.30)\quad \tau^E \alpha = i^{n+p(p-1)} *^E \alpha.$$

Of course, (3.30) still makes sense even if E is not spin.

In the sequel we will consider the ungraded tensor product $\Lambda(T^*B)\otimes\Lambda(E^*)$, on which the operators $*^{TB}$ and τ^E act naturally.

Recall that E is a flat vector bundle. Therefore the operator d acts naturally on smooth sections of $\Lambda(T^*B)\otimes\Lambda(E^*)$ over B.

DEFINITION 3.4. — Let $\alpha\in\Gamma((\Lambda^{2q}(T^*B)\otimes\Lambda(E^*))$. The signature operator $\tilde{D}^E$ acting on smooth sections of $\Lambda^{even}(T^*B)\otimes\Lambda(E^*)$ over B is defined by

$$\tilde{D}^E\alpha=i^k(-1)^q(d*^{TB}\tau^E-*^{TB}\tau^E d)\alpha \tag{3.31}$$

One easily verifies that $\tilde{D}^E$ is an elliptic operator which is formally self-adjoint with respect to the natural L_2 Hermitian product on $\Gamma((\Lambda^{even}(T^*B)\otimes\Lambda(E^*))$.

Since $(\tau^E)^2=1$, $d^2=0$, we get

$$(\tilde{D}^E)^2=d*^{TB}\tau^E d*^{TB}\tau^E-*^{TB}\tau^E d*^{TB}\tau^E d \tag{3.32}$$

Because d does not change the partial degree in $\Lambda(E^*)$, we deduce from (3.30) that

$$(\tilde{D}^E)^2=d(*^{TB})^{-1}(*^E)^{-1}d*^{TB}*^E-(*^{TB})^{-1}(*^E)^{-1}d*^{TB}*^E d \tag{3.33}$$

Let δ be the adjoint of d. The volume form of E is parallel with respect to the flat connection ∇. We thus easily deduce that if α is a smooth section of $\Lambda^q(T^*B)\otimes\Lambda(E^*)$, then

$$\delta\alpha=(-1)^q(*^{TB})^{-1}(*^E)^{-1}d*^{TB}*^E. \tag{3.34}$$

Using (3.33), (3.34), we get

$$(\tilde{D}^E)^2=d\delta+\delta d. \tag{3.35}$$

Let $H(B,\Lambda E^*)$ be the cohomology groups on B with coefficients in the flat $\mathbb{Z}$-graded vector bundle ΛE^*. From (3.35), we deduce that there is a canonical isomorphism of $\mathbb{Z}$-graded vector spaces

$$\operatorname{Ker}\tilde{D}^E\cong H^{even}(B,\Lambda E^*) \tag{3.36}$$

(*e*) THE RESTRICTION OF THE OPERATOR D'^H TO CONSTANT SECTIONS ALONG THE FIBRES OF M.

DEFINITION 3.5. — Set

$$K=\{\alpha\in\Gamma(\Lambda^{even}(T^*M)),\ D'\alpha=0\}. \tag{3.37}$$

By the proof of Theorem 2.7, it is clear that if $\alpha\in\Gamma(\Lambda^{even}(T^*M))$, then $D'\alpha=0$ if and only if α is constant along the fibres Z of M. Therefore K can be identified with the set $\Gamma(\Lambda^{even}(T^*B\otimes E^*))$ of smooth sections of $\Lambda^{even}(T^*B\otimes E^*)$ on B.

Using (3.23), we get a canonical isomorphism of vector bundles on B

$$\Lambda^{even}(T^*B\otimes E^*)\cong\Lambda^{even}(T^*B)\otimes\Lambda(E^*). \tag{3.38}$$

Clearly the operators D^H, D'^H preserve the vector space K. Let D_0^H, $D_0'^H$ be the restrictions of D^H, D'^H to K. By (3.38), D_0^H, $D_0'^H$ act on $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*))$.

PROPOSITION 3.6. — *The following identity of operators acting on* $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*))$ *holds*

$$D_0'^H=\tilde{D}^E. \tag{3.39}$$

Proof. — We use the notation of Definition 3.2. The action of $c(f_\beta)$ on $\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*)$ is given by $c(f_\beta)\,\tau^E$. Let $\tilde{\tau}^E$ be the operator defining the $\mathbb{Z}_2$-grading on $F^{E^*}=F_+^{E^*}\oplus F_-^{E^*}$. $\tilde{\tau}^E$ acts naturally on $\Lambda(E^*)=F^E\otimes F^{E^*}$. One verifies that if $\alpha\in\Lambda^p(E^*)$, then

$$\tilde{\tau}^E\alpha=i^{l+p(p+1)}*^E\alpha. \tag{3.40}$$

Equivalently

$$\tilde{\tau}^E\alpha=(-1)^p\tau^E\alpha. \tag{3.41}$$

Then $\hat{c}(f_\beta)$ acts on $\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*)$ by $\hat{c}(f_\beta)\tilde{\tau}^E$.

If $e\in E$, let $c(e)$ denote the natural action of e on F^E and let $\hat{c}(e)$ be the negative of the transpose of $c(e)$, which acts on F^{E^*}. Then if $e, f\in E$, $c(e)$ and $\hat{c}(f)$ both act on $\Lambda(E^*)=F^E\otimes F^{E^*}$ and commute. Also we find easily that if $e\in E$, $\alpha\in\Lambda(E^*)$, then

$$\begin{cases} c(e)\alpha=(e^*\wedge -i_e)\alpha \\ \hat{c}(e)\alpha=(-1)^{\deg\alpha+1}(e^*\wedge +i_e) \end{cases} \tag{3.42}$$

More generally $c(e)$, $\hat{c}(e)$ also act on $\Lambda(T^*B)\hat{\otimes}\Lambda(E^*)$.

Recall that ${}^0\theta$ is a 1-form on B taking values in self-adjoint elements of E which have zero trace. Thus, one easily finds that if $\alpha\in\Lambda^{\text{even}}(T^*B)\otimes\Lambda^p(E^*)$, then

$$\frac{1}{2}\sum_{\substack{1\leq i,j\leq 2n\\ 1\leq\beta\leq 2k-1}}\langle{}^0\theta(f_\beta)e_i,e_j\rangle\, c(e_i)\hat{c}(f_\beta)\hat{c}(e_j) =(-1)^{p+1}\sum_{\substack{1\leq i,j\leq 2n\\ 1\leq\beta\leq 2k-1}}\langle{}^0\theta(f_\beta)e_i,e_j\rangle\,\hat{c}(f_\beta)\,e_i\wedge i_{e_j}. \tag{3.43}$$

Also, one checks that

$$\tau^E e^i\wedge i_{e_j}=-e^j\wedge i_{e_i}\tau^E \tag{3.44}$$

Using (3.4), (3.25), (3.41), (3.43), we find that for $\alpha\in\Gamma(\Lambda^{2q}(T^*B)\otimes\Lambda^p(E^*))$, then

$$\begin{aligned} D'^H\alpha={}&i^k(-1)^q\sum_1^{2k-1}(f^\beta\wedge\nabla^u_{f_\beta}*^{TB}\tau^E-*^{TB}\tau^E f^\beta\wedge\nabla^u_{f_\beta})\alpha \\ &+i^k(-1)^{p+1+q}\Big(\sum_{\substack{1\leq i,j\leq 2n\\ 1\leq\beta\leq 2k-1}}(f^\beta\wedge *^{TB}+*^{TB}f^\beta\wedge)\tau^E\Big) \\ &(-1)^p\langle{}^0\theta(f_\beta)e_i,e_j\rangle\, e^i\wedge i_{e_j} \end{aligned} \tag{3.45}$$

From (3.44), (3.45), we get

$$(3.46)\quad D'^{H}\alpha = i^{k}(-1)^{q} \sum_{1\leq\beta\leq 2k-1} f^{\beta}\wedge\Big(\nabla^{u}_{f_\beta} + \sum_{1\leq i,j\leq 2n} \langle {}^{0}\theta(f_\beta)e_i, e_j\rangle e^{i}\wedge i_{e_j}\Big) *^{TB}\tau^{E}$$
$$- i^{k}(-1)^{q} *^{TB}\tau^{E} \sum_{1\leq\beta\leq 2k-1} f^{\beta}\wedge(\nabla^{u}_{f_\beta} + \langle {}^{0}\theta(f_\beta)e_i, e_j\rangle e^{i}\wedge i_{e_j})$$

By (1.18), it is clear that

$$(3.47)\qquad d = \nabla^{u} + \sum_{1\leq\beta\leq 2k-1} f^{\beta}\wedge \sum_{1\leq i,j\leq 2n} \langle {}^{0}\theta(f_\beta)e_i, e_j\rangle e^{i}\wedge i_{e_j}$$

From (3.31), (3.47), we get (3.39). □

Remark 3.7. – Set $\tilde{D} = \tilde{D}_1$. Let $*^{TM}$ be the Hodge operator associated to the metric g_1^{TM}. The operator $D_0'^{H}$ is exactly the restriction of the operator $\tilde{D}$ to the vector space K. By Proposition 3.6, the restriction of $\tilde{D}$ to K is then equal to $\tilde{D}^{E}$. However, when one compares formulas (3.8) and (3.31), it is not clear that these two operators really coincide. This is because of non trivial sign difficulties. Here, we will verify that our signs are indeed consistent. First observe that if $\alpha\in\Lambda(T^*B)$, $\beta\in\Lambda(E^*)$, then

$$(3.48)\qquad *^{TM}(\alpha\beta) = (-1)^{(\deg\alpha-1)\deg\beta} *^{TB}\alpha *^{E}\beta.$$

By (3.12), the operator $\tilde{D}^{tot} = \tilde{D}_1^{tot}$ acting on $\Gamma(\Lambda(T^*M))$ is such that if $\omega\in\Gamma(\Lambda^{r}(T^*M))$, then

$$(3.49)\qquad \tilde{D}^{tot}\omega = i^{n+k+r(r-1)}((-1)^{r} d *^{TM} - *^{TM} d)\omega$$

Let α, β be smooth sections of $\Lambda^{2q}(T^*B)$, $\Lambda^{p}(E^*)$ over B. As before, we regard $\alpha\beta$ as a smooth section of $\Lambda(T^*M)$ over M. By (3.48), (3.49), we get

$$(3.50)\qquad \tilde{D}^{tot}(\alpha\beta) = i^{n+k+(2q+p)(2q+p-1)}(d *^{TB} *^{E} - *^{TB} *^{E} d)(\alpha\beta).$$

Equivalently

$$(3.51)\qquad \tilde{D}^{tot}(\alpha\beta) = i^{n+k+p(p-1)}(-1)^{q}(d *^{TB} *^{E} - *^{TB} *^{E} d)(\alpha\beta).$$

Using (3.26), (3.30), (3.51), we get (3.39).

(*f*) THE ADIABATIC LIMIT OF THE ETA INVARIANT OF THE SIGNATURE OPERATOR OF M. – By [APS2, p. 83], [BF, Theorem 2.4], we know that as $t\to 0$

$$(3.52)\qquad \mathrm{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^{2})] = O(\sqrt{t})$$

Let $\eta_\varepsilon^{M}(s)$ and $\eta^{B,E}(s)$ be the eta functions associated to the self-adjoint elliptic operators $\tilde{D}_\varepsilon$ and $\tilde{D}^{E}$, as defined by Atiyah-Patodi-Singer [APS1]. For $s\in\mathbb{C}$,

$\operatorname{Re}(s) > \dim M$,

$$(3.53)\qquad \begin{cases} \eta_\varepsilon^{M}(s) = \dfrac{1}{\Gamma((s+1)/2)} \displaystyle\int_0^{+\infty} t^{(s-1)/2}\, \operatorname{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2)]\, dt \\ \eta^{B,E}(s) = \dfrac{1}{\Gamma((s+1)/2)} \displaystyle\int_0^{+\infty} t^{(s-1)/2}\, \operatorname{Tr}[\tilde{D}^E \exp(-t(\tilde{D}^E)^2)]\, dt \end{cases}$$

Then by [APS1, Theorem 4.2], $\eta_\varepsilon^M(s)$ and $\eta^{B,E}(s)$ extend to meromorphic functions of $s \in \mathbb{C}$, which are holomorphic at $s=0$.

By (3.10), we know that dim Ker $\tilde{D}_\varepsilon$, does not depend on $\varepsilon > 0$. Therefore by an easy adaptation of the argument of Bismut-Cheeger [BC, Proposition 4.3], we know that as $\varepsilon \to 0$, $\eta_\varepsilon^M(0)$ has a limit in $\mathbb{R}$, which we note $\eta_0^M(0)$.

Let $\mathscr{L}$ be the ad-invariant power series defined on $(2k-1, 2k-1)$ antisymmetric matrices such that if C has diagonal blocks $\begin{bmatrix} 0 & x_i \\ -x_i & 0 \end{bmatrix}$ $(1 \leq i \leq k-1)$ and 0, then

$$(3.54)\qquad \mathscr{L}(C) = \prod_1^{k-1} \frac{x_i/2}{\tanh(x_i/2)}$$

The series $\mathscr{L}$ defines a rescaled Hirzebruch genus.

Let R^{TB} be the curvature of the connection ∇^{TB}. Recall that the form ${}^0\gamma(0)$ is the form $\gamma(0)$ of Section (1 *d*) associated to the connection ${}^0\nabla^E$.

THEOREM 3.8. — *The following identity holds*

$$(3.55)\qquad \eta_0^M(0) = \eta^{B,E}(0) + (-1)^{(\dim E/2)+1}\, 2^{(\dim E + \dim B + 1)/2} \times \int_B \mathscr{L}\left(\frac{R^{TB}}{2\pi}\right) \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right) {}^0\gamma(0)$$

Also $\eta_0^M(0)$ *does not depend on the metrics* g^{TB} *and* g^E.

Proof. — We cannot apply directly the results of [BC, Section 4 *b*)] because the family of operators D′ is not fibrewise invertible. In fact, by (2.57), we know that for any $x \in B$

$$(3.56)\qquad \operatorname{Ker} D'_x = \Lambda(E^*)_x$$

Let P be the orthogonal projection operator from $\Gamma(\Lambda^{\text{even}}(T^*M))$ with respect to the standard Hermitian product on $\Gamma(\Lambda^{\text{even}}(T^*M))$ onto the vector space of smooth sections of $\Lambda^{\text{even}}(T^*M)$ which are fibrewise constant (*i.e.* those on which the family of operators D′ vanishes). Set $Q = 1 - P$. Then by (3.26)

$$(3.57)\qquad \operatorname{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2)] = \operatorname{Tr}[\sqrt{\varepsilon}\, D_0'^H \exp(-t\varepsilon(D_0'^H)^2) P] + \operatorname{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]$$

Using (3.39) and (3.57), we find that

$$(3.58)\qquad \operatorname{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2)] = \operatorname{Tr}[\sqrt{\varepsilon}\, \tilde{D}^E \exp(-t\varepsilon(\tilde{D}^E)^2)] + \operatorname{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]$$

Let $\tilde{D}_0^E$, V be the operators acting on $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*))$ given by

$$(3.59)\qquad \left\{\begin{aligned} &\tilde{D}_0^E=\sum_1^{2k-1} c(f_\beta)\nabla^u_{f_\beta},\\ &V=\frac{1}{2}\sum_{\substack{1\leq i,j\leq 2n\\ 1\leq\beta\leq 2k-1}}\langle {}^0\theta(f_\beta)e_i,e_j\rangle\, c(e_i)\hat{c}(f_\beta)\hat{c}(e_j).\end{aligned}\right.$$

By (3.25), (3.39), we get

$$(3.60)\qquad \tilde{D}^E=\tilde{D}_0^E+V.$$

Note that $\tilde{D}_0^E$ is a standard Dirac operator acting on $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*))$. Also $\tilde{D}_0^E$ preserves $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda_\pm(E^*))$. Since τ^E anticommutes with $c(e_i)$ $(1\leqq i\leqq 2n)$ and commutes with $\hat{c}(f_\beta)$ $(1\leqq\beta\leqq 2k-1)$ and $\hat{c}(e_j)$ $(1\leqq j\leqq 2n)$, it is clear that V and τ^E anticommute. Therefore V exchanges $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda_+(E^*))$ and $\Gamma(\Lambda^{\text{even}}(T^*B)\otimes\Lambda_-(E^*))$.

Now we are exactly in the situation considered in Bismut-Freed [BF, Theorem 2.4] and also in Bismut-Cheeger [BC, Lemma 2.11]. Hence, we find that as $t\to 0$

$$(3.61)\qquad \operatorname{Tr}[\tilde{D}^E\exp(-t(\tilde{D}^E)^2)]=O(\sqrt{t}).$$

From (3.52), (3.58), (3.61), we deduce in particular that for a given $\varepsilon>0$, as $t\to 0$

$$(3.62)\qquad \operatorname{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q]=O(\sqrt{t})$$

In fact, relation (3.62) can easily be proved directly. Recall that local cancellations in index theory come from algebraic manipulations on Clifford algebras. On the contrary, P and Q are essentially scalar operators, which do not interfer with Clifford algebras.

From (3.53), (3.58), (3.62), we deduce in particular that

$$(3.63)\qquad \eta_\varepsilon^M(0)=\eta^{B,E}(0)+\frac{1}{\sqrt{\pi}}\int_0^{+\infty} t^{-1/2}\operatorname{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q]\,dt$$

We now claim that we can study the limit as $\varepsilon\to 0$ of

$$(3.64)\qquad \frac{1}{\sqrt{\pi}}\int_0^{+\infty} t^{-1/2}\operatorname{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q]\,dt$$

by the same techniques as in [BC, proof of Theorem 4.35]. In fact:

(*a*) By using the same arguments as in [BC, eq. (4.74)], we find that for any $T\in\mathbb{R}_+$, $0<t\leqq T$

$$(3.65)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\pi t}}\operatorname{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)]=\left(\frac{1}{2\pi i}\right)^k 2^{k-1}\int_B \mathscr{L}(i R^{TB})\operatorname{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right]$$

and the convergence takes place boundedly for $0<t\leqq T$. More precisely by (3.52), for a given $\varepsilon>0$, the expression in the left-hand side of (3.65) has a limit as $t\to 0$. Also by the analogue of (2.48), the right-hand side of (3.65) tends to zero as $t\to 0$. The convergence in (3.65) is then uniform for $0\leqq t\leqq T$.

Note here that the situation is slightly different from that considered in [BC], since on M, we consider the vector bundle $F^{TM}\otimes F^{TM^*}$, and the connection on the twisting bundle $F^{TM^*}=\pi^* F^{TB^*}\otimes\pi^* F^{E^*}$ varies with ε. This was not the case in [BC], where the connection on the twisting bundle ξ is fixed. However by using the same arguments as in [BC], we obtain (3.65).

By (3.61), we see that for $0\leqq t\leqq T$

$$\mathrm{Tr}[\sqrt{\varepsilon}\,\tilde{D}^E\exp(-t\varepsilon(\tilde{D}^E)^2)]=\sqrt{\varepsilon}\,O(\sqrt{t\varepsilon}). \tag{3.66}$$

From (3.58), (3.65), (3.66), we find that for $0\leqq t\leqq T$

(3.67)
$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi t}}\mathrm{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q]=2^k\int_B\mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\frac{1}{(2i\pi)^{1/2}}\varphi\,\mathrm{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right]$$

Also, by using (3.58), (3.65), (3.66) and [BC, eq. (4.40), (4.81)] (note that the factor $1/2\sqrt{u}$ is missing in the left of [BC, eq. (4.40)], and that the term du should be cancelled on the right), we know there exists $N>0$ such that for any $T>0$, $0<t\leqq T$, and $0\leqq\varepsilon\leqq 1$, then

$$\frac{1}{\sqrt{\pi t}}\mathrm{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q] \cdot$$
$$2^k\int_B\mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\frac{1}{(2i\pi)^{1/2}}\varphi\,\mathrm{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right]+O(\varepsilon^{1/2}(1+T^N)). \tag{3.68}$$

(b) Let $\mathscr{I}$ be the set of smooth sections of $\Lambda^{\mathrm{even}}(T^*M)$ over M such that $Qs=s$. By (3.26), $\tilde{D}_\varepsilon$ acts on $\mathscr{I}$. We claim that there exist $\varepsilon_0\in]0,1]$, $\lambda_0>0$ such that if $0<\varepsilon\leqq\varepsilon_0$, $s\in\mathscr{I}$, then

$$\langle\tilde{D}_\varepsilon^2 s,s\rangle\geqq\lambda_0|s|^2. \tag{3.69}$$

In fact the spectrum of the family of operator D'^2 acting fibrewise on $\mathrm{Ker}\,D'^\perp$ has a strictly positive lower bound. We can then proceed exactly as in [BC, Proposition 4.41] and we obtain (3.69).

Clearly for $t\geqq 4$

$$|\mathrm{Tr}[\tilde{D}_\varepsilon\exp(-t\tilde{D}_\varepsilon^2)Q]|\leqq C\,\mathrm{Tr}\left[\exp\left(-\frac{t\tilde{D}_\varepsilon^2}{2}\right)Q\right]\leqq ce^{-\lambda_0 t/4}\,\mathrm{Tr}[\exp(-\tilde{D}_\varepsilon^2)] \tag{3.70}$$

Also [CGT, Theorem 1.4], we know that

$$\text{Tr}[\exp(-\tilde{D}_\varepsilon^2)] \leqq \frac{C}{\varepsilon^{(2k-1)/2}} \tag{3.71}$$

From (3.70), (3.71), we get

$$|\text{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]| \leqq \frac{ce^{-\lambda_0 t/4}}{\varepsilon^{(2k-1)/2}}. \tag{3.72}$$

On the other hand by using Duhamel's formula – or in this specific case the analogue of (2.47) – we know there exist $c'>0$, $c''>0$ such that for $0<\varepsilon\leqq 1$, $t\geqq 1$

$$\left|\varphi \text{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right]\right| \leqq c' \exp(-c'' t). \tag{3.73}$$

From (3.68), (3.73), we find that for $0<\varepsilon\leqq 1$, $t\geqq 1$

$$\left|\frac{1}{\sqrt{\pi t}}\text{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]\right| \leqq c' \exp(-c'' t) + O(\varepsilon^{1/2} t^N). \tag{3.74}$$

By using (3.72), (3.74), we deduce there exist $C>0$, $C'>0$ such that for $0<\varepsilon\leqq\varepsilon_0$, $t\geqq 1$

$$\left|\frac{1}{\sqrt{\pi t}}\text{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]\right| \leqq C \exp(-C' t). \tag{3.75}$$

From (3.67), (3.75), we get

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_0^{+\infty} t^{-1/2}\,\text{Tr}[\tilde{D}_\varepsilon \exp(-t\tilde{D}_\varepsilon^2) Q]\,dt = 2^k \int_B \mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\frac{1}{(2i\pi)^{1/2}}\int_0^\infty \varphi \text{Tr}_s\left[\frac{D'}{2\sqrt{t}}\exp(-A_t'^2)\right] dt \tag{3.76}$$

Using (3.63), (3.76), we find that

$$\lim_{\varepsilon\to 0}\eta_\varepsilon^M(0) = \eta^{B,E}(0) + 2^k \int_B \mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\tilde{\eta}'(0) \tag{3.77}$$

Then from Theorem 2.25 and (3.77), we get (3.55).

Let ∇_ε^{TM} be the Levi-Civita connection on TM associated to the metric g_ε^{TM}, and let R_ε^{TM} be its curvature. By proceeding as in [BF, eq. (3.196)], we find easily that as $\varepsilon\to 0$

$$\mathscr{L}\left(\frac{R_\varepsilon^{TM}}{2\pi}\right) \to \pi^*\left(\mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\mathscr{L}\left(\frac{{}^0R^E}{2\pi}\right)\right). \tag{3.78}$$

By using the variation formula for $d\eta_\varepsilon^{M}(0)$ of Atiyah-Patodi-Singer in [BF, Theorem 2.10], and also (3.78), one easily deduces that $\lim_{\varepsilon \to 0} \eta_\varepsilon^{M}(0)$ does not depend on the metrics g^{TB} or g^{E}.

The proof of Theorem 3.8 is completed. □

Remark 3.9. — It is not immediately clear from the form of the right-hand side of (3.55) that it does not depend on the metrics g^{TB} and g^{E}.

However, observe that because of (3.36), $\eta^{B,E}(0)$ depends continuously on the metrics g^{TB} and g^{E}. Let $\tilde{\eta}^{B,E}(s)$ be the eta function of the operator $\tilde{D}_0^{E}$ defined in (3.59). $\tilde{\eta}^{B,E}(s)$ is the difference of eta functions of classical Dirac operators acting on $\Lambda^{\text{even}}(T^*B) \otimes \Lambda_\pm(E^*)$. Also using (3.60), the considerations which follow and [BC, Theorem 2.7], we know that $\eta^{B,E}(0) = \tilde{\eta}^{B,E}(0) \bmod \mathbb{Z}$. By [BF, Theorem 2.10], we have an explicit formula for the variation of $\tilde{\eta}^{B,E}(0)$ in $\mathbb{R}/\mathbb{Z}$ when the metrics g^{TB} and g^{E} vary. The variation of $\eta^{B,E}(0)$ in R with respect to the metrics g^{TB} and g^{E} is then explicitly known. By using equation (1.62), we finally obtain a direct proof that the right-hand side of (3.55) does not depend on the metrics g^{TB} and g^{E}.

Remark 3.10. — Equation (3.77) is a special case of a more general result of Dai [D], where a similar problem is considered for general fibrations. The discussion of [D] is considerably more complicated than what is required for the relatively simple case considered here.

IV. Adiabatic limits of eta invariants, torus fibrations and solvmanifolds

The purpose at this Section is to specialize the results of Section 3 to torus fibrations over homogeneous vector bundles, and in particular to the solvmanifolds which are the cross sections of the cusps of Hilbert modular varieties. This extends and clarifies what was done in Atiyah [A, Section 5] and Cheeger [C, Appendice 3] for such varieties when their complex dimension is 2. We thus give a new proof of the Hirzebruch conjecture [H] on the signature of Hilbert modular varieties, which was first proved by Atiyah-Donnelly-Singer [ADS] and Müller [Mü1,2] (compare also [St]). The techniques of this Section could eventually be used in a broader context than the one which is considered here.

In (*a*), we describe the torus fibrations associated with homogeneous vector bundles, and we make more explicit the integrals of the form ${}^{0}\gamma_{[z]}(s)$ associated with orbits $[z]$ of Λ^* in this special situation.

In (*b*), we consider the special case of torus fibrations over tori. By using the results of Sections (3) and (4*a*), we calculate the adiabatic limit of the eta invariant of the signature operator of M in terms of the value at 0 of a certain L function. We deal in detail with some tricky question of signs to prove the compatibility of our results with the results of [ADS], [Mü1,2] and we thus give a new proof of the Hirzebruch conjecture [H].

In this Section, we use the notation of Sections 1, 2, and 3.

(*a*) TORUS FIBRATIONS ASSOCIATED TO HOMOGENEOUS VECTOR BUNDLES. — In the sequel $\mathbb{R}^{2n}$ will be equipped with its canonical orientation.

Let G be a connected Lie group, and let K be a compact subgroup of G. Set

$$X = G/K \tag{4.1}$$

Let τ be the projection $G \to G/K$. Let $\rho : G \to SL(2n, \mathbb{R})$ be a group homomorphism.

Let $\mathbb{R}^{2n*}$ be the dual of $\mathbb{R}^{2n}$, and let $\tilde{\rho}(g)$ be the transpose of $\rho(g)$ acting on $\mathbb{R}^{2n*}$.

Let $\langle\ ,\ \rangle$ be a $\rho(K)$-invariant scalar product on $\mathbb{R}^{2n}$ such that $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ has volume 1 with respect to the corresponding volume form on $\mathbb{R}^{2n}$. If $g \in G$, $x \in X$ are such that $\tau(g) = x$, let $\langle\ ,\ \rangle_x$ be the scalar product on $\mathbb{R}^{2n}$ such that if $U, V \in \mathbb{R}^{2n}$

$$\langle U, V \rangle_x = \langle \rho(g)^{-1} U, \rho(g)^{-1} V \rangle \tag{4.2}$$

Then, given $x \in X$, the vector space $\mathbb{R}^{2n}$ can equipped with the scalar product $\langle\ ,\ \rangle_x$. Let $g^{\mathbb{R}^{2n}}$ denote the corresponding smooth scalar product on the trivial vector bundle $\mathbb{R}^{2n}$ over X.

Note that G acts on X. More generally, if $g \in G$, $x \in X$, $U \in \mathbb{R}^{2n}$, set

$$g(x, U) = (gx, \rho(g) U). \tag{4.3}$$

Relation (4.3) defines a lift of the action of G on X to an action of G on the vector bundle $X \times \mathbb{R}^{2n}$ over X, which preserves the scalar product $g^{\mathbb{R}^{2n}}$.

The vector bundle $\mathbb{R}^{2n}$ over X can be equipped with the trivial connection, which is tautologically flat. Let ${}^0\nabla^{\mathbb{R}^{2n}}$ be the connection constructed in Definition 1.8 on the flat Euclidean vector bundle $(\mathbb{R}^{2n}, g^{\mathbb{R}^{2n}})$. Since the action of G on $X \times \mathbb{R}^{2n}$ preserves the flat structure of $\mathbb{R}^{2n}$ and is isometric, it is clear that the connection ${}^0\nabla^{\mathbb{R}^{2n}}$ is also G-invariant. In particular for any $s \in \mathbb{C}$, the form

$$\Gamma(s+1) \frac{(1/2)\, \mathrm{Pf}[{}^0R^{\mathbb{R}^{2n}}/2\pi] ({}^0R^{\mathbb{R}^{2n}})^{-1} Y}{(|Y|^2/2 + ({}^0R^{\mathbb{R}^{2n}})^{-1})^{s+1}} \tag{4.4}$$

on $X \times \mathbb{R}^{2n}$ is G-invariant.

Let now Γ be a discrete cocompact subgroup of G, which acts freely on X, and is such that $\rho(\Gamma) \subset SL(2n, \mathbb{Z})$. We make the fundamental assumption that for any $z \in \mathbb{Z}^{2n*}$, $z \neq 0$

$$\{\theta \in \Gamma; \tilde{\rho}(\theta) z = z\} = \{\mathrm{id}\}. \tag{4.5}$$

Set

$$B = \Gamma \backslash X. \tag{4.6}$$

Then $\pi : X \to B$ is Γ principal bundle. We will assume that the compact manifold B is oriented.

Set

$$E = X \times_{\Gamma} \mathbb{R}^{2n} \tag{4.7}$$

In other words, E is exactly the quotient of $X \times \mathbb{R}^{2n}$ by the equivalence relation $(x, U) = (\theta x, \rho(\theta) U)$, $\theta \in \Gamma$. Since $\rho(\Gamma) \subset SL(2n, \mathbb{Z})$, E is equipped with a lattice Λ, which is the equivariant image of the lattice $\mathbb{Z}^{2n}$ in $\mathbb{R}^{2n}$. Thus E is a flat oriented vector bundle on B.

Since the metric $g^{\mathbb{R}^{2n}}$ is G-invariant, it induces a Euclidean metric g^E on the vector bundle E over B. Also because ρ maps G into $SL(2n, \mathbb{R})$, the fibres E/Λ have constant volume 1 with respect to g^E.

Let ${}^0\nabla^E$ be the connection on E associated to g^E constructed in Definition 1.8. Tautologically

$$(\mathbb{R}^{2n}, g^{\mathbb{R}^{2n}}, {}^0\nabla^{\mathbb{R}^{2n}}) = \pi^* (E, g^E, {}^0\nabla^E). \tag{4.8}$$

Let $\rho^*(g)$ be the adjoint of $\rho(g)$ with respect to the scalar product $\langle\ ,\ \rangle$. If $k \in K$, then $\rho(k)\rho^*(k) = 1$. Therefore if $x = \tau g$, it follows that $\rho(g)\rho^*(g)$ depends only on $x \in X$.

Definition 4.1. – If $e \in \mathbb{R}^{2n*} \setminus \{0\}$, $s \in \mathbb{C}$, let ${}^0\delta_e(s)$ be the form on X

$${}^0\delta_e(s) = \Gamma(s+1)(i^{-1} e)^* \frac{(1/2)\,\mathrm{Pf}[{}^0R^{\mathbb{R}^{2n}}/2\pi]({}^0R^{\mathbb{R}^{2n}})^{-1} Y}{(|Y|^2/2 + ({}^0R^{\mathbb{R}^{2n}})^{-1})^{s+1}} \tag{4.9}$$

Proposition 4.2. – *For any $g \in G$, the following identity holds*

$${}^0\delta_{\tilde{\rho}(g)e}(s) = g^* \, {}^0\delta_e(s). \tag{4.10}$$

Proof. – Equation (4.10) is an easy consequence of the G-invariance of the form (4.4). □

Let h^X be a G-invariant scalar product on TX. Then h^X induces a metric g^{TB} on TB.

We use the notation of Section 3 in this special situation. Also, the objects constructed in Section 1 which are associated with the connection ${}^0\nabla^E$ will be marked with a ^{0}.

Let μ be the form on B

$$\mu = (-1)^{(\dim E/2)+1} 2^{(\dim E + \dim B + 1)/2} \mathscr{L}\left(\frac{R^{TB}}{2\pi}\right) \hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right) \tag{4.11}$$

Then the form $\pi^* \mu$ on B is G-invariant.

Using (4.5) and Proposition 1.34, we find that if $z \in \mathbb{Z}^{2n*}$, $z \neq 0$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > \dim E/2$, then

$$\int_B \mu \, {}^0\gamma_{[z]}(s) = \int_X (\pi^* \mu) \, {}^0\delta_z(s). \tag{4.12}$$

Tautologically, the left-hand side of (4.12) does not change if z is changed into $\tilde{\rho}(\theta)z$, with $\theta\in\Gamma$. Also, since $\pi^*\mu$ is G-invariant, we deduce from Proposition 4.2 that

$$(\pi^*\mu)\,{}^0\delta_{\tilde{\rho}(\theta)z}(s)=\theta^*((\pi^*\mu)\,{}^0\delta_z(s)) \tag{4.13}$$

Relation (4.13) also explains the invariance of (4.12) when z is changed into $\tilde{\rho}(\theta)z$.

More generally, since $\pi^*\mu$ is G-invariant, if $g\in G$, then

$$(\pi^*\mu)\,{}^0\delta_{\tilde{\rho}(g)z}(s)=g^*((\pi^*\mu)\,{}^0\delta_z(s)) \tag{4.14}$$

From (4.14), we deduce that

$$\int_X(\pi^*\mu)\,{}^0\delta_{\tilde{\rho}(g)z}(s)=\int_X(\pi^*\mu)\,{}^0\delta_z(s) \tag{4.15}$$

Therefore $\int_X(\pi^*\mu)\,{}^0\delta_z(s)$ only depends on the G-orbit of $z\in\mathbb{Z}^{2n*}$. Also if z is replaced by $cz\,(c>0)$, $\int_X(\pi^*\mu)\,{}^0\delta_z(s)$ is changed into $c^{-2s}\int_X(\pi^*\mu)\,{}^0\delta_z(s)$.

In many cases of interest, these considerations permit an explicit calculation of $\int_B\mu\,{}^0\gamma(s)$. In particular, this is true in the context of prehomogeneous vector spaces as discussed in [Mü2,3], [Sa], [Sat Shin]. Under adequate assumptions on the form μ and on the metric g^E, one sees immediately that there exists a function $h(s)$ and a G-invariant function $\varphi(z)$ on $\mathbb{R}^{2n*}$ which is homogeneous of degree $2n$ such that

$$\int_B\mu^0\gamma(s)=h(s)\sum_{\substack{[z]\in\Gamma\backslash\mathbb{Z}^{2n*}\\ [z]\neq 0}}\frac{\varphi(z)}{|\varphi(z)|^{1+s/n}} \tag{4.16}$$

One such case will be considered in more detail in the next Section.

(*b*) Torus bundles over tori (solvmanifolds). — We now specialize the results of Section (4*a*) to the case where G is the additive group $\mathbb{R}^{2n-1}$, $K=\{0\}$, $\Gamma=\mathbb{Z}^{2n-1}$. Then $X=\mathbb{R}^{2n-1}$, and G acts on X by translations. Also $X=\mathbb{R}^{2n-1}$ is canonically oriented and G preserves the orientation of X.

Let $\alpha^1,\dots,\alpha^{2n}$ be elements of $\mathbb{R}^{2n-1*}$ such that

$$\sum_1^{2n}\alpha^i=0. \tag{4.17}$$

Let $\rho:\mathbb{R}^{2n-1}\to SL(2n,\mathbb{R})$ be a group homomorphism such that ρ maps $\Gamma=\mathbb{Z}^{2n-1}$ into $SL(2n,\mathbb{Z})$. We also assume that there is an oriented base $v_1,\dots,v_{2n}$ of $\mathbb{R}^{2n}$ such that for any $x\in\mathbb{R}^{2n-1}$, the corresponding matrix of $\rho(x)$ is diagonal, with diagonal entries $e^{\langle\alpha^1,x\rangle},\dots,e^{\langle\alpha^{2n},x\rangle}$.

In this case $B=\mathbb{Z}^{2n-1}\diagdown\mathbb{R}^{2n-1}$ is an oriented torus.

The main purpose of this Section is to calculate $\eta_0^M(0)$. Recall that by Theorem 3.8, $\eta_0^M(0)$ does not depend on the metrics g^{TB} and g^E. Therefore we are free to make the most convenient choice of metrics.

Let $\langle\ ,\ \rangle$ be the scalar product on $\mathbb{R}^{2n}$ such that $v_1, \ldots, v_{2n}$ is an orthonormal base of $\mathbb{R}^{2n}$.

Let $g^{\mathbb{R}^{2n}}$ be the metric constructed on the trivial bundle $\mathbb{R}^{2n}$ over $\mathbb{R}^{2n-1}$ given by (4.2). Using the notation of Definition 1.8, we find that with respect to the base $v_1, \ldots, v_{2n}$, the matrix of the one form ${}^0\theta$ is given by

$$-\begin{bmatrix} \alpha^1 & & \\ & \ddots & \\ & & \alpha^{2n} \end{bmatrix}. \tag{4.18}$$

From (4.18), we deduce that $({}^0\theta)^2=0$. By Proposition 1.9, we get

$$ {}^0R^{\mathbb{R}^{2n}}=0 \tag{4.19}$$

Equivalently, the Euclidean connections ${}^0\nabla^{\mathbb{R}^{2n}}$ and ${}^0\nabla^E$ are flat.

A reformulation of this result is that $(e^{\langle \alpha^1, x\rangle} v_1, \ldots, e^{\langle \alpha^n, x\rangle} v_n)$ is an orthonormal base of $(\mathbb{R}^{2n}, g^{\mathbb{R}^{2n}})$ which is parallel with respect to the connection ${}^0\nabla^{\mathbb{R}^{2n}}$. The base $(e^{\langle \alpha^1, x\rangle} v_1, \ldots e^{\langle \alpha^n, x\rangle} v_n)$ induces a corresponding flat orthonormal base of $(E, g^E, {}^0\nabla^E)$ on B. Therefore $(E, g^E, {}^0\nabla^E)$ is a trivial vector bundle with metric and Euclidean connection over B.

We now make the *fundamental assumption* that $\alpha^1, \ldots, \alpha^{2n}$ span $\mathbb{R}^{2n-1*}$.

If $Y\in\mathbb{R}^{2n}$, we write Y in the form $Y=\sum_1^{2n} Y^i v_i$.

Then the orbits of $\mathbb{R}^{2n}$ under the action of G can be divided into two categories:

- A first category of orbits is contained in one or several of the hyperplanes $(Y^i=0)$, $1\leqq i\leqq 2n$.
- A second category of orbits is parametrized by a family $\beta=(\beta^1, \ldots, \beta^{2n})$, $\beta^i=\pm 1$, $1\leqq i\leqq 2n$ and by $r>0$. The orbit of $r\left(\sum_1^{2n}\beta^i v_i\right)$ is the hyperboloid $H_{r,\beta}$ given by

$$H_{r,\beta}=\{Y\in\mathbb{R}^{2n}, |Y^1\ldots Y^{2n}|=r^{2n};\ \mathrm{sign}\, Y^i=\beta_i, 1\leqq i\leqq n\}. \tag{4.20}$$

We identify $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n*}$ using the metric $g_0^{\mathbb{R}^{2n}}$, *i.e.* by the linear map i_0. Let N be the lattice $\mathbb{Z}^{2n}$ in $\mathbb{R}^{2n}$, $N^*=\mathbb{Z}^{2n*}$ the dual lattice in $\mathbb{R}^{2n*}$ and let N′ be the corresponding lattice in $\mathbb{R}^{2n}$, so that $N'=i_0^{-1}N^*$. The group Γ then acts on N′.

In the sequel, we assume that the nonzero elements in N′ only lie in the second category of orbits. In particular assumption (4.5) is verified, because $\alpha^1, \ldots, \alpha^{2n}$ span $\mathbb{R}^{2n-1*}$.

The hyperboloids $H_{r,\beta}$ inherit an orientation from the corresponding orientation of the sphere S_{2n-1} in $\mathbb{R}^{2n}$.

Let g^{TX} be the standard Euclidean metric on $X \cong \mathbb{R}^{2n-1}$. Note that g^{TX} is then G-invariant.

At this point, all the assumptions of Section (4 *a*) are verified.

We now introduce the L function of Shimizu [Sh].

DEFINITION 4.3. – For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, let $L(N', \Gamma, s)$ be the function

$$L(N', \Gamma, s) = \sum_{\substack{[z] \in \Gamma \backslash N' \\ [z] \neq 0}} \frac{z^1 \ldots z^{2n}}{|z^1 \ldots z^{2n}|^{1+s}} \tag{4.21}$$

Then by [H, p. 230], $L(N', \Gamma, s)$ extends to a holomorphic function of $s \in \mathbb{C}$. In particular $L(N', \Gamma, s)$ is holomorphic at $s = 0$.

Let θ_0 be the solid angle form in $\mathbb{R}^{2n}$ associated to the metric $g_0^{\mathbb{R}^{2n}}$. Then

$$\int_{S_{2n-1}} \theta_0 = 1 \tag{4.22}$$

Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{2n}$. On easily verifies that the function

$$s \in \mathbb{C}, \quad \mathrm{Re}(s) > -2n \to \int_{H_{1,1}} \theta_0 / |Y|^s$$

is holomorphic.

Recall that $\alpha^1, \ldots, \alpha^{2n-1}$ is a base of $\mathbb{R}^{2n-1}$. Set $\mathrm{sign}(\alpha^1, \ldots, \alpha^{2n-1}) = 1$ if this base is an oriented base of $\mathbb{R}^{2n-1}$ and -1 if it is not.

THEOREM 4.4. – *For any* $s \in \mathbb{C}$, $\mathrm{Re}(s) > -2$, *then*

$$\lim_{\varepsilon \to 0} \eta_\varepsilon^M(s) = \frac{\Gamma(1/2)}{\Gamma((s+1)/2)} \frac{\Gamma((s/2)+n)}{\Gamma(n)} (-1)^n \mathrm{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) \, 2^{2n} \int_{H_{1,1}} \frac{\theta_0}{|Y|^s} L(N', \Gamma, (s/2n)) \tag{4.23}$$

In particular

$$\eta_0^M(0) = (-1)^n \mathrm{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) L(N', \Gamma, 0). \tag{4.24}$$

Proof. – We use the notation of Section (3 *f*). We first claim that for any $s \in \mathbb{C}$

$$\eta^{B, E}(s) = 0. \tag{4.25}$$

Let us trivialize the vector bundle E over B by parallel transport with respect to the connection ${}^0\nabla^E$. Then $v_1, \ldots, v_n$ is an orthonormal base of E over B which is parallel with respect to ${}^0\nabla^E$. Let $f_1, \ldots, f_{2n-1}$ be an orthonormal base of TB. Here we are using the notation of (3.59). By (3.60) and (4.18), we find that

$$\tilde{D}^E = \tilde{D}_0^E - \frac{1}{2} \sum_{\substack{1 \leq \beta \leq 2n-1 \\ 1 \leq i \leq 2n}} \alpha_i(f_\beta)\, c(v_i)\, \hat{c}(f_\beta)\, \hat{c}(v_i) \tag{4.26}$$

For $\lambda \in \Gamma(\Lambda^{\text{even}}(T^*B) \otimes \Lambda(E^*))$, let $\psi\lambda \in \Gamma(\Lambda^{\text{even}}(T^*B) \otimes \Lambda(E^*))$ be defined by the formula

$$\psi\lambda(x) = \tau^E \lambda(-x). \tag{4.27}$$

The map τ^E commutes with $c(f_\beta)$ $(1 \leq \beta \leq 2n-1)$ and anticommutes with $c(v_i)\, \hat{c}(f_\beta)\, \hat{c}(v_i)$ $(1 \leq \beta \leq 2n-1,\ 1 \leq i \leq 2n)$. We deduce from (4.26), (4.27) that

$$\psi\, \tilde{D}^E \psi^{-1} = -\tilde{D}^E \tag{4.28}$$

From (4.28), we see that the spectrum of $\tilde{D}^E$ is symmetric with respect the origin. Therefore (4.25) follows.

On the other hand, since $R^{TB} = 0$, ${}^0R^E = 0$, we have

$$\begin{cases} \mathscr{L}\left(\dfrac{R^{TB}}{2\pi}\right) = 1 \\ \hat{A}^{-1}\left(2\dfrac{{}^0R^E}{2\pi}\right) = 1 \end{cases} \tag{4.29}$$

Using (4.25), (4.29) and proceeding exactly as in the proof of Theorem 3.8, we find easily that for $s \in \mathbb{C}$, $\operatorname{Re}(s) > -2$,

$$\lim_{\varepsilon \to 0} \eta_\varepsilon^M(s) = \frac{\sqrt{\pi}}{\Gamma((s+1)/2)} (-1)^{n+1} 2^{2n-s/2} \int_B {}^0\gamma\left(\frac{s}{2}\right) \tag{4.30}$$

Let ε be the volume form in the fibres of $\mathbb{R}^{2n}$ with respect to the metric $g^{\mathbb{R}^{2n}}$. As in Section (1 *a*), we identify ε with a $2n$-form on $X \times \mathbb{R}^{2n}$ which is vertical with respect to the connection ${}^0\nabla^{\mathbb{R}^{2n}}$. By proceeding as in the proof of Theorem 1.29, we find that for $s \in \mathbb{C}$, $\operatorname{Re}(s) > -n$

$${}^0\delta_e(s) = \frac{\Gamma(s+n)\,(i^{-1}e)^*}{(2\pi)^n} \frac{i_Y \varepsilon}{2(|Y|^2_{g^{\mathbb{R}^{2n}}}/2)^{s+n}} \tag{4.31}$$

By (1.68) and (1.42), we know that if $s \in \mathbb{C}$, $\operatorname{Re}(s) > n$

$$\int_B {}^0\gamma(s) = \sum_{\substack{[z] \in \Gamma \backslash N^* \\ [z] \neq 0}} \int_{\mathbb{R}^{2n-1}} {}^0\delta_z(s) \tag{4.32}$$

To evaluate (4.32), we first observe that

$$i^{-1} z = \sum_1^{2n} e^{2\langle \alpha^i, x\rangle} z_i v_i = \sum_1^{2n} (e^{\langle \alpha^i, x\rangle} z_i) e^{\langle \alpha^i, x\rangle} v_i \tag{4.33}$$

Now recall that $(e^{\langle \alpha^1, x\rangle} v_1, \ldots, e^{\langle \alpha^{2n}, x\rangle} v_n)$ is an orthonormal oriented base of $\mathbb{R}^{2n-1}$ with respect to $g^{\mathbb{R}^{2n-1}}$, which is flat with respect to the connection ${}^0\nabla^{\mathbb{R}^{2n-1}}$.

Let ε_0 be the given volume element in the oriented fibre $\mathbb{R}_0^{2n}$. Observe that $x \in \mathbb{R}^{2n-1} \to (e^{\langle \alpha^1, x\rangle} z_1, \ldots, e^{\langle \alpha^{2n}, x\rangle} z_n)$ is the orbit $\{i_0^{-1} z\}$ of $i_0^{-1} z \in \mathrm{N}'$ in $\mathbb{R}^{2n}$ under the action of $\mathrm{G} = \mathrm{R}^{2n-1}$. We give this orbit the orientation inherited from the orientation of $\mathbb{R}^{2n-1}$. Using (4.31), we find that if $z \in \mathrm{N}^*$, $z \neq 0$

$$\int_{\mathbb{R}^{2n-1}} {}^0\delta_z(s) = \int_{\{i_0^{-1} z\}} \frac{\Gamma(s+n)}{(2\pi)^n} \frac{i_{\mathrm{Y}} \varepsilon_0}{2(|\mathrm{Y}|^2_{g_0^{\mathbb{R}^{2n}}}/2)^{s+n}} \tag{4.34}$$

From (1.14), Remark 1.7 and (4.34), we get

$$\int_{\mathbb{R}^{2n-1}} {}^0\delta_z(s) = \frac{\Gamma(s+n)\, 2^s}{\Gamma(n)} \int_{\{i_0^{-1} z\}} \frac{\theta_0}{|\mathrm{Y}|^{2s}}. \tag{4.35}$$

Now by our fundamental assumption on the lattice N′, we know that there exists a unique $r > 0$, $\beta = (\beta^1, \ldots, \beta^{2n})$ such that the orbits $\{i_0^{-1} z\}$ and $\mathrm{H}_{r,\beta}$ coincide. Since $\{i_0^{-1} z\}$ and $\mathrm{H}_{r,\beta}$ are both oriented, with orientations which differ by $-\operatorname{sign}(\alpha^1, \ldots, \alpha^{2n-1})$, we deduce from (4.35) that

$$\int_{\mathbb{R}^{2n-1}} {}^0\delta_z(s) = -\frac{\Gamma(s+n)}{\Gamma(n)} 2^s \operatorname{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) \int_{\mathrm{H}_{1,1}} \frac{\theta_0}{|\mathrm{Y}|^{2s}} \frac{\beta^1 \ldots \beta^{2n}}{r^{2s}} \tag{4.36}$$

Equivalently, if $i_0^{-1} z = \sum_1^{2n} z^i v_i$, then

$$\int_{\mathbb{R}^{2n-1}} {}^0\delta_z(s) = -\frac{\Gamma(s+n)}{\Gamma(n)} 2^s \operatorname{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) \int_{\mathrm{H}_{1,1}} \frac{\theta_0}{|\mathrm{Y}|^{2s}} \frac{z^1 \ldots z^{2n}}{|z^1 \ldots z^{2n}|^{s/n+1}}. \tag{4.37}$$

Using (4.21), (4.32) and (4.37), we find that if $s \in \mathbb{C}$, $\operatorname{Re}(s) > n$, we get

$$\int_{\mathrm{B}} {}^0\gamma(s) = -\frac{\Gamma(s+n)}{\Gamma(n)} 2^s \operatorname{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) \int_{\mathrm{H}_{1,1}} \frac{\theta_0}{|\mathrm{Y}|^{2s}} \mathrm{L}(\mathrm{N}', \Gamma, (s/n)) \tag{4.38}$$

From (4.30), (4.38), it follows that if $s \in \mathbb{C}$, $\operatorname{Re}(s) > n$, then

$$(4.39)\quad \lim_{\varepsilon \to 0} \eta_\varepsilon^{\mathrm{M}}(s) = \frac{\Gamma(1/2)}{\Gamma((s+1)/2)} \frac{\Gamma((s/2)+n)}{\Gamma(n)} (-1)^n \operatorname{sgn}(\alpha^1, \ldots, \alpha^{2n-1}) 2^{2n} \int_{\mathrm{H}_{1,1}} \frac{\theta_0}{|\mathrm{Y}|^s} \mathrm{L}(\mathrm{N}', \Gamma, (s/2n))$$

Also by proceeding as in the proof of Theorem 3.8, we know that as $\varepsilon \to 0$, if $s \in \mathbb{C}$, $\operatorname{Re}(s) > -2$, $\lim_{\varepsilon \to 0} \eta_\varepsilon^{\mathrm{M}}(s)$ exists and depends holomorphically on s. Therefore (4.39) also holds for $s \in \mathbb{C}$, $\operatorname{Re}(s) > -2$.

Moreover, one has the trivial identity

$$(4.40)\quad \int_{\mathrm{H}_{1,1}} \theta_0 = \frac{1}{2^{2n}}.$$

Using (4.23) and (4.40), we get (4.24). This completes the proof. □

Remark 4.5. — As explained in Müller [Mü3, Section 8], using formula (4.24), we can give a new derivation of the result of Atiyah-Donnelly-Singer [ADS] and Müller [Mü1,2] on the signature defect for Hilbert modular varieties, *i.e.* give a new proof of a conjecture of Hirzebruch [H]. Incidently, it should be pointed out that although we ultimately obtain the same result as in [ADS], the signature operators $\tilde{\mathrm{D}}_\varepsilon$ which are considered here are not the same as the ones which appear in [ADS], which are associated with connections with non zero torsion.

However, the right-hand side of (4.24) coincides with the formula of [ADS] and [Mü1,2] up to a sign, the determination of which is non trivial. Thus we will briefly deal with this question of signs.

In fact recall that in [ADS] and [Mü1,2], the solvmanifold M appears as the boundary of a complex manifold. It thus inherits a natural orientation.

Let $\alpha_1, \ldots, \alpha_{2n-1}$ be the base of $\mathbb{R}^{2n-1}$ dual to the base $\alpha^1, \ldots, \alpha^{2n-1}$. From [ADS, eq. (6.2)], we find that $\alpha_1, \ldots, \alpha_{2n-1}$ have a norm equal to 1. From [ADS, eq. (6.2) and Lemma (10.2)] we see that in [ADS], the orientation of M is such that

$$(4.41)\quad \sqrt{-1}\, c(v_{2n}) \prod_1^{2n-1} \sqrt{-1}\, c(v_i)\, c(\alpha_i) = 1$$

Using (4.41), we find that

$$(4.42)\quad (\sqrt{-1})^{2n} \prod_1^{2n-1} c(\alpha_i) \prod_1^{2n} c(v_i) = (-1)^n$$

Comparing with [BF, eq. (1.9)], we find that with the conventions of [ADS], the orientation of M differs from the orientation given by $(\alpha_1, \ldots, \alpha_{2n-1}, v_1, \ldots, v_{2n})$ by a factor $(-1)^n$.

Let $\eta_\varepsilon^{M, ADS}(0)$ be the eta invariant of the signature operator $\tilde{D}_\varepsilon^{ADS}$ associated with the orientation of M given in [ADS]. By Theorem 4.4, we find in particular that

$$\lim_{\varepsilon \to 0} \eta_\varepsilon^{M, ADS}(0) = L(N', \Gamma, 0) \tag{4.43}$$

This fits with [ADS], [Mü1,2].

Remark 4.6. — If $\alpha^1, \ldots, \alpha^{2n}$ do not span $\mathbb{R}^{2n-1*}$, then for $s \in \mathbb{C}$, $\mathrm{Re}(s) > -2$

$$\lim_{\varepsilon \to 0} \eta_\varepsilon^M(s) = 0 \tag{4.44}$$

In fact the G-orbits have dimension strictly smaller than $2n-1$. The arguments in the proof of Theorem 4.4 show trivially that (4.44) holds.

Remark 4.7. — In the context of prehomogeneous vector spaces as discussed in [Mü,2], [Sa], [Sat Shin], one can use Theorem 3.8 and identity (4.16) to calculate $\eta_0^M(0)$. Therefore one can obtain a formula for the signature of more general manifolds than the Hilbert modular varieties. Note in this respect that the term $-(1/2)C_+(0)$, which appears in [Mü3, Theorem 4.8] is absent from our formula for $\eta_0^M(0)$.

V. Topological properties of adiabatic limits of eta invariants of torus bundles

Recall that by Theorem 3.8, $\eta_0^M(0)$ is a topological invariant. The purpose of this Section is to analyze in more detail some properties of this invariant.

In (*a*), we prove that $\eta_0^M(0) \in \mathbb{Q}$.

In (*b*), we show that if B bounds a manifold with similar properties, then $\eta_0^M(0) \in \mathbb{Z}$.

In (*c*) we consider $\eta_0^M(0)$ as defining a homomorphism η_0 from

$$KO^*(B_{SL(2n, \mathbb{Z})}) \otimes Z[1/2]$$

into $\mathbb{Q}/\mathbb{Z}$.

Finally in (*d*), we relate $\eta_0^M(0)$ to the differential characters of Cheeger and Simons [CS] and use this relation to define a secondary characteristic cohomology class $\hat{\chi} \in H^{2n-1}(B, \mathbb{R}/\mathbb{Z})$ for flat $SL(2n, \mathbb{Z})$-vector bundles.

(*a*) Rationality properties of adiabatic limits of the eta invariants of torus bundles. — We here make the same assumptions and use the same notation as in Section (3*f*).

Proposition 5.1. — *$\eta_0^M(0)$ lies in $\mathbb{Q}$.*

Proof. — For $m \in \mathbb{N}^*$, set

$$M_m = E/m\Lambda. \tag{5.1}$$

In particular $M_1 = M$. Then the map $\rho_m : e \in E/\Lambda \to me \in E/m\Lambda$ is a diffeomorphism from M into M_m. One then easily sees that

$$\eta_\varepsilon^{M_m}(0) = \eta_{\varepsilon m^2}^{M}(0) \tag{5.2}$$

From (5.2), we deduce that

$$\eta_0^{M_m}(0) = \eta_0^{M}(0). \tag{5.3}$$

Since the canonical map $\pi_m : M_m \to M$ is a m^{2n} covering, we deduce from Atiyah-Patodi-Singer [APS2, Theorem 2.9] that $m^{2n}\eta_\varepsilon^M(0) - \eta_\varepsilon^{M_m}(0)$ is a topological invariant which lies in $\mathbb{Q}$. Using (5.3), we find that $(m^{2n}-1)\ \eta_0^M(0) \in \mathbb{Q}$. Our Proposition follows. □

Remark 5.2. – From Theorem 4.4 and Proposition 5.1, we find in particular that $L(N', \Gamma, 0) \in \mathbb{Q}$. The observation was made in Atiyah-Donnelly-Singer [ADS, p. 138] than one could deduce the rationality of $L(N', \Gamma, 0)$ from the rationality of an eta invariant. The rationality of $L(N', \Gamma, 0)$ is a known result of Siegel [Si].

(*b*) The case where the torus fibration bounds. – We temporarily assume that the compact manifold B bounds a compact oriented manifold Y, and that the fibration $Z \to M \to B$ extends to a fibration $Z \to \tilde{M} \to Y$ of the same type.

Proposition 5.3. – *The invariant* $\eta_0^M(0)$ *lies in* $\mathbb{Z}$.

Proof. – Let g^{TY} be a metric on TY which is product near the boundary B, *i.e.* which is of the form $|du|^2 + g^{TB}$. If $x \in B$, and $u > 0$ is small enough, we identify $E_{(x,u)}$ with $E_{(x,0)}$ by using parallel transport with respect to the flat connection ∇ along the line $t \in [0,1] \to (x, tu)$. Let $T^H\tilde{M}$ be the subbundle of $T\tilde{M}$ associated with the flat connection ∇.

Let g^E be a metric on the vector bundle E over Y which is product near B with respect to the previous identifications, *i.e.* for u small enough, the identification of $E_{(x,u)}$ with $E_{(x,0)}$ $(x \in B)$ is an isometry. Let g^{TZ} be the induced metric on TZ.

One easily verifies that for any $\varepsilon > 0$, the metric $(1/\varepsilon)\pi^* g^{TB} \oplus g^{TZ}$ on $T\tilde{M} = T^H\tilde{M} \oplus TZ$ is also product near the boundary M. Let $R_\varepsilon^{T\tilde{M}}$ be the curvature of the Levi-Civita connection on $T\tilde{M}$ associated with the metric $g_\varepsilon^{T\tilde{M}}$.

Let $\eta_\varepsilon^M(s)$ be the eta function of the signature operator $\tilde{D}_\varepsilon$ on M associated with the metrics g^{TB}, $g^{E|B}$. By a result of Atiyah-Patodi-Singer [APS1, Theorem 4.14], the signature of Y is given by

$$\operatorname{sign}(Y) = 2^{(\dim E + \dim B + 1)/2} \int_Y \mathscr{L}\left(\frac{R_\varepsilon^{T\tilde{M}}}{2\pi}\right) - \eta_\varepsilon^M(0). \tag{5.4}$$

By making $\varepsilon \to 0$ in (5.4) and by proceeding as in (3.78), we get

$$\operatorname{sign}(Y) = -\eta_0^M(0). \tag{5.5}$$

Our Proposition follows. □

(*c*) $\eta_0^{\mathrm{M}}(0)$ AND THE ORIENTED BORDISM IN RING OF $\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$. — Let $\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$ be the classifying space for the group $\mathrm{SL}(2n,\mathbb{Z})$, let $\mathrm{E}\to\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$ be the universal $\mathrm{SL}(2n,\mathbb{Z})$ vector bundle, and let $\mathscr{M}\to\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$ be the corresponding universal flat torus bundle.

Let B be a compact oriented manifold. Let $f:\mathrm{B}\to\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$ be a continuous map, and let $f^*(\mathscr{M})\to\mathrm{B}$ be the induced flat torus bundle over B.

If B is odd dimensional, $\eta_0^{f^*(\mathscr{M})}(0)$ is defined as before. If B is even dimensional, set $\eta_0^{f^*(\mathscr{M})}(0)=0$ (this corresponds to the fact that in this case, the signature operator has symmetric spectrum with respect to the origin).

Let $\Omega_*(\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})})$ be the oriented bordism ring of $\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})}$. By Proposition 5.3, the assignment

$$(\mathrm{B},f)\to\eta_0^{f^*(\mathscr{M})}(0)\in\mathbb{Q}/\mathbb{Z} \tag{5.6}$$

defines a homomorphism $\Omega_*(\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})})\to\mathbb{Q}/\mathbb{Z}$.

Let W be a compact oriented manifold. If W is even dimensional, we define its signature sign(W) as usual. If W is odd dimensional, set sign(W)=0.

Let π_2 be the projection $\mathrm{W}\times\mathrm{B}\to\mathrm{B}$. Set

$$\mathrm{W}(\mathrm{B},f)=(\mathrm{W}\times\mathrm{B},f\pi_2). \tag{5.7}$$

Then one easily verifies that

$$\eta_0^{(f\pi_2)^*(\mathscr{M})}(0)=\mathrm{sign}(\mathrm{W})\,\eta_0^{f^*(\mathscr{M})}(0). \tag{5.8}$$

Let KO_* be the homology theory dual to KO^* theory. Also let $\mathbb{Z}[1/2]\subset\mathrm{Q}$ be the ring of rational numbers with denominator a power of 2.

In [Su2,3] (*see* also [CaSha, pp. 202, 208]), Sullivan showed there is a natural isomorphism between $\mathrm{KO}_*(\mathrm{X})\otimes\mathbb{Z}[1/2]$ and the quotient of the oriented bordism ring $\Omega_*(\mathrm{X})$, generated by the equivalence relation $(\mathrm{B},f)\sim(\mathrm{W}\times\mathrm{B},f\pi_2)$, where W varies over the manifolds with sign (W)=0.

It follows from the previous considerations that the map $(\mathrm{B},f)\to\eta_0^{f^*(\mathscr{M})}(0)$ defines a homomorphism

$$\eta_0:\quad \mathrm{KO}_*(\mathrm{B}_{\mathrm{SL}(2n,\mathbb{Z})})\otimes\mathrm{Z}\left[\frac{1}{2}\right]\to\mathbb{Q}/\mathbb{Z}. \tag{5.9}$$

(*d*) ETA INVARIANTS AND SECONDARY EULER CLASSES. — Let ${}^0\psi$ be the $2n-1$ current on the total space of E, associated to the connection ${}^0\nabla^{\mathrm{E}}$, which was defined in Definition 1.3.

PROPOSITION 5.4. — *If* $\dim\mathrm{B}<2n-1$, *then* $\eta_0^{\mathrm{M}}(0)\in\mathbb{Z}$.

Proof. — We use the notation of Remark 3.9. As explained in this Remark, we know that

$$\eta^{\mathrm{B},\mathrm{E}}(0)=\tilde{\eta}^{\mathrm{B},\mathrm{E}}(0)\bmod\mathbb{Z}. \tag{5.10}$$

Since B has dimension $\leqq 2n-1$, the vector bundle E has a smooth non vanishing section v. The Clifford multiplication operator $\sqrt{-1}\,c(v)$ acts on $\Lambda^{\text{even}}(T^*B)\otimes\Lambda(E^*)$ as a self-adjoint invertible operator and interchanges $\Lambda^{\text{even}}(T^*B)\otimes\Lambda_+(E^*)$ and $\Lambda^{\text{even}}(T^*B)\otimes\Lambda_-(E^*)$. By using a result of [BC, Theorems 2.7 and 2.28], (2.60), and proceeding as in [BGS2, Section 3 g)], we get

$$(5.11)\qquad \left\{\begin{aligned}&\tilde{\eta}^{B,E}(0)=(-1)^{\dim E/2}\,2^{(\dim E+\dim B+1)/2}\int_B \mathscr{L}\left(\frac{R^{TB}}{2\pi}\right)\\&\qquad\hat{A}^{-1}\left(2\frac{{}^0R^E}{2\pi}\right)v^*\,{}^0\psi\quad\text{in } \mathbb{R}/\mathbb{Z}.\end{aligned}\right.$$

Now $v^*\,{}^0\psi$ is a smooth form on B of degree $2n-1$. Since $\dim B<2n-1$ the right-hand side of (5.11) is equal to 0.

From (5.10), (5.11), we get

$$(5.12)\qquad \eta^{B,E}(0)=0\quad\text{in } \mathbb{R}/\mathbb{Z}.$$

Similarly the form ${}^0\gamma(0)$ has degree $2n-1$. Using Theorem 3.8 and (5.12), we have thus proved Proposition 5.4. □

THEOREM 5.5. — *If* $\dim B=2n-1$, *let v be a smooth non vanishing section of* E *over* B. *Then*

$$(5.13)\qquad \eta_0^M(0)=(-1)^n\,2^{2n}\int_B (v^*\,{}^0\psi-{}^0\gamma(0))\quad\text{in } \mathbb{R}/\mathbb{Z}$$

Proof. — By proceeding as in the proof of Proposition 5.4, we get (5.12). □

DEFINITION 5.6. — *If v* is a smooth non vanishing section of E over B, set

$$(5.14)\qquad \hat{\chi}(E)=v^*\,{}^0\psi-{}^0\gamma(0).$$

$\int_B v^*\,{}^0\psi\in\mathbb{R}/\mathbb{Z}$ is the Euler character of the Euclidean vector bundle with connection $(E, g^E, {}^0\nabla^E)$ as defined in Cheeger-Simons [CS, Section 3]. It does not depend on the section v. By (1.11), (1.62), $\hat{\chi}(E)$ is a closed form on B. It unambiguously defines an element of $H^{2n-1}(B, \mathbb{R}/\mathbb{Z})$ which does not depend on the metric g^E. $\hat{\chi}(E)$ behaves naturally under pull-backs. Therefore the class $\hat{\chi}(E)$ is defined universally on the classifying space $B_{SL(2n,\mathbb{Z})}$.

To the exact sequence of coefficients

$$(5.15)\qquad 0\to\mathbb{Z}\to\mathbb{R}\to\mathbb{R}/\mathbb{Z}\to 0$$

we can associate a long exact sequence in cohomology over $B_{SL(2n,\mathbb{Z})}$. Let b denote the Bockstein $H^*(B_{SL(2n,\mathbb{Z})}, \mathbb{R}/\mathbb{Z})\to H^{*+1}(B_{SL(2n,\mathbb{Z})}, \mathbb{Z})$.

Let $\chi(E) \in H^{2n}(B_{SL(2n, \mathbb{Z})}, \mathbb{Z})$ be the Euler class of E. By a result of Sullivan [S] (which here follows from Theorem 1.27), we know that the image of $\chi(E)$ in $H^{2n}(B_{SL(2n, \mathbb{Z})}, \mathbb{R})$ vanishes. In fact, one easily verifies that

$$b(\hat{\chi}(E)) = -\chi(E). \tag{5.16}$$

THEOREM 5.7. — *$\hat{\chi}(E) \in H^{2n-1}(B_{SL(2n, \mathbb{Z})}, \mathbb{R}/\mathbb{Z})$ is a nonzero torsion class.*

Proof. — Although $B_{SL(2n, \mathbb{Z})}$, does not have the homotopy type of a finite dimensional complex [since $SL(2n, \mathbb{Z})$ has torsion], it is known that there exists a model of this space for which the k-skeleton, Σ^k, is a finite complex for all $k < \infty$. In particular, in studying $H_{2n-1}(B_{SL(2n, \mathbb{Z})}, \mathbb{R}/\mathbb{Z})$, we can restrict attention to Σ^k.

Now Σ^n is homotopy equivalent to a smooth manifold with boundary Y, of dimension $N \gg 2n-1$. By the Pontrjagin-Thom construction, there exists $N(n)$ such that for any cycle $c \in H_{2n-1}(Y, \mathbb{Z})$, $N(n).c$ is represented by $f: B \to Y$, where B is a smooth manifold. Thus, by Proposition 5.1 and by (5.13), $\hat{\chi}(E)$ takes rational values with bounded denominator on the finitely generated group $H_{2n-1}(Y, \mathbb{Z})$. Also, by the universal coefficient theorem, $H^{2n-1}(Y, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(H_{2n-1}(Y, \mathbb{Z}), R/\mathbb{Z})$. Therefore $\hat{\chi}(E)$ is a torsion class.

Using the notation of Section (4*b*), we know that $L(N', \Gamma, 0)$ is not always an integer. By Theorems 4.4 and 5.5, we see that $\hat{\chi}(E)$ is nonzero. This completes the proof of Theorem 5.7.

Remark 5.8. — By [LeSc], we know that $H_3(B_{SL(4, \mathbb{Z})}, \mathbb{Z})$ has rank one. Thus $H^3(B_{SL(4, \mathbb{Z})}, \mathbb{R}/\mathbb{Z})$ does contain elements of infinite order. This shows that Theorem 5.7 is non empty.

Remark 5.9. — By the construction of Section (4*a*), the form ${}^0\gamma(0)$ is well-defined on $SL(2n, \mathbb{R})/SO(2n)$ and is $SL(2n, \mathbb{Z})$-invariant. Still $SL(2n, \mathbb{Z})$ does not act freely on $SL(2n, \mathbb{R})/SO(2n)$, and so there is no natural vector bundle on $SL(2n, \mathbb{Z}) \diagdown SL(2n, \mathbb{R})/SO(2n)$. This explains why the class $\hat{\chi}(E)$ cannot be constructed on this last space.

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