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MORITA EQUIVALENCE AND SYMPLECTIC REALIZATIONS OF POISSON MANIFOLDS

BY PING XU

Introduction

The role of symplectic realizations in the theory of Poisson manifolds is similar to that of representations in the study of noncommutative algebras, so their investigation is of basic importance in Poisson geometry. However, according to our knowledge, there is little work done in this direction because of the lack of a powerful general method. The main purpose of the present paper is to compute the realizations for several interesting Poisson manifolds by using the device of Morita equivalence developed in [X1] and [X2].

Three types of Poisson manifolds are of particular interest to us: locally trivial bundles of symplectic manifolds, semi-direct products of Poisson manifolds and reduced Poisson manifolds. Although it seems that these three kinds of Poisson manifolds are very different from each other, they have one thing in common, that is, they all can be considered as reduced Poisson manifolds. Therefore, it is not surprising that Poisson reduction plays a very special role in this work. The key result, roughly speaking, is that the reduced symplectic groupoids of Morita equivalent symplectic groupoids are still Morita equivalent. Many interesting examples can be obtained as a direct consequence of this fact, including the Morita equivalence between the symplectic groupoids of locally trivial bundles of symplectic manifolds and the cotangent bundle groupoids of the base spaces. Another example is the Morita equivalence between the symplectic groupoids of semi-direct products of Poisson manifolds and the symplectic groupoids of reduced Poisson manifolds.

We also note that complete symplectic realizations of Poisson manifolds are in fact the same as symplectic left modules of their α-simply connected symplectic groupoids, which are much more tractable. Hence, it is essential to find explicit constructions of symplectic groupoids for these Poisson manifolds, a problem to which we will devote the first section.

In Section 1, following a general construction of the symplectic groupoids of reduced Poisson manifolds by means of symplectic reduction for symplectic groupoids, we obtain the symplectic groupoids for locally trivial bundles of symplectic manifolds and semi-direct products of Poisson manifolds. The structures of these symplectic groupoids are also investigated. In particular, we prove that when the Lie group G is simply connected and the Poisson G-action on P has an equivariant momentum mapping J, the symplectic groupoid $\Gamma(-J)\times_s G^0$ of the corresponding semi-direct product of Poisson manifolds, constructed by this process, is isomorphic to a direct product of the symplectic groupoid Γ of P and the transformation groupoid $T^*G \rightrightarrows \mathcal{G}^*$, as expected.

In Section 2, we obtain several interesting Morita equivalence relations by using reduction, as described earlier in this introduction.

In Section 3, by using the machinery of Morita equivalence, we compute the complete symplectic realizations of locally trivial bundles of simply connected symplectic manifolds. Any such realization corresponds to a unique complete symplectic realization of the base space equipped with zero Poisson structure. The construction of such a realization from a given one of the base space is very similar to the usual Yang-Mills construction. Another example discussed in this section is the reduced space S/G of any simply connected symplectic manifold S.

Section 4 is devoted to the investigation of the complete symplectic realizations of semi-direct products of Poisson manifolds. The main result of this section is that the complete symplectic realizations of semi-direct products of Poisson manifolds $P\times_s\mathcal{G}^*$ are in one-to-one correspondence with the complete covariant symplectic realizations of P. Equivalently, on the level of symplectic modules, the symplectic left modules of $\Gamma(-J)\times_s G^0$, which is a symplectic groupoid of $P\times_s\mathcal{G}^*$, are in one-to-one correspondence with the covariant symplectic left Γ-modules. Here, by covariant symplectic realizations (or covariant symplectic modules), we mean symplectic realizations (or symplectic modules) that are also Hamiltonian G-spaces such that the G-actions commute with the realization morphisms.

In Section 5, we study the symplectic left modules of $(\Gamma)_0$, a symplectic groupoid of the reduced Poisson manifold P/G, by using Morita equivalence. As a consequence, we obtain a classification of complete symplectic realizations of reduced Poisson manifolds under certain assumptions.

Finally, we note that although all the discussions here are purely geometric, one can find counterparts in C*-algebra theory for many notations and techniques used here. In fact, it is the counterparts in C*-algebra theory that motivate some of our work. For instance, Poisson G-spaces correspond to C*-dynamical systems. The two basic constructions associated to a Poisson G-space, the semi-direct product of Poisson manifolds and the reduced Poisson manifold, correspond to the crossed product C*-algebra and the fixed point algebra. The notion of covariant realizations corresponds to that of covariant representations in C*-algebras. In C*-algebras, under certain reasonable conditions, one has a strong Morita equivalence between a certain ideal of the crossed product algebra and the generalized fixed point algebra [Rie4]. In particular, in the case of transformation C*-algebras, one obtains a strong Morita equivalence between the crossed product algebra $C_\infty(M)\propto G$ and the fixed point algebra $C_\infty(M/G)$, which is one of the basic examples in Morita equivalence theory of C*-algebras [Rie3]. Also, the one-to-one correspondence between the representations of crossed product C*-algebras and the covariant representations is a well-known fact in C*-algebras. All this once again indicates the similarities between these two subjects. Conversely, we hope that our discussions here are useful to the study of C*-algebras. In particular, by "quantizing" the results in this paper, we hope to obtain some results about C*-algebras. Indeed, Karasev-Maslov attempted much the same in their work on spectral theory [KaMa].

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1. Symplectic groupoids of reduced Poisson manifolds and semi-direct products of Poisson manifolds

The construction of symplectic groupoids for given Poisson manifolds is of particular importance in Poisson geometry. A great deal of progress has been made in this direction by many authors ([D1], [LW], [W3], [W5]). In this section, we will present a construction of symplectic groupoids for reduced Poisson manifolds and semi-direct products of Poisson manifolds, which is motivated by the construction in [W5]. Although the proofs are very easy, almost trivial in fact, this construction is very important to our later discussions. In the sequel, by $(X)_0$ we always denote the symplectic reduction at 0 [MW] of a Hamiltonian G-space X, *i.e.*, $(X)_0 = J^{-1}(0)/G$, where $J: X \to \mathcal{G}^*$ is an equivariant momentum mapping.

THEOREM 1.1. – *Let* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *be a symplectic groupoid,* G *a Lie group acting on* Γ *by symplectic groupoid automorphisms with an equivariant momentum mapping* $J \in Z^1(\Gamma, \mathcal{G}^*)$ (1). *Furthermore, assume that the action is free and proper. Then* $((\Gamma)_0 \rightrightarrows P/G, \alpha_1, \beta_1)$ *is a symplectic groupoid over* P/G, *where the source and target maps are defined by* $\alpha_1([\gamma]) = [\alpha(\gamma)]$, $\beta_1([\gamma]) = [\beta(\gamma)]$, *and the multiplication is defined by* $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$, *where* $\beta(\gamma_1) = \alpha(\gamma_2)$.

Proof. – By using the fact that the G-action preserves the groupoid structure and $J \in Z^1(\Gamma, \mathcal{G}^*)$, it is easy to check that the multiplication introduced above defines a well-defined groupoid structure on $(\Gamma)_0$. On the other hand, since the symplectic structure

(1) $Z^1(\Gamma, \mathcal{G}^*)$ denotes the set of all groupoid 1-cocycles of Γ with values in $\mathcal{G}^*$, where $\mathcal{G}^*$ is considered as an Abelian group with the usual addition. *I.e.*, $J \in Z^1(\Gamma, \mathcal{G}^*)$ if and only if $J: \Gamma \to \mathcal{G}^*$ satisfies $J(x \cdot y) = J(x) + J(y)$ for all composable pairs $(x, y) \in \Gamma^2$. *See* [WX] for some details.

on the reduced symplectic manifold $(\Gamma)_0$ is induced from that on Γ, and the graph of Γ is a Lagrangian submanifold of $\Gamma \times \Gamma \times \Gamma^-$, the graph of $(\Gamma)_0$ is a Lagrangian submanifold of $(\Gamma)_0 \times (\Gamma)_0 \times (\Gamma)_0^-$. Hence, $(\Gamma)_0$ is a symplectic groupoid over P/G.

Q.E.D.

Let (S, ω) be a symplectic G-space such that the G-action is free and proper. This action on S can be lifted in an evident way to an action on the fundamental groupoid $(\Pi_1(S) \rightrightarrows S, \alpha, \beta)$ by symplectic groupoid automorphisms, having an equivariant momentum mapping $J \in Z^1(\Pi_1(S), \mathscr{G}^*)$ given by

$$\langle J([\sigma]), \xi \rangle = -\int_\sigma \xi_S \lrcorner\, \omega, \qquad \forall [\sigma] \in \Pi_1(S), \quad \xi \in \mathscr{G}, \tag{1}$$

where $\mathscr{G}$ is the Lie algebra of G, $\mathscr{G}^*$ the dual of $\mathscr{G}$ and ξ_S the fundamental vector field on S defined by ξ [MiW]. As a consequence of the preceding theorem, we have the following corollary, where a symplectic groupoid over the reduced space S/G is obtained. Some form of this result has appeared in [Ka0], [Ka].

Corollary 1.1. — *$((\Pi_1(S))_0 \rightrightarrows S/G, \alpha_1, \beta_1)$ is a symplectic groupoid over S/G, where $\alpha_1([\sigma]) = [\sigma(0)]$, $\beta_1([\sigma]) = [\sigma(1)]$ and the multiplication is defined by $[\sigma_1] \cdot [\sigma_2] = [\sigma_1 \cdot \sigma_2]$ for any $\sigma, \sigma_1, \sigma_2 \in \Pi_1(S)$ such that $\sigma_1(1) = \sigma_2(0)$.*

As an immediate consequence of the above corollary, we have the following proposition describing the symplectic leaves of the reduced space S/G, a process that can be viewed as a generalization of the usual symplectic reduction.

Proposition 1.1. — *If S is a symplectic G-space, then $\beta(J^{-1}(0) \cap \alpha^{-1}(\mathscr{O}_{x_0}))/G$ is a symplectic manifold, where the symplectic structure is naturally induced from that on S, $J: \Pi_1(S) \to \mathscr{G}^*$ is the momentum mapping as given by equation (1) and $\mathscr{O}_{x_0}$ is the G-orbit through x_0.*

In fact, $\beta(J^{-1}(0) \cap \alpha^{-1}(\mathscr{O}_{x_0}))/G$ is the symplectic leaf of S/G passing through $\mathscr{O}_{x_0} \in S/G$.

In particular, if the G-action on S has an equivariant momentum mapping J′, then the momentum J on $\Pi_1(S)$ can be written as $J(r) = J'(r(0)) - J'(r(1))$. Therefore, $r \in J^{-1}(0) \cap \alpha^{-1}(\mathscr{O}_{x_0})$ if and only if $J'(r(0)) = J'(r(1))$ and $r(0) \in \mathscr{O}_{x_0}$, which is equivalent to saying that

$$\beta(r) = r(1) \in J'^{-1}(\mathscr{O}_u), \qquad \text{where} \quad u = J'(x_0).$$

Therefore, $\beta(J^{-1}(0) \cap \alpha^{-1}(\mathscr{O}_{x_0})) = J'^{-1}(\mathscr{O}_u)$. Hence, we obtain the usual symplectic reduction $J'^{-1}(\mathscr{O}_u)/G\,(\cong J'^{-1}(u)/G_u)$.

Example 1.1. (Weinstein [W5]). — Let $F \to P \overset{\pi}{\to} M$ be a locally trivial bundle of symplectic manifolds with structure group G. Suppose that F is a Hamiltonian G-space with an equivariant momentum mapping J. Let B be the associated principal bundle with zero Poisson structure; then the reduced Poisson manifold $(B \times F)/G$ is Poisson diffeomorphic to P. According to Theorem 1.1, $(T^*B \times F \times F^-)_0$ is a symplectic groupoid over $(B \times F)/G\,(\cong P)$. As a symplectic manifold, $(T^*B \times F \times F^-)_0$ is just the Yang-Mills-Higgs phase space for a classical particle with configuration space M and internal phase space $F \times F^-$ [W4], which is just a fibre product of the bundle $\tilde{P}(= P \times_M P) \to M$, with the cotangent bundle T^*M considered as an affine bundle rather than a vector bundle, and is usually denoted by $T^*M \times_M \tilde{P}$. The groupoid multiplication is given by $(\theta_1, x, y) \cdot (\theta_2, y, z) = (\theta_1 + \theta_2, x, z)$ with source and target maps $\alpha_1(\theta, x, y) = x$ and $\beta_1(\theta, x, y) = y$, for any (θ_1, x, y), (θ_2, y, z), $(\theta, x, y) \in T^*M \times_M \tilde{P}$.

Another useful construction of Poisson manifolds is the so-called semi-direct product [W4], which is defined as follows. Consider a Poisson manifold on which G acts by automorphisms. The quotient Poisson manifold $(P \times T^*G)/G$, where G acts on the factor T^*G by the lifts of right translations, is diffeomorphic to $P \times \mathscr{G}^*$ under the correspondence

$$\Phi: \quad [\theta, \alpha_g] \to (g \cdot \theta, r_g^* \alpha_g). \tag{2}$$

The induced Poisson structure on $P \times \mathscr{G}^*$ is in general not the product Poisson structure. This manifold $P \times \mathscr{G}^*$ together with the induced Poisson structure is called the semi-direct product of P and $\mathscr{G}^*$ with respect to the G-action on P and is denoted by $P \times_s \mathscr{G}^*$.

In order to construct symplectic groupoids over semi-direct products of Poisson manifolds, we begin with the following two well-known constructions for building up new groupoids from the old ones, namely skew product and semi-direct product of groupoids [Ren].

DEFINITION 1.1. – *Let $(\Gamma \rightrightarrows \Gamma_0, \alpha_0, \beta_0)$ be a groupoid, A a group and $c: \Gamma \to A$ a homomorphism. The skew product $\Gamma(c)$ is the groupoid $\Gamma \times A$ with unit space $\Gamma_0 \times A$, where*

$$\alpha(x, a) = (\alpha_0(x), a), \qquad \beta(x, a) = (\beta_0(x), a \cdot c(x)) \qquad \textit{and} \qquad (x, a) \cdot (y, b) = (xy, a),$$

if $\beta_0(x) = \alpha_0(y)$ and $b = a \cdot c(x)$.

DEFINITION 1.2. – *Let $(\Gamma \rightrightarrows \Gamma_0, \alpha_0, \beta_0)$ be a groupoid, on which the group A acts by automorphisms $\alpha: A \to \mathrm{Aut}(\Gamma)$. The semi-direct product $\Gamma \times_s A$ is the groupoid $\Gamma \times A$ with unit space Γ_0, where (x, a) and (z, b) are composable if $z = a^{-1} \cdot y$ with x, y composable, $(x, a) \cdot (a^{-1} \cdot y, b) = (xy, ab)$, and $(x, a)^{-1} = (a^{-1} \cdot x^{-1}, a^{-1})$. Then, $\alpha(x, a) = \alpha_0(x)$ and $\beta(x, a) = a^{-1} \cdot \beta_0(x)$.*

When Γ is a group, this semi-direct product is the usual semi-direct product of groups.

THEOREM 1.2. – *Let $(\Gamma \rightrightarrows P, \alpha, \beta)$ be a symplectic groupoid, on which G acts by symplectic groupoid automorphisms with an equivariant momentum mapping $J \in Z^1(\Gamma, \mathscr{G}^*)$. Then $\Gamma(-J) \times_s G^0$ is a symplectic groupoid over the semi-direct product $P \times_s \mathscr{G}^*$, where G^0, the opposite group of G, acts on the skew-product groupoid $\Gamma(-J)$ by $g \cdot (\gamma, u) = (g^{-1} \cdot \gamma, \mathrm{Ad}^*_{g^{-1}} u)$. Here $\mathrm{Ad}_g = l_{g^{-1}} \circ r_g$. The symplectic structure on $\Gamma(-J) \times_s G^0 (\cong \Gamma \times \mathscr{G}^* \times G)$ is given by $\varphi^*(\omega_\Gamma \oplus -\omega_{T^*G})$, where ω_{T^*G} is the standard symplectic structure on T^*G, ω_Γ is the symplectic structure on Γ, and φ is the map from $\Gamma \times \mathscr{G}^* \times G$ to $\Gamma \times T^*G$ defined by $\varphi(\gamma, u, g) = (\gamma, l^*_{g^{-1}}(u - J(\gamma)))$.*

Proof. – The groupoid direct product H of Γ with the coarse groupoid $T^*G \times (T^*G)^- \rightrightarrows T^*G$ is a symplectic groupoid over $P \times T^*G$. Consider the G-action on $T^* G$ by the lifts of right translations. Then, this action's equivariant momentum mapping $J_1 : T^* G \to \mathcal{G}^*$ is given by $J_1(\alpha_g) = -l_g^* \alpha_g$. Identify $T^* G$ with $\mathcal{G}^* \times G$ by right translations. Under this identification, the above G action on T^*G is given by $h \cdot (u, g) = (u, gh^{-1})$, for all $(u, g) \in \mathcal{G}^* \times G$, and the momentum mapping J_1 is given by $J_1(u, g) = -\mathrm{Ad}_{g^{-1}}^* u$.

Now G acts on H in a natural way; in fact, it is trivial to check that G acts on H by symplectic groupoid automorphisms. Therefore, $(H)_0$ is a symplectic groupoid over $(P \times T^* G)/G$, which is just the semi-direct product $P \times_s \mathcal{G}^*$, by definition. The momentum mapping $\bar{J}$ on $H = \Gamma \times T^* G \times (T^* G)^- \cong \Gamma \times \mathcal{G}^* \times G \times \mathcal{G}^* \times G$ is given by

$$\bar{J}(\gamma, u_1, g_1, u_2, g_2) = J(\gamma) - \mathrm{Ad}_{g_1^{-1}}^* u_1 + \mathrm{Ad}_{g_2^{-1}}^* u_2.$$

so $\bar{J}^{-1}(0)$ is defined by the equation $u_2 = \mathrm{Ad}_{g_2}^* (\mathrm{Ad}_{g_1^{-1}}^* u_1 - J(\gamma))$; therefore, $(H)_0 = \bar{J}^{-1}(0)/G$ is diffeomorphic to $\Gamma \times \mathcal{G}^* \times G$ under the correspondence

$$\Psi: \quad [\gamma, u_1, g_1, u_2, g_2] \to (g_1 \gamma, u_1, g_2 g_1^{-1}). \tag{3}$$

Hence, the induced symplectic structure on $\Gamma \times \mathcal{G}^* \times G$ is given by $\varphi^*(\omega_\Gamma \oplus -\omega_{T^* G})$, where $\varphi : \Gamma \times \mathcal{G}^* \times G \to \Gamma \times T^* G$ is defined by $\varphi(\gamma, u, g) = (\gamma, l_{g^{-1}}^* (u - J(\gamma)))$. In order to define the groupoid structure, first note that under the identification $T^* G \cong \mathcal{G}^* \times G$ by right translations, the correspondence Φ defined by equation (2) becomes

$$\Phi: \quad [\theta, u, g] \to (g\theta, u). \tag{4}$$

Let α_1 and β_1 denote the source and target maps of the groupoid $(H)_0$, respectively. By Theorem 1.1, $\beta_1[\gamma, u, 1_G, u', g] = [\beta(\gamma), u', g]$, where 1_G is the identity of the group G. To apply the natural isomorphism Ψ from $(H)_0$ to $\Gamma \times \mathcal{G}^* \times G$, we need to take $u' = \mathrm{Ad}_g^* (u - J(\gamma))$. Then $[\gamma, u, 1_G, u', g]$ goes to (γ, u, g) under the correspondence Ψ, while $[\beta(\gamma), u', g]$ goes to

$$(g\beta(\gamma), u') = (g\beta(\gamma), \quad \mathrm{Ad}_g^* (u - J(\gamma))).$$

Therefore, $\beta_1 : \Gamma \times \mathcal{G}^* \times G \to P \times \mathcal{G}^*$ is given by $\beta_1(\gamma, u, g) = (g\beta(\gamma), \mathrm{Ad}_g^* (u - J(\gamma)))$. Similarly, α_1 is given by $\alpha_1(\gamma, u, g) = (\alpha(\gamma), u)$. As for the multiplication, again according to Theorem 1.1, we have

$$\begin{aligned}
&[\gamma_1, u_1, 1_G, \mathrm{Ad}_{g_1}^* (u_1 - J(\gamma_1)), g_1] \cdot [\gamma_2, u_2, 1_G, \mathrm{Ad}_{g_2}^* (u_2 - J(\gamma_2)), g_2] \\
&\quad = [\gamma_1, u_1, 1_G, \mathrm{Ad}_{g_1}^* (u_1 - J(\gamma_1)), g_1] \cdot [g_1^{-1} \gamma_2, u_2, g_1, \mathrm{Ad}_{g_2}^* (u_2 - J(\gamma_2)), g_2 g_1] \\
&\quad\quad = [\gamma_1 \cdot (g_1^{-1} \gamma_2), u_1, 1_G, \mathrm{Ad}_{g_2}^* (u_2 - J(\gamma_2)), g_2 g_1],
\end{aligned}$$

if $\beta(\gamma_1) = g_1^{-1} \alpha(\gamma_2)$ and $u_2 = \mathrm{Ad}_{g_1}^* (u_1 - J(\gamma_1))$.

Now under the isomorphism Ψ from $(H)_0$ to $\Gamma \times \mathcal{G}^* \times G$, $[\gamma_1, u_1, 1_G, \mathrm{Ad}_{g_1}^* (u_1 - J(\gamma_1)), g_1]$ and $[\gamma_2, u_2, 1_G, \mathrm{Ad}_{g_2}^* (u_2 - J(\gamma_2)), g_2]$ go to (γ_1, u_1, g_1) and (γ_2, u_2, g_2), respectively, while $[\gamma_1 \cdot (g_1^{-1} \gamma_2), u_1, 1_G, \mathrm{Ad}_{g_2}^* (u_2 - J(\gamma_2)), g_2 g_1]$ goes to $(\gamma_1 \cdot (g_1^{-1} \gamma_2), u_1, g_2 g_1)$. In other words, the multiplication on $\Gamma \times \mathcal{G}^* \times G$ is given by $[(\gamma_1, u_1, g_1) \cdot (\gamma_2, u_2, g_2) = (\gamma_1 \cdot (g_1^{-1} \gamma_2), u_1, g_2 g_1)$ for all composable pairs (γ_1, u_1, g_1) and (γ_2, u_2, g_2)

satisfying $\alpha(\gamma_2)=g_1\beta(\gamma_1)$ and $u_2=\mathrm{Ad}^*_{g_1}(u_1-\mathrm{J}(\gamma_1))$. Finally, it is not difficult to check directly that the groupoid structure on $\Gamma\times\mathscr{G}^*\times\mathrm{G}$ arising from this coincides with that on $\Gamma(-\mathrm{J})\times_s\mathrm{G}^0$.

Q.E.D.

If P is a Hamiltonian G-space with an equivariant momentum mapping $\mathrm{J}_1:\mathrm{P}\to\mathscr{G}^*$, the semidirect product $\mathrm{P}\times_s\mathscr{G}^*$ is isomorphic to the direct product Poisson structure $\mathrm{P}\times\mathscr{G}^*$ under the isomorphism κ defined by $\kappa(\theta, u)=(\theta, u-\mathrm{J}_1(\theta))$ ([KM], [W4]).

On the other hand, if $(\Gamma\rightrightarrows\mathrm{P},\alpha,\beta)$ is a symplectic groupoid over P, it is easy to see that $\mathscr{A}:\mathscr{G}\to\mathscr{X}(\Gamma)$ given by $\mathscr{A}(\xi)=\mathrm{X}_{\alpha^*\hat{\mathrm{J}}_1(\xi)}-\mathrm{X}_{\beta^*\hat{\mathrm{J}}_1(\xi)}$, for all $\xi\in\mathscr{G}$, is a Lie algebra homomorphism, where $\hat{\mathrm{J}}_1(\xi)(p)=\langle\mathrm{J}_1(p),\xi\rangle\in\mathrm{C}^\infty(\mathrm{P})$. If G is simply connected, corresponding to $\mathscr{A}$, there is a natural G-action Φ_g on Γ, which is a lift of the given Poisson G-action on P (denoted by φ_g) by symplectic groupoid automorphisms [CDW] and has $\mathrm{J}=\alpha^*\mathrm{J}_1-\beta^*\mathrm{J}_1$ as its equivariant momentum mapping. Therefore, we can form the symplectic groupoid $\Gamma(-\mathrm{J})\times_s\mathrm{G}^0$ over $\mathrm{P}\times_s\mathscr{G}^*$. We shall show below that this symplectic groupoid $\Gamma(-\mathrm{J})\times_s\mathrm{G}^0$ is naturally isomorphic to the symplectic groupoid direct product $\Gamma\times\mathrm{T}^*\mathrm{G}$, as expected. By R_g and L_g, we denote the Hamiltonian G-actions on Γ corresponding to the Lie algebra homomorphisms

$$\{\xi\mapsto -\mathrm{X}_{\beta^*\hat{\mathrm{J}}_1(\xi)}\,|\,\forall\xi\in\mathscr{G}\}\qquad\text{and}\qquad\{\xi\mapsto\mathrm{X}_{\alpha^*\hat{\mathrm{J}}_1(\xi)}\,|\,\forall\xi\in\mathscr{G}\},$$

respectively. Here, the notation R_g and L_g is motivated by the fact that if $\Gamma=\mathrm{T}^*\mathrm{G}$, then these actions are just the lifts of right translations and left translations, respectively.

It is clear by definition that $\mathrm{R}_g\circ\mathrm{L}_g=\mathrm{L}_g\circ\mathrm{R}_g=\Phi_g$.

LEMMA 1.1. — (1) $\alpha\circ\mathrm{R}_g=\alpha$ *and* $\beta\circ\mathrm{R}_g=\varphi_g\circ\beta$; $\alpha\circ\mathrm{L}_g=\varphi_g\circ\alpha$ *and* $\beta\circ\mathrm{L}_g=\beta$.

(2) $\mathrm{R}_g(\gamma_1\cdot\gamma_2)=\gamma_1\cdot(\mathrm{R}_g\gamma_2)$; $\mathrm{L}_g(\gamma_1\cdot\gamma_2)=(\mathrm{L}_g\gamma_1)\cdot\gamma_2$.

(3) $(\mathrm{R}_g\gamma_1)\cdot\gamma_2=\gamma_1\cdot(\mathrm{L}_{g^{-1}}\gamma_2)$.

Proof. — (1) is quite obvious.

(2) Without loss of generality, assume that $g=\exp\xi$ for some $\xi\in\mathscr{G}$, and $\gamma_1=\varphi^\alpha(u)$, where φ^α is a product of Hamiltonian flows generated by the functions of the form α^*f for $f\in\mathrm{C}^\infty(\mathrm{P})$ and u is an element in P. Let ψ_t^β denote the flow generated by the Hamiltonian vector field $-\mathrm{X}_{\beta^*\hat{\mathrm{J}}_1(\xi)}$. Then $\mathrm{R}_g\gamma=\psi_1^\beta(\gamma)$ for all $\gamma\in\Gamma$. Therefore,

$$\mathrm{R}_g(\gamma_1\cdot\gamma_2)=\mathrm{R}_g(\varphi^\alpha(\gamma_2))=\psi_1^\beta(\varphi^\alpha(\gamma_2))=\varphi^\alpha\psi_1^\beta(\gamma_2)=\varphi^\alpha(\mathrm{R}_g\gamma_2)=\gamma_1\cdot(\mathrm{R}_g\gamma_2).$$

Similarly, we have $\mathrm{L}_g(\gamma_1\cdot\gamma_2)=(\mathrm{L}_g\gamma_1)\cdot\gamma_2$.

(3) For the multiplication to be defined, we must have $g\cdot\beta(\gamma_1)=\alpha(\gamma_2)$. From this, it follows that $R_g\beta(\gamma_1)=L_{g^{-1}}\alpha(\gamma_2)$. Then, by using (2), we have

$$\begin{aligned}(R_g\gamma_1)\cdot\gamma_2 &= [R_g(\gamma_1\cdot\beta(\gamma_1))]\cdot\gamma_2\\ &=[\gamma_1\cdot(R_g\beta(\gamma_1))]\cdot\gamma_2\\ &=\gamma_1\cdot[L_{g^{-1}}\alpha(\gamma_2)\cdot\gamma_2]\\ &=\gamma_1\cdot[L_{g^{-1}}(\alpha(\gamma_2)\cdot\gamma_2)]\\ &=\gamma_1\cdot(L_{g^{-1}}\gamma_2).\end{aligned}$$

Q.E.D.

LEMMA 1.2. — (1) *We have*

$$\omega_\Gamma(\delta(R_g\gamma),\delta'(R_g\gamma))=\omega_\Gamma(\delta\gamma,\delta'\gamma)+\langle T(J_1\circ\beta)\delta\gamma,g^{-1}\cdot\delta' g\rangle - \langle T(J_1\circ\beta)\delta'\gamma,g^{-1}\cdot\delta g\rangle-\langle J_1(\beta(\gamma)),[g^{-1}\cdot\delta g,g^{-1}\cdot\delta' g]\rangle.$$

(2) *The map* $\mathscr{D}:\Gamma\times(T^*G)^-(\cong\Gamma\times\mathscr{G}^*\times G)\to\Gamma\times T^*G(\cong\Gamma\times\mathscr{G}^*\times G)$ *given by*

$$(\gamma,u,g)\mapsto(R_g\gamma,\operatorname{Ad}^*_{g^{-1}}u-J_1(\beta(\gamma)),g^{-1})$$

is a symplectic diffeomorphism.

Proof. — (1) By definition,

$$\delta(R_g\gamma)=TR_g\cdot\delta\gamma-X_{\beta^*\hat{J}_1(\delta g\cdot g^{-1})}\quad\text{and}\quad\delta'(R_g\gamma)=TR_g\cdot\delta'\gamma-X_{\beta^*\hat{J}_1(\delta' g\cdot g^{-1})}.$$

Thus,

$$\omega_\Gamma(\delta(R_g\gamma),\delta'(R_g\gamma))=\omega_\Gamma(TR_g\cdot\delta\gamma,TR_g\cdot\delta'\gamma)+\omega_\Gamma(TR_g\cdot\delta\gamma,-X_{\beta^*\hat{J}_1(\delta' g\cdot g^{-1})}) +\omega_\Gamma(-X_{\beta^*\hat{J}_1(\delta g\cdot g^{-1})},TR_g\cdot\delta'\gamma)+\omega_\Gamma(X_{\beta^*\hat{J}_1(\delta g\cdot g^{-1})},X_{\beta^*\hat{J}_1(\delta' g\cdot g^{-1})}).$$

Now $\omega_\Gamma(TR_g\cdot\delta\gamma,TR_g\cdot\delta'\gamma)=\omega_\Gamma(\delta\gamma,\delta'\gamma)$, since R_g is a Hamiltonian G-action. On the other hand,

$$\begin{aligned}\omega_\Gamma(TR_g\cdot\delta\gamma,-X_{\beta^*\hat{J}_1(\delta' g\cdot g^{-1})}) &=(TR_g\cdot\delta\gamma)\cdot(\beta^*\hat{J}_1(\delta' g\cdot g^{-1}))\\ &=(T\beta\, TR_g\cdot\delta\gamma)(\hat{J}_1(\delta' g\cdot g^{-1}))\\ &=\langle T(J_1\circ\beta\circ R_g)\delta\gamma,\delta' g\cdot g^{-1}\rangle\quad\text{(by Lemma 1.1)}\\ &=\langle T(\operatorname{Ad}^*_g\circ J_1\circ\beta)\delta\gamma,\delta' g\cdot g^{-1}\rangle\\ &=\langle T(J_1\circ\beta)\delta\gamma,T(\operatorname{Ad}_g)(\delta' g\cdot g^{-1})\rangle\\ &=\langle T(J_1\circ\beta)\delta\gamma,g^{-1}\cdot\delta' g\rangle.\end{aligned}$$

Similarly, we have $\omega_\Gamma(-X_{\beta^*\hat{J}_1(\delta g\cdot g^{-1})},TR_g\cdot\delta'\gamma)=-\langle T(J_1\circ\beta)\delta'\gamma,g^{-1}\cdot\delta g\rangle$.

For the last term, we get

$$\begin{aligned}\omega_\Gamma(X_{\beta^*\hat{J}_1(\delta g\cdot g^{-1})}, X_{\beta^*\hat{J}_1(\delta' g\cdot g^{-1})}) &= \{\beta^*\hat{J}_1(\delta g\cdot g^{-1}), \beta^*\hat{J}_1(\delta' g\cdot g^{-1})\}(R_g\gamma)\\ &= -\{\hat{J}_1(\delta g\cdot g^{-1}), \hat{J}_1(\delta' g\cdot g^{-1})\}(\beta(R_g\gamma))\\ &= -\hat{J}_1([\delta g\cdot g^{-1}, \delta' g\cdot g^{-1}])(\beta(R_g\gamma))\\ &= -\langle (J_1\circ\beta\circ R_g)(\gamma), [\delta g\cdot g^{-1}, \delta' g\cdot g^{-1}]\rangle\\ &= -\langle (\mathrm{Ad}_g^*\circ J_1\circ\beta)(\gamma), [\delta g\cdot g^{-1}, \delta' g\cdot g^{-1}]\rangle\\ &= -\langle (J_1\circ\beta)(\gamma), [g^{-1}\cdot\delta g, g^{-1}\cdot\delta' g]\rangle.\end{aligned}$$

(2) Under the identification T^*G with $\mathscr{G}^*\times G$ by right translations, the lift i^*: $T^*G\to T^*G$ of the diffeomorphism i of G, $i(g)=g^{-1}$, is given by

$$(u, g)\mapsto(-\mathrm{Ad}_{g^{-1}}^* u, g^{-1}).$$

Hence, the map $i_1^*:(T^*G)^-\to T^*G$ defined by $i_1^*(u, g)=(\mathrm{Ad}_{g^{-1}}^* u, g^{-1})$ is a symplectic diffeomorphism. Using this fact, part (1) of this lemma, as well as the explicit formula describing the symplectic structure on $T^*G(\cong\mathscr{G}^*\times G)$ (*see* Example 3.1 of [MiW] for the formula), we can easily prove (2) by a routine computation.

Q.E.D.

With the two lemmas above, we can prove the following:

THEOREM 1.3. — *Suppose that* P *is a Hamiltonian* G*-space with an equivariant momentum mapping* $J_1:P\to\mathscr{G}^*$, *and* G *is simply connected so that* G *acts on* Γ *by symplectic groupoid automorphisms with momentum mapping* $J=\alpha^*J_1-\beta^*J_1$ *as in the observation preceding Lemma* 1.1. *Then the symplectic groupoid* $\Gamma(-J)\times_s G^0$ *is isomorphic to the symplectic groupoid direct product* $\Gamma\times T^*G$ *under the correspondence*

$$\mathscr{E}:\quad (\gamma, u, g)\to(R_g\gamma, u-J_1(\alpha(\gamma)), g^{-1}).$$

Proof. — Note that the symplectic groupoid structure on $T^*G(\cong\mathscr{G}^*\times G)$ can be described as follows [MiW]:

$$\alpha(u, g)=u,\qquad \beta(u, g)=\mathrm{Ad}_{g^{-1}}^* u,\qquad \varepsilon(u)=(u, e);$$
$$(u_1, g_1)\cdot(\mathrm{Ad}_{g_1^{-1}}^* u_1, g_2)=(u_1, g_1 g_2).$$

By using Lemma 1.1, it is simple to check that the diagram

$$\begin{array}{ccc}\Gamma(-J)\times_s G^0 & \xrightarrow{\varepsilon} & \Gamma\times T^*G\\ {\scriptstyle\alpha}\downarrow\downarrow{\scriptstyle\beta} & & {\scriptstyle\alpha}\downarrow\downarrow{\scriptstyle\beta}\\ P\times_s\mathscr{G}^* & \xrightarrow{\varkappa} & P\times\mathscr{G}^*\end{array}$$

commutes with respect to both α and β. As for the multiplication, on the one hand,

$$\begin{aligned}\mathscr{E}(\gamma_1, u_1, g_1)\cdot\mathscr{E}(\gamma_2, u_2, g_2) &= (R_{g_1}\gamma_1, u_1 - J_1(\alpha(\gamma_1)), g_1^{-1})\cdot(R_{g_2}\gamma_2, u_2 - J_1(\alpha(\gamma_2)), g_2^{-1})\\ &= (R_{g_1}\gamma_1\cdot R_{g_2}\gamma_2, u_1 - J_1(\alpha(\gamma_1)), g_1^{-1}g_2^{-1}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathscr{E}[(\gamma_1, u_1, g_1)\cdot(\gamma_2, u_2, g_2)] &= \mathscr{E}(\gamma_1\cdot(g_1^{-1}\gamma_2), u_1, g_2 g_1)\\ &= (R_{g_2 g_1}(\gamma_1\cdot(g_1^{-1}\gamma_2)), u_1 - J_1(\alpha(\gamma_1)), g_1^{-1}g_2^{-1}).\end{aligned}$$

So it remains to show that $R_{g_2 g_1}(\gamma_1\cdot(g_1^{-1}\gamma_2)) = R_{g_1}\gamma_1\cdot R_{g_2}\gamma_2$. By Lemma 1.1,

$$\begin{aligned}R_{g_2 g_1}(\gamma_1\cdot(g_1^{-1}\gamma_2)) &= \gamma_1\cdot R_{g_2 g_1}(g_1^{-1}\gamma_2)\\ &= \gamma_1\cdot(R_{g_2}R_{g_1}(L_{g_1}R_{g_1})^{-1}\gamma_2)\\ &= \gamma_1\cdot L_{g_1}^{-1}R_{g_2}\gamma_2\\ &= R_{g_1}\gamma_1\cdot R_{g_2}\gamma_2.\end{aligned}$$

Hence, $\mathscr{E}$ preserves the groupoid structure.

Finally, note that $\mathscr{E} = \mathscr{D}\circ\varphi$, where $\varphi: \Gamma(-J)\times_s G^0 \to \Gamma\times(T^*G)^-$ is as defined in Theorem 1.2. Therefore $\mathscr{E}$ is a symplectic diffeomorphism.

Q.E.D.

A Poisson manifold is called integrable in the sense of Dazord [D1] if it is the unit space of some symplectic groupoid.

If the symplectic groupoid Γ is α-simply connected, then any Poisson action on the base Poisson manifold P can be lifted naturally to an action on Γ by symplectic groupoid automorphisms with an equivariant momentum mapping $J\in Z^1(\Gamma, \mathscr{G}^*)$ (*cf.* [WX]). Hence, the conditions of Theorem 1.1 and Theorem 1.2 hold automatically. Consequently, we have the following:

COROLLARY 1.2. — *If* P *is an integrable Poisson* G-*space, then both* P/G *and* $P\times_s\mathscr{G}^*$ *are integrable.* (*Here, when we refer to* P/G, *we assume, of course, that* P/G *exists as a smooth manifold.*)

2. Morita equivalence in reduction

In the last section, we have provided a construction of symplectic groupoids for reduced Poisson manifolds and semi-direct product Poisson manifolds by means of symplectic reduction of symplectic groupoids. In this section, we will see that this process produces many interesting examples of Morita equivalence as well. Our main theorem is the following:

THEOREM 2.1. — *Let* $(\Gamma \rightrightarrows E, \alpha, \beta)$ *be a symplectic groupoid, on which* G *acts freely and properly by symplectic groupoid automorphisms with an equivariant momentum mapping*

$J_1 \in Z^1(\Gamma, \mathcal{G}^*)$. *Suppose that* F *is a Hamiltonian* G*-space with an equivariant momentum mapping* J_2. *Then the symplectic groupoid* $((\Gamma \times F \times F^-)_0 \rightrightarrows (E \times F)/G, \alpha_1, \beta_1)$ *is Morita equivalent* [XI] *to the symplectic groupoid* $((\Gamma)_0 \rightrightarrows E/G, \alpha_2, \beta_2)$ *with equivalence bimodule* $((\Gamma \times F)_0; \rho; \sigma)$, *provided both* ρ *and* σ *are surjective submersions, where*

$$\rho : (\Gamma \times F)_0 \to (E \times F)/G$$

is defined by $\rho([\gamma, x]) = [\alpha(\gamma), x]$, $\sigma : (\Gamma \times F)_0 \to E/G$ *is* $\sigma([\gamma, x]) = [\beta(\gamma)]$, *and the left and right groupoid actions are defined in an obvious way by* $[\gamma_1, x, y] \cdot [\gamma_2, y] = [\gamma_1 \gamma_2, x]$ *and* $[\gamma_2, y] \cdot [\gamma_3] = [\gamma_2 \gamma_3, y]$, *respectively.*

Remark. — It goes without saying that we need the usual assumptions of clean value, etc., to make the symplectic reduced spaces smooth manifolds.

Proof. — It is easy to check that both ρ and σ, and the left and right groupoid actions are all well-defined. In order to show that the left groupoid action on $(\Gamma \times F)_0$ is free, let us assume that $[\gamma_1, x, y] \cdot [\gamma_2, y] = [\gamma_2, y]$. That is, $[\gamma_1 \gamma_2, x] = [\gamma_2, y]$. Hence, there is an element $g \in G$ such that $x = gy$ and $\gamma_1 \gamma_2 = g \gamma_2$. Thus,

$$\beta(\gamma_2) = \beta(\gamma_1 \gamma_2) = \beta(g \gamma_2) = g \beta(\gamma_2).$$

Therefore, $g = \mathrm{id}$. From this, it follows immediately that $\gamma_1 \in E$ and $x = y$. So the left groupoid action is free. The freeness of the right groupoid action on $(\Gamma \times F)_0$ is obvious. As for the transitivity of the left groupoid action on σ-fibres, we assume that $[\gamma_1, x_1], [\gamma_2, x_2] \in (\Gamma \times F)_0$, such that

$$\sigma([\gamma_1, x_1]) = \sigma([\gamma_2, x_2]), \quad \textit{i.e.}, \qquad [\beta(\gamma_1)] = [\beta(\gamma_2)].$$

Hence, there is a $g \in G$ such that $\beta(\gamma_1) = g \beta(\gamma_2) = \beta(g \gamma_2)$. Take $\gamma_3 = (g \gamma_2) \cdot \gamma_1^{-1}$. Then $(\gamma_3, gx_2, x_1) \in J^{-1}(0)$, where $J : \Gamma \times F \times F^- \to \mathcal{G}^*$ is the momentum mapping given by $J(\gamma, x, y) = J_1(\gamma) + J_2(x) - J_2(y)$, since

$$\begin{aligned} J(\gamma_3, gx_2, x_1) &= J_1(\gamma_3) + J_2(gx_2) - J_2(x_1) \\ &= \mathrm{Ad}_g^* J_1(\gamma_2) - J_1(\gamma_1) + \mathrm{Ad}_g^* J_2(x_2) - J_2(x_1) \\ &= \mathrm{Ad}_g^* (J_1(\gamma_2) + J_2(x_2)) - (J_1(\gamma_1) + J_2(x_1)) \\ &= 0. \end{aligned}$$

It can be easily seen that $[\gamma_3, gx_2, x_1] \cdot [\gamma_1, x_1] = [\gamma_2, x_2]$. So the left groupoid action on $(\Gamma \times F)_0$ is transitive on each σ-fibre. The transitivity of the right groupoid action on ρ-fibres of $(\Gamma \times F)_0$ can be proved similarly.

Finally, it is straightforward to show directly that both the left and the right groupoid actions are symplectic, using the same techniques used repeatedly in [X1].

Q.E.D.

Remarks. — (1) It is worth noting that σ is always surjective, while ρ is not, in general. If ρ is not surjective, one can still get an equivalence relation as in the theorem by considering the symplectic subgroupoid of $((\Gamma \times F \times F^-)_0 \rightrightarrows (E \times F)/G, \alpha_1, \beta_1)$ over $\rho((\Gamma \times F)_0)$ instead of the whole groupoid.

(2) The above theorem is in fact a special case of a more general theorem in Section 4, which says, roughly speaking, that the reduced symplectic groupoids of Morita equivalent symplectic groupoids are still Morita equivalent. For a precise statement and the proof, *see* Theorem 4.1.

Theorem 2.1 has many interesting consequences, one of which applies to the case of the symplectic groupoid over a bundle of symplectic manifolds, constructed in Example 1.1. Let $F \to P \overset{\pi}{\to} M$ be a locally trivial bundle of symplectic manifolds. Take E to be its associated principal bundle with zero Poisson structure. Then E/G is diffeomorphic to M, and the symplectic groupoid $((\Gamma)_0 \rightrightarrows E/G, \alpha_2, \beta_2)$ is just the cotangent bundle groupoid $T^* M \rightrightarrows M$. On the other hand, we have indicated in Example 1.1 that the symplectic groupoid $((\Gamma \times F \times F^-)_0 \rightrightarrows (E \times F)/G, \alpha_1, \beta_1)$ is diffeomorphic to

$$(T^* M \times_M \tilde{P} \rightrightarrows P, \alpha_1, \beta_1).$$

Also, it is quite clear that the equivalence bimodule $(\Gamma \times F)_0$ is symplectically diffeomorphic to the Yang-Mills-Higgs phase space $T^* M \times_M P$. Moreover, the morphisms $\rho : T^* M \times_M P \to P$ and $\sigma : T^* M \times_M P \to M$ are the natural projections, and the left and right groupoid actions on $T^* M \times_M P$ are given by $(\theta_1, p_1, p_2) . (\theta_2, p_2) = (\theta_1 + \theta_2, p_1)$ and $(\theta_2, p_2) . \theta_3 = (\theta_2 + \theta_3, p_2)$, respectively. Hence, we have the following:

COROLLARY 2.1. — *Suppose that* $F \to P \overset{\pi}{\to} M$ *is a locally trivial bundle of symplectic manifolds. Then the symplectic groupoid* $(T^* M \times_M \tilde{P} \rightrightarrows P, \alpha_1, \beta_1)$ *is Morita equivalent to the symplectic groupoid* $T^* M \rightrightarrows M$ *with equivalence bimodule* $(T^* M \times_M P; \rho; \sigma)$, *where* ρ, σ *and the left and right groupoid actions are as described above.*

In particular, we have

COROLLARY 2.2. — *If* $P \overset{\pi}{\to} M$ *is a locally trivial bundle of simply connected symplectic manifolds, then* P *is Morita equivalent to* M *with zero Poisson structure.*

Remarks. – (1) *See* [X2] for a general discussion in the case where the symplectic strucrures along the fibres are not constant.

(2) The C*-version of this result is rather trivial. Any infinite dimensional Hilbert bundle is trivial according to the remarkable theorem of Kuiper [Ku]. Thus, its associated bundle of compact operator algebras must be isomorphic to a direct tensor product of the algebra of functions on the base space with the algebra of compact operators. Hence, this associated bundle is obviously Morita equivalent to the algebra of functions on the base space.

Our next corollary indicates that semi-direct products of Poisson manifolds and reduced Poisson manifolds are Morita equivalent under certain mild conditions. First of all, we need a lemma.

LEMMA 2.1 (Weinstein [W4]). — *Let* X *be a Hamiltonian* G*-space with momentum mapping* J. *Then the reduced space* $(X \times T^* G)_0$ *is naturally isomorphic to* X *under the mapping* $\mathscr{F} : [x, l^*_{g^{-1}} J(x)] \to gx$.

If we identify T^*G *with* $\mathscr{G}^* \times G$ *by right translations, then the correspondence* $\mathscr{F}$ *becomes* $[x, \mathrm{Ad}_g^* J(x), g] \mapsto gx$.

THEOREM 2.2. — *Let* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *be a symplectic groupoid, on which* G *acts by symplectic groupoid automorphisms with an equivariant momentum mapping* $J \in Z^1(\Gamma, \mathscr{G}^*)$. *Suppose that the map* $\rho: \Gamma \to P \times_s \mathscr{G}^*$ *given by* $\rho(\gamma) = (\alpha(\gamma), J(\gamma))$ *is a submersion. Then* $((\Gamma)_0 \rightrightarrows P/G, \alpha_2, \beta_2)$ *is Morita equivalent to the symplectic subgroupoid of* $(\Gamma(-J) \times_s G^0 \rightrightarrows P \times_s \mathscr{G}^*, \alpha_1, \beta_1)$ *over the Poisson submanifold* $\rho(\Gamma) \subseteq P \times_s \mathscr{G}^*$ *with equivalence bimodule* $(\Gamma; \rho; \sigma)$, *where* $\sigma: \Gamma \to P/G$ *is defined by* $\sigma(\gamma) = [\beta(\gamma)]$. *On* Γ, *the left groupoid action of* $\Gamma(-J) \times_s G^0$ *is given by* $(\gamma_1, u_1, g_1) \cdot \gamma = \gamma_1 \cdot g_1^{-1} \gamma$, *if* $\rho(\gamma) = \beta(\gamma_1, u_1, g_1)$, *while the right groupoid action of* $(\Gamma)_0$ *is given by* $\gamma \cdot [\gamma_2] = \gamma \cdot \gamma_2$, *where* $\beta(\gamma) = \alpha(\gamma_2)$.

Proof. — Again, we shall use Theorem 2.1. Take $E = P$ and $F = T^*G$ with G acting on T^*G by the lifts of right translations. As we have seen in Theorem 1.2, the symplectic groupoid $(\Gamma \times F \times F^-)_0 \rightrightarrows (E \times F)/G$ is isomorphic to $\Gamma(-J) \times_s G^0 \rightrightarrows P \times_s \mathscr{G}^*$. Now by Lemma 2.1, $(\Gamma \times F)_0 = (\Gamma \times T^*G)_0$ is naturally isomorphic to Γ under the correspondence $\mathscr{F}$. By Theorem 2.1, $\rho[\gamma, J(\gamma), 1_G] = [\alpha(\gamma), J(\gamma), 1_G]$. Now $[\gamma, J(\gamma), 1_G]$ goes to γ under $\mathscr{F}$, while $[\alpha(\gamma), J(\gamma), 1_G]$ goes to $(\alpha(\gamma), J(\gamma)) \in P \times_s \mathscr{G}^*$ under the correspondence Φ. Hence, we have $\rho(\gamma) = (\alpha(\gamma), J(\gamma))$. Similarly, we have $\sigma(\gamma) = [\beta(\gamma)]$. The right groupoid $(\Gamma)_0$-action on $(\Gamma \times T^*G)_0$ is given by $[\gamma, J(\gamma), 1_G] \cdot [\gamma_2] = [\gamma \cdot \gamma_2, J(\gamma), 1_G]$, if $\beta(\gamma) = \alpha(\gamma_2)$. Under the map $\mathscr{F}$, $[\gamma, J(\gamma), 1_G]$ corresponds to γ and $[\gamma \cdot \gamma_2, J(\gamma), 1_G]$ corresponds to $\gamma \cdot \gamma_2$. Hence, it follows immediately that the right groupoid action on Γ is given by $\gamma \cdot [\gamma_2] = \gamma \cdot \gamma_2$, if $\beta(\gamma) = \alpha(\gamma_2)$.

As for the left groupoid action of $\Gamma(-J) \times_s G^0$ on Γ, we leave the proof to Section 4, where a more general result will be proved (*see* Theorem 4.2).

Q.E.D.

COROLLARY 2.3. — *Under the same assumptions as in the preceding theorem, and if* G *and* $\alpha^{-1}(p) \cap J^{-1}(u)$ *are connected and simply connected for all* $p \in P$ *and* $u \in \mathscr{G}^*$, *then* P/G *is Morita equivalent to the Poisson submanifold* $\rho(\Gamma)$ *of the semi-direct product* $P \times_s \mathscr{G}^*$.

If G is simply connected and the Poisson G-action on P has an equivariant momentum mapping J_1, then the symplectic groupoid $\Gamma(-J) \times_s G^0$ is isomorphic to the symplectic groupoid direct product $\Gamma \times T^*G$. Therefore, in this case, we have

PROPOSITION 2.1. — *Let* P *be a Hamiltonian Poisson* G-*space with an equivariant momentum mapping* J_1 *and* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *a symplectic groupoid over* P. *Suppose that* G *is simply connected and* G *acts on* Γ *by symplectic groupoid automorphisms with momentum mapping* $J = \alpha^* J_1 - \beta^* J_1$. *If the morphism* $\rho: \Gamma \to P \times \mathscr{G}^*$ *given by* $\rho(\gamma) = (\alpha(\gamma), -J_1(\beta(\gamma)))$ *is a submersion, then as a symplectic groupoid,* $((\Gamma)_0 \rightrightarrows P/G, \alpha_2, \beta_2)$ *is Morita equivalent to* $(\Gamma \times T^*G \rightrightarrows P \times \mathscr{G}^*, \alpha_1, \beta_1)|_{\rho(\Gamma)}$ *with equivalence bimodule* $(\Gamma; \rho; \sigma)$, *where the groupoid* $(\Gamma \times T^*G \rightrightarrows P \times \mathscr{G}^*, \alpha_1, \beta_1)$ *is a symplectic groupoid direct product of* $\Gamma \rightrightarrows P$ *with* $T^*G \rightrightarrows \mathscr{G}^*$, *and* $(\Gamma \times T^*G \rightrightarrows P \times \mathscr{G}^*, \alpha_1, \beta_1)|_{\rho(\Gamma)}$ *is its subgroupoid over* $\rho(\Gamma) \subseteq P \times \mathscr{G}^*$.

In particular, we have

COROLLARY 2.4. — *Let* S *be a symplectic Hamiltonian* G*-space with an equivariant momentum mapping* J. *If* G *is simply connected and* $J: S \rightarrow \mathscr{G}^*$ *is a submersion, then*

1. *The symplectic groupoids*

$$(\Pi_1(S) \times T^*G \rightrightarrows S \times \mathscr{G}^*, \alpha_1, \beta_1)|_{\rho(\Pi_1(S))} \quad \text{and} \quad ((\Pi_1(S))_0 \rightrightarrows S/G, \alpha_2, \beta_2)$$

are Morita equivalent.

2. *If* $\tilde{J}^{-1}(u)$ *is simply connected for all* $u \in \mathscr{G}^*$, *where* $\tilde{J}$ *is the corresponding momentum mapping on the universal covering space* $\tilde{S}$, *then* S/G *is Morita equivalent to* $S \times (-J(S))$.

COROLLARY 2.5. — *If* S *is a simply connected symplectic Hamiltonian* G*-space with an equivariant momentum mapping* J such that J *is a submersion, and* G *and all* J*-fibres are simply connected, then* S/G *is Morita equivalent to* $-J(S)$, *a Poisson submanifold of* $\mathscr{G}^*$.

In fact, on the level of symplectic groupoids, we always have the following general fact.

THEOREM 2.3. — *Let* S *be a symplectic Hamiltonian* G*-space with an equivariant momentum mapping* J, *such that* J *is a submersion, then the symplectic groupoid* $((S \times S^-)_0 \rightrightarrows S/G, \alpha_1, \beta_1)$ *is Morita equivalent to* $(T^*G \rightrightarrows \mathscr{G}^*)|_{-J(S)}$, *the symplectic subgroupoid of* $T^*G \rightrightarrows \mathscr{G}^*$ *over* $-J(S)$, *with equivalence bimodule* $(S; \pi; -J)$ *and obvious left and right groupoid actions, where* π *is the natural projection from* S *onto* S/G.

Note that here the corresponding dual pair $S/G \overset{\pi}{\leftarrow} S \overset{-J}{\longrightarrow} \mathscr{G}^*$ is just the usual well-known dual pair studied by Weinstein [W1].

The proof of this theorem is rather straightforward and similar to those of the theorems above, so we omit it here [*see* the remark (2) following the proof of Theorem 2.1].

Finally, to end this section, we point out a direct consequence of Proposition 2.1 in the following:

COROLLARY 2.6. — *Under the assumption as in Proposition* 2.1, *any symplectic leaf of* P/G *is of the form* $\beta[\alpha^{-1}(L) \cap (J_1 \circ \beta)^{-1}(\mathcal{O}_u)]/G$ *for a symplectic leaf* L *of* P *and a coadjoint orbit* $\mathcal{O}_u$ *of* $\mathscr{G}^*$.

In particular, if P is a symplectic manifold, the symplectic leaf described above becomes the usual Marsden-Weinstein reduction.

3. The generalized Yang-Mills construction and examples of symplectic realizations

First recall that a symplectic realization of a Poisson manifold P is a pair (X, ρ), where X is a symplectic manifold and ρ is a Poisson morphism from X to P. A symplectic realization $\rho: X \rightarrow P$ is called complete if ρ is complete as a Poisson map and ρ is said to be full if it is a submersion. As an application of the machinery of Morita equivalence of Poisson manifolds, we can compute complete symplectic realizations of some particularly interesting Poisson manifolds, such as reduced Poisson manifolds and semi-direct products of Poisson manifolds. We will devote the rest of this paper to these computations. In this section, we start with a few simple examples.

Our first example will be a locally trivial bundle of simply connected symplectic manifolds $F \to P \xrightarrow{\pi} M$, as discussed in Example 1.1. Corollary 2.2 asserts that P is Morita equivalent to M with the zero Poisson structure. Hence, any complete symplectic realization of P corresponds to a unique complete symplectic realization of M, and vice versa [X2]. In order to describe complete symplectic realizations of P more precisely, we shall start with any complete symplectic realization $\rho : X \to M$ and then construct the corresponding realization of P. Through this process, we shall obtain all complete symplectic realizations of P. Note that $\rho : X \to M$ becomes a symplectic left module of the groupoid $(T^* M \rightrightarrows M, p, p)$ under the action defined as follows. For any $\theta \in T^*_m M$, let $(\rho^* \theta)^{\#}$ denote the vector field on $\rho^{-1}(m)$ defined by the equation:

$$(5) \qquad (\rho^* \theta)^{\#} \lrcorner\, \omega = \rho^* \theta,$$

where ω is the symplectic form on X. Let φ^ρ_t be the flow of $(\rho^* \theta)^{\#}$, and define the groupoid action by $\theta \cdot x = \varphi^\rho_1(x)$ for any $x \in \rho^{-1}(m)$. φ^ρ_1 always exists, since ρ is complete. This definition works when m is a regulr value. However, this action can always be defined without any conditions on regularity, as in the following. Take a smooth function f such that $\theta = (df)(m)$ and replace $(\rho^* \theta)^{\#}$ by $(\rho^* df)^{\#} = X_{\rho^* f}$. It can be checked that this definition does not depend on the choice of f.

According to [X1], for the realization $\rho : X \to M$, the corresponding complete symplectic realization of P is given by $\tilde{\rho} : ((T^* M \times_M P) \times_M X)/T^* M \to P$, where the groupoid $T^* M$ acts on $(T^* M \times_M P) \times_M X$ from the right by $(m, \xi, f, x) \cdot \theta = (m, \xi + \theta, f, (-\theta) \cdot x)$, when $p(\xi) = p(\theta) = \pi(f) = \rho(x) = m$. $((T^* M \times_M P) \times_M X)/T^* M$ is naturally isomorphic to $P \times_M X$ under the correspondence $\mathscr{K} : [m, \xi, f, x] \to (m, f, \xi \cdot x)$, under which $\tilde{\rho}$ goes to the natural projection from $P \times_M X$ to P. In order to analyze the symplectic structure on $P \times_M X$ more carefully, we need to recall the structure of the symplectic manifold $T^* M \times_M P$. According to Weinstein [W4], $T^* M \times_M P$ is a fiber bundle over M. If $\{(U_\alpha, \varphi_\alpha)\}$ is family of local coordinate systems of the fiber bundle $P \xrightarrow{\pi} M$, then over each U_α, $T^* M \times_M P$ is isomorphic to the symplectic direct product $T^* U_\alpha \times F$. The transition between two local coordinate systems is given by

$$\varphi_{\alpha\beta} : \quad (m, \xi, f) \to (m, \xi + [r'_{\alpha\beta}(m)]^* J(f), r_{\alpha\beta}(m) f),$$

where $J : F \to \mathscr{G}^*$ is an equivariant momentum mapping on F and $r_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the gauge transformations of the fiber bundle $P \xrightarrow{\pi} M$. Let $\tau : P \times_M X \to M$ be the natural projection. Then $\tau^{-1}(U_\alpha)$ is symplectically diffeomorphic to the symplectic manifold direct product $F \times \rho^{-1}(U_\alpha)$. As for transitions, notice that $\varphi_{\alpha\beta}$ induces a transition between the corresponding local coordinate charts of the fibre bundle $(T^* M \times_M P) \times_M X$, hence a transition $\tilde{\varphi}_{\alpha\beta}$ between the local coordinate charts of

$$((T^* M \times_M P) \times_M X)/T^* M.$$

Under $\tilde{\varphi}_{\alpha\beta}$, $[m, 0, f, x]$ goes to $[m, [r'_{\alpha\beta}(m)]^* J(f), r_{\alpha\beta}(m) f, x]$. By applying the isomorphism $\mathscr{K}$, $[m, 0, f, x]$ goes to (m, f, x), while $[m, [r'_{\alpha\beta}(m)]^* J(f), r_{\alpha\beta}(m) f, x]$ goes to $(m, r_{\alpha\beta}(m) f, ([r'_{\alpha\beta}(m)]^* J(f)) \cdot x)$. Hence, the transition of $\tau : P \times_M X \to M$ is given by

$$\bar{\varphi}_{\alpha\beta}: \quad (m, f, x) \to (m, r_{\alpha\beta}(m) f, ([r'_{\alpha\beta}(m)]^* J(f)) \cdot x).$$

For instance, if $X = T^* M$, with ρ being the natural projection from $T^* M$ onto M, the groupoid $T^* M \rightrightarrows M$ action on X is just the usual addition of cotangent vectors. Hence, we recover the usual Yang-Mills construction of the phase space $T^* M \times_M P$ [W4], which is just the complete symplectic realization of P corresponding to the symplectic realization $T^* M \to M$. For this reason, we call this construction of the symplectic manifold $P \times_M X$ the generalized Yang-Mills construction. We summarize the results above in the following theorem.

THEOREM 3.1. — *Let* $P \xrightarrow{\pi} M$ *be a locally trivial bundle of simply connected symplectic manifolds. If* $\rho : X \to M$ *is a complete symplectic realization of* M *considered as a zero Poisson manifold, then* $\tilde{\rho} : P \times_M X \to P$, *with* $\tilde{\rho}$ *being the natural projection and* $P \times_M X$ *being the symplectic manifold obtained from the generalized Yang-Mills construction above, is a complete symplectic realization of* P. *Conversely, any complete symplectic realization of* P *is of the above form for some complete symplectic realization of* M, $\rho : X \to M$. *Moreover,* $\tilde{\rho}$ *is full (or surjective) if and only if* ρ *is full (or surjective).*

In particular, we have

COROLLARY 3.1 *If* S *is a simply connected symplectic manifold, then any complete symplectic realization of* S *is of the form* $S \times X \xrightarrow{pr} S$ *for a symplectic manifold* X.

Therefore, the "category" of complete symplectic realizations of S is equivalent to that of all symplectic manifolds.

Remark. — It is essential that realizations be complete in the corollary above. For example, if $U \neq S$ is any open submanifold of S, then the natural inclusion $i : U \to S$ is a symplectic realization, which is obviously not of the form $S \times X$.

Our next example will be the reduced space S/G of a simply connected symplectic Hamiltonian G-space S. First of all, we need a lemma.

LEMMA 3.1 [CDW]. — *Let* G *be a simply connected Lie group with Lie algebra* $\mathscr{G}$, *and let* $\mathscr{G}^*$ *be the dual of* $\mathscr{G}$. *Suppose that* $J : X \to \mathscr{G}^*$ *is a symplectic realization of* $\mathscr{G}^*$. *Then* J *is complete if and only if* X *is a Hamiltonian* G*-space with* J *as its equivariant momentum mapping.*

Proof. — Suppose that $J : X \to \mathscr{G}^*$ is a complete symplectic realization. Consider the symplectic groupoid $T^* G \rightrightarrows \mathscr{G}^*$ over $\mathscr{G}^*$, which is obviously α-simply connected.

It follows from Theorem 3.1 in [X2] that X admits a symplectic action of the groupoid $T^* G$. In other words, X becomes a Hamiltonian G-space with equivariant momentum mapping J. The converse follows from Theorem 3.1 in [X2] as well.

Q.E.D.

THEOREM 3.2. — *Let* S *be a simply connected symplectic Hamiltonian* G*-space with an* *equivariant momentum mapping* J, *such that* J *is a submersion, and* G *and* J*-fibres are simply connected. For any Hamiltonian* G*-space* X, *the natural projection* $p:(S\times X^-)_0\to S/G$ *is a complete symplectic realization of* S/G. *Conversely, any complete symplectic realization of* S/G *is of the above form for some Hamiltonian* G*-space* X *satisfying* $J_1(X)\subseteq J(S)$, *where* J_1 *is an equivariant momentum mapping on* X. *Moreover, p is surjective if and only if* $J_1(X)=J(S)$; *p is full if and only if* J_1 *is a submersion.*

Proof. — According to Corollary 2.5, S^-/G is Morita equivalent to the Poisson submanifold $J(S)\subseteq\mathcal{G}^*$ with equivalence bimodule $\mathcal{G}^*\overset{J}{\leftarrow}S\overset{\pi}{\rightarrow}S^-/G$. Hence, the complete symplectic realizations of S^-/G and J(S) are in one-to-one correspondence. By Lemma 3.1, any complete symplectic realization of J(S) corresponds to a Hamiltonian G-space with an equivariant momentum mapping J_1 satisfying $J_1(X)\subseteq J(S)$. It follows from Theorem 3.1 in [X1] that the corresponding complete symplectic realization of S^-/G is $T^*G\backslash(S^-\times_{\mathcal{G}^*}X)\to S^-/G$, which is clearly isomorphic to $p:(S^-\times X)_0\to S^-/G$. Hence, the corresponding complete symplectic realization of S/G is just $p:(S\times X^-)_0\to S/G$. Finally, notice that for any given symplectic Hamiltonian G-space X with momentum mapping J_1, $X|_{J(S)}=J_1^{-1}(J(S))$ would be such a candidate of Hamiltonian G-space described above. However, it is easy to see that $(S\times X^-|_{J(S)})_0=(S\times X^-)_0$, by definition.

Q.E.D.

For example, corresponding to the Hamiltonian G-space T^*G, the complete symplectic realization of S/G can be easily seen to be $S\overset{\pi}{\rightarrow}S/G$, the most natural one.

4. Covariant symplectic realizations and covariant modules

The notion of covariant symplectic realizations is given by the following:

DEFINITION 4.1. — *Suppose that* P *is a Poisson* G*-space. A symplectic realization* $\rho:X\to P$ *is called covariant if* X *is a Hamiltonian* G*-space so that* ρ *is* G*-equivariant.*

The relation between covariant symplectic realizations and symplectic realizations of semi-direct products of Poisson manifolds is indicated by the following:

PROPOSITION 4.1. — *If* $\rho:X\to P$ *is a* G*-covariant symplectic realization, then the morphism* $\hat\rho:X\to P\times_s\mathcal{G}^*$ *defined by* $\hat\rho(x)=(\rho(x),J(x))$ *is a symplectic realization of* $P\times_s\mathcal{G}^*$, *where* J *is an equivariant momentum mapping on* X. *Conversely, if* $\hat\rho:X\to P\times_s\mathcal{G}^*$ *is a symplectic realization such that* $J=pr_2\circ\hat\rho:X\to\mathcal{G}^*$ *is complete, then* $\rho=pr_1\circ\hat\rho:X\to P$ *is a covariant symplectic realization of* P.

Proof. — By definition, both P and $\mathcal{G}^*$ are Poisson submanifolds of $P\times_s\mathcal{G}^*$. Since ρ and J are Poisson maps, it suffices to show that

$$\hat\rho^*(\{f,g\})=\{J^*f,\rho^*g\} \tag{6}$$

for all $f \in C^\infty(\mathcal{G}^*)$ and $g \in C^\infty(P)$. Since ρ is covariant, for any

$$\xi \in \mathcal{G}, \qquad \rho(\exp t\xi \cdot x) = \exp t\xi \cdot \rho(x).$$

By taking the derivative with respect to t at $t=0$, we obtain the following infinitesimal condition for a covariant realization:

$$T\rho(\xi_X) = \xi_P, \tag{7}$$

where by ξ_X and ξ_P, we denote the corresponding vector fields on X and P induced from the G-actions, respectively.

By l_ξ, we denote the linear function on $\mathcal{G}^*$ corresponding to $\xi \in \mathcal{G}$. Hence, the above equation is equivalent to the equation:

$$T\rho X_{J^* l_\xi} = \xi_P. \tag{8}$$

On the other hand, by the definition of the semi-direct Poisson structure, $\{l_\xi, g\} = (\xi_P) g$ for all $g \in C^\infty(P)$. Therefore,

$$\begin{aligned} \rho^*(\{l_\xi, g\}) &= \rho^*(\xi_P(g)) \\ &= \rho^*(T\rho X_{J^* l_\xi} g) \\ &= X_{J^* l_\xi}(\rho^* g) \\ &= \{J^* l_\xi, \rho^* g\}. \end{aligned}$$

That is, equation (6) holds for all linear functions $\{l_\xi\}$. Hence, it holds for all smooth functions $f \in C^\infty(\mathcal{G}^*)$. Conversely, if $\hat{\rho}: X \to P \times_s \mathcal{G}^*$ is a symplectic realization such that $J = pr_2 \circ \hat{\rho}: X \to \mathcal{G}^*$ is complete, by Lemma 3.1, X becomes a Hamiltonian G-space with J as its equivariant momentum mapping. From the equation

$$\hat{\rho}^*(\{l_\xi, g\}) = \{\hat{\rho}^* l_\xi, \hat{\rho}^* g\} = \{J^* l_\xi, \rho^* g\},$$

it follows immediately that $T\rho(\xi_X) = \xi_P$. Hence, the symplectic realization $\rho: X \to P$ is covariant.

Q.E.D.

Consequently, any complete symplectic realization $\hat{\rho}: X \to P \times_s \mathcal{G}^*$ of $P \times_s \mathcal{G}^*$ corresponds to a complete covariant symplectic realization $\rho = pr_1 \circ \hat{\rho}: X \to P$ of P. Then, it is natural to ask if the converse is still true, that is, if the completeness of both ρ and J can imply the completeness of $\hat{\rho} = \rho \times J$. At this moment, we do not know a direct proof of this. However, we will prove this fact for integrable Poisson manifolds by means of symplectic groupoids. For this purpose, as well as for studying symplectic left modules of the symplectic groupoid $\Gamma(-J) \times G^0$, we need to introduce the following notion of covariant modules.

DEFINITION 4.2. — *Let* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *be a symplectic groupoid. A symplectic left* Γ-*module is called covariant if its momentum mapping is covariant as a symplectic realization of* P.

DEFINITION 4.3. — *A symplectic groupoid* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *is called* G-*covariant if both* Γ *and* P *are Poisson* G-*spaces, such that* α, β *and* $i: P \to \Gamma$ *are* G-*equivariant*.

LEMMA 4.1. — *Let* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *be a* G-*covariant symplectic groupoid,* $\rho: X \to P$ *a* G-*covariant symplectic left* Γ-*module. Suppose that* J_1 *and* J_2 *are equivariant momentum mappings on* Γ *and* X, *respectively. Then*

1. *There is a* $\mu \in \mathscr{G}^*$ *such that* $J_2(r \cdot x) = J_1(r) + J_2(x) - \mu$, *for all* $r \in \Gamma$ *and* $x \in X$ *satisfying* $\beta(r) = \rho(x)$. *In particular, if* $J_1 \in Z^1(\Gamma, \mathscr{G}^*)$, *then* $J_2(r \cdot x) = J_1(r) + J_2(x)$.

2. *For any*

$$(r, x) \in \Gamma *_P X = \{(r, x) \in \Gamma \times X \mid \beta(r) = \rho(x)\} \quad \textit{and} \quad g \in G,$$

we have $g(r \cdot x) = (gr) \cdot (gx)$.

Proof. — It follows from the G-equivariance of the inclusion i that the G-action on Γ leaves P invariant. Therefore, the vector field ξ_Γ for every $\xi \in \mathscr{G}$ is always tangent to P. Since P is a Lagrangian submanifold and $\xi_\Gamma = X_{\hat{J}_1(\xi)}$, $\hat{J}_1(\xi)(r) = \langle J_1(r), \xi \rangle$ is constant along P, as is J_1. Let $\mu = J_1|_P \in \mathscr{G}$. Without loss of generality, we may always assume below that $J_1|_P = 0$, otherwise we only need to consider $J'_1 = J_1 - \mu$.

Let $\Lambda \subset \Gamma \times X \times X^-$ be the graph of the symplectic groupoid action and $J(r, x, y) = J_1(r) + J_2(x) - J_2(y)$. Then J is a momentum mapping on $\Gamma \times X \times X^-$ for the G-action naturally induced from that on Γ and X. For any $\xi \in \mathscr{G}$,

$$\hat{J}(\xi)(r, x, y) = \langle J(r, x, y), \xi \rangle = \hat{J}_1(\xi)(r) + \hat{J}_2(\xi)(x) - \hat{J}_2(\xi)(y).$$

Now consider L, the unique Lagrangian immersion in $\Gamma \times X \times X^-$, maximal among all the Lagrangian immersions contained in $K = \{(r, x, y) \mid \rho(y) = \alpha(r)\}$ and containing $I = \{(u, x, x) \mid \rho(x) = u \in P\}$ as a submanifold, as introduced in Theorem 1.1 in Chapter 3 of [CDW]. It is clear that $\Lambda \subset L$; therefore, it is sufficient to show that J vanishes on L. By construction, it suffices to show that J remains constant along the characteristic foliation of K, which is spanned by all the Hamiltonian vector fields generated by the functions in $\{(\alpha^* h)(r) - (\rho^* h)(y) \mid h \in C^\infty(P)\}$, since J already vanishes on I. However,

$$\begin{aligned}
X_{(\alpha^* h - \rho^* h)} \hat{J}(\xi) &= \{\alpha^* h - \rho^* h, \hat{J}(\xi)\} \\
&= \{\alpha^* h, \hat{J}_1(\xi)\}(r) - \{\rho^* h, \hat{J}_2(\xi)\}(y) \\
&= -\xi_\Gamma(\alpha^* h)(r) + \xi_X(\rho^* h)(y) \\
&= -(T\alpha\, \xi_\Gamma)(h)(\alpha(r)) + (T\rho\, \xi_X)(h)(\rho(y)) \\
&= -(\xi_P h)(\alpha(r)) + (\xi_P h)(\rho(y)) \\
&= 0 \quad (\text{on } K).
\end{aligned}$$

Therefore, we can conclude that $J = 0$ on L, hence on Λ. For the second part, it follows from $J = 0$ on the Lagrangian Λ that $(\xi_\Gamma, \xi_X, \xi_X)$ is tangent to Λ for any $\xi \in \mathscr{G}$. Therefore, its corresponding flow, which is generated by the one parameter group action of $\exp t\xi$, leaves Λ invariant. Since G is connected, the action of G leaves Λ invariant.

Q.E.D.

This lemma yields as an immediate consequence that the group actions on covariant symplectic groupoids with momentum mappings must preserve the groupoid structures. In fact, this is still true without the assumption on the existence of momentum mappings. Although we do not need this fact in the sequel, we would like to list it here as an interesting result.

PROPOSITION 4.2. — *Symplectic groupoid* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *is* G-*covariant if and only if* G *acts on it by symplectic groupoid automorphisms.*

Proof. — It is obvious that Γ is G-covariant if G acts on it by symplectic groupoid automorphisms. For the other direction, let $\Lambda \subset \Gamma \times \Gamma \times \Gamma^-$ be the graph of the groupoid multiplication and $L \subset \Gamma \times \Gamma \times \Gamma^-$ the maximal Lagrangian immersion as in the above proof. Since L is already a graph in this case [CDW], clearly $\Lambda = L$. It follows from the G-covariance of the groupoid that $g\Lambda$ is also contained in K and contains I as a submanifold. Hence, $g\Lambda \subseteq L = \Lambda$. Therefore, $g\Lambda = \Lambda$.

Q.E.D.

Remark. — This result follows essentially from the fact that the groupoid multiplication on Γ is determined by α, β and $i(P) \subset \Gamma$ [CDW].

Now we are in the position to prove the result claimed in the remarks following the proof of Theorem 2.1.

THEOREM 4.1. — *Suppose that symplectic groupoids*

$$(H \rightrightarrows H_0, \alpha_1, \beta_1) \qquad \textit{and} \qquad (K \rightrightarrows K_0, \alpha_2, \beta_2)$$

are Morita equivalent with an equivalence bimodule $(X; \rho; \sigma)$. *Suppose also that both* H *and* K *are* G-*covariant symplectic groupoids with equivariant momentum mappings* $J_1 \in Z^1(H, \mathcal{G}^*)$ *and* $J_2 \in Z^1(K, \mathcal{G}^*)$ *respectively, and both* $\rho: X \to H_0$ *and* $\sigma: X \to K_0$ *are* G-*covariant symplectic realizations. Then* $(H)_0 \rightrightarrows H_0/G$ *and* $(K)_0 \rightrightarrows K_0/G$ *are Morita equivalent with equivalence bimodule* $((X)_0; \tilde{\rho}; \tilde{\sigma})$, *provided that both* $\tilde{\rho}$ *and* $\tilde{\sigma}$ *are surjective submersions, where* $\tilde{\rho}: (X)_0 \to H_0/G$ *and* $\tilde{\sigma}: (X)_0 \to K_0/G$ *are morphisms naturally induced from* ρ *and* σ *respectively, and the left and right groupoid actions on* $(X)_0$ *are defined in an obvious way, respectively, by* $[h]\cdot[x] = [h\cdot x]$ *and* $[x]\cdot[k] = [x\cdot k]$ *if* $\beta_1(h) = \rho(x)$ *and* $\sigma(x) = \alpha_2(k)$.

Proof. — It follows from Lemma 4.1 that both the left and right groupoid actions on $(X)_0$ are well-defined. In order to show that the left $(H)_0$-action on $(X)_0$ is free, let us assume that $[h]\cdot[x] = [x]$, where $\beta_1(h) = \rho(x)$. That is, $[h\cdot x] = [x]$. Hence, there is an element $g \in G$ such that $h\cdot x = gx$. Thus, $\sigma(x) = \sigma(h\cdot x) = \sigma(gx) = g\sigma(x)$. Hence, $g = \mathrm{id}$. Since the left H-action on X is free, it follows immediately that $h \in H_0$. Hence, the left groupoid action on $(X)_0$ is free. Similarly, the right $(K)_0$-action on $(X)_0$ is also free.

As for the transitivity of the left groupoid action on $\tilde{\sigma}$-fibres, let us assume that $[x_1]$, $[x_2] \in (X)_0$ such that $\tilde{\sigma}([x_1]) = \tilde{\sigma}([x_2])$, *i.e.*, $[\sigma(x_1)] = [\sigma(x_2)]$. Hence, there is an element $g \in G$ such that $\sigma(x_1) = g\sigma(x_2) = \sigma(gx_2)$. As the left groupoid action of H on X is transitive on σ-fibres, there is a $h \in H$ such that $h\cdot x_1 = gx_2$. Suppose that $J: X \to \mathcal{G}^*$ is an equivariant momentum mapping on X. It follows from Lemma 4.1 that

$J(h\cdot x_1)=J_1(h)+J(x_1)=J_1(h)$. On the other hand, we have $J(gx_2)=Ad_g^* J(x_2)=0$. Hence, $h\in J_1^{-1}(0)$, and $[h]\cdot[x_1]=[x_2]$. Therefore, the left $(H)_0$-action on $(X)_0$ is transitive on each $\tilde{\sigma}$-fibre. The transitivity of the right groupoid action on $\tilde{\rho}$-fibres of $(X)_0$ can be proved similarly.

Finally, it is trivial to show directly that both the left $(H)_0$-action and the right $(K)_0$-action on $(X)_0$ are symplectic.

Q.E.D.

Remark. — The corresponding C*-version of this result should be interesting, too. Roughly speaking, the C*-version can be stated as follows: the fixed point algebras of strongly Morita equivalent C*-algebras are still strongly Morita equivalent. If this can be formulated rigorously, we believe that we can obtain some interesting Morita equivalence relations, including the one between a certain ideal of the crossed product algebra and the generalized fixed-point algebra [Rie4], and the one between the crossed products of Morita equivalent C*-algebras ([Co], [CMW]), at least in the case where the group is compact.

THEOREM 4.2. — *Suppose that* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *is a* G-*covariant symplectic groupoid with an equivariant momentum mapping* $J\in Z^1(\Gamma, \mathscr{G}^*)$, *and* $\rho: X\to P$ *is a covariant symplectic left* Γ-*module with momentum mapping* J_2. *Then* X *becomes a symplectic left module of the symplectic groupoid*

$$(\Gamma(-J)\times_s G^0 \rightrightarrows P\times_s \mathscr{G}^*, \alpha_1, \beta_1)$$

under the action $(\gamma, u, g)\cdot x=\gamma\cdot g^{-1}x$, *for any* $x\in X$, $\gamma\in\Gamma$, $u\in\mathscr{G}^*$ *and* $g\in G$ *such that* $\rho(x)=g\beta(\gamma)$ *and* $J_2(x)=Ad_g^*(u-J(\gamma))$, *with the momentum mapping* $\hat{\rho}: X\to P\times_s\mathscr{G}^*$ *being given by* $\hat{\rho}(x)=(\rho(x), J_2(x))$.

Proof. — As proved in Theorem 1.2, the symplectic groupoid

$$(\Gamma(-J)\times_s G^0 \rightrightarrows P\times_s \mathscr{G}^*, \alpha_1, \beta_1)$$

is isomorphic to the symplectic groupoid $((\Gamma\times T^*G\times(T^*G)^-)_0 \rightrightarrows (P\times T^*G)/G, \alpha_2, \beta_2)$. Consider the morphism $\tilde{\rho}: (X\times T^*G)_0\to(P\times T^*G)/G$ given by $\tilde{\rho}([x, \theta_g])=[\rho(x), \theta_g]$, and define an action of the groupoid $(\Gamma\times T^*G\times(T^*G)^-)_0$ on $(X\times T^*G)_0$ in an evident way, namely by $[\gamma, a, b]\cdot[x, b]=[\gamma\cdot x, a]$ for any composable $[\gamma, a, b]\in(\Gamma\times T^*G\times(T^*G)^-)_0$ and $[x, b]\in(X\times T^*G)_0$. By using Lemma 4.1, it is easy to check that this action is well-defined and in fact is a symplectic action. According to Lemma 2.1, $(X\times T^*G)_0$ is symplectically diffeomorphic to X under the correspondence

$$\mathscr{F}: [x, Ad_g^* J_2(x), g]\to gx.$$

Here again T^*G has been identified with $\mathscr{G}^*\times G$ by right translations. It can be easily checked that $\hat{\rho}\circ\mathscr{F}=\Phi\circ\tilde{\rho}$, where Φ is the natural isomorphism from $(P\times T^*G)/G$ to $P\times_s\mathscr{G}^*$, introduced by equation (4). As for the groupoid action, we map (r, u, g) under Ψ^{-1} [*cf.* equation (3)] to $[\gamma, u, 1_G, Ad_g^*(u-J(\gamma)), g]$ in $(\Gamma\times T^*G\times(T^*G)^-)_0$ and map x

under $\mathscr{F}^{-1}$ to $[g^{-1}x, J_2(x), g]$ in $(X \times T^* G)_0$. Therefore, if $\rho(x) = g\beta(\gamma)$ and $J_2(x) = \mathrm{Ad}_g^*(u - J(\gamma))$, we have

$$[r, u, 1_G, \mathrm{Ad}_g^*(u - J(\gamma)), g] \cdot [g^{-1}x, J_2(x), g] = [r \cdot g^{-1}x, u, 1_G],$$

which corresponds to $\gamma \cdot g^{-1}x$ in X under $\mathscr{F}$. Therefore, $(\gamma, u, g) \cdot x = \gamma \cdot g^{-1}x$.

Q.E.D.

In fact, the converse of Theorem 4.2 is also true. In order to show this, we need the following lemma describing explicitly the symplectic structure on $\Gamma(-J) \times_s G^0$.

LEMMA 4.2. – *Let Ω be the symplectic structure on $\Gamma(-J) \times_s G^0 (\cong \Gamma \times \mathscr{G}^* \times G)$*

(1) *For all δu, $\delta' u \in T_u \mathscr{G}^* (\cong \mathscr{G}^*)$, $\delta\gamma$, $\delta'\gamma \in T_\gamma \Gamma$ and δg, $\delta' g \in T_g G$, we have*

$$\Omega((\delta\gamma, \delta u, \delta g), (\delta'\gamma, \delta' u, \delta' g)) = \omega_\Gamma(\delta\gamma, \delta'\gamma) - \langle \delta' u_2, \delta g \cdot g^{-1} \rangle + \langle \delta u_2, \delta' g \cdot g^{-1} \rangle + \langle u, [\delta g \cdot g^{-1}, \delta' g \cdot g^{-1}] \rangle,$$

where $u_2 = \mathrm{Ad}_g^*(u - J(\gamma))$.

(2) *In particular, if $\delta g = \delta' g = 0$, then*

$$\Omega((\delta\gamma, \delta u, 0), (\delta'\gamma, \delta' u, 0)) = \omega_\Gamma(\delta\gamma, \delta'\gamma).$$

THEOREM 4.3. – *There is a one-to-one correspondence between symplectic left modules of the groupoid $\Gamma(-J) \times_s G^0$ and G-covariant symplectic left Γ-modules.*

Proof. – We only need to prove the other direction. Suppose that $\hat{\rho}: X \to P \times_s \mathscr{G}^*$ is a symplectic left module of $\Gamma(-J) \times_s G^0$. Let $\rho = pr_1 \circ \hat{\rho}$ and $J_2 = pr_2 \circ \hat{\rho}$, where pr_1 and pr_2 are the natural projections from $P \times_s \mathscr{G}^*$ onto P and $\mathscr{G}^*$, respectively. Define a G-action on X by

$$g \cdot x = (g \cdot \rho(x), \mathrm{Ad}_g^* J_2(x), g^{-1}) \cdot x,$$

where the dot on the right side means the groupoid $\Gamma(-J) \times_s G^0 (\cong \Gamma \times \mathscr{G}^* \times G)$ action. It is trivial to check that this is a well-defined group action. More precisely, it is a Hamiltonian action with momentum mapping J_2. In order to show this, we let $\Lambda \subset (\Gamma(-J) \times_s G^0) \times X \times X^-$ be the graph of the groupoid action and δx, $\delta' x$ be two arbitrary tangent vectors at $x \in X$. It is clear that both

$$[((T g \circ T \rho) \delta x, ((T \mathrm{Ad}_g^*) \circ TJ_2) \delta x, 0), \delta x, (Tg) \delta x]$$

and

$$[((T g \circ T \rho) \delta' x, ((T \mathrm{Ad}_g^*) \circ TJ_2) \delta' x, 0), \delta' x, (T g) \delta' x]$$

are tangent to Λ. According to Lemma 4.2,

$$\begin{aligned}\Omega(((T g \circ T \rho) \delta x, ((T \mathrm{Ad}_g^*) \circ TJ_2) \delta x, 0), ((T g \circ T \rho) \delta' x, ((T \mathrm{Ad}_g^*) \circ TJ_2) \delta' x, 0)) \\ = \omega_\Gamma((T g \circ T \rho) \delta x, (T g \circ T \rho) \delta' x) \\ = 0 \quad \text{(since P is a Lagrangian of } \Gamma).\end{aligned}$$

However, since Λ is a Lagrangian, it follows immediately that

$$\omega_X(\delta x, \delta' x) - \omega_X((Tg)\delta x, (Tg)\delta' x) = 0.$$

That is, the G-action on X is symplectic.

For any $\xi \in \mathscr{G}$ and $x \in X$, consider a smooth path

$$(\exp t\xi \cdot \rho(x), \operatorname{Ad}^*_{\exp t\xi} J_2(x), \exp(-t\xi))$$

in $\Gamma(-J) \times_s G^0 (\cong \Gamma \times \mathscr{G}^* \times G)$ and let v be its derivative at $t=0$, *i.e.*, $v = (\xi_P(\rho(x)), \operatorname{Ad}^*_\xi J_2(x), -\xi)$. As in Lemma 4.2, $u_2 = \operatorname{Ad}^*_{\exp(-t\xi)} \operatorname{Ad}^*_{\exp t\xi} J_2(x)$. Hence, $\delta u_2 = 0$. On the other hand, let $x(t)$ be any smooth path in X starting at x and $\delta x = x'(0)$, then $(\rho(x(t)), J_2(x(t)), 1_G)$ is a path in $\Gamma(-J) \times_s G^0 (\cong \Gamma \times \mathscr{G}^* \times G)$ and $v' = (T\rho\delta x, TJ_2 \delta x, 0)$ is its tangent vector at $t=0$. For this vector v', the corresponding u'_2 in Lemma 4.2 is given by $u'_2 = J_2(x(t)) - J(\rho(x(t))) = J_2(x(t))$ so that $\delta u'_2 = TJ_2 \delta x$. It follows from Lemma 4.2 that

$$\begin{aligned}\Omega(v, v') &= \omega_\Gamma(\xi_P(\rho(x)), T\rho\delta x) - \langle TJ_2 \delta x, -\xi \rangle \\ &= \langle TJ_2 \delta x, \xi \rangle \\ &= \delta x(\hat{J}_2(\xi)).\end{aligned}$$

Since $\exp t\xi \cdot x = (\exp t\xi \cdot \rho(x), \operatorname{Ad}^*_{\exp t\xi} J_2(x), \exp(-t\xi)) \cdot x$ by definition, $(v, 0, \xi_X(x))$ is tangent to Λ. Similarly, it follows from the equation $(\rho(x), J_2(x), 1_G) \cdot x = x$ that $(v', \delta x, \delta x)$ is also tangent to Λ. Hence,

$$\Omega(v, v') - \omega_X(\xi_X, \delta x) = 0, \quad i.e., \quad \delta x(\hat{J}_2(\xi)) - \omega_X(\xi_X, \delta x) = 0.$$

For any $f \in C^\infty(X)$, setting $\delta x = X_f$ leads to $\xi_X(f) = X_{\hat{J}_2(\xi)} f$. Therefore, J_2 is a momentum mapping of the G-action on X. Moreover, since

$$\begin{aligned}\rho(g \cdot x) &= (\operatorname{pr}_1 \circ \hat{\rho})[(g \cdot \rho(x), \operatorname{Ad}^*_g J_2(x), g^{-1}) \cdot x] \\ &= \operatorname{pr}_1[\alpha(g \cdot \rho(x)), \operatorname{Ad}^*_g J_2(x)] \\ &= g \cdot \rho(x),\end{aligned}$$

the G-action on X is covariant. Similarly, we can show that $J_2(g \cdot x) = \operatorname{Ad}^*_g J_2(x)$, in other words, J_2 is an equivariant momentum mapping.

Next we define a groupoid Γ-action on X by

$$\gamma \cdot x = (\gamma, J(\gamma) + J_2(x), 1_G) \cdot x, \qquad \text{if} \quad \beta(\gamma) = \rho(x).$$

It is trivial to check that this is a well-defined groupoid action. By Λ_1, we denote the graph of this groupoid action, which is a submanifold of $\Gamma \times X \times X^-$. Let $(\delta\gamma, \delta x, \delta(\gamma x))$ and $(\delta'\gamma, \delta' x, \delta'(\gamma x))$ be two arbitrary tangent vectors of Λ_1 at $(\gamma, x, \gamma x)$. Then both

$$((\delta\gamma, TJ\delta\gamma + TJ_2 \delta x, 0), \delta x, \delta(\gamma x)) \qquad \text{and} \qquad ((\delta'\gamma, TJ\delta'\gamma + TJ_2 \delta' x, 0), \delta' x, \delta'(\gamma x))$$

are tangent to Λ. By Lemma 4.2,

$$\Omega((\delta\gamma, TJ\delta\gamma + TJ_2 \delta x, 0), (\delta'\gamma, TJ\delta'\gamma + TJ_2 \delta' x, 0)) = \omega_\Gamma(\delta\gamma, \delta'\gamma).$$

Since Λ is a Lagrangian, it follows immediately that

$$\omega_\Gamma(\delta\gamma, \delta'\gamma) + \omega_X(\delta x, \delta' x) - \omega_X(\delta(\gamma x), \delta'(\gamma x)) = 0,$$

i.e., Λ_1 is a Lagrangian. So the groupoid Γ-action on X is symplectic.

Finally, it is quite easy to check directly that if

$$\beta(\gamma) = g^{-1}\rho(x) \qquad \text{and} \qquad \mathrm{Ad}_g^*(u - J(\gamma)) = J_2(x),$$

then $(\gamma, u, g)\cdot x = \gamma \cdot g^{-1} x$.

Q.E.D.

The following result is an immediate consequence of Theorem 4.2.

COROLLARY 4.1. — *If* P *is an integrable Poisson manifold and* $\rho : X \to P$ *is a complete covariant symplectic realization of* P, *then the symplectic realization* $\hat{\rho} : X \to P \times_s \mathcal{G}^*$ *defined by* $\hat{\rho}(x) = (\rho(x), J(x))$ *is also complete, where* $J : X \to \mathcal{G}^*$ *is an equivariant momentum mapping on* X.

Proof. — Let Γ be an α-simply connected symplectic groupoid of P. According to [WX], G has a natural action on Γ by symplectic groupoid automorphisms with an equivariant momentum mapping $J' \in Z^1(\Gamma, \mathcal{G}^*)$, and by Thorem 3.1 in [X2], X naturally becomes a covariant symplectic left Γ-module. It follows from Theorem 4.2 that $\hat{\rho} : X \to P \times_s \mathcal{G}^*$ becomes a symplectic left module of the groupoid $\Gamma(-J') \times_s G^0$. Hence, $\hat{\rho}$ must be complete [X2].

Q.E.D.

Combining Proposition 4.1 with Corollary 4.1 leads to the following:

THEOREM 4.4. — *If* P *is an integrable Poisson* G-*space, then there is a one-to-one correspondence between complete covariant symplectic realizations of* P *and complete symplectic realizations of* $P \times_s \mathcal{G}^*$.

Remark. — The C*-analogue of this result is the well-known theorem [Pe] that the covariant representations of a C*-algebra are in one-to-one correspondence with the representations of the crossed product C*-algebra.

5. Complete symplectic realizations of P/G

This section is devoted to the discussion of complete symplectic realizations of reduced Poisson manifolds, as well as symplectic modules of reduced symplectic groupoids $(\Gamma)_0$.

THEOREM 5.1. — *Suppose that* $(\Gamma \rightrightarrows P, \alpha, \beta)$ *is a symplectic groupoid, on which* G *acts freely and properly by symplectic groupoid automorphisms with an equivariant momentum mapping* $J \in Z^1(\Gamma, \mathcal{G}^*)$. *Assume that the map* $\rho : \Gamma \to P \times_s \mathcal{G}^*$ *given by* $\rho(\gamma) = (\alpha(\gamma), J(\gamma))$ *is a submersion. If* $\varphi : X \to P$ *is a covariant symplectic left* Γ-*module with an equivariant momentum mapping* J_1, *then* $\tilde{\varphi} : (X)_0 \to P/G$, *with* $\tilde{\varphi}$ *the map naturally induced from* φ, *is a symplectic left module of the symplectic groupoid* $((\Gamma)_0 \rightrightarrows P/G, \alpha_2, \beta_2)$ *under the natural action given by* $[\gamma]\cdot[x] = [\gamma \cdot x]$, *where* $\beta(\gamma) = \varphi(x)$. (*Here* $(X)_0$ *always exists as a* *manifold.) Conversely, every symplectic left module of* $((\Gamma)_0 \rightrightarrows P/G, \alpha_2, \beta_2)$ *is of the above form for some covariant symplectic left* Γ*-module* $\varphi : X \to P$. *Moreover,* $\tilde{\varphi}$ *is full if and only if* $\hat{\varphi} : X \to P \times_s \mathcal{G}^*$, $\hat{\varphi}(x) = (\varphi(x), J_1(x))$ *is a submersion;* $\tilde{\varphi}$ *is surjective if and only if* $\hat{\varphi}(X) = \rho(\Gamma)$.

Proof. — According to Theorem 2.2, the symplectic groupoid $((\Gamma)_0 \rightrightarrows P/G, \alpha_2, \beta_2)$ is Morita equivalent to $(\Gamma(-J) \times_s G^0)|_{\rho(\Gamma)}$, a symplectic subgroupoid of

$$(\Gamma(-J) \times_s G^0 \rightrightarrows P \times_s \mathcal{G}^*, \alpha_1, \beta_1).$$

Hence, every symplectic left module of $(\Gamma)_0$ corresponds to a unique symplectic left module of $(\Gamma(-J) \times_s G^0)|_{\rho(\Gamma)}$ and vice versa. However, it follows from Theorem 4.3 that there is a one-to-one correspondence between symplectic left modules of $(\Gamma(-J) \times_s G^0)|_{\rho(\Gamma)}$ and G-covariant symplectic left Γ-modules $\varphi : X \to P$ satisfying $\hat{\varphi}(X) \subseteq \rho(\Gamma)$, where $\hat{\varphi} : X \to P \times_s \mathcal{G}^*$ is given by $\hat{\varphi}(x) = (\varphi(x), J_1(x))$ with J_1 being the momentum mapping on X. Therefore, given any such G-covariant symplectic left Γ-module $\varphi : X \to P, \bar{\varphi} : (\Gamma(-J) \times_s G^0) \backslash (\Gamma^- *_{(P \times_s \mathcal{G}^*)} X) \to P/G$ becomes a symplectic left module of $(\Gamma)_0$, and conversely, any symplectic left module of $(\Gamma)_0$ must have such a form. It remains for us to show that $\bar{\varphi} : (\Gamma(-J) \times_s G^0) \backslash (\Gamma^- *_{(P \times_s \mathcal{G}^*)} X) \to P/G$ and $\tilde{\varphi} : (X)_0 \to P/G$ are isomorphic as symplectic left modules.

Introduce a map $\Phi : \Gamma^- *_{(P \times_s \mathcal{G}^*)} X \to (X)_0 (= J_1^{-1}(0)/G)$ by $\Phi(r, x) = [r^{-1} \cdot x]$. For any $x \in J_1^{-1}(0)$, Let $u = \varphi(x) \in P$. Then $(u, x) \in \Gamma^- *_{(P \times_s \mathcal{G}^*)} X$ since $\rho(u) = (u, 0) = \hat{\varphi}(x)$. Obviously, $\Phi(u, x) = [x]$, so Φ is surjective. Now it suffices to show that Φ-fibres coincide with the orbits of the groupoid $\Gamma(-J) \times_s G^0$ action. Suppose that $(r_1, x_1) = (r', u', g') \cdot (r, x)$. Then by Lemma 4.1,

$$\begin{aligned} r_1^{-1} \cdot x_1 &= [r' \cdot (g'^{-1} r)]^{-1} \cdot [r' \cdot (g'^{-1} x)] \\ &= (g'^{-1} r)^{-1} \cdot (g'^{-1} x) \\ &= g'^{-1} (r^{-1} \cdot x). \end{aligned}$$

Conversely, if $\Phi(r_1, x_1) = \Phi(r_2, x_2)$, then we may assume that $r_1^{-1} \cdot x_1 = g(r_2^{-1} \cdot x_2)$. Take $r' = r_2 \cdot (g^{-1} r_1^{-1})$, $u' = J_1(r_2)$ and $g' = g$. It is easy to check that

$$\beta_1(r', u', g') = \hat{\varphi}(x_1) = \rho(r_1).$$

Then, clearly

$$\begin{aligned} r' \cdot (g')^{-1} x_1 &= [r_2 \cdot (g^{-1} r_1^{-1})] \cdot g^{-1} x_1 \\ &= r_2 \cdot [g^{-1} (r_1^{-1} \cdot x_1)] \\ &= r_2 \cdot (r_2^{-1} \cdot x_2) \\ &= x_2. \end{aligned}$$

On the other hand, it is quite obvious that $r' \cdot (g')^{-1} r_1 = r_2$. Hence, we have

$$(r', u', g') \cdot (r_1, x_1) = (r_2, x_2).$$

The diffeomorphism $\hat{\Phi}$ from $(\Gamma(-J) \times_s G^0) \backslash (\Gamma^- *_{(P \times_s \mathcal{G}^*)} X)$ onto $(X)_0$ induced from Φ clearly preserves symplectic structures, since the symplectic structures on both

$(\Gamma(-J)\times_s G^0)\backslash(\Gamma^- *_{(P\times_s \mathscr{G}^*)} X)$ and $(X)_0$ are induced from that of X. It is quite easy to check that the following diagram

$$\begin{array}{ccc} (\Gamma(-J)\times_s G^0)\backslash(\Gamma^- *_{(P\times_s \mathscr{G}^*)} X) & \stackrel{\hat{\Phi}}{\rightarrow} & (X)_0 \\ \bar{\varphi}\downarrow & & \downarrow\tilde{\varphi} \\ P/G & \stackrel{\mathrm{id}}{\rightarrow} & P/G \end{array}$$

commutes and the groupoid action of $(\Gamma)_0$ on $(X)_0$ becomes the natural one defined in the theorem.

Finally, note that for any covariant symplectic left Γ-module

$$\varphi: X\rightarrow P, X|_{\rho(\Gamma)}=\hat{\varphi}^{-1}(\rho(\Gamma))$$

would be such a covariant symplectic left Γ-module satisfying $\hat{\varphi}(X|_{\rho(\Gamma)})\subseteq\rho(\Gamma)$. It is simple to see that $(X|_{\rho(\Gamma)})_0=(X)_0$.

Q.E.D.

With these preliminaries taken care of, we are ready to prove the main theorem of this section.

THEOREM 5.2. — *Let* P *be an integrable Poisson* G*-space where the action is proper and free, and let* $(\Gamma\rightrightarrows P, \alpha, \beta)$ *be its* α*-simply connected symplectic groupoid, on which* G *acts by symplectic groupoid automorphisms with an equivariant momentum mapping* $J\in Z^1(\Gamma, \mathscr{G}^*)$. *Assume that the map* $\rho:\Gamma\rightarrow P\times_s\mathscr{G}^*$ *given by* $\rho(\gamma)=(\alpha(\gamma), J(\gamma))$ *is a submersion, and* G *and* $\alpha^{-1}(p)\cap J^{-1}(u)$ *are connected and simply connected for all* $p\in P$ *and* $u\in\mathscr{G}^*$: *Then* $\tilde{\varphi}:Y\rightarrow P/G$ *is a complete symplectic realization if and only if* $Y=(X)_0$ *for some complete covariant symplectic realization* $\varphi:X\rightarrow P$, *and* $\tilde{\varphi}=[\pi\circ\varphi]$, *where* $\pi:P\rightarrow P/G$ *is the natural projection. Moreover,* $\tilde{\varphi}$ *is full if and only if* $\hat{\varphi}:X\rightarrow P\times_s\mathscr{G}^*$, $\hat{\varphi}(x)=(\varphi(x), J_1(x))$ *is a submersion, where* $J_1:X\rightarrow\mathscr{G}^*$ *is a momentum mapping on* X; $\tilde{\varphi}$ *is surjective if and only if* $\hat{\varphi}(X)=\rho(\Gamma)$.

Proof. — Under the assumption above, $(\Gamma)_0\rightrightarrows P/G$ is an α-simply connected symplectic groupoid over P/G. Thus, the conclusion follows immediately from Theorem 3.1 in [X2] and Theorem 5.1.

Q.E.D.

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